# The Bernstein-Landau paradox in an electrostatic plasma with an external magnetic field

A. Rege[1], F. Charles[1], B. Després[1], R. Weder[2]

[1]Laboratoire Jacques-Louis Lions
Sorbonne Université

[2]Universita Nacional Autonoma de Mexico

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# Outline

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# Kinetic formalism and Vlasov equations

- A plasma can be described statistically by considering the distribution function $f = f(t, x, v)$ = number density of particles located at position $x$, with velocity $v$ and at time $t$.
- If each particle is subject to an acceleration field $a = a(t, x, v)$ then we can deduce an equation on $f$ :

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + a(t,x,v) \nabla_v f(t,x,v) = 0 \qquad (1)$$

This is called a Vlasov equation when the vector field $(v, a(t, x, v))$ is coupled to another equation.

## The magnetized Vlasov-Ampère-Poisson system for electrons

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F \nabla_v f = 0, \\ F(t,x,v) = \frac{q}{m}(E(t,x) + v \wedge B), \\ \mathrm{rot}(B) = \mu_0 (j_{ion} - \int_{\mathbb{R}^3} v f dv + \epsilon_0 \partial_t E), \\ \partial_x E = \frac{q}{\epsilon_0}(\rho_{ion} - \int_{\mathbb{R}^3} f dv), \end{cases} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \omega_c \end{pmatrix}. \qquad (2)$$

## The Bernstein-Landau paradox

"In unmagnetized plasmas, waves exhibit Landau Damping, while in magnetized plasmas, waves perpendicular to the magnetic field are exactly undamped, independently of the strength of the magnetic field". [1]

- Several older physical papers [2] [1] and more recent mathematical papers [3] have studied the behaviour of magnetized plasmas.
- There seems to be a discontinuity between the theory of unmagnetized plasmas and the theory of magnetized plasmas.

1. A. I. Sukhorukov and P. Stubbe, On the Bernstein-Landau paradox, Phy. of Plasmas, 1997.
2. I. Bernstein, Waves in a Plasma in a Magnetic Field, Phy. Review, 1958.
3. J. Bedrossian and F. Wang, The linearized Vlasov and Vlasov-Fokker-Planck equations in a uniform magnetic field, Journal of Statistical Physics, 2020.

# Numerical illustration of the influence of $B$ : magnetic recurrence

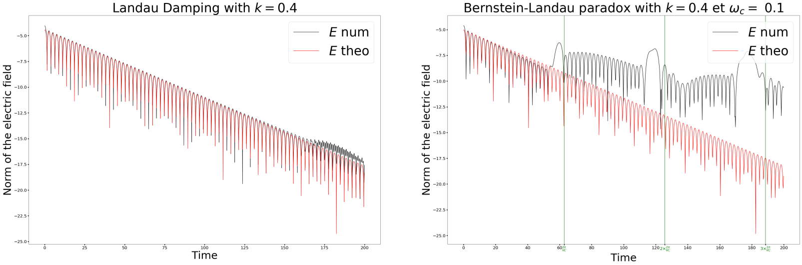


Figure – Damped and undamped electric field

Magnetic recurrence different from the numerical recurrence [4].

4. Recurrence phenomenon for Vlasov-Poisson simulations on regular finite element mesh, M. Mehrenberger, L. Navoret, N. Pham, Commun. Comput. Phys., 2020.

# Outline

# Spectral theory

- A general linear ordinary differential equation is given by $X'(t) = A(t)X(t) + B(t)$ with $X\colon I \to \mathbb{R}^n$, $B\colon I \to \mathbb{R}^n$ and $A\colon I \to \mathcal{M}_n(\mathbb{R})$ ($n \in \mathbb{N}^*$ and $I$ an interval of $\mathbb{R}$).
  If $A(t)$ is symmetric, one can solve the system by looking at the eigenvalues of $A(t)$ because

$$\mathbb{R}^n = \oplus_{\lambda \in Sp(A(t))} \ker(A(t) - \lambda).$$

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- More complicated for linear partial differential equations $\partial_t u(t,x) = H(t,x)u(t,x) + f(t,x)$ because now $u, f : I \to \mathcal{H}$ and $H\colon I \to \mathcal{B}(\mathcal{H})$.
  If $H(t)$ is self-adjoint (and $\mathcal{H}$ a Hilbert space), then we have the decomposition

$$\mathcal{H} = \mathcal{H}^{ac} \oplus \mathcal{H}^{sc} \oplus \mathcal{H}^{pp}$$

# Linearized Vlasov-Ampère system

$$\begin{cases} \partial_t u + v_1 \partial_x u + F v_1 e^{-\frac{v_1^2+v_2^2}{4}} + \omega_c(-v_2\partial_{v_1} + v_1 \partial_{v_2})u = 0, \\ \partial_t F = 1^* \int u e^{-\frac{v_1^2+v_2^2}{4}} v_1 dv_1 dv_2. \end{cases} \quad (3)$$

$$\text{with } 1^* g(x) = g(x) - \frac{1}{2\pi}\int_{\mathbb{T}} g(x)dx.$$

## Final formulation

$$\partial_t \begin{pmatrix} u \\ F \end{pmatrix} = iH \begin{pmatrix} u \\ F \end{pmatrix}, H = i \left( \begin{array}{c|c} v_1\partial_x + \omega_c(v_2\partial_{v_1} - v_1\partial_{v_2}) & v_1 e^{-\frac{v_1^2+v_2^2}{4}} \\ \hline -1^* \int v_1 e^{-\frac{v_1^2+v_2^2}{4}} \cdot dv_1 dv_2 & 0 \end{array} \right).$$

$$\mathcal{H} = \underbrace{(L^2(\mathbb{T}\times\mathbb{R}^2) \cap \left\{ \int u\sqrt{f_0}\, dx dv_1 dv_2 = 0 \right\})}_{=L_0^2(\mathbb{T}\times\mathbb{R}^2)} \times \underbrace{(L^2(\mathbb{T}) \cap \left\{ \int F dx = 0 \right\})}_{=L_0^2(\mathbb{T})}$$

# Spectral study : eigenvalues and eigenvectors

- We compute the eigenfunctions Fourier mode by Fourier mode.
- For a non-zero Fourier mode $n \neq 0$, the eigenspaces are as follows :

| Space | $\lambda$ | $m$ |
|---|---|---|
| $W_n^1 := \oplus_{m\in\mathbb{Z}^*} \left[ e^{mi\varphi - in\frac{v_2}{\omega_c}} V_{n,m} \times \{0\} \right]$ | $-m\omega_c$ | $m \neq 0$ |
| $W_n^2 := \oplus_{m\in\mathbb{Z}^*} \left\{ (e^{-in\frac{v_2}{\omega_c}} w_{n,m}, -ni) \right\}$ | $\lambda_m$ | $m \neq 0$ |
| $W_n^3 := \mathrm{Span}_\tau \left\{ (e^{-in\frac{v_2}{\omega_c}} \tau(r), 0) \right\} + \left\{ (e^{-\frac{r^2}{4}}, -in) \right\}$ | $0$ | |

- The eigenspaces corresponding to $n = 0$ are :

| Space | $\lambda$ | $m$ |
|---|---|---|
| $W_0^1 := \oplus_{m\in\mathbb{Z}^*} \left[ e^{mi\varphi} L^2(\mathbb{R}^+) \times \{0\} \right]$ | $-m\omega_c$ | $m \neq 0$ |
| $W_0^3 := \oplus_{p\in\mathbb{N}^*} \left[ \{\tau_p\} \times \{0\} \right]$ | $0$ | $p > 0$ |

# Spectral study : discrete spectrum

## Theorem

*We have completeness of the eigenspaces .*

$$L^2_0(\mathbb{T}\times\mathbb{R}^2)\times L^2_0(\mathbb{T}) = \oplus_{n\neq 0}\left[e^{inx}\left(W^1_n\oplus W^2_n\oplus W^3_n\right)\right]\oplus\left[L^2_0(\mathbb{R}^2)\times 0\right]$$

*and so the eigenvalues of $H$ are $0$, $-m\lambda_c$ and $\lambda_m$, $m\neq 0$.*

- This shows that $H$ can be fully diagonalized $\Rightarrow$ their is only discrete spectrum $V=\mathcal{H}^{pp}$.
- New result for this kind of system [5].

5. J. Bedrossian and F. Wang, The linearized Vlasov and Vlasov-Fokker-Planck equations in a uniform magnetic field, Journal of Statistical Physics, 2020.

## Spectral explanation for the Bernstein-Landau paradox [6]

1. The Vlasov-Ampère operator $H$ is self-adjoint and it has a complete set of eigenfunctions $\Rightarrow$ electric field is undamped. Expression of electric field with the eigenvectors and eigenvalues :

$$F_n(t) = -nie^{nix} \sum_{m \neq 0} \frac{\left\langle u_0, e^{-in\frac{v_2}{\omega_c}} w_{n,m} \right\rangle + niF_0}{\left\| e^{-in\frac{v_2}{\omega_c}} w_{n,m} \right\|^2 + n^2} e^{i\lambda_m t}$$

2. The Vlasov system without magnetic field has only absolutely continuous spectrum and a kernel $\Rightarrow$ electric field goes to 0.

6. F. Charles, B. Després, A. Rege, R. Weder, The Vlasov-Ampère system and the Bernstein-Landau paradox, submitted to Journal of Statistical Physics, 2020.

# Outline

1. Motivation : the Bernstein-Landau paradox
2. Spectral decomposition of a linearized Vlasov-Ampère system
3. Numerical study with a Semi-Lagrangian scheme : construction of reference solutions

**Objective** : compare the numerical and theoretical solutions of Vlasov-Ampère when initializing with an eigenvector.

- We consider an eigenvector $U_{n,m} = \begin{pmatrix} w_{n,m} \\ F_n \end{pmatrix}$ associated to the Fourier mode $n \neq 0$ and the eigenvalue $\lambda_m$.
- $w_{n,m}$ and $F_n$ are given by

$$w_{n,m} = e^{in(x - \frac{v_2}{\omega_c})} e^{-\frac{r^2}{4}} \sum_{p \in \mathbb{Z}^*} \frac{p\omega_c}{p\omega_c + \lambda_m} e^{pi\varphi} J_p\left(\frac{nr}{\omega_c}\right) \text{ and } F_n = -ine^{inx}$$

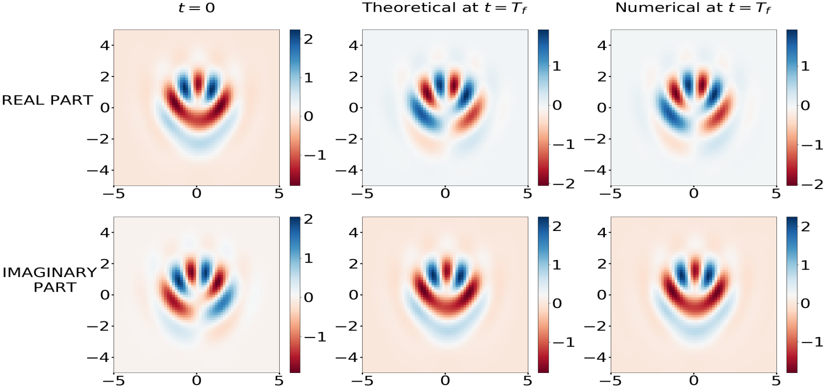
- $\lambda_m$ is one of the roots of a secular equation given by :

$$g(\lambda) = -1 - \frac{2\pi}{n^2} \sum_{m \in \mathbb{Z}^*} \frac{m\omega_c}{m\omega_c + \lambda} \int_0^\infty e^{-\frac{r^2}{2}} J_m\left(\frac{nr}{\omega_c}\right)^2 rdr = 0. \quad (4)$$

- $\partial_t U = i\lambda_m U \Rightarrow U = e^{i\lambda_m t} U_{n,m}$

# Linear Vlasov-Ampère : Numerical results for $u$ with $T_{end} = \frac{\pi}{2\lambda_m}$

For all of the simulations, $N_x = 33$, $N_{v_1} = N_{v_2} = 63$, $L_x = 2\pi$, $L_{v_1} = L_{v_2} = 10$ and we have taken $\omega_c = 0.5$ and $n = 1$.



# Linear Vlasov-Ampère : Numerical results for $u$ and $F$ for $T_{end} = \frac{\pi}{2\lambda_m}$

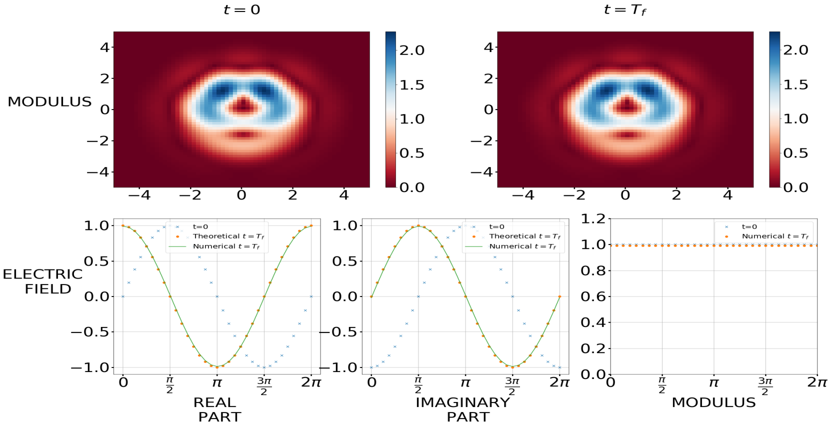



Figure – Module of $u$ in V1-V2 plane for $x = 0$ and real and imaginary parts of $F$.

# Non-linear Vlasov-Poisson : Numerical results for $u$ and $F$ with $T_{end} = \frac{\pi}{2\lambda_m}$

For all of the simulations, we have taken $\omega_c = 0.5$ and $n = 1$.

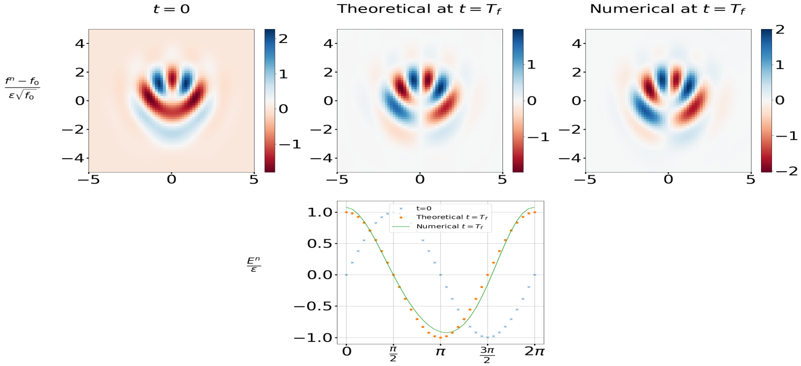



Figure – $\frac{f - f_0}{\varepsilon\sqrt{f_0}} \approx u$ in V1-V2 plane for $x = 0$ and $\frac{E}{\varepsilon} \approx F$.

# Summary and perspectives

- Spectral decomposition of the Vlasov-Ampère system
- Reinterpretation of the Bernstein-Landau paradox as a AC spectrum versus PP spectrum.
- Constructed new reference solutions that can be tested on linear and non-linear schemes.

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## Perspective

Limit $\omega_c \to 0$.
Mathematical difficulty :

$$w_{n,m} = e^{in(x - \frac{v_2}{\omega_c})} e^{-\frac{r^2}{4}} \sum_{p \in \mathbb{Z}^*} \frac{p\omega_c}{p\omega_c + \lambda_m} e^{pi\varphi} J_p\left(\frac{nr}{\omega_c}\right)$$

There is a singularity at $\omega_c = 0$.