An Extension of the Arithmetic Derivative

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Abstract

In this paper, we examine the product rule and the arithmetic derivative. We first find a closed-form formula for the arithmetic derivative over positive integers. We then extend the argument to negative numbers, rational numbers, power roots and complex numbers. Throughout our research, we also use *Mathematica* graphics to help visualize the behavior of the arithmetic derivative over different domains and explore boundary conditions and intermediate lines. Finally, we discuss the continuity of arithmetic derivative and give a continuous form that satisfies the product rule.

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1 Overview of the Product Rule

From calculus, we are familiar with the product rule, $\frac{d(f\cdot g)}{dx} = f \cdot \frac{dg}{dx} + g \cdot \frac{df}{dx}$, which allows us to compute the derivative of product of two functions with known derivatives.

This product rule can be applied to mathematical structure other than functions. When we apply it to rational numbers, meaning that $f(x) = af(b) + bf(a)$, where a and b are rational numbers and x is the product of a and b, we get similar properties encountered in calculus.

First of all, we need $f(1) = 0$ and $f(-1) = 0$ in order to preserve the product rule. By basic algebra, we find both the quotient rule $f(\frac{x}{y}) = \frac{f(x)y - xf(y)}{y^2}$, and the power rule $f(x^r) = rx^{r-1}f(x)$, where r is rational, may be applied over the rational field $\mathbb{Q}$.

In addition, the product rule shows that $f(x)$ is an odd function, since $f(-x) = f(-1)x - f(x) = -f(x)$.

2 Introduction to Arithmetic Derivative

The section above establishes the overarching structure of this special function preserving product rules over the rational numbers. To make the function more concrete, we need to assign certain initial values. By the unique prime factorization of all integers, we are naturally interested in what f(p), the image of prime numbers, would be. For convenience of explanation, we start by exploring the function over positive integers. The simplest value for $f(p)$ is 1. By adding this rule to the product rule, we discover the arithmetic derivative, defined formally below.

Definition 2.1. *The arithmetic derivative of x, denoted $f(x)$, is defined in the following way:*

$$\begin{gathered}\textit{if } x = 1, \textit{ then } f(x) = 0,\\ \textit{if } x = p, \textit{ then } f(x) = 1,\\ \textit{if } x = ab, \textit{ then } f(x) = af(b) + bf(a),\\ \textit{where } p \textit{ is some prime number, and } a, b \textit{ are any positive integers.}\end{gathered}$$

3 Arithmetic Derivative on Positive Integers

Looking at the recursive definition of the arithmetic derivative, we want to generalize it into a closed form formula that we can use to explore its properties. To find this formula, we will decompose a number x into its prime factors, and use induction to find a formula. The following theorem will provide a general way to calculate $f(x)$ where x is a positive integer.

Theorem 3.1. *$f(x)$ can be calculated by the following formula*

$$f(x) = x(\frac{1}{p_1} + \ldots + \frac{1}{p_n})$$

where $p_1, p_2, \ldots, p_n$ are non-distinct prime factors of x such that $x = p_1 \cdot p_2 \cdot \cdots \cdot p_n$

This is equivalent to the following formula:

$$f(x) = x(\frac{\alpha_1}{p_1} + \ldots + \frac{\alpha_n}{p_n})$$

in which

$$x = p_1^{\alpha_1} \ldots p_n^{\alpha_n}$$

where $p_1, \ldots, p_n$ are distinct prime numbers and $\alpha_1, \ldots, \alpha_n$ are natural numbers.

Proof 3.2. We can prove the theorem above by induction. We start with the base case:

$$x = p_1 p_2$$

where x can be written as the product of two non-distinct prime factors, which we will call as a two-factor in the rest of this paper.

By the product rule and the definition that all primes are mapped to one, we get

$$f(x) = f(p_1 p_2) = p_1 + p_2$$

$$= p_1 p_2(\frac{1}{p_1} + \frac{1}{p_2})$$

Now if we assume

$$f(x) = x(\frac{1}{p_1} + \ldots + \frac{1}{p_n})$$

is true for $x = p_1 p_2 \ldots p_n$, we have

$$f(x \cdot p_{n+1}) = xf(p_{n+1}) + p_{n+1}f(x)$$

$$= x + p_{n+1}x(\frac{1}{p_1} + \ldots + \frac{1}{p_n})$$

Therefore,

$$f(x \cdot p_{n+1}) = p_1 \ldots p_n + (p_1 \ldots p_{n+1})(\frac{1}{p_1} + \ldots + \frac{1}{p_n})$$
$$= (p_1 \ldots p_{n+1})(\frac{1}{p_1} + \ldots + \frac{1}{p_n} + \frac{1}{p_{n+1}})$$

and we have proved the general formula.

□

Corollary 3.3. *$f(x)$ is well defined.*

Proof 3.4. We know that $f(x) = x(\frac{1}{p_1} + \cdots + \frac{1}{p_n})$. Since we know there is a unique prime factorization of any integer x, we can find a unique set of $p_1, \ldots, p_n$. Thus $f(x)$ is well-defined.

□

The following corollary concerns powers of a positive integer, which is a special case we have for the product rule we mention in the first section.

Corollary 3.5. *For any integer $a > 1$, $f(x^a) = ax^{a-1}f(x)$.*

Proof 3.6. We prove by induction. Let a=2. We have

$$f(x^2) = xf(x) + xf(x) = 2xf(x)$$

Therefore, the base case holds.

Now assume

$$f(x^a) = ax^{a-1}f(x)$$

We have

$$f(x^{a+1}) = xf(x^a) + x^a f(x) = (a+1)x^a f(x)$$

Thus, the corollary holds. $\square$

4 Expansion to Negative and Rational Numbers

In this section we want to extend the arithmetic derivative to more domains than just the integers. We will start with negative numbers. First, we need to find the value of $f(-1)$ in order to work with negative numbers.

Definition 4.1. $f(-1) = 0$

Justification 4.2.

$$f(1) = -1 \cdot f(-1) - 1 \cdot f(-1) = -2f(-1) = 0$$

Thus, we see that $f(-x) = -f(x)$ for any x.

Now we seek to expand the function f to rational numbers. We will start by utilizing the fact that $f(\frac{p}{p}) = f(1)$.

$$f(1) = f(p \cdot \frac{1}{p}) = \frac{1}{p}f(p) + pf(\frac{1}{p}) = \frac{1}{p} + pf(\frac{1}{p}) = 0$$

Thus we get

$$f(\frac{1}{p}) = -\frac{1}{p^2}$$

Definition 4.3. *For rational number in the form of* $\frac{1}{p}$, $f(\frac{1}{p}) = -\frac{1}{p^2}$.

This definition is crucial as it is analogous to $f(p) = 1$ for the integers for all prime number p.

Theorem 4.4.

$$f(\frac{1}{y}) = -\frac{1}{y}(\frac{1}{q_1} + \ldots + \frac{1}{q_m})$$

in which $q_1, \ldots, q_m$ are non-distinct prime factors of y.

Proof 4.5. By induction, for the base case $m = 2$,

$$f(\frac{1}{q_1q_2}) = \frac{1}{q_1}f(\frac{1}{q_2}) + \frac{1}{q_2}f(\frac{1}{q_1})$$
$$= -\frac{1}{q_1q_2^2} - \frac{1}{q_1^2q_2} = -\frac{1}{y}(\frac{1}{q_1} + \frac{1}{q_2})$$

Then, assuming the theorem is true for $m = k$, the theorem when $m = k+1$ can be proved in the following way.

$$f(\frac{1}{q_1 \dots q_k}\frac{1}{q_{k+1}}) = \frac{1}{q_1 \dots q_k}f(\frac{1}{q_{k+1}}) + \frac{1}{q_{k+1}}f(\frac{1}{q_1 \dots q_k})$$
$$= \frac{1}{q_1 \dots q_k}(-\frac{1}{q_{k+1}^2}) + \frac{1}{q_{k+1}}(-\frac{1}{q_1 \dots q_k}(\frac{1}{q_1} + \dots + \frac{1}{q_k}))$$

which can be rewritten as

$$f(\frac{1}{q_1 \dots q_{k+1}}) = -\frac{1}{q_1 \dots q_{k+1}}(\frac{1}{q_1} + \dots + \frac{1}{q_k + 1})$$

□

Theorem 4.6. *The following theorem provides a compact way to calculate $f(\frac{x}{y})$. Using the definition of Theorem 1 and Theorem 2,*

$$f(\frac{x}{y}) = \frac{x}{y}(\frac{1}{p_1} + \dots + \frac{1}{p_n} - \frac{1}{q_1} - \dots - \frac{1}{q_m})$$

when x and y are integers such that $x = p_1 \dots p_n$ and $y = q_1 \dots q_m$.

Proof 4.7. Every fraction can be divided into an integer and an Egyptian function.

$$f(\frac{x}{y}) = xf(\frac{1}{y}) + \frac{1}{y}f(x)$$

Now, using, Theorems 1 and 2, the theorem naturally follows.

$$f(\frac{x}{y}) = xf(-\frac{1}{y}(\frac{1}{q_1} + \dots + \frac{1}{q_m})) + \frac{1}{y}x(\frac{1}{p_1} + \dots + \frac{1}{p_n})$$

which can be rearranged to

$$f(\frac{x}{y}) = \frac{x}{y}(\frac{1}{p_1} + \dots + \frac{1}{p_n} - \frac{1}{q_1} - \dots - \frac{1}{q_m})$$

□

5 Boundary Condition of the Integer Function

This section deals with the graphs of $f(x)$. Each graph is mapped over x values with n non-distinct prime factors for some n. For example, Figure 1 is a graph of $f(x)$ for all x with two factors.

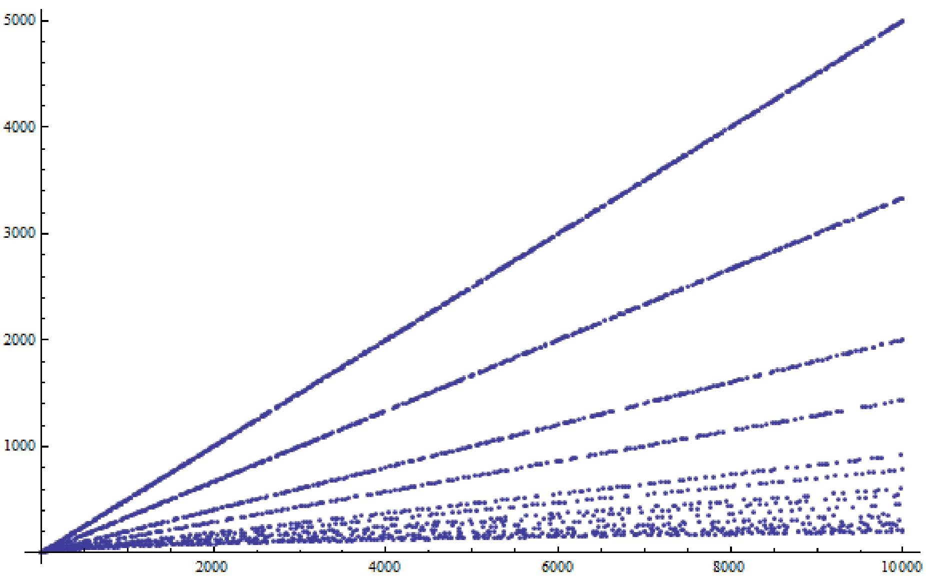

Figure 1: Integers with 2 non-distinct Prime Factors

Figure 2 is for $f(x)$ where x has 3 factors.

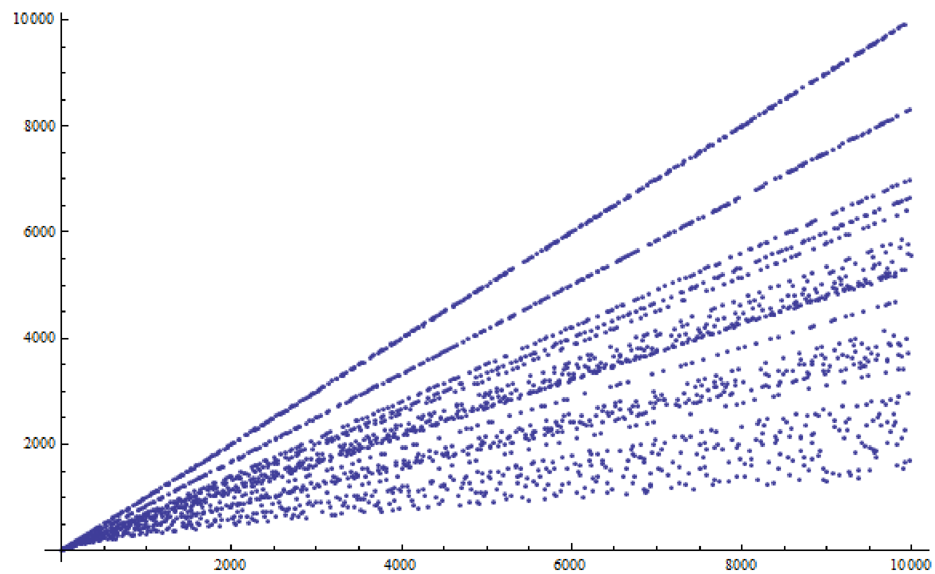

Figure 2: Integers with 3 non-distinct Prime Factors

For each of these graphs, we can determine an upper and lower bound based on n. As is clear, the graph of $f(x)$ also has discrete lines in it. These will be explored in the following section.

Theorem 5.1. *This theorem describes a lower bound for the graph of $f(x)$ where x has two factors.*

Let x be $x = p_1 p_2$. We have

$$f(x) \geq 2\sqrt{x}$$

Proof 5.2. Using formula (1), we have

$$f(x) = f(p_1 p_2) = \frac{x}{p_1} + \frac{x}{p_2} = \frac{x}{p_1 p_2}(p_1 + p_2) = (p_1 + p_2) \geq 2\sqrt{x}$$

by the AM-GM inequality

$$\frac{p_1 + p_2}{2} \geq \sqrt{p_1 p_2}$$

$\square$

Extending the inequality from the case of products of 2 factors to the case of products of k factors using similar inequality argument, we can then find a general formula for the lower bound of x, where $x = p_1 p_2 p_3 \ldots p_k$.

Theorem 5.3. *This theorem extends upon the previous one, and describes a lower bound for $f(x)$ for x with any number of factors.*

Let $x = p_1 p_2 p_3 \ldots p_k$. We have

$$f(x) \geq \frac{kx}{\sqrt[k]{x}}$$

Proof 5.4. Assume $x = p_1 p_2 p_3 \ldots p_n$ where all p_i are non-distinct prime factors. To minimize $f(x) = x(\frac{1}{p_1} + \frac{1}{p_2} + \ldots)$, we need to make the quantity $\frac{1}{p_1} + \frac{1}{p_2} + \ldots$ as small as possible. This implies every $p_i = \sqrt[n]{x}$. This is because similar to the proof for the lower bond with two factors, we can minimize $\frac{1}{p_1} + \frac{1}{p_2}$ by making p_1 and p_2 as close as possible. This yields $f(x) = \frac{kx}{\sqrt[k]{x}}$ as a lower bound.

□

The following graphs show this result for x with two and three factors. The yellow lines are the lower bounds.

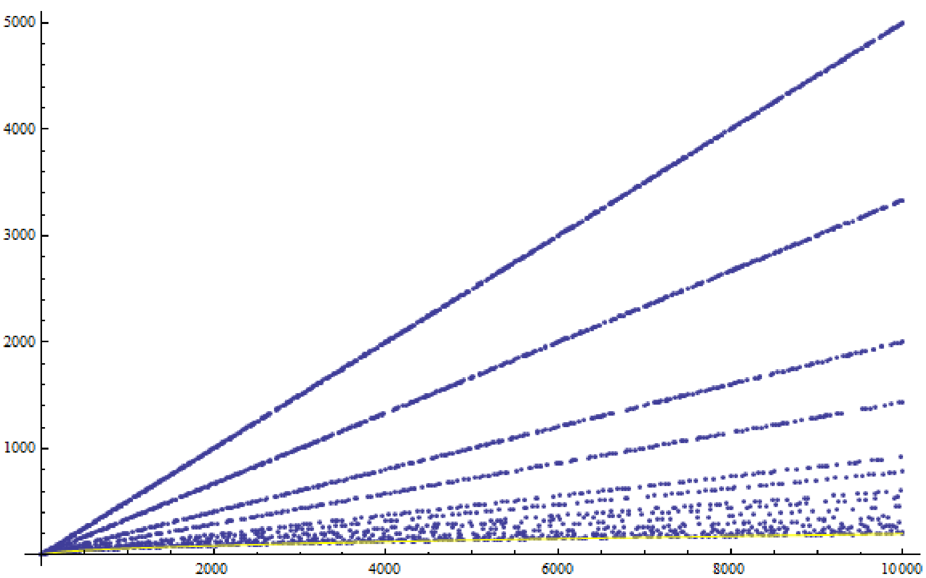

Figure 3: Integers with 2 non-distinct prime factors with the lower bound

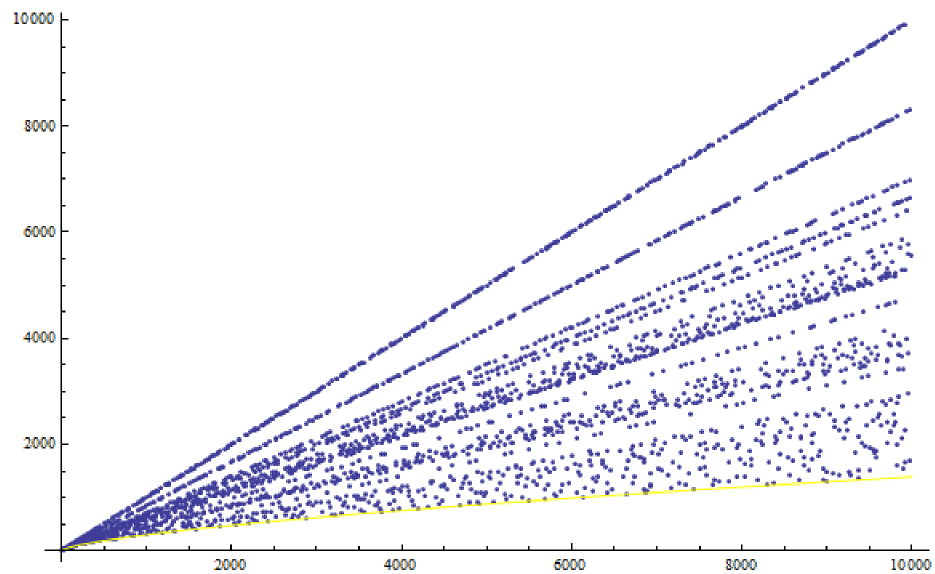

Figure 4: Integers with 3 non-distinct prime factors with the lower bound

Theorem 5.5. *This theorem describes the upper bound for $f(x)$. Let n be the number of non-distinct prime factors of x. Then, the upper bound line is*

$$y = \frac{(n-1)x}{2} + 2^{n-1}$$

Proof 5.6. To begin this proof, it is required that we define a new function, $g(x)$ such that

$$g(x) = x(\frac{1}{a_1} + \cdots + \frac{1}{a_n})$$

where $a_1 \cdots a_n = x$, and each a_i is greater or equal to 2, since the domain of $f(x)$ in this section is defined on the positive integers.

Thus, $g(x)$ behaves in exactly the same way as $f(x)$ if and only if each a_i is prime. In order to find an upper bound for $f(x)$, we may find an upper bound for $g(x)$ that can be transformed into an upper bound for $f(x)$.

Now, we need to show that if x has two factors greater than two, $\frac{g(x)}{x}$ can be made larger by changing one factor into 2 and the other into $\frac{ab}{2}$ in the factorization, preserving the total product x. Assume $x = p_1 \cdot \cdots \cdot p_n \cdot a \cdot b$. We want to prove the following inequality first

$$g(p_1 \cdot \cdots \cdot p_n \cdot a \cdot b) < g(p_1 \cdot \cdots \cdot p_n \cdot (\frac{ab}{2}) \cdot 2)$$

To prove this inequality, we start with the statement that

$$0 < (a-2)(b-2)$$

$$0 < 4 + ab - 2a - 2b$$

$$2a + 2b < 4 + ab$$

Multiplying both sides by $\frac{x}{2ab}$ gives

$$\frac{x}{a} + \frac{x}{b} < \frac{2x}{ab} + \frac{x}{2}$$

$$\frac{x}{p_1} + \frac{x}{p_2} + \cdots + \frac{x}{p_n} + \frac{x}{a} + \frac{x}{b} < \frac{x}{p_1} + \frac{x}{p_2} + \cdots + \frac{x}{p_n} + \frac{2x}{ab} + \frac{x}{2}$$

and therefore we complete the proof. We have shown that $\frac{g(x)}{x}$ can be maximized when we change two of the factors distinct from 2, into 2 and $\frac{ab}{2}$.

Now we need to show that $\frac{g(x)}{x}$, the slope, is maximized for all cases when all a_i are 2. Assume we have $x = 2^n \cdot a$, where only one factor of x is possibly not equal to 2. We want to show that in this case $\frac{g(x)}{x}$ is also maximized when $a = 2$. To do this, assume $\frac{g(x)}{x}$ with factor a defined above is greater than or equal to the expression with a equal to 2. We then get the following inequality

$$\frac{n2^{n-1} \cdot a + 2^n}{2^n \cdot a} \geq \frac{(n+1)2^n}{2^{n+1}}$$

$$\frac{n}{2} + \frac{1}{a} \geq \frac{n}{2} + \frac{1}{2}$$

$$a \leq 2$$

Since $a \geq 2$ by definition, a must be equal to 2. This implies that the way to maximize $\frac{g(x)}{x}$ is to rewrite x as factors of two. To translate this into an ideal solution to $f(x)$, we see that x must be of the form 2^n, and this gives us an upper bound line of

$$y = \frac{(n-1)x}{2} + 2^{n-1}$$

$\square$

The following are depictions of this result for x with two or three factors respectively.

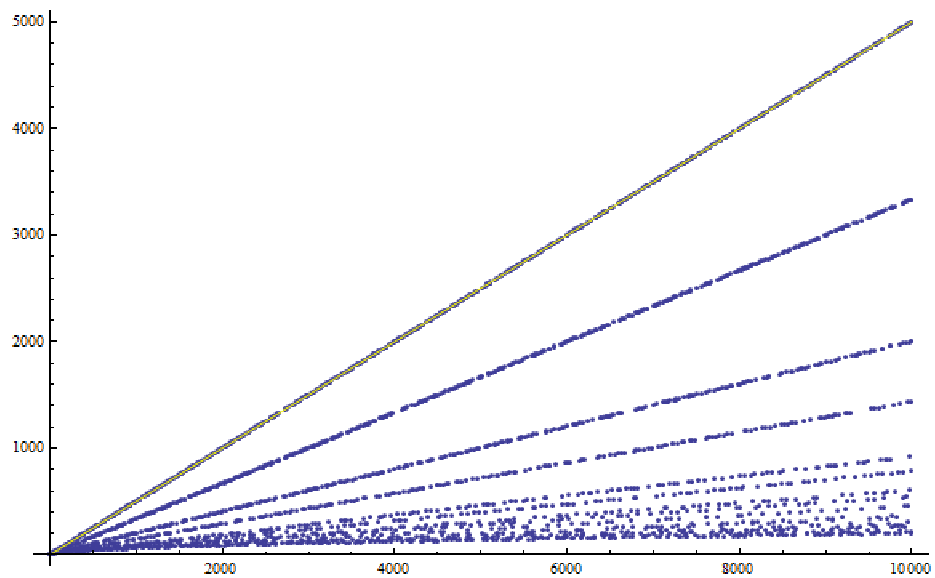

Figure 5: Integers with 2 non-distinct prime factors with the upper bound

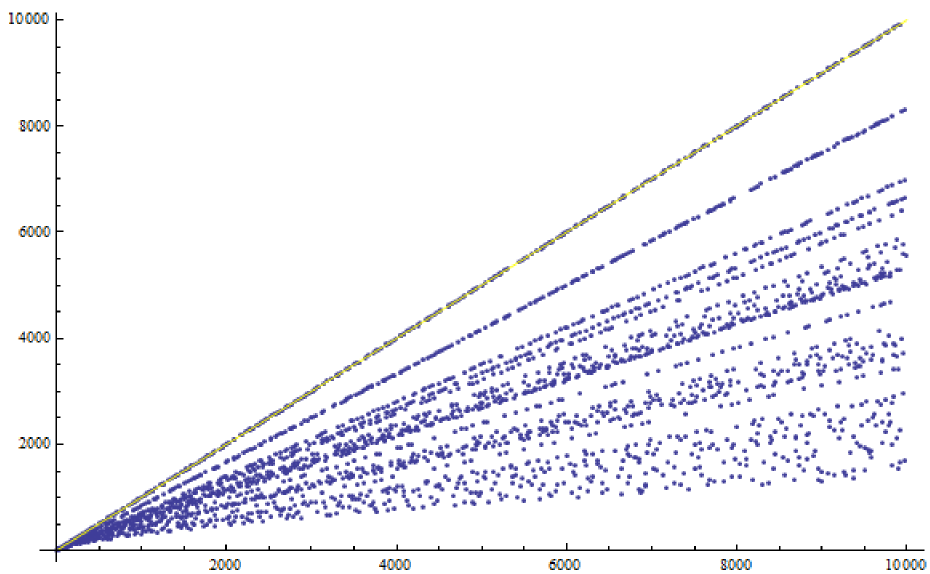

Figure 6: Integers with 3 non-distinct prime factors with the upper bound

6 Intermediate Lines within the Graph

Looking at a graph of $f(x)$ with respect to x, we observe intermediate lines within the graphs besides the boundaries we address in the previous section. We start with our investigation of these lines for x with two factors and x with three factors.

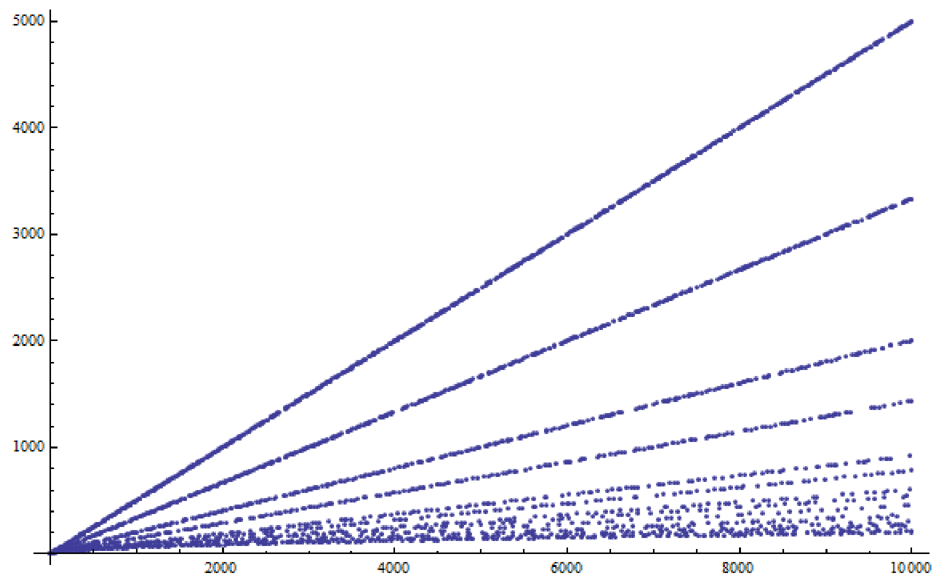

Figure 7: Integers with 2 non-distinct prime factors

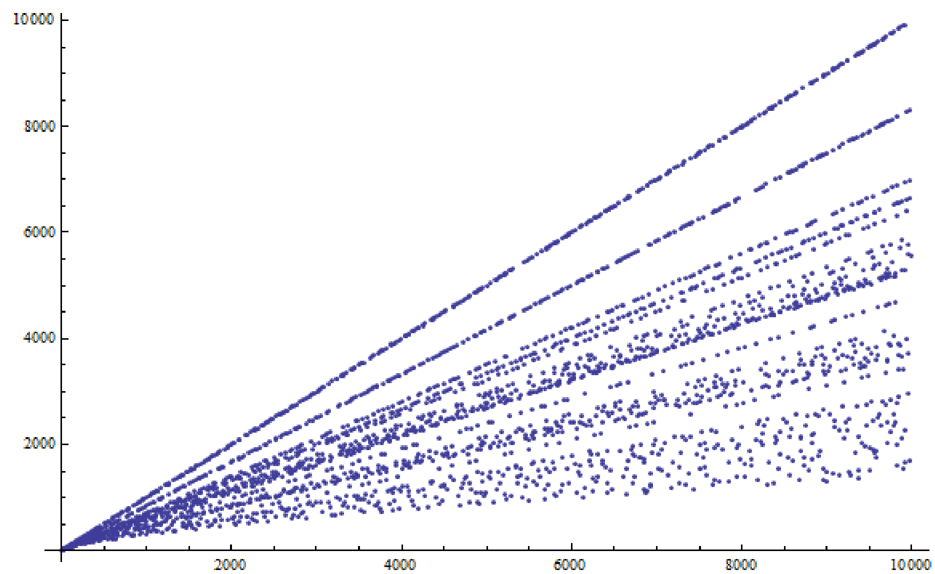

Figure 8: Integers with 3 non-distinct prime factors

The following theorems will give a generalized formula for these intermediate lines.

Theorem 6.1. *Let $x = p_1 p_2$. Each line within the graph of $f(x)$ is of the form*

$$y = \frac{x}{p_i} + p_i$$

where p is a prime.

Proof 6.2. To prove this theorem, we need to show that every $f(x)$ is on at least one of the intermediate lines of the form $y = \frac{x}{p_i} + p_i$ and that each line has infinite points on them. It is easy to see that if $x = p_1 p_2$ then $f(x) = p_1 + p_2$ will be on both the line $y = \frac{x}{p_1} + p_1$ and $y = \frac{x}{p_2} + p_2$.

Thus, it follows that for two-factors x, every point $f(x)$ is on two lines of this form. Since for any fixed p_1, there is a whole spectrum of p_2 we can choose, yet still yield $p_2 + p_1$ on $\frac{x}{p_1} + p_1$, we know that for each line of this form, there is a prime distribution of points, which is infinite, that constitute the line. $\square$

Theorem 6.3. *Extending the previous result to x with n factors, the lines in the function $f(x)$ will be of the form*

$$y = x\frac{f(p_1 \cdot \cdots \cdot p_{n-1})}{(p_1 \cdot \cdots \cdot p_{n-1})} + (p_1 \cdot \cdots \cdot p_{n-1})$$

where $p_1 \ldots p_{n-1}$ are primes.

Proof 6.4. Similar to the previous proof, we want to find the formula for line that contains a prime distribution of points. Let $x = p_1 \cdot \cdots \cdot p_n$. Then it follows that

$$f(x) = \frac{(p_1 \cdot \cdots \cdot p_n)}{p_1} + \cdots + \frac{(p_1 \cdot \cdots \cdot p_n)}{p_n}$$

Now we will create a line by holding one of the factors constant (WLOG p_n). This yields

$$y = \frac{f(p_1 \cdot \cdots \cdot p_{n-1})x}{p_1} + p_1 \cdot \cdots \cdot p_{n-1}$$

For every set of $p_1, \ldots, p_n$. Thus, every point will appear on n lines, and each line will contain a number of points equivalent to the distribution of primes among the integers. $\square$

The following are depictions of this result for x with two and three factors respectively.

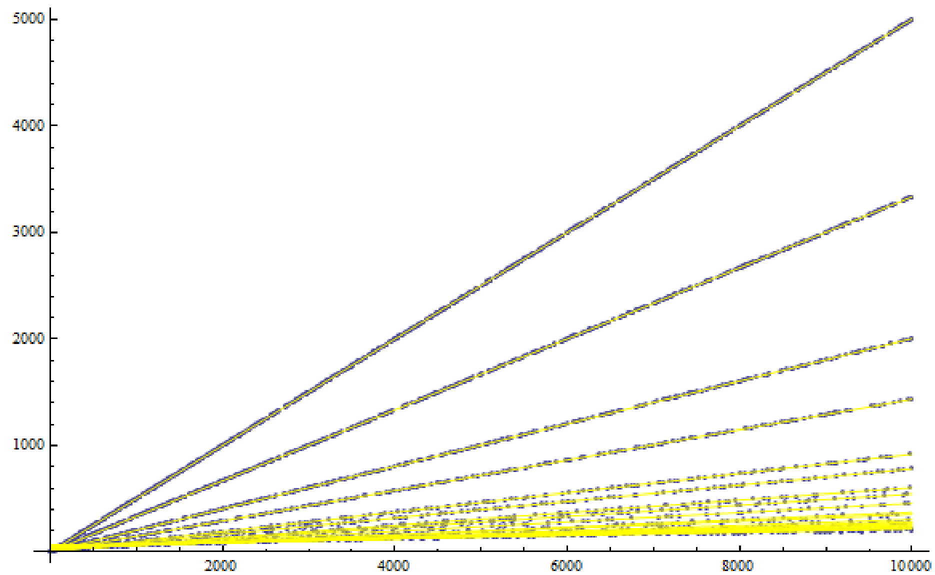

Figure 9: Integers with two non-distinct prime factors with lines added

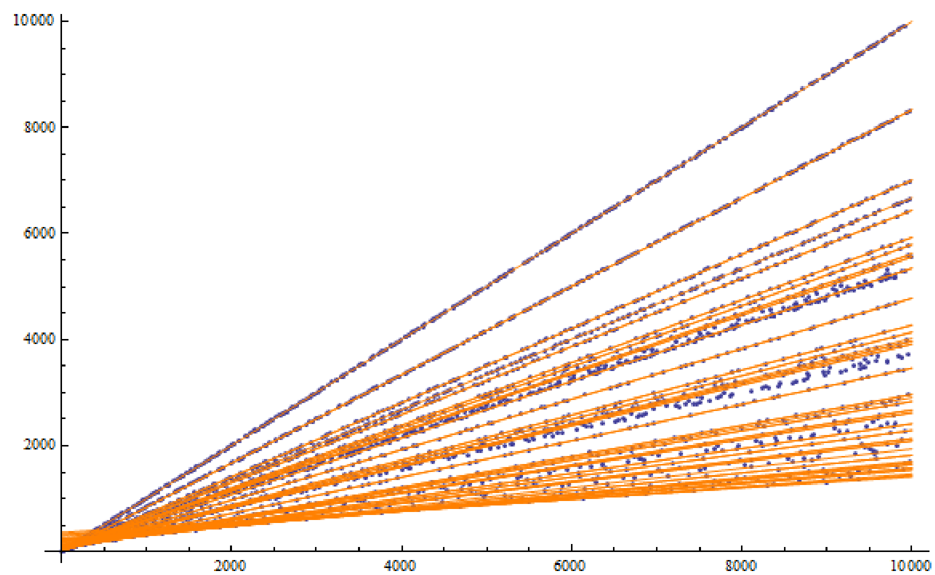

Figure 10: Integers with 3 non-distinct prime factors with lines added

7 Expansion to Graphs with Fractions

In this section, we will explore how $f(x)$ behaves when $x = \frac{a}{b}$, where a,b are positive integers. In this section, rather than working with a range of $\frac{a}{b}$, we will work with both a and b individually on certain ranges. We will establish a lower bound for $f(\frac{a}{b})$ in this section. The following is an example of $f(\frac{a}{b})$ with both a and b in the range $[1, 100]$.

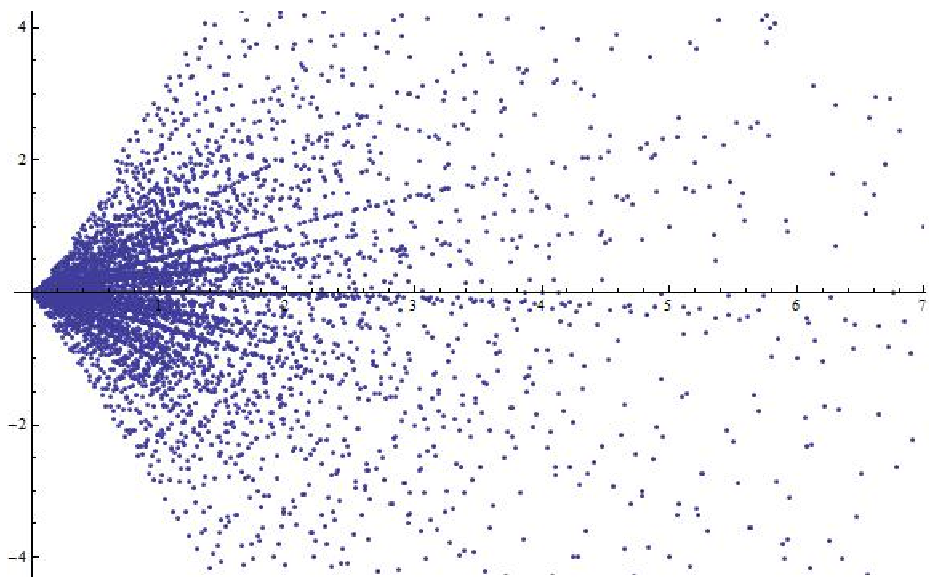

Figure 11: $f(\frac{a}{b})$ with a and b in range $[1, 100]$

Theorem 7.1. *Let $x = \frac{a}{b}$ where $a = [1, n]$ and $b = [1, m]$. The lower bound for $f(x)$ will be $y = -\frac{cx}{2}$ such that c is the largest number that satisfies the inequality $2^c < m$.*

Proof 7.2. In order to prove this, we want to minimize the quantity $\frac{f(x)}{x}$.

If we let $x = p_1 \cdot \cdots \cdot p_n$ the quantity above simplifies to

$$(p_1 \cdot \cdots \cdot p_n) \cdot (\frac{-1}{p_1} + \cdots + \frac{-1}{p_n}) \cdot (\frac{1}{p_1 \cdot \cdots \cdot p_n})$$

$$= -\frac{1}{p_1} + \cdots - \frac{1}{p_n}$$

Thus, to minimize this quantity, we are seeking to maximize

$$\frac{1}{p_1} + \cdots + \frac{1}{p_n}$$

This implies that each p_i should be as small as possible, which means p_i should be 2. It is easy to see that if one of $p_1, \ldots, p_n$ is greater than 2, $(\frac{1}{p_1} + \cdots + \frac{1}{p_n})$ can have a larger value by changing the value of that prime to 2. This follows from

$$\frac{1}{p_i} \leq \frac{1}{2},$$

Now we need to show that the solution that minimizes the slope of $f(\frac{a}{b})$ has $a = 1$. If a is not equal to one, then we get

$$\frac{f(\frac{a}{2^c})}{\frac{a}{2^c}} = \frac{a \cdot f(\frac{1}{2^c}) + \frac{1}{2^c} f(a)}{\frac{a}{2^c}}$$

To minimize this value based on a, we must minimize $\frac{1}{2^c} f(a)$. This occurs when $a = 1$. This minimizes our slope and completes the proof. $\square$

The following are depictions of this result with a and b in the ranges $[1, 100]$ and $[1, 500]$ respectively.

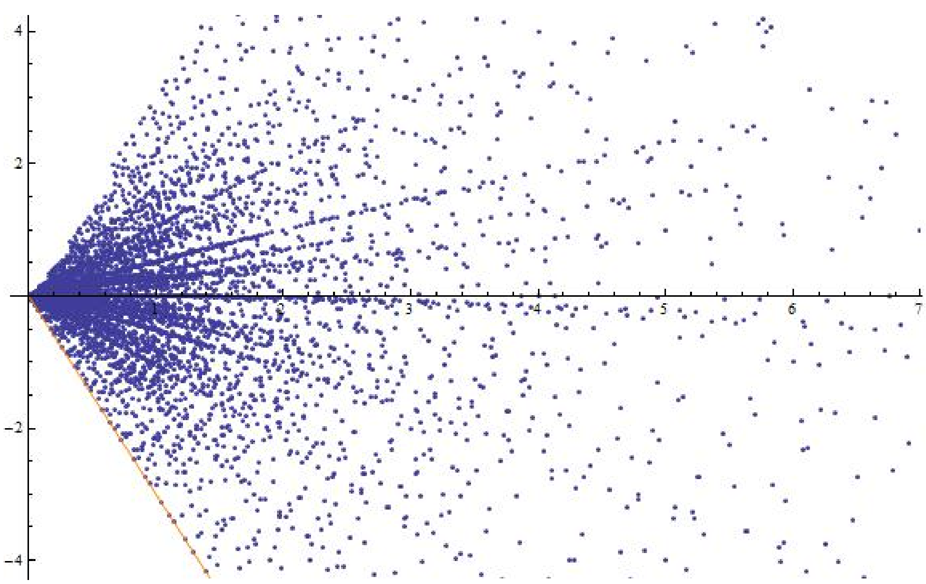

Figure 12: $f(\frac{a}{b})$ with a and b in range $[1, 100]$ with line added

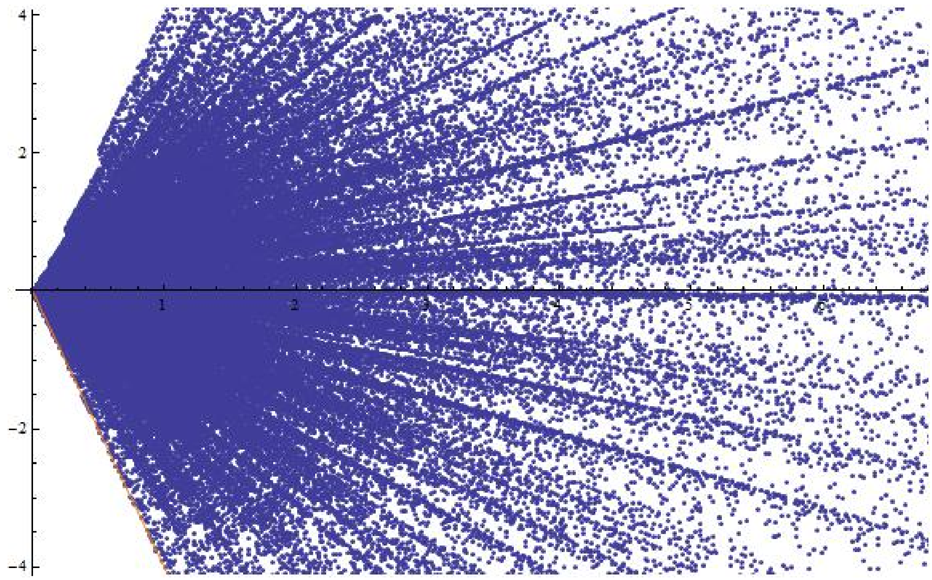

Figure 13: $f(\frac{a}{b})$ with a and b in range $[1, 500]$ with line added

Corollary 7.3. *There will only be one point on the lower bound line. This point is* $(\frac{1}{2^c}, -\frac{c}{2^{c+1}})$.

Proof 7.4. Any other point will either have b with less prime factors than 2^c or have the same number of prime factors with one of those factors greater than two. Either situation implies that $f(2^{-c})$ has a steeper slope than any other number in the range of a and b. $\square$

Corollary 7.5. $f(0) = -\infty$

Proof 7.6. As we increase the range for b such that b approaches ∞, then c in $\frac{1}{2^c}$ also approaches $-\infty$. Thus, the slope of the lower bound line approaches $-\infty$, implying that $f(0) = -\infty$. $\square$

Corollary 7.7. *If we graph* $f(\frac{a}{b})$ *with both* a *and* b *in range* $[1, n]$, *the points will be evenly distributed on both sides of the* $x - axis$.

Proof 7.8. $f(\frac{a}{b})$ is positive if and only if the sum of reciprocals of the primes of a is greater than that of b. Given that a and b are on the same range, this should occur just as many times as the opposite, giving the function an even distribution. $\square$

8 Continuity and an Expansion to Some Irrational Numbers

When working with a function, continuity is an obvious inquiry. From first insight, it seems that our function is not continuous. In this section, we will first complete a proof to show that arithmetic derivative defined on the rational numbers is not continuous using a counterexample. Then, we will address the implication of this discontinuity and propose other schemes of the mappings of primes in order to explore other options of our function. In the end we give a function which satisfies the product rule but is still continuous. First, we are going to work with the following series

$$1 + \frac{1}{2} + \frac{1}{4}...$$

We know f of that series should be $f(2) = 1$, but is it? Continuity entails such that f of a series as the number of elements approaches infinity is equal to f of the value the series converges to. Thus, $f(1 + \frac{1}{2} + \frac{1}{4}...)$ should be 1. However,

$$f(1 + \frac{1}{2} + \frac{1}{4}... + \frac{1}{2^x}) = f(2 - \frac{1}{2^x})$$

$$= f(\frac{2^{x+1} - 1}{2^x})$$

Using the general formula we have derived, this is equivalent to

$$f(1 + \frac{1}{2} + \frac{1}{4}... + \frac{1}{2^x}) = \frac{2^{x+1} - 1}{2^x} \cdot (\alpha - \frac{x}{2})$$

where α is the sum of the reciprocals of the prime factors of $2^{x+1} - 1$.

Now, we seek to show that the part $\alpha - \frac{x}{2}$ is always negative. This implies that $f(1 + \frac{1}{2} + \frac{1}{4}... + \frac{1}{2^x})$ can never approach 1 as x approaches infinity.

Since $2^{x+1} - 1$ is not divisible by two, in order to get the smallest prime factors to maximize α, we choose all prime factors to have a a value of 3. This implies we have $\log_3(2^{x+1} - 1)$ factors of 3. Thus, when maximized, $\alpha = \frac{1}{3}\log_3(2^{x+1} - 1)$.

Therefore,

$$\alpha - \frac{x}{2} \leq \frac{1}{3}\log_3(2^{x+1}-1) - \frac{x}{2} < \frac{1}{3}\log_3(2^{x+1}) - \frac{x}{2} = -\frac{x+1}{3}\log_3 2 - \frac{x}{2}$$

Since we know $\log_3 2 > 0.63$, we get

$$\alpha - \frac{x}{2} < \frac{x+1}{3} \cdot 0.63 - \frac{x}{2} = 0.21x + 0.21 - 0.5x = -0.29x + 0.21$$

From above, we see that since n is a non-negative integer, the expression $\alpha - \frac{x}{2}$ is always less than zero. Therefore, the limit as x goes to infinity of $f(1 + \frac{1}{2} + \frac{1}{4} ... + \frac{1}{2^x})$ is not equal to $f(2)$. This counterexample shows that f is not continuous.

Therefore, we have established that limits cannot be used to calculate f of irrational numbers, but is there another way? For a certain subset, it is true. If we start with the possibly irrational number $x^{\frac{a}{b}}$ we can in fact calculate its $f(x)$.

From the power rule we know that

$$f(x^b) = bx^{b-1}f(x)$$

Thus, by replacing x with $x^{\frac{1}{b}}$ and simplifying, we get

$$f(x^{\frac{1}{b}}) = \frac{f(x)}{bx^{\frac{b-1}{b}}}$$

Despite this result for mapping the roots, there are many irrational numbers for which f cannot be computed. In fact, we currently have no way to calculate any irrational numbers besides the subset of the form $x^{\frac{a}{b}}$ or a product of these numbers. In order to calculate these numbers, f needs to be continuous, or we need to find another technique to calculate them. This brings up a natural question. Can we make $f(p)$, the only elastic part of our function, be of a certain form such that it allows continuity and calculation of the irrationals?

Clearly, we have the trivial case $f(p) = 0$. In this case, f of any number equals 0 and thus is continuous. To find a non-trivial $f(p)$ we may focus on the product rule. Start with

$$f(xy) = xf(y) + yf(x)$$

Dividing both sides by xy yields

$$\frac{f(xy)}{xy} = \frac{f(y)}{y} + \frac{f(x)}{x}$$

Now define a new function $g(x)$, such that

$$g(x) = \frac{f(x)}{x}$$

Plugging into the formula above, we get

$$g(xy) = g(x) + g(y)$$

By calculus, we know that any logarithm, such as $\ln x$, will satisfy the property above that maps the product of two inputs to the sum of their individual outputs. Thus, since $g(x) = \ln x$, we see that $f(x) = x \ln x$ is the desired continuous function. Now to justify that we can choose $f(p)$ to give this result, if we set $f(p) = p \ln p$, we see that $f(x)$ for any positive rational number evaluates to $x \ln x$. This also allows calculating f of irrationals.

This continuous form by setting $f(p) = p \ln p$, while more inclusive, however, makes our function less interesting. In the following section, we will still work with $f(p) = 1$, and expand the arithmetic derivative to complex numbers.

9 Expansion to Complex Numbers

We can expand the arithmetic derivative with $f(p) = 1$ to imaginary numbers as well. The method is similar to that with which we have expanded the arithmetic derivative to the negative integers. We can use the following formula to derive $f(i)$.

$$f(-1) = f(i^2) = 2if(i) = 0$$

Thus,

$$f(i) = 0$$

By the same argument, it can be shown that $f(-i) = 0$. Similarly, we can find $f(ai)$

$$f(ai) = af(i) + if(a)$$

$$f(ai) = if(a)$$

However, we are still not able to find the function of every complex number. To do this, we need to start with the following lemma.

Lemma 9.1. *Any complex number $a+bi$ such that $a^2+b^2 = 1$ has $f(a+bi) = 0$.*

Definition 9.2. *For the purpose of this paper, every complex number has two square roots. We define the root with a positive angle θ closer to 0 to be positive (+), and the root with an angle θ larger than 180 degrees to be negative (-). Since the square roots of a number are 180 degrees apart, each number will have one positive and one negative root.*

Proof 9.3. Let S be the set of all numbers of the form $\cdots\sqrt{\pm\sqrt{\pm\sqrt{1}}}$. Any number of this form has $f(x) = 0$. This can be shown by induction, noting that if $f(a) = 0$ it implies $f(\sqrt{a}) = 0$. Thus, to show that all numbers on the complex unit circle have $f(x) = 0$, we seek to show they can all be written in this format.

To do this, we want to show a 1-1 correspondence between the set S and all numbers between 0 and 2π. By Cantor's results, this is the equivalent of showing a 1-1 correspondence between S and all numbers between -1 and 1.

In order to show this relation, we will start by showing a 1-1 correspondence between set S and a new set P, defined as follows.

Let P be the set of lists of numbers $(\pm1)(\pm1)(\pm1)$..., which can go on infinitely. An example would be the string $(1)(-1)(1)(-1)$....

To show a 1-1 correspondence, we must show that each element in S maps to one element in P and vice versa. This can be shown easily by mapping each $\pm$ is S to a $\pm$ in P and vice versa. For example $\cdots+\sqrt{+\sqrt{-\sqrt{1}}}$ would map to $(-1)(+1)(+1)$...

Now we need a 1-1 correspondence between set P and the set of all binary numbers between -1 and -1. We will define a map as follows: for any string $(\pm1)(\pm1)(\pm1)$... we will map it to a binary number with digits $0.(\pm1)(\pm1)(\pm1)$.... We will then evaluate this expression to get a binary number with digits 0 and 1, using an idea that a binary number can be written in the form $(\pm\frac{1}{2}) + (\pm\frac{1}{4}) + (\pm\frac{1}{8})$....

For example, $(-1)(+1)(-1)(-1)$ maps to $0.(-1)(+1)(-1)(-1)$, or $\frac{-1}{2} + \frac{1}{4} + \frac{-1}{8} + \frac{-1}{16}$. When evaluated, this yields -0.0111.

Now we start the proof by showing the injection, which means that every string in set P correspond to a unique binary number. This is simply taking a string and evaluating it using the above map, which is unique by arithmetic. This gives a binary decimal corresponding to that string.

Now, to show surjection, we must show that every binary number can be mapped from a string.

To do this, we must find a way to go backwards in the computation we previously did. We know any binary number between 0 and 1 can be created by strings of digits of the form ...0001 for some number of zeroes. For example, the number 0.001101 is made up of strings 001, 1, 01. To convert the binary number back into the form $(\pm1)(\pm1)(\pm1)$... we simply convert each string in the following manner. 0001 is equal to $(1)(-1)(-1)(-1)$, and similarly for any string of n zeroes with a 1 on the end, we can replace it with a 1 followed by n -1's. Numerically, it is clear that this doesn't change the value of the number, as we have one decimal digit countered by a series of its negative ensued. Once we have converted each string, we have a binary number consisting of all 1's and -1's. We can then turn this into a string by simply reading off the numbers. (Note: we can also account for negative binary numbers by switching the $\pm$ signs on each decimal digit in the string.)

To apply this method in an example, we will find the string of $+1$'s and -1's that maps to 0.00011100101.

$$= 0.(1)(-1)(-1)(-1)(1)(1)(0)(0)(1)(0)(1)$$

$$= 0.(1)(-1)(-1)(-1)(1)(1)(1)(-1)(-1)(0)(1)$$
$$= 0.(1)(-1)(-1)(-1)(1)(1)(1)(-1)(-1)(1)(-1)$$

Which corresponds to the string (1)(-1)(-1)(-1)(1)(1)(1)(-1)(-1)(1)(-1). It is easy to check this solution by evaluating this string by our initial mapping.

Now we have shown a clear map from P to all binary numbers between 1 and -1. Thus, there exists a 1-1 correspondence between set S and the set of all binary numbers between 1 and -1. Since all elements in S are mapped to zero, we can conclude that $f(a + bi) = 0$ for all complex numbers on the unit circle. □

Theorem 9.4. *Any complex number $a+bi$ can be written as $ce^{i\theta}$. $f(a+bi) = f(c)e^{i\theta}$.*

Proof 9.5. We know $f(a + bi) = f(ce^{i\theta})$. We will now use algebra to obtain our result.

$$f(ce^{i\theta}) = e^{i\theta} f(c) + cf(e^{i\theta})$$

By lemma 9.1, $f(e^{i\theta}) = 0$, since $e^{i\theta}$ lies on the unit circle. Thus, we can simplify our equation to

$$f(a + bi) = f(c)e^{i\theta}$$

□

Corollary 9.6. *The function $f(a + bi)$ has a result that is a scalar multiple of $a + bi$ for all a and b.*

Proof 9.7. Both the $a + bi$ and $f(a + bi)$ are in the direction of angle θ, assuming $a + bi = ce^{i\theta}$. Thus, both vectors are parallel to each other. □

10 Conclusion

Research on arithmetic derivative, that is when $f(p) = 1$, raises a number of interesting connections, including the Goldbach conjecture. Any number N that can be written as the product of two primes has $f(N) =$ the sum of those two primes. It may be possible to map the even numbers backwards to some N that is the product of two primes, and therefore show that all evens can be written as the sum of two primes.

There are also other aspects to this function which we have not explored. Looking at the graph of $f(\frac{a}{b})$ in the range $[1, 200]$, we see a jagged line as the upper bound.

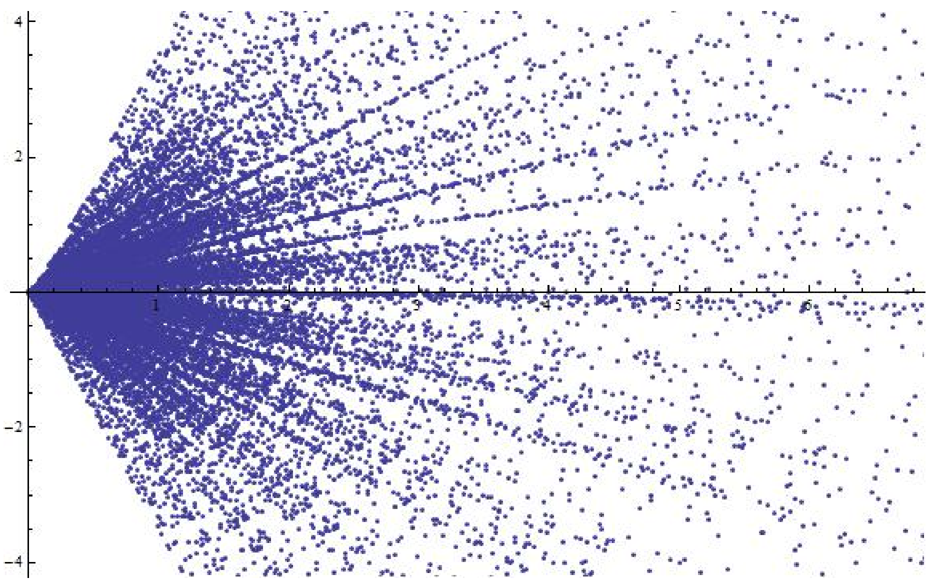

Figure 14: $f(\frac{a}{b})$ with a and b in range $[1, 200]$

We have not yet discovered the reason behind this jagged upper bound. Another question is to find a general relationship between a and b, such that $f(a) = f(b)$ when a is not equal to b.

These questions leave room for future exploration of the arithmetic derivative.

11 Codes used for the paper

Using *Mathematica* was an essential part of this research. Here we will present the program codes used to generate the graphs.

Listing 1: Predefined Functions

```
f[n_Integer] :=
 If[n == 1, 0, Sum[n/FactorInteger[n][[i]][[1]]* FactorInteger[n][[i]][[2]],
```

```
 {i, 1, Length[FactorInteger[n]]}]]

NumberofFactors[n_Integer] :=
 Sum[FactorInteger[n][[i ]][[2]],  {i, 1, Length[FactorInteger[n]]}]
```

Listing 2: Basic Plot

```
ListPlot[Table[{n, f[n]}, {n, 1, 10000}], ImageSize -> Large]
```

Listing 3: Figure 1 and 7

```
ListPlot[Table[
  If[NumberofFactors[n] == 2, {n, f[n]}, ## &[]] , {n, 1, 10000}],
 ImageSize -> Large]
```

Listing 4: Figure 2 and 8

```
ListPlot[Table[
  If[NumberofFactors[n] == 3, {n, f[n]}, ## &[]] , {n, 1, 10000}],
 ImageSize -> Large]
```

Listing 5: Figure 3

```
Show[ListPlot[
  Table[If[NumberofFactors[n] == 2, {n, f[n]}, ## &[]], {n, 1, 10000}],
  ImageSize -> Large],
 Plot[2*x^(1/2), {x, 1, 10000}, PlotStyle -> Yellow]]
```

Listing 6: Figure 4

```
Show[ListPlot[
  Table[If[NumberofFactors[n] == 3, {n, f[n]}, ## &[]] , {n, 1, 10000}],
  ImageSize -> Large],
 Plot[3*x^(2/3), {x, 1, 10000}, PlotStyle -> Yellow]]
```

Listing 7: Figure 5

```
Show[
 ListPlot[
  Table[If[NumberofFactors[n] == 2, {n, f[n]}, ## &[]] , {n, 1, 10000}],
  ImageSize -> Large],
 Plot[y = x/2 + 2, {x, 1, 10000}, PlotStyle -> Orange]
 ]
```

Listing 8: Figure 6

```
Show[ListPlot[
  Table[If[NumberofFactors[n] == 3, {n, f[n]}, ## &[]], {n, 1, 10000}],
  ImageSize -> Large],
 Plot[x+4, {x, 1, 10000}, PlotStyle -> Yellow]]
```

Listing 9: Figure 9

```
Show[
 ListPlot[
  Table[If[NumberofFactors[n] == 2, {n, f[n]}, ## &[]] , {n, 1, 10000}],
  ImageSize -> Large],
 Table[Plot[y = x/Prime[n] + Prime[n], {x, 1, 10000},
   PlotStyle -> Yellow], {n, 1, 20}]
 ]
```

Listing 10: Figure 10

```
Show[
 ListPlot[
  Table[If[NumberofFactors[n] == 3, {n, f[n]}, ## &[]] , {n, 1, 10000}],
  ImageSize -> Large],
 Table[Plot[
   y = x/Prime[a] + x/Prime[b] + Prime[a]*Prime[b], {x, 1, 10000},
   PlotStyle -> Orange], {a, 1, 8}, {b, 1, 8}]
 ]
```

Listing 11: Predefined Functions for Fractions

```
fEgyptFrac[b_Integer] :=
-1/b * Sum[1/FactorInteger[b][[i]][[1]]* FactorInteger[b][[i]][[2]],
  {i, 1, Length[FactorInteger[b]]}]

fTotFrac[a_Integer, b_Integer] := fInteger[a]/b + fEgyptFrac[b]*a
```

Listing 12: Figure 11

```
ListPlot[DeleteDuplicates[
  Flatten[Table[{a/b, fTotFrac[a, b]}, {a, 1, 100}, {b, 1, 100}], 1]],
  ImageSize -> Large]
```

Listing 13: Figure 12

```
Show[ListPlot[
  DeleteDuplicates[
```

```
  Flatten[Table[{a/b, fTotFrac[a, b]}, {a, 1, 100}, {b, 1, 100}], 1]],
   ImageSize -> Large],
 Plot[-3 x, {x, 0, 200}, PlotStyle -> Orange]
 ]
```

Listing 14: Figure 13

```
Show[ListPlot[
  DeleteDuplicates [
   Flatten[Table[{a/b, fTotFrac[a, b]}, {a, 1, 500}, {b, 1, 500}], 1]],
    ImageSize -> Large],
 Plot[-4 x, {x, 0, 200}, PlotStyle -> Orange]
 ]
```

Listing 15: Figure 14

```
ListPlot [ DeleteDuplicates [
  Flatten[Table[{a/b, fTotFrac[a, b]}, {a, 1, 200}, {b, 1, 200}], 1]],
  ImageSize -> Large]
```