1 Task 1

1.1

1.1.1

Finding the first order price sensitivity of Lower Bound of Arithmetic and Geometric Asian options using likelihood ratio method. The LR formulation is given as

$$\begin{aligned}\mu_{C,\theta} &= \mathbb{E}_\theta\left(C\right)\\ &= \mathbb{E}_\theta\left(f\left(S\right)\right)\\ &= \int_{R^n} f\left(s\right) ds\\ \mu'_{C,\theta} &= \int_{R^n} f\left(s\right)\frac{d\,g_\theta\left(s\right)}{d\theta} ds\end{aligned}$$

by multiplying and dividing $g_\theta\left(s\right)$ the expectation expression is obtained

$$\begin{aligned}\int_{R^n} f\left(s\right)\frac{d\,g_\theta\left(s\right)}{d\theta}\frac{1}{g_\theta\left(s\right)} g_\theta\left(s\right) ds &= \mathbb{E}_\theta\left(f\left(S\right)\frac{d\,g_\theta\left(S\right)}{d\theta}\frac{1}{g_\theta\left(S\right)}\right)\\ &= \mathbb{E}_\theta\left(f\left(S\right)\frac{d\,ln\,g_\theta\left(S\right)}{d\,\theta}\right)\end{aligned}$$

using the above expression, we take the expected value of the product of the discounted payoff function and the score as the LR first order sensitivity respected to S_0. We have the discounted payoff functions for the arithmetic Asian option Lower Bound and for the geometric Asian option in below.

$$\begin{aligned}LB_n &= e^{-rT}\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K\right)\mathbb{1}_{\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} > K}\\ G_n &= e^{-rT} max\left(\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} - K, 0\right)\end{aligned}$$

the score for computing an Asian option delta is given as

$$\frac{d\,ln\,g_{S_0}(S_1, ..., S_n)}{dS_0} = \frac{\xi\left(S_1|S_0\right)}{S_0\sigma\sqrt{t_1}}$$

$$\begin{aligned}\mu'_{LB,S_0} &= \mathbb{E}_{S_0}\left(e^{-rT}\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K\right)\mathbb{1}_{\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} > K}\cdot\frac{\xi\left(S_1|S_0\right)}{S_0\sigma\sqrt{t_1}}\right)\\ &= \mathbb{E}_{S_0}\left(e^{-rT}\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K\right)\mathbb{1}_{\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} > K}\cdot\frac{ln\frac{S_1}{S_0} - \left(r - \frac{1}{2}\sigma^2\right)\left(t_1 - t_0\right)}{\sigma\sqrt{t_1 - t_0}S_0\sigma\sqrt{t_1}}\right)\end{aligned}$$

$$\mu'_{G,S_0} = \mathbb{E}_{S_0}\left(e^{-rT}\left(\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} - K\right)^{+}\cdot\frac{\xi\left(S_i|S_{i-1}\right)}{S_0\sigma\sqrt{t_1}}\right)$$

1.1.2

No, as shown above, the lower bound payoff is discontinuous at $\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} = K$. *Such that it is not possible to obtain the PW estimator. LR first order sensitivity estimator should be used instead.*

1.2

The exact closed formed solutions for the expected Lower Bound and the Geometric payoff are given

$$\mathbb{E}\left(LB_n\right) = \frac{S_0 e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \mathcal{N}\left(b + a_k\right) - Ke^{-rT} \cdot \mathcal{N}\left(b\right)$$

$$\mathbb{E}\left(G_n\right) = S_0 e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \cdot \mathcal{N}\left(d\right) - Ke^{-rT} \cdot \mathcal{N}\left(d - \hat{\sigma}\sqrt{\hat{T}}\right)$$

by taking the derivative respected to S_0

$$\begin{aligned}\frac{d\,\mathbb{E}\left(LB_n\right)}{dS_0} &= \frac{e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \mathcal{N}\left(b + a_k\right) + \frac{S_0 e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \phi\left(b + a_k\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &- Ke^{-rT} \cdot \phi\left(b\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &= \frac{e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \mathcal{N}\left(b + a_k\right) + \frac{e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \phi\left(b + a_k\right) \cdot \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &- Ke^{-rT} \cdot \phi\left(b\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}}\end{aligned}$$

$$\begin{aligned}\frac{dG_n}{dS_0} &= e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \cdot \mathcal{N}\left(d\right) + S_0 e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \phi\left(d\right) \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &- Ke^{-rT} \cdot \phi\left(d - \hat{\sigma}\sqrt{\hat{T}}\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &= e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \cdot \mathcal{N}\left(d\right) + e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \phi\left(d\right) \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &- Ke^{-rT} \cdot \phi\left(d - \hat{\sigma}\sqrt{\hat{T}}\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}}\end{aligned}$$

where $\mathcal{N}$ *is the normal cdf and* ϕ *is the normal pdf.*