# Exercise 1 – Kinematics of a robotic leg

## Overview

In this exercise, we are going to analyze the kinematics of a single robot leg with a point foot. Starting with a set of generalized coordinates, we will elaborate relative rotations, translations, and homogeneous transformations. Subsequently, using foot point Jacobians, we will describe contact constraints and perform a trajectory tracking task using inverse kinematics.

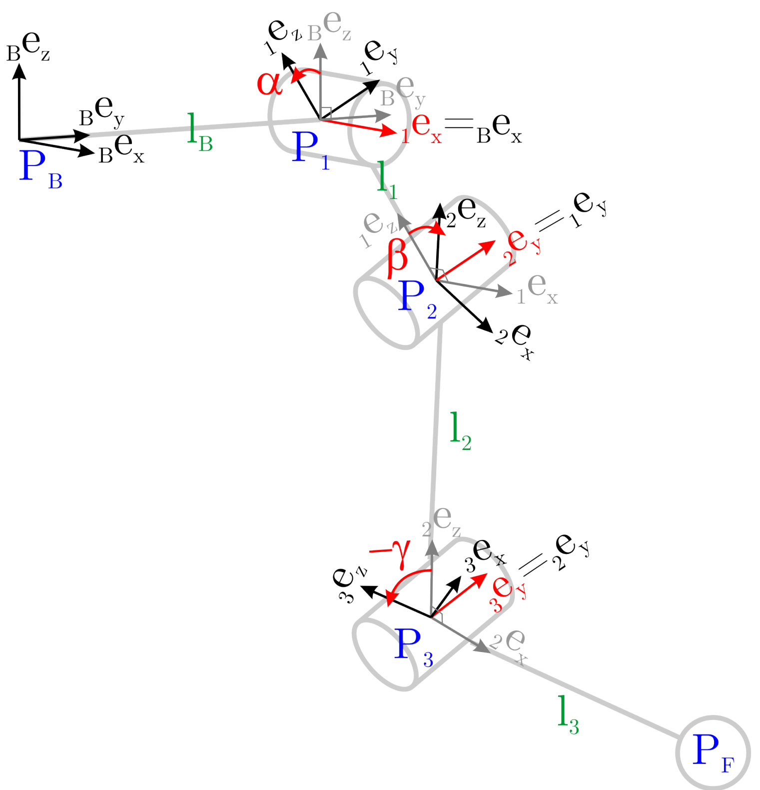


Figure 1: Kinematics of a single robot leg.

Figure 1 illustrates a single leg that is attached to a body fixed frame $B$. This leg has three degrees of freedom consisting of relative rotations about $\alpha$ (alpha) around $_B\mathbf{e}_x$, about $\beta$ (beta) around $_1\mathbf{e}_y$, and about $\gamma$ (gamma) around $_2\mathbf{e}_y$. Hence, the generalized coordinates are given by $\mathbf{q} := [\alpha\ \beta\ \gamma]^T$.

This exercise has an associated MATLAB script (`Ex1.m`) which provides a baseline skeleton of the code. Most questions can be solved by hand or through other methods, but may be easier via the scripts. Questions where MATLAB is necessary are indicated (MATLAB). This year, we also offer Python examples in 'early release' as an alternative to MATLAB (same content). The Python code should approximately mirror the MATLAB exercise, though naturally there are some differences due to code structure.

# Q1 Relative Rotation Matrix

Given the kinematic description of a single leg with three degrees of freedom ($\mathbf{q} := [\alpha\ \beta\ \gamma]^T$) we determine the relative rotation matrices $\mathbf{R}_{AC}$ rotating a vector $\mathbf{R}$ from an arbitrary coordinate system $C$ to $A$:

$$ {}_A\mathbf{r} = \mathbf{R}_{AC}\ {}_C\mathbf{r}. \tag{1}$$

As a function of the generalized coordinates $\alpha, \beta, \gamma$, what are the three relative rotation matrices $\mathbf{R}_{B1}$, $\mathbf{R}_{12}$, $\mathbf{R}_{23}$?

# Q2 Homogeneous Transformation

Given the relative rotation matrices from the previous problem and choosing unitary link lengths ($l_B = l_1 = l_2 = l_3 = 1$), we determine the homogeneous transformation $\mathbf{H}$ that transforms the footpoint position $\mathbf{r}_F$ represented in coordinate frame 3 to coordinate frame $B$:

$$\begin{bmatrix} {}_B\mathbf{r}_{BF} \\ 1 \end{bmatrix} = \mathbf{H}_{B3} \begin{bmatrix} {}_3\mathbf{r}_{3F} \\ 1 \end{bmatrix}. \tag{2}$$

As a function of the generalized coordinates $\alpha, \beta, \gamma$, what are:

1. the relative position vectors ${}_1\mathbf{r}_{12}$, ${}_2\mathbf{r}_{23}$, ${}_3\mathbf{r}_{3F}$, and ${}_B\mathbf{r}_{B1}$?
2. the homogeneous transformation matrices $\mathbf{H}_{B1}$, $\mathbf{H}_{12}$, and $\mathbf{H}_{23}$?
3. the foot point position vector in $B$, ${}_B\mathbf{r}_{BF}$?

# Q3 Jacobians and Differential Kinematics

Given the end effector position ${}_B\mathbf{r}_{BF}$ as a function of generalized coordinates $\mathbf{q}$:

1. determine the corresponding Jacobian $\mathbf{J}_{BF} = \frac{\partial_B \mathbf{r}_{BF}}{\partial \mathbf{q}}$.
2. determine the generalized coordinate velocity $\dot{\mathbf{q}}_t$ from a current configuration $\mathbf{q}_t = \begin{bmatrix} 0 & 60° & -120° \end{bmatrix}^T$ and a given target foot velocity ${}_B\dot{\mathbf{r}}_{BF} = \begin{bmatrix} 0 & 0 & -1\text{m/s} \end{bmatrix}^T$. (NOTE: This will require a matrix inversion, we recommend using a programming tool to help with this.)

# Q4 Numerical Inverse Kinematics

In this exercise, we will solve an inverse kinematics problem of finding the generalized coordinates $\mathbf{q}_{goal}$ for a given target pose ${}_B\mathbf{r}_{BF,goal}$.

1. (MATLAB) First, create a simple program to solve this problem numerically. We have an existing functional form of ${}_B\mathbf{r}_{BF} = {}_B\mathbf{r}_{BF}(\mathbf{q})$. Since the functional form of ${}_B\mathbf{r}_{BF}$ is often hard to analytically invert, we will implement a numerical approach. See the Q4 block in the associated `Ex1.m` template code to get started. HINT: A simple gradient descent is sufficient for this task. Initialise the estimate of the generalized coordinates $\mathbf{q}$ to the current state $\mathbf{q}_t$. Then, calculate the corresponding ${}_B\mathbf{r}_{BF}(\mathbf{q})$. In each step, calculate a new estimate for $\mathbf{q}$ using $\mathbf{q}' = \mathbf{q} + \mathbf{J}^{-1}(\mathbf{r}_{goal} - \mathbf{r})$, where the inversion can be approximated using the Moore-Penrose pseudo-invserse (`pinv(.)` in Matlab).
2. With unitary segment length, and ${}_B\mathbf{r}_{BF,goal} = \begin{bmatrix} 0.2 & 0.5 & -2 \end{bmatrix}^T$, how many different solutions $\mathbf{q}_{goal}$ exist?
3. The following goal position ${}_B\mathbf{r}_{BF,goal} = \begin{bmatrix} -1.5 & 1 & -2.5 \end{bmatrix}^T$ is not within the reaching range of our leg. What are the generalized coordinates that bring the leg as close as possible to the target position?

## Q5 Trajectory Following: Inverse Differential Kinematics

Extending from the previous parts, we will now build a working controller based on inverse differential kinematics to control a footpoint trajectory. Given a desired footpoint goal trajectory $\mathbf{r}_F(t)$ and a starting configuration $\mathbf{q}_0$, we have to determine the joint velocities $\dot{\mathbf{q}} = \begin{bmatrix}\dot{\alpha} & \dot{\beta} & \dot{\gamma}\end{bmatrix}^T$. The target trajectory is a basic circle in the $xz$-plane parameterized by time, $\mathbf{r}_F(t) = \mathbf{c} + \rho \begin{bmatrix}\sin(2\pi f t) & 0 & \cos 2\pi f t)\end{bmatrix}^T$, for a given center location $\mathbf{c}$, radius $\rho$ and frequency $f$. The target velocity is simply $\dot{\mathbf{r}}_F = \frac{d\mathbf{r}_F}{dt}$.

1. (MATLAB) Implement a simple proportional controller to determine the desired Cartesian foot point velocity (i.e. $\dot{\mathbf{r}}(t) = \dot{\mathbf{r}}_F(t) + K_p (\mathbf{r}_F(t) - \mathbf{r})$) and apply inverse differential kinematics to determine the joint velocities. The helper script `plotTrajectory.m` can be used to visualise the system response. Try different values for the controller gain and different parameters for the trajectory to see the effect on the system response.

2. Given a desired foot point velocity $\dot{\mathbf{r}}_F$ and generalized coordinates $\mathbf{q}$, which of the following statements are correct?

   (a) If the leg is in a non-singular configuration, there is a unique solution for $\dot{\mathbf{q}}$

   (b) In theory, $\dot{\mathbf{r}}_F$ can always be achieved

   (c) $\gamma = 0$ makes the foot point Jacobian rank deficient