General instructions

- Pre-assigned groups by the course office.
- Submitted report must be a pdf file (which you should keep as short as possible, bearing in mind that filling in pages with redundant information makes it more difficult to read and less efficient to mark!). Short extracts of Matlab code could go in the report (only in places where you feel that deserve special attention).
- Along with your report, submit working versions of your Matlab codes (m files). As these are the files that I will inspect and ultimately run to check the validity of your reported results, make sure that your lines of code appear neatly and not messed up; importantly, key lines of code should be accompanied by short comments (%) that provide necessary explanations of what is performed (similar to my codes in the lectures); give names to your m files that relate to particular exercise and task; within your reports, clearly refer to these m file names where relevant.
- The quality of the report is as important as the actual functioning of your codes to get the marks. Only your own codes are acceptable; no copies from textbooks or internet sources.
- Any questions regarding the coursework must be asked at the beginning of the lectures/computer labs in front of the whole class; I will not respond to e-mails or personal enquiries regarding the coursework.
- Submission must be made online through Moodle (`http://moodle.city.ac.uk/`) only: this must consist of a zip file containing the report (in pdf format) and the accompanying Matlab codes.
- **Submission deadline** for the coursework is on Wednesday, 21 March 2018; extensions are rarely granted and only by the Course Director, Dr Dirk Nitzsche.
- Feedback on marked coursework will be provided.

Some Matlab tips

- Make sure that you clear Matlab's memory every time before a new execution of a code (use command "`clear`"), so that variables' values stored from a previous session are all cleared and no conflicts are caused.
- To speed up executions, consider pre-allocating arrays before "`for`" loops (as discussed and demonstrated during lectures); not doing so will not cost you in marks (providing that the code generates the expected outcome), however it will take much longer to do the computations.
- Use "`tic-toc`" to measure the computational time where required.

Exercise 1

Theoretical background: consider the problem of pricing an *arithmetic* Asian call option on some underlying asset S in the geometric Brownian motion model framework. The time to maturity is T and the underlying is monitored discretely at n equally spaced dates throughout the lifespan of the option. The time interval between successive monitoring dates is $\delta = T/n$. The strike price is fixed at K. The arithmetic Asian call option has discounted payoff at time T

$$A_n = e^{-rT} \max\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K, 0\right), \tag{1}$$

where r is the continuously compounded risk-free interest rate. The price of the option is given by

$$E(A_n).$$

The following *lower bound* to the discounted payoff (1) applies:

$$\mathrm{LB}_n = e^{-rT}\left(\frac{1}{n}\sum_{k=1}^{n} S_k - K\right)\mathbf{1}_{\left\{\left(\prod_{k=1}^{n} S_k\right)^{1/n} > K\right\}} \tag{2}$$

where $\mathbf{1}_\Lambda = 1$, if event Λ holds, otherwise $\mathbf{1}_\Lambda = 0$. Curran (1994) derives the exact closed-form expression for $E(\mathrm{LB}_n)$:

$$E(\mathrm{LB}_n) = \frac{S_0 e^{-rT}}{n}\sum_{k=1}^{n} e^{\mu_k + \sigma_k^2/2} N(b + a_k) - K e^{-rT} N(b), \tag{3}$$

where $N(.)$ denotes the standard normal CDF and

$$\begin{aligned}
\mu_k &= (r - \sigma^2/2)k\delta, \qquad \sigma_k = \sigma\sqrt{k\delta}, \qquad a_k = \sigma\sqrt{\delta}\frac{k(n+1-(k+1)/2)}{\sqrt{n(n+1)(2n+1)/6}}, \\
b &= \frac{\ln(S_0/K) + (r - \sigma^2/2)\bar{T}}{\bar{\sigma}\sqrt{\bar{T}}}, \\
\bar{\sigma} &= \sigma\sqrt{\frac{2n+1}{3n}}, \qquad \bar{T} = (n+1)\delta/2.
\end{aligned} \tag{4}$$

In addition, consider the *geometric* Asian call option on the same asset S, strike K and time to maturity T. The discounted payoff at time T of the geometric Asian call option is

$$G_n = e^{-rT} \max\left(\left(\prod_{k=1}^{n} S_k\right)^{1/n} - K, 0\right). \tag{5}$$

Kemna and Vorst (1990) present the exact closed-form expression for $E(G_n)$, that is, the price of the geometric Asian call option,

$$E(G_n) = S_0 e^{(r - \sigma^2/2 + \bar{\sigma}^2/2)\bar{T} - rT} N(d) - K e^{-rT} N(d - \bar{\sigma}\sqrt{\bar{T}}), \tag{6}$$

where

$$d = \frac{\ln(S_0/K) + (r - \sigma^2/2 + \bar{\sigma}^2)\bar{T}}{\bar{\sigma}\sqrt{\bar{T}}}$$

and $\bar{\sigma}$ and $\bar{T}$ are given by (4).

Tasks

1) i) Present the expression for the likelihood ratio (LR) first-order sensitivity estimator with respect to S_0 of: (a) the arithmetic Asian option with discounted payoff (1); (b) the lower bound (2); (c) the geometric Asian option with discounted payoff (5). (Use the delta score without providing its proof.)

   ii) Does a pathwise (PW) first-order sensitivity estimator with respect to S_0 of the lower bound (2) exist? Explain briefly without providing any mathematical proof.

2) Derive exact formulae of the first sensitivities with respect to S_0 of the expected lower bound (3) and the geometric Asian option price (6) by differentiating the given formulae.

3) Consider the following parameter values: $S_0 = 100$, $r = 0.04$, $\sigma = 0.3$, $K = [90\ 100\ 110]$, $T = 1$, $n = [4\ 12\ 50]$. For all possible pairs (K, n) and using the exact formulae (3), (6) as well as the ones you have derived in Task 2, compute and report in a table: (a) the expected lower bounds and their first sensitivities with respect to S_0; (b) the geometric Asian option prices and their first sensitivities with respect to S_0.

4) For the same parameters given in Task 3, implement Monte Carlo simulation to compute:

   - the arithmetic Asian option prices for all possible pairs (K, n). For each pair (K, n), perform simulation using as control variate: (a) the lower bound (2); (b) the geometric Asian option with discounted payoff (5)
   - the arithmetic Asian option price (first-order) sensitivities with respect to S_0 for all possible pairs (K, n). For each pair (K, n), perform simulation using as control variate: (a) the sensitivity of the lower bound with respect to S_0; (b) the geometric Asian option sensitivity with respect to S_0

   In all experiments use $M = 10^5$ simulations.

   i) Report your arithmetic Asian option price and sensitivity results (in **two separate** tables), alongside their standard errors and 95% confidence intervals, for each (K, n) and both approaches (a) and (b). Refrain from splitting a table across two pages.

   ii) In the last column of each of the two tables above, report also the efficiency ratios

   $$\mathcal{E}(K, n) = \frac{\hat{t}_a \hat{\sigma}_a^2}{\hat{t}_b \hat{\sigma}_b^2},$$

   where $\hat{\sigma}^2$ is the squared standard error of the indicated approach (a) or (b) and $\hat{t}$ the corresponding computational time, noting that, when $\mathcal{E} > 1$, approach (b) is more efficient than (a), and vice versa when $\mathcal{E} < 1$.

iii) Comment on your results with references to the accuracy (in number of decimals) of your arithmetic Asian estimates based on your 95% confidence intervals; the effect of the two different control variates on the level of accuracy; the efficiency gains from the use of control variates and the impact of the correlation level between the arithmetic Asian discounted payoff and control variate samples; how the efficiency gains differ between prices and sensitivities (for given K and n); the effects of varying K (for given n) and varying n (for given K); how close the expected lower bound itself is to the arithmetic Asian option price estimate. Use your initiative and judgement in delivering the relevant information aiming to achieve the best readability.

5) Thompson (1999) derives the formula of the expected lower bound (3) for the case of **continuous monitoring** (that is, for $n = \infty$):

$$E(\mathrm{LB}_\infty) = \frac{S_0}{T}\int_0^T e^{-r(T-t)} N\left(\frac{-\gamma + \sigma t - \sigma t^2/(2T)}{\sqrt{T/3}}\right) dt - Ke^{-rT} N\left(\frac{-\gamma}{\sqrt{T/3}}\right), \tag{7}$$

where $N(.)$ denotes the standard normal CDF and

$$\gamma = \frac{\ln(K/S_0) - (r - \sigma^2/2)T/2}{\sigma}.$$

Formula (7) can be implemented in Matlab by numerical integration.

Consider the following parameter values: $S_0 = 100$, $r = 0.04$, $\sigma = 0.3$, $K = 100$, $T = 1$. Computing the expected lower bound (3) for increasing $n = 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, \ldots$, study and comment on the convergence to the expected lower bound (7).

Exercise 2

Theoretical background:

1) Consider the problem of pricing a **discretely monitored** *up-and-out barrier call* (UOC) option on some underlying asset S in the geometric Brownian motion model framework. The time to maturity is T and the underlying is monitored discretely at n equally spaced dates throughout the lifespan of the option. The time interval between successive monitoring dates is $\delta = T/n$. The option is knocked out when S hits some upper barrier level U; otherwise it pays off as a plain vanilla call option at maturity T. The strike price is K. In summary, the option has discounted payoff at time T

$$C_n = e^{-rT} \max(S_n - K, 0) \mathbf{1}_{\{\max\{S_0, S_1, \ldots, S_n\} < U\}},$$

where $\mathbf{1}_\Lambda = 1$, if event Λ holds, otherwise $\mathbf{1}_\Lambda = 0$. The price of this option

$$E(C_n)$$

is not known explicitly, hence it may be computed by Monte Carlo simulation.

Furthermore, it is proved in Shreve (2004) that the price of the **continuously monitored** up-and-out barrier call option (that is, the case of $n = \infty$) is given by the closed-form formula

$$\begin{aligned} & f_{UOC}(T, S_0, K, U, \sigma, r) \\ = \; & S_0 \left[N\left(d_+\left(\frac{S_0}{K}\right)\right) - N\left(d_+\left(\frac{S_0}{U}\right)\right)\right] - Ke^{-rT}\left[N\left(d_-\left(\frac{S_0}{K}\right)\right) - N\left(d_-\left(\frac{S_0}{U}\right)\right)\right] \\ & -U\left(\frac{S_0}{U}\right)^{-2r/\sigma^2}\left[N\left(d_+\left(\frac{U^2}{KS_0}\right)\right) - N\left(d_+\left(\frac{U}{S_0}\right)\right)\right] \\ & +Ke^{-rT}\left(\frac{S_0}{U}\right)^{1-2r/\sigma^2}\left[N\left(d_-\left(\frac{U^2}{KS_0}\right)\right) - N\left(d_-\left(\frac{U}{S_0}\right)\right)\right], \end{aligned} \tag{8}$$

where $N(.)$ denotes the standard normal CDF and

$$d_\pm(x) = \frac{\ln x + (r \pm \sigma^2/2)T}{\sigma\sqrt{T}}. \tag{9}$$

2) Consider the problem of pricing a **discretely monitored** *lookback call* option on S. The time to maturity is T and the underlying is monitored discretely at n equally spaced dates throughout the lifespan of the option. The time interval between successive monitoring dates is $\delta = T/n$. The strike price is fixed at K. The option has discounted payoff at time T

$$C_n = e^{-rT} \max(\max\{S_0, S_1, \ldots, S_n\} - K, 0).$$

The price of this option

$$E(C_n)$$

is not known explicitly, hence it may be computed by Monte Carlo simulation.

From Haug (2007), the price of the **continuously monitored** lookback call option (that is, the case of $n = \infty$) for $K \geq S_0$ is given by the closed-form formula

$$f_{LC}(T, S_0, K, \sigma, r) = S_0 N\left(d_+\left(\frac{S_0}{K}\right)\right) - Ke^{-rT} N\left(d_-\left(\frac{S_0}{K}\right)\right)$$
$$+\frac{S_0 e^{-rT}\sigma^2}{2r}\left[-\left(\frac{S_0}{K}\right)^{-2r/\sigma^2} N\left(d_+\left(\frac{S_0}{K}\right) - \frac{2r\sqrt{T}}{\sigma}\right) + e^{rT} N\left(d_+\left(\frac{S_0}{K}\right)\right)\right], \quad (10)$$

where $N(.)$ denotes the standard normal CDF and $d_\pm(x)$ are given by (9).

Tasks

1) Consider the following parameter values: $S_0 = 110$, $r = 0.05$, $\sigma = 0.1$, $K = 100$, $U = [160\ 170]$. Implement Monte Carlo simulation with antithetic variates to estimate the price of the discretely monitored UOC barrier option for increasing $n = 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, \ldots$ with $\delta = T/n$ and $T = 2$, using $M = 10^6$ for each n and U. (You may use $n > 2^8$ up to the point that computations become perceptibly time-consuming.)

   i) Report your results and comment on the performance of the antithetic variates and the accuracy (in number of decimals) of your price estimates for increasing n based on your 95% confidence intervals.

   ii) Study and comment on the convergence of the computed price estimates above to the price of the continuously monitored UOC barrier option given by (8).

2) Consider the following parameter values: $S_0 = 100$, $r = 0.05$, $\sigma = 0.1$, $K = [100\ 105\ 110]$. Implement Monte Carlo simulation with antithetic variates to estimate the price of the discretely monitored lookback call option for increasing $n = 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, \ldots$ with $\delta = T/n$ and $T = 2$, using $M = 10^6$ for each n and K. (You may use $n > 2^8$ up to the point that computations become perceptibly time-consuming.)

   i) Report your results and comment on the performance of the antithetic variates and the accuracy (in number of decimals) of your price estimates for increasing n based on your 95% confidence intervals.

   ii) Study and comment on the convergence of the computed price estimates to the price of the continuously monitored lookback call option given by (10).

Use your initiative and judgement in your study and organization of the information in tables and/or diagrams aiming to achieve the best readability.

References

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