Let x and y be random bitstrings and n be the SDM's number of bits; let x_i and y_i be the ith bit of x and y, respectivelly; and $d(x, y)$ be the Hamming distance.

Applying the law of total probability:

$$P(x_i = y_i | d(x, y) \leq r) = \sum_{k=0}^{r} P(x_i = y_i | d(x, y) = k \leq r) \cdot P(d(x, y) = k | d(x, y) \leq r) \tag{1}$$

We also know that

$$P(x_i = y_i | d(x, y) = k) = \frac{n-k}{n} \tag{2}$$

$$P(d(x, y) = k | d(x, y) \leq r) = \frac{\binom{n}{k}}{\sum_{j=0}^{r} \binom{n}{j}} \tag{3}$$

Hence,

$$P(x_i = y_i | d(x, y) \leq r) = \frac{\sum_{k=0}^{r} \frac{n-k}{n} \binom{n}{k}}{\sum_{j=0}^{r} \binom{n}{j}} \tag{4}$$

Finally, as $\frac{n-k}{n} \binom{n}{k} = \binom{n-1}{k}$,

$$P(x_i = y_i | d(x, y) \leq r) = \frac{\sum_{k=0}^{r} \binom{n-1}{k}}{\sum_{k=0}^{r} \binom{n}{k}} \tag{5}$$

This equation is valid for both "x at x" (autoassociative memory) and "random at x" (heteroassociative memory). When $n = 1,000$ and $r = 451$, $P(x_i = y_i | d(x, y) \leq r) = p = 0.552905498137$.

1 Counter bias

The bias begins in the counters. Let's analyze ith counter of a hard location.

Let s be the number of bitstrings written into memory (in our case, $s = 10,000$), h be the average number of activated hard locations ($h = 1,071.85$),

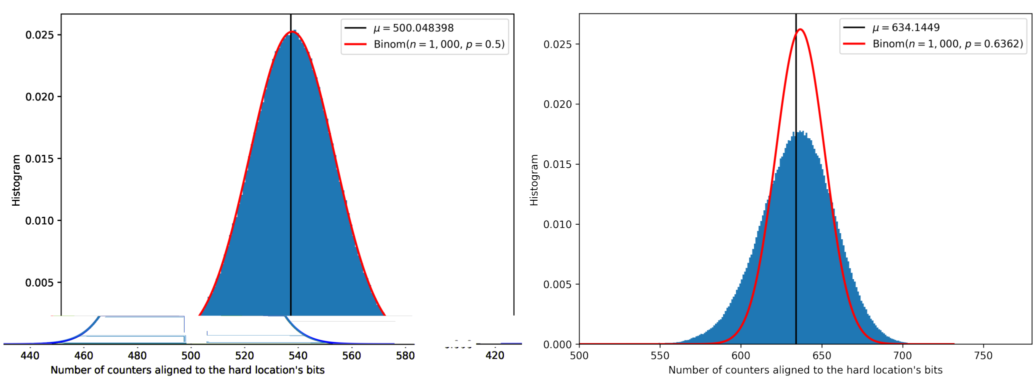
H be the number of hard locations ($H = 1,000,000$), and addr_i be the ith bit of the hard location's address. Then,

$$P(\text{cnt}_i > 0|\text{addr}_i = 1) = P(\text{cnt}_i < 0|\text{addr}i = 0) = \sum_{k=1}^{\theta} X_k > \frac{sh}{2H} \qquad (6)$$

Where $\theta = \frac{sh}{H}$ is the average number of bitstrings written in each hard location, and $X_k \sim \text{Bernoulli}(p)$ (where $p = P(x_i = y_i|d(x,y) \leq r)$). Thus, $\sum_{k=1}^{\theta} X_k \sim \text{Binomial}\,(\theta, p)$.

Finally, $P(\text{cnt}_i > 0|\text{addr}_i = 1) = P(\text{cnt}_i < 0|\text{addr}_i = 0) = \text{Binomial.cdf}\left(\frac{sh}{2H}\right)$.

For "random at x", $p = 0.5$, $P(\text{cnt}_i > 0|\text{addr}_i = 1) = P(\text{cnt}_i < 0|\text{addr}_i = 0) = 0.5$, independently of the parameters. But, for "x at x", $p = 0.5529$ and the probabilities depend on s. For $s = 10,000$, they are equal to 0.6362. For $s = 20,000$, they are equal to 0.6888. For $s = 30,000$, they are equal to 0.7268. The more random bitstrings are written into the memory, the more a hard location to point to itself. See Figure **??** — and notice that I still have to figure out why the mean is correct, but the standard deviation is not. As each of the n counters of a hard location may be equal or not with the same probability, I assumed it would follow a Binomial distribution (and it worked for "random at x").



(a) "random at x" (b) "x at x"

Figure 1: Autocorrelation in the counters in autoassociative memories ("x at x"). The histogram was obtained through simulation. The red curve is the

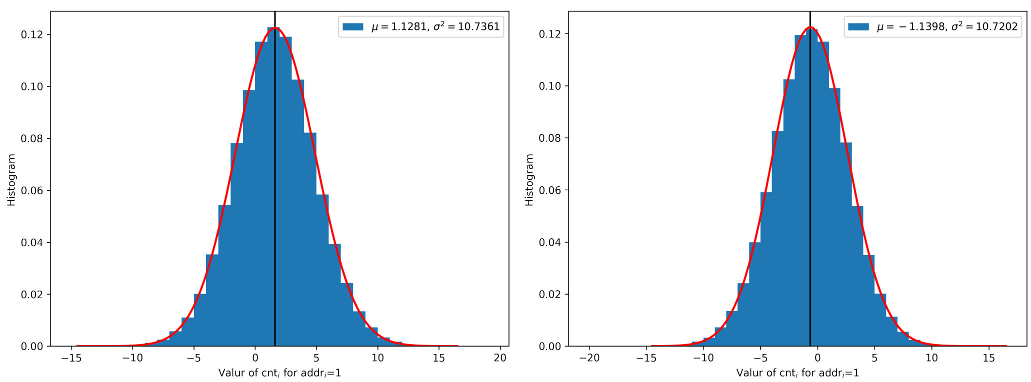
Not let's analyze the value of the counters for "x at x". In write operation, the counters are incremented for every bit 1 and decremented for every bit 0. So, after s writes, there will be θ bitstrings written in each hard location, $Y_i = \sum_{k=1}^{\theta} X_k$ bits 1, and $\theta - Y_i$ bits 0. Thus, when $\text{addr}_i = 1$, $\text{cnt}_i = (Y_i) - (\theta - Y_i) = 2Y_i - \theta$. And, when $\text{addr}_i = 0$, $\text{cnt}_i = \theta - 2Y_i$.

As $Y_i \sim \text{Binomial}(\theta, p)$, for θ sufficiently large, we may approximate $Y_i \sim \mathcal{N}(\mu = \theta p, \sigma^2 = \theta p(1-p))$. Hence, as $\text{cnt}_i = 2Y_i - \theta$, $\mathbf{E}[2Y_i - \theta] = 2\mathbf{E}[Y_i] - \theta$, and $\mathbf{V}[2Y_i - \theta] = 4\mathbf{V}[Y_i]$, then,

$$[\text{cnt}_i|\text{addr}_i = 1] \sim \mathcal{N}(\mu = (2p-1)\theta, \sigma^2 = 4\theta p(1-p)) \tag{7}$$

$$[\text{cnt}_i|\text{addr}_i = 0] \sim \mathcal{N}(\mu = -(2p-1)\theta, \sigma^2 = 4\theta p(1-p)) \tag{8}$$

In our case, $p = 0.5529$, $s = 10,000$, $h = 1,071.85$, and $H = 1,000,000$, so $\theta = 10.7185$ and $\text{cnt}_i \sim \mathcal{N}(\mu = 1.1341, \sigma^2 = 10.5985)$. For "random at x", $p = 0.5$, so $\mu = 0$ and $\sigma^2 = \theta$. See Figure **??**.



(a) $\text{addr}_i = 1$ (b) $\text{addr}_i = 0$

Figure 2: Autocorrelation in the counters in autoassociative memories ("x at x"). The histogram was obtained through simulation. The red curve is the theoretical normal distribution.

2 Read bias

Now that we know the distribution of $\text{cnt}_i|\text{addr}_i$, we may go to the read operation. During the read operation, on average, h

and their counters are summed up. So, for the ith bit,

$$\text{acc}_i = \sum_{k=1}^{h} \text{cnt}_k \tag{9}$$

Let η be the reading address and η_i the ith bit of it. Then,

$$[\text{acc}_i|\eta_i = 1] = \sum_{k=1}^{ph} [\text{cnt}_k|\text{addr}_k = 1] + \sum_{k=1}^{(1-p)h} [\text{cnt}_k|\text{addr}_k = 0] \tag{10}$$

$$[\text{acc}_i|\eta_i = 0] = \sum_{k=1}^{ph} [\text{cnt}_k|\text{addr}_k = 0] + \sum_{k=1}^{(1-p)h} [\text{cnt}_k|\text{addr}_k = 1] \tag{11}$$

We may analyze only one case, because the probability of the other is exactly the same.

Each sum is a sum of normally distributed random variables, so

$$\sum_{k=1}^{ph} [\text{cnt}_k|\text{addr}_k = 1] \sim \mathcal{N}(\mu = (2p-1)\theta ph, \sigma^2 = 4\theta p(1-p)ph) \tag{12}$$

$$\sum_{k=1}^{(1-p)h} [\text{cnt}_k|\text{addr}_k = 0] \sim \mathcal{N}(\mu = -(2p-1)\theta(1-p)h, \sigma^2 = 4\theta p(1-p)(1-p)h) \tag{13}$$

In our case, $\sum_{k=1}^{ph} [\text{cnt}_k|\text{addr}_k = 1] \sim \mathcal{N}(\mu = 672.12, \sigma^2 = 6281.00)$, and $\sum_{k=1}^{ph} [\text{cnt}_k|\text{addr}_k = 1] \sim \mathcal{N}(\mu = -543.49, \sigma^2 = 5078.99)$. See Figure **??** — we can notice there a small but significant difference between the theoretical and the simulated mean.

Hence,

$$[\text{acc}_i|\eta_i = 1] \sim \mathcal{N}(\mu = (2p-1)^2\theta h, \sigma^2 = 4\theta p(1-p)h) \tag{14}$$

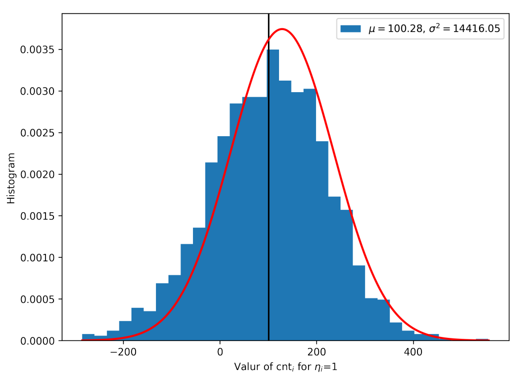
$$[\text{acc}_i|\eta_i = 0] \sim \mathcal{N}(\mu = -(2p-1)^2\theta h, \sigma^2 = 4\theta p(1-p)h) \tag{15}$$

In our case, $[\text{acc}_i|\eta_i = 1] \sim \mathcal{N}(\mu = 128.62, \sigma^2 = 11359.99)$, and $[\text{acc}_i|\eta_i = 0] \sim \mathcal{N}(\mu = -128.62, \sigma^2 = 11359.99)$. See Figure **??** — we can notice that the small difference in the means from Figure **??** has propagated to these images.

(a) Equation ?? ($\mathrm{addr}_k = 1$)

(b) Equation ?? ($\mathrm{addr}_k = 0$)

Figure 3: The histogram was obtained through simulation. The red curve is the theoretical normal distribution.



(a) Equation ?? (

= 1)

1 a

$$P(wrong) = P(\text{acc}_i < 0|\eta_i = 1) \cdot P(\eta_i = 1) + P(\text{acc}_i > 0|\eta_i = 0) \cdot P(\eta_i = 0) \tag{16}$$

$$= \frac{\mathcal{N}_{\eta_i=1}.\text{cdf}(0)}{2} + \frac{1 - \mathcal{N}_{\eta_i=0}.\text{cdf}(0)}{2} \tag{17}$$

$$= \frac{\mathcal{N}_{\eta_i=1}.\text{cdf}(0)}{2} + \frac{\mathcal{N}_{\eta_i=1}.\text{cdf}(0)}{2} \tag{18}$$

$$= \mathcal{N}_{\eta_i=1}.\text{cdf}(0) \tag{19}$$

In our case, $P(wrong) = 0.1137518032308093$.

In order to check this probability, I have run a simulation reading from 1,000 random bitstrings (which have never been written into memory) and calculate the distance from the result of a single read. As the $P(wrong) = 0.11375$, I expected to get an average distance of 113.75 with a standard deviation of 10.04. See Figure **??** — We can notice a big difference between the theoretical model and the simulation. Using $\mu = 102$ and $\sigma = 121$, the curves match. I'm still looking for the mistake in the equations. I believe the problem is in Equation **??**.