Real Analysis

Author: Alex Luo

Date: November 14, 2022

Contents

Chapter 1 Why We Need Lebesgue Integral **1**

Chapter 2 Lebesgue Measure **3**
2.1 Lebesgue Outer Measure 3
2.2 Lebesgue Measure 5
2.3 Outer and Inner Approximation of Measurable Sets 9

Chapter 3 Lebesgue Measuralbe Functions **12**
3.1 Definition and basic properties 12
3.2 Simple function approximation of measurable function 16
3.3 Egoroff's Theorem & Lusin's Theorem 19

Chapter 4 Lebesgue Integration **23**
4.1 Integral of Simple Functions 23
4.2 Integral for Bounded Measurable Functions over Bounded Sets 25
4.3 The Lebesgue Integral for Non-negative Functions 30
4.4 The Lebesgue Integral in General 34
4.5 Lebesgue vs Riemann 38

Chapter 5 Limit and Itergral **40**

Chapter 6 Lp Space **44**
6.1 Definition 44
6.2 Important Inequality 44
6.3 ? 44

Chapter 7 Appendix 1: Non-measurable Sets, Non-measurable Functions **45**

Chapter 1 Why We Need Lebesgue Integral

In the previous sections, we have already developed the theory of Riemann integral. But Riemann integral dose not close after taking limit. If f_n is a sequence of Riemann integrable functions, and it's limit function may not be Riemann integrable.

Example 1.1 Let f_n be a sequence of integrable function over $[0,1]$ and f be its limit function.

1. We already know that if f_n converge uniformly, then f must be integrable and
$$\lim_{n\to\infty}\int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n = \int_a^b f \tag{1.1}$$
by the theory of uniform Convergence.
2. f_n pointwise converge to f but f isn't integrable. Let r_k be the sequence which consits of all the rationals($\mathbb{Q}$ is countable, so this sequence exists). Let $f_n(x) = 1$ if $x = r_k$ for some $k < n$, and for other cases, $f_n(x) = 0$. For any n, f_n has only finite discontinuity, thus it is integrable. f_n converge to the dirichlet functions, which we know isn't integrable.
3. f_n pointwise converge to f and f is integrable. Let $f_n = x^n$, then f_n converge to $f : f(x) = 1$ if $x = 1$; $f(x) = 0$ if $x \in [0, 1)$. The integral exist and satisfies (1.1).

Riemann integral functions is in the exactly situation of rational number, if you take limit, you probably got something else. An integral like real number, with the completeness, is desired.

One of the motivation of the developing the theory of integration is to calculate the area, and the method we use basicly is to cut a whole thing into pieces. For Riemann integral, to calculate the area under a function, one take a partition the domain, and build up a "step function". If the step function converge to the original function as partition gets finer, we say the integral exist.

For Lebesgue integral, we also take partition, but instead of on it's domain, we take it on the range. Let $f : E \to \mathbb{R}$ be a real value function $P_{M_n} = \{\frac{2iM}{2^n} - M : i \in [0, 2^n] \cap \mathbb{Z}\}$ be a partition of $f(E) \in \mathbb{R}$. The integral under this partition will be look like
$$\int_E f(x) = \lim_{n\to\infty}\sum_{i=1}^{2^n}[(\frac{2iM}{2^n} - M)m(E_i)] = \lim_{n\to\infty}\sum_{i=1}^{2^n}[(\frac{2(i+1)M}{2^n} - M)m(E_i)] \tag{1.2}$$
where $E_i = \{x : f(x) \in [\frac{2iM}{2^n} - M, \frac{2(i+1)M}{2^n} - M]\}$, $m(E_i)$ is it's lenth.

One may find as $n \to \infty$, the last equal sign in (1.2) is true no matter what the intergrand is, but this doesn't means all the functions are Lebesgue integrable. Because without defining the lenth of a set(or volume, for higher dimension), the equation above is meaningless. So, in order to develop the theory of Lebesgue integral, we need to find a set function m, which is called Lebesgue measure, mapping a general set into $[0, +\infty]$. We also hope m, has the following properties(in order to fit in our notion of lenth):

1. (Monotonicity) If $E \subset F$, $m(E) \leq m(F)$. This is inherited from the intuitive notion of volume.
2. $m([a, b]) = b - a$. The lenth of integral is agree with its measure.
3. (Countable additivity) Let E_i be a disjoint collection of set, then
$$m(\bigcap_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} m(E_i)$$

which enable us doing integral.

We will first discuss Lebesgue's theory on $\mathbb{R}$, and for most sets that occurs will be subsets of $\mathbb{R}$. In next section, we will focus measure theory, which solves the problems above.

Chapter 2 Lebesgue Measure

To define a such measure function(the Lebesgue measure) we have discussed in the previous section, we need to first define a function which is called *Lebesgue outer measure*. This is because countable additivity is really a rigorous property. Outer measure is defined for all set, but is doesn't have the countable additivity. The true Lebesgue measure only is defined on a certain collection of sets, which we called measurable set. Basicly, we ristrict the Lebesgue outer measure on the measurable sets to obtain the Lebesgue measure. In the general measure theory, there are many measures, and Lebesgue measure is just one of them. Since we will only talk about Lebesgue measure, for convenience, we will just call the Lebesgue as measure(also the concepts measurable).

2.1 Lebesgue Outer Measure

Definition 2.1 (Lebesgue outer measure)

Given a set A, let $\{I_n\}$ be a countable open interval cover of A, namely $A \subset \bigcap_{n=1}^{\infty} I_n$ and I_n is a countable collection of open intervals. Let $m^*(A) = \inf\{\sum_{n=1}^{\infty} I_n : \{I_n\}$is a open interval cover of $A\}$, We call the function m^* Lebesgue outer measure. ♣

Since for every set such collection of number(can be infinity) is nonempty, so the outer measure of every set on $\mathbb{R}$ is well defined.

Proposition 2.1 (Properties of Lebesgue outer measure)

Let $A, B, \{A_n\}$ be any sets and countable collection of sets. The following properties is satisfied.

1. *Positivity: $m^*(A) \geq 0$.*
2. *Monotonicity: $B \subset A \Rightarrow m^*(A) \geq m^*(b)$*
3. *Countable Subadditivity: $\sum_{n=1}^{\infty} m^*(A_n) \geq m^*(\bigcap_{n=1}^{\infty} A_n)$*
4. *Translation Invariance: For any number y, $m^*(A+y) = m^*(A)$ where $A + y = a + y : a \in A$* ♠

Proof Positivity: Since for every open interval cover $\{I_n\}$, $\sum_{n=1}^{\infty} |I_n|$ is a positive series, $m^*(A) \geq 0$.

Monotonicity: If $B \subset A$, every open interval cover of A is also that of B. Thus by the property of inf, $m^*(B) \leq m^*(A)$.

Countable subadditivity: Suppose $\sum_{n=1}^{\infty} m^*(A_n) < +\infty$(the $+\infty$ case is trivial). For each $\{A_n\}$, let $\{I_{n_k}\}$ be its open interval cover, and we can choose a $\{I_{n_k}\}$ such that

$$\sum_{n=1}^{\infty}\{I_{n_k}\} - m^*(A_n) \leq \frac{\epsilon}{2^n}$$

Then we know

$$\bigcap_{n=1}^{\infty} A_n \subset \bigcap_{n=1}^{\infty}\{I_{n_k}\}, \quad \sum_{n,k=1}^{\infty} |I_{n_k}| \leq \sum_{n=1}^{\infty} m^*(A_n) + \epsilon$$

Let $\epsilon \to 0$, the desired proposition is proved.

(This is something called coset. You may have run into this concept in algebra, but if you haven't, never mind it)

Translation invariance: For every open interval cover $\{I_n\}$ of A, $\{I_n + y\}$ is a open interval cover of $A + y$, and at the same time, for every open cover $\{I_n\}$ of $A + y$, $\{I_n + y\}$ is a open cover of A. Since invervals is translation invariant,

$$\sum_{n=1}^{\infty} |I_n| = \sum_{n=1}^{\infty} I_n + y$$

Given any countable open interval cover $\{I_n\}$ of A,

$$m^*(A + y) \leq \sum_{n=1}^{\infty} I_n + y = \sum_{n=1}^{\infty} I_n \tag{2.1}$$

Then we can see that $m^*(A + y) \leq m^*(A + y)$, since the former one is a lower bound and the latter one is the greatest lower bound. The inequality in the other way can be proved in the came way, then we conclude that $m^*(A) = m^*(A + y)$.

Example 2.1The outer measure of $\varnothing$ Since $\varnothing \subset E$ for any set E, let $I_{n_k} = (0, \frac{1}{n2^k})$, then the collection $\{I_{n_k}\}_{k=1}^{\infty}$ is certain a open interval cover of $\varnothing$ and $\sum_{k=1}^{\infty} |I_{n_k}| = \frac{1}{n}$. Let $n \to \infty$, then $\sum_{k=1}^{\infty} |I_{n_k}| \to 0$. Thus $m^*(\varnothing) \leq 0$(remember m^* of a set is a lower bound). By the positivity of outer measure, $m^*(\varnothing) = 0$

Example 2.2The outer measure of countable set is 0 Give any countable set $A = \{a_k\}$, let $I_{n_k} = (a_k - \frac{1}{n2^{k+1}}, a_k + \frac{1}{n2^{k+1}})$. For each n, $\{I_{n_k}\}$ is a open interval cover of A, and

$$\sum_{k=1}^{\infty} |I_{n_k}| = \sum_{k=1}^{\infty} \frac{1}{n2^k} = \frac{1}{n}$$

Let $n \to \infty$, we arrive at the conclusion: $m^*(A) = 0$.

Proposition 2.2 (The outer measure of intervals)

The outer measure of an interval is its lenth. Let $I = [a, b]$, then

$$m^*(I) = b - a$$

The similar statements for $[a, b)$, $(a, b]$, (a, b) are also true. ♠

Proof For every close interval $[a, b]$, one can find an open interval $(a - \epsilon, b + \epsilon)$ cover it, which has the lenth $b - a + 2\epsilon$. Since epsilon can be chosen freely, $m^*([a, b]) \leq b - a$.

It remains to show $m^*([a, b]) \geq b - a$, which is equivalence to show for all countable open interval covers $\{I_n\}$,

$$\sum_{n=1}^{\infty} I_n \geq b - a \tag{2.2}$$

Because we are dealing a close interval on $\mathbb{R}$, which is a compat set, Heine-Borel theorem tells us it has a finite subcover for each open cover. So

$$\sum_{n=1}^{N} I_n \geq b - a \tag{2.3}$$

will indicate (2.2).

Since a is in $\bigcap_{n=1}^{N} I_n$, there exist a I_{n_1} contains a. If I_{n_1} contains b, $|I_{n_1}| > b-a$; if not, $I_{n_1} = (c,d)$, $c < a < d < b$, then d is in $[a,b]$, which means exist I_{n_2} such that $d \in I_{n_2}$; if $b \in I_{n_2}$, then $|I_{n_1}| + |I_{n_2}| > b-a$; if not, $I_2 = (f,g)$, $f < d, g < b$, ...There must be a I_{n_i} such that $b \in I_{n_i}$, and since we have only finite I_n, so this process will be eventually ended. After doing this for enough times, one will find out that

$$\sum_{i=1} N|I_{n_i}| > b-a$$

For other intervals, since they are subsets of closed interval, by monotonicity, we have $m^*(I) \leq b-a$. By the subadditivity and our previous conculsion on the outer measure of countable set,

$$m^*([a,b]) = m^*((a,b) \cap \{a,b\}) \leq m^*((a,b)) + m^*(\{a,b\}) = b-a$$

The proof for $[a,b)$ and $(a,b]$ is simialr.

2.2 Lebesgue Measure

Definition 2.2 (Measurable sets and Lebesgue measure)

Let E be a set, if for any set A,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \tag{2.4}$$

then we say E is a measuralbe set. The collection of all the measurable sets is denoted $\mathcal{M}$. If E is a measurable set, we define its Lebesgue measure $m(E) := m^*(E)$. ♣

Example 2.3The sets with outer measure zero are measurable First, notice that, if one wants to check whether a set E is measurable, the only thing is to show is

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

since the other inequality is guaranteened by the subadditivity.

For the set E whose outer measure is zero, and any set A, by monotonicity of outer measure, we have

$$m^*(A \cap E = 0) \ , \ \ m^*(A \cap E^c) \leq m^*(A)$$
$$\Rightarrow m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

And together the subadditivity of outer measure, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

which shows that E is measurable.

In our expectation of the measure function, it should have the countable additivity. So we know that the set must satisfied (2.4), because $(A \cap E) \cap (A \cap E^c) = \varnothing$ and $(A \cap E) \cup (A \cap E^c) = A$, which is a special case of finite additivity. The Lebesgue measure define above is just the ristriction of m^* on the collection of measurable sets. At last, we will show that the measure defined above have all the good properties we want. Before that, we first would like to prove some properties of the measurable sets.

Proposition 2.3 (Properties of measurable sets)

Let $E_1, E_2 \in \mathcal{M}$, and $\{E_i\} \subset \mathcal{M}$

1. $\varnothing \in \mathcal{M}$
2. $E_1^c \in \mathcal{M}$.
3. $E_1 \cap E_2,\ E_1 \cup E_2,\ E_1/E_2 \in \mathcal{M}$. *From this one can easily derive the closedness of finite intersection and union.*
4. *If for any* $i \neq j$, $E_i \cap E_j = \varnothing$, *then*
$$m(A \cap \bigcup_{i=1}^{N} E_i) = \sum_{i=1}^{N} m(A \cap E_i)$$
The finite additivity among measurable sets is a special case of this proposition, taking $A = \mathbb{R}$.
5. $\bigcap_{i=1}^{\infty} E_i,\ \bigcup_{i=1}^{\infty} E_i \in \mathcal{M}$

Proof 1. This is true because we have show that measure zero sets are measurable.

2. Since $(E^c)^c = E$, so
$$\begin{aligned} m^*(A) &= m^*(A \cap E) + m^*(A \cap E^c) \\ &= m^*(A \cap (E^c)^c) + m^*(A \cap E^c) \end{aligned}$$
which shows that E^c is measurable.

3. Here we first prove the union of measurable sets is measurable. Since $E_1,\ E_2 \in \mathcal{M}$, for any set A
$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap E_1^c) \\ &= m^*(A \cap E_1) + m^*((A \cap E_1^c) \cap E_2) + m^*((A \cap E_1^c) \cap E_2^c) \end{aligned}$$
By basic set identities,
$$\begin{aligned} (A \cap E_1^c) \cap E_2^c &= A \cap (E_1 \cup E_2)^c \\ [A \cap E_1] \cup [(A \cap E_1^c) \cap E_2] &= A \cap (E_1 \cup E_2) \end{aligned}$$
and by the subadditivity of outer measure,
$$\begin{aligned} m^*(A) &\geq m^*([A \cap E_1] \cup [(A \cap E_1^c) \cap E_2]) + m^*((A \cap E_1^c) \cap E_2^c) \\ &= m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \end{aligned}$$
is true. Thus the union of measurable set is measurable.

4. The proof proceeds by induction on N. It is clear true for $N = 1$. Suppose it is true for $N - 1$. By the measurability of E_N, and the fact that $\{E_n\}$ is disjoint sequence of sets,
$$\begin{aligned} m^*(A \cap (\bigcup_{n=1}^{N} E_n)) &= m^*([A \cap (\bigcup_{n=1}^{N} E_n)] \cap E_N) + m^*([A \cap (\bigcup_{n=1}^{N} E_n)] \cap E_N^c) \\ &= m^*(A \cap E_N) + m^*(A \cap \bigcup_{n=1}^{N-1} E_n) \\ &= \sum_{n=1}^{N} E_n \end{aligned}$$

5. Let E be the union of countable collection of measurable set, say $\{A_k\}$. By defining a new

sequence of sets,

$$E_1 = A_1, \quad E_n = A_n / E_{n-1}$$

one can express E as a union of disjoint sequence of measurable sets. Let A be any set. Observing that $E^c \subset (\bigcup_{k=1}^{N} E_k)^c$

$$\begin{aligned} m^*(A) &\geq m^*(A \cap \bigcup_{k=1}^{N} E_k) + m^*(A \cap (\bigcup_{k=1}^{N} E_k)^c \\ &\geq m^*(A \cap \bigcup_{k=1}^{N} E_k) + m^*(A \cap E^c) \\ &= \sum_{k=1}^{N} m^*(A \cap E_k) + m^*(A \cap E^c) \end{aligned}$$

Since the relation is regardless of N

$$\begin{aligned} m^*(A) &\geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap E^c) \\ &\geq m^*(A \cap E) + m^*(A \cap E^c) \end{aligned}$$

Then the desired conclusion is proved.

Proposition 2.4 (Intervals are measurable)

For every $a, \ b$ in $\mathbb{R}^*$, $[a, b], \ (a, b), \ [a, b), \ (a, b]$ are measurable.

Proof First, we show that (a, ∞) is measurable. Give any set A, we need to show that

$$m^*(A) \geq m^*(A \cap (a, +\infty)) + m^*(A \cap (-\infty, a))$$

since $m^*(A)$ is a infimum, it is surffice to show

$$m^*(A \cap (a, +\infty)) + m^*(A \cap (-\infty, a)) \leq \sum_{n=1}^{\infty} |I_n| \tag{2.5}$$

where I_n is a open interval cover of A. For each n, we define $J_n = I_n \cap (a, +\infty)$, $k_n = I_n \cap (-\infty, a)$. Then

$$\begin{aligned} |I_n| &= |J_n| + |K_n| \\ \sum_{n=1}^{\infty} |I_n| &= \sum_{n=1}^{\infty} |J_n| + |K_n| = \sum_{n=1}^{\infty} |J_n| + \sum_{n=1}^{\infty} |K_n| \end{aligned}$$

Observing that $\sum_{n=1}^{\infty} |J_n|$ and $\sum_{n=1}^{\infty} |K_n|$ are open interval covers of $A \cap (a, +\infty)$ and $A \cap (-\infty, a)$, So (2.5) is varified, then we proved that $(a, +\infty)$ is measurable.

Since $\mathcal{M}$ is closed under complement, $(-\infty, a]$ is measurable; Since $\mathcal{M}$ is closed under countable, union,

$$(-\infty, a) = \bigcup_{n=1}^{\infty} (-\infty, a - \frac{1}{n}]$$

is measurable, and also its complement $[a, \infty)$. Let $a \leq b$.

$$[a, b] = [a, \infty) \cap (-\infty, b)$$

is measurable. Other intervals are measurable can be shown in the same manner.

Definition 2.3 (Algebra and σ algebra)

Let A be a collection of sets, if $\forall F,\ E \in A$

1. $\varnothing \in \mathcal{M}$
2. $F^c \in \mathcal{M}$.
3. $F \cap E \in \mathcal{M}$

then, we say A is an algebra.

If a algebra A is closed under countable intersection, namely if a countable collection of sets $E_n \in A$, then $\bigcap_{n=1}^{\infty} E_n \in A$, we call A a σ algebra.

Given a collection of set A, we call the smallest algebra contains A(if you remove any part of it, it won't be an algebra or it won't contains A anymore) the algebra generate by A. ♣

By Demorgan identity, we can see that a algebra is also close under intersection and difference, and a σ algebra is close under countable intersection. The collection of all measurable set is a algebra. The collection of set with this particularstructure is quite useful in analysis. One need to be careful not to mix up this concept with algebra in Algebra.

Example 2.4Borel sets We call the σ algebra generated by the collection of open sets in $\mathbb{R}$ Borel sets. Before Lebesgue, Borel had already built up a measure theory, but not in the purpose of defining an integral. We use $\mathcal{B}$ to denote the collection of Borel sets.

Definition 2.4 (F_σ set and G_δ set)

For a countable collection of close sets $\{F_n\}$, we say $F = \bigcup_{n=1}^{\infty} F_n$ is a F_σ set; for any countable collection of open sets $\{G_n\}$, we call the set $G = \bigcap_{n=1}^{\infty}$ a G_δ set. Also you can have an $F_{\sigma\delta}$ set and $G_{\delta\sigma}$ set, which are intersection of countable F_σ sets and union of countable G_δ sets. ♣

Proposition 2.5 (The properties of Borel sets)

1. *Open sets, close sets, G_δ sets, F_σ sets ..., in $\mathbb{R}$ are all Borel sets.*
2. *$\mathcal{B} \subset \mathcal{M}$, the collection of Borel sets is contained in the collection of measuralbe set, namely, every Borel sets are measuralbe.* ♠

Proof

1. By definition, all the open sets should be Borel set, and so are close sets, which are complement of open sets. $G_\delta,\ F_\sigma$ are countable intersection, countable union of Borel sets, so they are also Borel sets.

2. By open set construction theorem, every open set $O = \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}$ is a countable collection of open intervals. Since I_n is measurable, so is open sets. Because Borel set is the smallest σ algebra contains all the open sets, thus $\mathcal{B} \subset \mathcal{M}$.

We end this section with the most important property of Lebesgue measure: countable additivity.

Proposition 2.6

Given a countable collection of disjoint measurable sets $\{E_n\}$,

$$m^*(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} E_n \tag{2.6}$$

Proof We know that $\bigcup_{n=1}^{\infty} E_n$ is measurable. Since outer measure has subadditivity, it remains to show

$$m^*(\bigcup_{n=1}^{\infty} E_n) \geq \sum_{n=1}^{\infty} E_n \tag{2.7}$$

For each number N, by finite additivity of measurable sets,

$$m^*(\bigcup_{n=1}^{N} E_n) = \sum_{n=1} NE_n$$

And we have $\bigcup_{n=1}^{N} E_n \subset \bigcup_{n=1}^{\infty} E_n$. By monotonicity of outer measure,

$$\begin{aligned} m^*(\bigcup_{n=1}^{\infty} E_n) &\geq m^*(\bigcup_{n=1}^{N} E_n) \\ &= \sum_{n=1} NE_n \end{aligned}$$

Let $N \to \infty$, then the proposition is proved.

2.3 Outer and Inner Approximation of Measurable Sets

One can find out a easy fact from the definition of measurable set, that is for a finite measuralbe set $E \subset B$ we have the excision property

$$m^*(B/E) = m^*(B) - m^*(E)$$

which is a quite useful obsevation.

Theorem 2.1 (Use open sets, close sets, G_δ, F_σ to approximate measurable set)

Let E be a subset of $\mathbb{R}$. The following assertions are equivalent to measurability of E.

1. $\forall \epsilon > 0$, $\exists O \supset E$ *is a open set, such that* $m^*(O/E) \leq \epsilon$.
2. $\exists G \supset E$ *is a* G_δ *set, such that* $m^*(G/E) = 0$
3. $\forall \epsilon > 0$, $\exists C \subset E$ *is a close set, such that* $m^*(E/C) \leq \epsilon$.
4. $\exists F \subset E$ *is a* F_σ *set, such that* $m^*(E/F) = 0$

Proof

1. Suppose E is measurable. Since the measure of E is a supremum, so given $\epsilon > 0$, there exists a open interval cover, such that

$$m^*(E) + \epsilon \geq \sum_{n=1}^{\infty} |I_n| \geq m^*(E) \tag{2.8}$$

Let $O = \bigcup_{n=1}^{\infty} I_n$, then O is a open set and $O \supset E$, it remains to show that

$$m^*(O/E) \leq \epsilon$$

By definition of outer measure

$$m^*(O) \leq \sum_{n=1}^{\infty} |I_n| \leq m^*(E) + \epsilon$$

which suggest that

$$m^*(O) - m^*(E) \leq \epsilon$$

From the observation we made in the beginning of this section, we know the proposition will be true if $m^*(E) < +\infty$. For the case that $m^*(E) = \infty$, oen can express E as countable disjoint collection of measurable sets of finite measure, say $\{E_K\}$. For instance, one construction of $\{E_k\}$ could be

$$E_k = [\frac{k}{2} - 1, \frac{k}{2}) \cap E \text{ while } k = 2z$$

$$E_k = [-\frac{k+1}{2}, -\frac{k+1}{2} + 1) \cap E \text{ while } k = 2z + 1$$

Then we can find a sequence of open sets $\{O_k\}$ such that $O_k \supset E_k$ and $m^*(O_k/E_k) \leq \frac{\epsilon}{2^k}$. Since O/E is a subset of $\bigcup_{k=1}^{\infty} O_k/E_k$, by monotonicity and subadditivity,

$$m^*(O/E) \leq m^*(\bigcup_{k=1}^{\infty} O_k/E_k) \leq \sum_{k=1}^{\infty} m^*(O_k/E_k) \leq \epsilon$$

Thus part 1 is proved.

2. By 1., for every n, we can find a open set $O_n \supset E$ such that $m^*(O/E) \leq \frac{1}{n}$. Let $G = \bigcap_{n=1}^{\infty} O_n$. It is clear that $G \supset E$, and G is a G_δ set. Since $G/E \subset O_n/E$ for every n,

$$\begin{aligned} m*(G/E) &\leq m^*(O_n/E) \\ &\leq \frac{1}{n} \end{aligned}$$

Let $n \to \infty$, we have $m^*(G/E) = 0$.

Suppose 2. holds. We know G and G/E are measurable (one is G_δ, one has out measure zero), so E must be measurable.

3. E^c is clearly measurable, so for every $\epsilon > 0$, we can find a $O \supset E^c$ which is a open set such that $m^*(O/E^c) \leq \epsilon$. It is not hard to see that $O^c \subset E$ is a close set, and it remains to show that $m^*(E/O^c) \leq \epsilon$.

$$E/O^c = E \cap O = O/E^c$$

Thus part 3 is proved.

4. For every n, we can find a $C_n \subset E$ which is a close set, such that $m^*(E/C_n) \leq \frac{1}{n}$. Let $F = \bigcup_{n=1}^{\infty} C_n$, then F is a F_σ set in E. Since $E/\bigcup_{n=1}^{\infty} C_n$ is a subset of E/C_n, $m^*(E/F) = 0$.

Suppose 4. holds. Then F and E/F are measurable, so is E.

This theorem indicated that every measurable set is nearly a simple set. Proving the next theorem, we push this idea futher.

Theorem 2.2 (Measurable sets are nearly finite disjoint union of open intervals)

Given a measuralbe set E of finite measure, for each $\epsilon > 0$, there is a finite collection of disjoint open intervals $\{I_n\}_{n=1}^N$, for which if $O = \bigcup_{n=1}^N I_n$,

$$m^*(E/O) + m^*(O/E) < \epsilon \tag{2.9}$$

Proof $\forall \epsilon > 0$, $\exists O'$ is a open set such that $O' \supset E$ and

$$m^*(O'/E) < \frac{\epsilon}{2}$$

Every open set can be expressed as a countable collection of disjoint open intervals, so is O'. Let $O' = \bigcup_{n=1}^\infty I_n$. From $m^*(E)$ is finite, we can know that $m^*(O')$ is finite. So

$$\sum_{n=1}^\infty |I_n| = m^*(O')$$

is a converge positive series. Thus there exists a N such that

$$\sum_{n=N}^\infty |I_n| < \frac{\epsilon}{2}$$

We define $O = \bigcup_{n=1}^N I_n$. Since $O/E \subset O'/E$, and $E/O \subset O'/O$

$$\begin{aligned} m^*(E/O) + m^*(O/E) &\le m^*(O'/O) + m^*(O'/E) \\ &< \sum_{n=N}^\infty |I_n| + \frac{\epsilon}{2} \\ &< \epsilon \end{aligned}$$

Notice that we used the excision property of measurable set($m^*(B/E) = m^*(B) - m^*(E)$) in the previous theorems, thus they only apply to finite measure sets. One should pay attention to the difference between $m^*(O/E) < \epsilon$ and $m^*(O) - m^*(E) < \epsilon$, as the later one is true for any set E. In the next proposition, we show that none measurable sets doesn't possess the excersion property.

Proposition 2.7

Suppose E is not measurable, and has finite out measure. Then there exist a open set O, such that

$$m^*(O/E) > m^*(O) - m^*(E)$$

Proof Since E is not measurable, there exists a set A, such that

$$m^*(A) < m^*(E \cap A) + m^*(A/E) \tag{2.10}$$

Given $\epsilon > 0$, let I_n be an open interval cover of $A \cup E$, and $O = \bigcup_{n=1}^\infty I_n$, where

$$m^*(A \cup E) \le m^*(O) \le m^*(A \cup E) + \epsilon$$

We only need to show $m^*(O) < m^*(O/E) + m^*(E)$. Since (2.10), $\exists \epsilon$ such that

$$\begin{aligned} m^*(O) &< m^*(A \cup E) + \epsilon \\ &< m^*(A) + m^*(E) \quad < m^*(O/E) + m^*(E) \end{aligned}$$

Thus the proposition is proved.

Chapter 3 Lebesgue Measuralbe Functions

Remembering that our purpose is to define Lebesgue integral, integral is an operator on functions, so the concept of measurable function is needed. In the first chapter, the range of the function f which we want to integrate is cut into small intervals, and we want to measure the size of their inverse image(of course using the Lebesgue measure). Generaly, the inverse image of these small intervals are not measurable, so we make compromise, by working with a specific type of function. In this chapter, we would see three interesting and importent properties of measurable function, called Littlewood's three principle. It says **measurable sets are nearly finite union of open intervals**; **measurable functions are nearly continuous functions**; **a sequence of measurable functions which converge pointwisely is almost converge uniformly**. The first Priciple has already shown by (2.2). We will show the latter two by Egoroff's theorem and Lusin's theorem. Although the utterance of the principles are not in the rigourous mathematic laguagues, it is still important since it provide a intuitive understanding of those theorems. In my opinion, this kind of understanding is more in need in the mathematic learning, rather than the rigourous proof.

3.1 Definition and basic properties

Although we want to have $f^{-1}[a, b]$ to be measurable, it turned out that the following condition is equivalent to that.

Definition 3.1 (Lebesgue measuralbe functions)

Given an extended real value function $f : E \to \mathbb{R}^*$ while $E \in \mathcal{M}$, We define

$$E\{f > \alpha\} := \{x \in E : f(x) > \alpha\}$$

.

If $\forall\alpha,\ E\{f > \alpha\} \in \mathcal{M}$ then we say f is measuralbe. We denote the set of all Lebesgue measuralbe functionson a measurable set E by $\mathcal{M}\mathfrak{F}\{E\}$

♣

Proposition 3.1

Let the function f have measurable domain. Then the following condition are equivalent to the measurability of f.

1. *For each α, $E\{f \geq \alpha\}$ is measurable.*
2. *For each α, $E\{f < \alpha\}$ is measurable.*
3. *For each α, $E\{f \leq \alpha\}$ is measurable.*

♠

Proof Suppose $f : E \to \mathbb{R}$ is measurable, which means $\forall\alpha \in \mathbb{R},\ E\{f > \alpha\}$ is measurable. Observing that

$$E\{f \geq \alpha\} = \bigcap_{n=1}^{\infty} E\{f > \alpha + \frac{1}{n}\}$$

which indicate $\forall \alpha \in \mathbb{R},\ E\{f \geq \alpha\}$ is measuralbe.

Suppose $\forall \alpha \in \mathbb{R},\ E\{f \geq \alpha\}$ are measuralbe.

$$E\{f < \alpha\} = E\{f \geq \alpha\}^c \cap E$$

Thus $\forall \alpha \in \mathbb{R},\ E\{f < \alpha\}$ is measurable.

Suppose $\forall \alpha \in \mathbb{R},\ E\{f < \alpha\}$ are measuralbe.

$$E\{f \leq \alpha\} = \bigcap_{n=1}^{\infty} E\{f < \alpha - \frac{1}{n}\}$$

Thus $\forall \alpha \in \mathbb{R},\ E\{f \leq \alpha\}$ is measurable.

And by the similar trick, one can show if $\forall \alpha \in \mathbb{R},\ E\{f \leq \alpha\}$ is measurable, then

$$E\{f > \alpha\} = E\{f \leq \alpha\}^c \cap E$$

is measurable.

Let's see some examples of measurable functions.

Proposition 3.2

f is define on $E \in \mathcal{M}$.

1. *If f is continued on E, $f \in \mathcal{M}\mathfrak{F}\{E\}$.*
2. *If f is monotonic on E, $f \in \mathcal{M}\mathfrak{F}\{E\}$.*
3. *Define $\chi_E : \mathbb{R} \to \mathbb{R}$ by following ristriction: if $x \in E$, $\chi_E(x) = 1$; otherwise, $\chi_E(x) = 0$. Then $E \in \mathcal{M} \iff \chi_E \in \mathcal{M}\mathfrak{F}\{E\}$.*

Proof 1. Since the inverse image of open set is open(relative open), given $O \in \mathbb{R}$, we have $f^{-1}(O) = E \cap U$, where U is open. Being the intersection of measurable sets, $f^{-1}(O)$ is measurable. Since each $\{x \in E : x > \alpha\}$ is open, thus $E\{f > \alpha\}$ is measurable.

2. Let $x_0 = \inf(E\{f > \alpha\})$, then for any $x > x_0,\ x \in E$,

$$f(x) \geq f(x_0) > \alpha$$

Thus $E\{f > \alpha\}$ can only be $[\alpha, \infty) \cap E$ or $(\alpha, \infty) \cap E$. Both of them are measurable.

3. Notice that the range of f only contains two points, 1 and 0. If $\alpha < 0$, $E\{f > \alpha\} = \mathbb{R}$; if $\alpha \geq 1$, $E\{f > \alpha\} = \varnothing$; if $0 \leq \alpha < 1$,$E\{f > \alpha\} = E$, thus χ_E is measurable if and only if E is measurable.

In some sense, a set with measure zero is very small. Suppose P is a property, E is a set and E_0 is a measure zero subset. It turned out that the condition *for all $x \in E - E_0$, $P(x)$ is true* is strong enough to derive many desirable conclusion. For conveniency, we say P *is true almost everywhere on* E to mean that for all $x \in E$ but a set of measure zero, $P(x)$ is true. And it's abbriviation is P *is true on E a.e.*

Proposition 3.3

Let $f : E \to \mathbb{R}^*$ be a extended real value function, while E is a measurable set.

1. *If f is measurable and $f = g$ a.e. on E, then g is measurable.*
2. *For a measurable subset $D \subset E$, f is measurable if and only if it's restriction on D and $E - D$ are measurable.*

Proof 1. Suppose $f \neq g$ on E_0, which is measure zero.

$$\begin{aligned} E\{g < \alpha\} &= (E - E_0)\{g < \alpha\} \cup E_0\{g < \alpha\} \\ &= (E - E_0)\{f < \alpha\} \cup (E\{g < \alpha\} \cap E_0) \\ &= (E\{f < \alpha\} - E_0) \cup (E\{g < \alpha\} \cap E_0) \end{aligned}$$

$(E\{g < \alpha\} \cap E_0)$ is a subset of E_0, thus it is measure zero, thus measurable; $E\{f < \alpha\} - E_0$ is the difference of two measurable set. Since the collection of all measurable sets is a σ algebra, so it is measurable. The union of two measurable set are still measurable, thus $E\{g < \alpha\}$ is measurable.

2. Suppose f is measurable. Then

$$D\{f|_D < \alpha\} = D \cap E\{f < \alpha\}$$
$$(E - D)\{f|_{D-E} < \alpha\} = (E - D) \cap E\{f < \alpha\}$$

So $f|_D$ and $f|_{E-D}$ are measurable.

On the other hand, if $f|_D$ and $f|_{E-D}$ are measurable,

$$\begin{aligned} E\{f < \alpha\} &= (E - D)\{f < \alpha\} \cup D\{f < \alpha\} \\ &= (E - D)\{f|_{E-D} < \alpha\} \cup D\{f|_D < \alpha\} \end{aligned}$$

Thus the proposition is proved.

Just as what is done when studing continuous function, we want to know that whether the sum, product and composition of measurable functions are still measurable. But in general, the sum of two extended real valued is not even well defined. For example, if $f(x) = +\infty$ while $g(x) = -\infty$, then $f(x) + g(x)$ is not well defined. If we ask f and g to be finite a.e. in there domain, then we can know the set of points which are not well defined is measure zero. By the former proposition, the value of a function in a measure zero set don't influence the fact whether it is measurable. So we are allowed to talk about whether $f + g$ is measurable in a slightly different condition.

Theorem 3.1 (Linear combinenation and mutlpilication)

Let $f : E \to \mathbb{R}^*$, $g : E \to \mathbb{R}^*$ be measurable and finite a.e. on E, $r \in \mathbb{R}$, then

1. *$r \cdot f$ is measurable.*
2. *$f + g$ is measurable.*
3. *$f \cdot g$ is measurable.*

♡

Proof One can easily find out that $f + g$ is well defined and finite a.e. on E. Let E_0 be the set that $f + g$ is well defined and finite a.e.. Then by the prec 1. Since f is measurable,

$$Ef > \frac{\alpha}{r} = Ef > \alpha$$

is measurable. 2. The element in $E\{f(x) + g(x) < c\}$ has the following property:

$$f(x) + g(x) < c$$
$$f(x) < c - g(x)$$

Since the rational number are dense in $\mathbb{R}$, so for each $x \in E$, we can find a $q \in \mathbb{Q}$ such that

$$f(x) < q < c - g(x)$$

which is equivalence to

$$f(x) < q$$
$$g(x) < c - q$$

So

$$E\{f(x) + g(x) < c\} = \bigcup_{q \in Q} E\{f(x) < c\} \cap E\{g(x) < c - q\}$$

The intersection and countable union of measurable sets are still measurable, thus $E\{f(x)+g(x) < c\}$ is measurable. 3. Observing that $fg = \frac{1}{2}(f+g)^2$, by 1. and 2., we only need to show that if f is measurable, then f^2 is measurable. $\forall x \in E\{f^2(x) < c\}$, by elementary algebra,

$$-c^{\frac{1}{2}} < f(x) < c^{\frac{1}{2}}$$

Thus $E\{f^2(x) < c\} = E\{f(x) > -c^{\frac{1}{2}}\} \cap E\{f(x) < c^{\frac{1}{2}}\}$, which is measurable.

Many properties such as continuity or differentiability, are preserved under the composition of functions. Howerver, in general,the composition of measurable functions are not measurable. The counter-example is provided in the appendix. The basic idea of the counter example is to prove that there exsit a measurable function which maps a measurable set to a non-measurable set, then composite it with a characteristic functio. To get the composition measurable, one more condition is needed.

Theorem 3.2 (Composition)

Let g be a measurable real valued function on a measurable E and f a continuous real valued function defined on $\mathbb{R}$ Then $f \circ g$ is measurable. ♡

We first prove a lemma.

Lemma 3.1

Let $f : E \to \mathbb{R}$ be a function defined on a measurable set E. f is measurable if and only if for each open sets O, $f^{-1}(O)$ is measurable. ♡

Proof Suppose the inverse image of each open sets are measurable, then since $(a, +\infty)$ is open,

$$E\{f(x) > a\} = f^{-1}(a, +\infty)$$

is measurable, which suggests that f is measurable.

Suppose f is measurable. Given any open sets, we can express it by a countable union of open intervals $\{I_n\}$. Suppose $I_n = (a_n, b_n)$. Then

$$\begin{aligned} f^{-1}(O) &= f^{-1}(\bigcup_{n=1}^{\infty} I_n) \\ &= \bigcup_{n=1}^{\infty}(f^{-1}(I_n)) \\ &= \bigcup_{n=1}^{\infty}(E\{f > a_n\} \cap E\{f < b_n\}) \end{aligned}$$

so the inverse image of O is measurable

Proof (Proof of the Composition Theorem) Let $O \subset \mathbb{R}$ be open, then we only need to show the inverse image of O under $f \circ g$ is measurable.

$$(f \circ g)^{-1}(O) = g^{-1}(f^{-1}(O))$$

Since f is continuous, $f^{-1}(O)$ is open. By the measurability of g, $g^{-1}(f^{-1}(O))$ is measurable. By the upper lemma, the proposition is proved.

At the last of this section, we show that for finite many functions, their maximum function is still measurable.

Proposition 3.4

For a set of finite functions $\{f_n\}_{n=1}^N$, $max(f_1,\ f_2, \dots f_n)$ *is measurable.*

Proof We prove the case of $n = 2$, and use the mathematic induction.

$$E\{max(f,\ f') < c\} = E\{f < c\} \cap E\{f' < c\}$$

So it is measurable. The part of induction is rather trivial.

The proposition for minimum can be proved in a simialr way. From the upper proposition, one can know

$$|f| = max(f, 0) - min(f, 0)$$

is measurable.

3.2 Simple function approximation of measurable function

Definition 3.2 (Characteristic functions)

Given a subset A of $\mathbb{R}$, the function $\chi_A : \mathbb{R} \to \mathbb{R}$ defined by

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

is called the characteristic function of A.

It is clear that χ_A is measurable if and only if A is measurable.

Proposition 3.5 (Some basic properties of characteristic function)

Let $A,\ B$ be sets, then

1. $\chi_{A\cap B} = \chi_A \cdot \chi_B$
2. $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$
3. $\chi_{A^c} = 1 - \chi_A$

Proof Here we prove only the first equality, the rest can be proved in the same manner, leaving as an exercise to the readers.

If $\chi_{A\cap B}(x) = 1$, then $x \in A$ and $x \in B$, so

$$\chi_A \cdot \chi_B = 1 \cdot 1 = 1$$

If $\chi_{A\cap B}(x) = 0$, then $x \notin A \cap B$. So one of χ_A, χ_B would be zero, which suggest

$$\chi_A \cdot \chi_B = 0$$

Definition 3.3 (Simple function)

*A real valued measurable function is **simple** if and only if it takes only finite values. Or in another word, a simple function is measurable and it's image is a finite subset of $\mathbb{R}$.* ♣

One can see immediately that χ_A is a simple function if it is measurable. We can express any simple function by sum of characteristic functions.

Proposition 3.6 (Canonical representation of simple function)

If $\phi : E \to \mathbb{R}$ is a simple function, then there exist a finite distinct sequence of numbers $\{a_n\}$ and a finite sequence of disjiont measurable sets $\{A_n\}$, such that

$$\phi(x) = \sum_{n=1}^{N} a_n \chi_{A_n}(x) \tag{3.1}$$

♠

Proof Since ϕ takes only finite real values, we arrange these number into a sequence $\{a_n\}$. Let A_n be the set contains all the $x \in E$ such that $\phi(x) = a_n$. Because ϕ is measurable, so is A_n. (3.1) is true for those sequence we have just found.

It isn't hard to see that finite sum of simple functions is still simple function, scaler multiple of simple function is still simple function. The proof of next proposition is left to the readers.

Proposition 3.7 (The linearity of simple functions)

Let $\varphi : E \to \mathbb{R}$, $\psi : E \to \mathbb{R}$ be simple functions, and r be a real number. Then

$$\varphi + \psi\ , \quad c \cdot \varphi \text{ are simple function} \tag{3.2}$$

♠

Lemma 3.2 (Simple approximation lemma)

Let $f : E \to \mathbb{R}$ be a bounded measurable function. For all $\epsilon > 0$, there exist simple functions $\varphi : E \to \mathbb{R}$, $\psi : E \to \mathbb{R}$, such that

$$\begin{gathered} \varphi \leq f \leq \psi \\ \psi - \varphi \leq \epsilon \end{gathered} \tag{3.3}$$

♡

Proof Let $[c, d]$ be a close bounded interval, whose interior contains $f(E)$ and let $P = \{c = y_0, y_1, y_2, \ldots y_n = d\}$ be a partition of $[c, d]$ such that $y_i - y_{i-1} < \epsilon$. We define $E_i = E\{y_{i-1} \leq f < y_i\}$, and

$$\begin{aligned} \varphi(x) &= \sum_{i=1}^{n} y_{i-1} \chi E_i(x) \\ \psi(x) &= \sum_{i=1}^{n} y_i \chi_{E_i}(x) \end{aligned} \tag{3.4}$$

Then it is easy to varify that these two functions satisfy (3.3)

Theorem 3.3 (Simple approximation Theorem)

A extended real value function on a measuralbe set, say $f : E \to \mathbb{R}^*$, is measurable if and only if there exists a sequence of simple functions $\{\varphi_n\}$ which converge to f on E pointwisely, and has the property that

$$|\varphi_n| \leq |f| \text{ on } E \text{ for all } n$$

More over, if the function is bounded, $\{\varphi_n\}$ can converge uniformly; if the function is nonegative, $\{\varphi_n\}$ can converge increasingly. ♡

Proof Since each simple functions are measurable, and sequence of measurable functions converge to measurable function, so the "if" is proved.

Assume f is measurable, and also assume $f \geq 0$ on E. Let $E_n = \{x \in E : f(x) \leq n\}$, then f is nonegative bounded measurable function on E_n. So by simple approximation lemma, there exist $\varphi_n,\ \psi_n$, such that

$$\varphi_n \leq f \leq \psi_n$$
$$\psi_n - \varphi_n \leq \frac{1}{n}$$

on E_n. So

$$f - \varphi_n \leq \frac{1}{n}$$

on E_n. We extend φ_n on all E by setting $\varphi_n(x) = n$ for x such that $f(x) \geq n$.

Claim: $\{\varphi_n\}$ converge to f.

If $f(x)$ is finite, then exists a N such that $f(x) < N$. Then

$$0 < f(x) - \varphi_n(x) < \frac{1}{n} \text{ for } n \geq N$$

Thus $\varphi_n(x)$ converge to $f(x)$.

If $f(x) = \infty$, then $\varphi_n(x) = n$, thus we also have $\lim_{n\to\infty} \varphi_n(x) = \infty$. Replacing each φ_n by $max(\{\varphi_i\}_{i=1}^n)$, then we have φ_n converge increasingly.

If f is not nonegative, let $f_+ = max(f, 0),\ f_- = -min(f, 0)$, then $f = f_+ - f_-$. Since $f_+,\ f_-$ are nonegative measurable functions, one can find two sequences of simple functions $\{\varphi_n\},\ \{\psi_n\}$, whose limit are f_+ and f_- separately. Thus

$$\begin{aligned} f &= f_+ - f_- \\ &= \lim_{n\to\infty} \varphi_n - lim_{n\to\infty}\psi_n \\ &= \lim_{n\to\infty} \varphi_n - \psi_n \end{aligned} \tag{3.5}$$

Since the sum of simple functions are still simple function, we find a sequence of simple function which converge to f pointwisely.

If f is bounded, there exist a $M > |f|$. So $f(E) \subset [-M, M]$. Given $n > 0$, We can choose a partition of $M,\ P = \{y_i\}_{i=1}^n$ which is finer enough, such that $y_i - y_{i-1} < \frac{1}{n}$, and define φ_n in the same manner. Then $\forall \epsilon > 0,\ \exists N$ such that $\forall n \geq N,\ x \in E,\ |\varphi_n - f| \leq \epsilon$.

3.3 Egoroff's Theorem & Lusin's Theorem

Now we are able to prove the last two principles which have been mentioned in the beginning of this chapter. The result is quite amazing. We will first deal with the Egoroff's theorem, which shows the thrid principle. In some sense, sequences of measurable functions which converge pointwisely are nearly converge uniformly. To prove this theorem, we first need to know more about the convergence of measurable functions and measurable sets.

Proposition 3.8 (Continuity of measure)

Let $\{A_n\}$ be a ascending sequence of sets, which means $A_1 \subset A_2 \subset \dots A_n \subset \dots$. Let $A = \bigcup_{n=1}^{\infty} A_n$. Then

$$m(A) = \lim_{n\to\infty} m(A_n)$$

If $\{B_n\}$ is descending, then let $B = \bigcap_{n=1}^{\infty} B_n$. We have

$$m(B) = \lim_{n\to\infty} m(B_n)$$

♠

Proof We first prove the part of ascending sequence. If there is a index k such that $m(A_k) = \infty$, then by the monotonicity of measure $m(A) = \infty$, thus the inequality holds. If all A_k are finite, we define $A_0 = \varnothing$, and $C_n = A_n - A_{n-1}$. Then

$$A = \bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} C_n$$

and $\{C_n\}$ are disjoint. For disjoint union of sets of finite measure, we can apply the countable additivity of measure, and the excirsion property of the measurable set,

$$\begin{aligned} m(A) = m(\bigcup_{n=1}^{\infty} C_n) &= \sum_{n=1}^{\infty} m(A_n - A_{n-1}) \\ &= \sum_{n=1}^{\infty} m(A_n) - m(A_{n-1}) \quad = \lim_{n\to\infty} m(A_n) - m(A_0) = \lim_{n\to\infty} m(A_n) \end{aligned}$$

For descending sequence $\{B_n\}$, if for all n, $m(B_n) = \infty$, then?????????????

Proposition 3.9 (Pointwise limit of measurable functions)

If a sequence of measurable functions $\{f_n\}$ converge pointwise a.e. on E, where E is their domain. Then the limit function f to which $\{f_n\}$ converge, is measurable.

♠

Proof Suppose the E_0 is the subset which f doesn't converge. By proposition 3.3, f is measurable if and only if it's ristriction on $E - E_0$ is measurable. So it is reasonable to assume that $\{f_n\}$ converge on E. For all $x \in E\{f < c\}$, since f is the pointwise limit, there exist a N and m such that

$$\forall n \geq N,\ f_n(x) < c - \frac{1}{m}$$

Since $E\{f_n < c - \frac{1}{m}\}$ is measurable for all $m,\ n$, thus

$$\bigcap_{n=N}^{\infty} E\{f_n < c - \frac{1}{m}\}$$

is measurable. Consequently,

$$E\{f(x) < c\} = \bigcup_{1 \le N, m < \infty} (\bigcap_{n=N}^{\infty} E\{f_n < c - \frac{1}{m}\})$$

is measurable.

Theorem 3.4 (Egoroff's Theorem)

If a sequence of measurable functions $\{f_n\}$ converge pointwise a.e. on E, where E is their domain which has finite measure. Then $\forall \epsilon > 0$, there exist a closed $F \subset E$, such that $\{f_n\}$ converge uniformly on F, and $m(E - F) < \epsilon$.

Since we want to prove the uniform convergence, the estimation $|f_n - f| < a$ will be important. Also, we need to find a closed set which is good enough. We will first find a measurable set which is good enough(which is easier), and then use closed set to approximate it. To these purposes, it is conveneint to establish the following lemma.

Lemma 3.3

Under the assumptions of Egoroff's theorem, for each $a, b > 0$, there is a measurable subset $A \subset E$ and a N' such that

$$|f_n - f| < a \text{ on } A \text{ for all } n > N \text{ where } m(E - A) < b$$

Proof We can simplely assume $f_n \to f$ on E. By the previous discussion, one can know that $|f_n - f|$ is measurable function. Let

$$E_N = \bigcap_{n=N}^{\infty} E\{|f_n - f| < a\}$$

which is a measurable subset of E. Since $\{f_n\}$ converge pointwisely to f, $\{E_N\}$ converge to E as n goes to infinity. So by excirsion property, there exists a N' such that $m(E - E_{N'}) = m(E) - m(E_{N'}) = b$. For the same $E_{N'}$, pick any x in $E_{N'}$, we have

$$\forall n \ge N', \; |f_n - f| < a$$

So $A = E_{N'}$ is the set we want.

Proof (Proof of the Egoroff's Theorem) Given $\epsilon > 0$, let $A = \bigcap_{m=1}^{\infty} A_m$, where A_m is the set in the above lemma such that there exists a N, for $n \ge N$, $|f_n - f| < \frac{1}{m}$ on A_m, and $m(E - A_m) < \frac{\epsilon}{2^{m+1}}$. It is easy to see that A is measurable subset of E. And we also have

$$m(E - A) = m(E - (\bigcap_{m=1}^{\infty} A_m)) = m(\bigcap_{m=1}^{\infty} (E - A_m)) \le \sum_{m=1}^{\infty} m(E - A_m) < \sum_{m=1}^{\infty} \frac{\epsilon}{2^{m+1}} = \frac{\epsilon}{2}$$

$$m(E - A) < \frac{\epsilon}{2}, \quad \forall \epsilon' > 0, \; \forall x \in A, \; \exists N \; s.t. \; \forall n > N, \; |f_n(x) - f(x)| < \epsilon'$$

which means f_n converge uniformly on A. Using the close inner approximation of measurable set, we can find a close subset of A, say F, such that, $m(A - F) = \frac{\epsilon}{2}$ Finally

$$m(E - F) = m((E - A) \cup (A - F)) = m(E - A) + m(A - F) = \epsilon$$

and $\{f_n\}$ converge uniformly on F.

This amazing theorem is very powerful. For pointwise converge sequence of functions, you can use Egoroff's theorem to get a strong condition on a very big subset. One thing need to be paid attention on is that we can only apply Egoroff's theorem on the functions which are defined on finite measure sets. In mathematic analysis, uniform converge can preserve the integrability, continuity. The next theorem is the proof of the Littlewood second principle, which is related to continuity. We will see how Egoroff's theorem work.

The main idea of the prove of Lusin's theorem is to first prove the case of simple function, then use simple functions to approximate measurable functions. Since for

Theorem 3.5 (Lusin's Theorem)

Let f be a measurable function which is finite a.e. on E, a measurable set. Then given $\delta > 0$, there exist a closed subset $F \subset E$, such that $m(E - F) < \delta$, f is conintuous on F.

Proof Suppose f is finite. First we consider the case of $f(x) = \sum_{n=1}^{N} a_n \chi_{E_n}(x)$ is a simple function, where the sum is the canonical representation. Let F_n be close subset of E_n, by inner approximation, we can require $m(E_n - F_n) = \frac{\delta}{n}$. On each E_n, f is a constant function, so is it on each F_n. Let $F = \bigcup_{n=1}^{N} F_n$. Being a finite union of close sets, f is closed. Then

$$\begin{aligned} m(E - F) &= m(\bigcup_{n=1}^{N} E_n - \bigcup_{n=1}^{N} F_n) \\ &\leq m(\bigcup_{n=1}^{N} E_n - F_n) \\ &= \sum_{n=1}^{N} m(E_n - F_n) \\ &\leq \sum_{n=1}^{N} \frac{\delta}{n} = \delta \end{aligned}$$

where the first inequality is come from the fact that $\bigcup_{n=1}^{N} E_n - \bigcup_{n=1}^{N} F_n \subset \bigcup_{n=1}^{N} E_n - F_n$. Thus we proved the case of f is simple functions.

Suppose $m(E) < +\infty$. By simple approximation of measurable function, we can find a sequence of simple functions $\{\varphi_i\}$, which converge to f pointwisely. By Egoroff's theorem, there exists a close subset $F \subset E$, such that $\{\varphi_i\}$ coneverge uniformly on F to f, while $m(E - F) < \frac{\epsilon}{2}$.

From the previous part of this proof, for every $\{\varphi_i\}$, we can find a closed set of E, say F_i, such that φ_i is continuous on F_i, while $m(E - F_i) < \frac{\epsilon}{2^{i+1}}$. We can see that, all φ_i is continuous on $\bigcap_{i=1}^{\infty} F_i$, which is a close set, and we have

$$\begin{aligned} m(E - (\bigcap_{i=1}^{\infty} F_i)) &= m(\bigcup_{i=1}^{\infty} E - F_i) \\ &\leq \sum_{i=1}^{\infty} m(E - F_i) \\ &\leq \sum_{i=1}^{\infty} \frac{\epsilon}{2^{i+1}} \\ &= \frac{\epsilon}{2} \end{aligned}$$

On $F \cap (\bigcap_{i=1}^{\infty} F_i)$, which is a close set, $\{\varphi_i\}$ is continuous and uniformly converge to f. By knowledge from mathematic analysis, f is continuous on $F \cap (\bigcap_{i=1}^{\infty} F_i)$. It only remains to show that $m(E - (F \cap (\bigcap_{i=1}^{\infty} F_i)))$ is small.

$$\begin{aligned} m(E - (F \cap (\bigcap_{i=1}^{\infty} F_i))) &= m((E - F) \cup (E - \bigcap_{i=1}^{\infty} F_i)) \\ &\leq m(E - F) + m(E - \bigcap_{i=1}^{\infty} F_i) \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Then we proved the case that E has finite measure.

For the case that $m(E) = +\infty$, we consider the ristirction of f on $E_n = E \cap [-n, n]$. Clearly, $f|_{E_n}$ are measurable functions defined on finite measure sets, so by the Lusin's theorem for finite case, we can find F_n for each E_n such that $f = f|_{E_n}$ is continuous on F_n, and $m(E_n - F_n) < \frac{\epsilon}{2^{n+1}}$.

Let $F = \bigcup_{n=1}^{\infty} F_n$, which is measurable. By our construction, f is continuous on F. With the fact that $\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{\infty} F_n \subset \bigcup_{n=1}^{\infty} E_n - F_n$ we can obtain the estimation below,

$$\begin{aligned} m(E - \bigcup_{n=1}^{\infty} F_n) &= m(\bigcup_{n=1}^{\infty} E_n - \bigcup_{n=1}^{\infty} F_n) \\ &\leq m(\bigcup_{n=1}^{\infty} E_n - F_n) \\ &\leq \sum_{n=1}^{\infty} m(E_n - F_n) \\ &\leq \sum_{n=1}^{\infty} \frac{\epsilon}{2^{n+1}} \\ &= \frac{\epsilon}{2} \end{aligned}$$

But this is not the end, since F is not a close set. According to the inner approximation of measurable function, we can find a close set of F, say F', such that $m(F - F') < \frac{\epsilon}{2}$. Then

$$\begin{aligned} m(E - F') &= m((E - F) \cup (F - F')) \\ &= m(E - F) + m(F - F') \\ &= \frac{\epsilon}{2} + \frac{\epsilon}{2} \qquad\qquad = \epsilon \end{aligned}$$

The continuous function on close set can be extended to $\mathbb{R}$, so we can have the following form of Lusin' theorem.

Theorem 3.6 (Lusin's theorem: another form)

Let f be a measurable function which is finite a.e. on E, a measurable set. Then given $\delta > 0$, there exist a closed subset $F \subset E$, and a continuous function $g : \mathbb{R} \to \mathbb{R}$, such that $m(E - F) < \delta$, $f = g$ on F.

Chapter 4 Lebesgue Integration

Now all the preparations have done, it is the time to define the integral. We will first define the the integral for simple functions on sets of finite measure; then for the bounded measurable functions on finite measure sets; then for nonegative measurable functions on measurable sets; finally the general measurable functions over measurable sets. Without other modifier, the word ”integral” in this chapter means Lebesgue integral.

4.1 Integral of Simple Functions

Remember that in the previous section, we have show that a simple function $\varphi : E \to \mathbb{R}$ can be written in this form

$$\varphi(x) = \sum_{n=1}^{N} a_n \chi E_n$$

where $E_n = E\{f = a_n\}$. This way of writting simple functions suggest a natrual way to define the integral for them.

Definition 4.1 (The integral for simple functions)

Let $\varphi : E \to \mathbb{R}$ be a simple function, where E is a measurable set. We define the integral of φ over E

$$\int_E \varphi = \sum_{n=0}^{N} a_n \cdot m(E_n) \tag{4.1}$$

where $\sum a_n \chi_{E_n}(x)$ is the canonical representation of φ. ♣

Lemma 4.1

Let $\{E_i\}$ be a finite disjoint of subsets of a finite measure set E. Let $\{a_i\}$ be a sequence of real number, then

$$\varphi = \sum_{i=1}^{n} a_i \chi_{E_i} \text{ on } E \;\Rightarrow\; \int_E \varphi = \sum_{i=1}^{n} a_i \cdot m(E_i)$$

Remark: *One may wonder why we should prove this(it just seems like have no difference to the definition of integral). Since $\{a_i\}$ is not distinct, $\sum_{i=1}^{n} a_i \chi_{E_i}$ may not be the canonical representation of φ.* ♡

Proof Let $\{\lambda_j\}$ be distinct values taken by φ and $A_j = \{x : \varphi(x) = \lambda_j\}$. By the definition of integral,

$$\int_E \varphi = \sum_{j=1}^{m} \lambda_j \cdot m(A_j)$$

Let $I_j = \{i : a_i = \lambda_j\}$. It isn't hard to see that $\sum_{i\in I_j} m(E_i) = A_j$

$$\begin{aligned}\sum_{i=1}^{n} a_i \cdot m(E_i) &= \sum_{j=1}^{n}(\lambda_j \cdot \sum_{i\in I_j} m(E_i)) \\ &= \sum_{j=1}^{n}(\lambda_j \cdot m(A_i)) \\ &= \int_E \varphi\end{aligned} \tag{4.2}$$

Of course we want to know the properties of the general Lebesgue integral, here is our first step in this direction.

Proposition 4.1 (The linearity of the intergration: simple function)

Let φ, ψ be simple functions on E, which is a set of finite measure. Then for $\alpha,\ \beta \in \mathbb{R}$,

$$\int_E (\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi \tag{4.3}$$

Proof For linearity, we first prove $\int_E \alpha\varphi = \alpha \int_E \varphi$(scaler can 'go though' the integral sign), then prove (4.3) for the case of $\alpha = \beta = 1$. We suppose $\varphi = \sum_{i=1}^{n} a_i\chi_{E_i}$ and $\psi = \sum_{i=1}^{m} b_i\chi_{F_i}$.

$$\begin{aligned}\int_E \alpha\varphi &= \sum_{i=1}^{n} \alpha \cdot a_i m(E_i) \\ &= \alpha \sum_{i=1}^{n} a_i m(E_i) \\ &= \alpha \int_E \varphi\end{aligned} \tag{4.4}$$

Let $A_{ij} = E_i \cap F_j$, where $1 \le i \le n$, $1 \le j \le m$. For all $x \in E$, $x \in E_i$ and $x \in F_j$ for some $i,\ j$, and since $\{E_i\}$ and $\{F_j\}$ are disjoint, so $\{A_{ij}\}$ is a disjoint finite collection of subset of E whose union is E. Let a_{ij}, b_{ij} be the value of $\varphi,\ \psi$ on A_{ij}. By lemma 4.1,

$$\begin{aligned}\int_E \varphi &= \sum_{1\le i\le n,\ 1\le j\le m} a_{ij} m(A_{ij}) \\ \int_E \psi &= \sum_{1\le i\le n,\ 1\le j\le m} b_{ij} m(A_{ij}) \\ \Rightarrow \int_E \varphi + \int_E \psi &= \sum_{1\le i\le n,\ 1\le j\le m} a_{ij} m(A_{ij}) + \sum_{1\le i\le n,\ 1\le j\le m} b_{ij} m(A_{ij}) \\ &= \sum_{1\le i\le n,\ 1\le j\le m} (a_{ij} + b_{ij}) m(A_{ij}) \\ &= \int_E \varphi + \psi\end{aligned}$$

Proposition 4.2 (Monotonicity of the integral)

Let φ, ψ be simple functions on E, which is a set of finite measure. Then

$$\varphi \leq \psi \Rightarrow \int_E \varphi \leq \int_E \psi$$

Proof Just like what we did in the previous proposition, we define $A_{ij},\ a_{ij},\ b_{ij}$ such that

$$\int_E \varphi = \sum_{1\leq i\leq n,\ 1\leq j\leq m} a_{ij} m(A_{ij})$$

$$\int_E \psi = \sum_{1\leq i\leq n,\ 1\leq j\leq m} b_{ij} m(A_{ij})$$

It is easy to see that $a_{ij} \leq b_{ij}$, so is their finite sum.

4.2 Integral for Bounded Measurable Functions over Bounded Sets

In this section, we define the integral over bounded sets for bounded measurable functions out of the integral for simple function.

Definition 4.2 (Upper integral, lower integral and integral)

Let $f : E \to \mathbb{R}$ be a Bounded measurable function over a finite measure set. We define

$$\sup \int_E \varphi : \varphi \text{ is simple and } \varphi \leq f$$

$$\sup \int_E \psi : \psi \text{ is simple and } \psi \geq f$$

to be the upper integral and the lower integral of f respectively, and denote them as

$$\sup \int_E f,\ \ \inf \int_E f$$

*If upper integral and lower integral of f on E are equal, we define there common value to be the **Lebesgue integral** of f on E.*

Theorem 4.1 (Lebesgue integral is a generalization of Riemann integral)

If a bounded function f defined on close bounded interval $[a, b]$ is Riemann integrable, then it is Lebesgue integrable, and two integrals are equal.

In this theorem, we use $(R)\int$ to represent Riemann integral.

Proof Since f is Riemann integrable on $[a, b]$, it means

$$\sup\{(R)\int_{[a,b]} \varphi(x) : \varphi(x) \leq f \text{ is a step function }\} = \inf\{(R)\int_{[a,b]} \psi(x) : \psi(x) \geq f \text{ is a step function }\}$$

Each step function is a simple function, and we can easily observe that the Riemann integral and Lebesgue integral are equal(since the lenth of the interval is the Lebesgue measure of the interval). We can find a sequence of step functions, which are measurable functions, converge to f pointwisely.

So f is a measurable function. The Riemann integralbility also indicate that f is bounded. Then by the definition of Lebesgue integral for bounded measurable functions, the Lebesgue integral exist and

$$\int_{[a,b]} f = (R)\int_{[a,b]} f$$

Theorem 4.2 (Bounded measurable functions are integralbe)

Let f be a bounded measurable function which is defined on a set with finite measure, say E. Then f is measurable. ♡

Proof Since f is bounded and measurable, given $\epsilon > 0$, we can find two sequence of simple functions $\{\varphi_n\},\ \{\psi_n\}$, such that

$$\varphi \leq f \leq \psi$$

and $\psi - \varphi < \frac{\epsilon}{m(E)}$. By the definition of upper and lower integral and the linearity of the integral for simlpe functions

$$\begin{aligned}\int_E \varphi \leq \inf \int_E f \leq \sup \int_E f \leq \int_E \psi \\ \Rightarrow \sup \int_E f - \inf \int_E f &\leq \int_E \psi - \int_E \varphi \\ &= \int_E (\psi - \varphi) \\ &\leq \int_E \frac{\epsilon}{m(E)} \\ &= \epsilon\end{aligned}$$

Then by the arbitrariness of ϵ, $\sup \int_E f = \inf \int_E f$, thus the theorem is proved.

In fact, a bounded function over finite measure set is integrable if and only if it is measurable, we will prove this later. Now we can establish the theorem for the linearity and monotonicity of the integral of bounded measurable functions.

Theorem 4.3 (Linearity)

Let E be a finite measure set, and $f,\ g$ be bounded measurable function define on E. Let $\alpha,\ \beta$ be real numbers. Then

$$\alpha \int_E f + \beta \int_E g = \int_E \alpha f + \beta g \tag{4.5}$$

♡

Proof We first prove that

$$\int_E \alpha f = \alpha \int_E f$$

, then prove

$$\int_E f + \int_E g = \int_E f + g$$

Suppose $\alpha > 0$(the argument for $\alpha < 0$ is similar), by simple approximate lemma, given ϵ, there exist simple functions $\varphi,\ \psi$, such that $\psi - \varphi < \frac{\epsilon}{\alpha \cdot m(E)}$

$$\varphi \leq f \leq \psi$$

$$\alpha\varphi \leq \alpha f \leq \alpha\psi$$

By the upper theorem, αf is integralbe. And by the definition of the integral, the linearity of the integral of simple functions,

$$\int_E \alpha\varphi \leq \int_E \alpha f \leq \int_E \alpha\psi$$

$$\Rightarrow \alpha \int_E \varphi \leq \int_E \alpha f \leq \alpha \int_E \psi$$

Notice that we also have

$$\alpha \int_E \varphi \leq \alpha \int_E f \leq \alpha \int_E \psi$$

So

$$\begin{aligned} |\alpha \int_E f - \int \alpha f| &\leq \alpha \int_E \psi - \alpha \int_E \varphi \\ &\leq \alpha \int_E \psi - \varphi \\ &\leq \alpha \int_E \frac{\epsilon}{\alpha \cdot m(E)} \\ &\leq \alpha \cdot m(E) \cdot \frac{\epsilon}{\alpha \cdot m(E)} = \epsilon \end{aligned}$$

Thus the first part of the theorem is proved.

Since the sum of measurable function is still measurable, $f + g$ is integralbe over E. Suppose $\varphi_1,\ \varphi_2,\ \psi_1,\ \psi_2$ are simple functions such that

$$\varphi_1 \leq f \leq \psi_1,\ \varphi_2 \leq g \leq \psi_2$$

We can see that

$$\begin{aligned} \int_E f + g &= \sup \int_E f + g \\ &\leq \int_E \psi_1 + \psi_2 \\ &= \int_E \psi_1 + \int_E \psi_2 \end{aligned}$$

Since the above inequality is true for all $\psi_1 > f$, $\psi_2 > g$, we have

$$\begin{aligned} \int_E f + g &\leq \inf_{\psi_1 \geq f} \int_E \psi_1 + \inf_{\psi_2 \geq g} \int_E \psi_2 \\ &= \int_E f + \int_E g \end{aligned}$$

On the other hand,

$$\begin{aligned}\int_E f+g &= \inf \int_E f+g \\ &\geq \int_E \varphi_1+\varphi_2 \\ &= \int_E \varphi_1 + \int_E \varphi_2\end{aligned}$$

This inequality is also true for all $\varphi_1 > f$, $\varphi_2 > g$, thus

$$\begin{aligned}\int_E f+g &\geq \sup_{\varphi_1 \leq f} \int_E \varphi_1 + \sup_{\varphi_2 \leq g} \int_E \varphi_2 \\ &= \int_E f + \int_E g\end{aligned}$$

Together the two inequality, we have

$$\int_E f+g = \int_E f + \int_E g$$

. Thus we prove the linearity of the integration for bounded measurable functions.

Theorem 4.4 (Monotonicity)

Let E be a finite measure set, and f, g be bounded measurable function define on E. If $f \leq g$ on E, then

$$\int_E f \leq \int_E g \tag{4.6}$$

Proof Let $h = g - f$, which is a non-negative measurable function. By linearity,

$$\int_E g - \int_E f = \int_E g - f = \int_E h$$

Since h is non-negative, $h \geq 0$. Let $\psi = 0$, we can see that

$$\int_E h \geq \int \psi = 0$$

which shows that

$$\int_E g - \int_E f \geq 0$$

Thus the monotonicity of the integration of bounded measurable functions is proved.

By the Monotonicity, and the fact that $-|f| \leq f \leq |f|$we can get the following useful conclusion.

Corollary 4.1 (Absolute value)

Let f be a bounded measurable function defined on a set of finite measure. Then

$$\int_E |f| \geq |\int_E f|$$

Proposition 4.3 (The integral over measure zero set is zero)

Suppose f is bounded measurable function on a measure zero set E, then

$$\int_E f = 0$$

Proof Since f is bounded, there exist a M, such that $|f| < M$ on E. So by monotonicity of the integral, we have

$$\begin{aligned}\int_E |f| &\leq \int_E M \\ &= M \cdot m(E) \quad = 0\end{aligned}$$

By previous corollary, the proposition is proved

Now we proved the last basic property of the integral: the additivity over domain.

Lemma 4.2

Let f be a bounded measurable function defined on E, and E_0 be a measurable subset. Then

$$\int_{E_0} f = \int_E f \cdot \chi_{E_0}$$

♡

Proof For all simple functions $\varphi \leq f$, we exetend φ to a new simple function on E by letting $\varphi'(x) = 0$ for $x \in E - E_0$. Notice that the integral of two functions are the same:

$$\int_{E_0} \varphi = \int_E \varphi'$$

For $x \in E_0$, $\varphi \leq f|_{E_0} = f \cdot \chi_{E_0}$; for $x \in E - E_0$, $\varphi = 0 \leq f \cdot \chi_{E_0}$. Thus

$$\begin{aligned}\int_{E_0} f &= \inf_{\varphi \leq f} \int_{E_0} \varphi \\ &= \inf_{\varphi' \leq f} \int_E \varphi' \\ &\leq \int_E f \cdot \chi_{E_0}\end{aligned}$$

For simple functions $\psi \geq f$, the extension ψ' on E with $\psi'(x) = 0$ for $x \in E - E_0$, is also a simple funcion. For $x \in E_0$, $\psi' \geq f|_{E_0} = f \cdot \chi_{E_0}$; for $x \in E - E_0$, $\psi' = 0 \geq f \cdot \chi_{E_0}$.

$$\begin{aligned}\int_{E_0} f &= \sup_{\psi \geq f} \int_{E_0} \psi \\ &= \inf_{\psi' \geq f} \int_E \psi' \\ &\geq \int_E f \cdot \chi_{E_0}\end{aligned}$$

Together the two inequality, we proved the theorem.

Theorem 4.5 (The additivity over domain)

Let f be bounded measurable functions over E, which is a finte measure set. Suppose A and B are disjoint subsets of E, then

$$\int_{A \cup B} f = \int_A f + \int_B f$$

♡

Proof By the linearity of the integral, and the lemma above,

$$\begin{aligned}\int_A f + \int_B f &= \int_{A\cup B} f\cdot\chi_A + \int_{A\cup B} f\cdot\chi_B \\ &= \int_{A\cup B} f\cdot\chi_A + f\cdot\chi_B \\ &= \int_{A\cup B} f\cdot\chi_{A\cup B} \\ &= \int_{A\cup B} f\end{aligned}$$

From the additivity over domain, we can derive a interesting result, which can enhance many propositions which involving integral.

Corollary 4.2 (Measure zero set does not influence the integral)

Let f, g be bounded measurable function on E, which is a finite measure set. If $f = g$ a.e. on E,

$$\int_E f = \int_E g$$

♡

Proof Let E_0 be the set that $f \neq g$. By additivity over domian,

$$\begin{aligned}\int_E f &= \int_{E_0} f + \int_{E-E_0} f \\ \int_E g &= \int_{E_0} g + \int_{E-E_0} g \\ \Rightarrow \int_E g - \int_E f &= \int_{E_0} g - \int_{E_0} f\end{aligned}$$

Since the integral over measure zero set is measure zero, the corollary is proved.

4.3 The Lebesgue Integral for Non-negative Functions

In this section, we push the definition of integral further, to non-negative measurable functions over general measurable sets. We allowed the function to take large value(not being bounded anymore, but don't take values at infinity), and define it on a larger space(without requiring E to have finite measure).

To this purpose, it is convenient to establish the concept of **finite support functions**.

Definition 4.3 (Support)

*For a function $f : E \to \mathbb{R}^*$, where E is a subset of $\mathbb{R}$, we say it's **support** E_0 is the closure of*

$$\{x \in E : f(x) \neq 0\}$$

If the support is finite, then we say f is a function has finite support, or compact support(notice that E_0 is closed and bounded, thus compact).

♣

It is not hard to see that the linear combinenation of finite support functions are still finite support.

We haven't defined the integral on a set which has infinite measure. Suppose $f : E \to \mathbb{R}$ is bounded, measurable and has a finite support E_0, then

$$\int_E f := \int_{E_0} f$$

is a very natrual definition. Any thing we proved in the last section can be applied on this new definition. For non-negative measurable function, we can define it's integral through the integral of bounded measurable function of finte support.

Definition 4.4 (Integral of non-negative measurable functions)

Let $f : E \to \mathbb{R}^*$ be a non-negative measurable function defined on a measurable set. The integral of f is defined as

$$\int_E f := \sup\{\int_E \varphi : \varphi \text{ is bounded measurable and with finite support}\,; 0 \leq \varphi \leq f\} \tag{4.7}$$

♣

To prove the properties of the integral of non-negative functions, **Chebychev's inequality** is needed.

Lemma 4.3 (Chebychev's inequality)

Let f be a non-negative function on E. Then for any $\lambda > 0$,

$$m(\{x \in E : f(x) \geq \lambda\}) \leq \frac{1}{\lambda} \cdot \int_E f \tag{4.8}$$

♡

Proof We denoted $\{x \in E : f(x) \geq \lambda\}$ as E_λ. Suppose g is defined by the following:

$$g(x) := \begin{cases} \lambda & \text{if } x \in E_\lambda \\ 0 & \text{if } x \in E - E_\lambda \end{cases}$$

Then $g \leq f$. If $m(E_{lambda})$, g is measurable bounded with finite support. By definition,

$$\int_E g \leq \int_E f$$

$$\lambda \cdot m(E_\lambda) \leq \int_E f$$

$$m(E_\lambda) \leq \frac{1}{\lambda} \int_E f$$

If $m(E_\lambda) = +\infty$, let $E_{\lambda,n} := E_\lambda \cap [-n, n]$. Then $\psi_n = \chi_{E_{\lambda,n}}$ is a bounded measurable function with finite support. And we have $\psi_n \leq f$. By continuity of the measure,

$$\begin{aligned} \lambda \cdot m(E_\lambda) &= \lambda \lim_{n\to\infty} E_{\lambda,n} \\ &= \int_E \psi_n \\ &\leq \int_E f \end{aligned}$$

Times $\frac{1}{\lambda}$ on the both side of the inequality, then we obtained the desirable conclusion.

Proposition 4.4 (Non-negative function and measure zero set)

Let f be a non-negative measurable function on E, then

$$\int_E f = 0$$

if and only if $f = 0$ a.e. on E.

Proof If $f = 0$ a.e. on E, it is bounded, measurable, finite support. Then by the conclusion in the last section,

$$\int_E f = 0$$

Suppose $\int_E f = 0$, and there exist a $E' \subset E$ such that $m(E') > 0$ and $f > 0$ on E'. We can see that

$$E' = \bigcup_{n=1}^{\infty} \{x \in E : f(x) \geq \frac{1}{n}\}$$

Since countable union of measure zero set is still measure zero, there exist N such that $m(\{x \in E : f(x) \geq \frac{1}{n}\}) > 0$. By Chebychev inequality,

$$m(\{x \in E : f(x) \geq \frac{1}{n}\}) \leq n \int_E f = 0$$

we arrive at a contradiction.

Theorem 4.6 (Linearity)

Let f and g be two non-negative measurable functions on a measurable set, say E. Let α, β be two non-negative real numbers. Then

$$\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$$

Proof With ordinary method, we first prove the part of scalar multiple, then the part of addition.

$\int_E \alpha f = \alpha \int_E f$: Notice that for $\alpha > 0$, $0 \leq h \leq f$ if and only if $0 < \alpha h < \alpha f$. Thus by the linearity of finite support,

$$\begin{aligned}
\int_E \alpha f &= \sup_{h' < \alpha} \int_E h' \\
&= \sup_{h < f} \int_E \alpha h \\
&= \alpha \sup_{h < f} \int_E h \\
&= \alpha \int_E f
\end{aligned}$$

where h is bounded and measurable, with finite support. Thus the part of scalar multiple is done.

Let h, k be bounded and measurable with finite support, then their sum is also bounded and measurable, with finite support. By the definition of the integral,

$$\begin{aligned}\int_E h + \int_E k &= \int_E h + k \\ &\leq \int_E f + g\end{aligned}$$

Thus for the supremum of the left side, the inequality still holds.

$$\int_E f + \int_E g \leq \int_E f + g$$

It remains to prove the inequality in the opposite side. Since $\int_E f + g$ is the supremum of $\int_E l$, where $l \leq f + g$ is a bounded measurable function with finite support. So it is surffice to prove the inequality for all such l in place of $f + g$.

For a such l, let $h = min(f, l)$, $k = l - h$. Then if $l(x) \leq f(x)$, $h(x) = l(x)$ and $k(x) = 0$; if $l(x) > f(x)$, $h(x) = f(x)$ and $k(x) = l(x) - f(x) < g(x)$. Then we can see that $h \leq f$, $k \leq g$. Thus by the additivity of the integral for bounded measurable fucntion of finite support,

$$\begin{aligned}\int_E l &= \int_E l + min(f, l) - min(f, l) \\ &= \int_E min(f, l) + \int_E l - min(f, l) \\ &= \int_E h + \int_E k \\ &\leq \int_E f + \int_E g\end{aligned}$$

Thus proved the the part of additivity.

Theorem 4.7 (Monotonicity)

Suppose f and g are non-negative measurable functions on measurable set, such that $f \leq g$ on their domain E. Then

$$\int_E f \leq \int_E g$$

♡

Proof Notice the fact that for all bounded measurable functions φ on E, which satisfy $\varphi \leq f$, we have $\varphi \leq g$. So we know that

$$\sup_{\varphi < f} \int_E \varphi \leq \sup_{\psi < g} \int_E \psi$$

$$\int_E f \leq \int_E g$$

Thus the theorem is proved.

Theorem 4.8 (Additivity over domain)

Let f be a non-negative measurable function over a measurble set E. Suppose E_1, E_2 are disjoint measurable subsets, then we have

$$\int_{E_1 \cup E_2} f = \int_{E_1} f + \int_{E_2} f$$

♡

Proof Observing that $f = f\chi_{E_1} + f\chi_{E_2}$ on $E_1 \cup E_2$, by linearity of the integral,

$$\begin{aligned}\int_{E_1 \cup E_2} f &= \int_{E_1 \cup E_2} f\chi_{E_1} + f\chi_{E_2} \\ &= \int_{E_1 \cup E_2} f\chi_{E_1} + \int_{E_1 \cup E_2} f\chi_{E_2}\end{aligned}$$

One can check that for all measurable bounded finite supported φ on $E_1 \cup E_2$, $\varphi \leq f$ on $E_1 \cup E_2$ if and only if $\varphi\chi_{E_1} \leq f\chi_{E_1}$ on E_1 and $\varphi\chi_{E_2} \leq f\chi_{E_2}$ on E_2. So

$$\begin{aligned}\int_{E_1 \cup E_2} f\chi_{E_1} + \int_{E_1 \cup E_2} f\chi_{E_2} &= \sup_{\varphi < f} \int_{E_1} \varphi\chi_{E_1} + \sup_{\varphi < f} \int_{E_2} \varphi\chi_{E_2} \\ &= \sup_{\varphi < f} \int_{E_1} \varphi + \sup_{\varphi < f} \int_{E_2} \varphi \\ &= \int_{E_1} f + \int_{E_2} f\end{aligned}$$

4.4 The Lebesgue Integral in General

For measurable function f in general, our mean to construct it's integral is to express f as the difference of two non-negative functions. In the chapter of Lebesgue measure functions, we have a conclusion that maximum and minimumu of two measurable functions is still measurable, which allowe us to split f into two part.

Definition 4.5 (Positive part and negative part)

Let f be a measurable function on E. We define it positive part f^+ and negative part f^- as the following:

$$\begin{aligned}f^+ &= max(f, 0) \\ f^- &= -min(f, 0)\end{aligned}$$

♣

It is not hard to find that f^+, f^- are non-negative functions. And the relation

$$f = f^+ - f^-$$

is also clear. But $\int f^+$, $\int f^-$ might be infinity at the same time, then $\infty - \infty$ is not a well defined value. To fix this problem, we introduce the concept of integralbe for non-negative measurable functions.

Definition 4.6 (Integralbe)

Let f be a non-negative function on E, f is integralbe on E if and only if

$$\int_E f$$

is a finite value.

♣

Proposition 4.5

A non-negative function f is integrable over E, then it is finite a.e. on E.

♠

Proof If f is integrable, by Chebychev inequality,

$$m(\{x \in E : f(x) \geq \infty\}) \leq m(\{x \in E : f(x) \geq \lambda\}) \frac{1}{\lambda} \cdot \int_E f$$

Let $\lambda \to \infty$, we see that the measure of the subset where $f = \infty$ is zero.

Proposition 4.6

For a measurable function f on E, f^+ and f^- are integralbe on E if and only if $|f|$ is integralbe on E.

Proof Suppose $|f|$ is integralbe on E. Notice that $|f| = f^+ + f^-$, suppose f^+ is not integralbe

$$\begin{aligned} +\infty &= \int_E f^+ \\ &\leq \int_E f^+ \int_E f^- \\ &= \int_E |f| \end{aligned}$$

Contradicting with the fact that $|f|$ is integralbe, f^+ is integralbe. One can show that f^- is integralbe in the exact same way. Then we proved one direction of the proposition.

Suppose f^+ and f^- are integrable, which means their integrals are finite. Then

$$\begin{aligned} \int_E |f| &= \int_E (f^+ + f^-) \\ &= \int_E f^+ + \int_E f^- \end{aligned}$$

So $|f|$ is integralbe.

Definition 4.7 (Lebesgue integral for measurable functions)

Let f be a measurable function on E. If $|f|$ is integrable on E, then we say f is integrable, and define the integral of f on E as

$$\int_E f = \int_E f^+ - \int_E f^- \tag{4.9}$$

Proposition 4.7

Let f be integrable over E, then f is finite a.e. on E, and

$$\int_E f = \int_{E-E_0} f \tag{4.10}$$

where E_0 is a measure zero subset of E.

Proof If f is not finite almost every where, then $|f| = \infty$ on $F \in E$ where $m(F) > 0$. Then by the additivity over domain, and the monotonicity of integration,

$$\begin{aligned}\int_E |f| &= \int_{E-F} f + \int_F f \\ &\geq \int_F f \\ &= \infty \cdot m(F) \\ &= \infty\end{aligned}$$

which contradict with the fact that f is integral.

Now we prove (4.10). Notice that since the proposition 4.3, being the supremum of $\{0\}$, $\int_{E_0} f^+ = \int_{E_0} f^- = 0$. By the definition of the integral,

$$\begin{aligned}\int_E f &= \int_{E-E_0} f^+ - \int_{E-E_0} f^- \\ &= \int_{E-E_0} f^+ + \int_{E_0} f^+ - \int_{E-E_0} f^- - \int_{E_0} f^- \\ &= \int_{E-E_0} f^+ - \int_{E-E_0} f^- \\ &= \int_{E-E_0} f\end{aligned}$$

Thus the proposition is proved.

Now we prove the familiar basic properties of integration in the most general extend. The function will be the general measurable functions, which can take value at infinity. Although we can't even defined the sum of two measurable functions if they have too much value of infinite, but if we ask them to be integrable, every thing turns out to be nice.

Theorem 4.9 (Linearity)

Let f, g be two measurable functions which are integrable over E, α and β are real numbers. Then $\alpha f + \beta g$ is integrable, and

$$\int_E \alpha f + \beta g = \int_E \alpha f + \int_E \beta g$$

♡

Proof As usual, we first prove αf is integrable and $\int_E \alpha f = \alpha \int_E f$. Since $|\alpha f| = |\alpha||f|$, by the linearity of the inftegral of non-negative functions,

$$\int_E |\alpha f| = \int_E |\alpha||f| = |\alpha| \int_E |f|$$

Since both $|\alpha|$ and $\int_E |f|$ are finite, αf is integrable. If $\alpha \geq 0$,

$$\begin{aligned}\int_E \alpha f &= \int_E (\alpha f)^+ + \int_E (\alpha f)^- \\ &= \alpha \int_E f^+ + \alpha \int_E f^- \\ &= \alpha \int_E f\end{aligned}$$

If $\alpha < 0$,

$$\begin{aligned}\int_E \alpha f &= \int_E (\alpha f)^+ + \int_E (\alpha f)^- \\ &= \alpha \int_E f^+ + \alpha \int_E f^- \\ &= \alpha \int_E f\end{aligned}$$

By the proposition above, f and g are finite a.e. on E, thus their sum is well defined. By triangle inequality of real numbers, $|f+g| \leq |f| + |g|$. Then by the monotonicity of the integral of non-negative functions, one can see that $\int_E |f+g|$ is finite, which indecated that $f+g$ is integrable.

We have the following relationship:

$$(f+g)^+ - (f+g)^- = f + g = (f^+ - f^-) + (g^+ - g^-)$$

Since the value on a measure zero set doesn't influence the integral, one can assuming that the above functions take finite value on E, so

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+ \text{ on } E$$

From the linearity of non-negative functions(notice that both side of the equality above is non-negative),

$$\int_E [(f+g)^+ + f^- + g^-] = \int_E [(f+g)^- + f^+ + g^+]$$

$$\int_E (f+g)^+ + \int_E f^- + \int_E g^- = \int_E (f+g)^- + \int_E f^+ + \int_E g^+$$

Since f, g, $f+g$ are all integrable, the integral of non-negative function above are all finite. So it is allowed to move some of them to the other side.

$$\int_E (f+g)^+ - \int_E (f+g)^- = \int_E f^+ - \int_E f^- + \int_E g^+ - \int_E g^-$$

$$\int_E f + g = \int_E f + \int_E g$$

Theorem 4.10 (Monotonicity)

Let f and g be integralbe over E. If $f \leq g$ on E, then

$$\int_E g \geq \int_E f$$

Proof Since $g \geq f$, it is not hard to see $g^+ \geq f^+$, and $g^- \leq f^-$. By monotonicity of the integral of non-negative function,

$$\int_E g^+ \geq \int_E f^+, \quad \int_E g^- \leq \int_E f^-$$

which indicated

$$\int_E g^+ - \int_E g^- \geq \int_E f^+ - \int_E f^-$$

$$\int_E g \geq \int_E f$$

Theorem 4.11 (Additivity over domain)

Let f be a integralbe function over E, and $E_1,\ E_2$ be measurable subset of E. Then f is integrable over E_1 and E_2, and we have

$$\int_{E_1\cup E_2} f = \int_{E_1} f + \int_{E_2} f$$

Proof By the symmetry of the proposition, we only need to prove that f is integralbe E_1. By the additivity over domian of the integral of non-negative functions, and the fact that the integral of any non-negative function is non-negative,

$$\begin{aligned}\int_E |f| &= \int_{E-E_1} |f| + \int_{E_1} |f| \\ &\geq \int_{E_1} f\end{aligned}$$

So it is clear that f is integrable over E_1.

By the additivity over domain of non-negative functions,

$$\begin{aligned}\int_{E_1\cup E_2} f &= \int_{E_1\cup E_2} f^+ - \int_{E_1\cup E_2} f^- \\ &= \int_{E_1} f^+ + \int_{E_2} f^+ - \int_{E_1} f^- - \int_{E_2} f^- \\ &= \int_{E_1} f + \int_{E_2} f\end{aligned}$$

Here we sum up the property of Lebesgue integral:

$$\begin{aligned}\text{Measure zero sets and integral: } & m(E_0) = 0 \Rightarrow \int_E f = \int_{E-E_0} f \\ \text{Linearity: } & \int_E f + \int_E g = \int_E (f+g) \\ \text{Monotonicity: } & f \leq g \Rightarrow \int_E f \leq \int_E g \\ \text{Additivity over domain: } & \int_{E_1} f + \int_{E_2} f = \int_{E_1\cup E_2} f\end{aligned}$$

The method of constructing the Lebesgue integral is very inspiring. When dealing with a rather gengeral mathematic concept, one can first consider some special case, and extend the proposition or property to the more general case.

4.5 Lebesgue vs Riemann

In this section we give some example, and discuss some difference between Lebesgue integral and Riemann integral. In the section of the integral for bounded measurabe function, we have show that any Riemann integralbe function is Lebesgue integralbe, but didn't show the converse is false. Now we give an counter-example.

Example 4.1 Let $f = \chi_{\mathbb{R}-\mathbb{Q}} \cap [0,1]$(the Dirichlet function). No matter how fine a Riemann partition is, the small interval determined by the partition will contains a rational and a irrational, which means the upper Riemann integral will be 1 and the lower will be 0. So f is not Riemann integralbe.

But with Lebesgue integral, f is just a simple function, so is integralbe and it's integral is 1.

Theorem 4.12 (The Lebesgue condition for Riemann integral funciton)

♡

By the upper theorem, a Riemann integrable function on a closed bounded interval is continuous a.e. on it's domain; hbounded functions on intervals is Lebesgue integrable if and only if it is measurable. By Lusin's theorem, given any $\epsilon > 0$, it is continouos on it's domain but a set whose measure is less than ϵ.

Chapter 5 Limit and Itergral

In the first section, we promised that Lebesgue integral will has better properties compare to Riemann integral. Now we have seen that Lebsegue integral have defined on more functions and sets. In this chapter, we will focus on the issue that under what condition, the following equation will hold:

$$\lim_{n\to\infty}\int_E f_n(x) = \int_E f(x)$$

where E is a measurable set, and $\{f_n\} \subset \mathcal{M}\mathfrak{F}\{E\}$ which converge to f.

Theorem 5.1 (Bounded convergence theorem)

Let $\{f_n\}$ be a sequence of measurable function on E, where $m(E) < +\infty$. Suppose f_n is uniform bounded on E. If $\{f_n\} \to f$ pointwisely a.e. on E, then

$$\lim_{n\to\infty}\int_E f_n(x) = \int_E f(x)$$

♡

Although we need to show that f is integrable, this part is rather trivial, so is it left to the reader to check. Our basic idea of the prove is by using Egoroff's theorem, to show that $\{f_n\}$ coverge uniformly on a very large subset. Then for the part on that subset, the integral and the limit sign could commute. Out of the subset, the difference of integral will be small enough.

Proof Since our result is the equation between integrals, we can assume $\{f_n\}$ converge pointwisely on E. Assuming $|f_n| < M$ for all possible n. Since $m(E) < +\infty$, Egoroff's theorem is available. Thus given $\epsilon > 0$, we can find a close subset of E, say F, such that $m(E - F) < \frac{\epsilon}{4M\cdot m(E)}$ and f_n converge uniform to f. Also because the uniform convergence, for the same ϵ we can find a N such that for $n \geq N$,

$$|\int_F f_n - \int_F f| \leq \frac{\epsilon}{2}$$

Together with the above estimation and properties of integral,

$$\begin{aligned}
|\int_E f_n - f| &= |\int_{E-F} f_n - \int_{E-F} f + \int_F f_n - \int_F f| \\
&\leq |\int_{E-F} f_n - \int_{E-F} f| + |\int_F f_n - \int_F f| \\
&\leq \int_{E-F} 2M + \frac{\epsilon}{2} \\
&\leq 2M \cdot m(E-F) + \frac{\epsilon}{2} \\
&\leq 2M \cdot \frac{\epsilon}{4M \cdot m(E)} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Thus we show that the limit sign could go though the integral sign.

Uniform bounded is weaker than uniform converge, but we can prove much more strong converge theorem under the context of Lebesgue integral. To prove the stronger converge theorems, it is conveneint to establish the following lemma.

Lemma 5.1 (Fatou's lemma)

Let $\{f_n\}$ be a sequence of non-negative measurable functions on E. If $\{f_n\}$ converge pointwisely a.e. over E. then

$$\int_E f \leq \liminf \int_E f_n$$

♡

Remember that $\liminf$ of a sequence $\{a_n\}$ is defined as $\sup\{\inf\{a_n\}_{n=k}^{\infty} : 1 \leq i < \infty\}$.

Proof First, measure zero sets doesn't influence the integral, thus one can assume $\{f_n\}$ converge pointwise on E. Since the integral of non-negative functions is defined as the supremum of the integral of bounded measurable function of finite support, we only need to show that for all such function $\varphi \leq f$,

$$\int_E \varphi \leq \liminf \int_E f_n$$

Let $\varphi_n = min(\varphi, f_n)$, then $\{\varphi_n\}$ coverge to φ pointwisely. There exist a set of finite measure E_0 on which φ and φ_n support on. So by bounded converge theorem,

$$\int_E \varphi = \int_{E_0} \lim_{n\to\infty} h_n = \lim_{n\to\infty} \int_{E_0} h_n$$

By the basic properties of limit and integral, and the fact that $h_n \leq f_n$we have

$$\begin{aligned}\int_E \varphi &= \lim_{n\to\infty} \int_E h_n \\ &\leq \liminf \int_E f_n\end{aligned}$$

The prove above illustrate that if you want to prove a proposition of non-egative function, you can first prove it for bounded measurable function of finite support. Also if yoou want to prove something for measurable function, it is always helpful to prove it for simple functions.

Theorem 5.2 (Monotone converge theorem)

Let $\{f_n\}$ be a sequence of nonegative measurable functions on E. If $f_m \leq f_n$ for $m < n$, and $\{f_n\} \to f$ pointwisely a.e. on E, then

$$\lim_{n\to\infty} \int_E f_n(x) = \int_E f(x)$$

♡

Proof We assume $\{f_n\}$ converge to f pointwisely on E. Since the sequence is increasing, by monotoniciy, if $m < n$, we have

$$\int_E f_m \leq \int_E f_n$$

Thus by Fatou's lemma

$$\int_E f \leq \liminf \int_E f_n = \lim_{n\to\infty} \int_E f_n$$

The last equality comes from the fact that $\{\int_E f_n\}$ is a increasing sequence. It remains to show the inequality from the other side. Since $\{f_n\}$ converge to f increasingly,$f > f_n$ for all n. Thus by the monotonicity of the integration,

$$\int_E f_n \leq \int_E f$$

$$\Rightarrow \lim_{n\to\infty} \int_E f_n \leq \int_E f$$

Notice that the existence of the limit of the integration of the sequence of function is guaranteened because the general limit exist for monotonic sequences.

Corollary 5.1

Let $\{f_n\}$ be non-negative measurable function on E. If $f = \sum_{n=1}^{\infty} f_n$ converge pointwisely a.e. on E,

$$\int_E f = \sum_{n=1}^{\infty} \int_E f_n$$

Proof Notice that the partial sum is coverge increasingly(since f_n are non-negative), so we can apply the monotone converge theorem. Let $F_k = \sum_{n=1}^{k} \int_E f_n$.

$$\begin{aligned} \int_E f &= \lim_{n\to\infty} \int_E F_k \\ &= \lim_{n\to\infty} \int_E \sum_{n=1}^{k} \int_E f_n \\ &= \sum_{n=1}^{\infty} \int_E f_n \end{aligned}$$

Lemma 5.2 (Beppo Levi's Lemma)

Let $\{f_n\}$ be a sequence of non-negative measurable function on E. If $\{\int_E f_n\}$ is bounded, then $\{f_n\}$ converge pointwisely to a f on E, which is finite a.e on E, and

$$\lim n \to \infty \int_E f_n = \int_E f < \infty$$

Proof Since every monotone sequence of real number converge to a extended real number, so for every $x \in E$, we can define

$$f(x) = \lim_{n\to\infty} f_n(x)$$

Then it is obvious that $\{f_n\}$ converge to f pointwisely, and increasingly. Thus we can use monotone converge theorem, which shows

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Since $\{\int_E f_n\}$ is bounded, $\int_E f$ is a finite value. So f is integralbe, which indicated f is finite a.e. on E (proposition 4.5).

Theorem 5.3 (Legesgue dominated converge theorem)

Let $\{f_n\}$ be a sequence of measurable functions on E, which converge pointwisely a.e. to f on E. If f_n is dominated by a integrable function g, in the sense of $|f_n| \leq g$, then

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof Again, the *a.e.* can be ignored. Notice that f is also dominated by g. By monotonicity of the integral,

$$\int_E |f| \leq \int_E g < \infty$$

f is integralbe. $g - f_n$ and $g - f$ are all non-negative functions, and $(g - f_n) \to (g - f)$ pointwisely as $n \to \infty$. Thus we can apply Fatou's lemma.

$$\begin{aligned}
\int_E g - f &\leq \liminf \int_E g - f_n \\
&= \int_E g - \limsup \int_E f_n \\
\Rightarrow \int_E g - \int_E f &\leq \int_E g - \limsup \int_E f_n \\
\Rightarrow \int_E f &\geq \limsup \int_E f_n
\end{aligned}$$

Observing that $g + f_n$ and $g + f$ and also non-negative, and $(g + f_n) \to (g + f)$ pointwisely as $n \to \infty$, using the same trick,

$$\begin{aligned}
\int_E g + f &\leq \liminf \int_E g + f_n \\
&= \int_E g + \lim inf \int_E f_n \\
\Rightarrow \int_E g + \int_E f &\leq \int_E g + \liminf \int_E f_n \\
\Rightarrow \int_E f &\leq \liminf \int_E f_n
\end{aligned}$$

Together the two inequality, we have

$$\int_E f \leq \liminf \int_E f_n \leq \limsup \int_E f_n \leq \int_E f$$

Thus $\lim_{n\to\infty} f_n$ exist and

$$\int_E f = \lim_{n\to\infty} f_n$$

Theorem 5.4 (General dominated converge theorem)

Definition 5.1 (Uniform integral)

Theorem 5.5 (Vetali's converge theorem)

Theorem 5.6 (General Vetali's converge theorem)

Chapter 6 Lp Space

6.1 Definition

6.2 Important Inequality

6.3 ?

Chapter 7 Appendix 1: Non-measurable Sets, Non-measurable Functions

For the purpose to illustrate the theory in a more clear way, we didn't talk about the existence of non-measurable sets, non-measurable functions and many other counter examples. Counter-examples are important. Without showing there exsistence, our theory is meaningless(mabey all the sets are measurable). In fact, many of them are quite hard to construct, which is the reason why we didn't mention them in the main body.