Preuves en logique du premier ordre

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Logique du premier ordre

Définitions préliminaires

- $\mathcal{V} \equiv$ ensemble de variables d'individu x, y, etc. ;
- $\mathcal{S}_{\mathcal{F}} \equiv$ ensemble de symboles de fonctions f, g, etc. ;
- $\mathcal{S}_{\mathcal{P}} \equiv$ ensemble de symboles de prédicats P, Q, etc. ;
- $\mathcal{S}_{\mathcal{F}} \cap \mathcal{S}_{\mathcal{P}} = \emptyset$;
- Arité $m : \mathcal{S}_{\mathcal{F}} \cup \mathcal{S}_{\mathcal{P}} \to \mathbb{N}$.

Termes du premier ordre

- Plus petit ensemble $\mathcal{T}$ t.q. :
  - Si $x \in \mathcal{V}$ alors $x \in \mathcal{T}$;
  - Si $f \in \mathcal{S}_{\mathcal{F}}$ d'arité n et $t_1, \ldots, t_n \in \mathcal{T}$, alors $f(t_1, \ldots, t_n) \in \mathcal{T}$.

Logique du premier ordre

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- Arité $m : \mathcal{S}_\mathcal{F} \cup \mathcal{S}_\mathcal{P} \to \mathbb{N}$.

Formules du premier ordre

- Plus petit ensemble $\mathcal{F}$ t.q. :
  - Si $P \in \mathcal{S}_\mathcal{P}$ d'arité n et $t_1, \ldots, t_n \in \mathcal{T}$, alors $P(t_1, \ldots, t_n) \in \mathcal{F}$;
  - $\perp, \top \in \mathcal{F}$;
  - Si $\Phi \in \mathcal{F}$ alors $\neg\Phi \in \mathcal{F}$;
  - Si $\Phi, \Phi' \in \mathcal{F}$ alors $\Phi \wedge \Phi', \Phi \vee \Phi', \Phi \Rightarrow \Phi', \Phi \Leftrightarrow \Phi' \in \mathcal{F}$;
  - Si $x \in \mathcal{V}$ et $\Phi \in \mathcal{F}$, alors $\forall x.\Phi, \exists x.\Phi \in \mathcal{F}$.

Logique du premier ordre

Associativité des connecteurs

- $\wedge$, $\vee$, et $\Leftrightarrow$ associent à gauche :
  - $A \wedge B \wedge C \equiv (A \wedge B) \wedge C$.
- $\Rightarrow$ associe à droite :
  - $A \Rightarrow B \Rightarrow C \equiv A \Rightarrow (B \Rightarrow C)$.

Précédence des connecteurs

- On a la précédence suivante : $\neg \succ \wedge \succ \vee \succ \Rightarrow \succ \Leftrightarrow$;
- Exemples :
  - $A \wedge B \Rightarrow C \equiv (A \wedge B) \Rightarrow C$;
  - $A \wedge \neg B \vee C \Rightarrow D \equiv ((A \wedge \neg B) \vee C) \Rightarrow D$;
  - $A \Rightarrow B \Leftrightarrow C \wedge D \equiv (A \Rightarrow B) \Leftrightarrow (C \wedge D)$.

Logique du premier ordre

Notation pointée pour les quantificateurs

- La portée d'un quantificateur va jusqu'à la parenthèse fermante de la formule du quantificateur ;
- Si la formule du quantificateur n'est pas parenthésée, la portée du quantificateur va jusqu'à la fin de la formule ;
- Donc, si on veut arrêter la portée d'un quantificateur, il suffit d'utiliser des parenthèses pour limiter explicitement la portée du quantificateur ;
- Exemple :
  - $\exists x.P(x) \Rightarrow P(a) \wedge P(b) \equiv \exists x.(P(x) \Rightarrow P(a) \wedge P(b))$;
  - Si on veut que le $\exists$ ne porte que sur $P(x)$, on doit écrire : $(\exists x.P(x)) \Rightarrow P(a) \wedge P(b)$.
- Notation : $\forall x, y.\Phi \equiv \forall x.\forall y.\Phi$ (idem pour $\exists$).

Sémantiques

Logique classique

- Une formule est toujours vraie ou fausse ;
- Que je puisse en démontrer la validité ou non ;
- Logique bi-valuée (vrai, faux) ;
- Logique du « tiers exclu » : $A \vee \neg A$.

Logique intuitionniste ou constructive

- Une formule est vraie, fausse, ou « on ne sait pas » ;
- Si on ne sait en démontrer la validité, alors « on ne sait pas » ;
- Logique tri-valuée d'une certaine manière ;
- Le « tiers exclu » n'est pas admis dans cette logique.

Systèmes de preuves

Plusieurs systèmes

- Systèmes à la Frege-Hilbert ;
- Systèmes à la Gentzen :
  - Déduction naturelle ;
  - Calcul des séquents.

Adéquation vis-à-vis de la sémantique

- Correction et complétude par rapport à la sémantique ;
- Correction : si je trouve une preuve de P alors P est vraie ;
- Complétude : si P est vraie alors il existe une preuve de P ;
- Preuve $\equiv$ moyen syntaxique de vérifier la validité d'une formule.

Calcul des séquents intuitionniste (LJ)

Règles

$$\frac{}{\Gamma, A \vdash A}\ \text{ax} \qquad \frac{\Gamma, A, A \vdash B}{\Gamma, A \vdash B}\ \text{cont}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \Rightarrow B \vdash C}\ \Rightarrow_{\text{left}} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \Rightarrow B}\ \Rightarrow_{\text{right}}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \Leftrightarrow B \vdash C}\ \Leftrightarrow_{\text{left1}}$$

$$\frac{\Gamma \vdash B \qquad \Gamma, A \vdash C}{\Gamma, A \Leftrightarrow B \vdash C}\ \Leftrightarrow_{\text{left2}} \qquad \frac{\Gamma, A \vdash B \qquad \Gamma, B \vdash A}{\Gamma \vdash A \Leftrightarrow B}\ \Leftrightarrow_{\text{right}}$$

Calcul des séquents intuitionniste (LJ)

Règles

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge_{\text{left}} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_{\text{right}}$$

$$\frac{\Gamma \vdash A}{\Gamma \vdash A \vee B}\ \vee_{\text{right1}}$$

$$\frac{\Gamma, A \vdash C \qquad \Gamma, B \vdash C}{\Gamma, A \vee B \vdash C}\ \vee_{\text{left}} \qquad \frac{\Gamma \vdash B}{\Gamma \vdash A \vee B}\ \vee_{\text{right2}}$$

$$\frac{\Gamma \vdash A}{\Gamma, \neg A \vdash B}\ \neg_{\text{left}} \qquad \frac{\Gamma, A \vdash \bot}{\Gamma \vdash \neg A}\ \neg_{\text{right}}$$

$$\frac{}{\Gamma, \bot \vdash A}\ \bot_{\text{left}} \qquad \frac{}{\Gamma \vdash \top}\ \top_{\text{right}}$$

Calcul des séquents intuitionniste (LJ)

Règles

$$\frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x.A(x) \vdash B}\ \forall_{\text{left}} \qquad \frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x.A(x)}\ \forall_{\text{right}},\ \ x \notin \Gamma$$

$$\frac{\Gamma, A(x) \vdash B}{\Gamma, \exists x.A(x) \vdash B}\ \exists_{\text{left}},\ \ x \notin \Gamma, B \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x.A(x)}\ \exists_{\text{right}}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}\ \text{cut}$$

Calcul des séquents classique (LJ_{em})

Règles

$$\frac{\Gamma, A(t) \vdash B}{\Gamma, \forall x.A(x) \vdash B}\ \forall_{\text{left}} \qquad \frac{\Gamma \vdash A(x)}{\Gamma \vdash \forall x.A(x)}\ \forall_{\text{right}},\ \ x \notin \Gamma$$

$$\frac{\Gamma, A(x) \vdash B}{\Gamma, \exists x.A(x) \vdash B}\ \exists_{\text{left}},\ \ x \notin \Gamma, B \qquad \frac{\Gamma \vdash A(t)}{\Gamma \vdash \exists x.A(x)}\ \exists_{\text{right}}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, A \vdash B}{\Gamma \vdash B}\ \text{cut} \qquad \frac{\Gamma \vdash \neg\neg A}{\Gamma \vdash A}\ \text{em}$$

Calcul des séquents classique (LK)

Règles

$$\frac{}{\Gamma, A \vdash \Delta, A}\ \text{ax} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, B}\ \text{cut}$$

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \text{cont}_{\text{left}} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \text{cont}_{\text{right}}$$

Calcul des séquents classique (LK)

Règles

$$\frac{\Gamma \vdash \Delta, A \qquad \Gamma, B \vdash \Delta}{\Gamma, A \Rightarrow B \vdash \Delta} \Rightarrow_{\text{left}} \qquad \frac{\Gamma, A \vdash \Delta, B}{\Gamma \vdash \Delta, A \Rightarrow B} \Rightarrow_{\text{right}}$$

$$\frac{\Gamma \vdash \Delta, A, B \qquad \Gamma, A, B \vdash \Delta}{\Gamma, A \Leftrightarrow B \vdash \Delta} \Leftrightarrow_{\text{left}}$$

$$\frac{\Gamma, A \vdash \Delta, B \qquad \Gamma, B \vdash \Delta, A}{\Gamma \vdash \Delta, A \Leftrightarrow B} \Leftrightarrow_{\text{right}}$$

Calcul des séquents classique (LK)

Règles

$$\frac{\Gamma, A, B \vdash \Delta}{\Gamma, A \wedge B \vdash \Delta}\ \wedge_{\text{left}} \qquad \frac{\Gamma \vdash \Delta, A \qquad \Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \wedge B}\ \wedge_{\text{right}}$$

$$\frac{\Gamma, A \vdash \Delta \qquad \Gamma, B \vdash \Delta}{\Gamma, A \vee B \vdash \Delta}\ \vee_{\text{left}} \qquad \frac{\Gamma \vdash \Delta, A, B}{\Gamma \vdash \Delta, A \vee B}\ \vee_{\text{right}}$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma, \neg A \vdash \Delta}\ \neg_{\text{left}} \qquad \frac{\Gamma, A \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \neg_{\text{right}}$$

$$\frac{}{\Gamma, \bot \vdash \Delta}\ \bot_{\text{left}} \qquad \frac{}{\Gamma \vdash \Delta, \top}\ \top_{\text{right}}$$

Calcul des séquents classique (LK)

Règles

$$\frac{\Gamma, A(t) \vdash \Delta}{\Gamma, \forall x.A(x) \vdash \Delta}\ \forall_{\text{left}} \qquad \frac{\Gamma \vdash \Delta, A(x)}{\Gamma \vdash \Delta, \forall x.A(x)}\ \forall_{\text{right}},\ \ x \notin \Gamma, \Delta$$

$$\frac{\Gamma, A(x) \vdash \Delta}{\Gamma, \exists x.A(x) \vdash \Delta}\ \exists_{\text{left}},\ \ x \notin \Gamma, \Delta \qquad \frac{\Gamma \vdash \Delta, A(t)}{\Gamma \vdash \Delta, \exists x.A(x)}\ \exists_{\text{right}}$$

Exemple de preuve propositionnelle dans LJ/LK

Une preuve simple

Règles LJ

$$\dfrac{\dfrac{\dfrac{A, B \vdash A \qquad A, B \vdash B}{A, B \vdash A \wedge B}}{A \vdash B \Rightarrow A \wedge B}}{\vdash A \Rightarrow B \Rightarrow A \wedge B}$$

Exemple de preuve propositionnelle dans LJ/LK

Une preuve simple

Règles LJ

$$\dfrac{\dfrac{\dfrac{A, B \vdash A \qquad A, B \vdash B}{A, B \vdash A \wedge B}}{A \vdash B \Rightarrow A \wedge B}}{\vdash A \Rightarrow B \Rightarrow A \wedge B} \Rightarrow_{\text{right}}$$

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Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

Règles LJ

$$P(x) \vdash P(x)$$

$$P(x) \vdash \exists x.P(x)$$

$$\neg\exists x.P(x), P(x) \vdash \bot$$

$$\neg\exists x.P(x) \vdash \neg P(x)$$

$$\neg\exists x.P(x) \vdash \forall x.\neg P(x)$$

$$\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)$$

Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

Règles LJ

$$P(x) \vdash P(x)$$

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$$\dfrac{\neg\exists x.P(x) \vdash \forall x.\neg P(x)}{\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)}\ \Rightarrow_{\text{right}}$$

Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

▸ Règles LJ

$$P(x) \vdash P(x)$$

$$P(x) \vdash \exists x.P(x)$$

$$\neg\exists x.P(x), P(x) \vdash \bot$$

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Exemple de preuve au premier ordre dans LJ/LK

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Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

Règles LJ

$$P(x) \vdash P(x)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash \exists x.P(x)}{\neg\exists x.P(x), P(x) \vdash \bot}\neg_{\text{left}}}{\neg\exists x.P(x) \vdash \neg P(x)}\neg_{\text{right}}}{\neg\exists x.P(x) \vdash \forall x.\neg P(x)}\forall_{\text{right}}}{\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)}\Rightarrow_{\text{right}}$$

Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}\ \exists_{\text{right}}}{\neg\exists x.P(x), P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg\exists x.P(x) \vdash \neg P(x)}\ \neg_{\text{right}}}{\neg\exists x.P(x) \vdash \forall x.\neg P(x)}\ \forall_{\text{right}}}{\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)}\ \Rightarrow_{\text{right}}$$

Exemple de preuve au premier ordre dans LJ/LK

Négation et quantificateurs

▸ Règles LJ

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\ \mathrm{ax}}{P(x) \vdash \exists x.P(x)}\ \exists_{\mathrm{right}}}{\neg\exists x.P(x), P(x) \vdash \bot}\ \neg_{\mathrm{left}}}{\neg\exists x.P(x) \vdash \neg P(x)}\ \neg_{\mathrm{right}}}{\neg\exists x.P(x) \vdash \forall x.\neg P(x)}\ \forall_{\mathrm{right}}}{\vdash \neg(\exists x.P(x)) \Rightarrow \forall x.\neg P(x)}\ \Rightarrow_{\mathrm{right}}
$$

Logiques classique/intuitionniste

Sémantique du « il existe »

- En logique classique : $\exists x.P(x) \equiv$ il existe n termes $t_1, t_2, \ldots, t_n$ tels que $P(t_1) \vee P(t_2) \vee \ldots \vee P(t_n)$ est vraie (théorème de Herbrand) ;
- En logique intuitionniste : $\exists x.P(x) \equiv$ il existe un terme t tel que $P(t)$ est vraie.
  On doit construire un témoin t qui vérifie P et en avoir l'intuition.
  D'où le nom de logique « intuitionniste » ou « constructive ».

Logique classique

- La logique classique est une logique assez « exotique » ;
- On peut démontrer une formule $\exists x.P(x)$ sans jamais montrer un seul témoin qui fonctionne (c'est-à-dire qui vérifie P) !
- De ce fait, c'est plus facile de faire des preuves en logique classique qu'en logique intuitionniste.

Exemple de preuve en logique classique

Petit théorème mathématique

- Il existe a et b irrationnels tels que a^b est rationnel ;
- Preuve :
  - Utilisation du tiers exclu : $\sqrt{2}^{\sqrt{2}}$ est rationnel ou non ; deux cas :
    - Si $\sqrt{2}^{\sqrt{2}}$ est rationnel, alors le théorème est vrai ;
    - Si $\sqrt{2}^{\sqrt{2}}$ est irrationnel, alors $(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} = \sqrt{2}^2 = 2$, qui est rationnel.

En logique intuitionniste

- Le théorème est vrai en logique intuitionniste ;
- Mais on doit montrer un a et b qui fonctionnent ;
- Plusieurs pages de théorie des nombres non triviales !

Un autre exemple de preuve en logique classique

Preuve dans LK

Démontrer : $\exists x.P(x) \Rightarrow P(a) \land P(b)$.

Cette formule est-elle valide ?

Un autre exemple de preuve en logique classique

Preuve dans LK

Règles LK

$$\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)$$

$$\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)$$

$$P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)$$

$$P(a) \vdash P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$$

$$\vdash P(a) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$$

$$\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$$

$$\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b)$$

Un autre exemple de preuve en logique classique

Preuve dans LK

Règles LK

$$\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)$$

$$\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)$$

$$P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)$$

$$P(a) \vdash P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)$$

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$$\frac{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\ \mathrm{cont}_{\mathrm{right}}$$

Un autre exemple de preuve en logique classique

Preuve dans LK

▸ Règles LK

$$\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)$$

$$\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)$$

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Un autre exemple de preuve en logique classique

Preuve dans LK

‣ Règles LK

$$\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)$$

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Un autre exemple de preuve en logique classique

Preuve dans LK

Règles LK

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\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma \vdash P(a), P(a) \wedge P(b) \qquad \Gamma \vdash P(b), P(a) \wedge P(b)}{\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)}}{P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)}}{P(a) \vdash P(a) \wedge P(b), \exists x. P(x) \Rightarrow P(a) \wedge P(b)}\ \exists_{\text{right}}}{\vdash P(a) \Rightarrow P(a) \wedge P(b), \exists x. P(x) \Rightarrow P(a) \wedge P(b)}\ \Rightarrow_{\text{right}}}{\dfrac{\vdash \exists x. P(x) \Rightarrow P(a) \wedge P(b), \exists x. P(x) \Rightarrow P(a) \wedge P(b)}{\vdash \exists x. P(x) \Rightarrow P(a) \wedge P(b)}\ \text{cont}_{\text{right}}}\ \exists_{\text{right}}
$$

Un autre exemple de preuve en logique classique

Preuve dans LK

Règles LK

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Preuve dans LK

▸ Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash P(a), P(a) \wedge P(b)}\text{ax} \qquad \Gamma \vdash P(b), P(a) \wedge P(b)}{\Gamma = P(a), P(b) \vdash P(a) \wedge P(b), P(a) \wedge P(b)}\wedge_{\text{right}}}{P(a) \vdash P(a) \wedge P(b), P(b) \Rightarrow P(a) \wedge P(b)}\Rightarrow_{\text{right}}}{P(a) \vdash P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\exists_{\text{right}}}{\vdash P(a) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b), \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow P(a) \wedge P(b)}\text{cont}_{\text{right}}
$$

Un autre exemple de preuve en logique classique

Preuve dans LK

▸ Règles LK

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{
          \dfrac{
            \dfrac{}{\Gamma \vdash P(a), P(a) \land P(b)}\,\mathrm{ax}
            \quad
            \dfrac{}{\Gamma \vdash P(b), P(a) \land P(b)}\,\mathrm{ax}
          }{\Gamma = P(a), P(b) \vdash P(a) \land P(b), P(a) \land P(b)}\,\land_{\mathrm{right}}
        }{P(a) \vdash P(a) \land P(b), P(b) \Rightarrow P(a) \land P(b)}\,\Rightarrow_{\mathrm{right}}
      }{P(a) \vdash P(a) \land P(b), \exists x.P(x) \Rightarrow P(a) \land P(b)}\,\exists_{\mathrm{right}}
    }{\vdash P(a) \Rightarrow P(a) \land P(b), \exists x.P(x) \Rightarrow P(a) \land P(b)}\,\Rightarrow_{\mathrm{right}}
  }{\vdash \exists x.P(x) \Rightarrow P(a) \land P(b), \exists x.P(x) \Rightarrow P(a) \land P(b)}\,\exists_{\mathrm{right}}
}{\vdash \exists x.P(x) \Rightarrow P(a) \land P(b)}\,\mathrm{cont}_{\mathrm{right}}
$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

$$
\begin{array}{c}
P(x), P(y) \vdash P(y), \forall y.P(y) \\
P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y) \\
P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y) \\
P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y) \\
\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y) \\
\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y) \\
\vdash \exists x.P(x) \Rightarrow \forall y.P(y)
\end{array}
$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

▸ Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$
$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$
$$P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$
$$\vdash \exists x.P(x) \Rightarrow \forall y.P(y)$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$

$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$

$$P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$\frac{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\ \mathrm{cont}_{\mathrm{right}}$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$

$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$

$$P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$\cfrac{\cfrac{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\ \exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\ \text{cont}_{\text{right}}$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

▸ Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$

$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$

$$P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$$

$$\dfrac{\dfrac{\dfrac{P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\text{cont}_{\text{right}}$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

▸ Règles LK

$$P(x), P(y) \vdash P(y), \forall y.P(y)$$

$$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}{P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\,\forall_{\text{right}}}{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\,\Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\,\exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\,\text{cont}_{\text{right}}$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

Règles LK

$$
\begin{prooftree}
\AxiomC{$P(x), P(y) \vdash P(y), \forall y.P(y)$}
\noLine
\UnaryInfC{$P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)$}
\RightLabel{$\exists_{\text{right}}$}
\UnaryInfC{$P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$}
\RightLabel{$\exists_{\text{right}}$}
\UnaryInfC{$\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)$}
\RightLabel{$\text{cont}_{\text{right}}$}
\UnaryInfC{$\vdash \exists x.P(x) \Rightarrow \forall y.P(y)$}
\end{prooftree}
$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x), P(y) \vdash P(y), \forall y.P(y)}{P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)}\Rightarrow_{\text{right}}}{P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\exists_{\text{right}}}{P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\forall_{\text{right}}}{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\text{cont}_{\text{right}}
$$

Une preuve classique un peu plus complexe

Paradoxe (pas si paradoxal) des buveurs

Énoncé : « Il y a quelqu'un dans un bar tel que, s'il boit alors tout le monde dans le bar boit ».

▸ Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x), P(y) \vdash P(y), \forall y.P(y)}\text{ax}}{P(x) \vdash P(y), P(y) \Rightarrow \forall y.P(y)}\Rightarrow_{\text{right}}}{P(x) \vdash P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\exists_{\text{right}}}{P(x) \vdash \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\forall_{\text{right}}}{\vdash P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\Rightarrow_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y), \exists x.P(x) \Rightarrow \forall y.P(y)}\exists_{\text{right}}}{\vdash \exists x.P(x) \Rightarrow \forall y.P(y)}\text{cont}_{\text{right}}
$$

Exercices en logique propositionnelle

Propositions à démontrer dans LJ et LK

1. $A \Rightarrow B \Rightarrow A$
2. $(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$
3. $A \wedge B \Rightarrow B$
4. $B \Rightarrow A \vee B$
5. $(A \vee B) \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$
6. $A \Rightarrow \perp \Rightarrow \neg A$
7. $\perp \Rightarrow A$
8. $(A \Leftrightarrow B) \Rightarrow A \Rightarrow B$
9. $(A \Leftrightarrow B) \Rightarrow B \Rightarrow A$
10. $(A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$

Correction

Preuve (1) dans LJ/LK

▸ Règles LJ

$$\cfrac{\cfrac{A, B \vdash A}{A \vdash B \Rightarrow A}}{\vdash A \Rightarrow B \Rightarrow A}$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{A, B \vdash A}{A \vdash B \Rightarrow A}}{\vdash A \Rightarrow B \Rightarrow A} \Rightarrow_{\text{right}}$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{A, B \vdash A}{A \vdash B \Rightarrow A}\Rightarrow_{\text{right}}}{\vdash A \Rightarrow B \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{}{A, B \vdash A}\,\text{ax}}{A \vdash B \Rightarrow A}\,\Rightarrow_{\text{right}}}{\vdash A \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{A, B \vdash B}{A \wedge B \vdash B}}{\vdash A \wedge B \Rightarrow B}$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$\dfrac{\dfrac{A, B \vdash B}{A \wedge B \vdash B}}{\vdash A \wedge B \Rightarrow B} \Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$\dfrac{\dfrac{A, B \vdash B}{A \wedge B \vdash B}\,\wedge_{\text{left}}}{\vdash A \wedge B \Rightarrow B}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{}{A, B \vdash B}\,\text{ax}}{A \wedge B \vdash B}\,\wedge_{\text{left}}}{\vdash A \wedge B \Rightarrow B}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{B \vdash B}{B \vdash A \vee B}}{\vdash B \Rightarrow A \vee B}$$

Correction

Preuve (4) dans LJ

Règles LJ

$$\dfrac{\dfrac{B \vdash B}{B \vdash A \vee B}}{\vdash B \Rightarrow A \vee B}\Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{B \vdash B}{B \vdash A \vee B}\ \vee_{\text{right2}}}{\vdash B \Rightarrow A \vee B}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{}{B \vdash B}\,\mathrm{ax}}{B \vdash A \vee B}\,\vee_{\mathrm{right2}}}{\vdash B \Rightarrow A \vee B}\,\Rightarrow_{\mathrm{right}}$$

Correction

Preuve (4) dans LK

▸ Règles LK

$$\dfrac{\dfrac{B \vdash A, B}{B \vdash A \vee B}}{\vdash B \Rightarrow A \vee B}$$

Correction

Preuve (4) dans LK

Règles LK

$$\dfrac{\dfrac{B \vdash A, B}{B \vdash A \vee B}}{\vdash B \Rightarrow A \vee B} \Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LK

Règles LK

$$\dfrac{\dfrac{B \vdash A, B}{B \vdash A \vee B}\ \vee_{\text{right}}}{\vdash B \Rightarrow A \vee B}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LK

Règles LK

$$\dfrac{\dfrac{\overline{B \vdash A, B}\ \text{ax}}{B \vdash A \vee B}\ \vee_{\text{right}}}{\vdash B \Rightarrow A \vee B}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ/LK

Règles LJ

$$A, \bot \vdash \neg A$$

$$A \vdash \bot \Rightarrow \neg A$$

$$\vdash A \Rightarrow \bot \Rightarrow \neg A$$

Correction

Preuve (6) dans LJ/LK

▸ Règles LJ

$$\dfrac{\dfrac{A, \bot \vdash \neg A}{A \vdash \bot \Rightarrow \neg A}}{\vdash A \Rightarrow \bot \Rightarrow \neg A} \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ/LK

Règles LJ

$$\cfrac{\cfrac{A, \bot \vdash \neg A}{A \vdash \bot \Rightarrow \neg A}\ \Rightarrow_{\text{right}}}{\vdash A \Rightarrow \bot \Rightarrow \neg A}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ/LK

Règles LJ

$$
\cfrac{\cfrac{\cfrac{}{A, \bot \vdash \neg A}\;\bot_{\text{left}}}{A \vdash \bot \Rightarrow \neg A}\;\Rightarrow_{\text{right}}}{\vdash A \Rightarrow \bot \Rightarrow \neg A}\;\Rightarrow_{\text{right}}
$$

Correction

Preuve (7) dans LJ/LK

Règles LJ

$$\dfrac{\perp \vdash A}{\vdash \perp \Rightarrow A}$$

Correction

Preuve (7) dans LJ/LK

Règles LJ

$$\frac{\bot \vdash A}{\vdash \bot \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (7) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{}{\bot \vdash A}\,\text{ax}}{\vdash \bot \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LJ

▸ Règles LJ

$$
\begin{array}{c}
A \vdash A \qquad\quad B \vdash B \\
A \Leftrightarrow B, A \vdash B \\
A \Leftrightarrow B \vdash A \Rightarrow B \\
\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B
\end{array}
$$

Correction

Preuve (8) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \Leftrightarrow B, A \vdash B}}{A \Leftrightarrow B \vdash A \Rightarrow B}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B} \Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \Leftrightarrow B, A \vdash B}}{A \Leftrightarrow B \vdash A \Rightarrow B} \Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B} \Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A \Leftrightarrow B, A \vdash B}\Leftrightarrow_{\text{left1}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\text{ax} \qquad B \vdash B}{A \Leftrightarrow B, A \vdash B}\Leftrightarrow_{\text{left1}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}\,\mathrm{ax} \qquad \dfrac{}{B \vdash B}\,\mathrm{ax}}{A \Leftrightarrow B, A \vdash B}\,\Leftrightarrow_{\mathrm{left1}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\,\Rightarrow_{\mathrm{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\,\Rightarrow_{\mathrm{right}}$$

Correction

Preuve (8) dans LK

Règles LK

$$A \vdash B, A, B \qquad A, A, B \vdash B$$
$$A \Leftrightarrow B, A \vdash B$$
$$A \Leftrightarrow B \vdash A \Rightarrow B$$
$$\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B$$

Correction

Preuve (8) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{A \vdash B, A, B \qquad A, A, B \vdash B}{A \Leftrightarrow B, A \vdash B}}{A \Leftrightarrow B \vdash A \Rightarrow B}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B} \Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{A \vdash B, A, B \qquad A, A, B \vdash B}{A \Leftrightarrow B, A \vdash B}}{A \Leftrightarrow B \vdash A \Rightarrow B}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LK

Règles LK

$$
\dfrac{\dfrac{\dfrac{A \vdash B, A, B \qquad A, A, B \vdash B}{A \Leftrightarrow B, A \vdash B}\Leftrightarrow_{\text{left}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\Rightarrow_{\text{right}}
$$

Correction

Preuve (8) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B, A, B}\,\text{ax} \qquad A, A, B \vdash B}{A \Leftrightarrow B, A \vdash B}\,\Leftrightarrow_{\text{left}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\,\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (8) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash B, A, B}\,\text{ax} \qquad \dfrac{}{A, A, B \vdash B}\,\text{ax}}{A \Leftrightarrow B, A \vdash B}\,\Leftrightarrow_{\text{left}}}{A \Leftrightarrow B \vdash A \Rightarrow B}\,\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow A \Rightarrow B}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A \Leftrightarrow B, B \vdash A}}{A \Leftrightarrow B \vdash B \Rightarrow A}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}$$

Correction

Preuve (9) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A \Leftrightarrow B, B \vdash A}}{A \Leftrightarrow B \vdash B \Rightarrow A}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A} \Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A \Leftrightarrow B, B \vdash A}}{A \Leftrightarrow B \vdash B \Rightarrow A}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{B \vdash B \qquad A \vdash A}{A \Leftrightarrow B, B \vdash A}\Leftrightarrow_{\text{left2}}}{A \Leftrightarrow B \vdash B \Rightarrow A}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash B}\,\text{ax} \qquad A \vdash A}{A \Leftrightarrow B, B \vdash A}\,\Leftrightarrow_{\text{left2}}}{A \Leftrightarrow B \vdash B \Rightarrow A}\,\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LJ

Règles LJ

$$\dfrac{\dfrac{}{B \vdash B}\,\text{ax} \qquad \dfrac{}{A \vdash A}\,\text{ax}}{\dfrac{A \Leftrightarrow B, B \vdash A}{\dfrac{A \Leftrightarrow B \vdash B \Rightarrow A}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}}}\,\Rightarrow_{\text{right}}}\,\Leftrightarrow_{\text{left2}}$$

Correction

Preuve (9) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{B \vdash A, A, B \qquad B, A, B \vdash A}{A \Leftrightarrow B, B \vdash A}}{A \Leftrightarrow B \vdash B \Rightarrow A}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}$$

Correction

Preuve (9) dans LK

Règles LK

$$B \vdash A, A, B \qquad B, A, B \vdash A$$

$$A \Leftrightarrow B, B \vdash A$$

$$\dfrac{A \Leftrightarrow B \vdash B \Rightarrow A}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{B \vdash A, A, B \qquad B, A, B \vdash A}{A \Leftrightarrow B, B \vdash A}}{A \Leftrightarrow B \vdash B \Rightarrow A} \Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A} \Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LK

Règles LK

$$\cfrac{\cfrac{\cfrac{B \vdash A, A, B \qquad B, A, B \vdash A}{A \Leftrightarrow B, B \vdash A}\Leftrightarrow_{\text{left}}}{A \Leftrightarrow B \vdash B \Rightarrow A}\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\Rightarrow_{\text{right}}$$

Correction

Preuve (9) dans LK

Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{}{B \vdash A, A, B}\,\text{ax} \qquad B, A, B \vdash A}{A \Leftrightarrow B, B \vdash A}\,\Leftrightarrow_{\text{left}}}{A \Leftrightarrow B \vdash B \Rightarrow A}\,\Rightarrow_{\text{right}}}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}}
$$

Correction

Preuve (9) dans LK

Règles LK

$$
\dfrac{\dfrac{\dfrac{}{B \vdash A, A, B}\,\text{ax} \qquad \dfrac{}{B, A, B \vdash A}\,\text{ax}}{A \Leftrightarrow B, B \vdash A}\,\Leftrightarrow_{\text{left}}}{\dfrac{A \Leftrightarrow B \vdash B \Rightarrow A}{\vdash (A \Leftrightarrow B) \Rightarrow B \Rightarrow A}\,\Rightarrow_{\text{right}}}\,\Rightarrow_{\text{right}}
$$

Correction

Preuve (10) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, A \vdash A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B} \qquad \dfrac{A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}$$

Correction

Preuve (10) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, A \vdash A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B} \qquad \dfrac{A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)} \Rightarrow_{\textbf{right}}$$

Correction

Preuve (10) dans LJ

Règles LJ

$$B \Rightarrow A, A \vdash A \qquad B \Rightarrow A, A, B \vdash B \qquad A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A$$

$$A \Rightarrow B, B \Rightarrow A, A \vdash B \qquad A \Rightarrow B, B \Rightarrow A, B \vdash A$$

$$\dfrac{\dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)} \Rightarrow_{\text{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)} \Rightarrow_{\text{right}}$$

Correction

Preuve (10) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, A \vdash A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B} \qquad \dfrac{A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\Leftrightarrow_{\mathbf{right}}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\mathbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\mathbf{right}}$$

Correction

Preuve (10) dans LJ

‣ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{B \Rightarrow A, A \vdash A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\Rightarrow_{\textbf{left}}
    \qquad
    \begin{array}{c} A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A \\ A \Rightarrow B, B \Rightarrow A, B \vdash A \end{array}
  }{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\Leftrightarrow_{\textbf{right}}
}{
  \dfrac{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\textbf{right}}
}\Rightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow A, A \vdash A}\,\text{ax} \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,{\Rightarrow_{\text{left}}} \qquad \begin{array}{c} A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A \\ A \Rightarrow B, B \Rightarrow A, B \vdash A \end{array}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\,{\Leftrightarrow_{\text{right}}}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,{\Rightarrow_{\text{right}}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,{\Rightarrow_{\text{right}}}$$

Correction

Preuve (10) dans LJ

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow A, A \vdash A}\,\text{ax} \qquad \dfrac{}{B \Rightarrow A, A, B \vdash B}\,\text{ax}}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}} \qquad \begin{array}{c} A \Rightarrow B, B \vdash B \qquad A \Rightarrow B, B, A \vdash A \\ A \Rightarrow B, B \Rightarrow A, B \vdash A \end{array}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\,\Leftrightarrow_{\textbf{right}}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LJ

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{B \Rightarrow A, A \vdash A}\,\text{ax}
      \qquad
      \dfrac{}{B \Rightarrow A, A, B \vdash B}\,\text{ax}
    }{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}}
    \qquad
    \dfrac{
      A \Rightarrow B, B \vdash B
      \qquad
      A \Rightarrow B, B, A \vdash A
    }{A \Rightarrow B, B \Rightarrow A, B \vdash A}\,\Rightarrow_{\textbf{left}}
  }{
    \dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
  }\,\Leftrightarrow_{\textbf{right}}
}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LJ

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{}{B \Rightarrow A, A \vdash A}\,\text{ax}
    \qquad
    \dfrac{}{B \Rightarrow A, A, B \vdash B}\,\text{ax}
  }{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}}
  \qquad
  \dfrac{
    \dfrac{}{A \Rightarrow B, B \vdash B}\,\text{ax}
    \qquad
    A \Rightarrow B, B, A \vdash A
  }{A \Rightarrow B, B \Rightarrow A, B \vdash A}\,\Rightarrow_{\textbf{left}}
}{
  \dfrac{
    \dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
  }{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
}\,\Leftrightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LJ

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{B \Rightarrow A, A \vdash A}\,\text{ax}
      \qquad
      \dfrac{}{B \Rightarrow A, A, B \vdash B}\,\text{ax}
    }{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}}
    \qquad
    \dfrac{
      \dfrac{}{A \Rightarrow B, B \vdash B}\,\text{ax}
      \qquad
      \dfrac{}{A \Rightarrow B, B, A \vdash A}\,\text{ax}
    }{A \Rightarrow B, B \Rightarrow A, B \vdash A}\,\Rightarrow_{\textbf{left}}
  }{
    \dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
  }\,\Leftrightarrow_{\textbf{right}}
}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{B \Rightarrow A, A \vdash B, A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B} \qquad \dfrac{A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}$$

Correction

Preuve (10) dans LK

Règles LK

$$B \Rightarrow A, A \vdash B, A \qquad B \Rightarrow A, A, B \vdash B \qquad A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A$$

$$A \Rightarrow B, B \Rightarrow A, A \vdash B \qquad\qquad A \Rightarrow B, B \Rightarrow A, B \vdash A$$

$$A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B$$

$$\frac{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)} \Rightarrow_{\textbf{right}}$$

Correction

Preuve (10) dans LK

▸ Règles LK

$$B \Rightarrow A, A \vdash B, A \qquad B \Rightarrow A, A, B \vdash B \qquad A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A$$

$$A \Rightarrow B, B \Rightarrow A, A \vdash B \qquad\qquad A \Rightarrow B, B \Rightarrow A, B \vdash A$$

$$\dfrac{\dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\mathbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\mathbf{right}}$$

Correction

Preuve (10) dans LK

Règles LK

$$B \Rightarrow A, A \vdash B, A \qquad B \Rightarrow A, A, B \vdash B \qquad A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A$$

$$\dfrac{\dfrac{\dfrac{A \Rightarrow B, B \Rightarrow A, A \vdash B \qquad\qquad A \Rightarrow B, B \Rightarrow A, B \vdash A}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\ \Leftrightarrow_{\mathbf{right}}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\ \Rightarrow_{\mathbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\ \Rightarrow_{\mathbf{right}}$$

Correction

Preuve (10) dans LK

Règles LK

$$
\dfrac{
  \dfrac{
    \dfrac{B \Rightarrow A, A \vdash B, A \qquad B \Rightarrow A, A, B \vdash B}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}}
    \qquad
    \dfrac{A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A}{A \Rightarrow B, B \Rightarrow A, B \vdash A}
  }{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\,\Leftrightarrow_{\textbf{right}}
}{
  \dfrac{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
}\,\Rightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LK

▸ Règles LK

$$
\dfrac{
  \dfrac{
    \dfrac{}{B \Rightarrow A, A \vdash B, A}\,\text{ax}
    \qquad
    B \Rightarrow A, A, B \vdash B
  }{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}}
  \qquad
  \dfrac{
    A \Rightarrow B, B \vdash A, B
    \qquad
    A \Rightarrow B, B, A \vdash A
  }{A \Rightarrow B, B \Rightarrow A, B \vdash A}
}{
  \dfrac{
    \dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
  }{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
}\,\Leftrightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LK

▸ Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow A, A \vdash B, A}\,\text{ax} \qquad \dfrac{}{B \Rightarrow A, A, B \vdash B}\,\text{ax}}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\Rightarrow_{\textbf{left}} \qquad \begin{array}{c} A \Rightarrow B, B \vdash A, B \qquad A \Rightarrow B, B, A \vdash A \\ A \Rightarrow B, B \Rightarrow A, B \vdash A \end{array}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\Leftrightarrow_{\textbf{right}}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\textbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\Rightarrow_{\textbf{right}}$$

Correction

Preuve (10) dans LK

Règles LK

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \Rightarrow A, A \vdash B, A$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$B \Rightarrow A, A, B \vdash B$}
\RightLabel{$\Rightarrow_{\textbf{left}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A, A \vdash B$}
\AxiomC{$A \Rightarrow B, B \vdash A, B$}
\AxiomC{$A \Rightarrow B, B, A \vdash A$}
\RightLabel{$\Rightarrow_{\textbf{left}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A, B \vdash A$}
\RightLabel{$\Leftrightarrow_{\textbf{right}}$}
\BinaryInfC{$A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B$}
\RightLabel{$\Rightarrow_{\textbf{right}}$}
\UnaryInfC{$A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$}
\RightLabel{$\Rightarrow_{\textbf{right}}$}
\UnaryInfC{$\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$}
\end{prooftree}
$$

Correction

Preuve (10) dans LK

Règles LK

$$
\dfrac{
  \dfrac{
    \dfrac{}{B \Rightarrow A, A \vdash B, A}\,\text{ax}
    \qquad
    \dfrac{}{B \Rightarrow A, A, B \vdash B}\,\text{ax}
  }{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}}
  \qquad
  \dfrac{
    \dfrac{}{A \Rightarrow B, B \vdash A, B}\,\text{ax}
    \qquad
    A \Rightarrow B, B, A \vdash A
  }{A \Rightarrow B, B \Rightarrow A, B \vdash A}\,\Rightarrow_{\textbf{left}}
}{
  \dfrac{
    \dfrac{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
  }{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
}\,\Leftrightarrow_{\textbf{right}}
$$

Correction

Preuve (10) dans LK

Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{B \Rightarrow A, A \vdash B, A}\,\text{ax} \qquad \dfrac{}{B \Rightarrow A, A, B \vdash B}\,\text{ax}}{A \Rightarrow B, B \Rightarrow A, A \vdash B}\,\Rightarrow_{\textbf{left}} \qquad \dfrac{\dfrac{}{A \Rightarrow B, B \vdash A, B}\,\text{ax} \qquad \dfrac{}{A \Rightarrow B, B, A \vdash A}\,\text{ax}}{A \Rightarrow B, B \Rightarrow A, B \vdash A}\,\Rightarrow_{\textbf{left}}}{A \Rightarrow B, B \Rightarrow A \vdash A \Leftrightarrow B}\,\Leftrightarrow_{\textbf{right}}}{A \Rightarrow B \vdash (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}}{\vdash (A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)}\,\Rightarrow_{\textbf{right}}
$$

Exercices en logique du premier ordre

Propositions à démontrer dans $LJ_{(em)}$ et LK

1. $\forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)$
2. $(\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))$
3. $(\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)$
4. $(\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))$
5. $(\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))$
6. $\neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$P(x) \vdash P(x), Q(x)$$

$$P(x) \vdash P(x) \vee Q(x)$$

$$P(x) \vdash \exists y.P(y) \vee Q(y)$$

$$\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)$$

$$\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$P(x) \vdash P(x), Q(x)$$

$$P(x) \vdash P(x) \lor Q(x)$$

$$P(x) \vdash \exists y.P(y) \lor Q(y)$$

$$\dfrac{\vdash P(x) \Rightarrow \exists y.P(y) \lor Q(y)}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \lor Q(y)}\ \forall_{\text{right}}$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$P(x) \vdash P(x), Q(x)$$

$$P(x) \vdash P(x) \vee Q(x)$$

$$\dfrac{\dfrac{P(x) \vdash \exists y.P(y) \vee Q(y)}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\Rightarrow_{\text{right}}}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\forall_{\text{right}}$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x), Q(x)}{P(x) \vdash P(x) \vee Q(x)}}{P(x) \vdash \exists y.P(y) \vee Q(y)}\ \exists_{\text{right}}}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\ \Rightarrow_{\text{right}}}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\ \forall_{\text{right}}$$

Correction

Preuve (1) dans LJ/LK

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x), Q(x)}{P(x) \vdash P(x) \vee Q(x)}\vee_{\text{right}}}{P(x) \vdash \exists y.P(y) \vee Q(y)}\exists_{\text{right}}}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\Rightarrow_{\text{right}}}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\forall_{\text{right}}$$

Correction

Preuve (1) dans LJ/LK

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), Q(x)}\ \mathrm{ax}}{P(x) \vdash P(x) \vee Q(x)}\ \vee_{\mathrm{right}}}{P(x) \vdash \exists y.P(y) \vee Q(y)}\ \exists_{\mathrm{right}}}{\vdash P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\ \Rightarrow_{\mathrm{right}}}{\vdash \forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)}\ \forall_{\mathrm{right}}
$$

Correction

Preuve (2) dans LJ

Règles LJ

$$P(x) \vdash P(x) \qquad\qquad Q(x) \vdash Q(x)$$

$$P(x) \vdash \exists x.P(x) \qquad\qquad Q(x) \vdash \exists x.Q(x)$$

$$P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x)) \qquad Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$$

$$P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$$

$$\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$$

$$\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))$$

Correction

Preuve (2) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}}{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \qquad \dfrac{\dfrac{Q(x) \vdash Q(x)}{Q(x) \vdash \exists x.Q(x)}}{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))} \Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LJ

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}}{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))} \quad \dfrac{\dfrac{Q(x) \vdash Q(x)}{Q(x) \vdash \exists x.Q(x)}}{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\ \exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\begin{array}{c}P(x) \vdash P(x) \\ P(x) \vdash \exists x.P(x) \\ P(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))\end{array} \qquad \begin{array}{c}Q(x) \vdash Q(x) \\ Q(x) \vdash \exists x.Q(x) \\ Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))\end{array}}{P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\lor_{\text{left}}}{\exists x.P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\exists_{\text{left}}}{\vdash (\exists x.P(x) \lor Q(x)) \Rightarrow (\exists x.P(x)) \lor (\exists x.Q(x))}\,\Rightarrow_{\text{right}}
$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}}{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right1}} \qquad \dfrac{\dfrac{Q(x) \vdash Q(x)}{Q(x) \vdash \exists x.Q(x)}}{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{\dfrac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\text{left}}}\,\vee_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}
$$

Correction

Preuve (2) dans LJ

‣ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{P(x) \vdash \exists x.P(x)}\,\exists_{\text{right}}}{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right1}} \qquad \dfrac{\dfrac{Q(x) \vdash Q(x)}{Q(x) \vdash \exists x.Q(x)}}{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{left}}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{P(x) \vdash P(x)}\,\mathsf{ax}
    }{P(x) \vdash \exists x.P(x)}\,\exists_{\mathsf{right}}
  }{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\mathsf{right1}}
  \qquad
  \begin{array}{c}
  Q(x) \vdash Q(x) \\
  Q(x) \vdash \exists x.Q(x) \\
  Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))
  \end{array}
}{
  \dfrac{
    \dfrac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\mathsf{left}}
  }{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\mathsf{right}}
}\,\vee_{\mathsf{left}}
$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\,\text{ax}}{P(x) \vdash \exists x.P(x)}\,\exists_{\text{right}}}{P(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\lor_{\text{right1}} \qquad \dfrac{\begin{array}{c} Q(x) \vdash Q(x) \\ Q(x) \vdash \exists x.Q(x) \end{array}}{Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\lor_{\text{right2}}}{P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\lor_{\text{left}}}{\exists x.P(x) \lor Q(x) \vdash (\exists x.P(x)) \lor (\exists x.Q(x))}\,\exists_{\text{left}}}{\vdash (\exists x.P(x) \lor Q(x)) \Rightarrow (\exists x.P(x)) \lor (\exists x.Q(x))}\,\Rightarrow_{\text{right}}
$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{P(x) \vdash P(x)}\,\text{ax}
      }{P(x) \vdash \exists x.P(x)}\,\exists_{\text{right}}
    }{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right1}}
    \qquad
    \dfrac{
      \dfrac{Q(x) \vdash Q(x)}{Q(x) \vdash \exists x.Q(x)}\,\exists_{\text{right}}
    }{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right2}}
  }{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{left}}
}{
  \dfrac{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}
}\,\exists_{\text{left}}
$$

Correction

Preuve (2) dans LJ

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\,\text{ax}}{P(x) \vdash \exists x.P(x)}\,\exists_{\text{right}}}{P(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right1}} \qquad \dfrac{\dfrac{\dfrac{}{Q(x) \vdash Q(x)}\,\text{ax}}{Q(x) \vdash \exists x.Q(x)}\,\exists_{\text{right}}}{Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right2}}}{\dfrac{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\text{left}}}\,\vee_{\text{left}}$$

$$\dfrac{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x), \exists x.Q(x)}{P(x) \vdash \exists x.P(x), \exists x.Q(x)} \quad \dfrac{Q(x) \vdash \exists x.P(x), Q(x)}{Q(x) \vdash \exists x.P(x), \exists x.Q(x)}}{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}$$

Correction

Preuve (2) dans LK

‣ Règles LK

$$P(x) \vdash P(x), \exists x.Q(x) \qquad Q(x) \vdash \exists x.P(x), Q(x)$$

$$P(x) \vdash \exists x.P(x), \exists x.Q(x) \qquad Q(x) \vdash \exists x.P(x), \exists x.Q(x)$$

$$P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)$$

$$P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$$

$$\dfrac{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))} \Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x), \exists x.Q(x)}{P(x) \vdash \exists x.P(x), \exists x.Q(x)} \qquad \dfrac{Q(x) \vdash \exists x.P(x), Q(x)}{Q(x) \vdash \exists x.P(x), \exists x.Q(x)}}{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LK

Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x), \exists x.Q(x)}{P(x) \vdash \exists x.P(x), \exists x.Q(x)} \qquad \dfrac{Q(x) \vdash \exists x.P(x), Q(x)}{Q(x) \vdash \exists x.P(x), \exists x.Q(x)}}{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\vee_{\text{right}}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\Rightarrow_{\text{right}}
$$

Correction

Preuve (2) dans LK

▸ Règles LK

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\begin{array}{cc}
P(x) \vdash P(x), \exists x.Q(x) & Q(x) \vdash \exists x.P(x), Q(x) \\
P(x) \vdash \exists x.P(x), \exists x.Q(x) & Q(x) \vdash \exists x.P(x), \exists x.Q(x)
\end{array}
}{
P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)
}\vee_{\text{left}}
}{
P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))
}\vee_{\text{right}}
}{
\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))
}\exists_{\text{left}}
}{
\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))
}\Rightarrow_{\text{right}}
$$

Correction

Preuve (2) dans LK

▸ Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x)\vdash P(x),\exists x.Q(x)}{P(x)\vdash \exists x.P(x),\exists x.Q(x)}\exists_{\text{right}} \qquad \begin{array}{c}Q(x)\vdash\exists x.P(x),Q(x)\\ Q(x)\vdash \exists x.P(x),\exists x.Q(x)\end{array}}{P(x)\lor Q(x)\vdash \exists x.P(x),\exists x.Q(x)}\lor_{\text{left}}}{P(x)\lor Q(x)\vdash(\exists x.P(x))\lor(\exists x.Q(x))}\lor_{\text{right}}}{\exists x.P(x)\lor Q(x)\vdash (\exists x.P(x))\lor(\exists x.Q(x))}\exists_{\text{left}}}{\vdash(\exists x.P(x)\lor Q(x))\Rightarrow(\exists x.P(x))\lor(\exists x.Q(x))}\Rightarrow_{\text{right}}$$

Correction

Preuve (2) dans LK

▸ Règles LK

$$
\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), \exists x.Q(x)}\text{ax}}{P(x) \vdash \exists x.P(x), \exists x.Q(x)}\exists_{\text{right}} \qquad \dfrac{Q(x) \vdash \exists x.P(x), Q(x)}{Q(x) \vdash \exists x.P(x), \exists x.Q(x)}}{\dfrac{\dfrac{\dfrac{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\vee_{\text{right}}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\Rightarrow_{\text{right}}}\vee_{\text{left}}
$$

Correction

Preuve (2) dans LK

‣ Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x), \exists x.Q(x)}\,\text{ax}}{P(x) \vdash \exists x.P(x), \exists x.Q(x)}\,\exists_{\text{right}} \qquad \dfrac{Q(x) \vdash \exists x.P(x), Q(x)}{Q(x) \vdash \exists x.P(x), \exists x.Q(x)}\,\exists_{\text{right}}}{P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)}\,\vee_{\text{left}}}{P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\vee_{\text{right}}}{\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))}\,\exists_{\text{left}}}{\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))}\,\Rightarrow_{\text{right}}
$$

Correction

Preuve (2) dans LK

Règles LK

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x) \vdash P(x), \exists x.Q(x)$}
\RightLabel{$\exists_{\text{right}}$}
\UnaryInfC{$P(x) \vdash \exists x.P(x), \exists x.Q(x)$}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$Q(x) \vdash \exists x.P(x), Q(x)$}
\RightLabel{$\exists_{\text{right}}$}
\UnaryInfC{$Q(x) \vdash \exists x.P(x), \exists x.Q(x)$}
\RightLabel{$\vee_{\text{left}}$}
\BinaryInfC{$P(x) \vee Q(x) \vdash \exists x.P(x), \exists x.Q(x)$}
\RightLabel{$\vee_{\text{right}}$}
\UnaryInfC{$P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$}
\RightLabel{$\exists_{\text{left}}$}
\UnaryInfC{$\exists x.P(x) \vee Q(x) \vdash (\exists x.P(x)) \vee (\exists x.Q(x))$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash (\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))$}
\end{prooftree}
$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$P(x), \forall x.Q(x) \vdash P(x) \qquad \forall x.P(x), Q(x) \vdash Q(x)$$

$$\forall x.P(x), \forall x.Q(x) \vdash P(x) \qquad \forall x.P(x), \forall x.Q(x) \vdash Q(x)$$

$$\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)$$

$$(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)$$

$$(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)$$

$$\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)$$

Correction

Preuve (3) dans LJ/LK

‣ Règles LJ

$$P(x), \forall x.Q(x) \vdash P(x) \qquad \forall x.P(x), Q(x) \vdash Q(x)$$

$$\forall x.P(x), \forall x.Q(x) \vdash P(x) \qquad \forall x.P(x), \forall x.Q(x) \vdash Q(x)$$

$$\forall x.P(x), \forall x.Q(x) \vdash P(x) \land Q(x)$$

$$(\forall x.P(x)) \land (\forall x.Q(x)) \vdash P(x) \land Q(x)$$

$$\frac{(\forall x.P(x)) \land (\forall x.Q(x)) \vdash \forall x.P(x) \land Q(x)}{\vdash (\forall x.P(x)) \land (\forall x.Q(x)) \Rightarrow \forall x.P(x) \land Q(x)} \Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x), \forall x.Q(x) \vdash P(x)}{\forall x.P(x), \forall x.Q(x) \vdash P(x)} \quad \dfrac{\forall x.P(x), Q(x) \vdash Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\ \forall_{\text{right}}}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$P(x), \forall x.Q(x) \vdash P(x) \qquad \forall x.P(x), Q(x) \vdash Q(x)$$

$$\forall x.P(x), \forall x.Q(x) \vdash P(x) \qquad \forall x.P(x), \forall x.Q(x) \vdash Q(x)$$

$$\dfrac{\dfrac{\dfrac{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}\wedge_{\text{left}}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\forall_{\text{right}}}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

‣ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x), \forall x.Q(x) \vdash P(x)}{\forall x.P(x), \forall x.Q(x) \vdash P(x)} \qquad \dfrac{\forall x.P(x), Q(x) \vdash Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)} \wedge_{right}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)} \wedge_{left}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)} \forall_{right}}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)} \Rightarrow_{right}
$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{P(x), \forall x.Q(x) \vdash P(x)}{\forall x.P(x), \forall x.Q(x) \vdash P(x)}\ \forall_{\text{left}}
\qquad
\begin{array}{c}\forall x.P(x), Q(x) \vdash Q(x) \\ \forall x.P(x), \forall x.Q(x) \vdash Q(x)\end{array}
}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}\ \wedge_{\text{right}}
}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}\ \wedge_{\text{left}}
}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\ \forall_{\text{right}}
}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (3) dans LJ/LK

‣ Règles LJ

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x), \forall x.Q(x) \vdash P(x)$}
\RightLabel{$\forall_{\text{left}}$}
\UnaryInfC{$\forall x.P(x), \forall x.Q(x) \vdash P(x)$}
\AxiomC{$\forall x.P(x), Q(x) \vdash Q(x)$}
\noLine
\UnaryInfC{$\forall x.P(x), \forall x.Q(x) \vdash Q(x)$}
\RightLabel{$\wedge_{\text{right}}$}
\BinaryInfC{$\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)$}
\RightLabel{$\wedge_{\text{left}}$}
\UnaryInfC{$(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)$}
\end{prooftree}
$$

Correction

Preuve (3) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x), \forall x.Q(x) \vdash P(x)}\,\text{ax}}{\forall x.P(x), \forall x.Q(x) \vdash P(x)}\,\forall_{\text{left}} \qquad \dfrac{\forall x.P(x), Q(x) \vdash Q(x)}{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}\,\forall_{\text{left}}}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}\,\wedge_{\text{right}}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}\,\wedge_{\text{left}}}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\,\forall_{\text{right}}}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (3) dans LJ/LK

▸ Règles LJ

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{\dfrac{}{P(x), \forall x.Q(x) \vdash P(x)}\,\mathsf{ax}}{\forall x.P(x), \forall x.Q(x) \vdash P(x)}\,\forall_{\mathsf{left}}
\qquad
\dfrac{\dfrac{}{\forall x.P(x), Q(x) \vdash Q(x)}\,\mathsf{ax}}{\forall x.P(x), \forall x.Q(x) \vdash Q(x)}\,\forall_{\mathsf{left}}
}{\forall x.P(x), \forall x.Q(x) \vdash P(x) \wedge Q(x)}\,\wedge_{\mathsf{right}}
}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash P(x) \wedge Q(x)}\,\wedge_{\mathsf{left}}
}{(\forall x.P(x)) \wedge (\forall x.Q(x)) \vdash \forall x.P(x) \wedge Q(x)}\,\forall_{\mathsf{right}}
}{\vdash (\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)}\,\Rightarrow_{\mathsf{right}}
$$

Correction

Preuve (4) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)}}{\forall x.P(x) \wedge Q(x) \vdash P(x)}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)} \quad \dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}}{\dfrac{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}}$$

Correction

Preuve (4) dans LJ/LK

▸ Règles LJ

$$P(x), Q(x) \vdash P(x) \qquad\qquad P(x), Q(x) \vdash Q(x)$$

$$P(x) \wedge Q(x) \vdash P(x) \qquad\qquad P(x) \wedge Q(x) \vdash Q(x)$$

$$\forall x.P(x) \wedge Q(x) \vdash P(x) \qquad\qquad \forall x.P(x) \wedge Q(x) \vdash Q(x)$$

$$\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x) \qquad\qquad \forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)$$

$$\dfrac{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)}}{\forall x.P(x) \wedge Q(x) \vdash P(x)}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)} \qquad \dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))} \wedge_{\text{right}}}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))} \Rightarrow_{\text{right}}$$

Correction

Preuve (4) dans LJ/LK

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \begin{array}{c}
      P(x), Q(x) \vdash P(x) \\
      P(x) \wedge Q(x) \vdash P(x) \\
      \forall x.P(x) \wedge Q(x) \vdash P(x)
      \end{array}
    }{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}\ \forall_{\text{right}}
    \qquad
    \begin{array}{c}
    P(x), Q(x) \vdash Q(x) \\
    P(x) \wedge Q(x) \vdash Q(x) \\
    \forall x.P(x) \wedge Q(x) \vdash Q(x) \\
    \forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)
    \end{array}
  }{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}\ \wedge_{\text{right}}
}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (4) dans LJ/LK

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)}}{\forall x.P(x) \wedge Q(x) \vdash P(x)}\ \forall_{\text{left}}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}\ \forall_{\text{right}} \qquad \dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash Q(x)}{P(x) \wedge Q(x) \vdash Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash Q(x)}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}}{\dfrac{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}\ \Rightarrow_{\text{right}}}\ \wedge_{\text{right}}
$$

Correction

Preuve (4) dans LJ/LK

▸ Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{P(x), Q(x) \vdash P(x)}{P(x) \wedge Q(x) \vdash P(x)}\wedge_{\text{left}}}{\forall x.P(x) \wedge Q(x) \vdash P(x)}\forall_{\text{left}}}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}\forall_{\text{right}} \qquad \begin{array}{c} P(x), Q(x) \vdash Q(x) \\ P(x) \wedge Q(x) \vdash Q(x) \\ \forall x.P(x) \wedge Q(x) \vdash Q(x) \\ \forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x) \end{array}}{\dfrac{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}\Rightarrow_{\text{right}}}\wedge_{\text{right}}$$

Correction

Preuve (4) dans LJ/LK

▸ Règles LJ

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{P(x), Q(x) \vdash P(x)}\,\text{ax}
}{P(x) \land Q(x) \vdash P(x)}\,\land_{\text{left}}
}{\forall x.P(x) \land Q(x) \vdash P(x)}\,\forall_{\text{left}}
}{\forall x.P(x) \land Q(x) \vdash \forall x.P(x)}\,\forall_{\text{right}}
\qquad
\begin{array}{c}
P(x), Q(x) \vdash Q(x) \\
P(x) \land Q(x) \vdash Q(x) \\
\forall x.P(x) \land Q(x) \vdash Q(x) \\
\forall x.P(x) \land Q(x) \vdash \forall x.Q(x)
\end{array}
}{\forall x.P(x) \land Q(x) \vdash (\forall x.P(x)) \land (\forall x.Q(x))}\,\land_{\text{right}}
}{\vdash (\forall x.P(x) \land Q(x)) \Rightarrow (\forall x.P(x)) \land (\forall x.Q(x))}\,\Rightarrow_{\text{right}}
$$

Correction

Preuve (4) dans LJ/LK

Règles LJ

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{}{P(x), Q(x) \vdash P(x)}\,\text{ax}
}{P(x) \wedge Q(x) \vdash P(x)}\,\wedge_{\text{left}}
}{\forall x.P(x) \wedge Q(x) \vdash P(x)}\,\forall_{\text{left}}
}{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}\,\forall_{\text{right}}
\qquad
\dfrac{
\begin{array}{c}
P(x), Q(x) \vdash Q(x) \\
P(x) \wedge Q(x) \vdash Q(x) \\
\forall x.P(x) \wedge Q(x) \vdash Q(x)
\end{array}
}{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}\,\forall_{\text{right}}
}{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}\,\wedge_{\text{right}}
}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}\,\Rightarrow_{\text{right}}
$$

Correction

Preuve (4) dans LJ/LK

▸ Règles LJ

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x), Q(x) \vdash P(x)$}
\RightLabel{$\wedge_{\text{left}}$}
\UnaryInfC{$P(x) \wedge Q(x) \vdash P(x)$}
\RightLabel{$\forall_{\text{left}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash P(x)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)$}
\AxiomC{$P(x), Q(x) \vdash Q(x)$}
\noLine
\UnaryInfC{$P(x) \wedge Q(x) \vdash Q(x)$}
\RightLabel{$\forall_{\text{left}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash Q(x)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)$}
\RightLabel{$\wedge_{\text{right}}$}
\BinaryInfC{$\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))$}
\end{prooftree}
$$

Correction

Preuve (4) dans LJ/LK

‣ Règles LJ

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{ax}
\UnaryInfC{$P(x), Q(x) \vdash P(x)$}
\RightLabel{$\wedge_{\text{left}}$}
\UnaryInfC{$P(x) \wedge Q(x) \vdash P(x)$}
\RightLabel{$\forall_{\text{left}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash P(x)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)$}
\AxiomC{$P(x), Q(x) \vdash Q(x)$}
\RightLabel{$\wedge_{\text{left}}$}
\UnaryInfC{$P(x) \wedge Q(x) \vdash Q(x)$}
\RightLabel{$\forall_{\text{left}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash Q(x)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)$}
\RightLabel{$\wedge_{\text{right}}$}
\BinaryInfC{$\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))$}
\end{prooftree}
$$

Correction

Preuve (4) dans LJ/LK

▸ Règles LJ

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{P(x), Q(x) \vdash P(x)}\,\mathrm{ax}
      }{P(x) \wedge Q(x) \vdash P(x)}\,\wedge_{\mathrm{left}}
    }{\forall x.P(x) \wedge Q(x) \vdash P(x)}\,\forall_{\mathrm{left}}
  }{\forall x.P(x) \wedge Q(x) \vdash \forall x.P(x)}\,\forall_{\mathrm{right}}
  \quad
  \dfrac{
    \dfrac{
      \dfrac{
        \dfrac{}{P(x), Q(x) \vdash Q(x)}\,\mathrm{ax}
      }{P(x) \wedge Q(x) \vdash Q(x)}\,\wedge_{\mathrm{left}}
    }{\forall x.P(x) \wedge Q(x) \vdash Q(x)}\,\forall_{\mathrm{left}}
  }{\forall x.P(x) \wedge Q(x) \vdash \forall x.Q(x)}\,\forall_{\mathrm{right}}
}{
  \dfrac{\forall x.P(x) \wedge Q(x) \vdash (\forall x.P(x)) \wedge (\forall x.Q(x))}{\vdash (\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))}\,\Rightarrow_{\mathrm{right}}
}\,\wedge_{\mathrm{right}}
$$

Correction

Preuve (5) dans LJ/LK

Règles LJ

$$P(x) \vdash P(x)$$

$$\neg P(x), P(x) \vdash \bot$$

$$\forall x.\neg P(x), P(x) \vdash \bot$$

$$\forall x.\neg P(x), \exists x.P(x) \vdash \bot$$

$$\forall x.\neg P(x) \vdash \neg(\exists x.P(x))$$

$$\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))$$

Correction

Preuve (5) dans LJ/LK

▸ Règles LJ

$$P(x) \vdash P(x)$$

$$\neg P(x), P(x) \vdash \bot$$

$$\forall x.\neg P(x), P(x) \vdash \bot$$

$$\forall x.\neg P(x), \exists x.P(x) \vdash \bot$$

$$\dfrac{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))} \Rightarrow_{\text{right}}$$

Correction

Preuve (5) dans LJ/LK

Règles LJ

$$P(x) \vdash P(x)$$

$$\neg P(x), P(x) \vdash \bot$$

$$\forall x.\neg P(x), P(x) \vdash \bot$$

$$\dfrac{\dfrac{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\ \neg_{\text{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (5) dans LJ/LK

Règles LJ

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} P(x) \vdash P(x) \\ \neg P(x), P(x) \vdash \bot \end{array}}{\forall x.\neg P(x), P(x) \vdash \bot}}{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}\exists_{\text{left}}}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\neg_{\text{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}\Rightarrow_{\text{right}}$$

Correction

Preuve (5) dans LJ/LK

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\neg P(x), P(x) \vdash \bot}}{\forall x.\neg P(x), P(x) \vdash \bot}\ \forall_{\text{left}}}{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}\ \exists_{\text{left}}}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\ \neg_{\text{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (5) dans LJ/LK

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\neg P(x), P(x) \vdash \bot}\ \neg_{\text{left}}}{\forall x.\neg P(x), P(x) \vdash \bot}\ \forall_{\text{left}}}{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}\ \exists_{\text{left}}}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\ \neg_{\text{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (5) dans LJ/LK

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\ \mathrm{ax}}{\neg P(x), P(x) \vdash \bot}\ \neg_{\mathrm{left}}}{\forall x.\neg P(x), P(x) \vdash \bot}\ \forall_{\mathrm{left}}}{\forall x.\neg P(x), \exists x.P(x) \vdash \bot}\ \exists_{\mathrm{left}}}{\forall x.\neg P(x) \vdash \neg(\exists x.P(x))}\ \neg_{\mathrm{right}}}{\vdash (\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))}\ \Rightarrow_{\mathrm{right}}
$$

Correction

Preuve (6) dans LJ_{em}

Règles LJ

$$\neg P(x) \vdash \neg P(x)$$
$$\neg P(x) \vdash \exists x.\neg P(x)$$
$$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$$
$$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$$
$$\neg\exists x.\neg P(x) \vdash P(x)$$
$$\neg\exists x.\neg P(x) \vdash \forall x.P(x)$$
$$\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot$$
$$\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)$$
$$\neg\forall x.P(x) \vdash \exists x.\neg P(x)$$
$$\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$$

Correction

Preuve (6) dans LJ_{em}

Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x.\neg P(x)$$

$$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$$

$$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$$

$$\neg\exists x.\neg P(x) \vdash P(x)$$

$$\neg\exists x.\neg P(x) \vdash \forall x.P(x)$$

$$\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot$$

$$\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)$$

$$\dfrac{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)} \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x.\neg P(x)$$

$$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$$

$$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$$

$$\neg\exists x.\neg P(x) \vdash P(x)$$

$$\neg\exists x.\neg P(x) \vdash \forall x.P(x)$$

$$\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot$$

$$\dfrac{\dfrac{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,\text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x.\neg P(x)$$

$$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$$

$$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$$

$$\neg\exists x.\neg P(x) \vdash P(x)$$

$$\neg\exists x.\neg P(x) \vdash \forall x.P(x)$$

$$\dfrac{\dfrac{\dfrac{\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot}{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}\ \neg_{\text{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\ \text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x.\neg P(x)$$

$$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$$

$$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$$

$$\neg\exists x.\neg P(x) \vdash P(x)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg\exists x.\neg P(x) \vdash \forall x.P(x)}{\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot}\,\neg\text{left}}{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}\,\neg\text{right}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,\text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\,\Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LJ_{em}

‣ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x. \neg P(x)$$

$$\neg \exists x. \neg P(x), \neg P(x) \vdash \bot$$

$$\neg \exists x. \neg P(x) \vdash \neg\neg P(x)$$

$$
\begin{prooftree}
\AxiomC{$\neg \exists x. \neg P(x) \vdash P(x)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$\neg \exists x. \neg P(x) \vdash \forall x. P(x)$}
\RightLabel{$\neg_{\text{left}}$}
\UnaryInfC{$\neg \forall x. P(x), \neg \exists x. \neg P(x) \vdash \bot$}
\RightLabel{$\neg_{\text{right}}$}
\UnaryInfC{$\neg \forall x. P(x) \vdash \neg\neg \exists x. \neg P(x)$}
\RightLabel{em}
\UnaryInfC{$\neg \forall x. P(x) \vdash \exists x. \neg P(x)$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash \neg(\forall x. P(x)) \Rightarrow \exists x. \neg P(x)$}
\end{prooftree}
$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x.\neg P(x)$$

$$\neg \exists x.\neg P(x), \neg P(x) \vdash \bot$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\neg \exists x.\neg P(x) \vdash \neg\neg P(x)}{\neg \exists x.\neg P(x) \vdash P(x)}\ \text{em}}{\neg \exists x.\neg P(x) \vdash \forall x.P(x)}\ \forall_{\text{right}}}{\neg \forall x.P(x), \neg \exists x.\neg P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg \forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}\ \neg_{\text{right}}}{\neg \forall x.P(x) \vdash \exists x.\neg P(x)}\ \text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}}{}$$

Correction

Preuve (6) dans LJ_{em}

‣ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$\neg P(x) \vdash \exists x.\neg P(x)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg\exists x.\neg P(x), \neg P(x) \vdash \bot}{\neg\exists x.\neg P(x) \vdash \neg\neg P(x)}\ \neg_{\text{right}}}{\neg\exists x.\neg P(x) \vdash P(x)}\ \text{em}}{\neg\exists x.\neg P(x) \vdash \forall x.P(x)}\ \forall_{\text{right}}}{\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}\ \neg_{\text{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\ \text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}}{}$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$\neg P(x) \vdash \neg P(x)$$

$$
\begin{prooftree}
\AxiomC{$\neg P(x) \vdash \exists x.\neg P(x)$}
\RightLabel{$\neg_{\text{left}}$}
\UnaryInfC{$\neg\exists x.\neg P(x), \neg P(x) \vdash \bot$}
\RightLabel{$\neg_{\text{right}}$}
\UnaryInfC{$\neg\exists x.\neg P(x) \vdash \neg\neg P(x)$}
\RightLabel{em}
\UnaryInfC{$\neg\exists x.\neg P(x) \vdash P(x)$}
\RightLabel{$\forall_{\text{right}}$}
\UnaryInfC{$\neg\exists x.\neg P(x) \vdash \forall x.P(x)$}
\RightLabel{$\neg_{\text{left}}$}
\UnaryInfC{$\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot$}
\RightLabel{$\neg_{\text{right}}$}
\UnaryInfC{$\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)$}
\RightLabel{em}
\UnaryInfC{$\neg\forall x.P(x) \vdash \exists x.\neg P(x)$}
\RightLabel{$\Rightarrow_{\text{right}}$}
\UnaryInfC{$\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$}
\end{prooftree}
$$

Correction

Preuve (6) dans LJ_{em}

Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\neg P(x) \vdash \neg P(x)}{\neg P(x) \vdash \exists x. \neg P(x)}\ \exists_{\text{right}}}{\neg \exists x. \neg P(x), \neg P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg \exists x. \neg P(x) \vdash \neg\neg P(x)}\ \neg_{\text{right}}}{\neg \exists x. \neg P(x) \vdash P(x)}\ \text{em}}{\neg \exists x. \neg P(x) \vdash \forall x. P(x)}\ \forall_{\text{right}}}{\neg \forall x. P(x), \neg \exists x. \neg P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg \forall x. P(x) \vdash \neg\neg \exists x. \neg P(x)}\ \neg_{\text{right}}}{\neg \forall x. P(x) \vdash \exists x. \neg P(x)}\ \text{em}}{\vdash \neg(\forall x. P(x)) \Rightarrow \exists x. \neg P(x)}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (6) dans LJ_{em}

▸ Règles LJ

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\neg P(x) \vdash \neg P(x)}\ \text{ax}}{\neg P(x) \vdash \exists x.\neg P(x)}\ \exists_{\text{right}}}{\neg\exists x.\neg P(x), \neg P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg\exists x.\neg P(x) \vdash \neg\neg P(x)}\ \neg_{\text{right}}}{\neg\exists x.\neg P(x) \vdash P(x)}\ \text{em}}{\neg\exists x.\neg P(x) \vdash \forall x.P(x)}\ \forall_{\text{right}}}{\neg\forall x.P(x), \neg\exists x.\neg P(x) \vdash \bot}\ \neg_{\text{left}}}{\neg\forall x.P(x) \vdash \neg\neg\exists x.\neg P(x)}\ \neg_{\text{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\ \text{em}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (6) dans LK

Règles LK

$$P(x) \vdash P(x)$$

$$\vdash P(x), \neg P(x)$$

$$\vdash P(x), \exists x.\neg P(x)$$

$$\vdash \forall x.P(x), \exists x.\neg P(x)$$

$$\neg\forall x.P(x) \vdash \exists x.\neg P(x)$$

$$\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$$

Correction

Preuve (6) dans LK

Règles LK

$$P(x) \vdash P(x)$$

$$\vdash P(x), \neg P(x)$$

$$\vdash P(x), \exists x.\neg P(x)$$

$$\vdash \forall x.P(x), \exists x.\neg P(x)$$

$$\dfrac{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)} \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LK

Règles LK

$$P(x) \vdash P(x)$$

$$\vdash P(x), \neg P(x)$$

$$\vdash P(x), \exists x. \neg P(x)$$

$$\cfrac{\cfrac{\vdash \forall x. P(x), \exists x. \neg P(x)}{\neg \forall x. P(x) \vdash \exists x. \neg P(x)}\ \neg_{\text{left}}}{\vdash \neg(\forall x. P(x)) \Rightarrow \exists x. \neg P(x)}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LK

Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\vdash P(x), \neg P(x)}}{\vdash P(x), \exists x.\neg P(x)}}{\vdash \forall x.P(x), \exists x.\neg P(x)}\ \forall_{\text{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\ \neg_{\text{left}}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}$$

Correction

Preuve (6) dans LK

Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\vdash P(x), \neg P(x)}}{\vdash P(x), \exists x.\neg P(x)}\,\exists_{\text{right}}}{\vdash \forall x.P(x), \exists x.\neg P(x)}\,\forall_{\text{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,\neg_{\text{left}}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\,\Rightarrow_{\text{right}}
$$

Correction

Preuve (6) dans LK

▸ Règles LK

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(x) \vdash P(x)}{\vdash P(x), \neg P(x)}\ \neg_{\text{right}}}{\vdash P(x), \exists x.\neg P(x)}\ \exists_{\text{right}}}{\vdash \forall x.P(x), \exists x.\neg P(x)}\ \forall_{\text{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\ \neg_{\text{left}}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\ \Rightarrow_{\text{right}}
$$

Correction

Preuve (6) dans LK

▸ Règles LK

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{P(x) \vdash P(x)}\,\mathrm{ax}}{\vdash P(x), \neg P(x)}\,\neg_{\mathrm{right}}}{\vdash P(x), \exists x.\neg P(x)}\,\exists_{\mathrm{right}}}{\vdash \forall x.P(x), \exists x.\neg P(x)}\,\forall_{\mathrm{right}}}{\neg\forall x.P(x) \vdash \exists x.\neg P(x)}\,\neg_{\mathrm{left}}}{\vdash \neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)}\,\Rightarrow_{\mathrm{right}}$$

Outil d'aide à la preuve Coq

Caractéristiques

- Développement par l'équipe Inria πr^2 ;
- Preuve de programmes fonctionnels ;
- Théorie des types (calcul des constructions inductives) ;
- Isomorphisme de Curry-Howard (objets preuves).

Implantation

- Premières versions milieu des années 80 ;
- Implantation actuelle en OCaml ;
- Preuve interactive (peu d'automatisation) ;
- En ligne de commande ou avec l'interface graphique CoqIDE ;
- Installation : `https://coq.inria.fr/`.

Logique propositionnelle

Exemples de preuves

- Implication :

```
Coq < Parameter A : Prop.
A is assumed

Coq < Goal A -> A.
1 subgoal

  ============================
   A -> A
```

Logique propositionnelle

Exemples de preuves

- Implication :

```
Coq < intro.
1 subgoal

  H : A
  ============================
   A
```

Logique propositionnelle

Exemples de preuves

- Implication :

```
Coq < assumption.
No more subgoals.

Coq < Save my_thm.
intro.
assumption.
my_thm is defined
```

Logique propositionnelle

Exemples de preuves

- Application (modus ponens) :

```
Coq < Parameters A B : Prop.
A is assumed
B is assumed

Coq < Goal (A -> B) -> A -> B.
1 subgoal

  ============================
   (A -> B) -> A -> B
```

Logique propositionnelle

Exemples de preuves

- Application (modus ponens) :

```
Coq < intros.
1 subgoal

  H : A -> B
  H0 : A
  ============================
   B

Coq < apply (H H0).
No more subgoals.
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < Parameters A B : Prop.
A is assumed
B is assumed

Coq < Goal A /\ B -> A.
1 subgoal

  ============================
   A /\ B -> A
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < intro.
1 subgoal

  H : A /\ B
  ============================
   A
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < elim H.
1 subgoal

  H : A /\ B
  ============================
   A -> B -> A
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < intros.
1 subgoal

  H : A /\ B
  H0 : A
  H1 : B
  ============================
   A

Coq < assumption.
No more subgoals.
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < Parameters A B : Prop.
A is assumed
B is assumed

Coq < Goal A -> A \/ B.
1 subgoal

  ============================
   A -> A \/ B
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < intro.
1 subgoal

  H : A
  ============================
   A \/ B
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\wedge$ et $\vee$:

```
Coq < left.
1 subgoal

  H : A
  ============================
   A

Coq < assumption.
No more subgoals.
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\neg$:

```
Coq < Parameters A B : Prop.
A is assumed
B is assumed

Coq < Goal A -> ~A -> False.
1 subgoal

  ============================
   A -> ~ A -> False
```

Logique propositionnelle

Exemples de preuves

- Connecteurs $\neg$:

```
Coq < intros.
1 subgoal

  H : A
  H0 : ~ A
  ============================
   False

Coq < apply (H0 H).
No more subgoals.
```

Exercices

Propositions à démontrer en Coq

1. $A \Rightarrow B \Rightarrow A$
2. $(A \Rightarrow B \Rightarrow C) \Rightarrow (A \Rightarrow B) \Rightarrow A \Rightarrow C$
3. $A \wedge B \Rightarrow B$
4. $B \Rightarrow A \vee B$
5. $(A \vee B) \Rightarrow (A \Rightarrow C) \Rightarrow (B \Rightarrow C) \Rightarrow C$
6. $A \Rightarrow \bot \Rightarrow \neg A$
7. $\bot \Rightarrow A$
8. $(A \Leftrightarrow B) \Rightarrow A \Rightarrow B$
9. $(A \Leftrightarrow B) \Rightarrow B \Rightarrow A$
10. $(A \Rightarrow B) \Rightarrow (B \Rightarrow A) \Rightarrow (A \Leftrightarrow B)$

Logique du premier ordre

Exemples de preuves

- Quantificateur ∀ :

```
Coq < Parameter E : Set.
E is assumed

Coq < Parameter P : E -> Prop.
P is assumed

Coq < Goal forall x : E, (P x) -> (P x).
1 subgoal

  ============================
   forall x : E, P x -> P x
```

Logique du premier ordre

Exemples de preuves

- Quantificateur ∀ :

```
Coq < intros.
1 subgoal

  x : E
  H : P x
  ============================
   P x

Coq < assumption.
No more subgoals.
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\forall$:

```
Coq < Parameter E : Set.
E is assumed

Coq < Parameter a : E.
a is assumed

Coq < Parameter P : E -> Prop.
P is assumed

Coq < Goal (forall x : E, (P x)) -> (P a).
1 subgoal

  ============================
   (forall x : E, P x) -> P a
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\forall$:

```
Coq < intro.
1 subgoal

  H : forall x : E, P x
  ============================
   P a

Coq < apply H.
No more subgoals.
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < Parameter E : Set.
E is assumed

Coq < Parameter a : E.
a is assumed

Coq < Parameter P : E -> Prop.
P is assumed

Coq < Goal (P a) -> exists x : E, (P x).
1 subgoal

  ============================
   P a -> exists x : E, P x
```

Logique du premier ordre

Exemples de preuves

- Quantificateur ∃ :

```
Coq < intro.
1 subgoal

  H : P a
  ============================
   exists x : E, P x
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < exists a.
1 subgoal

  H : P a
  ============================
   P a

Coq < assumption.
No more subgoals.
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < Parameter E : Set.
E is assumed

Coq < Parameter a : E.
a is assumed

Coq < Parameter P : E -> Prop.
P is assumed

Coq < Goal (exists x : E, ~(P x)) ->
           ~(forall x : E, (P x)).
1 subgoal

  ============================
   (exists x : E, ~ P x) -> ~ (forall x : E, P x)
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < intros.
1 subgoal

  H : exists x : E, ~ P x
  ============================
   ~ (forall x : E, P x)

Coq < red.
1 subgoal

  H : exists x : E, ~ P x
  ============================
   (forall x : E, P x) -> False
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < intro.
1 subgoal

  H : exists x : E, ~ P x
  H0 : forall x : E, P x
  ============================
   False
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < elim H.
1 subgoal

  H : exists x : E, ~ P x
  H0 : forall x : E, P x
  ============================
   forall x : E, ~ P x -> False
```

Exemples de preuves

- Quantificateur ∃ :

```
Coq < intros.
1 subgoal

  H : exists x : E, ~ P x
  H0 : forall x : E, P x
  x : E
  H1 : ~ P x
  ============================
   False
```

Logique du premier ordre

Exemples de preuves

- Quantificateur $\exists$:

```
Coq < apply H1.
1 subgoal

  H : exists x : E, ~ P x
  H0 : forall x : E, P x
  x : E
  H1 : ~ P x
  ============================
   P x

Coq < apply H0.
No more subgoals.
```

Exercices

Propositions à démontrer en Coq

1. $\forall x.P(x) \Rightarrow \exists y.P(y) \vee Q(y)$
2. $(\exists x.P(x) \vee Q(x)) \Rightarrow (\exists x.P(x)) \vee (\exists x.Q(x))$
3. $(\forall x.P(x)) \wedge (\forall x.Q(x)) \Rightarrow \forall x.P(x) \wedge Q(x)$
4. $(\forall x.P(x) \wedge Q(x)) \Rightarrow (\forall x.P(x)) \wedge (\forall x.Q(x))$
5. $(\forall x.\neg P(x)) \Rightarrow \neg(\exists x.P(x))$
6. $\neg(\forall x.P(x)) \Rightarrow \exists x.\neg P(x)$

Guide de survie du petit Coq-uin

Correspondance LJ/Coq

Logique propositionnelle		Logique du premier ordre	
Règle LJ	Tactique Coq	Règle LJ	Tactique Coq
ax	`assumption`	$\forall_{right}$	`intro`
cut	`cut`	$\forall_{left}$	`apply`
$\Rightarrow_{right}$	`intro`	$\exists_{right}$	`exists`
$\Rightarrow_{left}$	`apply`	$\exists_{left}$	`elim`
$\Leftrightarrow_{right}$	`split`		
$\Leftrightarrow_{lefti}$	`elim`		
$\wedge_{right}$	`split`		
$\wedge_{left}$	`elim`		
$\vee_{right1}$	`left`		
$\vee_{right2}$	`right`		
$\vee_{left}$	`elim`		
$\neg_{right}$	`intro`		
$\neg_{left}$	`elimtype False` + `apply`		
$\top_{right}$, $\bot_{left}$	`auto`		