Propagation of velocity moments for the magnetized Vlasov–Poisson system

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ETH

D MATH

The magnetized Vlasov–Poisson system

Constant B

Non-constant B

The magnetized Vlasov–Poisson system

Constant B

Non-constant B

The magnetized Vlasov–Poisson system for electrons

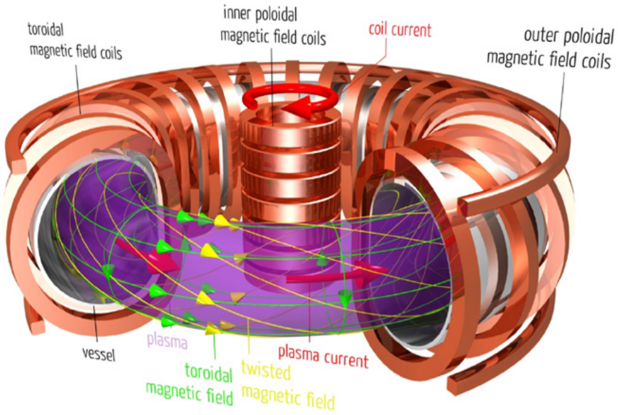
At the time scale of electrons:

$$
\left\{
\begin{array}{l}
\partial_t f + v \cdot \partial_x f + \frac{q_e}{m_e} \left(E + v \wedge B \right) \cdot \partial_v f = 0, \\
\mathrm{div}_x E(t,x) = \frac{q_{ion}}{\epsilon_0} \underbrace{\int_{\mathbb{R}^3} f_{ion}(x,v) dv}_{=:\rho_{ion}} + \frac{q_e}{\epsilon_0} \underbrace{\int_{\mathbb{R}^3} f(t,x,v) dv}_{=:\rho(t,x)}, \\
E(t,x) = -\partial_x \phi(t,x), \\
-\Delta_x \phi = \frac{q_e}{\epsilon_0} \rho + \frac{q_{ion}}{\epsilon_0} \rho_{ion}.
\end{array}
\right.
\tag{VPB}
$$

with $f \equiv f(t,x,v)$ the distribution function of electrons, $f_{ion}(x,v)$ the constant ion distribution, $E \equiv E(t,x)$ the self-consistent electric field, ϕ the electrostatic potential and $B \equiv B(t,x)$ the external magnetic field.

Magnetic confinement fusion

- Use an intense external (not self-induced) magnetic field B to confine the hot plasma.
- Feasibility of controlled nuclear fusion: ITER tokamak under construction in Cadarache, France.



A priori estimates

- Conservation of L^p norms of f:
  $\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v\left((E + v \wedge B)f\right) = 0$, so for $1 \leq p \leq \infty$ and $t \geq 0$,
  $$\|f(t)\|_p = \left\|f^{in}\right\|_p .$$
- Local conservation of charge:
  $$\partial_t \rho(t,x) + \mathrm{div}_x \underbrace{\left(\int_{\mathbb{R}^3} vf(t,x,v)dv\right)}_{=:j(t,x)} = 0.$$
- Conservation of energy:
  $$\frac{d}{dt}\left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{2}|v|^2 f(t,x,v)dxdv + \int_{\mathbb{R}^3} \frac{1}{2}|E(t,x)|^2 dx\right) = 0.$$

Results on existence of solutions in the unmagnetized case

- Existence of weak solutions in dimension 3 [Arsenev, 1975]
- Small initial data in dimension 3 [Bardos, Degond, 1985]
- Existence of smooth solutions in dimension 3 [Pfaffelmoser, 1992]
- Propagation of velocity moments with Eulerian approach in dimension 3 [Lions, Perthame, 1991]
- Propagation of velocity moments with Lagrangian approach in dimension 3 [Pallard, 2012]

We will first consider a constant B and then a non-constant uniform B

$$B := (0, 0, \omega), \quad B := B(t).$$

The magnetized Vlasov–Poisson system

Constant B

Non-constant B

Propagation of velocity moments

Theorem (Lions, Perthame 1991)

Let $k_0 > 3$, $T > 0, f^{in} = f^{in}(x, v) \geq 0$ *a.e. with* $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and assume that*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{k_0} f^{in} dxdv < \infty.$$

Then there exists a weak solution f *to the Vlasov–Poisson system and*

$$C = C\left(T, k_0, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}^{in}, \iint |v|^{k_0} f^{in}\right)$$

such that

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{k_0} f(t, x, v) dxdv \leq C < +\infty, \quad 0 \leq t \leq T.$$

What happens for (VPB) with $B := (0, 0, \omega)$?

Differential inequality on M_k

For $k \geq 0$, we write

$$M_k(t) := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t,x,v) dvdx.$$

$$\begin{aligned}
|\frac{d}{dt} M_k(t)| &= |\iint |v|^k (-v \cdot \partial_x f - (E + v \wedge B) \cdot \partial_v f) dvdx|, \\
&= |\iint |v|^k \, \mathrm{div_v} \left((E + v \wedge B) f \right) dvdx|, \\
&= |\iint k|v|^{k-2} v \cdot (E + v \wedge B) f dvdx|, \\
&\leq \int \left(\int k|v|^{k-1} f dv \right) |E| dx \\
&\leq C \, \|E(t)\|_{k+3} \, M_k(t)^{\frac{k+2}{k+3}}.
\end{aligned}$$

Next step: we need to control of $\|E(t)\|_{k+3}$ with $M_k(t)^\alpha$ with $\alpha \leq \frac{1}{k+3}$.

A representation formula for ρ (I)

We rewrite the Vlasov equation

$$\partial_t f + v \cdot \partial_x f + (v \wedge B) \cdot \partial_v f = -E \cdot \partial_v f.$$

The associated characteristic system is given by

$$\begin{cases} \dot{X}(s) = V(s), \dot{V}(s) = V(s) \wedge B = (\omega V_2(s), -\omega V_1(s), 0), \\ (X(t), V(t)) = (x, v). \end{cases}$$

$$\begin{cases} V(s) = \begin{pmatrix} v_1 \cos(\omega(s-t)) + v_2 \sin(\omega(s-t)) \\ -v_1 \sin(\omega(s-t)) + v_2 \cos(\omega(s-t)) \\ v_3 \end{pmatrix}, \\ X(s) = \begin{pmatrix} x_1 + \frac{v_1}{\omega} \sin(\omega(s-t)) + \frac{v_2}{\omega}(1 - \cos(\omega(s-t))) \\ x_2 + \frac{v_1}{\omega}(\cos(\omega(s-t)) - 1) + \frac{v_2}{\omega} \sin(\omega(s-t)) \\ x_3 + v_3(s-t)) \end{pmatrix}. \end{cases}$$

A representation formula for ρ (II)

We apply the Duhamel formula

$$f(t,x,v) = f^{in}(X(0)), V(0)) - \int_0^t \mathrm{div}_v(fE)(s, X(s), V(s))ds,$$

We also have

$$\begin{aligned} &\mathrm{div}_v \int_0^t (fG_t)(s, X(s), V(s))ds \\ &= \int_0^t \mathrm{div}_v(fE)(s, X(s), V(s))ds + \mathrm{div}_x \int_0^t (fH_t)(s, X(s), V(s))ds, \end{aligned}$$

so that

$$\begin{aligned} f(t,x,v) = f^{in}(X(0), V(0)) + \mathrm{div}_x \int_0^t (fH_t)(s, X(s), V(s))ds \\ - \mathrm{div}_v \int_0^t (fG_t)(s, X(s), V(s))ds. \end{aligned}$$

$$\rho(t,x) = \underbrace{\int_v f^{in}(X(0), V(0))dv}_{=:\rho_0(t,x)} + \mathrm{div}_x \int_0^t \underbrace{\int_v (fH_t)\,(s, X(s), V(s))\,dv}_{=:\sigma(s,t,x)} ds.$$

Controlling E with σ

We can thus split E

$$\begin{aligned} E(t,x) &= -\left(\partial_x \frac{1}{|x|} \star \rho\right)(t,x) \\ &= E_{\rho_0}(t,x) + E_\sigma(t,x). \end{aligned}$$

Thanks to the Calderón–Zygmund inequality we have

$$\|E_\sigma(t)\|_{k+3} \leq \int_0^t \|\sigma(s,t,x)\|_{k+3}\, ds.$$

And we can bound E^0 uniformly

$$\|E_{\rho_0}(t)\|_{k+3} \leq C(k, \left\|f_1^{in}\right\|, M_k(f^{in})).$$

Singularities at multiples of the cyclotron frequency

We have

$$|\sigma(s,t,x)| \le c \left\| H_t(s,X(s),V(s)\right\|_{L_v^{\frac{3}{2},w}} \|f\|_\infty^{\frac{2}{3}} \|f(t-s,X(s),\cdot)\|_{L_v^1}^{\frac{1}{3}},$$

which yields

$$\|\sigma(s,t,\cdot)\|_{k+3} \le C\frac{\sqrt{2}}{s}\left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))}\right)^{\frac{2}{3}} M_k(t-s)^{\frac{1}{k+3}}.$$

Proposition (Propagation of moments on a fixed interval, R. 2021)

For all $0 \le t \le T_\omega := \frac{\pi}{\omega}$ *we have*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f(t,x,v)dxdv \le C < +\infty,$$

with $C = C(k,\omega,\left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

Dealing with the singularity in 0 (I)

We deal with the singularity at 0 by writing

$$\left\| \int_0^t \sigma(s,t,x)ds \right\|_{k+3} = \left\| \int_0^{t_0} \dots \right\|_{k+3} + \left\| \int_{t_0}^{t} \dots \right\|_{k+3}$$

with t_0 a "small time".
Rough estimate in small time

$$\left\| \int_0^{t_0} \sigma(s,t,x)ds \right\|_{k+3} \leq (1+t)^{\delta} t_0^{\beta} \left(1 + \sup_{0 \leq s \leq t} M_k(s) \right)^{\alpha}, \alpha > \frac{1}{k+3}, \beta > 0.$$

Dealing with the singularity in 0 (II)

Precise estimate in large time

$$\left\| \int_{t_0}^{t} \sigma(s,t,\cdot) ds \right\|_{k+3} \le C \ln\left(\frac{t}{t_0}\right) \sup_{0\le s\le t} M_k(s)^{\frac{1}{k+3}}.$$

Optimizing in t_0, we obtain

$$\|E(t,\cdot)\|_{k+3} \le C \left(1 + \sup_{0\le s\le t} M_k(s)\right)^{\frac{1}{k+3}} \left(1 + \ln\left(1 + \sup_{0\le s\le t} M_k(s)\right)\right).$$

Propagation of moments for all time

We have that

- $\left\|f^{in}\right\|_1 = \|f(T_\omega)\|_1$ and $\left\|f^{in}\right\|_\infty = \|f(T_\omega)\|_\infty$,
- $\mathcal{E}(T_\omega) \leq \mathcal{E}_{in}$,
- $M_k(f(T_\omega)) \leq C(k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

This means $f(T_\omega)$ verifies the same assumptions as $f^{in} \implies$ we can show propagation of moments for all time by induction.

The magnetized Vlasov–Poisson system

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Non-constant B

Lagrangian formulation for propagation of velocity moments

Characteristic system of the Vlasov–Poisson system

$$\dot{X}(s) = V(s), \dot{V}(s) = E(s, X(s)).$$

Consider the quantity

$$Q(t,\delta) := \sup\left\{ \int_{t-\delta}^{t} |E(s, X(s;0,x,v))|ds, (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \right\},$$

then

$$\begin{aligned} M_k(t) &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} |V(t;0,x,v)|^k f^{in}(x,v)dvdx, \\ &\leq \iint_{\mathbb{R}^3\times\mathbb{R}^3} \left(|v| + \left(\sup_{0\leq t\leq T} Q(t,t)\right)\right)^k f^{in}(x,v)dvdx \\ &\leq 2^{k-1}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f^{in}(x,v)dvdx + \left(\sup_{0\leq t\leq T} Q(t,t)\right)^k \left\|f^{in}\right\|_1\right) \end{aligned}$$

Propagation of moments of order $k > 2$

Theorem (Pallard, 2012)

Let $T > 0, k > 2$, and f^{in} such that $M_k(f^{in}) < +\infty$, then for all $0 \leq t \leq T$,

$$Q(t,t) \leq C(T^{\frac{1}{2}} + T^{\frac{7}{5}}),$$

with $C = C(k, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

The good, the bad, and the ugly 2

$$\int_{t-\delta}^{t} |E(s, X(s; t, x_*, v_*))| ds \le \int_{t-\delta}^{t} \iint \frac{f(s,x,v) dv dx}{|x - X_*(s)|^2} ds.$$

$$\begin{aligned} G &= \{(s,x,v) : \min(|v|, |v - V_*(s)|) < P\}, \\ B &= \{(s,x,v) : |x - X_*(s)| \le \Lambda_\varepsilon(s,v)\} \setminus G, \\ U &= [t-\delta, t] \times \mathbb{R}^3 \times \mathbb{R}^3 \setminus (G \cup B), \end{aligned}$$

with $P = 2^{10} Q(t,\delta)$ and $\Lambda_\varepsilon(s,v) = L(1 + |v|^{2+\varepsilon})^{-1} |v - V_*(s)|^{-1}$.
Estimate on U:

$$I_U^*(t,\delta) \le L^{-1}(1 + M_{2+\varepsilon}(T)).$$

Adding a general magnetic field

Now we have $B := B(t,x)$ such that

$$B \in L^\infty(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}^3)),$$

$$\dot{X}(s) = V(s), \dot{V}(s) = E(s, X(s)) + V(s) \wedge B(s, X(s)).$$

So

$$V(t;0,x,v) = v + \int_0^t \left(E(s, X(s;0,x,v)) + V(s;0,x,v) \wedge B(s, X(s;0,x,v))\right) ds$$

$$\implies |V(t;0,x,v)| \leq (|v| + \sup_{0\leq t\leq T} Q(t,t)) \exp(t \|B\|_\infty), \text{ which yields}$$

$$M_k(t) \leq 2^{k-1} \exp(kt \|B\|_\infty) \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f^{in} x + \left(\sup_{0\leq t\leq T} Q(t,t)\right)^k \left\|f^{in}\right\|_1 \right).$$

Non-constant uniform B

Assume

$$B \coloneqq B(t).$$

Theorem

For all time t such that $0 \leq t \leq T_B$,

$$Q(t,t) \leq C \exp(T_B \left\|B\right\|_\infty)^{\frac{2}{5}} (T_B^{\frac{1}{2}} + T_B^{\frac{7}{5}}),$$

with $C = C(k, \left\|B\right\|_\infty, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

$$\begin{aligned}
G &= \{(s,x,v) : \min(|v|, |v - V_*(s)|) < P\},\\
B &= \{(s,x,v) : |x - X_*(s)| \leq \Lambda_\varepsilon(s,v)\} \setminus G,\\
U &= [t-\delta, t] \times \mathbb{R}^3 \times \mathbb{R}^3 \setminus (G \cup B),
\end{aligned}$$

with
$P = 2^{10} Q(t,\delta) \exp(\delta \left\|B\right\|_\infty)$ and $\Lambda_\varepsilon(s,v) = L(1+|v|^{2+\varepsilon})^{-1} |v - V_*(s)|^{-1}$.

Estimate on the ugly set

Lemma

Let $s_1 \in [t-\delta\,;t]$ such that $(s_1, X(s_1), V(s_1)) \in U$, then for all $s \in [t-\delta\,;t]$ we have

$$2^{-1}|v| \leq |V(s)| \leq 2|v|,$$

and

$$2^{-1}|v-v_*| \leq |V(s)-V_*(s)| \leq 2|v-v_*|.$$

which yields

$$\int_{t-\delta}^{t} \frac{\mathbf{1}_U(s, X(s), V(s))}{|X(s)-X_*(s)|^2}\, ds \leq C\left(\frac{1+|v|^{2+\varepsilon}}{L}\right).$$

Non constant, non-uniform B

Assume

$$B = B(t, x)$$

Difficulty to control the difference between velocity characteristics $|V(s) - V_*(s)|$ in terms of $Q(t, \delta)$ and $|V(s) - V_*(s)|$:

$$\begin{aligned} |V(s) - V_*(s)| \leq {} & |v - v_*| + 2Q(t, \delta) \\ & + \int_s^t |V(s) \wedge B(s, X(s)) - V_*(s) \wedge B(s, X_*(s))| ds. \end{aligned}$$

Uniqueness for bounded ρ

Uniqueness with bounded charge density (Loeper, 2006)

Let $T > 0$ and $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, then there exists at most one solution to the Vlasov–Poisson system such that

$$\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3)} < +\infty.$$

Example: $f^{in}(x, v) = \frac{\phi(x)}{1+|v|^4}$ with $\phi \in L^1 \cap L^\infty(\mathbb{R}^3)$.

Uniqueness for unbounded ρ

Uniqueness with unbounded charge density (Miot, 2016)

Let $T > 0$ and $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, then there exists at most one solution to the Vlasov–Poisson system such that

$$\sup_{[0,T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty.$$

Example: $f^{in}(x,v) = \phi\left(|v|^2 - (\ln_-|x|)^{2/3}\right)$ with $\phi \in L^\infty(\mathbb{R}, \mathbb{R}_+)$.

Characteristics with a general B

Now we have $B := B(t, x)$ such that

$$B \in L^\infty_{loc}\left(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}^3)\right). \tag{1}$$

$$\begin{cases} \dot{X}(s; t, x, v) = V(s; t, x, v), \\ \dot{V}(s; t, x, v) = E(s, X(s; t, x, v)) + V(s; t, x, v) \wedge B(s, X(s; t, x, v)). \end{cases}$$

Consider two solutions f_1, f_2 with $(X_1, V_1), (X_2, V_2)$ the corresponding characteristics. We study the distance

$$D(t) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |X_1(t, x, v) - X_2(t, x, v)| f^{in}(x, v) dx dv. \tag{2}$$

Additional terms in the magnetized case

$$\begin{aligned}
D(t) \leq & \int_0^t \int_0^s \int_{\mathbb{R}^6} |E_1(\tau, X_1(\tau)) - E_2(\tau, X_2(\tau))| \\
& + |V_1(\tau) \wedge B(\tau, X_1(\tau)) - V_2(\tau) \wedge B(\tau, X_2(\tau))| f^{in}(x,v) dx dv d\tau ds, \\
& \leq \int_0^t \int_0^s \int_{\mathbb{R}^6} |E_1(\tau, X_1(\tau)) - E_2(\tau, X_2(\tau))| f^{in}(x,v) dx dv d\tau ds \\
& + \|B\|_\infty \int_0^t \int_0^s \int_{\mathbb{R}^6} |V_1(\tau) - V_2(\tau)| f^{in}(x,v) dx dv d\tau ds \\
& + \int_0^t \int_0^s \int_{\mathbb{R}^6} |V_2(\tau)| |B(\tau, X_1(\tau)) - B(\tau, X_2(\tau))| f^{in}(x,v) dx dv d\tau ds, \\
& = I(t) + J(t) + K(t).
\end{aligned}$$

The additional terms $J(t), K(t)$ are controlled with $\|\rho_1\|_p, \|\rho_2\|_p$ and velocity moments of f^{in}.

Uniqueness criterion with a general magnetic field

Theorem (R.)

Let $T > 0$ and $B := B(t, x)$ such that $B \in L^{\infty}_{loc}\left(\mathbb{R}_+, W^{1,\infty}(\mathbb{R}^3)\right)$.
If f^{in} satisfies

$$\forall k \geq 1, \quad \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f^{in}(x, v) dx dv \leq (C_0 k)^{\frac{k}{3}}, \text{[1]} \tag{3}$$

with C_0 a constant independent of k, then there exists at most one solution $f \in L^{\infty}([0\,,T], L^1 \cap L^{\infty}(\mathbb{R}^3 \times \mathbb{R}^3))$ to the Cauchy problem for the magnetized Vlasov–Poisson system. If such a solution exists then it will verify

$$\sup_{[0\,,T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty. \tag{4}$$

[1] E. Miot, A uniqueness criterion for unbounded solutions to the Vlasov–Poisson system, Comm. Math. Phys., 2016.

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Weak solutions

A function $f : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ is a weak solution of (VPB) if we have

- $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^2(\mathbb{R}^3 \times \mathbb{R}^3))$.
- $|v|^2 f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$.
- $\partial_t f + v \cdot \partial_x f + (E + v \wedge B) \cdot \partial_v f = 0$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$.

We will consider solutions such that

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{k_0} f^{in} dx dv < +\infty,$$

$$\implies \mathcal{E}^{in} = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{2}|v|^2 f^{in} dx dv + \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{2}|E^{in}|^2 dx dv < +\infty.$$

- Existence of smooth solutions in dimension 1 [Iordanskii, 1964]
- Existence of weak solutions in dimension 3 [Arsenev, 1975]
- Existence of smooth solutions in dimension 2 [Okabe, Ukai, 1978]
- Small initial data in dimension 3 [Bardos, Degond, 1985]
- Existence of smooth solutions in dimension 3 [Pfaffelmoser, 1992]
- Propagation of velocity moments with Eulerian approach in dimension 3 [Lions, Perthame, 1991]
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Propagation of regularity

Theorem (Propagation of regularity)

Let $h \in C^1(\mathbb{R})$ such that

$$h \geq 0, h' \leq 0 \text{ and } h(r) = \mathcal{O}(r^{-\alpha}) \text{ with } \alpha > 3,$$

and let $f^{in} \in C^1(\mathbb{R}^3)$ be a probability density on $\mathbb{R}^3 \times \mathbb{R}^3$ such that $f^{in}(x, v) \leq h(|v|)$ for all x, v and which verifies

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{k_0}) f^{in}(x, v) dxdv < \infty$$

with $k_0 > 6$.
Then there exists a unique classical solution f to (VPB) *with $f(0) = f^{in}$.*