1 Task 1

1.1

1.1.1

Finding the first order price sensitivity of Lower Bound of Arithmetic and Geometric Asian options using likelihood ratio method. The LR formulation is given as

$$\begin{aligned}\mu_{C,\theta} &= \mathbb{E}_\theta\left(C\right)\\ &= \mathbb{E}_\theta\left(f\left(S\right)\right)\\ &= \int_{R^n} f\left(s\right)ds\\ \mu'_{C,\theta} &= \int_{R^n} f\left(s\right)\frac{d\,g_\theta\left(s\right)}{d\theta}ds\end{aligned}$$

by multiplying and dividing $g_\theta\left(s\right)$ the expectation expression is obtained

$$\begin{aligned}\int_{R^n} f\left(s\right)\frac{d\,g_\theta\left(s\right)}{d\theta}\frac{1}{g_\theta\left(s\right)}g_\theta\left(s\right)ds &= \mathbb{E}_\theta\left(f\left(S\right)\frac{d\,g_\theta\left(S\right)}{d\theta}\frac{1}{g_\theta\left(S\right)}\right)\\ &= \mathbb{E}_\theta\left(f\left(S\right)\frac{d\,ln\,g_\theta\left(S\right)}{d\,\theta}\right)\end{aligned}$$

using the above expression, we take the expected value of the product of the discounted payoff function and the score as the LR first order sensitivity respected to S_0. We have the discounted payoff functions for the arithmetic Asian option Lower Bound and for the geometric Asian option below.

$$\begin{aligned}LB_n &= e^{-rT}\left(\frac{1}{n}\sum_{k=1}^{n}S_k - K\right)\mathbb{1}_{\left(\prod_{k=1}^{n}S_k\right)^{\frac{1}{n}}>K}\\ G_n &= e^{-rT}max\left(\left(\prod_{k=1}^{n}S_k\right)^{\frac{1}{n}} - K, 0\right)\end{aligned}$$

the score for computing an Asian option delta is given as

$$\frac{d\,ln\,g_{S_0}(S_1,....,S_n)}{dS_0} = \frac{\xi\left(S_1|S_0\right)}{S_0\sigma\sqrt{t_1}}$$

$$\begin{aligned}\mu'_{LB,S_0} &= \mathbb{E}_{S_0}\left(e^{-rT}\left(\frac{1}{n}\sum_{k=1}^{n}S_k - K\right)\mathbb{1}_{\left(\prod_{k=1}^{n}S_k\right)^{\frac{1}{n}}>K}\cdot\frac{\xi\left(S_1|S_0\right)}{S_0\sigma\sqrt{t_1}}\right)\\ &= \mathbb{E}_{S_0}\left(e^{-rT}\left(\frac{1}{n}\sum_{k=1}^{n}S_k - K\right)\mathbb{1}_{\left(\prod_{k=1}^{n}S_k\right)^{\frac{1}{n}}>K}\cdot\frac{ln\frac{S_1}{S_0} - \left(r - \frac{1}{2}\sigma^2\right)\left(t_1 - t_0\right)}{\sigma\sqrt{t_1 - t_0}S_0\sigma\sqrt{t_1}}\right)\end{aligned}$$

$$\mu'_{G,S_0} = \mathbb{E}_{S_0}\left(e^{-rT}\left(\left(\prod_{k=1}^{n}S_k\right)^{\frac{1}{n}} - K\right)^{+}\cdot\frac{\xi\left(S_i|S_{i-1}\right)}{S_0\sigma\sqrt{t_1}}\right)$$

1.1.2

Now, as shown above, the lower bound payoff is discontinuous at $\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} = K$. Hence it is not possible to obtain the PW estimator. LR first order sensitivity estimator should be used instead.

1.2

The exact closed formed solutions for the expected Lower Bound and the Geometric payoff are given

$$\mathbb{E}\left(LB_n\right) = \frac{S_0 e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \mathcal{N}\left(b + a_k\right) - Ke^{-rT} \cdot \mathcal{N}\left(b\right)$$
$$\mathbb{E}\left(G_n\right) = S_0 e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \cdot \mathcal{N}\left(d\right) - Ke^{-rT} \cdot \mathcal{N}\left(d - \hat{\sigma}\sqrt{\hat{T}}\right)$$

by taking the derivative respected to S_0

$$\begin{aligned}
\frac{d\,\mathbb{E}\left(LB_n\right)}{dS_0} &= \frac{e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \mathcal{N}\left(b + a_k\right) + \frac{S_0 e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \phi\left(b + a_k\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\
&- Ke^{-rT} \cdot \phi\left(b\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\
&= \frac{e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \mathcal{N}\left(b + a_k\right) + \frac{e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \phi\left(b + a_k\right) \cdot \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\
&- Ke^{-rT} \cdot \phi\left(b\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}}
\end{aligned}$$

$$\begin{aligned}
\frac{dG_n}{dS_0} &= e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \cdot \mathcal{N}\left(d\right) + S_0 e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \phi\left(d\right) \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\
&- Ke^{-rT} \cdot \phi\left(d - \hat{\sigma}\sqrt{\hat{T}}\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\
&= e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \cdot \mathcal{N}\left(d\right) + e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \phi\left(d\right) \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\
&- Ke^{-rT} \cdot \phi\left(d - \hat{\sigma}\sqrt{\hat{T}}\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}}
\end{aligned}$$

where $\mathcal{N}$ is the normal cdf and ϕ is the normal pdf. And $b = \frac{ln(S_0/K) + \left(r - \sigma^2/2\right)\hat{T}}{\hat{\sigma}\sqrt{\hat{T}}}$, $d = \frac{ln(S_0/K) + \left(r - \sigma^2/2 + \hat{\sigma}^2\right)\hat{T}}{\hat{\sigma}\sqrt{\hat{T}}}$.

1.3

1.3.1

Table 1 describes the computed values for the expected lower bound using different values for equally spaced dates and strike prices.

Table 2 describes the computed values for the first sensitivities of the expected lower bounds using different values for equally spaced dates and strike prices.

K

n	90	100	110
4	15.032	9.2449	5.291
12	14.136	8.2345	4.3702
50	13.798	7.849	4.0268

Table 1: Expected Lower Bound Values

K

n	90	100	110
4	0.75587	0.57648	0.39682
12	0.76575	0.56715	0.36875
50	0.77045	0.56335	0.35684

Table 2: Sensitivity of Expected Lower Bound Values

1.3.2

Table 3 displays the geometric asian call option values for different n's and K's.

Table 3: Geometric Asian Call Option Values

Table 4 shows the values for the first sensitivities with respect to S_0 for the geometric Asian option prices.

Table 4: Sensitivity of Geometric Asian Call Option Values

1.4

1.4.1

Table 5 approximates the value of the Asian call option prices using Monte Carlo simulations and the lower bound as a control variate. The standard error (S.E.) and the confidence intervals (C.I) can also be viewed.

K

	90			100			110		
	Price	S.E.	C.I.	Price	S.E..	C.I.	Price	S.E.	C.I
4	15.0371	0.00020261	15.0367-15.0375	9.2492	0.00020476	9.2488-9.2496	5.2960	0.00020476	5.2956-5.2965
12	14.1400	0.00017697	14.1396-14.1403	8.2382	0.00017074	8.2379-8.2385	4.3749	0.00022525	4.3745-4.3754
50	13.8019	0.00017443	13.8016-13.8023	7.8525	0.00014983	7.8522-7.8528	4.0315	0.0002129	4.0311-4.0319

Table 5: Approximation with Lower Bound as Control Variate

Table 6 approximates the Asian call option prices using Monte Carlo simulations and the geometric asian call option with discounted payoff as a control variate. It also includes the standard error (S.E) and the confidence intervales (C.I.)

K

	90			100			110		
	Price	S.E.	C.I.	Price	S.E..	C.I.	Price	S.E.	C.I
4	15.0388	0.0020434	15.0348-15.0428	9.2480	0.0018979	9.2443-9.2517	5.2960	0.0018398	5.2924-5.2996
12	14.1386	0.0017756	14.1352-14.1421	8.2395	0.0016818	8.2362-8.2428	4.3736	0.0018398	4.3706-4.3766
50	13.8030	0.0017397	13.7996-13.8064	7.8525	0.0015716	7.8494-7.8556	4.0326	0.0014452	4.0298-4.0354

Table 6: Approximation with Geometric Asian Call Option as Control Variate

K

n	90	100	110
4	14.525	8.8315	4.971
12	13.602	7.8021	4.0382
50	13.263	7.4155	3.6945

K

n	90	100	110
4	0.74248	0.56288	0.38356
12	0.75131	0.55277	0.35439
50	0.7558	0.54899	0.34226

Table 7 shows the epsilon values (efficiency ratios) for the different strike prices and equally spaced dates. The values display the ratio between the two approximations with different control variates. The numerator is the efficiency of the approximation with the lower bound as a control variate and the denominator is the efficiency of the approximation with the geometric Asian option as a control variate. It is clear from Table 7 that using the expected lower bound as a control variate is more efficient than using the geometric asian call option.

K

n	90	100	110
4	0.0147	0.0162	0.0218
12	0.0120	0.0146	0.0146
50	0.0126	0.0124	0.0236

Table 7: Efficiency Values for the arithmetic Asian option price

1.4.2

Table 8 approximates the Asian call option sensitivity with respect to S_0 using Monte Carlo simulations and the sensitivity of the lower bound as a control variate. The table also displays the standard errors (S.E) and the confidence intervals (C.I).

K

	90			**100**			**110**		
	Δ	S.E.	C.I.	Δ	S.E..	C.I.	Δ	S.E.	C.I
4	0.7639	0.00027227	0.7633-0.7644	0.5856	0.00029519	0.5850-0.5861	0.4071	0.00032565	0.4064-0.4077
12	0.7750	0.00027437	0.7744-0.7755	0.5768	0.00030602	0.5762-0.5774	0.3799	0.00033711	0.3793-0.3806
50	0.7793	0.00028271	0.7788-0.7798	0.5730	0.00030121	0.5724-0.5736	0.3673	0.00033988	0.3667-0.3680

Table 8: Approximation with Lower Bound as Control Variate

Table 9 approximates the Asian call option sensitivity with respect to S_0 using Monte Carlo simulations and the sensitivity of the geometric Asian option as a control variate. The table also displays the standard errors (S.E) and the confidence intervals (C.I).

K

	90			**100**			**110**		
	Δ	S.E.	C.I.	Δ	S.E..	C.I.	Δ	S.E.	C.I
4	0.7560	0.00026752	0.7554-0.7565	0.5761	0.00029418	0.5755-0.5767	0.3966	0.00032115	0.3959-0.3972
12	0.7662	0.00028428	0.7656 -0.7667	0.5671	0.00030736	0.5664-0.5677	0.3686	0.00033728	0.3679-0.3692
50	0.7710	0.00028924	0.7704-0.7716	0.5630	0.00030418	0.5624-0.5636	0.3577	0.0003561	0.3570-0.3584

Table 9: Approximation with Geometric Asian Call Option as Control Variate

Table 10 shows the epsilon values (efficiency ratios) for the different strike prices and equally spaced dates. The values display the ratio between the two approximations with different control variates. The numerator is the efficiency of the approximation with the sensitivity of the expected lower bound as a control variate and the denominator is the efficiency of the approximation with the sensitivity of the geometric Asian option as a control variate. It is clear from Table 10 that using the sensitivity of the expected lower bound as a control variate is more efficient than using the geometric asian call option. It is clear from Table 10 that the using the sensitivity of the expected lower bound is more efficient than using the geometric asian call option.

n	K: **90**	**100**	**110**
4	0.9858	0.97608	0.97934
12	0.87348	0.92373	0.95308
50	0.90616	0.93272	0.71096

Table 10: Efficiency Values for the sensitivity of the arithmetic Asian option price

1.4.3

The accuracy of the 95% confidence intervals of the control variate estimations can be displayed by the tightness that exists. This is true for all the approximations that were executed, both for the arithmetic asian call option, but also for the sensitivities. More concretely, observing tables 5,6,8 and 9 and subtracting the confidence intervals, the difference between the two values is of the 10^{-4} for the more accurate approximations and 10^{-3} for the less efficient ones.
Control variates provide a better estimate for the estimation of the Asian Call Options and their sensitivities. In general, the higher the correlation between the variables, the more accurate the approximated value. In this case, the correlation is similar and very high between the two control variates and the payoff as well as the sensitivities, which means that the selection of the control variate is not very important.
It is also clear from the efficiency Table 7 that there is a lot more efficiency when using the lower bound as a control variate with respect to the geometric option. Since the standard errors seem to be similar, the reason could be assigned to the construction of the control variate.
As it is to be expected, the price of the option and of the sensitivities is more accurate the higher the n (when controlling for K). The reason for that would be that there are more values computed and a more accurate coefficient and bstarhat can be created. This is also apparent in Tables 5 and 6, where there is the standard error is less with higher n's. The downside of these is that it is more computationally expensive as more commands need to be executed. The same goes for higher K's as far as prices are concerned.
Finally, observing Tables 1,5 and 6, it is clear that all prices are very close with the value of the expected lower bound for the price. This is to be expected, as it uses a version of the Black-Scholes formula with the Asian option payoff embedded.

1.5

Running the expected lower bound with increasing values of n gives a good approximation for the precise value of the expected value. As expected the higher the value of n the better the convergence to the precise value.
Concretely, the precise value for the expected lower bound is **7.726961356596341**. Figure 1, shows the convergence of the value with respect to increasing n's.

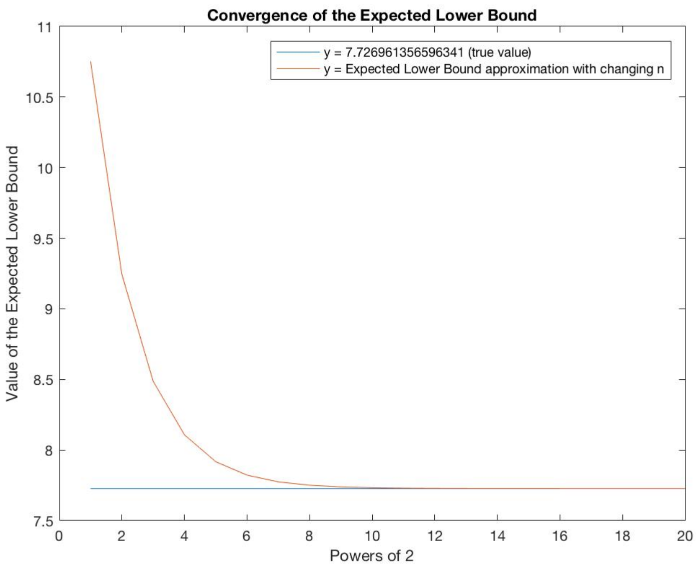

Figure 1: Convergence of the approximation

i (Power of 2)	Value
1	10.752849553377715
2	9.244949514019062
3	8.487670434364354
4	8.107824667550155
5	7.917532175125288
6	7.822283197949488
7	7.774631600352421
8	7.750798836772930
9	7.738880689741741
10	7.732921171871205
11	7.729941301464144
12	7.728451338344790
13	7.727706349800059
14	7.727333853780493
15	7.727147605333769
16	7.727054481001673
17	7.727007918808091
18	7.726984637705648
19	7.726972997149829
20	7.726967176874716

Table 11: Approximations of the lower bound

Table SSSSSS can be reviewed for further analysis of the approximation method precision.