1 Task 1

1.1

1.1.1

Finding the first order price sensitivity of Lower Bound of Arithmetic and Geometric Asian options using likelihood ratio method. The LR formulation is given as

$$\begin{aligned}
\mu_{C,\theta} &= \mathbb{E}_\theta\left(C\right) \\
&= \mathbb{E}_\theta\left(f\left(S\right)\right) \\
&= \int_{R^n} f\left(s\right) ds \\
\mu'_{C,\theta} &= \int_{R^n} f\left(s\right) \frac{d\, g_\theta\left(s\right)}{d\theta} ds
\end{aligned}$$

by multiplying and dividing $g_\theta\left(s\right)$ the expectation expression is obtained

$$\begin{aligned}
\int_{R^n} f\left(s\right) \frac{d\, g_\theta\left(s\right)}{d\theta} \frac{1}{g_\theta\left(s\right)} g_\theta\left(s\right) ds &= \mathbb{E}_\theta\left(f\left(S\right) \frac{d\, g_\theta\left(S\right)}{d\theta} \frac{1}{g_\theta\left(S\right)}\right) \\
&= \mathbb{E}_\theta\left(f\left(S\right) \frac{d\, ln\, g_\theta\left(S\right)}{d\, \theta}\right)
\end{aligned}$$

using the above expression, we take the expected value of the product of the discounted payoff function and the score as the LR first order sensitivity respected to S_0. We have the discounted payoff functions for the arithmetic Asian option Lower Bound and for the geometric Asian option in below.

$$\begin{aligned}
LB_n &= e^{-rT}\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K\right) \mathbb{1}_{\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} > K} \\
G_n &= e^{-rT} max\left(\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} - K, 0\right)
\end{aligned}$$

the score for computing an Asian option delta is given as

$$\frac{d\, ln\, g_{S_0}(S_1, ..., S_n)}{dS_0} = \frac{\xi\left(S_1|S_0\right)}{S_0 \sigma \sqrt{t_1}}$$

$$\begin{aligned}
\mu'_{LB,S_0} &= \mathbb{E}_{S_0}\left(e^{-rT}\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K\right) \mathbb{1}_{\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} > K} \cdot \frac{\xi\left(S_1|S_0\right)}{S_0 \sigma \sqrt{t_1}}\right) \\
&= \mathbb{E}_{S_0}\left(e^{-rT}\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K\right) \mathbb{1}_{\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} > K} \cdot \frac{ln\frac{S_1}{S_0} - \left(r - \frac{1}{2}\sigma^2\right)\left(t_1 - t_0\right)}{\sigma\sqrt{t_1 - t_0} S_0 \sigma \sqrt{t_1}}\right)
\end{aligned}$$

$$\mu'_{G,S_0} = \mathbb{E}_{S_0}\left(e^{-rT}\left(\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} - K\right)^{+} \cdot \frac{\xi\left(S_i|S_{i-1}\right)}{S_0 \sigma \sqrt{t_1}}\right)$$

1.1.2

Now, as shown above, the lower bound payoff is discontinuous at $\left(\prod_{k=1}^{n} S_k\right)^{\frac{1}{n}} = K$. Such that it is not possible to obtain the PW estimator. LR first order sensitivity estimator should be used instead.

1.2

The exact closed formed solutions for the expected Lower Bound and the Geometric payoff are given

$$\mathbb{E}\left(LB_n\right) = \frac{S_0 e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \mathcal{N}\left(b + a_k\right) - Ke^{-rT} \cdot \mathcal{N}\left(b\right)$$
$$\mathbb{E}\left(G_n\right) = S_0 e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \cdot \mathcal{N}\left(d\right) - Ke^{-rT} \cdot \mathcal{N}\left(d - \hat{\sigma}\sqrt{\hat{T}}\right)$$

by taking the derivative respected to S_0

$$\begin{aligned}\frac{d\,\mathbb{E}\left(LB_n\right)}{dS_0} &= \frac{e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \mathcal{N}\left(b + a_k\right) + \frac{S_0 e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \phi\left(b + a_k\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &\quad - Ke^{-rT} \cdot \phi\left(b\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &= \frac{e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \mathcal{N}\left(b + a_k\right) + \frac{e^{-rT}}{n} \sum_{k=1}^{n} e^{\mu_k + \sigma_k^2} \cdot \phi\left(b + a_k\right) \cdot \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &\quad - Ke^{-rT} \cdot \phi\left(b\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}}\end{aligned}$$

$$\begin{aligned}\frac{dG_n}{dS_0} &= e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \cdot \mathcal{N}\left(d\right) + S_0 e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \phi\left(d\right) \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &\quad - Ke^{-rT} \cdot \phi\left(d - \hat{\sigma}\sqrt{\hat{T}}\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &= e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \cdot \mathcal{N}\left(d\right) + e^{\left(r - \frac{\sigma^2}{2} + \frac{\hat{\sigma}^2}{2}\right)\hat{T} - rT} \phi\left(d\right) \frac{1}{\hat{\sigma}\sqrt{\hat{T}}} \\ &\quad - Ke^{-rT} \cdot \phi\left(d - \hat{\sigma}\sqrt{\hat{T}}\right) \cdot \frac{1}{S_0} \frac{1}{\hat{\sigma}\sqrt{\hat{T}}}\end{aligned}$$

where $\mathcal{N}$ is the normal cdf and ϕ is the normal pdf.

1.3

1.3.1

The table for the expected lower bounds is the following:

		K		
		90	**100**	**110**
n	**4**	15.032	9.2449	5.291
	12	14.136	8.2345	4.3702
	50	13.798	7.849	4.0268

Table 1: Expected Lower Bound Values

The table for the first sensitivities of the expected lower bounds with respect to S_0 is the following:

	K: 90	100	110
n = 4	0.75587	0.57648	0.39682
12	0.76575	0.56715	0.36875
50	0.77045	0.56335	0.35684

Table 2: Sensitivity of Expected Lower Bound Values

1.3.2

The table for the geometric Asian call option prices is the following:

	K: 90	100	110
n = 4	14.525	8.8315	4.971
12	13.602	7.8021	4.0382
50	13.263	7.4155	3.6945

Table 3: Geometric Asian Call Option Values

The table for the first sensitivities with respect to S_0 for the geometric Asian option prices is the following:

	K: 90	100	110
n = 4	0.74248	0.56288	0.38356
12	0.75131	0.55277	0.35439
50	0.7558	0.54899	0.34226

Table 4: Sensitivity of Geometric Asian Call Option Values

1.4

1.4.1

The table for the Asian call option prices approximated using Monte Carlo simulations using the expected lower bound as a control variate. The table also displays the standard errors and the confidence intervals:

	K: 90			100			110		
	Price	S.E.	C.I.	Price	S.E..	C.I.	Price	S.E.	C.I
4	15.0371	0.00020261	15.0367-15.0375	9.2492	0.00020476	9.2488-9.2496	5.2960	0.00020476	5.2956-5.2965
12	14.1400	0.00017697	14.1396-14.1403	8.2382	0.00017074	8.2379-8.2385	4.3749	0.00022525	4.3745-4.3754
50	13.8019	0.00017443	13.8016-13.8023	7.8525	0.00014983	7.8522-7.8528	4.0315	0.0002129	4.0311-4.0319

Table 5: Approximation with Expected Lower Bound as Control Variate

The table for the Asian call option prices approximated using Monte Carlo simulations using the geometric asian call option with discounted payoff as a control variate. The table also displays the standard errors and the confidence intervals:

K

	90			100			110		
	Price	S.E.	C.I.	Price	S.E..	C.I.	Price	S.E.	C.I
4	15.0388	0.0020434	15.0348-15.0428	9.2480	0.0018979	9.2443-9.2517	5.2960	0.0018398	5.2924-5.2996
12	14.1386	0.0017756	14.1352-14.1421	8.2395	0.0016818	8.2362-8.2428	4.3736	0.0018398	4.3706-4.3766
50	13.8030	0.0017397	13.7996-13.8064	7.8525	0.0015716	7.8494-7.8556	4.0326	0.0014452	4.0298-4.0354

Table 6: Approximation with Geometric Asian Call Option as Control Variate

The table with the epsilon values for each strike price and n is the following. It measures which of the two control variates is more efficient:

K

n		**90**	**100**	**110**
	4	0.0147	0.0162	0.0218
	12	0.0120	0.0146	0.0146
	50	0.0126	0.0124	0.0236

Table 7: Efficiency Values

It is clear from Table 7 that the using the expected lower bound is more efficient than using the geometric asian call option.

1.4.2

The table for the Asian call option sensitivity with respect to S_0 approximated using Monte Carlo simulations using the sensitivity of the lower bound as a control variate. The table also displays the standard errors and the confidence intervals:

K

	90			100			110		
	Price	S.E.	C.I.	Price	S.E..	C.I.	Price	S.E.	C.I
4	0.7639	0.00027227	0.7633-0.7644	0.5856	0.00029519	0.5850-0.5861	0.4071	0.00032565	0.4064-0.4077
12	0.7750	0.00027437	0.7744-0.7755	0.5768	0.00030602	0.5762-0.5774	0.3799	0.00033711	0.3793-0.3806
50	0.7793	0.00028271	0.7788-0.7798	0.5730	0.00030121	0.5724-0.5736	0.3673	0.00033988	0.3667-0.3680

Table 8: Approximation with Expected Lower Bound as Control Variate

The table for the Asian call option sensitivity with respect to S_0 approximated using Monte Carlo simulations using the sensitivity of the geometric Asian option as a control variate. The table also displays the standard errors and the confidence intervals:

K

	90			100			110		
	Price	S.E.	C.I.	Price	S.E..	C.I.	Price	S.E.	C.I
4	0.7560	0.00026752	0.7554-0.7565	0.5761	0.00029418	0.5755-0.5767	0.3966	0.00032115	0.3959-0.3972
12	0.7662	0.00028428	0.7656 -0.7667	0.5671	0.00030736	0.5664-0.5677	0.3686	0.00033728	0.3679-0.3692
50	0.7710	0.00028924	0.7704-0.7716	0.5630	0.00030418	0.5624-0.5636	0.3577	0.0003561	0.3570-0.3584

Table 9: Approximation with Geometric Asian Call Option as Control Variate

The table with the epsilon values for each strike price and n is the following. It measures which of the two control variates is more efficient:

n \ K	**90**	**100**	**110**
4	0.9858	0.97608	0.97934
12	0.87348	0.92373	0.95308
50	0.90616	0.93272	0.71096

Table 10: Efficiency Values

It is clear from Table 10 that the using the sensitivity of the expected lower bound is more efficient than using the geometric asian call option.

1.4.3

1.5

Running the expected lower bound with increasing values of n gives a good approximation for the precise value of the expected value. As expected the higher the value of n the bigger the convergence to the precise value.
Concretely, the precise value for the expected lower bound is **7.726961356596341**. Plotting the approximations the following figure can be presented.

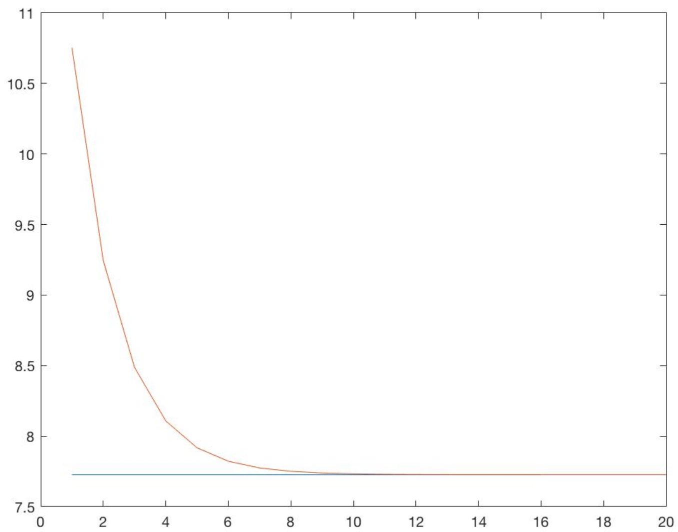

Figure 1: Convergence of the approximation

As it is visible from Figure 1, the approximation gets better when the n is close to the power of 10, and the precision improves as it gets closer to higher n's.

The table can be reviewed for further analysis of the precision of the approximation method.

i (Power of 2)	**Value**
1	10.752849553377715
2	9.244949514019062
3	8.487670434364354
4	8.107824667550155
5	7.917532175125288
6	7.822283197949488
7	7.774631600352421
8	7.750798836772930
9	7.738880689741741
10	7.732921171871205
11	7.729941301464144
12	7.728451338344790
13	7.727706349800059
14	7.727333853780493
15	7.727147605333769
16	7.727054481001673
17	7.727007918808091
18	7.726984637705648
19	7.726972997149829
20	7.726967176874716

Table 11: Approximations of the lower bound