Applying spectral theory to the study of the Bernstein-Landau paradox

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Outline

1. Motivation : the Bernstein-Landau paradox
2. Spectral decomposition of a linearized Vlasov-Ampère system
3. Numerical study with a Semi-Lagrangian scheme : construction of reference solutions

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The paradox

The Bernstein-Landau paradox

"In unmagnetized plasmas, waves exhibit Landau Damping, while in magnetized plasmas, waves perpendicular to the magnetic field are exactly undamped, independently of the strength of the magnetic field". [1]

- Several older physical papers [2] [1] and more recent mathematical papers [3] have studied the behaviour of magnetized plasmas.
- There seems to be a discontinuity between the theory of unmagnetized plasmas and the theory of magnetized plasmas.

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1. A. I. Sukhorukov and P. Stubbe, On the Bernstein-Landau paradox, Phy. of Plasmas, 1997.
2. I. Bernstein, Waves in a Plasma in a Magnetic Field, Phy. Review, 1958.
3. J. Bedrossian and F. Wang, The linearized Vlasov and Vlasov-Fokker-Planck equations in a uniform magnetic field, Journal of Statistical Physics, 2020.

Linear Landau damping

Definition of Landau damping

Exponential decrease in time of longitudinal waves in plasma.

- First predicted by Landau with the linearized Vlasov-Poisson system where he showed damping of the electric field E.[4]

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{q}{m} E \cdot \nabla_v f_0 = 0, \\ \partial_x E = \frac{q}{\epsilon_0} \int f dv_1 dv_2. \end{cases} \tag{1}$$

  with $f = f(t, x, v)$, $(t, x, v) \in \mathbb{R}^+ \times [0\,;2\pi]^3_{per} \times \mathbb{R}^3$ the distribution of electrons and $f_0 = e^{-\frac{|v|^2}{2}}$ the equilibrium Maxwellian distribution.
- Using a Laplace transform, Landau showed the damping of the electric field.

$$|E_k(t)| \leq Ce^{-\alpha_k t} \tag{2}$$

- Irreversible behavior observed in a reversible in time system.

4. L. Landau, On the vibration of the electronic plasma, J. Phys. USSR, 1946.

Numerical illustration of the influence of B : the model

1d-2v Vlasov-Poisson system with magnetic field

$$\begin{cases} \partial_t f + v_1 \partial_x f - E \partial_{v_1} f + \underbrace{\omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2} \right) f}_{= -v \wedge B} = 0, \\ \\ \partial_x E = 2\pi - \int f dv_1 dv_2. \end{cases} \tag{3}$$

Here $\omega_c > 0$ is the constant cyclotron frequency for electrons ($B = (0, 0, \omega_c)$). The unknowns are the density of electrons $f(t, x, v_1, v_2) \geq 0$ and the electric field $E(t, x)$. The domain is $\Omega = \mathbb{T} \times \mathbb{R}^2, \quad \mathbb{T} = [0, 2\pi]_{\text{per}}$ is the 1D-torus.

Ions are considered as a motionless background of neutralizing positive charge.

Numerical illustration of the influence of B : magnetic recurrence

Initialization : $f_0(x, v_1, v_2) = (1 + \varepsilon \cos kx) \exp\left(-\frac{v_1^2+v_2^2}{2}\right)$ with $\varepsilon = 0.001$.

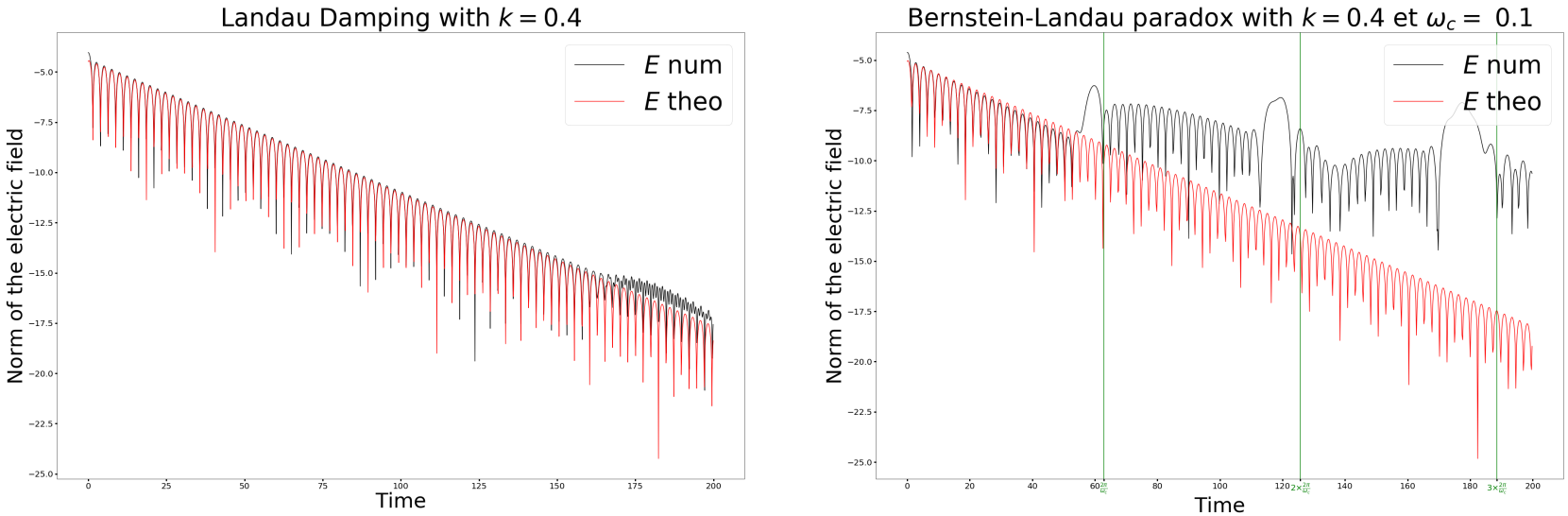

Figure – Damped and undamped electric field

Magnetic recurrence different from the numerical recurrence [5].

5. Recurrence phenomenon for Vlasov-Poisson simulations on regular finite element mesh, M. Mehrenberger, L. Navoret, N. Pham, Commun. Comput. Phys., 2020.

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Linearized system

We linearize (3) by writing

$$f = f_0 + \varepsilon\sqrt{f_0}u + O(\varepsilon^2) \text{ and } E = \varepsilon F + O(\varepsilon^2).$$

where $(f_0, E_0) = (\exp(-\frac{v_1^2+v_2^2}{2}), 0)$ is a stationary solution of (3).

Linearized Vlasov-Poisson with magnetic field

$$\begin{cases} \partial_t u + v_1\partial_x u + Fv_1 e^{-\frac{v_1^2+v_2^2}{4}} + \omega_c\left(-v_2\partial_{v_1} + v_1\partial_{v_2}\right)u = 0, \\ \partial_x F = -\int u e^{-\frac{v_1^2+v_2^2}{4}} dv_1 dv_2. \end{cases} \tag{4}$$

- $\int u e^{-\frac{v_1^2+v_2^2}{4}} dx dv_1 dv_2 = 0$. (total mass equal zero)
- $\int F dx = 0$. (F is derived from a potential)

Scattering theory

Scattering theory : consider two self-adjoint operators H_0 and H on a Hilbert space $\mathcal{H}$ that are "close" in some sense, then we expect the spectral properties of H to also be close to those of $H_0 \Rightarrow$ the dynamics of $U'(t) = iHU$ similar to the dynamics of $U'(t) = iH_0U$.

- For the self-adjoint operator H, we have the following decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}^{ac} \oplus \mathcal{H}^{sc} \oplus \mathcal{H}^{pp}$. [6] [7]
- Scattering structures of inhomogeneous linear Vlasov equations are studied in [8] [9] where it was shown that for Vlasov-Poisson without magnetic field, there is only the absolutely continuous part $\mathcal{H}^{ac}$ and a kernel $\Rightarrow E \to 0$ weakly.

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6. T. Kato, Perturbation theory for linear operators, 1966.
7. D.R. Yafaev, Scattering theory : Some old and new problems, 2000.
8. B. Després, Scattering structure and Landau damping for linearized Vlasov eq. with inhomogeneous Boltzmannian states, Ann. IHP, 2019.
9. B. Després, Trace class properties of the linear Vlasov-Poisson equation, accepted in Journal of Spectral theory.

No self-adjointness with Vlasov-Poisson

- We write the linearized system in the framework of scattering theory :

$$\partial_t u = iH_{vp}u, \quad i^2 = -1$$

with $H_{vp}u = i(v_1\partial_x + \omega_c(-v_2\partial_{v_1} + v_1\partial_{v_2}))u + iFv_1 e^{-\frac{v_1^2+v_2^2}{4}}$

and $\partial_x F = -\int u e^{-\frac{v_1^2+v_2^2}{4}} dv_1 dv_2$.

- Unfortunately,

$$H_{vp}^* \neq H_{vp}.$$

- Solution : rewrite the system with the Ampère equation (both systems are equivalent).

Linearized Vlasov-Ampère system

$$\begin{cases} \partial_t u + v_1 \partial_x u + F v_1 e^{-\frac{v_1^2+v_2^2}{4}} + \omega_c(-v_2\partial_{v_1} + v_1\partial_{v_2})u = 0, \\ \partial_t F = 1^* \int u e^{-\frac{v_1^2+v_2^2}{4}} v_1 dv_1 dv_2. \end{cases} \tag{5}$$

$$\text{with } 1^* g(x) = g(x) - \frac{1}{2\pi}\int_{\mathbb{T}} g(x)dx.$$

Final formulation

$$\partial_t \begin{pmatrix} u \\ F \end{pmatrix} = iH \begin{pmatrix} u \\ F \end{pmatrix}, H = i \left(\begin{array}{c|c} v_1\partial_x + \omega_c(v_2\partial_{v_1} - v_1\partial_{v_2}) & v_1 e^{-\frac{v_1^2+v_2^2}{4}} \\ \hline -1^* \int v_1 e^{-\frac{v_1^2+v_2^2}{4}} \cdot dv_1 dv_2 & 0 \end{array} \right).$$

$$\mathcal{H} = \underbrace{(L^2(\mathbb{T}\times\mathbb{R}^2) \cap \left\{ \int u\sqrt{f_0} dx dv_1 dv_2 = 0 \right\})}_{=L^2_0(\mathbb{T}\times\mathbb{R}^2)} \times \underbrace{(L^2(\mathbb{T}) \cap \left\{ \int F dx = 0 \right\})}_{=L^2_0(\mathbb{T})}$$

Self-adjointness with the Vlasov-Ampère formulation

$$H = i \left(\begin{array}{c|c} v_1 \partial_x + \omega_c (v_2 \partial_{v_1} - v_1 \partial_{v_2}) & v_1 e^{-\frac{v_1^2+v_2^2}{4}} \\ \hline -1^* \int v_1 e^{-\frac{v_1^2+v_2^2}{4}} \cdot dv_1 dv_2 & 0 \end{array} \right) = H_0 + V$$

with

$$\left\{ \begin{array}{l} H_0 = i \left(\begin{array}{c|c} v_1 \partial_x + \omega_c (v_2 \partial_{v_1} - v_1 \partial_{v_2}) & 0 \\ \hline 0 & 0 \end{array} \right), \\ V = i \left(\begin{array}{c|c} 0 & v_1 e^{-\frac{v_1^2+v_2^2}{4}} \\ \hline -1^* \int v_1 e^{-\frac{v_1^2+v_2^2}{4}} \cdot dv_1 dv_2 & 0 \end{array} \right). \end{array} \right.$$

- H_0 is unitarily equivalent to a maximal symmetric multiplication operator $\Rightarrow$ H_0 is self-adjoint
- V is bounded in $\mathcal{H}$, symmetric and $D[V] = \mathcal{H} \Rightarrow H$ is self-adjoint thanks to the Kato-Rellich theorem.

Spectral study : eigenvalues and eigenvectors

- We compute the eigenfunctions Fourier mode by Fourier mode.
- For a non-zero Fourier mode $n \neq 0$, the eigenspaces are as follows :

Space	λ	m
$W_n^1 := \oplus_{m\in\mathbb{Z}^*} \left[e^{mi\varphi - in\frac{v_2}{\omega_c}} V_{n,m} \times \{0\} \right]$	$-m\omega_c$	$m \neq 0$
$W_n^2 := \oplus_{m\in\mathbb{Z}^*} \left\{ (e^{-in\frac{v_2}{\omega_c}} w_{n,m}, -ni) \right\}$	λ_m	$m \neq 0$
$W_n^3 := \mathrm{Span}_\tau \left\{ (e^{-in\frac{v_2}{\omega_c}} \tau(r), 0) \right\} + \left\{ (e^{-\frac{r^2}{4}}, -in) \right\}$	0	

- The eigenspaces corresponding to $n = 0$ are :

Space	λ	m
$W_0^1 := \oplus_{m\in\mathbb{Z}^*} \left[e^{mi\varphi} L^2(\mathbb{R}^+) \times \{0\} \right]$	$-m\omega_c$	$m \neq 0$
$W_0^3 := \oplus_{p\in\mathbb{N}^*} \left[\{\tau_p\} \times \{0\} \right]$	0	$p > 0$

Spectral study : discrete spectrum

Theorem

We have completeness of the eigenspaces .

$$\mathcal{H} = \oplus_{n\neq 0} \left[e^{inx} \left(W_n^1 \oplus W_n^2 \oplus W_n^3 \right) \right] \oplus \left[L_0^2(\mathbb{R}^2) \times 0 \right]$$

and so the eigenvalues of H are 0, $-m\lambda_c$ and λ_m, $m \neq 0$.

- This shows that H can be fully diagonalized $\Rightarrow$ their is only discrete spectrum $\mathcal{H} = \mathcal{H}^{pp}$.
- New result for this kind of system [10].

10. J. Bedrossian and F. Wang, The linearized Vlasov and Vlasov-Fokker-Planck equations in a uniform magnetic field, Journal of Statistical Physics, 2020.

Back to the Bernstein-Landau paradox

Spectral explanation for the Bernstein-Landau paradox

1. The Vlasov-Ampère operator H is self-adjoint and it has a complete set of eigenfunctions $\Rightarrow$ electric field is undamped. Expression of electric field with the eigenvectors and eigenvalues :

$$F_n(t) = -nie^{nix} \sum_{m \neq 0} \frac{\left\langle u_0, e^{-in\frac{v_2}{\omega_c}} w_{n,m} \right\rangle + niF_0}{\left\| e^{-in\frac{v_2}{\omega_c}} w_{n,m} \right\|^2 + n^2} e^{i\lambda_m t}$$

2. The Vlasov system without magnetic field has only absolutely continuous spectrum and a kernel $\Rightarrow$ electric field goes to 0.

In the context of spectral theory, this can be interpreted as a sharp change of the domain of the operator when $\omega_c = 0$.

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Initialization : back to the spectral study

Objective : compare the numerical and theoretical solutions of Vlasov-Ampère when initializing with an eigenvector.

- We consider an eigenvector $\begin{pmatrix} w_{n,m} \\ F_n \end{pmatrix}$ associated to the Fourier mode $n \neq 0$ and the eigenvalue λ_m.
- $w_{n,m}$ and F_n are given by

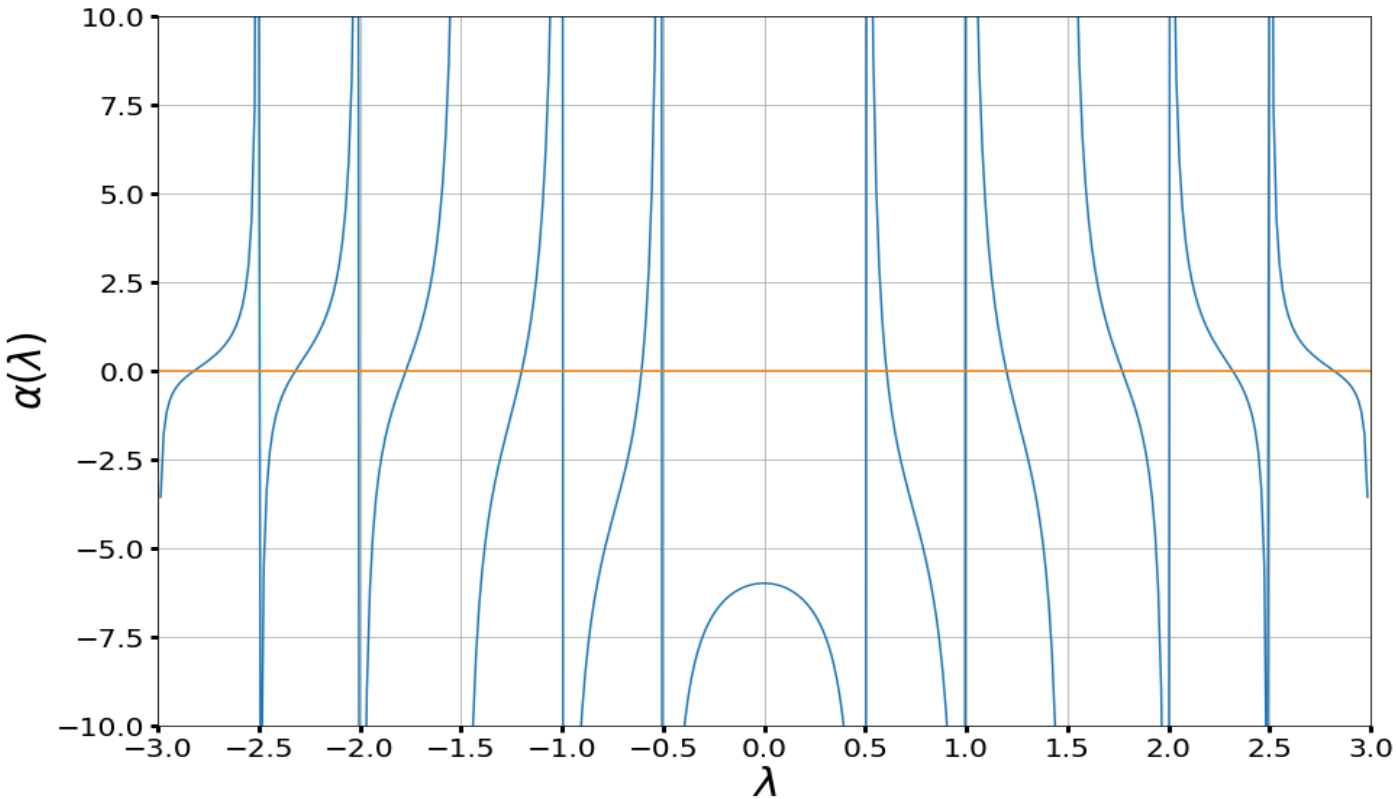
$$w_{n,m} = e^{in(x-\frac{v_2}{\omega_c})} e^{-\frac{r^2}{4}} \sum_{p \in \mathbb{Z}^*} \frac{p\omega_c}{p\omega_c + \lambda_m} e^{pi\varphi} J_p\left(\frac{nr}{\omega_c}\right) \text{ and } F_n = -ine^{inx}$$

- λ_m is one of the roots of a secular equation given by :

$$g(\lambda) = -1 - \frac{2\pi}{n^2} \sum_{m \in \mathbb{Z}^*} \frac{m\omega_c}{m\omega_c + \lambda} \int_0^\infty e^{-\frac{r^2}{2}} J_m\left(\frac{nr}{\omega_c}\right)^2 r dr = 0. \quad (6)$$

Secular equation

g has a unique root in $]m\omega_c\,;(m+1)\omega_c[$ for $m \geq 1$ and $](m-1)\omega_c\,;m\omega_c[$ for $m \leq -1$.



For $(n, m) = (1, 2)$, we compute $\lambda_2 \approx 1.19928$ with a numerical procedure.

Semi-Lagrangian scheme with splitting

Principle of the classical (backward) semi-lagrangian method

The aim is to find an approximation f_n of the solution of $\partial_t f + E(x,t)\partial_x f = 0$ at all discrete time t_n.

- For every point x_i of the grid, we compute the foot at time t_n of the characteristic $X(t_n)$ which is equal to x_i at time t_{n+1}.

$$\begin{cases} \dot{X} = E(X,t), \\ X(t_{n+1}) = x_i. \end{cases} \tag{7}$$

- We compute f_{n+1} by interpolation thanks to the relation $f_{n+1}(x_i) = f_n(X(t_n))$.

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Splitting

We approximate the solution of $\partial_t f + (\mathcal{A} + \mathcal{B})f = 0$ by solving $\partial_t f + \mathcal{A}f = 0$ and $\partial_t f + \mathcal{B}f = 0$ one after the other.

Splitting for Vlasov-Ampère

We split the Vlasov-Ampère so as to only solve transport equations with constant advection terms

$$\partial_t \begin{pmatrix} u \\ F \end{pmatrix} + (\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}) \begin{pmatrix} u \\ F \end{pmatrix} = 0$$

with

$$\mathcal{A} = \begin{pmatrix} v_1 \partial_x \\ 0 \end{pmatrix}, \mathcal{B} = \begin{pmatrix} F v_1 e^{-\frac{v_1^2+v_2^2}{4}} \\ 1^* \int u e^{-\frac{v_1^2+v_2^2}{4}} v_1 dv_1 dv_2 \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} -\omega_c v_2 \partial_{v_1} \\ 0 \end{pmatrix}, \mathcal{D} = \begin{pmatrix} -\omega_c v_1 \partial_{v_2} \\ 0 \end{pmatrix}$$

Algorithm to solve linearized Vlasov-Ampère

1. **Initialization** $U_0 = \begin{pmatrix} w_{n,m} \\ F_n \end{pmatrix}$ given.
2. **Going from t_n to t_{n+1}**
   Assume we know U_n, the approximation of U at time t_n.
   - We compute U^* by solving $\partial_t U + \mathcal{A}U = 0$ with a SL scheme during one time step Δt with initial condition U^n.
   - We compute $\hat{U}$ by solving $\partial_t U + \mathcal{B}U = 0$ with a Runge-Kutta 2 scheme during one time step Δt with initial condition U^*.
   - We compute U^{**} by solving $\partial_t U + \mathcal{C}U = 0$ with a SL scheme during one time step Δt with initial condition $\hat{U}$.
   - We compute U^{n+1} by solving $\partial_t U + \mathcal{D}U = 0$ with a SL scheme during one time step Δt with initial condition U^{**}.

Numerical results for u with $T_f = \frac{\pi}{2\lambda_m}$

For all of the simulations, $N_x = 33$, $N_{v_1} = N_{v_2} = 63$, $L_x = 2\pi$, $L_{v_1} = L_{v_2} = 10$ and we have taken $\omega_c = 0.5$ and $n = 1$.

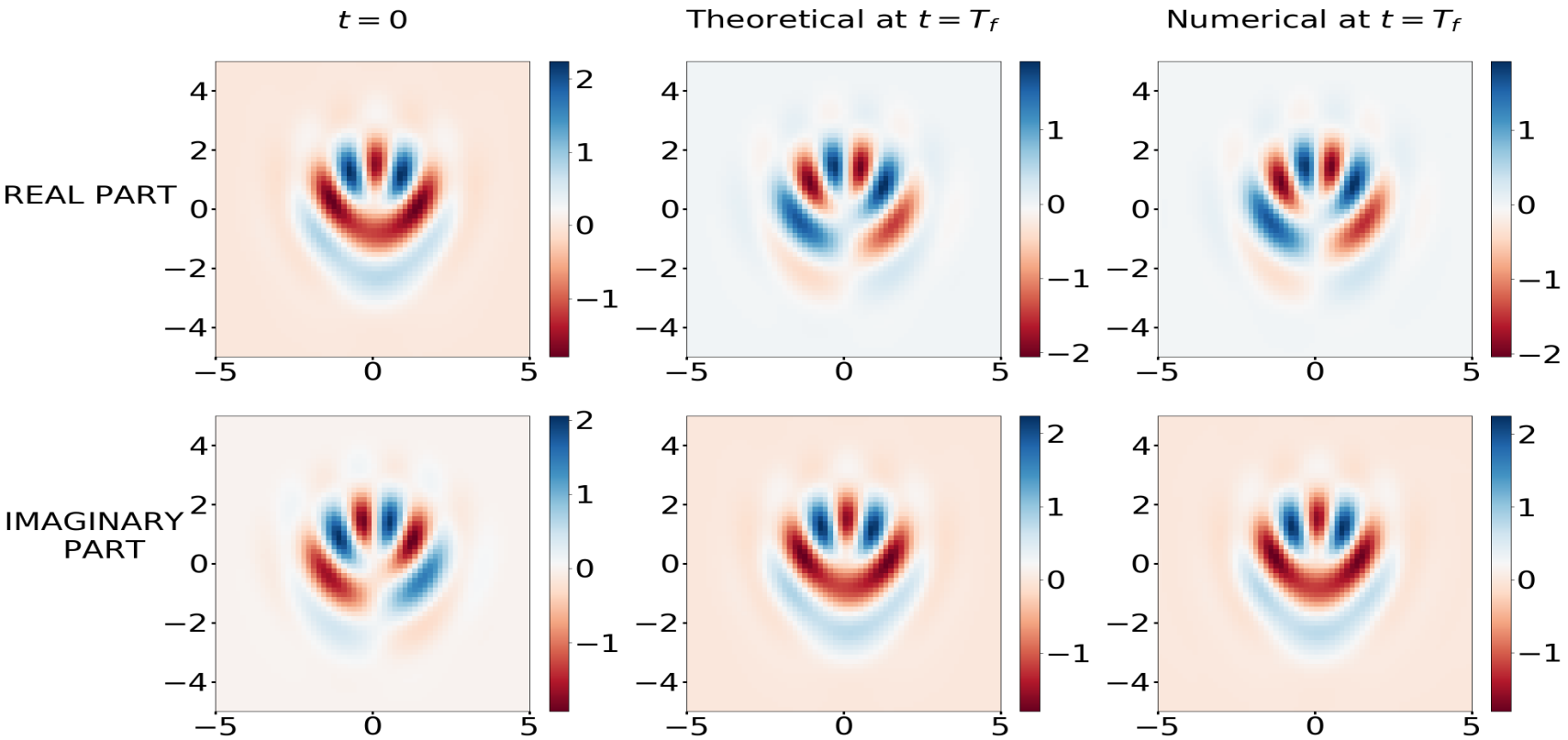

Figure – Real and imaginary parts of u in V1-V2 plane for $x = 0$.

Numerical results for u and F for $T_f = \frac{\pi}{2\lambda_m}$

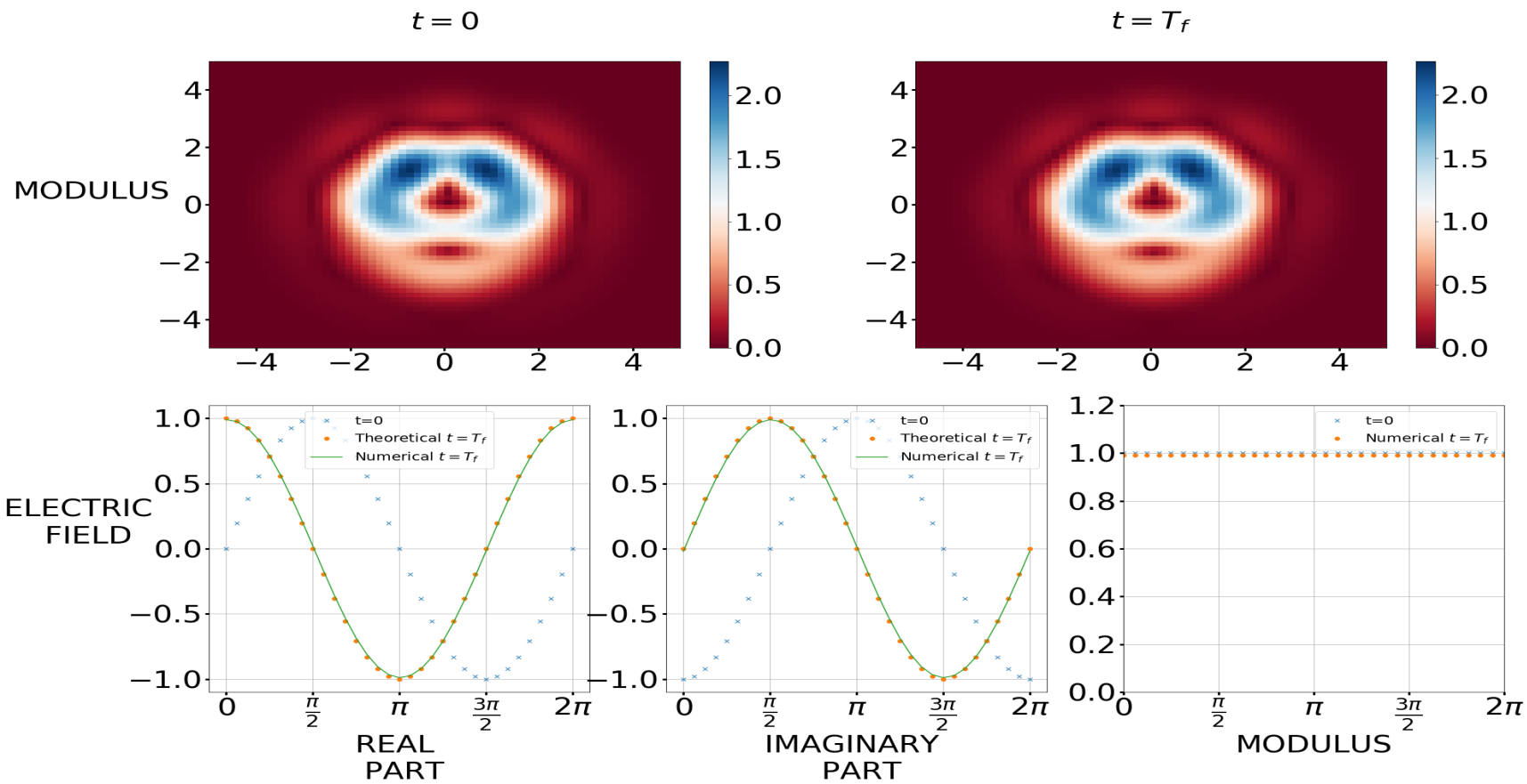

Figure – Module of u in V1-V2 plane for $x = 0$ and real and imaginary parts of F.

Algorithm to solve linearized Vlasov-Poisson

$U_0 = (w_{n,m}, F_n)$ is also a solution of the linearized Vlasov-Poisson system.

1. **Initialization** $u_0 = w_{n,m}$ and $F_0 = F_n$ given.
2. **Going from** t_n **to** t_{n+1}
   Assume we know u_n and F_n, the approximations of u and F at time t_n.
   - We compute u^* by solving $\partial_t u + v_1 \partial_x u = 0$ with a SL scheme during one time step Δt with initial condition u^n.
   - We compute F_{n+1} by solving the Poisson equation with u^*.
   - We compute u^{**} by solving $\partial_t u + F v_1 e^{-\frac{v_1^2+v_2^2}{4}} = 0$ with an Euler explicit scheme during one time step Δt with initial condition u^*.
   - We compute $\hat{u}$ by solving $\partial_t u - \omega_c v_2 \partial_{v_1} u = 0$ with a SL scheme during one time step Δt with initial condition u^{**}.
   - We compute u^{n+1} by solving $\partial_t u + \omega_c v_1 \partial_{v_2} u = 0$ with a SL scheme during one time step Δt with initial condition $\hat{u}$.

Numerical results for u with $T_f = \frac{\pi}{2\lambda_m}$

For all of the simulations, we have taken $\omega_c = 0.5$ and $n = 1$.

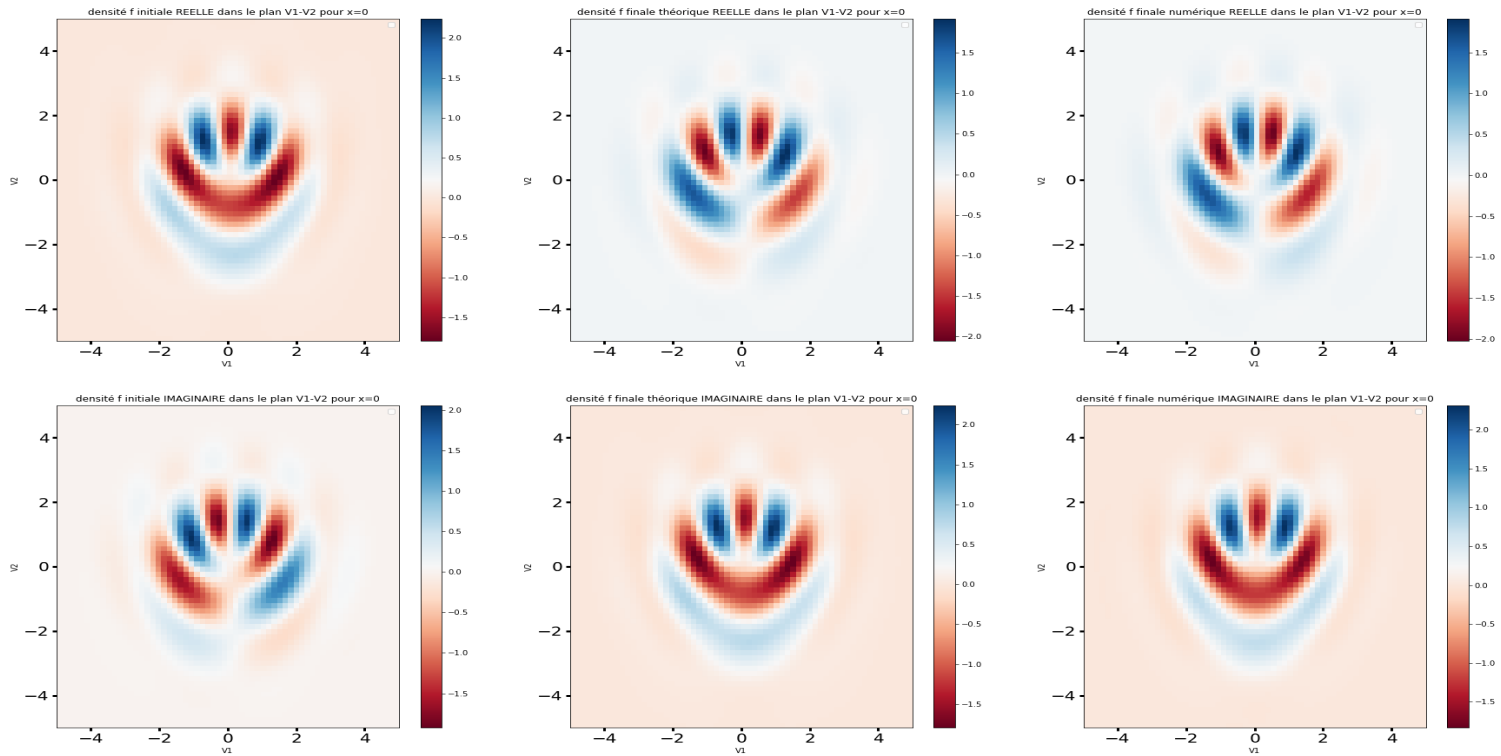

Figure – Real and imaginary parts of u in V1-V2 plane for $x = 0$.

Numerical results for u and F for $T_f = \frac{\pi}{2\lambda_m}$

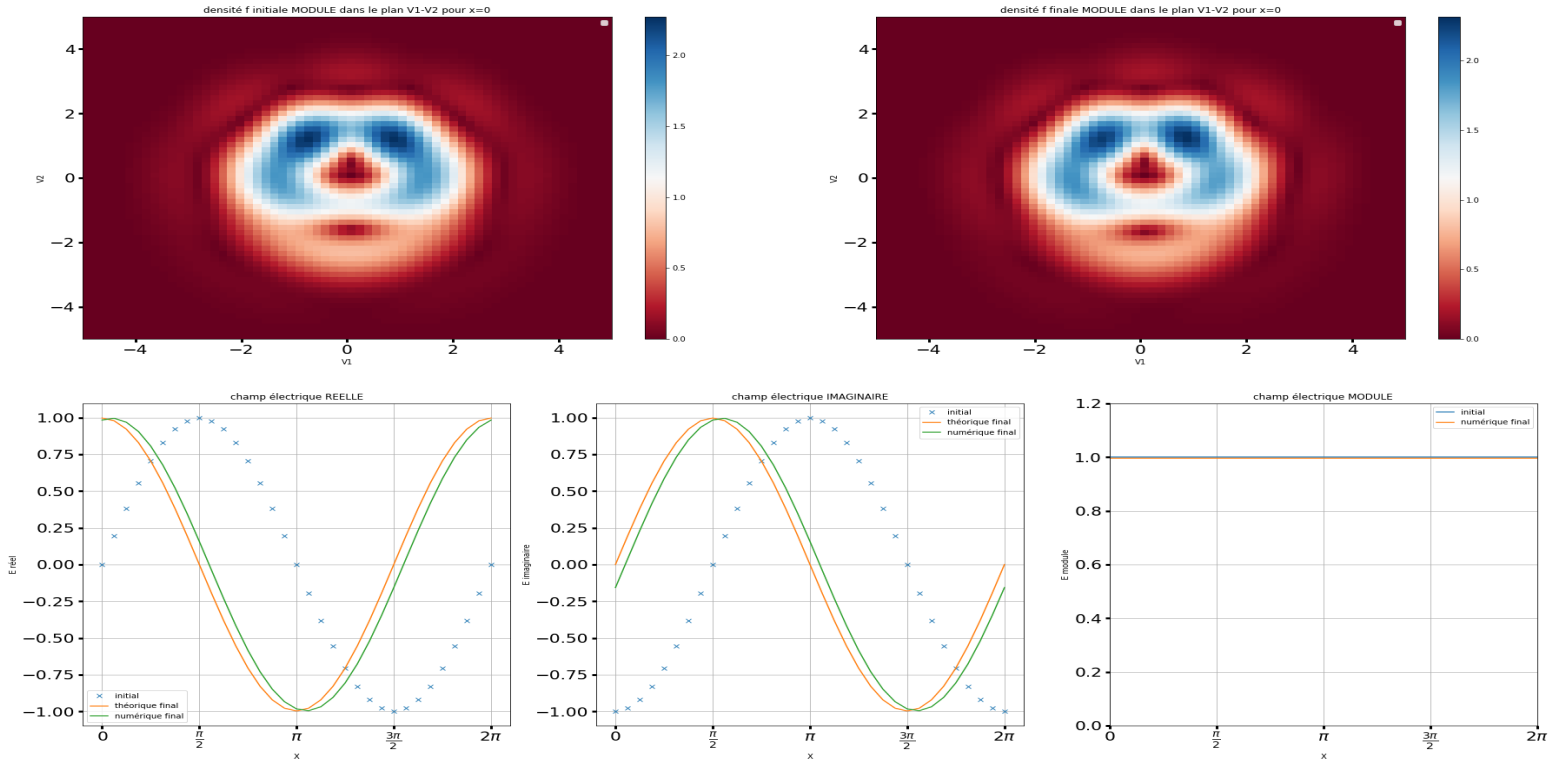

Figure – Module of u in V1-V2 plane for $x = 0$ and real and imaginary parts of F.

Algorithm for Vlasov-Poisson (non linear code)

We can also test the eigenvectors in the non-linear code for Vlasov-Poisson.

1. **Initialization** $f_{ini} = f_0 + \varepsilon\sqrt{f_0}\,\mathrm{Re}(w_{n,m})$ and $E_{ini} = \varepsilon\,\mathrm{Re}(F_n)$ given.
2. **Going from** t_n **to** t_{n+1}
   Assume we know f_n and E_n, the approximations of u and F at time t_n.
   - We compute f^* by solving $\partial_t f + v_1 \partial_x f = 0$ with a SL scheme during one time step Δt with initial condition f^n.
   - We compute E_{n+1} by solving the Poisson equation with f^*.
   - We compute $\hat{f}$ by solving $\partial_t f - (E_{n+1} + \omega_c v_2)\partial_{v_1} f = 0$ with a SL scheme during one time step Δt with initial condition f^*.
   - We compute f^{n+1} by solving $\partial_t f + \omega_c v_1 \partial_{v_2} f = 0$ with a SL scheme during one time step Δt with initial condition $\hat{f}$.

Numerical results for u and F with $T_f = \frac{\pi}{2\lambda_m}$

For all of the simulations, we have taken $\omega_c = 0.5$ and $n = 1$.

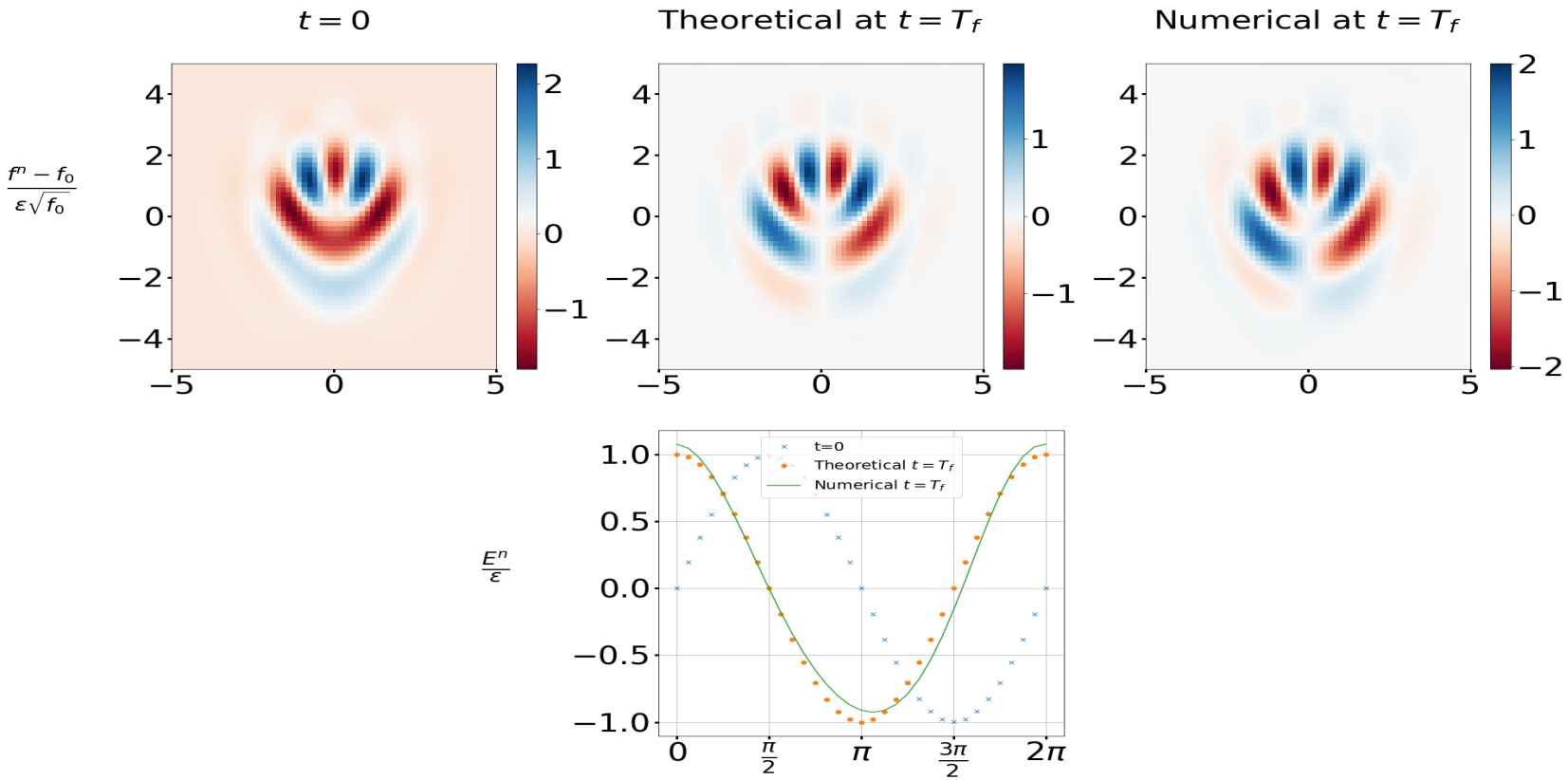

Figure – $\frac{f - f_0}{\varepsilon\sqrt{f_0}} \approx u$ in V1-V2 plane for $x = 0$ and $\frac{E}{\varepsilon} \approx F$.

Summary and perspectives

- Spectral decomposition of the Vlasov-Ampère system
- Reinterpretation of the Bernstein-Landau paradox as AC spectrum versus PP spectrum.
- Constructed new reference solutions that can be tested on linear and non-linear schemes.

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Perspective

Limit $\omega_c \to 0$.
Mathematical difficulty :

$$w_{n,m} = e^{in(x-\frac{v_2}{\omega_c})} e^{-\frac{r^2}{4}} \sum_{p\in\mathbb{Z}^*} \frac{p\omega_c}{p\omega_c + \lambda_m} e^{pi\varphi} J_p\left(\frac{nr}{\omega_c}\right)$$

There is a singularity at $\omega_c = 0$.