[Thesis text....] ...As one can easily notice, we have got exactly the same curve as Kanerva. Both his and our model expect that, after reading, say, from 550 bits of distance from a written bitstring, we should obtain the expected $n/2$ equator distance. This question has intrigued us, and here we look for a more analytic explanation than merely interference from the other written attractors. Let us turn back to mathematics to study this anomaly.

1 A deviation from the equator distance?

Dr. Kanerva writes:

> You have done an incredibly thorough analysis of SDM. I like the puzzle in your message and believe that your simulations are correct and to be learned from. So what to make of the difference compared to my Figure 7.3 (and your Figure **??**)? I think the difference comes from my not having accounted fully for the effect of the other 9,999 vectors that are stored in the memory. You say it in
>
> "Our results show that the theoretical prediction is not accurate. There are interaction effects from one or more of the attractors created by the 10,000 writes, and these attractors seem to raise the distance beyond 500 bits (Figure **??**)."
>
> I think that is correct. It also brings to mind a comment Louis Jaeckel made when we worked at NASA Ames. He pointed out that autoassociative storage (each vector is stored with itself as the address) introduces autocorrelation that my formula for Figure 7.2 did not take into account. When we read from memory, each stored vector exerts a pull toward itself, which also means that each bit of a retrieved vector is slightly biased toward the same bit of the read address, regardless of the read address. We never worked out the math.

This is an important insight: a hard location is activated because it shares many dimensions with the items read from or written onto it. Imagine the ‘counter’s eye view’: each individual counter ‘likes’ to write on its own corresponding bit-address more than it likes the opposite; as each hard-location has a say in its own area (and nowhere else).

Let x and y be random bitstrings and n be the number of dimensions in the memory; let x_i and y_i be the i-th bit of x and y, respectively; and $d(x, y)$ be the Hamming distance. Whilst the probability of a shared bit-value between same dimension-bits in two random addresses is 1/2, an address only activates hard-locations close to it.

So what is the probability of shared bit-values given that we know the access radius r between the address and a hard-location? Applying the law of total probability:

$$P(x_i = y_i | d(x, y) \leq r) = \sum_{k=0}^{r} P(x_i = y_i | d(x, y) = k \leq r) \cdot P(d(x, y) = k | d(x, y) \leq r) \tag{1}$$

We also know that

$$P(x_i = y_i | d(x, y) = k) = \frac{n-k}{n} \tag{2}$$

$$P(d(x, y) = k | d(x, y) \leq r) = \frac{\binom{n}{k}}{\sum_{j=0}^{r} \binom{n}{j}} \tag{3}$$

Hence,

$$P(x_i = y_i | d(x, y) \leq r) = \frac{\sum_{k=0}^{r} \frac{n-k}{n} \binom{n}{k}}{\sum_{j=0}^{r} \binom{n}{j}} \tag{4}$$

Finally, as $\frac{n-k}{n}\binom{n}{k} = \binom{n-1}{k}$,
THIS STEP IS A LITTLE WTF TO ME...
$\frac{n-k}{n}\binom{n}{k} = \frac{n-k}{n}\binom{n}{n-k} = \frac{n-k}{n}\frac{n!}{k!(n-k)!}$

$= \frac{(n-k)(n-1)!}{k!(n-k)!}$
$= \frac{(n-1)!}{k!(n-k-1)!}$

$$P(x_i = y_i | d(x,y) \leq r) = \frac{\sum_{k=0}^{r} \binom{n-1}{k}}{\sum_{k=0}^{r} \binom{n}{k}} \tag{5}$$

This equation is valid for both “x at x” (autoassociative memory) and “random at x” (heteroassociative memory). When $n = 1,000$ and $r = 451$, $P(x_i = y_i | d(x,y) \leq r) = p = 0.552905498137$.

Hence, let Z be the number of activated hard location with the same bit as the reading address, then: $Z = \sum_{i=1}^{h} X_i$, where $\mathbf{E}[h] = h$, $\mathbf{V}[h] = Hp_1(1-p_1)$, $p_1 = 2^{-n} \sum_{k=0}^{r} \binom{n}{k}$, and $X_i \sim \text{Bernoulli}(p)$. By the central limit theorem, Z is normally distributed with $\mathbf{E}[Z] = \mathbf{E}[\mathbf{E}[W_i|h]] = \mathbf{E}[ph] = p\mathbf{E}[h] = ph$, and $\mathbf{V}[Z] = \mathbf{E}[\mathbf{V}[X_i|h]] + \mathbf{V}[\mathbf{E}[X_i|h]] = \mathbf{E}[hp(1-p)] + \mathbf{V}[ph] = p(1-p)\mathbf{E}[h] + p^2\mathbf{V}[h] = hp(1-p) + p^2Hp_1(1-p_1)$. See Figure 1 for a comparison between the theoretical model and a simulation.

2 Counter bias

The bias begins in the counters. Let’s analyze the ith counter of a hard location.

Let s be the number of bitstrings written into memory (in our case, $s = 10,000$), h be the average number of activated hard locations ($h = 1,071.85$), H be the number of hard locations ($H = 1,000,000$), and addr_i be the ith bit of the hard location’s address.

Let $\theta = \frac{sh}{H}$ be the average number of bitstrings written in each hard location, and $X_k \sim \text{Bernoulli}(p)$ (where $p = P(x_i = y_i | d(x,y) \leq r)$). Thus, $Y_i = \sum_{k=1}^{\theta} X_k \sim \mathcal{N}(\mu_1, \sigma_1^2)$ with $\mu_1 = p\theta$, and $\sigma_1^2 = p(1-p)\theta + p^2s^2p_1(1-p_1)/H$.

During a write operation, the counters are incremented for every bit 1 and decremented for every bit 0. So, after s writes, there will be θ bitstrings written in each hard location, Y_i bits 1, and $\theta - Y_i$ bits 0. Thus, $[\text{cnt}_i | \text{addr}_i = 1] = (Y_i) - (\theta - Y_i) = 2Y_i - \theta$; and $[\text{cnt}_i | \text{addr}_i = 0] = \theta - 2Y_i$.

Hence, as $\text{cnt}_i = 2Y_i - \theta$, $\mathbf{E}[2Y_i - \theta] = 2\mathbf{E}[Y_i] - \theta$, and $\mathbf{V}[2Y_i - \theta] = 4\mathbf{V}[Y_i]$, then,

$$[\text{cnt}_i | \text{addr}_i = 1] \sim \mathcal{N}(\mu_2 = (2p-1)\theta, \sigma_2^2 = 4\sigma_1^2) \tag{6}$$

$$[\text{cnt}_i | \text{addr}_i = 0] \sim \mathcal{N}(\mu_2 = -(2p-1)\theta, \sigma_2^2 = 4\sigma_1^2) \tag{7}$$

./images02/autocorrelation/same-bit-histogram.png

Figure 1: The histogram was obtained through simulation. The red curve is the theoretical normal distribution.

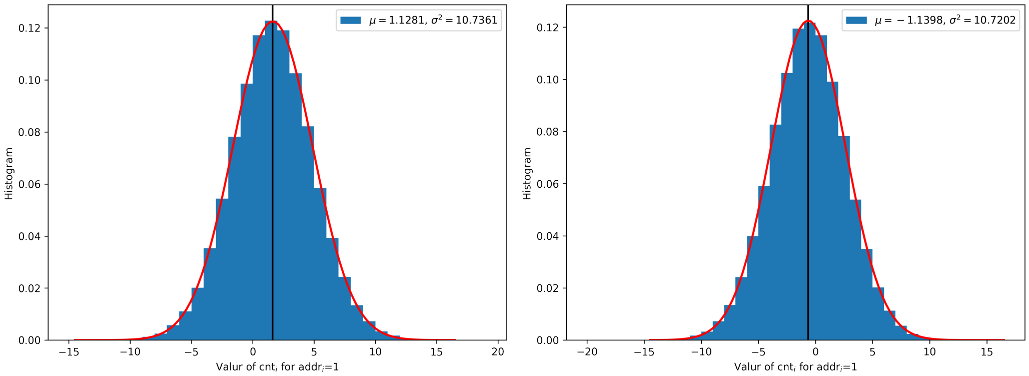
In our case, $p = 0.5529$, $s = 10,000$, $h = 1,071.85$, and $H = 1,000,000$, so $\theta = 10.7185$ and $\text{cnt}_i \sim \mathcal{N}(\mu = 1.1341, \sigma^2 = 10.7294)$. For "random at x", $p = 0.5$, so $\mu = 0$ and $\sigma^2 = 10.8255$. See Figure 2.

Finally,

$$P(\text{cnt}_i > 0|\text{addr}_i = 1) = P(\text{cnt}_i < 0|\text{addr}_i = 0) = 1 - \mathcal{N}.\text{cdf}(0) \qquad (8)$$

For "random at x", $p = 0.5$ implies $\mu_2 = 0$, which implies $P(\text{cnt}_i > 0|\text{addr}_i = 1) = P(\text{cnt}_i < 0|\text{addr}_i = 0) = 0.5$, independently of the parameters because they will only affect the variance and the normal distribution is symmetrical around the average.

However, for "x at x", $p = 0.5529$ and the probabilities depend on s. For $s = 10,000$, they are equal to 0.6354. For $s = 20,000$, they are equal



(a) $\mathrm{addr}_i = 1$

(b) $\mathrm{addr}_i = 0$

Figure 2: The value of the counters after $s = 10,000$ writes shows the autocorrelation in the counters in autoassociative memories ("x at x"). The histogram was obtained through simulation. The red curve is the theoretical normal distribution.

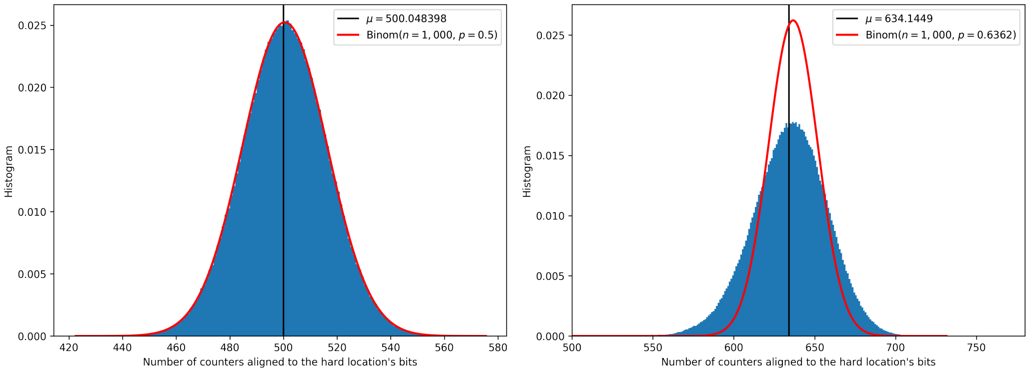
to 0.6867. For $s = 30,000$, they are equal to 0.7232. The more random bitstrings are written into the memory, the more a hard location to point to itself. See Figure 3 — and notice that I still have to figure out why the mean is correct, but the standard deviation is not. As each of the n counters of a hard location may be equal or not with the same probability, I assumed it would follow a Binomial distribution (and it worked for "random at x").

3 Read bias

Now that we know the distribution of $\mathrm{cnt}_i|\mathrm{addr}_i$, we may go to the read operation. During the read operation, on average, h hard locations are activated and their counters are summed up. So, for the ith bit,

$$\mathrm{acc}_i = \sum_{k=1}^{h} \mathrm{cnt}_k \tag{9}$$

Let η be the reading address and η_i the ith bit of it. Then,



(a) “random at x”

(b) “x at x”

Figure 3: Autocorrelation in the counters in autoassociative memories (“x at x”). The histogram was obtained through simulation. The red curve is the theoretical distribution.

$$[\text{acc}_i|\eta_i = 1] = \sum_{k=1}^{ph} [\text{cnt}_k|\text{addr}_k = 1] + \sum_{k=1}^{(1-p)h} [\text{cnt}_k|\text{addr}_k = 0] \tag{10}$$

$$[\text{acc}_i|\eta_i = 0] = \sum_{k=1}^{ph} [\text{cnt}_k|\text{addr}_k = 0] + \sum_{k=1}^{(1-p)h} [\text{cnt}_k|\text{addr}_k = 1] \tag{11}$$

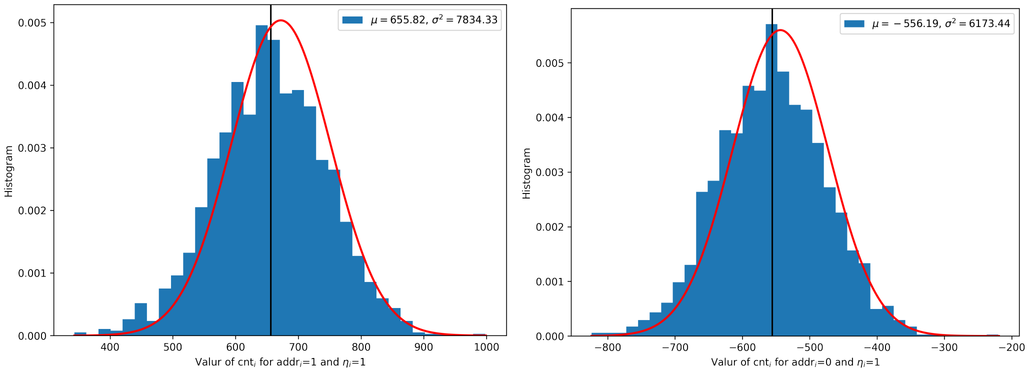
We may analyze only one case, because the probability of the other is exactly the same.

Each sum is a sum of normally distributed random variables, so

$$\sum_{k=1}^{ph} [\text{cnt}_k|\text{addr}_k = 1] \sim \mathcal{N}(\mu_3 = \mu_2 ph, \sigma^2 = \sigma_1^2 ph) \tag{12}$$

$$\sum_{k=1}^{(1-p)h} [\text{cnt}_k|\text{addr}_k = 0] \sim \mathcal{N}(\mu = -\mu_2(1-p)h, \sigma^2 = \sigma_1^2(1-p)h) \tag{13}$$

In our case, $\sum_{k=1}^{ph} [\text{cnt}_k|\text{addr}_k = 1] \sim \mathcal{N}(\mu = 672.12, \sigma^2 = 6281.00)$, and $\sum_{k=1}^{ph} [\text{cnt}_k|\text{addr}_k = 1] \sim \mathcal{N}(\mu = -543.49, \sigma^2 = 5078.99)$. See Figure 4 —



(a) Equation 12 ($\text{addr}_k = 1$)

(b) Equation 13 ($\text{addr}_k = 0$)

Figure 4: The histogram was obtained through simulation. The red curve is the theoretical normal distribution.

we can notice there a small but significant difference between the theoretical and the simulated mean.

Hence,

$$[\text{acc}_i|\eta_i = 1] \sim \mathcal{N}(\mu = (2p-1)^2\theta h, \sigma^2 = 4\theta p(1-p)h) \tag{14}$$

$$[\text{acc}_i|\eta_i = 0] \sim \mathcal{N}(\mu = -(2p-1)^2\theta h, \sigma^2 = 4\theta p(1-p)h) \tag{15}$$

In our case, $[\text{acc}_i|\eta_i = 1] \sim \mathcal{N}(\mu = 128.62, \sigma^2 = 11359.99)$, and $[\text{acc}_i|\eta_i = 0] \sim \mathcal{N}(\mu = -128.62, \sigma^2 = 11359.99)$. See Figure 5 — we can notice that the small difference in the means from Figure 4 has propagated to these images.

Finally,

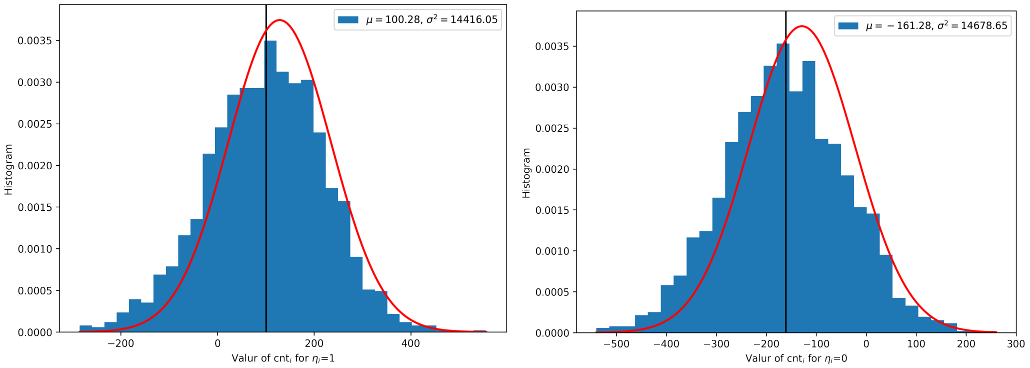
$$P(wrong) = P(\text{acc}_i < 0|\eta_i = 1) \cdot P(\eta_i = 1) + P(\text{acc}_i > 0|\eta_i = 0) \cdot P(\eta_i = 0) \tag{16}$$

$$= \frac{\mathcal{N}_{\eta_i=1}.\text{cdf}(0)}{2} + \frac{1 - \mathcal{N}_{\eta_i=0}.\text{cdf}(0)}{2} \tag{17}$$

$$= \frac{\mathcal{N}_{\eta_i=1}.\text{cdf}(0)}{2} + \frac{\mathcal{N}_{\eta_i=1}.\text{cdf}(0)}{2} \tag{18}$$

$$= \mathcal{N}_{\eta_i=1}.\text{cdf}(0) \tag{19}$$

In our case, $P(wrong) = 0.1137518032308093$.



(a) Equation 14 ($\eta_k = 1$)

(b) Equation 15 ($\eta_k = 0$)

Figure 5: The histogram was obtained through simulation. The red curve is the theoretical normal distribution.

In order to check this probability, I have run a simulation reading from 1,000 random bitstrings (which have never been written into memory) and calculate the distance from the result of a single read. As the $P(wrong) = 0.11375$, I expected to get an average distance of 113.75 with a standard deviation of 10.04. See Figure 6 — We can notice a big difference between the theoretical model and the simulation. Using $\mu = 102$ and $\sigma = 121$, the curves match. I'm still looking for the mistake in the equations. I believe the problem is in Equation 12.

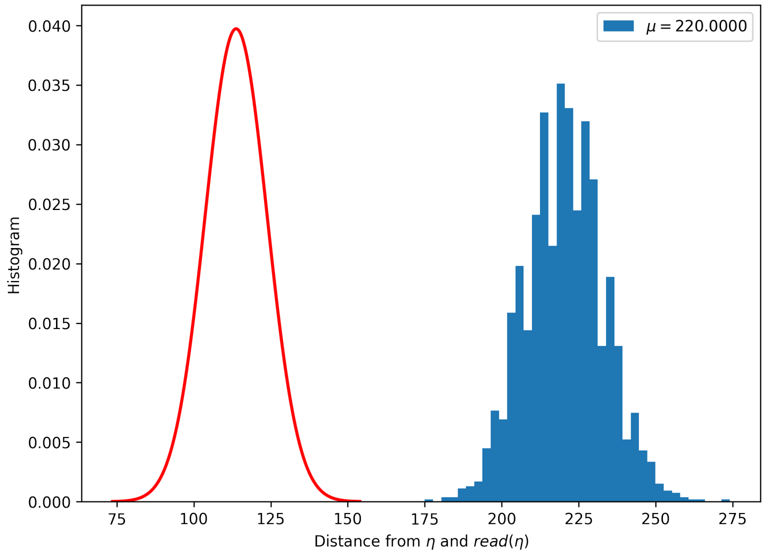


Figure 6: The histogram was obtained through simulation. The red curve is the theoretical normal distribution.