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Kinetic models for magnetized plasmas

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à mon grand-père Raphaël,

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1 General Introduction

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Summary

This chapter gives an overview of the physics at stake and of the main contributions in this thesis. We begin with some important physical definitions, namely those of plasmas, nuclear fusion and tokamaks. Then we introduce the kinetic formalism and give several examples of kinetic models for plasmas. Finally, we summarize the main contributions of this thesis, first presenting our results regarding linear magnetized Vlasov systems and then focusing on the nonlinear magnetized Vlasov–Poisson system.

1.1 Physical motivation

1.1.1 Plasmas and nuclear fusion

When a gas is heated to a very high temperature, electrons are ripped from the orbit of the nucleus of the atom to which they are attached. The result is a globally neutral mixture of charged particles, ions and electrons, which is called a plasma. Plasmas are considered as the fourth state of matter, next to solids, liquids and gases.

The term plasma was first introduced by the Nobel prize winning physicist and chemist Irving Langmuir in 1928 [83] to describe a ionized gas. However, even though a plasma is similar to a gas in the sense that it has no definite shape or volume, there are fundamental differences between these two states of matter, notably with respect to the interactions between particles. Indeed, in a gas the main interactions are binary collisions, which means a single particle can interact with only one other particle at a time. In a plasma, even though collisions still occur, long range electric and magnetic forces impose a collective behavior, which means a single charged particle is constantly interacting with all the other charged particles that form the plasma. At the same time, the local non-neutrality of the plasma generates self-induced electromagnetic forces that also influence how the particles evolve. Furthermore, electrons and ions possess opposite charges and vastly different masses, which means that they behave differently in many circumstances, with various types of plasma-specific waves and instabilities emerging as a result. These different physical realities make plasmas very complicated objects to study and understand.

The use of plasmas in everyday life has now become frequent. We can quote neon tubes or plasma screens. There are also a certain number of industrial applications: amplifiers in telecommunication satellites, plasma etching in microelectronics, and X-ray production. Even though it is almost absent in its natural state on Earth it's important to highlight that plasma constitutes 99 of the mass of the universe. Notably stars are made of plasmas and the energy they give off comes from the process of fusion of light nuclei like protons. This process of nuclear fusion releases huge amounts of energy: for the same mass, the fusion of light atoms releases an energy nearly four million times greater than that of a chemical reaction such as the combustion of coal, oil or gas, and four times greater than that of nuclear fission reactions which are currently powering commercial nuclear power plants. Thus, harnessing the energy from this transformation has now arguably become one of the most important potential applications of plasma physics, and various research projects around the world have been launched in order to master nuclear fusion.

The most important (in terms of scale and funding) of these projects is the ITER (originally International Thermonuclear Experimental Reactor) program, an international collaboration that aims to demonstrate the feasibility of electricity production via controlled nuclear fusion [74]. To achieve this, ITER is currently building the largest tokamak in the world in Cadarache in the south of France. A tokamak is a large toroidal chamber where the nuclear reactions inside the plasma using deuterium-tritium fuel will take place. This fuel has been chosen because the most accessible fusion reaction consists of fusing nuclei of deuterium and tritium (isotopes of hydrogen) to obtain a helium atom and a very energetic neutron which will be used to produce the heat necessary to make electricity.

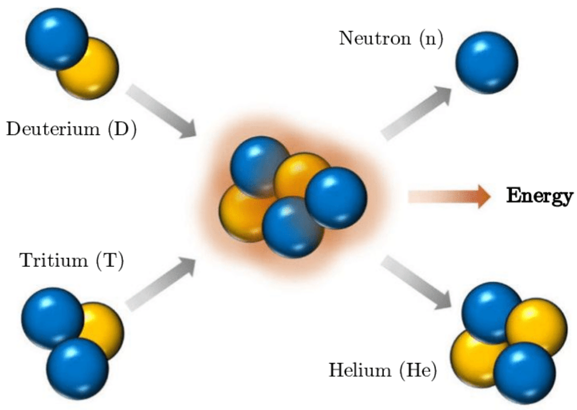

Figure 1.1 – Deuterium-tritium nuclear reaction [49].

In order for this reaction to take place, the plasma needs to be heated to extreme temperatures, approximately 10^8 degrees Celsius or ten times the temperature at the core of the sun. At the same time the nuclear reaction doesn't require a huge amount of material to be initiated, illustrated by the fact that the density inside the tokamak will be of the order of 10^{20} particles per cubic meter or around 10^5 less dense than air. Heating the material inside the plasma to such temperatures requires enormous amounts of energy, and thus far no tokamak has ever managed to generate a positive net energy output. Until now, the best energy input-energy output ratio has been obtained by the Joint European Torus (JET) which achieved a ratio $Q = 0.67$. The first main objective of the ITER tokamak is to achieve "plasma energy breakeven" ($Q = 1$) and scientists hope that the Q ratio for ITER will eventually exceed 10. Furthermore, the extreme temperatures inside the tokamak, a machine only a dozen meters in diameter, means that the temperature gradient inside the torus will be enormous and that the particles forming the plasma will rapidly escape if unconfined. To remedy this, important magnetic fields are generated by coils around the chamber in order to create a helicoidal field that confines the plasma [73].

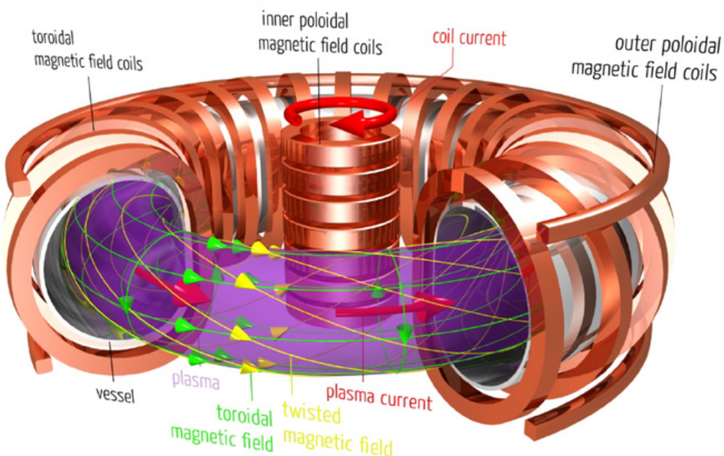

Figure 1.2 – Helicoidal field confining a plasma in a tokamak [51].

Hence, it is important to consider models that take into account both the inherent collective dynamics of the plasma as well as these external magnetic forces that heavily influence the evolution of the system.

1.1.2 The Maxwell–Boltzmann distribution function

In this thesis, we are going to be looking at models often used to describe the evolution of a plasma: kinetic models. These models describe matter at what is called a "mesoscopic scale" between the microscopic and macroscopic scales. This is relevant for plasmas where the microscopic description of the system, particle by particle, is too complex to be used in practice (due in part to the collective dynamics mentioned above) and where macroscopic models like fluid models are too coarse to capture the essential mechanisms and obtain a realistic description of the plasma.

Historically, the mathematical basis for kinetic models was introduced by Maxwell in [92, 93] to describe the behavior of an ideal gas. In these works, he introduced a function describing the distribution of particle speeds of the molecules in the gas. Boltzmann later conducted a more theoretical study [17] in order to derive this distribution function from physical considerations, laying the foundations for statistical physics in the process. This Maxwell–Boltzmann distribution function f quantifies the number of particles moving with a velocity between v and $v + dv$.

$$f(v) = \rho \left(\frac{m}{2\pi kT}\right)^{\frac{3}{2}} \exp\left(-\frac{m\,|v-u|^2}{2kT}\right) \tag{1.1.1}$$

with ρ the total number of particles, m the mass of a single particle, k the Boltzmann constant, T the absolute temperature and u the macroscopic or collective velocity.

When a plasma is at thermodynamic equilibrium, its distribution function is given by the Maxwell–Boltzmann distribution function. In this context, fluid models are relevant to describe the system. However, when the plasma is far from thermodynamic equilibrium, a kinetic description of the system is necessary and the plasma is fully characterized by the Vlasov–Maxwell system.

1.2 Kinetic formalism and the Vlasov equation

1.2.1 A hierarchy of kinetic models

In a kinetic model for plasmas, each species of particle in the plasma is characterized by a positive distribution function $f_\alpha(t, x, v)$. This function f_α quantifies the probability at time t that a particle will be at position x with velocity v. To put it another way, the quantity $f_\alpha(t, x, v)dxdv$ corresponds to the number of particles at time t that are in box of volume $dxdv$ centered around (x, v).

Now we present a hierarchy of kinetic models increasing in complexity. In the rest of this subsection, we will denote by P the set of different species of particles inside the plasma, q_α the charge of the particles of species α, m_α the mass of of particles of species α, ϵ_0 the vacuum permittivity, μ_0 the magnetic permeability and c the speed of light.

Free transport If we consider free particles that aren't subject to any kind of internal or external force, then this system is characterized by the free transport equation

$$\partial_t f_\alpha + v \cdot \nabla_x f_\alpha = 0. \tag{1.2.1}$$

Solving this sort of transport equation is straightforward and solutions are given by the expression

$$f_\alpha(t,x,v) = f_\alpha^{ini}(x - tv, v) \tag{1.2.2}$$

where f_α^{ini} is the initial distribution function for the species of particle α.

Linear Vlasov When we consider an added external electromagnetic field (E_{ext}, B_{ext}), than the particles in the plasma are subject to an electromagnetic or Lorentz force

$$F_{ext}(t,x,v) = q_\alpha(E_{ext}(t,x) + v \wedge B_{ext}(t,x)) \tag{1.2.3}$$

and the system is described by the linear Vlasov equation given by

$$\partial_t f_\alpha + v \cdot \nabla_x f_\alpha + \frac{F_{ext}}{m_\alpha} \cdot \nabla_v f_\alpha = 0. \tag{1.2.4}$$

We can explicitly write the solution to this equation

$$f_\alpha(t,x,v) = f_\alpha^{ini}(X(0;t,x,v), V(0;t,x,v)) \tag{1.2.5}$$

where (X,V) is the solution to the following Cauchy problem

$$\begin{cases} \dfrac{d}{ds}X(s;t,x,v) = V(s;t,x,v), \\ \dfrac{d}{ds}V(s;t,x,v) = \dfrac{q_\alpha}{m_\alpha}F_{ext}(s, X(s;t,x,v), V(s;t,x,v)), \end{cases} \tag{1.2.6}$$

with

$$(X(t;t,x,v), V(t;t,x,v)) = (x,v). \tag{1.2.7}$$

This equation and its solution are more complex than for (1.2.1), because even though (1.2.4) is a linear partial differential equation (PDE) and E_{ext} and B_{ext} are given external fields that don't depend on the system, one needs to solve a nonlinear ordinary differential equation (ODE) to solve the equation.

Vlasov–Poisson Instead of considering outside forces acting on the system, one can consider that the particles in the plasma are in self-interaction through the Coulombian force F_α exerted at time t and position x on a particle of charge q_α

$$F_\alpha(t,x) = q_\alpha \sum_{\beta \in P} \frac{q_\beta}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho_\beta(t,y)dy = q_\alpha E(t,x) \tag{1.2.8}$$

where $\rho_\beta(t,x) = \int_{\mathbb{R}^3} f_\beta(t,x,v)dv$ is the macroscopic density for the species of particle β and E is the electrostatic field.

The electrostatic field E derives from a potential Φ

$$E(t,x) = \sum_{\beta \in P} \frac{q_\beta}{4\pi\epsilon_0} \int_{\mathbb{R}^3} -\nabla_x \left(\frac{1}{|x-y|}\right) \rho_\beta(t,y)dy = -\nabla_x \Phi(t,x), \tag{1.2.9}$$

with

$$\Phi(t,x) = \sum_{\beta\in P} \frac{q_\beta}{4\pi\epsilon_0} \int_{\mathbb{R}^3} \left(\frac{1}{|x-y|}\right) \rho_\beta(t,y)dy. \tag{1.2.10}$$

Thus, since $\mathcal{G}_3(x) := \frac{1}{4\pi|x|}$ is the Green function for $-\Delta$, Φ is the solution to the Poisson equation

$$-\Delta_x \Phi = \sum_{\alpha\in P} \frac{q_\alpha}{\epsilon_0} \rho_\alpha. \tag{1.2.11}$$

Hence, the system is modeled by the so-called Vlasov–Poisson system for Coulombian interaction

$$\begin{cases} \partial_t f_\alpha + v\cdot\nabla_x f_\alpha - \dfrac{q_\alpha}{m_\alpha}\nabla_x\Phi\cdot\nabla_v f_\alpha = 0, \quad \alpha\in P, \\ -\Delta_x\Phi = \displaystyle\sum_{\alpha\in P}\frac{q_\alpha}{\epsilon_0}\rho_\alpha. \end{cases} \tag{1.2.12}$$

It is important to highlight that the force F_α exerted on the particle of charge q_α is expressed using the distribution function for the entire system, including the particle in question. This approximation is valid because the system or plasma we consider is made up of a large number of particles and so the effect of a single particle is considered negligible compared to the collective effect of the whole system.

The Vlasov–Poisson given by (1.2.12) describes the behavior of particles in Coulombian interaction. In truth, this system was first derived in 1915 by Jeans [75] in the context of stellar dynamics. Indeed, when one considers a system of identical points of mass m characterized by the distribution function f, each particle is subject to the gravitational force F_g generated by all the other particles, given by the following expression:

$$F_g(t,x) = -\Gamma m^2 \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} f(t,y,v)dvdy. \tag{1.2.13}$$

F_g is an attractive force that also derives from a potential Φ_g. Hence, this system of points is modeled by the Vlasov–Poisson system for Newtonian interaction

$$\begin{cases} \partial_t f + v\cdot\nabla_x f + \dfrac{1}{m}\nabla_x\Phi\cdot\nabla_v f = 0, \\ -\Delta_x\Phi_g(t,x) = 4\pi\Gamma m\displaystyle\int_{\mathbb{R}^3} f(t,x,v)dv. \end{cases} \tag{1.2.14}$$

Since the systems of PDEs (1.2.12) and (1.2.14) are nonlinear, the well-posedness problem is much more complicated than for the previous equation. This problem was solved at the beginning of the nineties using two different methods that lead to two distinct papers [87, 103].

Vlasov–Maxwell In the previous model given by (1.2.12), the speed of the charged particles in the plasma was considered small enough so that the contribution of the magnetic field to the electromagnetic Lorentz force (1.2.3) became negligible. Now we consider a plasma where this assumption is no longer valid, so that we have to take into account the self-induced magnetic force resulting from the motion of the charged particles. The electromagnetic field (E,B) generated is given by the Maxwell equations, first introduced (in part) in [94]

$$\begin{cases} \mathrm{div}_x E = \displaystyle\sum_{\alpha\in P} \frac{q_\alpha}{\epsilon_0}\rho_\alpha, \\ \mathrm{rot}_x E = -\partial_t B, \\ \mathrm{div}_x B = 0, \\ -\dfrac{1}{c^2}\partial_t E + \mathrm{rot}_x B = \mu_0 \displaystyle\sum_{\alpha\in P} q_\alpha j_\alpha, \end{cases} \tag{1.2.15}$$

where $j_\alpha(t,x) \int_{\mathbb{R}^3} f(t,x,v)v dv$ is the current density for the species of particle α.

Hence, the plasma is completely modeled by the Vlasov–Maxwell system of PDEs which couples the Maxwell equations with the Vlasov equation

$$\begin{cases} \partial_t f_\alpha + v\cdot\nabla_x f_\alpha + \dfrac{q_\alpha}{m_\alpha}(E + v\wedge B)\cdot\nabla_v f_\alpha = 0, \quad \alpha\in P, \\ \mathrm{div}_x E = \displaystyle\sum_{\alpha\in P} \frac{q_\alpha}{\epsilon_0}\rho_\alpha, \\ \mathrm{rot}_x E = -\partial_t B, \\ \mathrm{div}_x B = 0, \\ -\dfrac{1}{c^2}\partial_t E + \mathrm{rot}_x B = \mu_0 \displaystyle\sum_{\alpha\in P} q_\alpha j_\alpha. \end{cases} \tag{1.2.16}$$

As explained above, this system of PDEs captures all the properties of the plasma and provides a complete description of the system. The well-posedness for (1.2.16) is still an open problem. However, weak solutions to the Vlasov–Maxwell system were shown to exist in [43] by DiPerna and Lions.

Vlasov–Landau For this paragraph on collisional kinetic models, to simplify the discussion and the notations we will only consider a single species of particles. In certain configurations, one needs to take into account the collisions between particles in the plasma. In practice, this means adding a collision operator in the r.h.s. of the Vlasov equation. We begin by presenting the Boltzmann collision operator Q, introduced in [17] to model the collisions in a gas, and given by the following expression:

$$\begin{cases} Q(f,f) = \displaystyle\int_{\mathbb{R}^3}\int_{S^2} B(v-v_*,\omega)(f'f'_* - ff_*)dv_* d\omega, \\ f = f(t,x,v), f_* = f(t,x,v_*), f' = f(t,x,v'_*), f'_* = f(t,x,v'_*). \end{cases} \tag{1.2.17}$$

v and v_* are the velocities of the incoming particles (before collision), v' and v'_* are the velocities of the outgoing particles (after collision) and are given in terms of v and v^* by

$$\begin{cases} v' = v - ((v-v_*)\cdot\omega)\omega, \\ v'_* = v_* + ((v-v_*)\cdot\omega)\omega, \end{cases} \tag{1.2.18}$$

for some $\omega\in S^2$ which parametrizes the set of admissible outgoing velocities under the constraints $v + v_* = v' + v'_*$ and $|v|^2 + |v_*|^2 = |v'|^2 + |v'_*|^2$, and $B(v-v_*,\omega)$ is the differential cross-section, which measures the probability of the collision process $(v,v_*) \mapsto (v',v'_*)$. If we assume that collisions are the only interactions inside the gas, then the system is modeled by the Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f, f). \tag{1.2.19}$$

With regards to the existence of solutions to the Boltzmann equation, using their innovative theory of renormalized solutions [45] just like for the Vlasov–Maxwell system, DiPerna and Lions showed existence of weak solutions to (1.2.19) in [44, 47]. More recently, a very complete review on the results related to collisional kinetic theory was assembled by Villani in [124].

For plasmas, since the interaction between particles is Coulombian, the Boltzmann operator isn't necessarily well-defined. To remedy this we use a Boltzmann type operator called the Landau collision operator Q_L [81, 82] given by

$$Q_L(f, f) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} a(v - v_*)[f_*(\nabla f) - f(\nabla f)_*] dv_* \right). \tag{1.2.20}$$

Here $a(z)$ is a symmetric degenerate nonnegative matrix, proportional to the orthogonal projection onto $z^\perp$:

$$a_{ij}(z) = \frac{L}{|z|} \left(\delta_{ij} - \frac{z_{ij}}{|z|^2} \right). \tag{1.2.21}$$

Finally, if we take into account both electromagnetic interactions and collisions in the plasma, the system is described by the Landau equation:

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = Q_L(f, f), \tag{1.2.22}$$

with the electromagnetic field F determined by the Poisson equation in the electrostatic case or the Maxwell equations in the electrodynamic setting.

Like for the Vlasov–Maxwell system, for many Vlasov type models with collisions, including the Landau equation, the well-posedness is still an open problem (see [66, 123] for recent advances). Progress has also been made when the solutions are assumed to be perturbations of Maxwellians [22, 65, 120, 119, 129, 130].

1.2.2 Vlasov–Poisson with an external magnetic field

In this thesis, the model that will be at the center of most of our discussions will be the Vlasov–Poisson system with an external magnetic field or magnetized Vlasov–Poisson system. This model will be studied in both the full space framework, where the phase space is $\mathbb{R}^d \times \mathbb{R}^d$, and in the periodic setting, where the phase space is $\mathcal{T}^d \times \mathbb{R}^d$ with $\mathcal{T}^d = [0\,, 1]_{\text{per}} = \mathbb{R}^d/\mathbb{Z}^d$ the d-dimensional torus. Furthermore, to simplify the discussion, in the rest of the thesis, we make the assumption that the ion distribution in our system is constant. This is a physically sensible assumption given that ions are much heavier than electrons (the ion-to-electron mass ratio is of order 10^3), which means that at the time-scale of the electron dynamics ions can be considered static. Hence we will consider the magnetized Vlasov–Poisson system for electrons given by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \dfrac{q}{m}(E + v \wedge B) \cdot \nabla_v f = 0, \\ -\Delta_x \Phi = \dfrac{q_{ion}}{\epsilon_0} \rho_0 + \dfrac{q}{\epsilon_0} \rho, \\ \rho(t, x) = \displaystyle\int_{\mathbb{R}^3} f(t, x, v) dv, \\ E(t, x) = -\nabla_x \Phi(t, x), \end{cases} \tag{1.2.23}$$

where $f = f(t, x, v)$ is the electron distribution function, $q = -1,602 \times 10^{-19}$ C is the negative charge of the electron, $m = 9,109 \times 10^{-31}$ kg is the mass of the electron, E is the self-generated electric field, Φ is the electric potential and B is an external magnetic field which we will consider either constant or bounded throughout the manuscript. q_{ion} corresponds to the positive charge of the ions in the system and ρ_0 is the constant ion macroscopic density. In the full space framework we can take $\rho_0 = 0$ because $(f, \Phi) = (0, 0)$ is a solution to (1.2.23), whereas in the periodic setting we need ρ_0 to be strictly positive and we take $\rho_0 = \int_{\mathcal{T}^d} \int_{\mathbb{R}^d} f(0, x, v)dxdv$. This is due to the nature of the Poisson equation on the torus.

As mentioned above, when a plasma is at thermodynamic equilibrium, the distribution of particles follows (1.1.1). Hence, it is natural that one uses the Maxwell-Boltzmann distribution to find a stationary solution to (1.2.23). In chapter 2, we will linearize (1.2.23) by considering that the solution (f, Φ) is a perturbation of such a stationary solution (f_0, Φ_0), so we write $(f, \Phi) = (f_0, \Phi_0) + \varepsilon(u, \varphi)$ for $\varepsilon > 0$. To the first order in ε, (u, φ) is the solution to the following linearized Vlasov–Poisson system with magnetic field:

$$\begin{cases} \partial_t u + v \cdot \nabla_x u + \dfrac{q}{m} \left(F \cdot \nabla_v f_0 + v \wedge B \cdot \nabla_v u \right) = 0, \\ -\Delta_x \varphi = \dfrac{q}{\epsilon_0} \displaystyle\int_{\mathbb{R}^3} u(t, x, v)dv, \\ F(t, x) = -\nabla_x \Phi(t, x), \end{cases} \tag{1.2.24}$$

with F the perturbation of the electric field.

For the rest of this chapter, we will normalize all the physical constants except the charge q to 1. With regards to the charge q we normalize q to -1 in the subsection on the linear system (section 1.3) and q to 1 in the subsection on the nonlinear system (section 1.4). This is just for cosmetic purposes because the normalization of q doesn't influence the results in the following chapters, meaning we could have normalized q to any constant.

As explained above, fusion plasmas are systems that are subject to an external magnetic field whose role is to confine the plasma despite the enormous thermal agitation and gradient of temperature inside the tokamak. Motivated by this important physical and industrial application, this thesis aims to investigate how the kinetic models that describe the system are modified in the presence of an added external magnetic field. We summarize the contributions of this thesis in the next two sections.

1.3 Spectral theory for magnetized linear kinetic models

The first question that arose was one regarding the linear magnetized Vlasov–Poisson system (1.2.24). In a seminal paper [80] published in 1946 where he studied the linearized Vlasov–Poisson system, Lev Landau showed that longitudinal space charge waves in plasma are damped (decrease exponentially as a function of time), predicting what is now known as Landau damping. This phenomenon was confirmed experimentally by Malmberg and Wharton in 1964 [90]. In another seminal paper [14] published in 1958, Ira Bernstein proved that when a plasma is subject to a constant external magnetic field, certain electrostatic waves (now known as Bernstein modes) perpendicular to the magnetic field are undamped. A crucial point is that these waves exist independently of the strength of the magnetic field, which means there is a discontinuity between Landau's theory on unmagnetized electrostatic plasmas and Bernstein's theory on magnetized plasmas, known as the Bernstein–Landau paradox.

In chapter 2, we investigate this phenomenon by studying (1.2.24) in a periodic 1d-2v configuration, meaning that the distribution function u depends on one coordinate x in position and two coordinates $v = (v_1, v_2)$ in velocity with $(x, v) \in \mathcal{T} \times \mathbb{R}^2$. We reformulate system (1.2.24) using the Maxwell–Ampère equation (written with the unknowns of (1.2.24))

$$\partial_t F = \int uv dv. \tag{1.3.1}$$

With this reformulation, we succeed in interpreting the Bernstein–Landau paradox in terms of the spectrum of a self-adjoint operator associated to the Vlasov–Ampère system. Furthermore, we build semi-Lagrangian schemes to show how the eigenfunctions constructed in our analysis can be considered as new test functions for these magnetized kinetic models.

In this subsection we summarize these results on our new interpretation of the Bernstein–Landau paradox. Here when we refer to magnetized Vlasov type systems, we are referring to the linearized systems, because in the next subsection we study the nonlinear Vlasov–Poisson system. First we explain why the symmetric operator associated to the magnetized Vlasov–Poisson system isn't self-adjoint, which is why we study the magnetized Vlasov–Ampère system instead. Then we detail the spectral decomposition of the magnetized Vlasov–Ampère operator. As said above, this allows us to give a spectral interpretation of the Bernstein–Landau paradox. Finally we explicate the semi-Lagrangian schemes used to test these eigenfunctions.

1.3.1 From the magnetized Vlasov–Poisson system to the magnetized Vlasov–Ampère system

The main objective of the work in chapter 2 is to try to understand the Bernstein–Landau paradox. Our approach consists in studying the magnetized Vlasov–Poisson system in the framework of spectral theory. Such an approach is justified by the fact that there exists a wide literature devoted to the spectral properties of the linear (unmagnetized) Vlasov–Poisson system [7, 23, 34, 36, 37, 38]. As said above, we will be working in a 1d-2v configuration which means the position coordinate for the distribution function f is given by the real variable $x \in \mathcal{T}$ and the velocity v variable is two-dimensional $v = (v_1, v_2)$. Hence, we consider the distribution function $f := f(t, x, v)$ which is solution the 1d-2v nonlinear Vlasov–Poisson system with constant magnetic field $B_0 = (0, 0, \omega_c)$

$$\begin{cases} \partial_t f + v_1 \partial_x f - E \partial_{v_1} f + \omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) f = 0, \\ \partial_x E(t, x) = 2\pi - \displaystyle\int_{\mathbb{R}^2} f dv. \end{cases} \tag{1.3.2}$$

We linearize (1.3.2) around the stationary solution (f_0, E_0) with $f_0(v) := \exp(\frac{v^2}{2})$ (with $v^2 = v_1^2 + v_2^2$) a Maxwellian equilibrium and $E_0 = 0$ a null reference electric field. This corresponds to writing the following expansion:

$$f(t, x, v) = f_0(v) + \varepsilon \sqrt{f_0(v)} u(t, x, v) + O(\varepsilon^2), \tag{1.3.3}$$

and

$$E(t, x) = E_0 + \varepsilon F(t, x) + O(\varepsilon^2). \tag{1.3.4}$$

Then we only keep the terms linear in ε to obtain the (linear) magnetized Vlasov–Poisson system

$$\begin{cases} \partial_t u + v_1 \partial_x u + F v_1 \sqrt{f_0} + \omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) u = 0, \\ \partial_x F = -\displaystyle\int_{\mathbb{R}^2} u \sqrt{f_0} dv, \\ \displaystyle\int_{[0,2\pi]} F = 0, \end{cases} \tag{1.3.5}$$

where the unknowns are u and F the perturbations of the distribution function and the electric field. The condition

$$\int_{[0,2\pi]} F = 0 \tag{1.3.6}$$

implies

$$\int_{[0,2\pi]} \int_{\mathbb{R}^2} u \sqrt{f_0} dx dv = 0. \tag{1.3.7}$$

Hence, we naturally define the Hilbert space $L^2_0(\mathcal{T} \times \mathbb{R}^2)$ (endowed with the canonical L^2 scalar product $(\cdot,\cdot)_{\mathcal{H}}$) defined by

$$L^2_0(\mathcal{T} \times \mathbb{R}^2) := \left\{ u \in L^2(\mathcal{T} \times \mathbb{R}^2) \text{ with } \int_{[0,2\pi]} \int_{\mathbb{R}^2} u \sqrt{f_0} dx dv = 0 \right\}. \tag{1.3.8}$$

Note that the unknowns u, F are assumed to be periodic in x. Now we rewrite system (1.3.5) in the framework of spectral theory highlighting the magnetized Vlasov–Poisson operator H

$$i \partial_t u = H u \tag{1.3.9}$$

and H given by

$$H = H_0 u - i v_1 \sqrt{f_0} F \quad \text{ with } \partial_x F = -\int_{\mathbb{R}^2} u \sqrt{f_0} dv. \tag{1.3.10}$$

H_0 is the only self-adjoint extension (this is proved in chapter 2) of h_0 the free transport operator associated to the Vlasov equation without coupling with the electric field

$$h_0 = i \left(-v_1 \partial_x + \omega_c (v_2 \, \partial_{v_1} - v_1 \, \partial_{v_2})\right) \tag{1.3.11}$$

and defined for smooth functions.

Unfortunately the operator H defined on $D[H] = D[H_0]$ isn't self-adjoint, which means studying the magnetized Vlasov–Poisson system isn't the right setting to use spectral theory. To remedy this, we replace the Poisson equation with the Ampère equation given by

$$\partial_t F = I^* \int_{\mathbb{R}^2} v_1 \sqrt{f_0} \, u \, dv, \tag{1.3.12}$$

where I^* is the space operator such that $I^* g = g - [g]$ and the mean value in space of a function g is denoted by $[g]$. We need to modify the Ampère equation (1.3.1) by adding this operator I^* to guarantee the equivalence between the Vlasov–Poisson and Vlasov–Ampère systems (with some added assumptions on the initial condition). Now we can write the Vlasov–Ampère system,

$$\begin{cases} \partial_t u + v_1 \partial_x u + F v_1 \sqrt{f_0} + \omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) u = 0, \\ \partial_t F = I^* \displaystyle\int_{\mathbb{R}^2} v_1 \, \sqrt{f_0} \, u dv. \end{cases} \tag{1.3.13}$$

Just like for the magnetized Vlasov–Poisson system, we rewrite this system in the framework of spectral theory, with a Hilbert space $\mathcal{H}$ (endowed with the canonical L^2 scalar product $(\cdot,\cdot)_{\mathcal{H}}$) defined by

$$\mathcal{H} := L^2((0,2\pi)\times\mathbb{R}^2) \oplus L^2_0(0,2\pi), \tag{1.3.14}$$

with

$$L^2_0(0,2\pi) := \left\{ F \in L^2(0,2\pi) : \int_0^{2\pi} F(x)\,dx = 0 \right\}. \tag{1.3.15}$$

Now we can write

$$i\partial_t \begin{pmatrix} u \\ F \end{pmatrix} = \mathbf{H} \begin{pmatrix} u \\ F \end{pmatrix}, \tag{1.3.16}$$

where $\mathbf{H}$ is the magnetized Vlasov–Ampère operator given by

$$\mathbf{H} = \begin{bmatrix} H_0 & -iv_1\, e^{\frac{-v^2}{4}} \\ iI^* \int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}} \cdot dv & 0 \end{bmatrix} \tag{1.3.17}$$

and defined on

$$D[\mathbf{H}] := D[H_0] \oplus L^2_0(0,2\pi). \tag{1.3.18}$$

We prove that H_0 is self-adjoint because it is unitarily equivalent to a multiplication operator. Then we write

$$\mathbf{H} = \begin{bmatrix} H_0 & 0 \\ 0 & 0 \end{bmatrix} + \mathbf{V}, \tag{1.3.19}$$

and we can show $\mathbf{V}$ is a symmetric operator that sends $\mathcal{H}$ into $\mathcal{H}$. This implies that $\mathbf{H}$ is self-adjoint thanks to the Kato–Rellich theorem [76, Chapter 5, Theorem 4.3].

Finally, we emphasize that this approach is optimal because the L^2 regularity on the solution (u, F) is the minimum requirement to properly pose the problem. This L^2 framework is further justified by the fact that the L^2 regularity is propagated in time, because the magnetized Vlasov–Ampère system is endowed with the following energy relation:

$$\frac{d}{dt}\left(\int_{[0,2\pi]\times\mathbb{R}^2} \frac{u^2}{2} dxdv + \int_{[0,2\pi]} \frac{F^2}{2} dx \right) = 0. \tag{1.3.20}$$

1.3.2 Interpretation of the Bernstein–Landau paradox with the spectrum of the magnetized Vlasov–Ampère operator

We have shown that working with the Vlasov–Ampère system (1.3.13) is the right framework because we can find a self-adjoint operator $\mathbf{H}$ associated to this system. Now we study the spectrum and eigenfunctions of $\mathbf{H}$. We begin by presenting the orthogonal decomposition of the Hilbert space $\mathcal{H}$ into subspaces $\mathcal{H}^{pp}, \mathcal{H}^{ac}, \mathcal{H}^{sc}$ defined according to the spectrum of $\mathbf{H}$ (a classical result found in [107, 128]),

$$\mathcal{H} = \mathcal{H}^{pp} \oplus \mathcal{H}^{ac} \oplus \mathcal{H}^{sc}. \tag{1.3.21}$$

where $\mathcal{H}^{pp}$ (resp. $\mathcal{H}^{ac}$, $\mathcal{H}^{sc}$) corresponds to the pure point (resp. absolutely continuous, resp. singular continuous) part of the spectrum. The pure point subspace is spanned by the eigenfunctions of $\mathbf{H}$

$$\mathcal{H}^{pp} = \{\varphi \in \mathcal{H} : \mathbf{H} = \lambda\varphi \text{ for some } \lambda \in \mathbb{R}\}. \tag{1.3.22}$$

We refer to [107] for precise definitions of $\mathcal{H}^{ac}$ and $\mathcal{H}^{sc}$ because we won't be manipulating these subspaces.

In our study, we are interested in how the spectrum evolves with the cyclotron frequency ω_c. For $\omega_c = 0$, it was proved in [36, 37] that the spectrum of the Vlasov–Ampère operator $\mathbf{H}$ is made up of an absolutely continuous part and a kernel. Landau damping is a direct consequence of this result because for a self-adjoint operator V we have that $\exp(-itV)P_0$ goes weakly to 0 (with P_0 the projection on the subspace $\mathcal{H}^{ac}$). For $\omega_c \neq 0$, we are going to compute the eigenvalues and eigenfunctions of $\mathbf{H}$. One of the main results of chapter 2 is that when $\omega_c \neq 0$ the operator $\mathbf{H}$ only has pure point spectrum.

Theorem 1.3.1 (Charles, Després, Rege and Weder, [26])**.** *We assume that $\omega_c > 0$. Then the Vlasov–Ampère operator $\mathbf{H}\colon D[\mathbf{H}] \to \mathcal{H}$ given by (1.3.17) satisfies*

$$\mathcal{H} = \mathcal{H}^{pp}. \tag{1.3.23}$$

One way to prove this result is simply by explicitly computing the eigenvalues and eigenfunctions of $\mathbf{H}$, which in practice means finding the solutions to the system

$$\begin{cases} H_0 u - i v_1\, e^{\frac{-v^2}{4}} F = \lambda u, \\ i I^* \displaystyle\int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}}\, u\, dv = \lambda F, \end{cases} \tag{1.3.24}$$

with $\lambda \in \mathbb{R}$ and $\begin{pmatrix} u \\ F \end{pmatrix} \in D[\mathbf{H}]$. To perform these computations we rely on three natural decompositions of $\mathcal{H}^{pp}$. The first one is based on Fourier decomposition in the position variable x (because the unknowns u, F are periodic in x), while the second one relies on the classical relation involving the kernel of $\mathbf{H}$ and its orthogonal $\mathcal{H} = \mathrm{Ker}[\mathbf{H}] \oplus \mathrm{Ker}[\mathbf{H}]^{\perp}$. The third one is based on distinguishing the eigenfunctions with vanishing electric field F. In the following table we enumerate the different eigenfunctions of $\mathbf{H}$ and distinguish them using these three decompositions.

Eigenfunctions	Ker	Electric field F	Fourier mode n	λ	Ref.
$\mathbf{V}_n^{(0)}$	$\mathrm{Ker}[\mathbf{H}]$	$\neq 0$	$n \in \mathbb{Z}^*$	0	(2.5.18)
$\mathbf{M}_{n,j}^{(0)}$	$\mathrm{Ker}[\mathbf{H}]$	$= 0$	$n \in \mathbb{Z}$	0	(2.5.18), (2.4.2)
$\mathbf{V}_{m,j}$	$\mathrm{Ker}[\mathbf{H}]^{\perp}$	$= 0$	$n = 0$	$m\omega_c$ with $m \neq 0$	(2.5.26), (2.4.2)
$\mathbf{W}_{n,m,j}$	$\mathrm{Ker}[\mathbf{H}]^{\perp}$	$= 0$	$n \in \mathbb{Z}^*$	$m\omega_c$ with $m \neq 0$	(2.5.37)
$\mathbf{Y}_{n,m}$	$\mathrm{Ker}[\mathbf{H}]^{\perp}$	$\neq 0$	$n \in \mathbb{Z}^*$	$\lambda_{n,m}$ with $m \neq 0$	(2.5.54), (2.5.51)

In chapter 2, we prove that the eigenfunctions enumerated above form a complete set of $\mathcal{H}$.

Theorem 1.3.2 (Charles, Després, Rege and Weder, [26])**.** *Let $\mathbf{H}$ be the magnetized Vlasov–Ampère operator defined in (1.3.17) and (1.3.18). Then, the eigenfunctions of $\mathbf{H}$ are a complete set in $\mathcal{H}$.*

Theorem 1.3.1 is a corollary of this result and thanks to our computations we can also enumerate the different eigenvalues of $\mathbf{H}$ with their multiplicities.

Theorem 1.3.3 (Charles, Després, Rege and Weder, [26])**.** *Let $\mathbf{H}$ be the magnetized Vlasov–Ampère operator defined in (1.3.17) and (1.3.18). Then $\mathbf{H}$ is self-adjoint and it has pure point spectrum. The eigenvalues of $\mathbf{H}$ are given by:*

1. *The infinite multiplicity eigenvalues* $\lambda_m^{(0)} := m\omega_c, m \in \mathbb{Z}$.

2. *The simple eigenvalues* $\lambda_{n,m}, n, m \in \mathbb{Z}^*$, *given by the roots to equation* (2.5.51).

Thanks to this result, for any initial data (u, F) we can express the solution of (1.3.13) as a sum of terms $(u_\lambda, F_\lambda) = \exp(-it\lambda)\left((u, F), (v_\lambda, G_\lambda)\right)_{\mathcal{H}} (v_\lambda, G_\lambda)$ where (v_λ, G_λ) is a normalized eigenfunction of $\mathbf{H}$ associated to the eigenvalue λ. This means that for non trivial initial data, the solution to the magnetized Vlasov–Ampère system is the sum of an oscillatory function ($\lambda \neq 0$) and of a constant function ($\lambda = 0$). Hence, for $\omega_c > 0$ Landau damping does not occur.

To conclude this subsection, we give an interpretation of the Bernstein–Landau paradox using the results detailed above. This interpretation actually shows that there is no paradox, because it can be explained by a classical property of spectral theory, namely that for a self-adjoint operator depending on a parameter, the domain of the operator can change sharply at certain values of the parameter.

Interpretation of the Bernstein–Landau paradox: In the limit when the magnetic field goes to zero the spectrum of the magnetized Vlasov–Ampère operator changes drastically from pure point to absolutely continuous, due to a sharp change on its domain.

1.3.3 Numerical tests using semi-Lagrangian schemes

In chapter 2, we use a numerical method to approximate the solutions of the magnetized Vlasov–Ampère system, with the aim to test the eigenfunctions given above. As we have already said, we rely on a semi-Lagrangian method with splitting (and we use cubic spline interpolation). Semi-Lagrangian schemes were first introduced by Cheng and Knorr in 1976 [30] to approximate solutions to the 1d-1v Vlasov–Poisson system. Since these schemes require a grid for the phase space, they are quite slow and inefficient in large dimensions. For plasma simulations in large dimensions (2d-2v and above) Particle-in-Cell (PIC) methods [32] are used far more extensively. Here, since we are working in a 1d-2v setting (our phase space is of dimension 3), semi-Lagrangian schemes are appropriate tools to approximate the solutions of (1.3.13). The originality of semi-Lagrangian schemes, compared to other numerical methods, is that they use the characteristics of the approximated system to compute the solution at each time step. In this subsection, we are going to present the backward semi-Lagrangian method, showing how this method exploits the characteristics of the system. Then we will detail the different steps of the numerical method from chapter 2 used to approximate the magnetized Vlasov–Ampère system.

We consider a 1d transport equation $\partial_t f + b \cdot \partial_x f = 0$ with $b : \mathbb{R}^d \to \mathbb{R}$, then we set $X := X(s; t; x)$ for the characteristic (X is the characteristic equal to $x \in \mathbb{R}$ at time $t \geq 0$). We call x_i the points of the regular grid in position with h the position step and we call t_n the points of the discretization in time with Δt the time step. The classical semi-Lagrangian scheme is called "backward" because we compute the approximation f^{n+1} of the solution f at time t_{n+1} using the approximation f^n at the previous time t_n. Now we detail the three steps that make up this semi-Lagrangian scheme:

1. For every point x_i of the grid in position, we compute the value at time t_n of the characteristic $X(t_n; t_{n+1}, x_i)$ equal to x_i at time t_{n+1}.

2. Thanks to the fact that f is the solution to a transport equation, we have the following

relation:

$$f(t,x) = f(s, X(s;t,x))$$

for all $x \in \mathbb{R}^d$ and $t, s \geq 0$. In the discretized setting, this means we can express f^{n+1} using f^n and the characteristic calculated at the previous step because for each x_i we have

$$f^{n+1}(x_i) = f^n(X(t_n; t_{n+1}, x_i)). \tag{1.3.25}$$

Hence we obtain the desired approximation at the next time $f^{n+1}(x_i)$ by computing $f^n(X(t_n; t_{n+1}, x_i))$.

3. However, since $X(t_n; t_{n+1}, x_i)$ isn't necessarily a point x_i of the grid, we compute $f^n(X(t_n; t_{n+1}, x_i))$ with the values $f^n(x_i)$ using cubic spline interpolation, which is a very common interpolation technique for semi-Lagrangian schemes. In practise, this consists in finding the interpolation function f^n_S of the solution f by using the approximation f^n. Since this is a cubic spline interpolation function, we can decompose f^n_S on the basis of the cubic B-splines:

$$f^n_S(x) = \sum_j \alpha^n_j S^3(x - x_j) \tag{1.3.26}$$

with S^3 given by

$$S^3(x) = \frac{1}{6h} \begin{cases} \left(2 - \frac{|x|}{h}\right)^3 & \text{if } h \leq |x| < 2h, \\ 4 - 6\left(\frac{x}{h}\right)^2 + 3\left(\frac{|x|}{h}\right)^3 & \text{if } 0 \leq |x| < h, \\ 0 & \text{otherwise.} \end{cases} \tag{1.3.27}$$

The coefficients α^n_j are determined by evaluating f^n_S at the points of the grid, for all x_i we have

$$f^n_S(x_i) = f^n(x_i). \tag{1.3.28}$$

Thanks to the expression of S^3 (the support of S^3 is of length $4h$) and (1.3.28), we can compute the coefficients α^n_j by solving a linear system.

Finally, the approximation of f at time t_{n+1} is given by

$$f^{n+1}(x_i) = f^n_S(X(t_n; t_{n+1}, x_i)) = \sum_j \alpha^n_j S^3\left(X(t_n; t_{n+1}, x_i) - x_j\right) \tag{1.3.29}$$

with the α^n_j calculated with the values $f^n(x_i)$.

To finish this paragraph we give a figure representing the different steps of the backward semi-Lagrangian scheme in the case of the free transport equation (1.2.1).

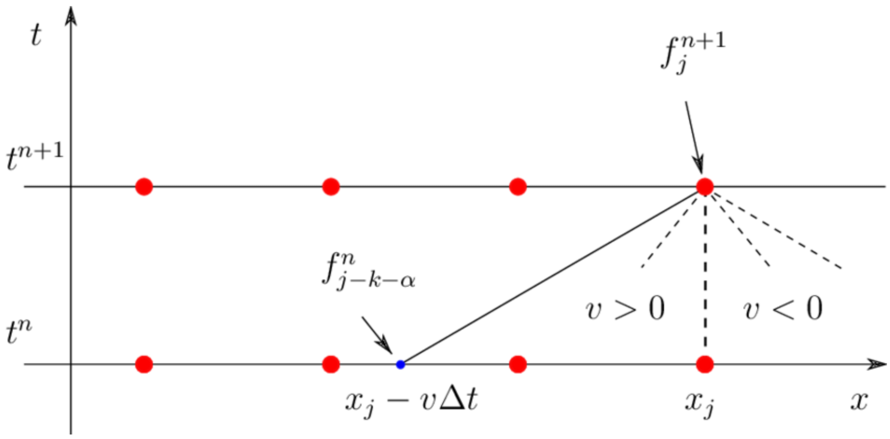

Figure 1.3 – Representation of the semi-Lagrangian approach [41].

As mentioned above, to approximate the magnetized Vlasov–Ampère system, we combine the semi-Lagrangian method with a splitting procedure. Such a procedure corresponds to approximating the solution of $\partial_t f + (\mathcal{A} + \mathcal{B})f = 0$ by solving $\partial_t f + \mathcal{A}f = 0$ and $\partial_t f + \mathcal{B}f = 0$ one after the other.

Hence, we split the magnetized Vlasov–Ampère system so as to only solve transport equations with constant advection terms.

$$\partial_t \begin{pmatrix} u \\ F \end{pmatrix} + (\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}) \begin{pmatrix} u \\ F \end{pmatrix} = 0,$$

with $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ given in (2.8.5).

Combining this splitting procedure with the backward semi-Lagrangian method, we construct the following scheme to solve the magnetized Vlasov–Ampère system.

1. **Initialization:**

   The initial distribution function and electric field $\mathbf{U}_{ini} = \begin{pmatrix} u_{ini} \\ F_{ini} \end{pmatrix}$ are given functions.

2. **Going from t_n to t_{n+1}:**

   Assume we know $\mathbf{U}^n$, the approximation of $\mathbf{U} = \begin{pmatrix} u \\ F \end{pmatrix}$ at time t_n.

   - We compute $\mathbf{U}^*$ by solving $\partial_t \mathbf{U} + \mathcal{A}\mathbf{U} = 0$ with a semi-Lagrangian scheme during one time step Δt with initial condition $\mathbf{U}^n$.
   - We compute $\hat{\mathbf{U}}$ by solving $\partial_t \mathbf{U} + \mathcal{B}\mathbf{U} = 0$ with a Runge-Kutta 2 scheme during one time step Δt with initial condition $\mathbf{U}^*$. Here we don't use a semi-Lagrangian scheme because $\mathcal{B}\mathbf{U}$ is just a source term.
   - We compute $\mathbf{U}^{**}$ by solving $\partial_t \mathbf{U} + \mathcal{C}\mathbf{U} = 0$ with a semi-Lagrangian scheme during one time step Δt with initial condition $\hat{\mathbf{U}}$.
   - We compute $\mathbf{U}^{n+1}$ by solving $\partial_t \mathbf{U} + \mathcal{D}\mathbf{U} = 0$ with a semi-Lagrangian scheme during one time step Δt with initial condition $\mathbf{U}^{**}$.

The numerical results for this scheme can be found in chapter 2, notably by consulting fig. 2.2 and fig. 2.3.

1.4 Mathematical properties of the magnetized Vlasov–Poisson system

The second question that was addressed in this thesis concerns certain mathematical properties for the (nonlinear) magnetized Vlasov–Poisson system (1.2.23). The existence of classical solutions to the (unmagnetized) Vlasov–Poisson system for general initial data (1.2.12) was proved at the beginning of the nineties in two separate papers. The first work by Pfaffelmoser [103] relied on a careful analysis of the characteristics of the system (that we will define later on). This enabled him to obtain a uniform (in x, v) estimate on the support in velocity of the solution f on a local interval of existence (which exists thanks to the results in [79]), which is sufficient to obtain global existence. This method was simplified in a subsequent paper by Schäffer [114] where the estimate on the support in velocity of f was also improved. The second work by Lions and Perthame [87] is based on the propagation of velocity moments of the distribution function defined for $k \geq 0$ by

$$M_k(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |v|^k f(t, x, v) dv dx. \tag{1.4.1}$$

In [87], this propagation property is obtained by controlling a certain norm of the electric field in order to get a Grönwall inequality on the velocity moment. This approach also guarantees propagation of the regularity of the initial data, proving the existence of classical solutions for Vlasov–Poisson. An interesting point is that even though these two approaches differ greatly, in both of these methods a fundamental assumption is to limit the influence of high velocities on the dynamics of the system. Finally in [99], Pallard combined the two approaches to prove an optimal version of the main result from [87].

With regards to uniqueness, Robert [48] first proved uniqueness for compactly supported weak solutions. Then Loeper [88] managed to extend this result by proving uniqueness for solutions with bounded macroscopic density, using techniques from optimal transportation. Finally, in a recent paper [95], Miot generalized Loeper's uniqueness condition, and in the process proved that for certain initial data with unbounded macroscopic density, the Vlasov–Poisson system is well-posed.

In chapter 4 and chapter 5, we study propagation of velocity moments, propagation of regularity and uniqueness for the magnetized Vlasov–Poisson system. We highlight how in the case of a constant magnetic field $B = (0, 0, \omega)$ (chapter 4), we can combine the method of propagation of moments from [87] and an induction procedure involving the cyclotron period $T_c = \frac{2\pi}{\omega}$ to prove the propagation of velocity moments for the magnetized Vlasov–Poisson system. For a general bounded and Lipschitz magnetic field, we combine the results by Pallard [99] and an analogous induction procedure involving the cyclotron period $T_c = \frac{1}{\|B\|_\infty}$ to show propagation of velocity moments. Finally, we show how the uniqueness conditions found in [88, 95] can be adapted to our setting.

In this subsection we summarize these results, first by highlighting how, as said above, the proofs of our main results regarding propagation of moments combine the main methods developed for the unmagnetized Vlasov–Poisson system [87, 99, 103, 114] with an induction argument that involves the cyclotron period. Then, we show how the uniqueness conditions found in [88, 95], which are conditions on the L^p norms of the macroscopic density ρ, are modified in the magnetized setting to become conditions involving the velocity moments of the initial data f^{in}.

To finish this introduction, we mention an important point related to the method of propagation of velocity moments. This method relies on the regularization procedure introduced by Arsenev in [4] to prove the existence of weak solutions for Vlasov–Poisson. This is due to the fact that in [87, 99], all the a priori estimates in the proofs are true only for smooth solutions. However, since these estimates depend only on constants and quantities that are conserved by the regularization procedure, we can pass to the limit in the approximate or regularized system from [4]. This is also true for the estimates that we find in chapter 4 and chapter 5, which means we can also use Arsenev's approximate system. Since this is an important part in the proofs, in chapter 3 we detail exactly what we mean by "conserved quantities" and present the approximate Vlasov–Poisson system from [4].

1.4.1 Propagation of velocity moments with an induction procedure depending on the cyclotron period

Propagation of moments with a constant magnetic field

In chapter 4, we prove the propagation of velocity moments for the three-dimensional magnetized Vlasov–Poisson system for electrons (1.2.23) with a constant magnetic field given by $B = (0, 0, \omega)$ and $\omega > 0$ (theorem 1.4.2 written below), which we rewrite here without the electric potential and with the physical constants normalized to 1:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E + v \wedge B) \cdot \nabla_v f = 0, \\ E(t,x) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} f(t,y,v) dv dy. \end{cases} \tag{1.4.2}$$

As said above, we are going to rely on the main result from [87] (given in the following theorem), where the propagation of velocity moments for weak solutions to Vlasov–Poisson was proved.

Theorem 1.4.1 (Lions and Perthame, [87]). *Let $f^{in} \geq 0$, $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$. We assume that*

$$\int_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^m f^{in}(x,v) dx dv < +\infty \quad \text{if } m < m_0 \tag{1.4.3}$$

where $3 < m_0$. Then there exists a solution $f \in C(\mathbb{R}_+; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+; L^p(\mathbb{R}^3 \times \mathbb{R}^3))$ (for all $1 \leq p < +\infty$) of the Vlasov–Poisson system ((1.4.2) with $\omega = 0$) satisfying

$$\sup_{t \in [0,T]} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^m f(t,x,v) dx dv < +\infty, \tag{1.4.4}$$

for all $T < +\infty$, $m < m_0$.

First, following the different steps from [111] where the ideas of [87] were simplified, we are going to summarize the proof of theorem 1.4.1. Then we will present the main result from chapter 4 which is analogous to theorem 1.4.1 for (1.4.2) and show how these different steps change in the proof of our result. By doing this, we hope to illustrate why an induction procedure involving the cyclotron period is necessary in the magnetized case. In the following discussions, the quantity C

will denote a constant that depends only on conserved quantities and that can change from one line to the other.

- ☐ **Step 1: A differential inequality on M_k** We differentiate the velocity moment of order k M_k, and by using the Vlasov equation and integrating by parts, we can write the following estimate:
$$\left|\frac{d}{dt}M_k(t)\right| = \left|\iint k\,|v|^{k-2}\,v\cdot Ef dvdx\right|. \tag{1.4.5}$$
Thanks to a classical moment inequality, this last estimate yields the following differential inequality:
$$\left|\frac{d}{dt}M_k(t)\right| \leq C\,\|E(t)\|_{k+3}\,M_k(t)^{\frac{k+2}{k+3}}. \tag{1.4.6}$$
This inequality shows us that, in order to obtain a satisfactory Grönwall inequality on M_k, we need to control $\|E(t)\|_{k+3}$ with the quantity $M_k(t)^{\frac{1}{k+3}}$.

- ☐ **Step 2: A representation formula for ρ** We go back to the Vlasov equation and write it like a linear transport equation with the nonlinear term $E\cdot\nabla_v f$ considered as a source term. This means we can write the solution of the Vlasov equation using this "new" source term and the characteristics of the free transport equation (1.2.1) which are defined by
$$\begin{cases} \dfrac{d}{ds}X(s;t,x,v) = V(s;t,x,v), \\ \dfrac{d}{ds}V(s;t,x,v) = 0. \end{cases} \tag{1.4.7}$$
We immediately get $(X(s;t,x,v),V(s;t,x,v)) = (x+(s-t)v,v)$, and so we can write the distribution function f using this other point of view
$$f(t,x,v) = f^{in}(x-tv,v) - \int_0^t E\cdot\nabla_v f(s,x+(s-t)v,v)ds. \tag{1.4.8}$$
The second integral term can be written as $\mathrm{div}_x\cdots+\mathrm{div}_v\ldots$ thanks to the characteristics and the fact that E is independent of v. After integration in v, this results in the following representation formula for ρ:
$$\rho(t,x) = \int_{\mathbb{R}^3} f^{in}(x-tv,v)dv + \mathrm{div}_x\int_0^t\int_{\mathbb{R}^3}(s-t)(fE)(s,x+(s-t)v,v)dvds. \tag{1.4.9}$$

- ☐ **Step 3: Control of the electric field** This is actually the most difficult step. Thanks to (1.4.9), we can estimate $\|E(t)\|_{k+3}$ simply because $E = -\nabla_x\mathcal{G}_3\star\rho$. With the help of different classical functional inequalities, we show that to estimate $\|E(t)\|_{k+3}$ we need to estimate the $k+3$-norm of $\Sigma(t,x) := \int_0^t\int_{\mathbb{R}^3}(s-t)(fE)(s,x+(s-t)v,v)dvds = \int_0^t\sigma(s,t,x)ds$. The analysis of the term σ yields an estimate which depends on $M_k^{\frac{1}{k+3}}$
$$\|\sigma(s,t,\cdot)\|_{k+3} \leq \|\sigma(s,t,\cdot)\|_{k+3} \leq \frac{C}{s}M_k(t-s)^{\frac{1}{k+3}}. \tag{1.4.10}$$
This is almost the desired estimate, except for the singularity at $t=0$. We deal with this issue by splitting the integral $\int_0^t\sigma = \int_0^{t_0}\sigma + \int_{t_0}^t\sigma$ where $\int_0^{t_0}\sigma$ is a "small time" part that

we estimate roughly and $\int_{t_0}^{t}\sigma$ a "large time" part that we estimate with (1.4.10). Then we can optimize the parameter t_0 in order to obtain the following estimate on $\|E(t)\|_{k+3}$:

$$\|E(t)\|_{k+3} \leq C(1+\sup_{0\leq s\leq t} M_k(s))^{\frac{1}{k+3}} \ln\left(1+\sup_{0\leq s\leq t} M_k(s)\right). \tag{1.4.11}$$

- ☐ **Step 4: A Grönwall inequality for the moments on** $[0\,,T]$ Thanks to (1.4.11), we get a linear Grönwall inequality for $y(t) = 1 + \sup\limits_{0\leq s\leq t} M_k(s)$ on $[0\,,T]$ which concludes the proof of theorem 1.4.2.

Now we give the main result from chapter 4.

Theorem 1.4.2 (Rege, [110]). *Let $k_0 > 3, T > 0, f^{in} = f^{in}(x,v) \geq 0$ a.e. with $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and assume that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^{k_0} f^{in} dxdv < \infty. \tag{1.4.12}$$

Then there exists a weak solution

$$f \in C(\mathbb{R}_+; L^p(\mathbb{R}^3\times\mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+; L^p(\mathbb{R}^3\times\mathbb{R}^3)) \tag{1.4.13}$$

$(1 \leq p < +\infty)$ to the Cauchy problem for the Vlasov–Poisson system with magnetic field (1.4.2) in $\mathbb{R}^3 \times \mathbb{R}^3$ such that

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^{k} f(t,x,v) dxdv \leq C < +\infty, \quad 0 \leq t \leq T, \tag{1.4.14}$$

for all k such that $0 \leq k \leq k_0$, and with $C = C(T, k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in})) > 0$.

As we already mentioned, we use the same steps as in the proof of theorem 1.4.1 and we show why an added induction procedure depending on T_c is necessary.

- ☐ **Step 1: A differential inequality on** M_k We differentiate M_k in the same way and obtain the same estimate (1.4.6) because the magnetic part of the Lorentz force vanishes in the computations (this is physically coherent because $M_k(t)$ is a macroscopic quantity and the magnetic force doesn't work on the system)

$$\left|\frac{d}{dt}M_k(t)\right| = \left|\iint k\,|v|^{k-2}\,\underbrace{v\cdot(E+v\wedge B)}_{=v\cdot E} f dvdx\right|. \tag{1.4.15}$$

  Hence, just like in the unmagnetized setting, we need to control $\|E(t)\|_{k+3}$ with the quantity $M_k(t)^{\frac{1}{k+3}}$.

- ☐ **Step 2: A representation formula for** ρ We follow the same procedure by considering the nonlinear term $E\cdot\nabla_v f$ as a source term. However now we have to deal with the characteristics of the following transport equation:

$$\begin{cases} \dfrac{d}{ds}X(s;t,x,v) = V(s;t,x,v), \\ \dfrac{d}{ds}V(s;t,x,v) = V(s;t,x,v)\wedge B. \end{cases} \tag{1.4.16}$$

The characteristics (which we will write $X(s), V(s)$ in the rest of this discussion) are a bit more complicated than for (1.4.7), but we can still compute them explicitly, and since $\partial_x V = 0$ and $\partial_x X = \mathrm{Id}$, we find a decomposition for f given by

$$f(t,x,v) = f^{in}(x - tv, v) + \mathrm{div}_x \int_0^t fH ds + \mathrm{div}_v \int_0^t fG ds, \tag{1.4.17}$$

with the coordinates of G, H being linear combinations of the coordinates of E. This means that after integration in v, we have a similar representation formula for ρ compared to (1.4.9):

$$\rho(t,x) = \int_{\mathbb{R}^3} f^{in}(x - tv, v) dv + \mathrm{div}_x \int_0^t \int_{\mathbb{R}^3} (fH)(s, X(s), V(s)) dv ds. \tag{1.4.18}$$

☐ **Step 3: Control of the electric field** We proceed to estimate $\|E(t)\|_{k+3}$ using the same tools, which means we need to estimate $\Sigma(t,x) := \int_0^t \int_{\mathbb{R}^3} (fH)(s, X(s), V(s)) dv ds = \int_0^t \sigma(s,t,x) ds$. The analysis of the term σ yields the following estimate

$$\|\sigma(s,t,\cdot)\|_{k+3} \leq C \frac{\sqrt{2}}{s} \left(\frac{\omega^2 s^2}{2(1 - \cos(\omega s))} \right)^{\frac{2}{3}} M_k (t-s)^{\frac{1}{k+3}}, \tag{1.4.19}$$

where we see a new term $\left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))} \right)^{\frac{2}{3}}$ compared to (1.4.10) in the unmagnetized case. This means we now have singularities at multiples of the cyclotron period $t = \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \ldots$ and we notice that the singularity in $t = 0$ that was present in the unmagnetized case still remains. We can deal with this singularity in $t = 0$ like in the unmagnetized case but unfortunately that's not the case for the others. Hence the first step is to show propagation of moments on $[0, T_\omega]$ with $T_\omega = \frac{T_c}{2} = \frac{\pi}{\omega}$ (in fact we can take any time $< T_c$).

☐ **Step 4: A Grönwall inequality for the moments on** $[0, T_\omega]$ For $t \in [0, T_\omega]$, thanks to the previous analysis we can show the following estimate

$$\|E(t)\|_{k+3} \leq C(1 + \sup_{0 \leq s \leq t} M_k(s))^{\frac{1}{k+3}} \ln \left(1 + \sup_{0 \leq s \leq t} M_k(s) \right). \tag{1.4.20}$$

This is the same as (1.4.11) except here the constant C depends on ω. Thanks to (1.4.20), we get a linear Grönwall inequality for $y(t) = 1 + \sup_{0 \leq s \leq t} M_k(s)$ on $[0, T_\omega]$ which proves propagation of moments on $[0, T_\omega]$.

☐ **Step 5: Propagation of moments for all time by induction** The main argument here is that the constants in (1.4.20) depend only on T_ω, k, ω which are constants, $\|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in}$ ($\mathcal{E}_{in}$ is the initial energy of the system) which are conserved quantities, and $M_k(f^{in})$ which is finite. Since we are looking to prove by induction that $M_k(t)$ is bounded for all time, the assumption $M_k(f^{in}) < +\infty$ is the initialization in our induction. With these observations, we can follow the previous steps on any time interval $[nT_\omega, (n+1)T_\omega]$, and this concludes the proof of theorem 1.4.2.

Propagation of moments with a general bounded magnetic field

We present one of the main results of chapter 5, which extends the main theorem of chapter 4, where we show propagation of velocity moments for weak solutions to the magnetized Vlasov–Poisson system (1.4.2) with a general magnetic field. However, we note that in chapter 5 we succeed in showing this result only in the case of a magnetic field $B := B(t)$ that depends on time and that is independent of the position x. In chapter 5, we explain why our proof only works for such a magnetic field and also present a lead to adapt our proof to the case of a more general magnetic field $B := B(t, x)$ that also depends on x.

We also assume that B verifies

$$B \in W^{1,\infty}(\mathbb{R}_+). \tag{1.4.21}$$

This is a natural regularity assumption on B which enables us to have well-posed characteristics in the case of smooth solutions.

As mentioned above, our method relies on the paper by Pallard [99], where propagation of velocity moments for weak solutions to Vlasov–Poisson was proved for any velocity moment of order $k > 2$, improving theorem 1.4.1 from [87]. An important idea in [99] is to combine the Eulerian point of view developed in [87], where the authors studied the behavior of velocity moments, and the Lagrangian point of view developed by Pfaffelmoser in [103] and improved by Schäffer in [114], where the existence of classical solutions to Vlasov–Poisson was shown by studying the characteristics of the system. More precisely, in [99] Pallard exploited this second Lagrangian approach to obtain an a priori bound depending on the characteristics. This approach involves studying the following quantity defined for any $t > 0$ and $\delta \in]0\,, t[$:

$$Q(t, \delta) := \sup \left\{ \int_{t-\delta}^{t} |E(s, X(s; 0, x, v))| \, ds, (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \right\}. \tag{1.4.22}$$

Indeed, $Q(t, \delta)$ quantifies the evolution of the characteristics of the Vlasov–Poisson system given by

$$\begin{cases} \dfrac{d}{ds} X(s; t, x, v) = V(s; t, x, v), \\ \dfrac{d}{ds} V(s; t, x, v) = E(s, X(s; t, x, v)), \end{cases} \tag{1.4.23}$$

with

$$(X(t; t, x, v), V(t; t, x, v)) = (x, v). \tag{1.4.24}$$

All the following results and inequalities in this subsection will be written for smooth solutions in both unmagnetized and magnetized frameworks, mainly because the quantity $Q(t, \delta)$ isn't necessarily well defined for functions in Lebesgue spaces. However the result on propagation of velocity moments is true for general initial data $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ which verifies the assumptions of theorem 1.4.1 (with $2 < m_0$) thanks to the approximation procedure mentioned above. The main contribution in [99] was to show a uniform bound for Q, which is given in the next theorem.

Theorem 1.4.3 (Pallard, [99]). *Let $f^{in} \in C^1_c(\mathbb{R}^3 \times \mathbb{R}^3)$ with $f^{in} \geq 0$ and let $f \in C^1_c(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ be the unique classical solution to the Vlasov–Poisson system. Then for any $k > 2, T > 0$ we have*

$$\sup_{0 \leq t \leq T} Q(t, t) \leq C(T^{\frac{1}{2}} + T^{\frac{7}{5}}), \tag{1.4.25}$$

with C that only depends on

$$k, \mathcal{E}(0), \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \iint |v|^k f^{in}(x,v)dvdx. \tag{1.4.26}$$

The propagation of velocity moments is a direct consequence of (1.4.25) because we can write

$$\begin{aligned}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f(t,x,v)dvdx &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} |V(t;0,x,v)|^k f^{in}(x,v)dvdx \\ &\leq \iint_{\mathbb{R}^3\times\mathbb{R}^3} (|v| + Q(t,t))^k f^{in}(x,v)dvdx \\ &\leq 2^{k-1}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f^{in}(x,v)dvdx + Q(t,t)^k \left\|f^{in}\right\|_1\right).\end{aligned}$$

As said above, the estimate (1.4.25) on $Q(t,t)$ is obtained by studying the characteristics (1.4.23). Hence we fix $(t, x_*, v_*) \in [0\,, T] \times \mathbb{R}^3 \times \mathbb{R}^3$ and set $(X_*, V_*)(s) = (X_*, V_*)(s; t, x_*, v_*)$. For any $\delta \in [0\,, t]$ we can estimate $Q(t,\delta)$ by controlling $\int_{t-\delta}^t E(s, X_*(s))ds$ independently of (x_*, v_*). Thanks to the definition of E we can write

$$\int_{t-\delta}^t E(s,X_*(s))ds \leq \int_{t-\delta}^t \int \frac{\rho(s,x)dx}{4\pi\left|x - X_*(s)\right|^2}ds = \int_{t-\delta}^t \iint \frac{f(s,x,v)dxdv}{4\pi\left|x - X_*(s)\right|^2}ds = \frac{I_*(t,\delta)}{4\pi}.$$

In [99], $I_*(t,\delta)$ is estimated by splitting the space of integration $[t-\delta\,, t] \times \mathbb{R}^3 \times \mathbb{R}^3$ in three parts. This method is inspired from Schäffer's paper [114] where such a splitting was used to show the existence of smooth solutions to the Vlasov–Poisson system. The partition from [114] is modified in [99] by introducing a small parameter $\varepsilon > 0$ (specified in chapter 5) and the three parts of the partition are given by

$$\begin{aligned} G &= \{(s,x,v) : \min(|v|\,, |v - V_*(s)|) < P\}\,, \\ B &= \{(s,x,v) : |x - X_*(s)| \leq \Lambda_\varepsilon(s,v)\} \setminus G, \\ U &= [t-\delta\,, t] \times \mathbb{R}^3 \times \mathbb{R}^3 \setminus (G \cup B), \end{aligned}$$

with

$$P = 2^{10} Q(t,\delta) \text{ and } \Lambda_\varepsilon(s,v) = L(1 + |v|^{2+\varepsilon})^{-1} \left|v - V_*(s)\right|^{-1} \text{ with } L > 0. \tag{1.4.27}$$

Using obvious notations, we write $I_* = I_*^G + I_*^B + I_*^U$. The set G is considered the "good" set because it's a part of the integration space where the velocities are bounded. The set B is the "bad" set because the velocities are large and the distance $|x - X_*(s)|$ is small which means the singularity in the integral is stronger. Finally U is considered the "ugly" set. In [99, 114], I_*^G and I_*^B are estimated using static estimates, while the time integral is exploited in a crucial way to obtain a sharp estimate for I_*^U. Similarly, in the magnetized case I_*^G and I_*^B are estimated using the same static estimates as in the unmagnetized case. However the estimate for I_*^U inherently depends on the characteristics of the system, which adds a difficulty in the magnetized case because of the modified characteristics.

As said above, in both unmagnetized and magnetized frameworks, we exploit the time integral to estimate I_*^U, so with the change of variables $(y,w) = (X(s;t,x,v), V(s;t,x,v)) = (X(s), V(s))$ we can write

$$I_*^U = \int_{t-\delta}^t \iint \frac{f(s,y,w)\mathbf{1}_U(s,y,w)dydw}{\left|x - X_*(s)\right|^2}ds = \iint \int_{t-\delta}^t \frac{\mathbf{1}_U(s,X(s),V(s))}{\left|X(s) - X_*(s)\right|^2}ds f(t,x,v)dxdv.$$

This last inequality is true because f is the solution to a transport equation and so is constant along the characteristics.

Now we give an important lemma from [99].

Lemma 1.4.4 (Pallard, [99]). *For any $(x, v) \in \mathbb{R}^6$ we have:*

$$\int_{t-\delta}^{t} \frac{\mathbf{1}_U(s, X(s), V(s))}{|X(s) - X_*(s)|^2} ds \leq C \frac{(1 + |v|^{2+\varepsilon})}{L} \tag{1.4.28}$$

with $C = C(\mathcal{E}(0), \|f^{in}\|_1, \|f^{in}\|_\infty)$.

A crucial point in the proof of lemma 1.4.4 is that if there exists $s_1 \in [t - \delta, t]$ such that $(s_1, X(s_1), V(s_1)) \in U$ then for all $s \in [t - \delta, t]$ we have the following estimates for the velocity characteristic (in the unmagnetized case):

$$\begin{aligned} 2^{-1} |v| \leq |V(s)| \leq 2 |v|, \\ 2^{-1} |v - v_*| \leq |V(s) - V_*(s)| \leq 2 |v - v_*|. \end{aligned} \tag{1.4.29}$$

This result is intuitive because $V(s_1) \geq P = 2^{10} Q(t, \delta)$ which means $V(s_1)$ is 2^{10} larger than the variation of $V(\cdot)$ on $[t - \delta, t]$, which implies that $V(\cdot)$ doesn't vary much on $[t - \delta, t]$. Since $V(t) = v$, we can estimate $|V(s)|$ taking v as a reference point to obtain (1.4.29).

In the magnetized case, we try to obtain similar estimates using the characteristics now defined by

$$\begin{cases} \dfrac{d}{ds} X(s; t, x, v) = V(s; t, x, v), \\ \dfrac{d}{ds} V(s; t, x, v) = E(s, X(s; t, x, v)) + V(s; t, x, v) \wedge B(s, X(s; t, x, v)), \end{cases} \tag{1.4.30}$$

with

$$(X(t; t, x, v), V(t; t, x, v)) = (x, v). \tag{1.4.31}$$

The evolution of the velocity characteristic in this framework is quantified by the following inequality for $s \in [t - \delta, t]$:

$$|V(s)| \leq (|v| + Q(t, \delta)) \exp(\delta \|B\|_\infty). \tag{1.4.32}$$

This estimate is obtained via a classical Grönwall inequality because thanks to (1.4.30) we have

$$|V(s)| \leq |v| + \int_s^t |E(\tau, X(\tau; t, x, v))| \, d\tau + \int_s^t |V(\tau; t, x, v)| \, \|B\|_\infty \, d\tau.$$

The inequality (1.4.32) highlights how the evolution of $V(\cdot)$ is influenced by the strength of the magnetic field. Indeed, the estimates (1.4.29) are no longer verified in the magnetized case if the magnetic field is large enough. One way around this is to limit our analysis to an interval $[0, T_B]$ with T_B (defined more precisely in chapter 5) depending on the cyclotron period $T_c = \frac{1}{\|B\|_\infty}$, exactly like in the constant magnetic case presented previously. Hence, on $[0, T_B]$ we manage to adapt the analysis from [99] and we obtain an estimate on $Q(t, t)$ analogous to (1.4.25).

Proposition 1.4.5 (Rege, [109]). *For all $T > 0$ such that $T \leq T_B$ (T_B defined in chapter 5), we have the following estimate on $Q(t, t)$*

$$Q(t, t) \leq C \exp(T \|B\|_\infty)^{\frac{2}{5}} (T^{\frac{1}{2}} + T^{\frac{7}{5}}), \tag{1.4.33}$$

with C that depends on

$$k, \mathcal{E}(0), \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \iint |v|^k f^{in}(x,v)dvdx.$$

Like in the constant magnetic field case, we rely on an induction procedure to prove the boundedness of $Q(t,t)$ for all time. First we can write for all $n \in \mathbb{N}$

$$Q(nT_B, nT_B) = \sum_{p=0}^{n-1} Q((p+1)T_B, T_B).$$

Then for all p we can use the same analysis as previously to estimate $Q((p+1)T_B, T_B)$

$$Q((p+1)T_B, T_B) \leq C \exp(T \|B\|_\infty)^{\frac{2}{5}} (T_B^{\frac{1}{2}} + T_B^{\frac{7}{5}}) \tag{1.4.34}$$

but with the constant C now depending on

$$k, \mathcal{E}(0), \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \iint |v|^k f((p-1)T_B, x, v)dvdx.$$

However by an immediate induction we can show that for all p, $M_k(pT_B)$ verifies

$$M_k(pT_B) \leq C_p(k, \|B\|_\infty, \mathcal{E}(0), \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, M_k(0)). \tag{1.4.35}$$

This induction procedure is illustrated by the following implications

$$M_k(0) \leq C \Rightarrow Q(T_B, T_B) \leq C(M_k(0)) \Rightarrow M_k(T_B) \leq C(Q(T_B, T_B)) = C(M_k(0)) \ldots$$

Hence, thanks to this proof by induction, we obtain an estimate on $Q(t,t)$ for $0 \leq t \leq T$ with $T > 0$

$$Q(t,t) \leq C, \tag{1.4.36}$$

with C that depends on

$$T, k, \|B\|_\infty, \mathcal{E}(0), \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \iint |v|^k f^{in}(x,v)dvdx.$$

Using an inequality similar to (1.4.32) also obtained thanks to a Grönwall inequality, we prove that this estimate on $Q(t,t)$ implies the propagation of velocity moments of order $k > 2$ for the magnetized Vlasov–Poisson system.

1.4.2 Uniqueness via conditions on the moments

To finish this section on the results regarding the magnetized Vlasov–Poisson system, we present some uniqueness conditions for solutions to (1.4.2). Contrary to the previous subsection, these uniqueness conditions are valid for a general magnetic field $B := B(t,x)$ that depends on time and position, and that verifies

$$B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3). \tag{1.4.37}$$

As mentioned above, we mainly rely on two papers by Loeper [88] and Miot [95] in which new uniqueness conditions for the Vlasov–Poisson system were proved. To illustrate our contributions, we first present some ideas from these papers. The main result from [88] is given by the following theorem:

Theorem 1.4.6 (Loeper, [88]). *Given $f^{in} \geq 0$ with $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, given $T > 0$, there exists at most one weak solution to the Vlasov–Poisson system such that*

$$\rho \in L^\infty([0\,, T] \times \mathbb{R}^3). \tag{1.4.38}$$

This theorem shows that the boundedness of the macroscopic density is a sufficient condition for uniqueness. The proof is based on the control of a certain distance $Q(t)$ (not to be confused with $Q(t,t)$ from the previous subsection) between two solutions f_1, f_2 (with ρ_1, ρ_2 and E_1, E_2 the corresponding macroscopic densities and electric fields) to the Vlasov–Poisson system which is given by

$$Q(t) := \frac{1}{2} \int_{\mathbb{R}^6} f^{in}(x,v)\, |Y_1(t,x,v) - Y_2(t,x,v)|\, dxdv, \tag{1.4.39}$$

with $Y_i(t,x,v) = (X_i(t;0,x,v), V_i(t;0,x,v))$ and (X_i, V_i) $i = 1,2$ the characteristics defined by (1.4.7). Using optimal transportation techniques and the assumption $\rho_1, \rho_2 \in L^\infty([0\,,T] \times \mathbb{R}^3)$, Loeper proves in [88] that Q verifies a Grönwall inequality. More precisely, this Grönwall inequality is obtained thanks to two estimates involving the Wasserstein distances of order 2 between f_1 and f_2, and between ρ_1 and ρ_2, which we write $W_2(f_1, f_2)$ and $W_2(\rho_1, \rho_2)$. We recall that the Wasserstein distance of order $p \geq 1$ corresponds to a distance on the space of Borel probability measures and its definition is given in [88, Definition 2.1].

The first estimate involves controlling the H^{-1} norm of $\rho_1 - \rho_2$ using optimal transportation techniques to obtain the following estimate:

$$\|\rho_1 - \rho_2\|_{H^{-1}} \leq \max(\|\rho_1\|_\infty\,, \|\rho_2\|_\infty)^{\frac{1}{2}} W_2(\rho_1, \rho_2). \tag{1.4.40}$$

The second estimate on the other hand is a consequence of the main assumption of theorem 1.4.6 $\rho \in L^\infty([0\,,T] \times \mathbb{R}^3)$. This condition yields loglipschitz regularity for the electric field

$$|E(t,x) - E(t,y)| \leq (\|\rho\|_1 + \|\rho\|_\infty)\, |x-y|\, (1 + |\ln|x-y||). \tag{1.4.41}$$

Thanks to these two estimates, we obtain the following first order differential inequality on Q:

$$\frac{d}{dt} Q(t) \leq CQ(t) \left(1 + \ln \frac{1}{Q(t)}\right) \tag{1.4.42}$$

where C depends only on $\|\rho_i\|_\infty$, $i = 1,2$. Thanks to this Grönwall type inequality, we have that if $Q(0) = 0$ then $Q(t) = 0$ on $[0\,,T]$, which yields uniqueness because $Q(t)$ controls $W_2(f_1, f_2)$ [88, Lemma 3.6].

In [95], using similar techniques, Miot generalized theorem 1.4.6 by showing a new uniqueness condition for the Vlasov–Poisson system involving the L^p norms ($p \geq 1$) of ρ.

Theorem 1.4.7 (Miot, [95]). *Let $T > 0$. There exists at most one weak solution $f \in L^\infty([0\,,T], L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ to the Vlasov–Poisson system on $[0\,,T]$ such that*

$$\sup_{[0\,,T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty. \tag{1.4.43}$$

Using this condition, Miot also proved that for certain initial data with unbounded macroscopic density the Vlasov–Poisson system is well-posed.

Theorem 1.4.8 (Miot, [95]). *There exists $f^{in} \geq 0$ a.e. with $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ such that*

$$\rho_0(x) = C \ln_-(|x|), \quad \forall x \in \mathbb{R}^3 \tag{1.4.44}$$

and which defines a unique weak solution to the Vlasov–Poisson system.

A crucial point in [95], contrary to the method used in [88], is to exploit the second-order structure of the characteristics of the Vlasov–Poisson system. This approach yields a second-order differential inequality on the distance $D(t) = \iint_{\mathbb{R}^6} |X_1(t,x,v) - X_2(t,x,v)| \, f^{in}(x,v) dx dv$ which in turn implies uniqueness.

When we add an external magnetic field, we show in chapter 4 and chapter 5 that these results can be adapted to find new uniqueness conditions for solutions to the magnetized Vlasov–Poisson system. In truth, if we assume that the magnetic field is constant, then the uniqueness conditions from [88, 95] stay the same (this is proved in chapter 4). However, if one considers a general magnetic field verifying (1.4.37), then we need additional conditions on certain velocity moments of the solution (1.4.46) to guarantee uniqueness compared to the unmagnetized case. These conditions are necessary to control the extra terms in $D(t)$ that appear because of the added magnetic field.

Theorem 1.4.9 (Rege, [109]). *Let $T > 0$ and B verify (1.4.37). Furthermore, let $f^{in} \geq 0$ a.e. with $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and such that*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^m \, f^{in}(x,v) dx dv < +\infty, \tag{1.4.45}$$

for some $m > 6$.

If f^{in} also satisfies

$$\forall k \geq 1, \quad \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k \, f^{in}(x,v) dx dv \leq (C_0 k)^{\frac{k}{3}}, \tag{1.4.46}$$

for some constant C_0 independent of k, then there exists at most one solution f to (5.1.1). If such a solution f exists then it will verify the following bound on the L^p norms of ρ:

$$\sup_{[0\,,T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty. \tag{1.4.47}$$

1.5 List of publications, preprints and numerical codes

This thesis has resulted in the following publications, preprints and numerical codes:

- [26] F. Charles, B. Després, A. Rege, and R. Weder, *The magnetized Vlasov–Ampère system and the Bernstein–Landau paradox*, Journal of Statistical Physics, 183:23, 2021.
- [108] A. Rege, Numerical codes in Python to implement semi-Lagrangian methods for the linearized Vlasov–Ampère, linearized Vlasov–Poisson and Vlasov–Poisson systems. These codes were used for the numerical results in [26].

- [110] A. Rege, *The Vlasov–Poisson system with a uniform magnetic field: propagation of moments and regularity*, SIAM Journal on Mathematical Analysis, 53:2452–2475, 2021.
- [109] A. Rege, *Propagation of velocity moments and uniqueness for the magnetized Vlasov–Poisson system*, preprint.

I Spectral theory for magnetized linear kinetic models

2 The magnetized Vlasov–Ampère system and the Bernstein–Landau paradox

Contents

Summary

This chapter corresponds to a joint work with Frédérique Charles, Bruno Després and Ricardo Weder and has been published in the Journal of Statistical Physics [26].

We study the Bernstein–Landau paradox in an electrostatic plasma, which consists in that in the presence of the magnetic field, the electric field and the charge density fluctuation have an oscillatory behavior in time. We consider this problem from a new point of view. Instead of analyzing the linear magnetized Vlasov–Poisson system, as is usually done, we study the linear magnetized Vlasov–Ampère system because we can reformulate this system as a Schrödinger equation with a self-adjoint magnetized Vlasov–Ampère operator. This self-adjoint operator has a complete set of orthonormal eigenfunctions. In the limit when the magnetic field goes to zero the spectrum of the magnetized Vlasov–Ampère operator changes drastically from pure point to absolutely continuous in the orthogonal complement to its kernel, due to a sharp change on its domain. This spectral property explains the Bernstein–Landau paradox. Finally, we present numerical simulations that illustrate the Bernstein–Landau paradox using semi-Lagrangian schemes.

2.1 Introduction

Collisionless motion of an electrostatic plasma can exhibit wave damping, a phenomenon identified by Landau in [80] which is now called Landau damping. It consists in the decay for large times of the electric field. There is a very extensive literature on Landau damping. See for example, [34, 37, 36, 112, 118, 122], and the references quoted there. For a recent deep mathematical study of Landau damping in the nonlinear case see [96]. On the contrary, it is known that magnetized plasmas can prevent Landau damping [14]. In fact, it was shown by Bernstein [14] that in the presence of a constant magnetic field the electric field does not decay for large times, and that, actually, it has an oscillatory behavior as a function of time. This phenomenon is called the Bernstein–Landau paradox, see for example [121], because it seems paradoxical that even an arbitrarily small, but nonzero, value of the external constant magnetic field can be the cause of this radical change in the behaviour of the electric field for large times. The standard theory of the Bernstein–Landau paradox in the physics literature is based on the representation of the solutions to the magnetized Vlasov–Poisson system of equations in terms of a series of Bernstein modes. See, for example, [118, section 9.16] and [122, section 4.4.1].

It is the purpose of the present work to revisit the Bernstein–Landau paradox from a new point of view. Instead of considering the magnetized Vlasov–Poisson system we study the magnetized Vlasov–Ampère system. We write the magnetized Vlasov–Ampère system as a Schrödinger equation where the magnetized Vlasov–Ampère operator plays the role of the Hamiltonian. We construct a realization of the magnetized Vlasov–Ampère operator as a self-adjoint operator in the Hilbert space, that we call $\mathcal{H}$, that consists of the charge density functions that are square integrable and of the electric fields that are square integrable and of mean zero. Actually, the square of the norm of $\mathcal{H}$ is the energy. From the physical point of view this permits us to use the conservation of the energy in a very explicit way. On the mathematical side, this allows us to bring into the fore the powerful methods of the spectral theory of self-adjoint operators in a Hilbert space. There is a very extensive literature in spectral theory, see for example [76, 107, 104, 105, 106]. This approach has previously been used in the case without magnetic field to analyze Landau damping in [36, 37]. Within this framework the study of the Bernstein–Landau paradox reduces to the proof that the magnetized Vlasov–Ampère operator only has pure point spectrum, i.e., that its spectrum consists only of eigenvalues. Then, the fact that the magnetized Vlasov–Ampère operator has a complete set of orthonormal eigenfunctions follows from the abstract spectral theory of self-adjoint operators. We expand the general solutions to the magnetized Vlasov–Ampère system in the orthonormal basis of eigenfunctions of the magnetized Vlasov–Ampère operator. The coefficients of this expansion are the product of the scalar product of the initial state with the corresponding eigenfunction, and of the phase $e^{-it\lambda}$, where t is time and λ is the eigenvalue of the eigenfunction. This representation of the solution shows the oscillatory behavior in time for $\lambda \neq 0$, or constant in time for $\lambda = 0$, that is to say the Bernstein–Landau paradox. Moreover, our representation of the solution as an expansion in the orthonormal basis of eigenfunctions of the magnetized Vlasov–Ampère operator converges strongly in $\mathcal{H}$ for any initial data in $\mathcal{H}$, that is to say for any square integrable initial state without any further restriction in regularity and decay. Note that our result is optimal, since square integrability is the minimum that we can require, even to pose precisely the problem. A physical state has to have finite energy, i.e., it has to be square integrable.Note moreover, that we prove that the Bernstein modes are not complete. In fact, we prove in theorem 2.6.3 that for general initial data with finite energy, and that satisfy the Gauss law, the charge density fluctuation is a sum of two terms. One of them is oscillatory in time (see (2.6.16)) and it coincides with the standard series of Bernstein modes given in [13, 14]. The other term, given in (2.6.15), is constant in time, and it is a series of eigenfunctions with eigenvalue zero, of the magnetized Vlasov-Ampére system, and that satisfy the Gauss law. It appears that the fact that the Bernstein modes are not enough to expand the general charge density fluctuation, and that one has also to consider static modes is a new result, that has not been observed previously in the literature. We prove that the spectrum of the magnetized Vlasov–Ampère operator is pure point in two different ways.

In the first one, we actually compute the eigenvalues and we explicitly construct a orthonormal basis of eigenfunctions, i.e., a complete set of orthonormal eigenfunctions. This, of course, gives us much more than just that the expansion of the charge density fluctuation, and is interesting in its own right, because it can be used for many other purposes. As we mentioned above, our analysis shows that the Bernstein modes alone are not a complete orthonormal system. In addition to the eigenfunctions with eigenvalue zero that contribute to the static part of the charge density fluctuation, to have a complete orthonormal system in the Hilbert space, $\mathcal{H}$ of configurations with finite energy, it is necessary to add other eigenfunctions that are associated with eigenvalues at all the integer multiples of the cyclotron frequency, including the zero eigenvalue. These other eigenfunctions have nontrivial density function, but the electric field and the charge density fluctuation are zero. Recall that the charge density fluctuation is obtained averaging the density function over the velocities. In consequence, these other eigenfunctions do not appear in the expansion of the charge density fluctuation. Anyhow, these eigenfunctions are physically interesting because they show that there are plasma oscillations such that at each point the charge density fluctuation and the electric field are zero. Some of them are time independent. Note that since our eigenfunctions are orthonormal, these special plasma oscillations actually exist on their own, without the excitation of the other modes. It appears to us that this fact, or at least the exact form of these eigenfunctions, with zero charge density fluctuation and zero electric field, has not been observed previously in the literature.

In the second one we use an abstract operator theoretical argument based on the celebrated Weyl theorem on the invariance of the essential spectrum of a self-adjoint operators in Hilbert space. This argument allows us to prove that the magnetized Vlasov–Ampère operator has pure point spectrum. It gives a less detailed information about where the eigenvalues are located, and it tells us nothing about the eigenfunctions. However, it is enough for the proof of the existence of the Bernstein–Landau paradox without going through the detailed calculations of the first approach. It also tells us why the Bernstein–Landau paradox exists from a general principle in spectral theory.

On the contrary in the case where the magnetic field is zero, it was proven in [36, 37] that the spectrum of the magnetized Vlasov–Ampère operator is made of an absolutely continuous part and of a kernel. The Landau damping follows from the well known fact that for a self-adjoint operator $\mathbf{H}$, the operator $e^{-it\mathbf{H}}P_0$ goes weakly to zero as $t \to \pm\infty$ (here P_0 is the projection on the absolutely continuous part of the spectrum). It has been remarked in [34] that there are "interesting analogies with Lax and Phillips scattering theory " [84]. In fact, the results of [36, 37] prove that it is not just an analogy, but the consequence of a convenient reformulation of Landau damping in terms of the magnetized Vlasov–Ampère system. The sharp change in the spectrum of the magnetized Vlasov–Ampère operator when the magnetic field goes to zero, i.e. from pure point to absolutely continuous in the orthogonal complement to its kernel, may appear to be paradoxical because the formal magnetized Vlasov–Ampère operator is formally analytic in the magnetic field. The issue is that the domain of the self-adjoint realization of the magnetized Vlasov–Ampère operator changes abruptly when the magnetic field is zero. It is a well known fact in the spectral theory of families of linear operators that the spectrum can change sharply at values of the parameter where the domain of the operator sharply changes. For a comprehensive presentation of these results the reader can consult, for example, [76]. Summing up, this shows that there is no paradox in the Bernstein–Landau paradox, just a well known fact of spectral theory, but, of course, in the physics literature the domains of the operators are usually not taken into account. Perhaps the reason why the absence of Landau damping for arbitrarily small magnetic fields is considered as paradoxical is related to the fact that the magnetized Vlasov–Poisson system somehow hides the underlying mathematical physics structure of our problem, in spite of the fact that it is a convenient tool, particularly for computational purposes. Let us explain what we mean. The full Maxwell equations consist of the Maxwell-Faraday equation, the Ampère equation, the Gauss law, and the Gauss law for magnetism, i.e., the divergence of the magnetic field is zero. In our case the Maxwell-Faraday equation and the Gauss law for magnetism are automatically satisfied. So, of the Vlasov-Maxwell equations, only the Vlasov equation remains, as well as the Ampère equation, and the Gauss law. Furthermore, the Gauss law is a constraint that is only necessary to impose at the initial time, since it is propagated by the magnetized Vlasov–Ampère system. Further, both the Vlasov and the Ampère equations are evolution equations. So, the natural way to proceed is to solve the magnetized Vlasov–Ampère system as an evolution problem, and to restrict the initial data to those who satisfy the Gauss law. The situation with the magnetized Vlasov–Poisson system is somewhat different because the Ampère equation is not explicitly taken into account. So, one could think that the magnetized Vlasov–Poisson system is incomplete. The remedy is that instead of imposing the Gauss law only at the initial time, it is required at all times. We actually prove in section 2.2 that the magnetized Vlasov–Poisson system is indeed equivalent to the magnetized Vlasov–Ampère system plus the validity of the Gauss law at the initial time. However, the magnetized Vlasov–Poisson system is a hybrid one where the Vlasov equation is an evolution equation and the Poisson equation is an elliptic equation, without time derivative. This is one way to understand why in the magnetized Vlasov–Poisson system the basic mathematical physics of our problem is not so apparent. On the contrary, as we mentioned above, the magnetized Vlasov–Ampère system is an evolution problem, that moreover, as we already mentioned, and as we explain in section 2.2, has a conserved energy that is explicitly expressed in terms of the density function and the electric field that appear in the magnetized

Vlasov–Ampère system. These two facts are the reasons why the magnetized Vlasov–Ampère system has a self-adjoint formulation in Hilbert space, and then, it is clear that there is no paradox in the Bernstein–Landau paradox, as we explained above.

Once the self-adjointness of the magnetized Vlasov–Ampère formulation is established, it is a matter of explicit calculations to determine the eigenfunctions. The technicalities of the calculations are related to the fact that three different natural decompositions are combined. The first one is based on Fourier decomposition (factors e^{inx}), the second one is based on a direct sum of the kernel of the operator and its orthogonal (it will be denoted as $\mathcal{H} = \mathrm{Ker}[\mathbf{H}] \oplus \mathrm{Ker}[\mathbf{H}]^{\perp}$) and the third one starts from the determination of the eigenfunctions with a vanishing electric field (it will be denoted as $F = 0$). The combination of these three decompositions is made compatible with convenient notations.

Passing to the limit $\omega_c \to 0$ in the representation formulae is possible in principle. This requires a careful analysis. We do not consider this problem in this work except in the very last remark in appendix B. Nevertheless, in lemma 2.5.3 two consecutive eigenvalues are different by the constant ω_c and in lemma 2.5.6 two consecutive eigenvalues are different by a value that is smaller than $2\,\omega_c$. Therefore, at the limit $\omega_c \to 0$, the discrete spectrum fills the entire real line with density and so it approaches the spectrum of the limit problem without magnetic field. A recent mathematical work [13] studied the Vlasov and the Vlasov-Fokker-Planck equations in a box, in three dimensions in configuration and in velocity space, and with a constant background magnetic field. They consider the Landau damping, the Bernstein–Landau paradox and the enhanced collisional relaxation in the limit when the collision frequency goes to zero. Since in this chapter we study the case when there are no collisions, we will only comment on the results of [13] when the collision frequency is zero. We denote by $\mathcal{T}^3$ the three-dimensional torus, ie., $\mathcal{T}^3 := [0, 2\pi]^3_{\mathrm{per}}$. In the collisionless case, [13] considers the following linearized magnetized Vlasov–Poisson system for $(\mathbf{x}, \mathbf{v}) \in \mathcal{T}^3 \times \mathbb{R}^3$, where $\mathbf{x} = (x, y, z)$, and $\mathbf{v} = (v_1, v_2, v_3)$,

$$\begin{cases} \partial_t \mathcal{G}(t,\mathbf{x},\mathbf{v}) + \mathbf{v}\cdot\nabla_{\mathbf{x}}\mathcal{G}(t,\mathbf{x},\mathbf{v}) + \dfrac{q}{m}\mathcal{F}(t,\mathbf{x})\cdot\nabla_{\mathbf{v}} f^0(\mathbf{v}) + \dfrac{q}{m}\mathbf{v}\times\mathbf{B}_0\cdot\nabla_{\mathbf{v}}\mathcal{G}(t,\mathbf{x},\mathbf{v}) = 0, \\ \mathcal{F}(t,\mathbf{x}) = -\nabla_{\mathbf{x}} \displaystyle\int_{\mathcal{T}^3} W(\mathbf{x}-\mathbf{y})\,\rho(t,\mathbf{y})\,d^3\mathbf{y}, \\ \displaystyle\int_{\mathcal{T}^3\times\mathbb{R}^3} \mathcal{G}(t,\mathbf{x},\mathbf{v})\,d^3\mathbf{x}\,d^3\mathbf{v} = 0, \\ \mathcal{G}(0,\mathbf{x},\mathbf{v}) = \mathcal{G}_{\mathrm{in}}(\mathbf{x},\mathbf{v}), \end{cases} \tag{2.1.1}$$

where,

$$\rho(t,\mathbf{x}) := \int_{\mathbb{R}^3} \mathcal{G}(t,\mathbf{x},\mathbf{v})\,d^3\mathbf{v},$$

and

$$W(\mathbf{x}) := \frac{q}{4\pi}\,\frac{1}{(2\pi)^3} \sum_{\mathbf{k}\in\mathbb{Z}^3\setminus\{0\}} \frac{1}{|\mathbf{k}|^2}\,e^{ik\cdot\mathbf{x}},$$

$q, m > 0$, and $\mathbf{B}_0 = (0, 0, B_0), B_0 > 0$. Moreover,

$$f^0(v) = \frac{1}{2\pi}\,e^{-\frac{v_1^2+v_2^2}{2}} \left(\frac{1}{\sqrt{2\pi T_{\|}}}\,e^{-\frac{-v_3^2}{2T_{\|}}} + \tilde{f}(v_3^2) \right), \tag{2.1.2}$$

with $T_{\|} > 0$.

To state the results of [13] we first introduce some notations. Let us define the Fourier coefficients of $f \in L^2(\mathcal{T}^3)$ as follows,

$$\hat{f}(\mathbf{k}) := \int_{\mathcal{T}^3} e^{-i\mathbf{k}\cdot\mathbf{x}}\, f(\mathbf{x})\, d^3\mathbf{x}, \qquad \mathbf{k} = (k_1, k_2, k_3) \in \mathbb{Z}^3.$$

Then,

$$f(\mathbf{x}) = \frac{1}{(2\pi)^3} \sum_{\mathbf{k}\in\mathbb{Z}^3} e^{i\mathbf{k}\cdot\mathbf{x}}\, \hat{f}(\mathbf{k}).$$

Denoting $< \mathbf{v} >:= \sqrt{1+|\mathbf{v}|^2}$, the H^σ_m norm of a function f is defined as follows [13],

$$\|f\|^2_{H^\sigma_m} := \int_{\mathcal{T}^3\times\mathbb{R}^3} < \mathbf{v} >^{2m} | < \nabla >^\sigma \; f(\mathbf{x},\mathbf{v})|^2\, d^3\mathbf{x} d^3\mathbf{v},$$

with $\nabla = \nabla_{\mathbf{x},\mathbf{v}}$ the differential operator both in $\mathbf{x}$ and $\mathbf{v}$.

In [13] the following theorem is stated.

Theorem 2.1.1 (Bedrossian and Wang, Theorem 1 [13]). *Let $< \mathbf{v} >^m \mathcal{G}_{\text{in}} \in H^\sigma$, for $\sigma \geq 0$, and $m > 2$. Suppose that $\|\tilde{f}\|_{H^{\sigma'}_m} \leq \delta_0$, with $\sigma' > \sigma + 5/2$, and $|T_\| - 1| + \delta_0$ sufficient small depending on universal constants and B_0. Then, the following holds.*

a) The Landau damping for $z-$dependent modes:

$$\||\partial_z|^{1/2} < \nabla, \partial_z t >^\sigma \rho(t)\|_{L^2_t L^2_x} \lesssim_{\sigma,\sigma',m} \|\mathcal{G}_{\text{in}}\|_{H^\sigma_m}.$$

b) If $k_3 = 0$,and we additionally have $\sigma > 5/2$, then for all $\mathbf{k}_\perp := (k_1, k_2, 0)$ and $n \in \mathbb{N}$, $\exists!$ $b_{n,\mathbf{k}} = b_{n,\mathbf{k}}(\frac{q}{m}B_0) \in (n, n+1)$ and coefficients $r_{\pm n,\mathbf{k}}$ depending on $\mathcal{G}_{\text{in}}$ such that (with the convention that $r_{-0,\mathbf{k}}$ is distinct from $r_{0,\mathbf{k}}$),

$$\hat{\rho}(t, \mathbf{k}_\perp, 0) = \sum_{n=0}^{\infty} r_{n,\mathbf{k}}\, e^{i\frac{b_{n,\mathbf{k}} q B_0}{m} t} + r_{-n,\mathbf{k}}\, e^{-i\frac{b_{n,\mathbf{k}} q B_0}{m} t}, \tag{2.1.3}$$

and further, there holds,

$$|r_{\pm n,\mathbf{k}}| \leq \frac{1}{< \mathbf{k} >^\alpha < n >^\gamma} \|\mathcal{G}_{\text{in}}\|_{H^\sigma_m},$$

where α, β, γ are such that $\gamma = \min(\beta+1, 2\beta-1), \alpha + 2\beta - \frac{1}{2} \leq \sigma$, and $\beta + 1 < m$.

Item a) of Theorem 1 of [13] (theorem 2.1.1 above) is concerned with the Landau damping of the $z-$dependent modes (see also [14]), that is to say, of the modes that depend on the coordinate z, along the direction of the magnetic field $\mathbf{B}_0$. In this work we do not consider this problem, since as we work in $1+2$ dimensions our modes only depend on the coordinate x that is orthogonal to the direction of the magnetic field $\mathbf{B}_0$.

Item b) of Theorem 1 of [13] (theorem 2.1.1 above) considers the expansion of the Fourier coefficients, $\hat{\rho}(t, k_\perp, 0)$ of the charge density fluctuation in terms of the Bernstein modes, that is to say the Bernstein–Landau paradox, in the case of $3+3$ dimensions. This is the problem that we consider in $1+2$ dimensions in this work.

We now proceed to discuss this result. Let us first show that if $\hat{\rho}(t, k_\perp, 0)$ in (2.1.3) is time independent, then it has to be identically zero. Assume then, that $\hat{\rho}(t, \mathbf{k}_\perp, 0) = \hat{\rho}(\mathbf{k}_\perp, 0)$. Since all the $b_{n,\mathbf{k}}$ are different from each other, and different from zero, then one deduces

$$\begin{aligned} 0 &= \lim_{T\to\infty} \int_0^T \hat{\rho}(\mathbf{k}_\perp, 0)\, e^{\mp i \frac{b_{j,\mathbf{k}} q B_0}{m} t}\, dt \\ &= \lim_{T\to\infty} \int_0^T \left(\sum_{n=0}^{\infty} r_{n,\mathbf{k}}\, e^{i \frac{b_{n,\mathbf{k}} q B_0}{m} t} + r_{-n,\mathbf{k}}\, e^{-i \frac{b_{n,\mathbf{k}} q B_0}{m} t} \right) e^{\mp i \frac{b_{j,\mathbf{k}} q B_0}{m} t}\, dt \\ &= r_{\pm j,\mathbf{k}}. \end{aligned} \tag{2.1.4}$$

Hence, $r_{\pm n,\mathbf{k}} = 0$ for all $n \in \mathbb{N}$, and all $\mathbf{k} = (\mathbf{k}_\perp, 0)$. Therefore, item b) of Theorem 1 of [13] (theorem 2.1.1 above) implies that for the time independent solutions to (2.1.1) the Fourier coefficient of the charge density, $\hat{\rho}(\mathbf{k}_\perp, 0)$, is identically zero for all $\mathbf{k}_\perp$. Recall that $\hat{\rho}(\mathbf{k})$ is the Fourier coefficient of $\rho(\mathbf{x})$, that is to say,

$$\hat{\rho}(\mathbf{k}) := \int_{\mathcal{T}^3} \rho(\mathbf{x})\, e^{-i\mathbf{k}\mathbf{x}} d\mathbf{x}.$$

Then, inverting the Fourier series,

$$\rho(\mathbf{x}) = \frac{1}{(2\pi)^3} \sum_{\mathbf{k}\in\mathbb{Z}^3} e^{i\mathbf{k}\mathbf{x}}\, \hat{\rho}(\mathbf{k}). \tag{2.1.5}$$

Further, by (2.1.5)

$$\int_{\mathcal{T}} \rho(x, y, z)\, dz = \frac{1}{(2\pi)^2} \sum_{(k_1,k_2)\in\mathbb{Z}^2} e^{i(k_1 x + k_2 y)}\, \hat{\rho}(k_1, k_2, 0).$$

Hence, as for time independent solutions, if (2.1.3) holds, $\hat{\rho}(\mathbf{k}_\perp, 0) = 0$, we have proven that for the solutions given in Item b) of Theorem 1 of [13] (theorem 2.1.1 above), if their are time independent, necessarily

$$\int_{\mathcal{T}} \rho(x_\perp, x_3)\, dx_3 = 0. \tag{2.1.6}$$

However, in appendix B we construct an explicit family of time-independent solutions to (2.1.1), that satisfy the hypotheses of item b) of Theorem 1 of [13] (theorem 2.1.1 above) with $\hat{\rho}(k_\perp, 0)$ that is not identically zero, and where (2.1.6) does not hold. This shows that the result of item b) of Theorem 1 of [13] (theorem 2.1.1 above) is devoted to the behaviour of a special class of solutions. This is clear by comparison with the physical literature [6, 14, 121] which is based on the study of a dispersion relation. The dispersion relation is mathematically justified in [13], in particular with a summability argument of the contributions of all poles of the dispersion relation. Nevertheless the pole $1/z$, that corresponds to time-independent solutions, is discarded in equation (2.14) in [13], and in this sense, the works [6, 13, 14] focus on a subclass, or special class, of solutions. In our work, we do not make such hypothesis or restriction and that is why we recover a stationary solution as in [121, eq. (55)]. Note that in appendix B we write the model for ions like in [13], for the purpose of making the comparison with the results of [13] more transparent. In the rest of this chapter we write the model for electrons. Actually, the models for ions and electrons are the same, up to a change in the sign of the cyclotron frequency and of the electric field (see remark 2.2.1). This is actually in agreement with our theoretical expansions in (2.6.14)-(2.6.16) that show that in general the charge density fluctuations have a time-independent part and a time-dependent part. Further, our result in (2.6.14)-(2.6.16) solves, in the case of one dimension in space and two dimensions in velocity, the problem posed in Remark 3 of [13] of justifying the expansion in the Bernstein modes of the charge density fluctuation, without the regularity in space and decay in velocity that they assume in Theorem 1 of [13] (theorem 2.1.1 above). Our model is one dimensional in space and two dimensional in velocity. However, there is no difficulty to write it in three dimensions in space and velocity, because Maxwellian functions have a natural compatibility with separation of variables techniques. In principle, the extension of our results to three dimensions in space and velocity is possible with due attention paid to the anisotropy introduced by the magnetic field. Indeed, it is known [13, 14] that the physics is different in the orthogonal direction (treated in this work) and in the parallel direction (not treated in his work). In fact, in [13, 14] it is proved that there is damping in the direction of the magnetic field, and no damping perpendicular to the magnetic field. We leave this 3-dimensional study for further research.

The organization of this work is as follows. In section 2.2 we introduce the magnetized Vlasov–Poisson and the magnetized Vlasov–Ampère systems, and we prove their equivalence. In section 2.3 we give the notations and definitions that we use. In section section 2.4 we consider the case of a pure magnetized Vlasov equation without coupling. We construct a self-adjoint realization of the magnetized Vlasov operator, we explicitly compute the eigenvalues and we explicitly construct an orthonormal system of eigenfunctions that is complete, i.e., it is a basis of the Hilbert space. In section 2.5 we construct a self-adjoint realization of the magnetized Vlasov–Ampère operator, we compute the eigenvalues, and we construct an orthonormal systems of eigenfunctions that is complete, that is to say that is a basis of the Hilbert space. In section 2.6 we obtain a representation of the general solution to the magnetized Vlasov–Ampère system as an expansion in our orthonormal basis of eigenfunctions. In particular we prove the convergence of the Bernstein expansion [13, 14], under optimal conditions on the initial state. In section 2.7 we give an operator theoretical proof of the existence of the Bernstein–Landau paradox, with an argument based on the Weyl theorem for the invariance of the essential spectrum. In section 2.8 we illustrate our results, using semi-Lagrangian schemes to highlight how the eigenfunctions constructed in our study can be considered as new test functions for this kind of kinetic system. In appendix A we study the properties of the secular equation. Finally, in appendix B we construct explicit families of time-independent solutions to the linearized magnetized Vlasov–Poisson system.

2.2 The magnetized Vlasov–Poisson and the magnetized Vlasov–Ampère systems

We adopt the Klimontovitch approach [77, 62] where the Newton equation of a very large number of charged particles with velocity v moving in an electromagnetic field is approximated by a continuous density function $f(t,x,v) \geq 0$. The variable t is time. We assume that the charged particles undergo a one dimensional motion, and that the real variable x is the position of the charged particles. Furthermore, we suppose that the velocity, v, of the charged particles is two dimensional, i.e., $v = (v_1, v_2) \in \mathbb{R}^2$. Further, we take the motion of the charged particles along the first coordinate axis of the velocity of the charged particles. The density function is a solution of a Vlasov equation,

$$\partial_t f + v_1 \partial_x f + \mathbf{F} \cdot \nabla_v f = 0. \tag{2.2.1}$$

We assume, for simplicity, that the motion of the charged particles is a 2π-periodic oscillation, that is a usual assumption [34]. Hence, we look for solutions to (2.2.1), $f(t,x,v)$, for $t \in \mathbb{R}, x \in [0, 2\pi], v = (v_1, v_2) \in \mathbb{R}^2$, that are periodic in x, i.e., $f(t,0,v) = f(t,2\pi,v)$. The electromagnetic

Lorentz force,

$$\mathbf{F}(t,x) = \frac{q}{m}\left(\mathbf{E}(t,x) + v \times \mathbf{B}(t,\mathbf{x})\right), \tag{2.2.2}$$

is divergence free with respect to the velocity variable, that is $\nabla_v \cdot \mathbf{F} = 0$. The Maxwell's equations are simplified, assuming that the magnetic field $\mathbf{B}(t,\mathbf{x}) = \mathbf{B}_0$ is constant in space-time. Following the convention adopted in [6, 121], we suppose that the two dimensional velocity v is perpendicular to the constant magnetic field, i.e., $\mathbf{B}_0 = (0,0,B_0), B_0 > 0$. Moreover, we assume that the electric field is directed along the first coordinate axis, $\mathbf{E}(t,x) = (E(t,x),0,0)$. We adopt a convenient normalization adapted to electrons, that is $q_{\text{ref}} = -1$ and $m_{\text{ref}} = 1$, where q_{ref} is the charge of the electron, and m_{ref} is the mass of the electron. The electric field satisfies the Gauss law,

$$\partial_x E(t,x) = 2\pi - \int_{\mathbb{R}^2} f dv, \tag{2.2.3}$$

where 2π is the constant density of the heavy ions, that do not move. We take the density of the ions equal to 2π to simplify some of the calculations below. The term $-\int_{\mathbb{R}^2} f\, dv$ is the charge density of the particles with charge -1.

With these notations and normalizations (2.2.1), and (2.2.3) are written as the following system,

$$\begin{cases} \partial_t f + v_1 \partial_x f - E\partial_{v_1} f + \omega_c\left(-v_2\partial_{v_1} + v_1\partial_{v_2}\right) f = 0, \\ \partial_x E(t,x) = 2\pi - \displaystyle\int_{\mathbb{R}^2} f dv. \end{cases} \tag{2.2.4}$$

We denote the cyclotron frequency by $\omega_{\text{c}} := B_0$.

Remark 2.2.1. The model written for positive ions instead of (negatively charged) electrons is similar to ((2.2.4)), with the only modification that the sign in front of the electric field and in front of ω_c is changed in both equations.

We retain the potential part of the electric field

$$E(t,x) = -\partial_x \varphi(t,x), \tag{2.2.5}$$

where the potential $\varphi(t,\mathbf{x})$ is a solution to the Poison equation,

$$-\Delta\varphi = 2\pi - \int_{\mathbb{R}^2} f dv. \tag{2.2.6}$$

The electric field and the potential are assumed to be periodic with period 2π, i.e. $E(t,0) = E(t,2\pi), \varphi(t,0) = \varphi(t,2\pi)$. Note that since the potential $\varphi(t,x)$ is periodic it follows from (2.2.5) that the mean value of the electric field is zero,

$$\int_0^{2\pi} E(t,x)\, dx = 0. \tag{2.2.7}$$

Two important properties of the magnetized Vlasov–Poisson system (2.2.4), (2.2.5), and (2.2.6) are that the density function satisfies the maximum principle

$$\inf_{(x,v)\in[0,2\pi]\times\mathbb{R}^2} f_{\text{ini}}(x,v) \le f(t,x,v) \le \sup_{(x,v)\in[0,2\pi]\times\mathbb{R}^2} f_{\text{ini}}(x,v),$$

where f_{ini} is the initial value of the solution, $f(t,x,v)$, and that the total energy is constant in time,

$$\frac{d}{dt}\left(\int_{[0,2\pi]\times\mathbb{R}^2}\frac{|v|^2}{2}f\,dxdv+\int_{[0,2\pi]}\frac{|E|^2}{2}dx\right)=0. \tag{2.2.8}$$

Following [14], a linearization of the equations around a homogeneous Maxwellian equilibrium state $f_0(v)$, where, $f_0(v):=e^{\frac{-v^2}{2}}$ is performed. Here the Maxwellian distribution is normalized for $T_{\text{ref}}\,k_{\text{B}}=1$, where T_{ref} is the reference temperature and k_{B} is Boltzmann's constant. It corresponds to the expansion

$$f(t,x,v)=f_0(v)+\varepsilon\sqrt{f_0(v)}u(t,x,v)+O(\varepsilon^2), \tag{2.2.9}$$

and

$$E(t,x)=E_0+\varepsilon F(t,x)+O(\varepsilon^2), \tag{2.2.10}$$

with a null reference electric field $E_0=0$. Inserting (2.2.9) and (2.2.10) into (2.2.4), and keeping the terms up to linear in ε, one gets the linearized magnetized Vlasov–Poisson system written as,

$$\begin{cases}\partial_t u+v_1\partial_x u+Fv_1\sqrt{f_0}+\omega_c\left(-v_2\partial_{v_1}+v_1\partial_{v_2}\right)u=0,\\ \partial_x F=-\displaystyle\int_{\mathbb{R}^2}u\sqrt{f_0}dv,\\ \displaystyle\int_{[0,2\pi]}F=0,\end{cases} \tag{2.2.11}$$

where in the third equation we have added the constraint that the mean value of the electric field F is zero, as in (2.2.7). Moreover, the electric field $F(t,x)=-\partial_x\varphi(t,x)$ is obtained from a potential as in (2.2.5), where the potential is periodic, $\varphi(t,0)=\varphi(t,2\pi)$, and it solves the Poisson equation,

$$-\Delta\varphi=-\int_{\mathbb{R}^2}u\,\sqrt{f_0}dv. \tag{2.2.12}$$

Observe that the second equation in (2.2.11) is the Gauss law,

$$\partial_x F(t,x)=\rho(t,x), \tag{2.2.13}$$

where $\rho(t,x)$ is the charge density fluctuation of the perturbation of the Maxwellian equilibrium state,

$$\rho(t,x):=-\int_{\mathbb{R}^2}u(t,x,v)\,\sqrt{f_0}(v)dv. \tag{2.2.14}$$

The study of the solutions to the magnetized Vlasov–Poisson system is the standard method to analyze the dynamics of a very large number of charged particles moving in the presence of a constant external magnetic field. For the case of the Bernstein–Landau paradox see, for example, [14], [121], [118, section 9.16], [122, section 4.4.1] and [13]. We now present an alternate method to study this problem. One of Maxwell's equations is called the Ampère equation and is given by:

$$\partial_t F=\int_{\mathbb{R}^2}v_1\,u\,\sqrt{f_0}dv, \tag{2.2.15}$$

where we have taken the dielectric constant $\varepsilon_0=1$. We consider here the following modified Ampère equation

$$\partial_t F=I^*\int_{\mathbb{R}^2}v_1\sqrt{f_0}\,u\,dv, \tag{2.2.16}$$

where I^* is the space operator such that $I^*g = g - [g]$ and the mean value in space of a function g is denoted by $[g]$, that is to say, $I^*g(x) := g(x) - \frac{1}{2\pi}\int_0^{2\pi} g(y)\,dy$. With this convention the magnetized Vlasov–Ampère system is written as follows,

$$\begin{cases} \partial_t u + v_1\partial_x u + Fv_1\sqrt{f_0} + \omega_c\left(-v_2\partial_{v_1} + v_1\partial_{v_2}\right)u = 0, \\ \partial_t F = I^* \displaystyle\int_{\mathbb{R}^2} v_1\sqrt{f_0}\,u dv. \end{cases} \tag{2.2.17}$$

To the magnetized Vlasov–Ampère system (2.2.17), we add conditions for $F_{\text{ini}} := F(0,\cdot)$ and $u_{\text{ini}} = u(0,\cdot,\cdot)$ the integral constraint,

$$\int_0^{2\pi} F_{\text{ini}}\,dx = 0, \tag{2.2.18}$$

is satisfied at initial time, and the Gauss law (2.2.13), (2.2.14) is also satisfied at the initial time,

$$\frac{d}{dx}F_{\text{ini}} = -\int_{\mathbb{R}^2} u_{\text{ini}}\sqrt{f_0}dv. \tag{2.2.19}$$

Lemma 2.2.2. *The linearized magnetized Vlasov–Poisson system* (2.2.11) *is equivalent to the magnetized Vlasov–Ampère system* (2.2.17) *with initial conditions that satisfies* (2.2.18), (2.2.19).

Proof. Let (u, F) be a solution to the magnetized Vlasov–Ampère system (2.2.17) that satisfy (2.2.18), (2.2.19). It follows from the Ampère equation that

$$\partial_t \int_0^{2\pi} F(t,x)\,dx = \int_0^{2\pi} I^* \int_{\mathbb{R}^2} v_1\sqrt{f_0}\,u = 0$$

and consequently the integral constraint (2.2.18) is propagated for all times. The Gauss law (2.2.19) is also propagated to all times by the magnetized Vlasov–Ampère system, as we proceed to prove. Multiplying the first equation in (2.2.17) by $\sqrt{f_0}$, integrating in v over $\mathbb{R}^2$, using that f_0 is an even function of $|v|$ and using integration by parts, we prove the following continuity equation,

$$\partial_t \int_{\mathbb{R}^2} u\sqrt{f_0}dv + \partial_x \int_{\mathbb{R}^2} v_1 u\sqrt{f_0}dv = 0. \tag{2.2.20}$$

Deriving (2.2.16) with respect to x we obtain, $0 = \partial_x\left(\partial_t F - I^*\int_{\mathbb{R}^2} v_1 u\sqrt{f_0}dv\right) = \partial_x\left(\partial_t F - \int_{\mathbb{R}^2} v_1 u\sqrt{f_0}dv\right)$, because $\partial_x = \partial_x I^*$. Then, by (2.2.20)

$$0 = \partial_t\left(\partial_x F + \int_{\mathbb{R}^2} u\sqrt{f_0}\,dv\right),$$

from which the Gauss law follows for all times. We have proven that a solution to the magnetized Vlasov–Ampère system (2.2.17) that satisfies the initial conditions (2.2.18), (2.2.19) solves the magnetized Vlasov–Poisson system (2.2.11).

Conversely let (u, F) be a solution to the magnetized Vlasov–Poisson system (2.2.11). Then by the second equation in (2.2.11) and (2.2.20),

$$0 = \partial_x\partial_t F + \partial_t \int_{\mathbb{R}^2} u\sqrt{f_0}dv = \partial_x\left(\partial_t F - \int_{\mathbb{R}^2} v_1 u\sqrt{f_0}dv\right).$$

So $\partial_t F = \int_{\mathbb{R}^2} v_1 u\sqrt{f_0}dv + C(t)$, where $C(t)$ is constant in space. Then, $\partial_t I^*F = I^*\int_{\mathbb{R}^2} v_1 u\sqrt{f_0}dv$. But F has zero mean value, so $I^*F = F$, and it follows that the Ampère law in (2.2.16) holds. Hence, the magnetized Vlasov–Poisson system implies the magnetized Vlasov–Ampère system (2.2.17) and the initial conditions (2.2.18), (2.2.19). □

From now on we only consider the magnetized Vlasov–Ampère system (2.2.17) with conditions (2.2.18), (2.2.19). A fundamental energy relation is easily shown for solutions to the magnetized Vlasov–Ampère formulation (2.2.17)

$$\frac{d}{dt}\left(\int_{[0,2\pi]\times\mathbb{R}^2}\frac{u^2}{2}dxdv+\int_{[0,2\pi]}\frac{F^2}{2}dx\right)=0. \tag{2.2.21}$$

It is the counterpart of the energy identity (2.2.8), so the term $\int_{[0,2\pi]\times\mathbb{R}^2}\int_{\mathbf{v}}\frac{u^2}{2}dxdv$ is identified with the kinetic energy of the negatively charged particles, and the term $\int_{[0,2\pi]}\frac{F^2}{2}dx$ is the energy of the electric field. This identity is known since [3, 78]. As we show in the next sections, the identity (2.2.21) is the basis of our formulation of the magnetized Vlasov–Ampère system as a Schrödinger equation in a Hilbert space, where the magnetized Vlasov–Ampère operator plays the role of the self-adjoint Hamiltonian.

Once the reformulation of (2.10.31) within the Lax and Philipps abstract scattering theory [84] has been performed with a convenient self-adjoint operator, linear Landau damping is implied by the fact that, in the case $\omega_c=0$, the spectrum of the operator is made up of an absolutely continuous part and a kernel [37]. It is clear that the magnetic operator $\omega_c\left(-v_2\partial_{v_1}+v_1\partial_{v_2}\right)$ is formerly analytic with respect to ω_c, but it is also a singular perturbation. Explained in terms of the dynamics of the system (2.10.31), our results will recover the seminal results of [14] and the recent mathematical works of [13]. But we will take the full benefit of having a rigorous scattering structure. In particular, in our context the questions asked in Remark 3 in [13], "how to justify the Bernstein mode expansion", are automatically answered by the spectral decomposition of self-adjoint operators and by the fact that the transport operator $v_1\partial_x u+\omega_c\left(-v_2\partial_{v_1}+v_1\partial_{v_2}\right)$ with non zero cyclotron frequency has only pure point spectrum. This will be explained in due place.

2.3 Notations and Definitions

We will write the magnetized Vlasov–Ampère system as a Schrödinger equation with a self-adjoint Hamiltonian in an appropriate Hilbert space. We find it convenient to borrow some terminology from quantum mechanics. For this purpose, we first introduce some notations and definitions. We designate by $\mathbb{R}^+$ the positive real semi-axis, i.e., $\mathbb{R}^+:=[0\,,+\infty[$, and by $\mathbb{R}^2$ the real plane. The set of all integers is denoted by $\mathbb{Z}$ and the set of all nonzero integers by $\mathbb{Z}^*$. The positive natural numbers are designated by $\mathbb{N}^*$. By $\mathbb{C}$ we designate the complex numbers. We denote by C a generic constant whose value does not have to be the same when it appears in different places. By $C^\infty([0,2\pi])$ we designate the set of all infinitely differentiable functions in $[0,2\pi]$, and by $C_0^\infty(\mathbb{R}^2)$ we denote the set of all infinitely differentiable functions in $\mathbb{R}^2$ with compact support. Let $\mathcal{B}$ be a set of vectors in a Hilbert space, $\mathbb{H}$. We denote by $\mathrm{Span}[\mathcal{B}]$ the closure for strong convergence in $\mathbb{H}$ of all finite linear combinations of elements of $\mathcal{B}$, in other words,

$$\mathrm{Span}[\mathcal{B}]:=\text{closure}\left\{\sum_{j=1}^{N}\alpha_jX_j:\alpha_j\in\mathbb{C},X_j\in\mathcal{B},N\in\mathbb{N}^*\right\}.$$

Let $\mathcal{M}$ be a subset of a Hilbert space $\mathbb{H}$. We define the orthogonal complement of $\mathcal{M}$, in symbol, $\mathcal{M}^\perp$, as follows,

$$\mathcal{M}^\perp:=\{f\in\mathbb{H}:(f,u)_{\mathbb{H}}=0,\ \text{ for all}\,u\in\mathcal{M}\}.$$

Let $\mathbb{H}$ be a Hilbert space, and let $\mathbb{H}_j, j = 1, \ldots, N, 2 \leq N \leq \infty$, be mutually orthogonal closed subspaces of $\mathbb{H}$, that is to say,

$$\mathbb{H}_j \subset \mathbb{H}_m^{\perp}, \text{ and } \mathbb{H}_m \subset \mathbb{H}_j^{\perp}, \qquad j \neq m, 1 \leq j, m \leq N.$$

Note that if $\mathbb{H}_j$ and $\mathbb{H}_m$ are mutually orthogonal, then one has $(f, u)_{\mathbb{H}} = 0$, $f \in \mathbb{H}_j$, $u \in \mathbb{H}_m$. We say that $\mathbb{H}$ is the direct sum of the $\mathbb{H}_j, j = 1, \ldots, N, 2 \leq N \leq \infty$, mutually orthogonal closed subspaces of $\mathbb{H}$, and we write,

$$\mathbb{H} = \oplus_{j=1}^{N} \mathbb{H}_j,$$

if for any $f \in \mathbb{H}$, there are $f_j \in \mathbb{H}_j, j = 1, \ldots, N$, such that, $f = \sum_{j=1}^{N} f_j$. Note that the $f_j, j = 1, \ldots, N$ are unique for a given f, and that $\|f\|_{\mathbb{H}}^2 = \sum_{j=1}^{N} \|f_j\|_{\mathbb{H}}^2$.

Let A be an operator in a Hilbert space $\mathbb{H}$, and let us denote by $D[A]$ the domain of A. We say that the operator B is an extension of the operator A, in symbol, $A \subset B$, if $D[A] \subset D[B]$, and if $Au = Bu$, for all $u \in D[A]$. Suppose that the domain of A is dense in $\mathbb{H}$. We denote by $A^{\dagger}$ the adjoint of A, that is defined as follows,

$$D[A^{\dagger}] := \{v \in \mathbb{H} : (Au, v)_{\mathbb{H}} = (u, f)_{\mathbb{H}}, \text{for some}\, f \in \mathbb{H}, \text{and for all}\, u \in D[A]\},$$

and

$$A^{\dagger} v = f, \qquad v \in D[A^{\dagger}].$$

We say that A is symmetric if $A \subset A^{\dagger}$, and that A is self-adjoint if $A = A^{\dagger}$, that is to say if $D[A] = D[A^{\dagger}]$, and $Au = A^{\dagger} u$, $u \in D[A] = D[A^{\dagger}]$. An essentially self-adjoint operator has only one self-adjoint extension. For any operator A we denote by $\mathrm{Ker}[A] := \{u \in D[A] : Au = 0\}$ the set of all eigenvectors of A with eigenvalue zero. For more information on the theory of operators in Hilbert space the reader can consult [76] and [107].

We denote by $L^2(0, 2\pi)$ the standard Hilbert space of functions that are square integrable in $(0, 2\pi)$. Furthermore, we designate by $L_0^2(0, 2\pi)$ the closed subspace of $L^2(0, 2\pi)$ consisting of all functions with zero mean value, i.e.,

$$L_0^2(0, 2\pi) := \left\{ F \in L^2(0, 2\pi) : \int_0^{2\pi} F(x)\, dx = 0 \right\}. \tag{2.3.1}$$

Note that since all the functions in $L^2(0, 2\pi)$ are integrable over $(0, 2\pi)$ the space $L_0^2(0, 2\pi)$ is well defined. Further, we denote by $L^2(\mathbb{R}^2)$ the standard Hilbert space of all functions that are square integrable in $\mathbb{R}^2$. Let us denote by $\mathcal{A}$ the tensor product of $L^2(0, 2\pi)$ and of $L^2(\mathbb{R}^2)$, namely,

$$\mathcal{A} := L^2(0, 2\pi) \otimes L^2(\mathbb{R}^2). \tag{2.3.2}$$

For the definition and the properties of tensor products of Hilbert spaces the reader can consult [107, Section 4, Chapter 2]. We often make use of the fact that the tensor product of an orthonormal basis in $L^2(0, 2\pi)$ and an orthonormal basis in $L^2(\mathbb{R}^2)$ is an orthonormal basis in $\mathcal{A}$. As shown in [107, Section 4, Chapter 2], the space $\mathcal{A}$ can be identified with the standard Hilbert space $L^2((0, 2\pi) \times \mathbb{R}^2)$ of square integrable functions in $(0, 2\pi) \times \mathbb{R}^2$ with the scalar product,

$$(u, f)_{L^2((0,2\pi)\times\mathbb{R}^2)} := \int_{(0,2\pi)\times\mathbb{R}^2} u(x, v)\, \overline{f(x, v)}\, dx\, dv,$$

where $x \in (0, 2\pi)$ and $v = (v_1, v_2) \in \mathbb{R}^2$. Our space of physical states, that we denote by $\mathcal{H}$, is defined as the direct sum of $\mathcal{A}$ and $L_0^2(0, 2\pi)$.

$$\mathcal{H} := \mathcal{A} \oplus L_0^2(0, 2\pi). \tag{2.3.3}$$

We find it convenient to write $\mathcal{H}$ as the space of the column vector-valued functions, $\begin{pmatrix} u \\ F \end{pmatrix}$ where $u(x,v) \in \mathcal{A}$ and $F(x) \in L^2_0(0,2\pi)$. The scalar product in $\mathcal{H}$ is given by,

$$\left(\begin{pmatrix} u \\ F \end{pmatrix}, \begin{pmatrix} f \\ G \end{pmatrix} \right)_{\mathcal{H}} := (u,f)_{\mathcal{A}} + (F,G)_{L^2(0,2\pi)}.$$

Note that by the identity (2.2.21) the $\mathcal{H}$-norm of the solutions to the magnetized Vlasov–Ampère system is constant in time. This is the underlying reason why we will be able in later sections to formulate the magnetized Vlasov–Ampère system as a Schrödinger equation in $\mathcal{H}$ with a self-adjoint realization of the magnetized Vlasov–Ampère operator playing the role of the Hamiltonian. Moreover, the square of the norm of $\mathcal{H}$ is the constant energy of the solutions to the magnetized Vlasov–Ampère system.

Let us denote by $H^1(0,2\pi)$ the standard Sobolev space [2] of all functions in $L^2(0,2\pi)$ such that its derivative in the distribution sense is a function in $L^2(0,2\pi)$, with the scalar product,

$$(F,G)_{H^1(0,2\pi)} := (F,G)_{L^2(0,2\pi)} + (\partial_x F, \partial_x G)_{L^2(0,2\pi)}.$$

We designate by $H^{1,0}(0,2\pi)$ the closed subspace of $H^1(0,2\pi)$ that consists of all functions in $F \in H^1(0,2\pi)$ such that $F(0) = F(2\pi)$ and that have mean zero. Namely,

$$H^{1,0}(0,2\pi) := \left\{ F \in H^1(0,2\pi) : F(0) = F(2\pi), \text{ and } \int_0^{2\pi} F(x)\, dx = 0 \right\}.$$

Note [2] that as the functions in $H^1(0,2\pi)$ have a continuous extension to $[0,2\pi]$, the space $H^{1,0}(0,2\pi)$ is well defined.

We denote by $L^2(\mathbb{R}^+, rdr)$ the standard Hilbert space of functions defined on $\mathbb{R}^+$ with the scalar product,

$$(\tau,\eta)_{L^2(\mathbb{R}^+,rdr)} := \int_0^\infty \tau(r)\, \overline{\eta(r)}\, r\, dr.$$

2.4 The magnetized Vlasov equation without coupling

In this section we consider the case without electric field, i.e. the magnetized Vlasov equation. The results of this section will be useful in the study of the full magnetized Vlasov–Ampère system, that we carry over in section 2.5.

The magnetized Vlasov equation can be written as the following Schrödinger equation in $\mathcal{A}$,

$$i\partial_t u = i\left(-v_1 \partial_x + \omega_c (v_2\, \partial_{v_1} - v_1\, \partial_{v_2})\right) u. \tag{2.4.1}$$

In the following proposition we obtain a complete orthonormal system of eigenfunctions for the magnetized Vlasov equation (2.4.1). To this end, we introduce the polar coordinates (r,φ) of the velocity $v \in \mathbb{R}^2$.

Proposition 2.4.1. *Let $\{\tau_j\}_{j=1}^\infty$ be an orthonormal basis of $L^2(\mathbb{R}^+, rdr)$. Let $\varphi \in [0\,, 2\pi[, r > 0$, be polar coordinates in $\mathbb{R}^2$, $v_1 = r\cos\varphi, v_2 = r\sin\varphi$. For $(n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*$ we define,*

$$u_{n,m,j} := \frac{e^{in(x - \frac{v_2}{\omega_c})}}{\sqrt{2\pi}} \frac{e^{im\varphi}}{\sqrt{2\pi}}\, \tau_j(r). \tag{2.4.2}$$

Then, the $u_{n,m,j}, (n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*$ *are an orthonormal basis in* $\mathcal{A}$. *Furthermore, each* $u_{n,m,j}$ *is an eigenfunction for the magnetized Vlasov equation* (2.4.1) *with eigenvalue* $\lambda_m^{(0)} = m\,\omega_c$,

$$i\,(-v_1\partial_x + \omega_c(v_2\,\partial_{v_1} - v_1\,\partial_{v_2}))\,u_{n,m,j} = \lambda_m^{(0)}\,u_{n,m,j}, \qquad (n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*. \tag{2.4.3}$$

Moreover, the eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$, *have infinite multiplicity.*

Proof. We first prove that the $u_{n,m,j}, (n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*$ are an orthonormal basis in $\mathcal{A}$. Clearly, it is an orthonormal system. To prove that it is a basis it is enough to prove that if a function in $\mathcal{A}$ is orthogonal to all the $u_{n,m,j}, (n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*$, then, it is the zero function. Hence, assume that $u \in \mathcal{A}$ satisfies,

$$(u, u_{n,m,j})_{\mathcal{A}} = 0, \qquad (n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*. \tag{2.4.4}$$

Denote $g_n(v) := \int_0^{2\pi} e^{-inx}\,u(x,v)\,dx$. By the Cauchy-Schwarz inequality, one has $|g_n(v)|^2 \leq 2\pi \int_0^{2\pi} |u(x,v)|^2\,dx$. Further, since $u \in \mathcal{A}$, it follows that $g_n \in L^2(\mathbb{R}^2)$. By (2.4.4), for each fixed $n \in \mathbb{Z}$,

$$\int_{(0,2\pi)\times\mathbb{R}^+} g_n(v)\,e^{in\frac{v_2}{\omega_c}}\,e^{-im\varphi}\,\overline{\tau_j(r)}\,d\varphi\,r\,dr = 0, \qquad (m,j) \in \mathbb{Z} \times \mathbb{N}^*.$$

As the functions $\frac{1}{\sqrt{2\pi}}\,e^{im\varphi}\,\tau_j(r), m \in \mathbb{Z}, j \in \mathbb{N}^*$ are an orthonormal basis in $L^2(\mathbb{R}^2)$, one has that $g_n(v)\,e^{in\frac{v_2}{\omega_c}} = 0$ for a.e. $v \in \mathbb{R}^2$. Moreover, as $e^{in\frac{v_2}{\omega_c}}$ is never zero, we obtain, $g_n(v) = 0$, for a.e. $v \in \mathbb{R}^2$, i.e., $\int_0^{2\pi} e^{-inx}\,u(x,v)\,dx = 0, n \in \mathbb{Z}$. As the functions $\frac{1}{\sqrt{2\pi}}e^{inx}, n \in \mathbb{Z}$ are an orthonormal basis in $L^2(0,2\pi)$, it follows that $u(x,v) = 0$. This completes the proof that the $u_{n,m,j}, (n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*$, are an orthonormal basis of $\mathcal{A}$. Equation (2.4.3) follows from a simple calculation using that $\partial_{v_1} = \frac{v_1}{r}\,\partial_r - \frac{v_2}{r^2}\,\partial_\varphi$, $\partial_{v_2} = \frac{v_2}{r}\,\partial_r + \frac{v_1}{r^2}\,\partial_\varphi$, and $v_2\,\partial_{v_1} - v_1\,\partial_{v_2} = -\partial_\varphi$. Note that the eigenvalues $\lambda_m^{(0)}$ have infinite multiplicity because all the $u_{n,m,j}$ with m fixed and $n \in \mathbb{Z}, j \in \mathbb{N}^*$ are orthogonal eigenfunctions for $\lambda_m^{(0)}$. □

Let us denote by h_0 the formal magnetized Vlasov operator with periodic boundary conditions in x, that we define as follows,

$$h_0 u := i\,(-v_1\partial_x + \omega_c(v_2\,\partial_{v_1} - v_1\,\partial_{v_2}))\,u, \tag{2.4.5}$$

with domain,

$$D[h_0] := \mathcal{D}, \tag{2.4.6}$$

where by $\mathcal{D}$ we denote the following space of test functions,

$$\mathcal{D} := \{u \in C_0^\infty([0,2\pi] \times \mathbb{R}^2) : \frac{d^j}{dx^j}u(0,v) = \frac{d^j}{dx^j}u(2\pi,v), j = 1, \ldots\}, \tag{2.4.7}$$

where by $C_0^\infty([0,2\pi] \times \mathbb{R}^2)$ we designate the space of all infinitely differentiable functions, defined in $[0,2\pi] \times \mathbb{R}^2$, and that have compact support in $[0,2\pi] \times \mathbb{R}^2$.

We will construct a self-adjoint extension of h_0. For this purpose, we first introduce some definitions. Let us denote by $l^2(\mathbb{Z}^2 \times \mathbb{N}^*)$ the standard Hilbert space of square summable sequences, $s = \left\{s_{n,m,j}, (n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\}$ with the scalar product,

$$(s,d)_{l^2(\mathbb{Z}^2\times\mathbb{N}^*)} := \sum_{(n,m,j)\in\mathbb{Z}^2\times\mathbb{N}^*} s_{n,m,j}\,\overline{d_{n,m,j}}.$$

Let $\mathbf{U}$ be the following unitary operator from $\mathcal{A}$ onto $l^2(\mathbb{Z}^2 \times \mathbb{N}^*)$,

$$\mathbf{U}u := \left\{(u, u_{n,m,j})_{\mathcal{A}}, (n, m, j) \in \mathbb{Z}^2 \times \mathbb{Z}\right\}. \tag{2.4.8}$$

We denote by $\widehat{H_0}$ the following operator in $l^2(\mathbb{Z}^2 \times \mathbb{N}^*)$,

$$\left\{(\widehat{H_0}\, s)_{n,m,j}, (n, m, j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\} := \left\{\lambda_m^{(0)} s_{n,m,j}, (n, m, j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\}, \tag{2.4.9}$$

with domain, $D[\widehat{H_0}]$, given by,

$$D[\widehat{H_0}] := \left\{\left\{s_{n,m,j}, (n, m, j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\} \in l^2(\mathbb{Z}^2 \times \mathbb{N}^*) : \left\{\lambda_m^{(0)}\, s_{n,m,j}, (n, m, j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\} \in l^2(\mathbb{Z}^2 \times \mathbb{N}^*)\right\}. \tag{2.4.10}$$

The operator $\widehat{H_0}$ is self-adjoint because it is the multiplication operator by the real eigenvalues $\lambda_m^{(0)}$ defined on its maximal domain.

Proposition 2.4.2. *Let us define*

$$H_0 = \mathbf{U}^\dagger\, \widehat{H_0}\, \mathbf{U}, \quad D[H_0] := \{u \in \mathcal{A} : \mathbf{U}u \in D[\widehat{H_0}]\}. \tag{2.4.11}$$

Then, H_0 is self-adjoint. Its spectrum is pure point, and it consists of the eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$. Moreover, each eigenvalue $\lambda_m^{(0)}, m \in \mathbb{Z}$, has infinite multiplicity. Further, $h_0 \subset H_0$.

Proof. H_0 is unitarily equivalent to the self-adjoint operator $\widehat{H_0}$, and in consequence H_0 is self-adjoint. Let us prove that $h_0 \subset H_0$. Suppose that $u \in D[h_0]$. Integrating by parts we obtain,

$$(h_0 u, u_{n,m,j})_{\mathcal{A}} = (u, h_0\, u_{n,m,j})_{\mathcal{A}} = \lambda_m^{(0)}\, (u, u_{n,m,j})_{\mathcal{A}}, \qquad (n, m, j) \in \mathbb{Z}^2 \times \mathbb{N}^*.$$

Hence,

$$\mathbf{U}h_0 u := \left\{(h_0 u, u_{n,m,j})_{\mathcal{A}}, (n, m, j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\} = \left\{\lambda_m^{(0)}\, (u, u_{n,m,j})_{\mathcal{A}}, (n, m, j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\} \in l^2\left(\mathbb{Z}^2 \times \mathbb{N}^*\right), \tag{2.4.12}$$

where we used that $h_0 u \in \mathcal{A}$, Hence,

$$\mathbf{U}u \in D[\widehat{H_0}].$$

Moreover,

$$H_0 u = \mathbf{U}^\dagger \widehat{H_0} \mathbf{U}u = \mathbf{U}^\dagger \left\{\lambda_m^{(0)}\, (u, u_{n,m,j})_{\mathcal{A}}, (n, m, j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\} = \mathbf{U}^\dagger\, \mathbf{U}h_0 u = h_0 u.$$

This completes the proof that $h_0 \subset H_0$. As $h_0 \subset H_0$ and one has the completeness of the eigenfunctions of h_0 by proposition 2.4.1, it follows that the spectrum of H_0 is pure point, it consists of the eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$, and each eigenvalue $\lambda_m^{(0)}, m \in \mathbb{Z}$, has infinite multiplicity. $\square$

We write the magnetized Vlasov equation (2.4.1) as a Schrödinger equation with a self-adjoint Hamiltonian as follows,

$$i\partial_t u = H_0 u.$$

We call H_0 the magnetized Vlasov operator.

Actually, we can give more information on h_0.

Proposition 2.4.3. *Let h_0 be the formal magnetized Vlasov operator defined in (2.4.5) and (2.4.6), and let H_0 be the magnetized Vlasov operator defined in (2.4.11). We have that,*

$$h_0^{\dagger} = H_0,$$

and, furthermore, h_0 is essentially self-adjoint, i.e., H_0 is the only self-adjoint extension of h_0.

Proof. Suppose that $f \in D[h_0^{\dagger}]$. Then

$$(h_0 u, f)_{\mathcal{A}} = (u, h_0^{\dagger} f)_{\mathcal{A}}. \tag{2.4.13}$$

Hence, by (2.4.12) and (2.4.13)

$$\begin{aligned}(h_0 u, f)_{\mathcal{A}} = (\mathbf{U} h_0 u, \mathbf{U} f)_{l^2(\mathbb{Z}^2\times\mathbb{N}^*)} = \textstyle\sum_{(n,m,j)\in\mathbb{Z}^2\times\mathbb{N}^*} \lambda_m^{(0)} (u, u_{n,m,j})_{\mathcal{A}} \overline{(f, u_{n,m,j})_{\mathcal{A}}} = \\ \textstyle\sum_{(n,m,j)\in\mathbb{Z}^2\times\mathbb{N}^*} (u, u_{n,m,j})_{\mathcal{A}} \overline{(h_0^{\dagger} f, u_{n,m,j})_{\mathcal{A}}}.\end{aligned} \tag{2.4.14}$$

Since (2.4.14) holds for all u in the dense set $D[h_0]$ we obtain,

$$\left\{\lambda_m^{(0)} (f, u_{n,m,j})_{\mathcal{A}}, (n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\} = \left\{(h_0^{\dagger} f, u_{n,m,j})_{\mathcal{A}}, (n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\} \in l^2(\mathbb{Z}^2 \times \mathbb{N}^*). \tag{2.4.15}$$

It follows that,

$$\left\{\lambda_m^{(0)} (f, u_{n,m,j})_{\mathcal{A}}, (n,m,j) \in \mathbb{Z}^2 \times \mathbb{N}^*\right\} \in l^2(\mathbb{Z}^2 \times \mathbb{N}^*). \tag{2.4.16}$$

This implies that $f \in D[H_0]$ and that $h_0^{\dagger} f = H_0 f$. Then, $h_0^{\dagger} \subset H_0$. We prove in a similar way that if $f \in D[H_0]$, then $f \in D[h_0^{\dagger}]$ and that, $H_0 f = h_0^{\dagger} f$. This implies that $H_0 \subset h_0^{\dagger}$. Hence the proof that $h_0^{\dagger} = H_0$ is complete. Finally let A be a self-adjoint operator such that $h_0 \subset A$. Then, $A^{\dagger} \subset h_0^{\dagger} = H_0$. But as $A = A^{\dagger}$, we obtain that $A \subset H_0$, and then, $H_0^{\dagger} \subset A^{\dagger}$, but as $A = A^{\dagger}, H_0 = H_0^{\dagger}$, we have $H_0 \subset A$, and finally $A = H_0$. This proves that H_0 is the only self-adjoint extension of h_0. ☐

2.5 The full magnetized Vlasov–Ampère system with coupling

In this section we consider the full magnetized Vlasov–Ampère system. We write the system as a Schrödinger equation in the Hilbert space $\mathcal{H}$ as follows

$$i\partial_t \begin{pmatrix} u \\ F \end{pmatrix} = \mathbf{H} \begin{pmatrix} u \\ F \end{pmatrix}, \tag{2.5.1}$$

where the magnetized Vlasov–Ampère operator $\mathbf{H}$ is the following operator in $\mathcal{H}$,

$$\mathbf{H} = \begin{bmatrix} H_0 & -iv_1 e^{\frac{-v^2}{4}} \\ iI^* \int_{\mathbb{R}^2} v_1 e^{\frac{-v^2}{4}} \cdot dv & 0 \end{bmatrix} \quad \left(\text{where we use the notation } e^{\frac{-v^2}{4}} = e^{-\frac{|v|^2}{4}} = e^{-\frac{v_1^2+v_2^2}{4}}\right). \tag{2.5.2}$$

In a more detailed way, the right-hand side of (2.5.1) is defined as follows,

$$\mathbf{H}\begin{pmatrix} u \\ F \end{pmatrix} := \begin{pmatrix} H_0 u - i v_1\, e^{\frac{-v^2}{4}} F \\ iI^* \int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}}\, u\, dv \end{pmatrix}. \tag{2.5.3}$$

We recall that I^* gives zero when applied to constant functions in $L^2(0, 2\pi)$. The domain of $\mathbf{H}$ is defined as follows,

$$D[\mathbf{H}] := D(H_0) \oplus L^2_0(0, 2\pi). \tag{2.5.4}$$

We write $\mathbf{H}$ in the following form,

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{V}, \tag{2.5.5}$$

where

$$\mathbf{H}_0 := \begin{bmatrix} H_0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{2.5.6}$$

and

$$\mathbf{V} := \begin{bmatrix} 0 & -i v_1\, e^{\frac{-v^2}{4}} \\ iI^* \int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}} \cdot dv & 0 \end{bmatrix}. \tag{2.5.7}$$

Clearly, $\mathbf{H}_0$ is self-adjoint with $D[\mathbf{H}_0] = D[\mathbf{H}]$. Moreover, $\mathbf{V}$, with $D[\mathbf{V}] = \mathcal{H}$, is bounded in $\mathcal{H}$. Observe that the presence of I^* in $\mathbf{V}$ assures us that $\mathbf{V}$ sends $\mathcal{H}$ in to $\mathcal{H}$. Further, it follows from a simple calculation that $\mathbf{V}$ is symmetric in $\mathbf{H}$. Then, by the Kato–Rellich theorem, see theorem 4.3 in page 287 of [76], the operator $\mathbf{H}$ is self-adjoint. We proceed to prove that $\mathbf{H}$ has pure point spectrum. Actually, we will explicitly compute the eigenvalues and a basis of eigenfunctions. We do this in several steps.

Remark 2.5.1. The Gauss law in strong sense for a function $\begin{pmatrix} u(x,v) \\ F(x) \end{pmatrix} \in \mathcal{H}$ reads,

$$\int_{\mathbb{R}^2} u(x,v)\, e^{\frac{-v^2}{4}}\, dv + F'(x) = 0. \tag{2.5.8}$$

Later, in remark 2.6.1, we write the Gauss law in weak sense, and we show that it can, equivalently, be expressed as a orthogonality relation with a subset of the eigenfunctions in the kernel of the magnetized Vlasov–Ampère operator $\mathbf{H}$.

2.5.1 The kernel of H

In this subsection we compute a basis for the kernel of the magnetized Vlasov–Ampère operator $\mathbf{H}$. We have to solve the equation

$$\mathbf{H}\begin{pmatrix} u \\ F \end{pmatrix} = 0. \tag{2.5.9}$$

Inserting (2.5.3) in (2.5.9) we obtain,

$$\begin{cases} i\,(-v_1\partial_x + \omega_c(v_2\,\partial_{v_1} - v_1\,\partial_{v_2}))\, u - i v_1\, e^{\frac{-v^2}{4}} F = 0, \\ iI^* \displaystyle\int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}}\, u\, dv = 0. \end{cases} \tag{2.5.10}$$

Denote,

$$\psi(x) := \int_x^{2\pi} F(y)\, dy - \frac{1}{2\pi}\int_0^{2\pi} yF(y)dy. \tag{2.5.11}$$

Then, as $F \in L_0^2(0,2\pi)$, we have that $\psi \in H^{1,0}(0,2\pi)$. Further,

$$F(x) = -\psi'(x). \tag{2.5.12}$$

Let us designate $\gamma(x,v) := u(x,v) - e^{\frac{-v^2}{4}}\,\psi(x)$. Hence, the first equation in (2.5.10) is equivalent to the following equation

$$H_0\,\gamma = 0. \tag{2.5.13}$$

Then, the general solution to the first equation in (2.5.10) can be written as

$$u(x,v) = e^{\frac{-v^2}{4}}\,\psi(x) + \gamma(x,v), \tag{2.5.14}$$

with $F = -\psi'$, where $\psi \in H^{1,0}(0,2\pi)$, and γ solves (2.5.13). Furthermore, by (2.5.14) the second equation is (2.5.10) is equivalent to,

$$I^* \int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}}\,\gamma\, dv = 0. \tag{2.5.15}$$

Then, we have proven that the general solution to (2.5.10) can be written as,

$$\begin{pmatrix} u \\ F \end{pmatrix} = \begin{pmatrix} e^{\frac{-v^2}{4}}\,\psi(x) + \gamma(x,v) \\ -\psi'(x) \end{pmatrix}, \tag{2.5.16}$$

where $\psi \in H^{1,0}(0,2\pi), F = -\psi'$, and γ solves (2.5.13). By proposition 2.4.1 the general solution can be written as

$$\gamma = \sum_{(n,j)\in\mathbb{Z}\times\mathbb{N}^*} (\gamma, u_{n,0,j})_{\mathcal{A}}\, u_{n,0,j}. \tag{2.5.17}$$

Using (2.4.2) we prove by explicit calculation that $u_{n,0,j}$, $n \in \mathbb{Z}$ and $j \in \mathbb{N}^*$, satisfies (2.5.15). So the general solution (2.5.17) satisfies (2.5.13) and (2.5.15).

In the following lemma we construct a basis of $\mathrm{Ker}[\mathbf{H}]$, using the results above.

Lemma 2.5.2. *Let* $\mathbf{H}$ *be the magnetized Vlasov–Ampère operator defined in* (2.5.3) *and,* (2.5.4)*. Let* $u_{n,0,j}$ *be the eigenfunctions defined in* (2.4.2)*. Then, the following set of eigenfunctions of* $\mathbf{H}$ *with eigenvalue zero,*

$$\left\{\mathbf{V}_n^{(0)} := \frac{1}{\sqrt{2\pi+n^2}}\frac{e^{inx}}{\sqrt{2\pi}}\begin{pmatrix} e^{\frac{-v^2}{4}} \\ -i\,n\end{pmatrix}, n \in \mathbb{Z}^*\right\} \cup \left\{\mathbf{M}_{n,j}^{(0)} := \begin{pmatrix} u_{n,0,j} \\ 0 \end{pmatrix}, (n,j) \in \mathbb{Z}\times\mathbb{N}^*\right\}, \tag{2.5.18}$$

is linearly independent and it is a basis of $\mathrm{Ker}[\mathbf{H}]$.

Proof. Let us first prove the linear independence of the sets of functions (2.5.18). We have to prove that if a linear combination of the eigenfunctions (2.5.18) is equal to zero then, each of the coefficients in the linear combination is equal to zero. For this purpose we write the general linear combination of the eigenfunctions in (2.5.18) with a convenient notation. Let $\mathbb{M}_1$ be any finite subset of $\mathbb{Z}^*$ and let $\mathbb{M}_2$ be any finite subset of $\mathbb{Z}\times\mathbb{N}^*$. Then, the general linear combination of the eigenfunctions in (2.5.18) can be written as follows,

$$\sum_{n\in\mathbb{M}_1}\alpha_n\,\frac{1}{\sqrt{2\pi+n^2}}\,\frac{e^{inx}}{\sqrt{2\pi}}\begin{pmatrix}e^{\frac{-v^2}{4}}\\ -i\,n\end{pmatrix}+\sum_{(l,p)\in\mathbb{M}_2}\beta_{(l,p)}\begin{pmatrix}u_{l,0,p}\\ 0\end{pmatrix},$$

for some complex numbers $\alpha_n, n\in\mathbb{M}_1$, and $\beta_{(l,p)}, (l,p)\in\mathbb{M}_2$. Suppose that,

$$\sum_{n\in\mathbb{M}_1}\alpha_n\,\frac{1}{\sqrt{2\pi+n^2}}\,\frac{e^{inx}}{\sqrt{2\pi}}\begin{pmatrix}e^{\frac{-v^2}{4}}\\ -i\,n\end{pmatrix}+\sum_{(l,p)\in\mathbb{M}_2}\beta_{(l,p)}\begin{pmatrix}u_{l,0,p}\\ 0\end{pmatrix}=0.$$

Suppose that for some $\alpha_{l_n}, n=1,\dots,N$, and some $\beta_{q_l,j_p}, l=1,\dots,M, p=1,\dots,P$, where $l_n\in\mathbb{Z}\setminus\{0\}, n=1,\dots,N, l_n\neq l_m$, if $n\neq m$, and $q_l\in\mathbb{Z}, l=1,\dots,M, j_p\in\mathbb{N}^*, p=1,\dots,P$, $(q_l,j_p)\neq(q_m,j_q)$, if $(l,p)\neq(m,q)$,

$$\sum_{n=1}^{N}\alpha_{l_n}\,\frac{1}{\sqrt{2\pi+l_n^2}}\,\frac{e^{il_nx}}{\sqrt{2\pi}}\begin{pmatrix}e^{\frac{-v^2}{4}}\\ -i\,l_n\end{pmatrix}+\sum_{l=1}^{M}\sum_{p=1}^{P}\beta_{q_l,j_p}\begin{pmatrix}u_{q_l,0,j_p}\\ 0\end{pmatrix}=0. \tag{2.5.19}$$

Since the second component of the functions in the second sum is zero, we have $\sum_{n\in\mathbb{M}_1}\alpha_n\,\frac{1}{\sqrt{2\pi+n^2}}\,\frac{e^{inx}}{\sqrt{2\pi}}\,n=0$. Further, as the $\dfrac{e^{inx}}{\sqrt{2\pi}}, n\in\mathbb{M}_1$ are orthogonal to each other, we have that, $\alpha_n=0, n\in\mathbb{M}_1$. Furthermore, as the $\alpha_n, n\in\mathbb{M}_1$ are equal to zero, we obtain $\sum_{(l.p)\in\mathbb{M}_2}\beta_{(l,p)}u_{l,0,p}=0$. Moreover, since the $u_{l,0,p}, (l,p)\in\mathbb{M}_2$ are an orthonormal set, $\beta_{(l,p)}=0, (l,p)\in\mathbb{M}_2$. This proves the linear independence of the set (2.5.18). Moreover, by (2.5.16) with $\psi(x)=\frac{e^{inx}}{\sqrt{2\pi}}, n\in\mathbb{Z}^*$, and $f=0$, each of the functions

$$\frac{1}{\sqrt{2\pi+n^2}}\,\frac{e^{inx}}{\sqrt{2\pi}}\begin{pmatrix}e^{\frac{-v^2}{4}}\\ -i\,n\end{pmatrix}\qquad n\in\mathbb{Z}^*,$$

is an eigenvector of $\mathbf{H}$ with eigenvalue zero. Similarly, by (2.5.16) with $\psi(x)=0$, and $f=u_{n,0,j}$, one has that each of the functions,

$$\begin{pmatrix}u_{n,0,j}\\ 0\end{pmatrix},\qquad (n,j)\in\mathbb{Z}\times\mathbb{N}^*,$$

is an eigenfunctions of $\mathbf{H}$ with eigenvalue zero. By the Fourier transform, the set of functions, $\frac{e^{inx}}{\sqrt{2\pi}}$, $n\in\mathbb{Z}$, is a complete orthonormal set in $L^2(0,2\pi)$. Then, in particular, any $\psi\in H^{1,0}(0,2\pi)$, can be represented as follows,

$$\psi(x)=\sum_{n\in\mathbb{Z}^*}\left(\psi,\frac{e^{inx}}{\sqrt{2\pi}}\right)_{L^2(0,2\pi)}\frac{e^{inx}}{\sqrt{2\pi}}, \tag{2.5.20}$$

where the series converges in the norm of $L^2(0,2\pi)$. Note that there is no term with $n=0$ because the mean value of ψ is zero. Then, by (2.5.20),

$$\begin{pmatrix}e^{\frac{-v^2}{4}}\,\psi(x)\\ -\psi'(x)\end{pmatrix}=\sum_{n\in\mathbb{N}^*}\left(\psi,\frac{e^{inx}}{\sqrt{2\pi}}\right)_{L^2(0,2\pi)}\frac{e^{inx}}{\sqrt{2\pi}}\begin{pmatrix}e^{\frac{-v^2}{4}}\\ -i\,n\end{pmatrix}. \tag{2.5.21}$$

Finally, it follows from (2.5.16), (2.5.17) and (2.5.21) that the set (2.5.18) is a basis of the kernel of $\mathbf{H}$. □

2.5.2 The eigenvalues of H different from zero and their eigenfunctions

In this subsection we compute the non-zero eigenvalues of $\mathbf{H}$ and we give explicit formulae for the eigenfunctions that correspond to each eigenvalue. By (2.5.3) we have to solve the system of equations

$$\begin{cases} H_0 u - i v_1\, e^{\frac{-v^2}{4}}\, F = \lambda u, \\ i I^* \displaystyle\int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}}\, u\, dv = \lambda F, \end{cases} \tag{2.5.22}$$

with $\lambda \in \mathbb{R} \setminus \{0\}$, and $\begin{pmatrix} u \\ F \end{pmatrix} \in D[\mathbf{H}]$. We first consider the case where the electric field, F, is zero, and then, when it is different from zero.

The case with zero electric field

We have to compute solutions to (2.5.22) of the form,

$$\begin{pmatrix} u \\ 0 \end{pmatrix} \in D[\mathbf{H}], \tag{2.5.23}$$

with $u \in D[H_0]$. Introducing (2.5.23) into the system (2.5.22) we obtain,

$$\begin{cases} H_0 u = \lambda u, \\ i I^* \displaystyle\int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}}\, u\, dv = 0. \end{cases} \tag{2.5.24}$$

We seek for eigenfunctions of the form,

$$u(x, v) := \frac{1}{\sqrt{2\pi}}\, e^{in(x - \frac{v_2}{\omega_c})}\, \frac{1}{\sqrt{2\pi}}\, e^{im\varphi}\, \tau(r), \qquad (n, m) \in \mathbb{Z}^2, \tag{2.5.25}$$

where (r, φ) are the polar coordinates of $v \in \mathbb{R}^2$, and the function τ will be specified later. We first consider the case when $n = 0$. In this case the second equation in (2.5.24) is satisfied because the operator I^* gives zero when applied to functions that are independent of x. Hence, we are left with the first equation only, that is the problem that we solved in section 2.4. Then, as we seek non zero eigenvalues we have to have $m \neq 0$ in (2.5.25). Using the results of section 2.4 we obtain the following lemma.

Lemma 2.5.3. *Let $\mathbf{H}$ be the magnetized Vlasov–Ampère operator defined in (2.5.3) and, (2.5.4). Let $\{\tau_j\}_{j=1}^{\infty}$ be an orthonormal basis of $L^2(\mathbb{R}^+, r dr)$. Let $\varphi \in [0, 2\pi), r > 0$, be polar coordinates in $\mathbb{R}^2, v_1 = r\cos\varphi, v_2 = r\sin\varphi$. For $(m, j) \in \mathbb{Z}^* \times \mathbb{N}^*$ let $u_{0,m,j}$ be the eigenfunction defined in (2.4.2). Then, the set*

$$\mathbf{V}_{m,j} := \left\{ \begin{pmatrix} u_{0,m,j} \\ 0 \end{pmatrix}, (m, j) \in \mathbb{Z}^* \times \mathbb{N}^* \right\}, \tag{2.5.26}$$

is an orthonormal set in $\mathcal{H}$. Furthermore, each function on this set is an eigenvector of $\mathbf{H}$ corresponding the eigenvalue $\lambda_m^{(0)} = m\,\omega_c \neq 0$,

$$\mathbf{H}\mathbf{V}_{m,j} = \lambda_m^{(0)}\, \mathbf{V}_{m,j}, \qquad (m, j) \in \mathbb{Z}^* \times \mathbb{N}^*. \tag{2.5.27}$$

Moreover, each eigenvalue $\lambda_m^{(0)}$ has infinite multiplicity.

Proof. The lemma follows from proposition 2.4.1 and since the second equation in (2.5.24) is always satisfied for functions that are independent of x. $\square$

Let us now study the second case, namely $n \neq 0$. We have to consider the second equation in the system (2.5.24). We first prepare some results. For $m \in \mathbb{Z}$ let $J_m(z), z \in \mathbb{C}$, be the Bessel function. We have that

$$J_m(-z) = (-1)^m J_m(z), \qquad J_{-m}(-z) = J_m(z). \tag{2.5.28}$$

For the first equation see formula 10.4.1 in page 222 of [53] and for the second see formula 9.1.5 in page 358 of [1]. The Jacobi-Anger formula, given in equation 10.12.1, page 226 of [53], yields,

$$e^{iz\sin\varphi} = \sum_{m\in\mathbb{Z}} e^{im\varphi} J_m(z). \tag{2.5.29}$$

The Parseval identity for the Fourier series applied to (2.5.29) gives,

$$\sum_{m\in\mathbb{Z}} J_m(z)^2 = 1, \qquad z \in \mathbb{R}. \tag{2.5.30}$$

Differentiating the Jacobi-Anger formula with respect to φ we obtain,

$$z\cos\varphi\, e^{iz\sin\varphi} = \sum_{m\in\mathbb{Z}} m\, e^{im\varphi} J_m(z). \tag{2.5.31}$$

Taking in (2.5.31) $z = -nr/\omega_c$, with $n \neq 0$, recalling that $v_1 = r\cos\varphi, v_2 = r\sin\varphi$, and using the first equation in (2.5.28) we get,

$$v_1\, e^{-i\frac{nv_2}{\omega_c}} = -\frac{\omega_c}{n}\sum_{m\in\mathbb{Z}} m\, e^{im\varphi}(-1)^m J_m\left(\frac{nr}{\omega_c}\right), \qquad n \neq 0. \tag{2.5.32}$$

From (2.5.32) we obtain,

$$\int_0^{2\pi} v_1\, e^{-i\frac{nv_2}{\omega_c}}\, e^{im\varphi}\, d\varphi = 2\pi\frac{m\omega_c}{n}(-1)^m J_{-m}\left(\frac{nr}{\omega_c}\right) = 2\pi\frac{m\omega_c}{n} J_m\left(\frac{nr}{\omega_c}\right), \qquad n \neq 0, \tag{2.5.33}$$

where in the last equality we used both equations in (2.5.28). Using (2.5.32) and taking $n, m \neq 0$ we prove that the second equation in (2.5.24) with u given by (2.5.25) is equivalent to,

$$\int_0^\infty e^{-\frac{r^2}{4}} J_m\left(\frac{nr}{\omega_c}\right) \tau(r)\, r\, dr = 0. \tag{2.5.34}$$

Taking $m = 0$ is possible, but it will be discarded below in lemma 2.5.4. Let us denote by $V_{n,m}$ the orthogonal complement in $L^2(\mathbb{R}^+, rdr)$ to the function, $e^{-\frac{r^2}{4}} J_m\left(\frac{nr}{\omega_c}\right)$, that is to say,

$$V_{n,m} := \left\{ f \in L^2(\mathbb{R}^+, rdr) : \left(f, e^{-\frac{r^2}{4}} J_m\left(\frac{nr}{\omega_c}\right)\right)_{L^2(\mathbb{R}^+, rdr)} = 0 \right\}, n, m \in \mathbb{Z}^*. \tag{2.5.35}$$

Note that $V_{n,m}$ is an infinite dimensional subspace of $L^2(\mathbb{R}^+, rdr)$ of codimension equal to one. We prove the following lemma using the results above.

Lemma 2.5.4. *Let* $\mathbf{H}$ *be the magnetized Vlasov–Ampère operator defined in* (2.5.3) *and* (2.5.4). *Let* $\tau_{n,m,j}, n, m \in \mathbb{Z}^*, j \in \mathbb{N}^*$ *be an orthonormal basis in* $V_{n,m}$ *and define,*

$$f_{n,m,j} := \frac{1}{\sqrt{2\pi}}\, e^{in(x-\frac{v_2}{\omega_c})}\, \frac{1}{\sqrt{2\pi}}\, e^{im\varphi}\, \tau_{n,m,j}(r), n, m \in \mathbb{Z}^*, j \in \mathbb{N}^*. \tag{2.5.36}$$

Then, the set

$$\left\{ \mathbf{W}_{n,m,j} := \begin{pmatrix} f_{n,m,j} \\ 0 \end{pmatrix}, n, m \in \mathbb{Z}^*, j \in \mathbb{N}^*. \right\} \tag{2.5.37}$$

is an orthonormal set in $\mathcal{H}$. *Furthermore, each function on this set is an eigenvector of* $\mathbf{H}$ *corresponding the eigenvalue* $\lambda_m^{(0)} = m\,\omega_c \neq 0$,

$$\mathbf{H}\mathbf{W}_{n,m,j} = \lambda_m^{(0)}\, \mathbf{W}_{n,m,j} \qquad n, m \in \mathbb{Z}^*, j \in \mathbb{N}^*. \tag{2.5.38}$$

Moreover, each eigenvalue $\lambda_m^{(0)}$ has infinite multiplicity.

Proof. The lemma follows from (2.5.24), (2.5.34), (2.5.35) and (2.5.36). Note that the case $m = 0$ does not appear because we are looking for eigenfunctions with eigenvalue different from zero. Furthermore, the eigenvalues $\lambda_m^{(0)}$ have infinite multiplicity because all the eigenfunctions $\mathbf{W}_{n,m,j}$ with a fixed m and all $n \in \mathbb{Z}^*, j \in \mathbb{N}^*$, are orthogonal eigenfunctions for the eigenvalue, $\lambda_m^{(0)}$.

□

The case with electric field different from zero

From the physical point of view this is the most interesting situation, since it describes the interaction of the electrons with the electric field. Moreover, it is the most involved technically. We look for eigenfunctions of the form,

$$\frac{1}{\sqrt{2\pi}} \begin{pmatrix} e^{in(x-\frac{v_2}{\omega_c})}\,\tau(v) \\ e^{inx} G \end{pmatrix}, \tag{2.5.39}$$

where G is a constant. Since we wish that the electric field is nonzero we must have $G \neq 0$. Hence, to fulfill that $\int_0^{2\pi} F(x)\,dx = 0$, we must have $n \neq 0$. The eigenvalue system (2.5.22) recasts as,

$$\begin{cases} (-i\omega_c\partial_\varphi - \lambda)\tau = iG\,v_1\,e^{\frac{-v^2}{4}}\,e^{in\frac{v_2}{\omega_c}}, \\ \lambda G = i\displaystyle\int_{\mathbb{R}^2} v_1\,e^{\frac{-v^2}{4}}\,e^{-in\frac{v_2}{\omega_c}}\,\tau(v)\,dv. \end{cases} \tag{2.5.40}$$

Changing n into $-n$ in (2.5.32) and using the first equation in (2.5.28) we obtain,

$$v_1\,e^{i\frac{nv_2}{\omega_c}} = \frac{\omega_c}{n} \sum_{m\in\mathbb{Z}} m\,e^{im\varphi}\,J_m\left(\frac{nr}{\omega_c}\right), \qquad n \neq 0. \tag{2.5.41}$$

Plugging (2.5.41) into the first equation in the system (2.5.40) we get,

$$(-i\omega_c\partial_\varphi - \lambda)\tau(r,\varphi) = iG\,e^{\frac{-r^2}{4}}\,\frac{\omega_c}{n} \sum_{m\in\mathbb{Z}^*} m\,e^{im\varphi}\,J_m\left(\frac{nr}{\omega_c}\right), \qquad n \neq 0. \tag{2.5.42}$$

A solution to (2.5.42) is given by

$$\tau(r,\varphi) = iG\, e^{\frac{-r^2}{4}}\, \frac{1}{n} \sum_{m\in\mathbb{Z}^*} \frac{m\,\omega_c}{m\omega_c - \lambda}\, e^{im\varphi}\, J_m\left(\frac{nr}{\omega_c}\right), \qquad n \neq 0, \tag{2.5.43}$$

for $\lambda \neq m\omega_c, m \in \mathbb{Z}^*$. Introducing (2.5.43) into the second equation in the system (2.5.40), and simplifying by $G \neq 0$ we get,

$$\lambda = -\frac{1}{n} \sum_{m\in\mathbb{Z}^*} \frac{m\,\omega_c}{m\omega_c - \lambda} \int_{\mathbb{R}^2} e^{\frac{-r^2}{2}}\, e^{im\varphi}\, e^{-in\frac{v_2}{\omega_c}}\, J_m\left(\frac{nr}{\omega_c}\right)\, v_1\, dv, \qquad n \neq 0, \quad \lambda \neq m\omega_c, \quad m \in \mathbb{Z}^*. \tag{2.5.44}$$

Plugging (2.5.33) into (2.5.44) and using that $dv = rdrd\varphi$, we obtain,

$$\lambda = -\frac{2\pi}{n^2} \sum_{m\in\mathbb{Z}^*} \frac{m^2\,\omega_c^2}{m\omega_c - \lambda}\, a_{n,m}, \qquad n \neq 0, \quad \lambda \neq m\omega_c, \quad m \in \mathbb{Z}^*. \tag{2.5.45}$$

where we denote

$$a_{n,m} := \int_0^\infty e^{\frac{-r^2}{2}}\, J_m\left(\frac{nr}{\omega_c}\right)^2 rdr > 0, \quad m \in \mathbb{Z}. \tag{2.5.46}$$

Equation (2.5.45) is a secular equation that we will study to determine the possible values of λ. Remark that (2.5.45) coincides with the secular equation obtained by [13] and [14]. First we write it in a more convenient form. Note that thanks to the two equations in (2.5.28) we have $J_{-m}(z) = (-1)^m J_m(z)$ and then $a_{n,-m} = a_{n,m}$. Using also $\frac{m\,\omega_c}{m\omega_c - \lambda} = 1 + \frac{\lambda}{m\omega_c - \lambda}$, we can write

$$\sum_{m\in\mathbb{Z}^*} \frac{m^2\,\omega_c^2}{m\omega_c - \lambda}\, a_{n,m} = \sum_{m\in\mathbb{Z}^*} \left(m\omega_c + \frac{m\,\omega_c\lambda}{m\omega_c - \lambda} \right) a_{n,m} = \lambda \sum_{m\in\mathbb{Z}^*} \frac{m\,\omega_c}{m\omega_c - \lambda}\, a_{n,m}. \tag{2.5.47}$$

Simplifying by $\lambda \neq 0$ and using (2.5.47) we write (2.5.45) as

$$1 = -\frac{2\pi}{n^2} \sum_{m\in\mathbb{Z}^*} \frac{m\,\omega_c}{m\omega_c - \lambda}\, a_{n,m}, \qquad n \neq 0, \quad \lambda \neq m\omega_c, \quad m \in \mathbb{Z}^*. \tag{2.5.48}$$

By (2.5.30) we have $\sum_{m\in\mathbb{Z}^*} a_{n,m} < +\infty$ and thus the series in (2.5.48) is absolutely convergent. Secondly we proceed to write (2.5.48) in another form that we find convenient. Using again $a_{n,-m} = a_{n,m}$ we have

$$\sum_{m\in\mathbb{Z}^*} \frac{m\,\omega_c}{m\omega_c - \lambda}\, a_{n,m} = 2 \sum_{m=1}^{\infty} \frac{m^2\,\omega_c^2}{m^2\omega_c^2 - \lambda}\, a_{n,m}. \tag{2.5.49}$$

Let us denote

$$g(\lambda) := 4\pi \sum_{m=1}^{\infty} \frac{m^2\,\omega_c^2}{m^2\omega_c^2 - \lambda^2} a_{n,m}, \qquad \lambda \neq m\omega_c, \qquad m \in \mathbb{Z}^*. \tag{2.5.50}$$

Then using (2.5.49), (2.5.48) is equivalent to

$$g(\lambda) = -n^2, n \in \mathbb{Z}^*, \qquad \lambda \neq m\omega_c, \qquad m \in \mathbb{Z}^*. \tag{2.5.51}$$

Since the function g is even it is enough to study it for $\lambda \geq 0$. It has simple poles as $\lambda = m\omega_c, m \in \mathbb{N}^*$. It is well defined for $\lambda \in \cup_{m=0}^{\infty} I_m$, where,

$$I_0 := [0\,, \omega_c[, \qquad I_m :=]m\omega_c\,, (m+1)\omega_c[, \qquad m \in \mathbb{N}^*. \tag{2.5.52}$$

Lemma 2.5.5. *The function g is positive in I_0. For $m \geq 1$, g is monotone increasing in the interval I_m and the following limits hold,*

$$\lim \lambda \to (m\omega_c)^- = +\infty, \qquad \lim \lambda \to \ (m\omega_c)^+ = -\infty. \tag{2.5.53}$$

Proof. The fact that g is positive in I_0 follows from the definition of g in (2.5.50). Furthermore, since $a_{n,m} > 0, m \geq 1$, and the functions $\lambda \mapsto \frac{m^2\,\omega_c^2}{m^2\omega_c^2 - \lambda^2}$ are monotone increasing away from the poles, we have that g is increasing in $I_m, m \geq 1$, and that the limits in (2.5.53) hold. □

In the following lemma we obtain the solutions to (2.5.51)

Lemma 2.5.6. *For $n \in \mathbb{Z}^*$, the equation (2.5.51) has a countable number of real simple roots, $\lambda_{n,m}$ in $]m\omega_c\,, (m+1)\omega_c[, m \geq 1$. By parity $\lambda_{n,m} := -\lambda_{n,-m}$, $m \leq -1$ is also a root. There is no root in $]-\omega_c\,, \omega_c[$. Furthermore, $\lambda_{n_1,m_1} = \lambda_{n_2,m_2}$ if and only if $n_1 = n_2$, and $m_1 = m_2$,*

Proof. The first two items follow from lemma 2.5.5 and the parity of g. The third point is true because g is positive in $]-\omega_c\,, \omega_c[$. Finally, if $\lambda_{n_1,m_1} = \lambda_{n_2,m_2}$, we have, $m_1 = m_2$, because $\lambda_{n_1,m_1} \in\,]m_1\omega_c\,, (m_1+1)\omega_c[$ and $\lambda_{n_2,m_2} \in\,]m_2\omega_c\,, (m_2+1)\omega_c[$. Furthermore, if $n_1 \neq n_2$, then, $\lambda_{n_1,m} \neq \lambda_{n_2,m}$, because, otherwise, $-n_1^2 = g(\lambda_{n_1,m}) = g(\lambda_{n_2,m}) = -n_2^2$, and this is impossible. □

Using (2.5.39) and (2.5.43) we define,

$$\mathbf{Y}_{n,m} := \frac{1}{\sqrt{2\pi}}\, e^{inx} \begin{pmatrix} e^{-in\frac{v_2}{\omega_c}}\, \eta_{n,m}(v) \\ -n\, i \end{pmatrix}, \qquad n, m \in \mathbb{Z}^*, \tag{2.5.54}$$

where

$$\eta_{n,m}(v) := e^{\frac{-r^2}{4}} \sum_{q \in \mathbb{Z}^*} \frac{q\,\omega_c}{q\omega_c - \lambda_{n,m}}\, e^{iq\varphi}\, J_q\left(\frac{nr}{\omega_c}\right), \qquad n, m \in \mathbb{Z}^*. \tag{2.5.55}$$

For $m \in \mathbb{Z}^*, \lambda_{n,m}$ is the root given in lemma 2.5.6. Note that we have simplified the factor $\frac{i}{n}$ in (2.5.43) and we have taken $G = 1$. Remark that, formally, $\mathbf{Y}_{n,m}$ is an eigenfunction of $\mathbf{H}$,

$$\mathbf{H}\mathbf{Y}_{n,m} = \lambda_{n,m}\, \mathbf{Y}_{n,m}. \tag{2.5.56}$$

However, we have to verify that $\mathbf{Y}_{n,m} \in \mathcal{H}$. We have

$$\|Y_{n,m}\|_{\mathcal{H}} = \sqrt{\|\eta_{n,m}\|^2_{L^2(\mathbb{R}^2)} + n^2},$$

and

$$\|\eta_{n,m}(v)\|^2_{L^2(\mathbb{R}^2)} = 2\pi \sum_{q \in \mathbb{Z}^*} \left(\frac{q\omega_c}{q\omega_c - \lambda_{n,m}}\right)^2 a_{n,q}, \tag{2.5.57}$$

where we used the first equation in (2.5.28). We now prove that $\|\eta_{n,m}(v)\|^2_{L^2(\mathbb{R}^2)} < +\infty$ and exhibit an asymptotic expansion of this quantity which will be used later.

Lemma 2.5.7. *We have,*

$$\|\eta_{n,m}(v)\|^2_{L^2(\mathbb{R}^2)} = \frac{n^4}{2\pi}\, \frac{1}{a_{n,m}} \left(1 + O\left(\frac{1}{m^2}\right)\right) + O\left(\frac{1}{m^2}\right), \qquad m \to \pm\infty. \tag{2.5.58}$$

Proof. Recall that for $m \leq -1, \lambda_{n,m} = -\lambda_{n,-m}$. Then,

$$\|\eta_{n,m}(v)\|^2_{L^2(\mathbb{R}^2)} = \|\eta_{n,-m}(v)\|^2_{L^2(\mathbb{R}^2)}, \qquad m \leq -1. \tag{2.5.59}$$

Hence, it is enough to consider the case $m \geq 1$. We decompose the sum in (2.5.57) as follows,

$$\|\eta_{n,m}(v)\|^2_{L^2(\mathbb{R}^2)} := \sum_{j=1}^{4} h^{(j)}(\lambda_{n,m}), \tag{2.5.60}$$

where,

$$h^{(1)}(\lambda_{n,m}) := 2\pi \sum_{q\leq -1} \left(\frac{q\omega_{\rm c}}{q\omega_{\rm c} - \lambda_{n,m}}\right)^2 a_{n,q}, \tag{2.5.61}$$

$$h^{(2)}(\lambda_{n,m}) := 2\pi \sum_{1\leq q\leq m-1} \left(\frac{q\omega_{\rm c}}{q\omega_{\rm c} - \lambda_{n,m}}\right)^2 a_{n,q}, \tag{2.5.62}$$

$$h^{(3)}(\lambda_{n,m}) := 2\pi \left(\frac{m\,\omega_{\rm c}}{m\omega_{\rm c} - \lambda_{n,m}}\right)^2 a_{n,m}, \tag{2.5.63}$$

and

$$h^{(4)}(\lambda_{n,m}) := 2\pi \sum_{m+1\leq q} \left(\frac{q\omega_{\rm c}}{q\omega_{\rm c} - \lambda_{n,m}}\right)^2 a_{n,q}. \tag{2.5.64}$$

Since $\left(\frac{q\omega_{\rm c}}{q\omega_{\rm c}-\lambda_{n,m}}\right)^2 \leq \frac{q^2}{(m+1)^2}$, $q \leq -1$, we have,

$$\left|h^{(1)}(\lambda_{n,m})\right| \leq 2\pi \sum_{q\leq -1} \frac{q^2}{(m+1)^2}\, a_{n,-q} \leq C\frac{1}{(m+1)^2}, \tag{2.5.65}$$

where in the last inequality we used (2.9.2). Assuming that m is even, we decompose $h^{(2)}(\lambda_{n,m})$ as follows,

$$h^{(2)}(\lambda_{n,m}) := h^{(2,1)}(\lambda_{n,m}) + h^{(2,2)}(\lambda_{n,m}), \tag{2.5.66}$$

where,

$$h^{(2,1)}(\lambda_{n,m}) := 2\pi \sum_{1\leq q\leq m/2} \left(\frac{q\omega_{\rm c}}{q\omega_{\rm c} - \lambda_{n,m}}\right)^2 a_{n,q}, \tag{2.5.67}$$

and

$$h^{(2,2)}(\lambda_{n,m}) := 2\pi \sum_{m/2< q\leq m-1} \left(\frac{q\omega_{\rm c}}{q\omega_{\rm c} - \lambda_{n,m}}\right)^2 a_{n,q}, \tag{2.5.68}$$

Since $\left(\frac{q\omega_{\rm c}}{q\omega_{\rm c}-\lambda_{n,m}}\right)^2 \leq 4\frac{q^2}{m^2}$, $1 \leq q \leq \frac{m}{2}$,

and, using (2.9.2) we obtain,

$$\left|h^{(2,1)}(\lambda_{n,m})\right| \leq 2\pi \sum_{1\leq q\leq m/2} 4\,\frac{q^2}{m^2}\, a_{n,q} \leq C\frac{1}{m^2}. \tag{2.5.69}$$

Furthermore, as, $\left(\frac{q\omega_c}{q\omega_c-\lambda_{n,m}}\right)^2 \leq q^2$, $m/2 < q \leq m-1$, and by (2.9.2), we have

$$\left|h^{(2,2)}(\lambda_{n,m})\right| \leq 2\pi \sum_{m/2<q\leq m-1} q^2\, a_{n,q} \leq C_p\, \frac{1}{m^p} \tag{2.5.70}$$

for all $p > 0$. When m is odd we decompose $h^{(2)}(\lambda_{n,m})$ as in (2.5.66) with

$$h^{(2,1)}(\lambda_{n,m}) := 2\pi \sum_{1\leq q\leq (m-1)/2} \left(\frac{q\omega_c}{q\omega_c-\lambda_{n,m}}\right)^2 a_{n,q}, \tag{2.5.71}$$

and

$$h^{(2,2)}(\lambda_{n,m}) := 2\pi \sum_{(m-1)/2<q\leq m-1} \left(\frac{q\omega_c}{q\omega_c-\lambda_{n,m}}\right)^2 a_{n,q}, \tag{2.5.72}$$

and we prove that (2.5.69) and (2.5.70) hold arguing as in the case where m is even. This proves that,

$$\left|h^{(2)}(\lambda_{n,m})\right| \leq C\, \frac{1}{m^2}. \tag{2.5.73}$$

The technical result (2.9.22) in appendix A is $\lambda_{n,m} = m\omega_c + 2\pi m\, \omega_c\, \frac{a_{n,|m|}}{n^2} + a_{n,|m|}\, O\left(\frac{1}{|m|}\right)$. It yields

$$h^{(3)}(\lambda_{n,m}) = \frac{n^4}{2\pi}\, \frac{1}{a_{n,m}} \left(1 + O\left(\frac{1}{m^2}\right)\right), \qquad m \to \infty. \tag{2.5.74}$$

Moreover, by (2.9.2) and (2.9.22) there is an m_0 such that

$$\left(\frac{q\omega_c}{q\omega_c-\lambda_{n,m}}\right)^2 \leq 2\, q^2, \qquad q \geq m+1, m \geq m_0.$$

Then, using (2.9.2) we obtain for all $p > 0$,

$$\left|h^{(4)}(\lambda_{n,m})\right| \leq 4\pi \sum_{m+1\leq q} q^2\, a_{n,q} \leq C_p\, \frac{1}{m^p}, \qquad m \geq m_0. \tag{2.5.75}$$

Equation (2.5.58) follows from (2.5.59), (2.5.60), (2.5.65), (2.5.66), (2.5.73), (2.5.74), and, (2.5.75). □

Since $\mathbf{Y}_{n,m} \in \mathcal{H}$ we can define the associated normalized eigenfunctions as follows. Let us denote,

$$b_{n,m} := \sqrt{\|\eta_{n,m}\|^2_{L^2(\mathbb{R}^2)} + n^2} = \|Y_{n,m}\|_{\mathcal{H}}. \tag{2.5.76}$$

The normalized eigenfunctions are given by,

$$\mathbf{Z}_{n,m} := \frac{1}{b_{n,m}}\, \mathbf{Y}_{n,m}, \qquad n, m \in \mathbb{Z}^*. \tag{2.5.77}$$

The normalized eigenfunctions (2.5.77) are the Bernstein modes [14].

Then, we have,

Lemma 2.5.8. *Let* $\mathbf{H}$ *be the magnetized Vlasov–Ampère operator defined in* (2.5.3) *and* (2.5.4). *Let* $\lambda_{n,m}, n, m \in \mathbb{Z}^*$, *be the roots to equation* (2.5.51) *obtained in lemma 2.5.6. Then, each* $\lambda_{n,m}, n, m \in \mathbb{Z}^*$, *is an eigenvalue of* $\mathbf{H}$ *with eigenfunction* $\mathbf{Z}_{n,m}$.

Proof. The fact that the $\lambda_{n,m}, n, m \in \mathbb{Z}^*$, are eigenvalues of $\mathbf{H}$ with eigenfunction $\mathbf{Z}_{n,m}$ follows from (2.5.54), (2.5.56), and (2.5.58). □

In preparation for lemma 2.5.9 below, we briefly study the asymptotic expansion for large $|m|$ of the normalized eigenfunction. By (2.5.55), (2.5.65), (2.5.73), and (2.5.75), we have,

$$\left\| \eta_{n,m} - e^{\frac{-r^2}{4}} \frac{m\omega_c}{m\omega_c - \lambda_{n,m}} e^{im\varphi} J_m\left(\frac{nr}{\omega_c}\right) \right\|_{L^2(\mathbb{R}^2)} = O\left(\frac{1}{|m|}\right), \qquad m \to \pm\infty. \tag{2.5.78}$$

Note that (2.5.65), (2.5.73), and (2.5.75) were only proven for $m \geq 1$, and then, they only imply (2.5.78) for $m \to \infty$. However, using both equations in (2.5.28) and as $\lambda_{n,-m} = -\lambda_{n,m}$ we prove that (2.5.78) with $m \to \infty$ implies (2.5.78) with $m \to -\infty$. Then, by (2.5.58),

$$\left\| \frac{1}{b_{n,m}} \eta_{n,m} - \frac{1}{b_{n,m}} e^{\frac{-r^2}{4}} \frac{m\omega_c}{m\omega_c - \lambda_{n,m}} e^{im\varphi} J_m\left(\frac{nr}{\omega_c}\right) \right\|_{L^2(\mathbb{R}^2)} = \sqrt{a_{n,|m|}}\, O\left(\frac{1}{m^2}\right), \qquad m \to \pm\infty. \tag{2.5.79}$$

Let us denote,

$$\eta^{(a)}_{n,m} := -\frac{1}{\sqrt{2\pi a_{n,|m|}}} e^{\frac{-r^2}{4}} e^{im\varphi} J_m\left(\frac{nr}{\omega_c}\right), \qquad n, m \in \mathbb{Z}^*. \tag{2.5.80}$$

By (2.9.22) and (2.5.58),

$$\frac{1}{b_{n,m}} \frac{m\omega_c}{m\omega_c - \lambda_{n,m}} = -\frac{1}{\sqrt{2\pi a_{n,|m|}}} \left(1 + O\left(\frac{1}{|m|}\right)\right), \qquad m \to \pm\infty. \tag{2.5.81}$$

Then, by (2.5.80) and, (2.5.81)

$$\left\| \frac{1}{b_{n,m}} e^{\frac{-r^2}{4}} \frac{m\omega_c}{m\omega_c - \lambda_{n,m}} e^{im\varphi} J_m\left(\frac{nr}{\omega_c}\right) - \eta^{(a)}_{n,m} \right\|_{L^2(\mathbb{R}^2)} = O\left(\frac{1}{|m|}\right), \qquad m \to \pm\infty. \tag{2.5.82}$$

Let us now define the asymptotic function that is the dominant term for large m of the normalized eigenfunction $\mathbf{Z}_{n,m}$

$$\mathbf{Z}^{(a)}_{n,m} := \frac{1}{b_{n,m}} \frac{1}{\sqrt{2\pi}} e^{inx} \begin{pmatrix} e^{-in\frac{v_2}{\omega_c}} \eta^{(a)}_{n,m} \\ -in \end{pmatrix}, \qquad n, m \in \mathbb{Z}^*. \tag{2.5.83}$$

In the next lemma we show that for large m the eigenfunction $\mathbf{Z}_{n,m}$ is concentrated in $\mathbf{Z}^{(a)}_{n,m}$.

Lemma 2.5.9. *Let* $a_{n,m}$ *be the quantity defined in* (2.5.46), *let* $\mathbf{Z}_{n,m}$ *be the eigenfunction defined in* (2.5.77), *and let* $\mathbf{Z}^{(a)}_{n,m}$ *be the asymptotic function defined in* (2.5.83). *We have that,*

$$\left\| \mathbf{Z}_{n,m} - \mathbf{Z}^{(a)}_{n,m} \right\|_{\mathcal{H}} \leq C \frac{1}{|m|}, \qquad m \to \pm\infty, n \in \mathbb{Z}^*. \tag{2.5.84}$$

Proof. The lemma follows from (2.9.2), (2.5.79), and (2.5.82) □

2.5.3 The completeness of the eigenfunctions of H

In this subsection we prove that the eigenfunctions of the magnetized Vlasov–Ampère operator $\mathbf{H}$ are a complete set in $\mathcal{H}$. That is to say, that the closure of the set of all finite linear combinations of eigenfunctions of $\mathbf{H}$ is equal to $\mathcal{H}$, or in other words, that $\mathcal{H}$ coincides with the span of the set of all the eigenfunctions of $\mathbf{H}$. For this purpose we first introduce some notations. By (2.5.20)

$$L^2(0,2\pi) = \oplus_{n\in\mathbb{Z}} \text{Span}\left[\frac{e^{inx}}{\sqrt{2\pi}}\right], \tag{2.5.85}$$

and

$$L^2_0(0,2\pi) = \oplus_{n\in\mathbb{Z}^*} \text{Span}\left[\frac{e^{inx}}{\sqrt{2\pi}}\right]. \tag{2.5.86}$$

Furthermore, by (2.5.85) and (2.5.86),

$$\mathcal{H} = \oplus_{n\in\mathbb{Z}}\, \mathcal{H}_n, \tag{2.5.87}$$

where

$$\mathcal{H}_0 := L^2(\mathbb{R}^2) \oplus \{0\}, \tag{2.5.88}$$

and

$$\mathcal{H}_n := \text{Span}\left[\frac{e^{inx}}{\sqrt{2\pi}}\right] \otimes \left(L^2(\mathbb{R}^2) \oplus \mathbb{C}\right), \qquad n \in \mathbb{Z}^*. \tag{2.5.89}$$

Alternatively, $\mathcal{H}_0$ can be written as the Hilbert space of all vector valued functions of the form $(u,0)^T$, $u \in L^2(\mathbb{R}^2)$, where the injection of $L^2(\mathbb{R}^2)$ onto the subspace of $\mathcal{A}$ consists of all the functions in $\mathcal{A}$ that are independent of x. In other words, we identify $f(v) \in L^2(\mathbb{R}^2)$ with the same function $f(v) \in \mathcal{A}$ that is independent of x. Moreover, $\mathcal{H}_n$ can be written as the Hilbert space of all vector valued functions of the form,

$$\frac{e^{inx}}{\sqrt{2\pi}} \begin{pmatrix} u(v) \\ \alpha \end{pmatrix}, u \in L^2(\mathbb{R}^2), \alpha \in \mathbb{C}.$$

Furthermore, $\mathcal{H}$ can be written as the Hilbert space of all vector valued functions of the form

$$\begin{pmatrix} u_0(v) \\ 0 \end{pmatrix} + \sum_{n\in\mathbb{Z}^*} \frac{e^{inx}}{\sqrt{2\pi}} \begin{pmatrix} u_n(v) \\ \alpha_n \end{pmatrix},$$

where $u_n \in L^2(\mathbb{R}^2), \alpha_n \in \mathbb{C}, n \in \mathbb{Z}^*$, and, further, $\sum_{n\in\mathbb{Z}^*} \|u_n\|^2_{L^2(\mathbb{R}^2)} < \infty$, and $\sum_{n\in\mathbb{Z}^*} |\alpha_n|^2 < \infty$. The strategy of the proof that the eigenfunctions of $\mathbf{H}$ are complete in $\mathcal{H}$ will be to prove that the eigenfunctions of a given n are complete on the corresponding $\mathcal{H}_n$. For this purpose we introduce the following convenient spaces. A first space is defined as follows,

$$\mathbf{W}_0 := \text{Span}\left[\left\{\mathbf{M}^{(0)}_{0,j}\right\}_{j\in\mathbb{N}^*}\right] \oplus \text{Span}\left[\{\mathbf{V}_{m,j}\}_{m\in\mathbb{Z}^*, j\in\mathbb{N}^*}\right] \subset \mathcal{H}_0 \tag{2.5.90}$$

where the eigenfunctions $\mathbf{M}^{(0)}_{0,j}$, $j \in \mathbb{N}^*$, are defined in (2.5.18) and the eigenfunctions $\mathbf{V}_{m,j}, m \in \mathbb{Z}^*, j \in \mathbb{N}^*$ are defined in (2.5.26). Next we introduce the space,

$$\mathbf{W}^{(1)}_n := \text{Span}\left[\{\mathbf{W}_{n,m,j}\}_{n,m\in\mathbb{Z}^*, j\in\mathbb{N}^*}\right] \subset \mathcal{H}_n, \quad n \neq 0, \tag{2.5.91}$$

where the eigenfunctions $\mathbf{W}_{n,m,j}, n,m \in \mathbb{Z}^\star, j \in \mathbb{N}^*$ are defined in (2.5.37). We also need the following space,

$$\mathbf{W}_n^{(2)} := \text{Span}\left[\{\mathbf{Z}_{n,m}\}_{n,m\in\mathbb{Z}^*}\right] \subset \mathcal{H}_n, \quad n \neq 0, \tag{2.5.92}$$

where the eigenfunctions $\mathbf{Z}_{n,m}, n,m \in \mathbb{Z}^*$ are defined in (2.5.77). Finally, we define the space,

$$\mathbf{W}_n^{(3)} := \text{Span}\left[\left\{\mathbf{V}_n^{(0)}\right\}_{n\in\mathbb{Z}^*} \cup \left\{\mathbf{M}_{n,j}^{(0)}\right\}_{n\in\mathbb{Z}^*, j\in\mathbb{N}^*}\right] \subset \mathcal{H}_n \cap \text{Ker}[\mathbf{H}], \quad n \neq 0, \tag{2.5.93}$$

where the eigenfunctions $\mathbf{V}_n^{(0)}$ and $\mathbf{M}_{n,j}^{(0)}$ are defined in (2.5.18).

Theorem 2.5.10. *Let* $\mathbf{H}$ *be the magnetized Vlasov–Ampère operator defined in* (2.5.3) *and* (2.5.4)*. Then, the eigenfunctions of* $\mathbf{H}$ *are a complete set in* $\mathcal{H}$*. Namely,*

$$\mathcal{H}_0 = \mathbf{W}_0, \tag{2.5.94}$$

$$\mathcal{H}_n = \mathbf{W}_n^{(1)} \oplus \mathbf{W}_n^{(2)} \oplus \mathbf{W}_n^{(3)}, \qquad n \in \mathbb{Z}^*. \tag{2.5.95}$$

Furthermore,

$$\mathcal{H} = \mathbf{W}_0 \oplus_{n\in\mathbb{Z}^*} \left(\mathbf{W}_n^{(1)} \oplus \mathbf{W}_n^{(2)} \oplus \mathbf{W}_n^{(3)}\right). \tag{2.5.96}$$

Proof. Note that $\mathbf{W}_0$ is orthogonal to $\mathbf{W}_n^{(1)}, \mathbf{W}_n^{(2)}$, and $\mathbf{W}_n^{(3)}$ because $\mathbf{W}_0$ is the span of eigenfunctions with $n = 0$ and $\mathbf{W}_n^{(1)}, \mathbf{W}_n^{(2)}$, and $\mathbf{W}_n^{(3)}$ are the span of eigenfunctions with n different from zero. Furthermore, the $\mathbf{W}_n^{(1)}, \mathbf{W}_n^{(2)}$, and $\mathbf{W}_n^{(3)}$ are orthogonal among themselves because they are the span of eigenfunctions with different eigenvalues. Furthermore the $\mathbf{W}_n^{(1)}, \mathbf{W}_q^{(1)}$, with $n \neq q$, are orthogonal to each other because they are the span of eigenfunctions that contain the factor, respectively, e^{inx}, e^{iqx}. Similarly, $\mathbf{W}_n^{(2)}, \mathbf{W}_q^{(2)}, n \neq q$ are orthogonal to each other and $\mathbf{W}_n^{(3)}, \mathbf{W}_q^{(3)}, n \neq q$ are also orthogonal to each other. Equation (2.5.94) is immediate because the span of $u_{0,m,j}, m \in \mathbb{Z}, j \in \mathbb{N}^*$ is equal to $L^2(\mathbb{R}^2)$. We proceed to prove (2.5.95). We clearly have,

$$\mathbf{W}_n^{(1)} \oplus \mathbf{W}_n^{(2)} \oplus \mathbf{W}_n^{(3)} \subset \mathcal{H}_n, \qquad n \in \mathbb{Z}^*. \tag{2.5.97}$$

Our goal is to prove the opposite embedding, i.e.,

$$\mathcal{H}_n \subset \mathbf{W}_n^{(1)} \oplus \mathbf{W}_n^{(2)} \oplus \mathbf{W}_n^{(3)}, \qquad n \in \mathbb{Z}^*. \tag{2.5.98}$$

Consider the decomposition,

$$\mathcal{H}_n = \mathbf{W}_n^{(1)} \oplus \left(\mathbf{W}_n^{(1)}\right)^\perp, \tag{2.5.99}$$

where $\left(\mathbf{W}_n^{(1)}\right)^\perp$ denotes the orthogonal complement of $\mathbf{W}_n^{(1)}$ in $\mathcal{H}_n$. Recall that

$$\mathbf{W}_n^{(2)} \oplus \mathbf{W}_n^{(3)} \subset \left(\mathbf{W}_n^{(1)}\right)^\perp \qquad n \in \mathbb{Z}^*. \tag{2.5.100}$$

Our strategy to prove (2.5.98) will be to establish,

$$\left(\mathbf{W}_n^{(1)}\right)^\perp \subset \mathbf{W}_n^{(2)} \oplus \mathbf{W}_n^{(3)}, \qquad n \in \mathbb{Z}^*. \tag{2.5.101}$$

It follows from the definition of $\mathbf{W}_n^{(2)}$ in (2.5.92) and of $\mathbf{W}_n^{(3)}$ in (2.5.93) that the following set of eigenfunctions is a basis of $\mathbf{W}_n^{(2)} \oplus \mathbf{W}_n^{(3)}$,

$$\begin{cases} \mathbf{Z}_{n,m}, n, m \in \mathbb{Z}^*, \\ \mathbf{M}_{n,j}^{(0)}, n \in \mathbb{Z}^*, j \in \mathbb{N}^*, \\ \mathbf{V}_n^{(0)}, n \in \mathbb{Z}^*. \end{cases} \tag{2.5.102}$$

Furthermore, it is a consequence of the definition of $\mathbf{W}_n^{(1)}$ in (2.5.91) and of the definition of $\mathbf{Z}_{n,m}^{(a)}$ in (2.5.83) that the following set of functions is an orthonormal basis of $\left(\mathbf{W}_n^{(1)}\right)^\perp$

$$\begin{cases} \mathbf{Z}_{n,m}^{(a)}, n, m \in \mathbb{Z}^*, \\ \mathbf{M}_{n,j}^{(0)}, n \in \mathbb{Z}^*, j \in \mathbb{N}^*, \quad , \\ \mathbf{Q}, \end{cases} \tag{2.5.103}$$

where the asymptotic functions $\mathbf{Z}_{n,m}^{(a)}, n, m \in \mathbb{Z}^*$ are defined in (2.5.83), the eigenfunctions $\mathbf{M}_{n,j}^{(0)}, n \in \mathbb{Z}^*, j \in \mathbb{N}^*$ are defined in (2.5.18), and $\mathbf{Q}$ is given by,

$$\mathbf{Q} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.5.104}$$

Any $\mathbf{X} \in \left(\mathbf{W}_n^{(1)}\right)^\perp$ can be uniquely written as,

$$\mathbf{X} = \sum_{m \in \mathbb{Z}^*} \left(\mathbf{X}, \mathbf{Z}_{n,m}^{(a)}\right)_{\mathcal{H}} \mathbf{Z}_{n,m}^{(a)} + \sum_{j \in \mathbb{N}^*} \left(\mathbf{X}, \mathbf{M}_{n,j}^{(0)}\right)_{\mathcal{H}} \mathbf{M}_{n,j}^{(0)} + (\mathbf{X}, \mathbf{Q})_{\mathcal{H}}\, \mathbf{Q}. \tag{2.5.105}$$

We define the following operator from $\left(\mathbf{W}_n^{(1)}\right)^\perp$ into $\left(\mathbf{W}_n^{(1)}\right)^\perp$,

$$\mathbf{\Lambda X} := \sum_{m \in \mathbb{Z}^*} \left(\mathbf{X}, \mathbf{Z}_{n,m}^{(a)}\right)_{\mathcal{H}} \mathbf{Z}_{n,m} + \sum_{j \in \mathbb{N}^*} \left(\mathbf{X}, \mathbf{M}_{n,j}^{(0)}\right)_{\mathcal{H}} \mathbf{M}_{n,j}^{(0)} + (\mathbf{X}, \mathbf{Q})_{\mathcal{H}}\, \mathbf{V}_n^{(0)}. \tag{2.5.106}$$

We will prove that (2.5.101) holds by showing that $\mathbf{\Lambda}$ is onto, $\left(\mathbf{W}_n^{(1)}\right)^\perp$. We write $\mathbf{\Lambda}$ as follows,

$$\mathbf{\Lambda} = I + \mathbf{K}, \tag{2.5.107}$$

where $\mathbf{K}$ is the operator,

$$\mathbf{KX} := \sum_{m \in \mathbb{Z}^*} \left(\mathbf{X}, \mathbf{Z}_{n,m}^{(a)}\right)_{\mathcal{H}} \left(\mathbf{Z}_{n,m} - \mathbf{Z}_{n,m}^{(a)}\right) + (\mathbf{X}, \mathbf{Q})_{\mathcal{H}} \left(\mathbf{V}_n^{(0)} - \mathbf{Q}\right). \tag{2.5.108}$$

We will prove that $\mathbf{K}$ is Hilbert–Schmidt. For information about Hilbert–Schmidt operators see [107, Section 6, Chapter 6]. For this purpose we have to prove that $\mathbf{K}^* \mathbf{K}$ is trace class. Since the functions in (2.5.103) are an orthonormal basis of $\left(\mathbf{W}_n^{(1)}\right)^\perp$, we can verify the trace class criterion under the form,

$$\sum_{m \in \mathbb{Z}^*} \left(\mathbf{KZ}_{n,m}^{(a)}, \mathbf{KZ}_{n,m}^{(a)}\right)_{\mathcal{H}} + \sum_{j \in \mathbb{N}^*} \left(\mathbf{KM}_{n,j}^{(0)}, \mathbf{KM}_{n,j}^{(0)}\right)_{\mathcal{H}} + (\mathbf{KQ}, \mathbf{KQ})_{\mathcal{H}} < \infty. \tag{2.5.109}$$

However, by (2.5.108) $\sum_{m\in\mathbb{Z}^*}\left(\mathbf{K}\mathbf{Z}^{(a)}_{n,m},\mathbf{K}\mathbf{Z}^{(a)}_{n,m}\right)_{\mathcal{H}}=\sum_{m\in\mathbb{Z}^*}\left\|(\mathbf{Z}_{n,m}-\mathbf{Z}^{(a)}_{n,m})\right\|^2_{\mathcal{H}}<\infty$, where we used, (2.5.84). Moreover, $\sum_{j\in\mathbb{N}^*}\left(\mathbf{K}\mathbf{M}^{(0)}_{n,j},\mathbf{K}\mathbf{M}^{(0)}_{n,j}\right)_{\mathcal{H}}=0$, and, clearly, $(\mathbf{K}\mathbf{Q},\mathbf{K}\mathbf{Q})_{\mathcal{H}}<\infty$. Hence, $\mathbf{K}$ is Hilbert–Schmidt, and then, it is compact. It follows from the Fredholm alternative, see the Corollary in page 203 of [107], that to prove that $\mathbf{\Lambda}$ is onto it is enough to prove that it is invertible. Suppose that $\mathbf{X}\in\left(\mathbf{W}^{(1)}_n\right)^{\perp}$ satisfies $\mathbf{\Lambda}\mathbf{X}=0$. Then, by (2.5.106)

$$\sum_{m\in\mathbb{Z}^*}\left(\mathbf{X},\mathbf{Z}^{(a)}_{n,m}\right)_{\mathcal{H}}\mathbf{Z}_{n,m}+\sum_{j\in\mathbb{N}^*}\left(\mathbf{X},\mathbf{M}^{(0)}_{n,j}\right)_{\mathcal{H}}\mathbf{M}^{(0)}_{n,j}+(\mathbf{X},\mathbf{Q})_{\mathcal{H}}\,\mathbf{V}^{(0)}_n=0. \tag{2.5.110}$$

However, as the eigenfunctions $\mathbf{Z}_{n,m}$ are orthogonal to the $\mathbf{M}^{(0)}_{n,j}$and to $\mathbf{V}^{(0)}_n$, we have,

$$\sum_{m\in\mathbb{Z}^*}\left(\mathbf{X},\mathbf{Z}^{(a)}_{n,m}\right)_{\mathcal{H}}\mathbf{Z}_{n,m}=0, \tag{2.5.111}$$

and,

$$\sum_{j\in\mathbb{N}^*}\left(\mathbf{X},\mathbf{M}^{(0)}_{n,j}\right)_{\mathcal{H}}\mathbf{M}^{(0)}_{n,j}+(\mathbf{X},\mathbf{Q})_{\mathcal{H}}\,\mathbf{V}^{(0)}_n=0. \tag{2.5.112}$$

Since the eigenfunctions $\mathbf{Z}_{n,m}$ are mutually orthogonal, it follows from (2.5.111) that $\left(\mathbf{X},\mathbf{Z}^{(a)}_{n,m}\right)_{\mathcal{H}}=0$, $m\in\mathbb{Z}^*$. Moreover, by lemma 2.5.2 the eigenfunctions $\mathbf{M}^{(0)}_{n,j}, j\in\mathbb{N}^*$ and , $\mathbf{V}^{(0)}_n$ are linearly independent, and, then (2.5.112) implies $\left(\mathbf{X},\mathbf{M}^{(0)}_{n,j}\right)_{\mathcal{H}}=0, \quad j\in\mathbb{N}^*$, and $(\mathbf{X},\mathbf{Q})_{\mathcal{H}}=0$. Finally, as the set (2.5.103) is an orthonormal basis of $\left(\mathbf{W}^{(1)}_n\right)^{\perp}$ we have that $\mathbf{X}=0$. Then, $\mathbf{\Lambda}$ is onto $\left(\mathbf{W}^{(1)}_n\right)^{\perp}$ and (2.5.101) holds. Since also (2.5.100) is satisfied we obtain $\mathbf{W}^{(2)}_n\oplus\mathbf{W}^{(3)}_n=\left(\mathbf{W}^{(1)}_n\right)^{\perp}$, $n\in\mathbb{Z}^*$. This completes the proof of the theorem ☐

Theorem 2.5.11. *Let* $\mathbf{H}$ *be the magnetized Vlasov–Ampère operator defined in* (2.5.3) *and* (2.5.4)*. Then* $\mathbf{H}$ *is self-adjoint and it has pure point spectrum. The eigenvalues of* $\mathbf{H}$ *are given by.*

1. *The infinite multiplicity eigenvalues,* $\lambda^{(0)}_m:=m\omega_{\mathrm{c}}, m\in\mathbb{Z}$.
2. *The simple eigenvalues* $\lambda_{n,m}, n,m\in\mathbb{Z}^*$*, given by the roots to equation* (2.5.51) *obtained in lemma 2.5.6.*

Proof. We have already proven that $\mathbf{H}$ is self-adjoint below (2.5.7). The spectrum of $\mathbf{H}$ is pure point because it has a complete set of eigenfunctions, as we proved in theorem 2.5.10. The fact that the eigenvalues of $\mathbf{H}$ are equal to the $\lambda^{(0)}_m, m\in\mathbb{Z}$, and the $\lambda_{n,m}, n,m\in\mathbb{Z}^*$ follows from lemma 2.5.2, lemma 2.5.3, lemma 2.5.4 and lemma 2.5.8. The $\lambda^{(0)}_m, m\in\mathbb{Z}$ have infinite multiplicity because by lemma 2.5.2, lemma 2.5.3, and lemma 2.5.4 each $\lambda^{(0)}_m$ has a countable set of orthogonal eigenfunctions. Let us prove that the eigenvalues $\lambda_{n,m}$ are simple. Suppose that for some $n,m\in\mathbb{Z}^*$ the eigenvalue $\lambda_{n,m}$ has multiplicity bigger than one. Then, there is an eigenfunction, $\mathbf{P}$, such that $\mathbf{H}\mathbf{P}=\lambda_{n,m}\,\mathbf{P}$, and with $\mathbf{P}$ orthogonal to $\mathbf{Z}_{n,m}$. However since by lemma 2.5.6 $\lambda_{n_1,m_1}=\lambda_{n_2,m_2}$ if and only if $n_1=n_2$, and $m_1=m_2$, it follows that $\mathbf{P}$ is orthogonal to the right hand side of (2.5.96), but hence, $\mathbf{P}$ is orthogonal to $\mathcal{H}$, and then $\mathbf{P}=0$. This completes the proof that the $\lambda_{n,m}$ are simple eigenvalues. ☐

2.5.4 Orthonormal basis for the kernel of H

In section 2.5.1 we constructed a linear independent basis for the kernel of the magnetized Vlasov–Ampère operator **H**. In this subsection we prove that, for an appropriate choice of the orthonormal basis of $L^2(\mathbb{R}^+, rdr)$ that appears in the definition of the eigenfunctions $\mathbf{M}_{n,j}^{(0)}$ in (2.5.18), we can construct an orthonormal basis for the kernel of **H**. The choice of the orthonormal basis is n dependent. For $n \in \mathbb{Z}^*$, let $\tau_j^{(n)}, j = 1, \dots$ be any orthonormal basis of $L^2(\mathbb{R}^+, rdr)$ where the first basis function is

$$\tau_1^{(n)}(r) := \frac{1}{\sqrt{a_{n,0}}} e^{\frac{-r^2}{4}} J_0\left(\frac{nr}{\omega_c}\right), \qquad n \in \mathbb{Z}^*, \tag{2.5.113}$$

with $a_{n,0}$ defined in (2.5.46). Note that this implies that the $\tau_j^{(n)}, j = 2, \dots$ is an orthonormal basis of the subspace $V_{n,0}$ that we defined in (2.5.35). Moreover, in the definition of the $u_{n,0,j}$ in (2.4.2) let us use this basis. In particular it yields

$$u_{n,0,1} := \frac{e^{in(x - \frac{v_2}{\omega_c})}}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{a_{n,0}}} e^{\frac{-r^2}{4}} J_0\left(\frac{nr}{\omega_c}\right), \qquad n \in \mathbb{Z}^*. \tag{2.5.114}$$

The eigenfunctions $\mathbf{M}_{n,j}^{(0)} = \begin{pmatrix} u_{n,0,j} \\ 0 \end{pmatrix}, n \in \mathbb{Z}^*, j \in \mathbb{N}^*$, of **H** precised with (2.5.113) are now a particular case of the ones defined in (2.5.18). However, we keep the same notation for $\mathbf{M}_{n,j}^{(0)}$ for a sake of readability.

For the other eigenfunctions we can use different orthonormal basis of $L^2(\mathbb{R}^+, rdr)$, if we find it convenient. It follows from simple calculations that the eigenfunctions $\mathbf{V}_n^{(0)}, n \in \mathbb{Z}^*$, defined in (2.5.18) are mutually orthogonal and that the eigenfunctions $\mathbf{M}_{n,j}^{(0)}, (n,j) \in \mathbb{Z}^* \times \mathbb{N}^*$, are also mutually orthogonal. Moreover, since the functions $e^{inx}, n \in \mathbb{Z}^*$ are orthogonal in $L^2(0, 2\pi)$ to the function equal to one, the eigenfunctions $\mathbf{V}_n^{(0)}, n \in \mathbb{Z}^*$, and $\mathbf{M}_{0,j}^{(0)}, j \in \mathbb{N}^*$ are orthogonal. Let us compute the scalar product of the $\mathbf{V}_n^{(0)}, n \in \mathbb{Z}^*$, and the $\mathbf{M}_{n,j}^{(0)}, n \in \mathbb{Z}^*, j = 1, \dots$.

$$\left(\mathbf{V}_n^{(0)}, \mathbf{M}_{m,j}^{(0)}\right)_{\mathcal{H}} = \delta_{n,m} \frac{1}{\sqrt{2\pi + n^2}} \left(e^{\frac{-v^2}{4}}, \frac{e^{-in\frac{v_2}{\omega_c}}}{\sqrt{2\pi}} \tau_j(r)\right)_{\mathcal{A}}, \qquad n \in \mathbb{Z}^*, m \in \mathbb{Z}^*, j \in \mathbb{N}^*. \tag{2.5.115}$$

Moreover, by the Jacobi-Anger formula (2.5.29), with $z = \frac{-nr}{\omega_c}$,

$$\left(e^{\frac{-v^2}{4}}, \frac{e^{-in\frac{v_2}{\omega_c}}}{\sqrt{2\pi}} \tau_j(r)\right)_{\mathcal{A}} = \left(e^{\frac{-v^2}{4}}, \left(\sum_{m \in \mathbb{Z}} e^{im\varphi} J_m\left(\frac{-nr}{\omega_c}\right)\right) \frac{1}{\sqrt{2\pi}} \tau_j(r)\right)_{\mathcal{A}}, \qquad n \in \mathbb{Z}^*, j \in \mathbb{N}^*.$$

Hence, by (2.5.113) and the second equation in (2.5.28)

$$\left(e^{\frac{-v^2}{4}}, \frac{e^{-in\frac{v_2}{\omega_c}}}{\sqrt{2\pi}} \tau_j(r)\right)_{\mathcal{A}} = \delta_{j,1} \sqrt{2\pi} \sqrt{a_{n,0}}, \qquad n \in \mathbb{Z}^*. \tag{2.5.116}$$

By (2.5.115) and (2.5.116),

$$\left(\mathbf{V}_n^{(0)}, \mathbf{M}_{m,j}^{(0)}\right)_{\mathcal{H}} = \delta_{n,m} \, \delta_{j,1} \frac{\sqrt{2\pi a_{n,0}}}{\sqrt{2\pi + n^2}}, \qquad n \in \mathbb{Z}^*, m \in \mathbb{Z}^*, j \in \mathbb{N}^*. \tag{2.5.117}$$

This proves that the $\mathbf{V}_n^{(0)}, n \in \mathbb{Z}^*$, and the $\mathbf{M}_{n,j}^{(0)}, n \in \mathbb{Z}^*, j = 2, \dots$, are orthogonal to each other, and also that $\mathbf{V}_n^{(0)}$, and $\mathbf{M}_{n,1}^{(0)}, n \in \mathbb{Z}^*$, are not orthogonal. We apply the Gramm–Schmidt orthonormalization process to $\mathbf{V}_n^{(0)}$, and $\mathbf{M}_{n,1}^{(0)}, n \in \mathbb{Z}^*$, and we define the eigenfunctions,

$$\mathbf{E}_n^{(0)} := \mathbf{M}_{n,1}^{(0)} - \left(\mathbf{M}_{n,1}^{(0)}, \mathbf{V}_n^{(0)}\right)_{\mathcal{H}} \mathbf{V}_n^{(0)}, \qquad n \in \mathbb{Z}^*, \tag{2.5.118}$$

and the normalized eigenfunctions,

$$\mathbf{F}_n^{(0)} := \frac{\mathbf{E}_n^{(0)}}{\|\mathbf{E}_n^{(0)}\|_{\mathcal{H}}}, n \in \mathbb{Z}^*. \tag{2.5.119}$$

By (2.5.117), (2.5.118), and (2.5.119),

$$\mathbf{F}_n^{(0)} = \frac{2\pi + n^2}{2\pi(1 - a_{n,0}) + n^2} \left(\mathbf{M}_{n,1}^{(0)} - \frac{\sqrt{2\pi a_{n,0}}}{\sqrt{2\pi + n^2}} \mathbf{V}_n^{(0)} \right), \qquad n \in \mathbb{Z}^*. \tag{2.5.120}$$

Note that by (2.5.30) and (2.5.46) $a_{n,0} < 1$, and then $1 - a_{n,0} > 0$.

Using the results above we prove the following theorem.

Theorem 2.5.12. *Let* $\mathbf{H}$ *be the magnetized Vlasov–Ampère operator defined in* (2.5.3) *and* (2.5.4)*. Then, the following set of eigenfunctions of* $\mathbf{H}$ *with eigenvalue zero,*

$$\left\{\mathbf{V}_n^{(0)}, n \in \mathbb{Z}^*\right\} \cup \left\{\mathbf{M}_{0,j}^{(0)}, j \in \mathbb{N}^*\right\} \cup \left\{\mathbf{M}_{n,j}^{(0)}, n \in \mathbb{Z}^*, j = 2, \dots\right\} \cup \left\{\mathbf{F}_n^{(0)}, n \in \mathbb{Z}^*\right\}, \tag{2.5.121}$$

is an orthonormal basis of $\mathrm{Ker}[\mathbf{H}]$*. The eigenfunctions* $\mathbf{V}_n^{(0)}$*, and* $\mathbf{M}_{0,j}^{(0)}$*, are defined in* (2.5.18)*, and the eigenfunctions,* $\mathbf{M}_{n,j}^{(0)}$*, and* $\mathbf{F}_n^{(0)}$*, are defined, respectively, in* (2.5.18) *with* (2.5.114)*, and* (2.5.119)*.*

Proof. The lemma follows from lemma 2.5.2 □

2.5.5 Orthonormal basis with eigenfunctions of H

In this subsection we show how to assemble an orthonormal basis for $\mathcal{H}$ with eigenfunctions of $\mathbf{H}$, using the eigenfunctions that we have already computed. We first obtain an orthonormal basis for $\mathrm{Ker}[\mathbf{H}]^{\perp}$, with the eigenfunctions of $\mathbf{H}$ with eigenvalue different from zero.

Theorem 2.5.13. *Let* $\mathbf{H}$ *be the magnetized Vlasov–Ampère operator defined in* (2.5.3) *and,* (2.5.4)*. Then, the following set of eigenfunctions of* $\mathbf{H}$ *with eigenvalue different from zero,*

$$\{\mathbf{V}_{m,j}, m \in \mathbb{Z}^*, j \in \mathbb{N}^*\} \cup \{\mathbf{W}_{n,m,j}, n, m \in \mathbb{Z}^*, j \in \mathbb{N}^*\} \cup \{\mathbf{Z}_{n,m}, n, m \in \mathbb{Z}^*\}, \tag{2.5.122}$$

is an orthonormal basis of $\mathrm{Ker}[\mathbf{H}]^{\perp}$*. Moreover, the eigenfunctions* $\mathbf{V}_{m,j}, \mathbf{W}_{n,m,j}$*, and* $\mathbf{Z}_{n,m}$ *are defined, respectively in* (2.5.26)*,* (2.5.37)*, and* (2.5.77)*.*

Proof. Equation (2.5.96) can be written as follows,

$$\mathcal{H} = \left[\mathrm{Span}\left[\left\{\mathbf{M}_{0,j}^{(0)}\right\}_{j\in\mathbb{N}^*}\right] \oplus_{n\in\mathbb{Z}^*} \mathbf{W}_n^{(3)}\right] \oplus \left[\mathrm{Span}\left[\{\mathbf{V}_{m,j}\}_{m\in\mathbb{Z}^*, j\in\mathbb{N}^*}\right] \oplus \mathbf{W}_n^{(1)} \oplus \mathbf{W}_n^{(2)}\right]. \tag{2.5.123}$$

Moreover, by lemma 2.5.2

$$\operatorname{Ker}[\mathbf{H}] = \operatorname{Span}\left[\left\{\mathbf{M}_{0,j}^{(0)}\right\}_{j\in\mathbb{N}^*}\right] \oplus_{n\in\mathbb{Z}^*} \mathbf{W}_n^{(3)}. \tag{2.5.124}$$

Further, as $\mathcal{H} = \operatorname{Ker}[\mathbf{H}] \oplus \operatorname{Ker}[\mathbf{H}]^\perp$, it follows from (2.5.123), (2.5.124)

$$\operatorname{Ker}[\mathbf{H}]^\perp = \operatorname{Span}\left[\{\mathbf{V}_{m,j}\}_{m\in\mathbb{Z}^*,j\in\mathbb{N}^*}\right] \oplus \mathbf{W}_n^{(1)} \oplus \mathbf{W}_n^{(2)}. \tag{2.5.125}$$

Finally, using the definitions of $\mathbf{W}_n^{(1)}$ in (2.5.91) and of $\mathbf{W}_n^{(2)}$ in (2.5.92) we obtain that the set (2.5.122) is an orthonormal basis of $\operatorname{Ker}[\mathbf{H}]^\perp$. □

In the following theorem we present an orthonormal basis for $\mathcal{H}$ with eigenfunctions of $\mathbf{H}$.

Theorem 2.5.14. *Let* $\mathbf{H}$ *be the magnetized Vlasov–Ampère operator defined in* (2.5.3), *and* (2.5.4). *Then, the following set of eigenfunctions of* $\mathbf{H}$,

$$\begin{aligned}&\left\{\mathbf{V}_n^{(0)}, n\in\mathbb{Z}^*\right\} \cup \left\{\mathbf{M}_{0,j}^{(0)}, j\in\mathbb{N}^*\right\} \cup \left\{\mathbf{M}_{n,j}^{(0)}, n\in\mathbb{Z}^*, j=2,\dots\right\} \cup \left\{\mathbf{F}_n^{(0)}, n\in\mathbb{Z}^*\right\} \cup \\ &\{\mathbf{V}_{m,j}, m\in\mathbb{Z}^*, j\in\mathbb{N}^*\} \cup \{\mathbf{W}_{n,m,j}, n,m\in\mathbb{Z}^*, j\in\mathbb{N}^*\} \cup \{\mathbf{Z}_{n,m}, n,m\in\mathbb{Z}^*\},\end{aligned} \tag{2.5.126}$$

is an orthonormal basis of $\mathcal{H}$. *The eigenfunctions,* $\mathbf{V}_n^{(0)}$, *and* $\mathbf{M}_{0,j}^{(0)}$ *are defined in* (2.5.18). *The eigenfunctions,* $\mathbf{M}_n^{(0)}$, *and* $\mathbf{F}_n^{(0)}$ *are defined, respectively in* (2.5.18) *with* (2.5.114), *and* (2.5.119). *Moreover, the eigenfunctions* $\mathbf{V}_{m,j}$, $\mathbf{W}_{n,m,j}$, *and* $\mathbf{Z}_{n,m}$ *are defined, respectively in* (2.5.26), (2.5.37), *and* (2.5.77).

Proof. The result follows from theorem 2.5.12, and theorem 2.5.13. □

2.6 The general solution to the magnetized Vlasov–Ampère system, and the Bernstein–Landau paradox

In this section we give an explicit formula for the general solution of the Vlasov–Ampère system with the help of the orthonormal basis of $\mathcal{H}$ with eigenfunctions of $\mathbf{H}$. Let us take a general initial state,

$$\mathbf{G}_0 = \begin{pmatrix} u \\ F \end{pmatrix} \in \mathcal{H}.$$

Then, by theorem 2.5.14, the general solution to the magnetized Vlasov–Ampère system with initial value at $t=0$ equal to $\mathbf{G}_0$ is given by,

$$\mathbf{G}(t) := e^{-it\mathbf{H}}\,\mathbf{G}_0, \tag{2.6.1}$$

and, furthermore,

$$\mathbf{G}(t) = \mathbf{G}_1 + \mathbf{G}_2(t), \tag{2.6.2}$$

where the static part $\mathbf{G}_1$ is time independent, and the dynamical part $\mathbf{G}_2(t)$ is oscillatory in time. They are given by,

$$\begin{aligned}\mathbf{G}_1 = &\sum_{n\in\mathbb{Z}^*}\left(\mathbf{G}_0, \mathbf{V}_n^{(0)}\right)_{\mathcal{H}} \mathbf{V}_n^{(0)} + \sum_{j\in\mathbb{N}^*}\left(\mathbf{G}_0, \mathbf{M}_{0,j}^{(0)}\right)_{\mathcal{H}} \mathbf{M}_{0,j}^{(0)} + \sum_{n\in\mathbb{Z}^*,j\geq 2}\left(\mathbf{G}_0, \mathbf{M}_{n,j}^{(0)}\right)_{\mathcal{H}} \mathbf{M}_{n,j}^{(0)} \\ &+ \sum_{n\in\mathbb{Z}^*}\left(\mathbf{G}_0, \mathbf{F}_n^{(0)}\right)_{\mathcal{H}} \mathbf{F}_n^{(0)},\end{aligned} \tag{2.6.3}$$

and

$$\begin{aligned}\mathbf{G}_2(t) = &\sum_{m\in\mathbb{Z}^*, j\in\mathbb{N}^*} e^{-it\lambda_m^{(0)}} \left(\mathbf{G}_0, \mathbf{V}_{m,j},\right)_{\mathcal{H}} \mathbf{V}_{m,j} + \sum_{n,m\in\mathbb{Z}^*, j\in\mathbb{N}^*} e^{-it\lambda_m^{(0)}} \left(\mathbf{G}_0, \mathbf{W}_{n,m,j}\right)_{\mathcal{H}} \mathbf{W}_{n,m,j} \\ &+ \sum_{n,m\in\mathbb{Z}^*} e^{-it\lambda_{n,m}} \left(\mathbf{G}_0, \mathbf{Z}_{n,m}\right)_{\mathcal{H}} \mathbf{Z}_{n,m}.\end{aligned} \tag{2.6.4}$$

We still have to impose the Gauss law (2.2.13), (2.2.14), or equivalently (2.5.8), to our general solution to the magnetized Vlasov–Ampère system (2.2.17). For the eigenfunction $\mathbf{M}_{n,j}^{(0)}, n \in \mathbb{Z}^*, j \geq 2$, the Gauss law (2.5.8) is equivalent to

$$\left(\mathbf{M}_{n,j}^{(0)}, \mathbf{V}_n^{(0)}\right)_{\mathcal{H}} = 0,$$

that is valid by the orthogonality of the $\mathbf{M}_{n,j}^{(0)}$ and the $\mathbf{V}_n^{(0)}$. We prove in the same way that the Gauss law (2.5.8) holds for the eigenfunctions $\mathbf{F}_n^{(0)}$, $\mathbf{W}_{n,m,j}$, and $\mathbf{Z}_{n,m}$. We prove that $\mathbf{V}_{m,j}$ satisfies the Gauss law by direct computation. It remains to consider the eigenfunctions $\mathbf{M}_{0,j}^{(0)}, j \in \mathbb{N}^*$, defined in (2.5.18). For the $\mathbf{M}_{0,j}^{(0)}$, the Gauss law (2.5.8) reads,

$$\int_0^\infty e^{\frac{-v^2}{4}} \tau_j \, dv = 0, \qquad j \in \mathbb{N}^*. \tag{2.6.5}$$

We can make sure that (2.6.5) holds for all but one j by choosing the orthonormal basis in $L^2(\mathbb{R}^+, rdr)$ that we use in the definition of the $\mathbf{M}_{0,j}^{(0)}, j \in \mathbb{N}^*$, as follows. As we proceed in (2.5.113)-(2.5.114) for $n \in \mathbb{Z}^*$, we specify the choice of the orthonormal basis $(\tau_j)_{j\in\mathbb{N}^*}$ in (2.4.2) and (2.5.18) for $n = 0$. We take an orthonormal basis, $\tau_j^{(0)}, j \in \mathbb{N}^*$, in $L^2(\mathbb{R}^+, rdr)$, such that,

$$\tau_1^{(0)}(r) := e^{\frac{-v^2}{4}}. \tag{2.6.6}$$

With this choice of the $\tau_j^{(0)}, j \in \mathbb{N}^*$, the Gauss law (2.5.8) holds for $\mathbf{M}_{0,j}^{(0)}, j = 2, \ldots$. Hence, with this choice, the general solution of the magnetized Vlasov–Ampère system given in (2.6.1) and that satisfies the Gauss law (2.5.8) can be written as in (2.6.2) with the dynamical part $\mathbf{G}_2(t)$ as in (2.6.4), but with the static part $\mathbf{G}_1$ given by

$$\mathbf{G}_1 = \sum_{j\geq 2} \left(\mathbf{G}_0, \mathbf{M}_{0,j}^{(0)}\right) \mathbf{M}_{0,j}^{(0)} + \sum_{n\in\mathbb{Z}^*, j\geq 2} \left(\mathbf{G}_0, \mathbf{M}_{n,j}^{(0)}\right) \mathbf{M}_{n,j}^{(0)} + \sum_{n\in\mathbb{Z}^*} \left(\mathbf{G}_0, \mathbf{F}_n^{(0)}\right) \mathbf{F}_n^{(0)}. \tag{2.6.7}$$

This exhibits the Bernstein–Landau paradox. Namely, the general solution contains a time independent part and a part that is oscillatory time. There is no part of the solution that tends to zero as $t \to \pm\infty$, that is to say, there is no Landau damping in the presence of the magnetic field.

Remark 2.6.1. This remark concerns the space $\mathcal{H}_{\mathrm{G}}$ for the Gauss law and its orthogonal complement.

Let us denote,

$$\mathcal{H}_{\mathrm{G}} := \mathrm{Span}\left[\left\{\mathbf{V}_n^{(0)}, n \in \mathbb{Z}^*\right\} \cup \mathbf{M}_{0,1}^{(0)}\right],$$

where the eigenfunctions $\mathbf{V}_n^{(0)}$ are defined in (2.5.18) and the eigenfunction $\mathbf{M}_{0,1}^{(0)}$ is defined in (2.5.18), (2.6.6). Note that it follows from the results above that the condition that each one of the eigenfunctions that appear in (2.6.3), and (2.6.4) satisfies the Gauss law is equivalent to ask that the eigenfunction is orthogonal to $\mathcal{H}_{\rm G}$. Then, it follows from (2.6.1), (2.6.2), (2.6.3), and (2.6.4), that general solution to the magnetized Vlasov–Ampère system given in (2.6.1) satisfies the Gauss law (2.5.8) if and only if $\mathbf{G}_0 \in \mathcal{H}_{\rm G}^{\perp}$.

The Hilbert space $\mathcal{H}_{\rm G}$ is a closed subspace of the kernel of $\mathbf{H}$. So, the Gauss law is equivalent to have the initial state in the orthogonal complement to a closed subspace of the kernel of $\mathbf{H}$. Actually, it is usually the case that when the Maxwell equations are formulated as a self-adjoint Schrödinger equation in the Hilbert space of electromagnetic fields with finite energy, the Gauss law is equivalent to have the initial data in the orthogonal complement of the kernel of the Maxwell operator. See for example [127]. Let us further elaborate on the condition $\mathbf{G}_0 \in \mathcal{H}_{\rm G}^{\perp}$. We introduce the space of test functions $\mathcal{D}_{\mathbb{T}} := \{\varphi \in C^\infty[0,2\pi] : \frac{d^l}{dx^l}\varphi(0) = \frac{d^l}{dx^l}\varphi(2\pi), l = 0, \dots\}$. Let us expand $\varphi \in \mathcal{D}_{\mathbb{T}}$ in Fourier series

$$\varphi(x) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2\pi}} e^{inx}\varphi_n, \qquad \text{where} \qquad \varphi_n := \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \varphi(x)\, e^{-inx}\, dx, n \in \mathbb{Z}. \tag{2.6.8}$$

Integrating by parts we prove that

$$|\varphi_n| \leq \frac{C_l}{|n|^l}, \quad l \in \mathbb{N}, n \in \mathbb{Z}^*. \tag{2.6.9}$$

By a simple calculation, and using (2.6.8) and (2.6.9) we prove that,

$$\begin{pmatrix} \varphi(x)\, e^{\frac{-v^2}{4}} \\ -\frac{d}{dx}\varphi(x) \end{pmatrix} = \sum_{n\in\mathbb{Z}^*} \varphi_n \sqrt{2\pi + n^2}\, V_n^{(0)} + \sqrt{2\pi}\, \varphi_0\, \mathbf{M}_{0,1}^{(0)} \in \mathcal{H}_{\rm G}. \tag{2.6.10}$$

Suppose that

$$\begin{pmatrix} u(x,v) \\ F(x) \end{pmatrix} \in \mathcal{H}_{\rm G}^{\perp}. \tag{2.6.11}$$

Then, by (2.6.10)

$$\left(\begin{pmatrix} u(x,v) \\ F(x) \end{pmatrix}, \begin{pmatrix} \varphi(x)\, e^{\frac{-v^2}{4}} \\ -\frac{d}{dx}\varphi(x) \end{pmatrix} \right)_{\mathcal{H}} = \int_0^{2\pi} \rho(x)\, \varphi(x)\, dx - \int_0^{2\pi} F(x) \frac{d}{dx}\varphi(x)\, dx = 0, \qquad \varphi \in \mathcal{D}_{\mathbb{T}}, \tag{2.6.12}$$

where $\rho(x)$ is defined in (2.2.14). By (2.6.12) we see $(u, F)^T$ satisfies the Gauss law (2.2.13), (2.2.14), or equivalently (2.5.8), in weak sense, where the weak derivatives are defined with respect to the test space $\mathcal{D}_{\mathbb{T}}$. Conversely, if $(u, F)^T$ satisfies (2.6.12) for all $\varphi \in \mathcal{D}_{\mathbb{T}}$, we prove in a similar way that (2.6.11) holds taking $\varphi(x) = e^{inx}, n \in \mathbb{Z}$.

Remark 2.6.2. Observe that the general solution of the magnetized Vlasov–Ampère system, $\mathbf{G}(t) = (u(t,x,v), F(t,x))^T$ given in (2.6.1) and that satisfies the Gauss law (2.5.8) fulfills the condition that the total charge fluctuation is equal to zero,

$$\int_{[0,2\pi]\times\mathbb{R}^2} u(t,x,v) e^{\frac{-v^2}{4}}\, dx\, dv = 0. \tag{2.6.13}$$

This is true because each one of the the eigenfunctions that appear in the expansion (2.6.2), with $G_2(t)$ as in (2.6.4) and $G_1(t)$ as in (2.6.7) satisfy this condition.

□

Let us now consider the expansion of the charge density fluctuation of the perturbation to the Maxwellian equilibrium state, $\rho(t,x)$, that we defined in (2.2.14). We compute the expansion of $\rho(t,x)$ multiplying the first component of the left- and right- hand sides of (2.6.2), by $e^{-v^2/4}$, integrating both sides of the resulting equation over $v \in \mathbb{R}^2$, and using (2.6.4), and (2.6.7). For this purpose, note that for a function in $(u,0)^T \in \mathcal{H}$ with electric field zero the Gauss law (2.2.13), (2.2.14) implies that the charge density fluctuation of the function is zero. In particular the charge density fluctuation of the eigenfunctions $\mathbf{M}^{(0)}_{0,j}, j \geq 2, \mathbf{M}^{(0)}_{n,j}, n \in \mathbb{Z}^*, j \geq 2, \mathbf{V}_{m,j}, m \in \mathbb{Z}^*, j \in \mathbb{N}^*, \mathbf{W}_{n,m,j}, n,m \in \mathbb{Z}^*, j \in \mathbb{N}^*$ is equal to zero.

Then, if we apply the expansion (2.6.2), with $G_1(t)$ given in (2.6.7) and $G_2(t)$ given in (2.6.4) to the charge density fluctuation of the general solution to the magnetized Vlasov–Ampère system (2.6.1) that satisfies the Gauss law, only the terms with , $\mathbf{F}^{(0)}_n, n \in \mathbb{Z}^*$, and $\mathbf{Z}_{n,m}, n,m \in \mathbb{Z}^*$ survive. and we obtain,

$$\rho(t,x) = \rho_{\text{stat}}(x) + \rho_{\text{din}}(t,x), \tag{2.6.14}$$

where,

$$\rho_{\text{stat}} := -\sum_{n\in\mathbb{Z}^*} \left(\mathbf{G}_0, \mathbf{F}^{(0}_n\right) \int_{\mathbb{R}^2} \mathbf{F}^{(0,1)}(x,v)\, e^{\frac{-v^2}{4}}\, dv, \tag{2.6.15}$$

is the static part of the charge density fluctuation, and where $\mathbf{F}^{(0,1)}(x,v)$ is the first component of $\mathbf{F}^{(0)}_n$. Moreover,

$$\rho_{\text{din}}(t,x) = \sum_{n,m\in\mathbb{Z}^*} e^{-it\lambda_{n,m}}\, (\mathbf{G}_0, \mathbf{Z}_{n,m})_{\mathcal{H}}\, \rho_{n,m}(x), \tag{2.6.16}$$

is the time dependent part of the charge density fluctuation. Here, $\rho_{n,m}(x)$ is the charge density fluctuation of the eigenfunction $\mathbf{Z}_{n,m}$ that is given by

$$\rho_{n,m}(x) = -\frac{1}{b_{n,m}\sqrt{2\pi}}\, e^{inx} \int_{\mathbb{R}^2} e^{\frac{-r^2}{4}}\, e^{-in\frac{v_2}{\omega_c}}\, \eta_{n,m}(v)\, dv, n,m \in \mathbb{Z}^*, \tag{2.6.17}$$

where we used (2.5.77). The right-hand side of (2.6.16) is the expansion of the charge density fluctuation in the Bernstein modes, [14], [13]. Note however, that for general initial data there is also the static part of the charge density fluctuation (2.6.15), that is not reported in [14], [13]. This means that the Bernstein modes $\mathbf{Z}_{n,m}, n,m \in \mathbb{Z}^*$ are not complete, and that to expand the charge density fluctuation, $\rho(t,x)$ with the general initial data, $\mathbf{G}_0$ that has finite energy and that satisfies the Gauss law, one has to add the contribution of the static part $\rho_{\text{stat}}(x)$ given by the modes, $\mathbf{F}^{(0)}_n, n \in \mathbb{Z}^*$. It appears that this fact has not been observed before.

In the following theorem we prove that the expansion (2.6.14), (2.6.15), (2.6.16) of the charge density fluctuation converges for initial data in $\mathcal{H}$.

Theorem 2.6.3. *Let $\rho(t,x)$ be the charge density fluctuation defined in* (2.2.14). *Then, for any initial state, $\mathbf{G}_0 \in \mathcal{H}$, that satisfies the Gauss law, the expansion,* (2.6.14), (2.6.15), (2.6.16) *converges strongly in the norm of $L^2(0,2\pi)$.*

Proof. We denote by $G(t,x,v)$ the following quantity,

$$G := \sum_{n\in\mathbb{Z}^*,} \left(\mathbf{G}_0, \mathbf{F}^{(0)}_n\right) \mathbf{F}^{(0,1)}_n + \sum_{n,m\in\mathbb{Z}^*} e^{-it\lambda_{n,m}}\, (\mathbf{G}_0, \mathbf{Z}_{n,m})_{\mathcal{H}}\, \mathbf{Z}^{(1)}_{n,m}, \tag{2.6.18}$$

where $\mathbf{Z}^{(1)}_{n,m}$ is the first component of the eigenfunction $\mathbf{Z}_{n,m}$. Then,

$$\rho(t,x) = -\int_{\mathbb{R}^2} G(t,x,v)\, e^{\frac{-v^2}{4}} dv. \tag{2.6.19}$$

Hence, since $G(t,x,v) \in \mathcal{A}$, it follows from Fubini's theorem that for a.e. $x \in (0,2\pi)$, $G(t,x,\cdot) \in L^2(\mathbb{R}^2)$, and as also $e^{\frac{-v^2}{4}} \in L^2(\mathbb{R}^2)$, the integral in the right-hand side of (2.6.19) exists, and then, the charge density fluctuation $\rho(t,x)$ is well defined. Furthermore, by the Cauchy-Schwarz inequality $\rho(t,x) \in L^2(0,2\pi)$. We denote,

$$\begin{aligned}\rho_N(t,x) := -\sum_{n\in\mathbb{Z}^*,|n|\le N} \left(\mathbf{G}_0, \mathbf{F}^{(0)}_n\right)_{\mathcal{H}} \int_{\mathbb{R}^2} \mathbf{F}^{(0,1)}(x,v)\, e^{\frac{-v^2}{4}}\, dv + \\ \sum_{n,m\in\mathbb{Z}^*,|n|+|m|\le N} e^{-it\lambda_{n,m}} \left(\mathbf{G}_0, \mathbf{Z}_{n,m}\right)_{\mathcal{H}} \rho_{n,m}(t,x).\end{aligned} \tag{2.6.20}$$

We will prove that $\rho_N(t,x)$ converges to $\rho(t,x)$ in norm in $L^2(0,2\pi)$, i.e. that the series, (2.6.14), (2.6.15) and (2.6.16) converges strongly in $L^2(\mathbb{R}^2)$. We designate,

$$G_N := \sum_{n\in\mathbb{Z}^*,|n|\le N} \left(\mathbf{G}_0, \mathbf{F}^{(0)}_n\right) \mathbf{F}^{(0,1)}_n + \sum_{n,m\in\mathbb{Z}^*,|n|+|m|\le N} e^{-it\lambda_{n,m}} \left(\mathbf{G}_0, \mathbf{Z}_{n,m}\right)_{\mathcal{H}} \mathbf{Z}^{(1)}_{n,m}. \tag{2.6.21}$$

We have that

$$\lim_{N\to\infty} \|G - G_N\|_{\mathcal{A}} = 0. \tag{2.6.22}$$

Furthermore,

$$-\int_{\mathbb{R}^2} G_N(t,x,v)\, e^{\frac{-v^2}{4}} dv = \rho_N(t,x). \tag{2.6.23}$$

Hence,

$$\rho(t,x) - \rho_N(t,x) = -\int_{\mathbb{R}^2} (G(t,x,v) - G_N(t,x,v))\, e^{\frac{-v^2}{4}}\, dv. \tag{2.6.24}$$

Finally, by (2.6.22), (2.6.24), and the Cauchy- Schwarz inequality,

$$\begin{aligned}\int_0^{2\pi} |\rho(t,x) - \rho_N(t,x)|^2\, dx &= \int_0^{2\pi} \left| \int_{\mathbb{R}^2} (G(t,x,v) - G_N(t,x,v)) e^{\frac{-v^2}{4}}\, dv \right|^2 dx \\ &\le 2\pi \int_0^{2\pi} \int_{\mathbb{R}^2} |(G(t,x,v) - G_N(t,x,v)|^2\, dx\, dv = 2\pi\, \|G - G_N\|^2_{\mathcal{A}} \to 0, \text{ as } N \to \infty.\end{aligned} \tag{2.6.25}$$

This completes the proof that the expansion (2.6.14), (2.6.15), (2.6.16) converges strongly in the norm of $L^2(0,2\pi)$. □

Remark 2.6.4. The eigenfunctions $\mathbf{M}^{(0)}_{0,j}, j \ge 2, \mathbf{M}^{(0)}_{n,j}, n \in \mathbb{Z}^*, j \ge 2, \mathbf{V}_{m,j}, m \in \mathbb{Z}^*, j \in \mathbb{N}^*, \mathbf{W}_{n,m,j}, n,m \in \mathbb{Z}^*, j \in \mathbb{N}^*$, do not appear in the expansion (2.6.14), (2.6.15), (2.6.16) of the charge density fluctuation. Still, as we mentioned in the introduction, these eigenfunctions are physically interesting because they show that there are plasma oscillations such that at each point the charge density fluctuation is zero and the electric field is also zero. Some of them are time independent. Note that since our eigenfunctions are orthonormal, these special plasma oscillations actually exist on their own, without the excitation of the other modes. It appears that this fact has not been observed previously in the literature.

2.7 Operator theoretical proof of the Bernstein–Landau paradox

We first study the operator $\mathbf{H}_0$ that appears in the formula for $\mathbf{H}$ that we gave in (2.5.5),(2.5.6), and (2.5.7). Let us recall the representation of $\mathcal{H}$ as the direct sum of the $\mathcal{H}_n$ given in (2.5.87). Using proposition 2.4.1 we see that the functions $(u_n, \alpha_n)^T$ in $\mathcal{H}_n$ can be written as

$$\begin{pmatrix} u_n(x,v) \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \sum_{m\in\mathbb{Z}, j\in\mathbb{N}^*} u_{n,m,j}(x,v)\,(u_n, u_{n,m,j})_{\mathcal{A}} \\ \alpha_n \end{pmatrix}, \tag{2.7.1}$$

where for $n = 0, \alpha_n = 0$. Then, by proposition 2.4.1

$$\mathbf{H}_0 \begin{pmatrix} u_n(x,v) \\ \alpha_n \end{pmatrix} = \mathbf{H}_{0,n} \begin{pmatrix} u_n(x,v) \\ \alpha_n \end{pmatrix}, \tag{2.7.2}$$

where by $\mathbf{H}_{0,n}$ we denote the operator in $\mathcal{H}_n$ given by,

$$\mathbf{H}_{0,n} \begin{pmatrix} u_n(x,v) \\ \alpha_n \end{pmatrix} := \sum_{m\in\mathbb{Z}, j\in\mathbb{N}^*} \begin{pmatrix} \lambda_m^{(0)} u_{n,m,j}(x,v)\,(u_n, u_{n,m,j})_{\mathcal{A}} \\ 0 \end{pmatrix},$$

with domain $D[\mathbf{H}_{0,n}] := \{(u_n, \alpha_n)^T : \sum_{m\in\mathbb{Z}, j\in N} (\lambda_m^{(0)})^2 |(u_n, u_{n,m,j})_{\mathcal{A}}|^2 < \infty$. Observe that $\mathbf{H}_{0,n}$ is the restriction of $\mathbf{H}_0$ to $\mathcal{H}_n$, and that,

$$\mathbf{H}_0 = \oplus_{n\in\mathbb{Z}} \mathbf{H}_{0,n}. \tag{2.7.3}$$

Further, the spectrum of $\mathbf{H}_{0,n}$ is pure point and it consists of the infinite multiplicity eigenvalue $\lambda_m^{(0)}, m \in \mathbb{Z}$. Then, the spectrum of $\mathbf{H}_0$ is also pure point and it consists of the infinite multiplicity eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$. Recall that the discrete spectrum of a self-adjoint operator consists of the isolated eigenvalues of finite multiplicity, and that the essential spectrum is the complement in the spectrum of the discrete spectrum. So, we have reached the conclusion that the spectrum of $\mathbf{H}_0$ coincides with the essential spectrum and it is given by the infinite multiplicity eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$. Let us now consider the operator $\mathbf{V}$ that appears in (2.5.7). For $e^{inx}\,(\tau(v), \alpha_n)^T \in \mathcal{H}_n$,

$$\mathbf{V} e^{inx} \begin{pmatrix} \tau(v) \\ \alpha_n \end{pmatrix} = e^{inx} \begin{pmatrix} -iv_1 e^{\frac{-v^2}{4}}\,\alpha_n \\ iI^* \int_{\mathbb{R}^2} v_1\, e^{\frac{-v^2}{4}}\,\tau(v)\,dv \end{pmatrix}.$$

Then, $\mathbf{V}$ sends $\mathcal{H}_n$ into $\mathcal{H}_n$, and that it acts in the same way in all the $\mathcal{H}_n$. Let us denote by $\mathbf{V}_n$ the restriction of $\mathbf{V}$ to $\mathcal{H}_n$. Then, we have,

$$\mathbf{V} = \oplus_{n\in\mathbb{Z}} \mathbf{V}_n. \tag{2.7.4}$$

furthermore, by (2.5.5), (2.7.3), and (2.7.4),

$$\mathbf{H} = \oplus \mathbf{H}_n, \tag{2.7.5}$$

where $\mathbf{H}_n = \mathbf{H}_{0,n} + \mathbf{V}_n$. Further, it follows from (2.7.4) that $\mathbf{V}_n$ is a rank two operator, hence, it is compact. Then, it is a consequence of the Weyl theorem for the invariance of the essential spectrum, see Theorem 3, in page 207 of [16], that the essential spectrum of $\mathbf{H}_n, n \in \mathbb{Z}$ is given by the infinite multiplicity eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$. Hence, by (2.7.5) the essential spectrum of $\mathbf{H}$

is given by the infinite multiplicity eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$. However, since the complement of the essential spectrum is discrete, we have that the spectrum of $\mathbf{H}$ consists of the infinite multiplicity eigenvalues $\lambda_m^{(0)}, m \in \mathbb{Z}$, and of a set of isolated eigenvalues of finite multiplicity that can only accumulate at the essential spectrum and at $\pm\infty$. We know from the results of section 2.5 that these eigenvalues are the $\lambda_{n,m}, n, m \in \mathbb{Z}^*$, and that they are of multiplicity one. However, the operator theoretical argument does not tell us that. However, it tells us that the spectrum of $\mathbf{H}$ is pure point and that $\mathbf{H}$ has a complete orthonormal set of eigenfunctions. This implies that the Bernstein -Landau paradox exists. Let us elaborate on this point. As we mentioned in the introduction, it was shown by [36, 37] that the Landau damping can be characterized as the fact that when the magnetic field is zero $e^{-it\mathbf{H}}$ goes weakly to zero as $t \to \pm\infty$. Let us prove that when the magnetic field is non zero this is not true. We prove this fact using only the operator theoretical results of this section, i.e. without using the detailed calculations of section 2.5. Let us denote by γ_j, $j = 1, \dots,$ the eigenvalues of $\mathbf{H}$, repeated according to their multiplicity, and let $\mathbf{X}_j$, $j = 1, \dots$ be a complete set of orthonormal eigenfunctions, where the eigenfunction $\mathbf{X}_j$, is associated with the eigenvalue, $\gamma_j, j = 1, \dots$. We know explicitly from section 2.5 the eigenvalues and an orthonormal basis of eigenvectors, but we do not need this information here. Suppose that $e^{-it\mathbf{H}}$ goes weakly to zero as $t \to \pm\infty$. Then, for any $\mathbf{X}, \mathbf{Y} \in \mathcal{H}$,

$$\lim_{t \to \pm\infty} \left(e^{-it\mathbf{H}} \mathbf{X}, \mathbf{Y} \right)_{\mathcal{H}} = 0. \tag{2.7.6}$$

Let us prove that there is no non trivial $\mathbf{X} \in \mathcal{H}$ such that (2.7.6) holds for all $\mathbf{Y} \in \mathcal{H}$. We have that,

$$\left(e^{-it\mathbf{H}} \mathbf{X}, \mathbf{Y} \right)_{\mathcal{H}} = \sum_{l=1}^{\infty} e^{-it\gamma_l} (\mathbf{X}, \mathbf{X}_l)_{\mathcal{H}} (\mathbf{X}_l, \mathbf{Y})_{\mathcal{H}}.$$

However, let us take $\mathbf{Y} = \mathbf{X}_j, j = 1, \dots$. Then, $\lim_{t \to \pm\infty} \left(e^{-it\mathbf{H}} \mathbf{X}, \mathbf{Y}_j \right)_{\mathcal{H}} = \lim_{t \to \pm\infty} e^{-it\gamma_j} (\mathbf{X}, \mathbf{X}_j)_{\mathcal{H}}$, $j = 1, \dots,$ is a non-zero constant if $\gamma_j = 0$, and it is oscillatory if $\gamma_j \neq 0$, unless $(\mathbf{X}, \mathbf{X}_j)_{\mathcal{H}} = 0, j = 1, \dots$. However, if $(\mathbf{X}, \mathbf{X}_j)_{\mathcal{H}} = 0, j = 1, \dots,$ then, $\mathbf{X} = 0$. It follows that (2.7.6) only holds for $\mathbf{X} = 0$.

2.8 Numerical results

The objective of this section is to illustrate the numerical behavior of the eigenfunctions constructed previously. More precisely, we will construct a numerical scheme that approximates the solution of the magnetized Vlasov–Ampère system initialized with an eigenfunction and compare this numerical solution with the theoretical dynamics of the system. The numerical results below show that the difference between the theoretical and numerical solutions is small, confirming the theoretical analysis. Furthermore, we will use the eigenfunctions to initialize a code solving the non-linear magnetized Vlasov–Poisson system showing how we can approximate the solution of the non-linear system with our linear theory. Finally, using the same non-linear code, we will illustrate the Bernstein–Landau paradox, as in the spirit of [52, 91], by initializing with a standard test function traditionally used to highlight Landau damping and show how the damping is lost when we add a constant magnetic field.

2.8.1 Computing the eigenvalues

As in (2.5.54), we consider an eigenfunction

$$\begin{pmatrix} w_{n,m} \\ F_n \end{pmatrix}, \tag{2.8.1}$$

of the operator $\mathbf{H}$ associated to the Fourier mode $n \neq 0$ and the eigenvalue $\lambda_{n,-m} = -\lambda_{n,m}$ where $w_{n,m}$ and F_n are given by

$$w_{n,m} = e^{in(x - \frac{v_2}{\omega_c})} e^{-\frac{r^2}{4}} \sum_{p \in \mathbb{Z}^*} \frac{p\omega_c}{p\omega_c + \lambda_{n,m}} e^{ip\varphi} J_p\left(\frac{nr}{\omega_c}\right) \text{ and } F_n = -ine^{inx}. \tag{2.8.2}$$

Furthermore, $\lambda_{n,m}$ is one of the roots of a secular equation (2.5.48), which could be written as

$$\alpha(\lambda) = 0,$$

where the *secular function* $\alpha(\lambda)$ is given by

$$\alpha(\lambda) = -1 - \frac{2\pi}{n^2} \sum_{m \in \mathbb{Z}^*} \frac{m\omega_c}{m\omega_c + \lambda} a_{n,m}. \tag{2.8.3}$$

In (2.8.3) $a_{n,m}$ is defined by (2.5.46). The secular function $\alpha(\lambda)$ is a convergent series with poles at the multiples of the cyclotron frequency ω_c. Note that the function α in (2.8.3) and the function g in (2.5.51) are linked by the relation

$$\alpha(\lambda) = -1 - \frac{1}{n^2} g(\lambda). \tag{2.8.4}$$

and consequently one can easily deduce properties of α from those of g proved in lemma 2.5.5.

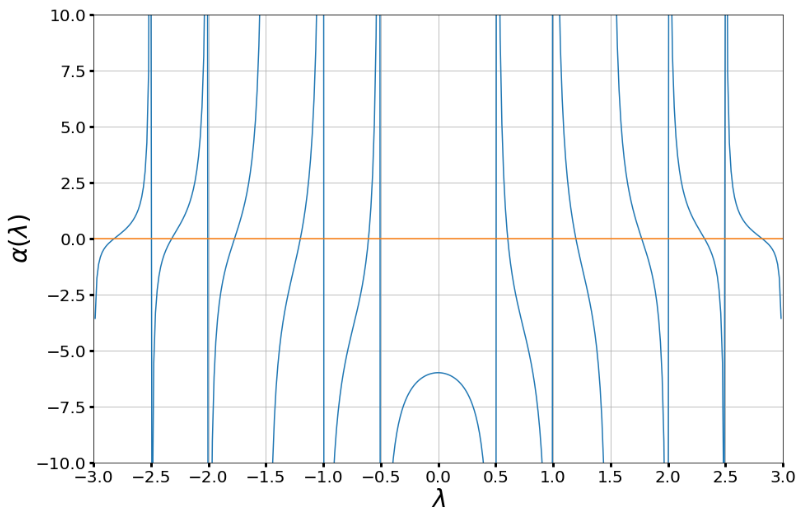

Figure 2.1 – Secular function for $\omega_c = 0.5$ and $n = 1$

The plot in fig. 2.1 illustrates the properties of α (deduced from lemma 2.5.5 and relation (2.8.4)), most notably that there is unique root (hence an eigenvalue for $\mathbf{H}$) in $(m\omega_c, (m+1)\omega_c)$ for $m \geq 1$, and $((m-1)\omega_c, m\omega_c)$ for $m \leq -1$. With a standard numerical method (dichotomy or Newton), we can determine the roots of α. For example, with $(n, m) = (1, 2)$, we find $\lambda_{1,2} \approx 1.19928$. This eigenvalue $\lambda_{1,2}$ will be used in all the following numerical tests.

2.8.2 Solving the linear magnetized Vlasov–Ampère system with a Semi-Lagrangian scheme with splitting

To approximate the linear system (2.2.17) or (2.5.1)-(2.5.2), we use a semi-Lagrangian scheme [30, 15], which is a classical method to approximate transport equations of the form $\partial_t f + E(x,t)\partial_x f = 0$, coupled with a splitting procedure. A splitting procedure corresponds to approximating the solution of $\partial_t f + (\mathcal{A} + \mathcal{B})f = 0$ by solving $\partial_t f + \mathcal{A}f = 0$ and $\partial_t f + \mathcal{B}f = 0$ one after the other.

Hence, the magnetized Vlasov–Ampère system is split so as to only solve transport equations with constant advection terms.

$$\partial_t \begin{pmatrix} u \\ F \end{pmatrix} + (\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}) \begin{pmatrix} u \\ F \end{pmatrix} = 0,$$

with

$$\mathcal{A} = \begin{pmatrix} v_1 \partial_x \\ 0 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} F v_1 e^{-\frac{v_1^2+v_2^2}{4}} \\ 1^* \int u e^{-\frac{v_1^2+v_2^2}{4}} v_1 dv_1 dv_2 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} -\omega_c v_2 \partial_{v_1} \\ 0 \end{pmatrix}, \quad \mathcal{D} = \begin{pmatrix} \omega_c v_1 \partial_{v_2} \\ 0 \end{pmatrix}. \tag{2.8.5}$$

The algorithm used to solve the linearized magnetized Vlasov–Ampère system can thus be summarized as follows

1. **Initialization** $\mathbf{U}_{ini} = \begin{pmatrix} w_{n,m} \\ F_n \end{pmatrix}$ given in (2.8.1).

2. **Going from t_n to t_{n+1}**

   Assume we know $\mathbf{U}^n$, the approximation of $\mathbf{U} = \begin{pmatrix} u \\ F \end{pmatrix}$ at time t_n.

   - We compute $\mathbf{U}^*$ by solving $\partial_t \mathbf{U} + \mathcal{A}\mathbf{U} = 0$ with a semi-Lagrangian scheme during one time step Δt with initial condition $\mathbf{U}^n$.
   - We compute $\hat{\mathbf{U}}$ by solving $\partial_t \mathbf{U} + \mathcal{B}\mathbf{U} = 0$ with a Runge-Kutta 2 scheme during one time step Δt with initial condition $\mathbf{U}^*$.
   - We compute $\mathbf{U}^{**}$ by solving $\partial_t \mathbf{U} + \mathcal{C}\mathbf{U} = 0$ with a semi-Lagrangian scheme during one time step Δt with initial condition $\hat{\mathbf{U}}$.
   - We compute $\mathbf{U}^{n+1}$ by solving $\partial_t \mathbf{U} + \mathcal{D}\mathbf{U} = 0$ with a semi-Lagrangian scheme during one time step Δt with initial condition $\mathbf{U}^{**}$.

2.8.3 Results for the magnetized Vlasov–Ampère system

The solution of the magnetized Vlasov–Ampère system initialized with an eigenfunctions $\mathbf{U}_{\text{ini}} = \begin{pmatrix} w_{n,m} \\ F_n \end{pmatrix}$ as in (2.8.1) is simply given by

$$\mathbf{U}(t) = e^{i\lambda_{n,m}t}\,\mathbf{U}_{\text{ini}}. \tag{2.8.6}$$

Recall that (2.8.1) is an eigenfunction of $\mathbf{H}$ with eigenvalue $\lambda_{n,-m} = -\lambda_{n,m}$. In the following results, we have taken $(n,m) = (1,2)$, $\omega_c = 0.5$, $N_x = 33$ (number of points of discretization in position), $N_{v_1} = N_{v_2} = 63$ (number of points of discretization in both velocity variables), $L_{v_1} = L_{v_2} = 10$ (numerical truncation in both velocity variables) and, most importantly, $T_{\text{f}} = \frac{\pi}{\lambda_{1,2}}$. This means that $\mathbf{U}(T_{\text{f}}) = \exp\left(i\frac{\pi}{2}\right)\mathbf{U}_{\text{ini}} = i\mathbf{U}_{\text{ini}}$, and then, the solution of the system at $t = T_{\text{f}}$ corresponds to the initial condition where the real and imaginary parts have been exchanged (up to a sign).

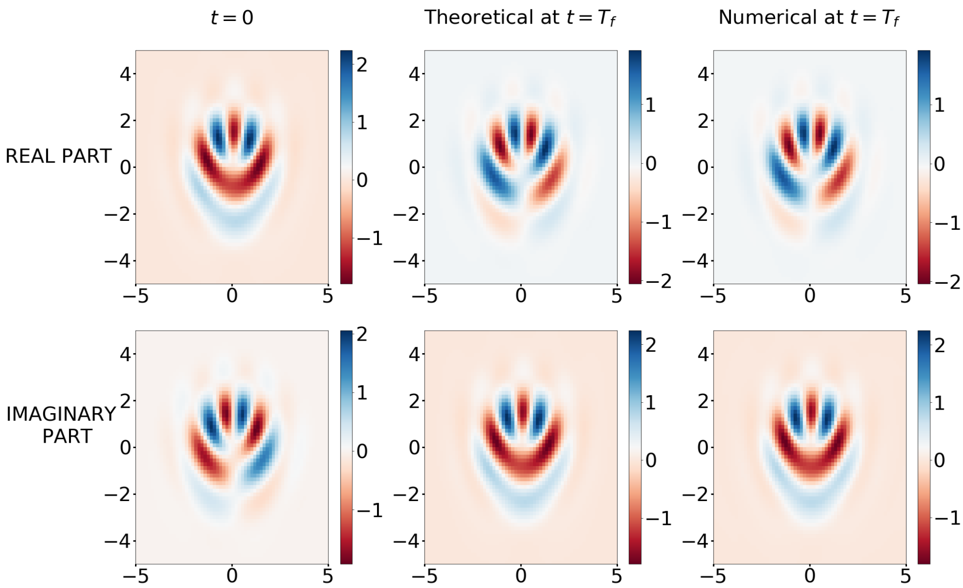

Figure 2.2 – Real and imaginary parts of the first component of $\mathbf{U}(t)$ given by (2.8.6) in $v_1 - v_2$ plane for $x = 0$.

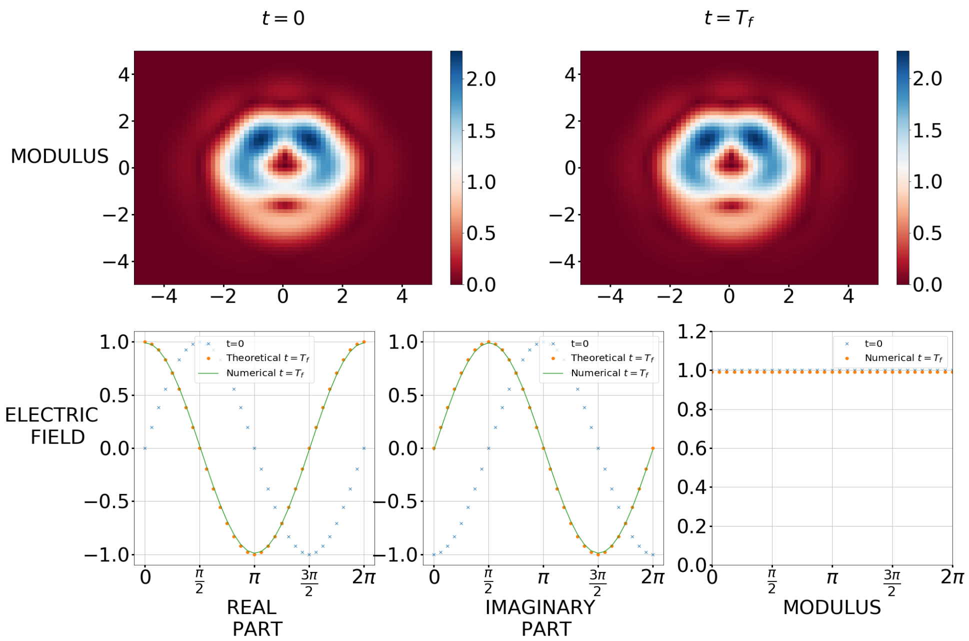

Figure 2.3 – Modulus of the first component of $\mathbf{U}(t)$ given by (2.8.6) in $v_1 - v_2$ plane for $x = 0$, and real and imaginary parts of F.

The figures show that the solution of the system behaves according to the theory.

2.8.4 Results for the non-linear magnetized Vlasov–Poisson system

We now look at how the solution of the non-linear magnetized Vlasov–Poisson system (2.2.4) behaves when initialized with an eigenfunction of the Hamiltonian $\mathbf{H}$ of the magnetized Vlasov–Ampère system. The idea is that for a certain time, the solution for the non-linear magnetized Vlasov–Poisson system follows the same dynamics as the solution for the linearized magnetized Vlasov–Poisson system. We consider the magnetized Vlasov–Poisson system because it is more convenient for numerical purposes. Recall that the linearized magnetized Vlasov–Poisson and the magnetized Vlasov–Ampère systems are equivalent. Furthermore, the articles [14, 52, 121, 91] have studied the Bernstein–Landau paradox using the magnetized Vlasov–Poisson system. We use almost the same numerical scheme as in the previous subsection to approximate the solution of the system.

The Vlasov equation, namely the first equation in (2.2.4), in the non-linear magnetized Vlasov–Poisson system is split so as to only solve transport equations with constant advection terms,

$$\partial_t u + (\mathcal{A} + \mathcal{B} + \mathcal{C})u = 0,$$

with $\mathcal{A} = v_1 \partial_x$, $\mathcal{B} = -(E + \omega_c v_2)\,\partial_{v_1}$ and $\mathcal{C} = \omega_c v_1 \partial_{v_2}$. To update the electric field, the strategy adopted is the same as in [30] where the Poisson equation is solved at each time step. On this numerical computation we consider real valued solutions f, E.

Let us denote by u the perturbation of the charge density function, f, and by F be the perturbation of the electric field, E. The functions u, F solve the linearized magnetized Vlasov–Poisson system (2.2.11). Recall that we proven in section 2.5 that the linearized magnetized Vlason-Poisson and magnetized Vlasov–Ampère systems are equivalent. Then, we can use the real part of (2.8.6) to write the expression of u, F when initializing with, $u_{\text{ini}}, F_{\text{ini}}$, with $u_{\text{ini}} = \text{Re}(w_{n,m})$, $F_{\text{ini}} = \text{Re}(F_n)$. Recall that $w_{n,m}$, and F_n are defined in (2.8.2). Then, we have,

$$\begin{pmatrix} u(t) \\ F(t) \end{pmatrix} = \text{Re}(\mathbf{U}(t)) = \begin{pmatrix} \cos(\lambda_m t)\,\text{Re}(w_{n,m}) - \sin(\lambda_m t)\,\text{Im}(w_{n,m}) \\ \cos(\lambda_m t)\,\text{Re}(F_n) - \sin(\lambda_m t)\,\text{Im}(F_n) \end{pmatrix} \tag{2.8.7}$$

where $\mathbf{U}(t)$ is given by (2.8.6).

The objective of this subsection is to show that we can approximate the solution of the non-linear system using (2.8.7), which means that the solutions of both linear and non-linear systems are close to each other for a certain time.

The algorithm used to solve the non-linear magnetized Vlasov–Poisson system can be summarized as follows:

1. **Initialization** $f_{ini} = f_0 + \varepsilon\sqrt{f_0}\,\text{Re}(w_{n,m})$ and $E_{\text{ini}} = \varepsilon\,\text{Re}(F_n)$ are given, where ε is a scalar which controls the amplitude of the perturbation. We take $\varepsilon = 0.1$.

2. **Going from t_n to t_{n+1}**

   Assume we know f^n and E_n, the approximations of f and E at time t_n.

   - We compute f^* by solving $\partial_t f + v_1\partial_x f = 0$ with a semi-Lagrangian scheme during one time step Δt with initial condition f_n.
   - We compute E_{n+1} by solving the Poisson equation with f^*.
   - We compute $\hat{f}$ by solving $\partial_t f - (E_{n+1} + \omega_c v_2)\partial_{v_1} f = 0$ with a semi-Lagrangian scheme during one time step Δt with initial condition f^*.
   - We compute f^{n+1} by solving $\partial_t f + \omega_c v_1 \partial_{v_2} f = 0$ with a semi-Lagrangian scheme during one time step Δt with initial condition $\hat{f}$.

As in section 2.8.3 we take $(n, m) = (1, 2)$, $\omega_c = 0.5$, $N_x = 33$ (number of points of discretization in position), $N_{v_1} = N_{v_2} = 63$ (number of points of discretization in both velocity variables), $L_{v_1} = L_{v_2} = 10$ (numerical truncation in both velocity variables) and, $T_{\text{f}} = \frac{\pi}{\lambda_{1,2}}$. In the following figures, we are comparing respectively the theoretical perturbations, u, F, that are given by (2.8.7), and the numerical perturbations,

$$u^n = \frac{f^n - f_0}{\varepsilon\sqrt{f_0}}, \text{and } F^n = \frac{E^n}{\varepsilon},$$

where f^n and E^n are given by the above algorithm.

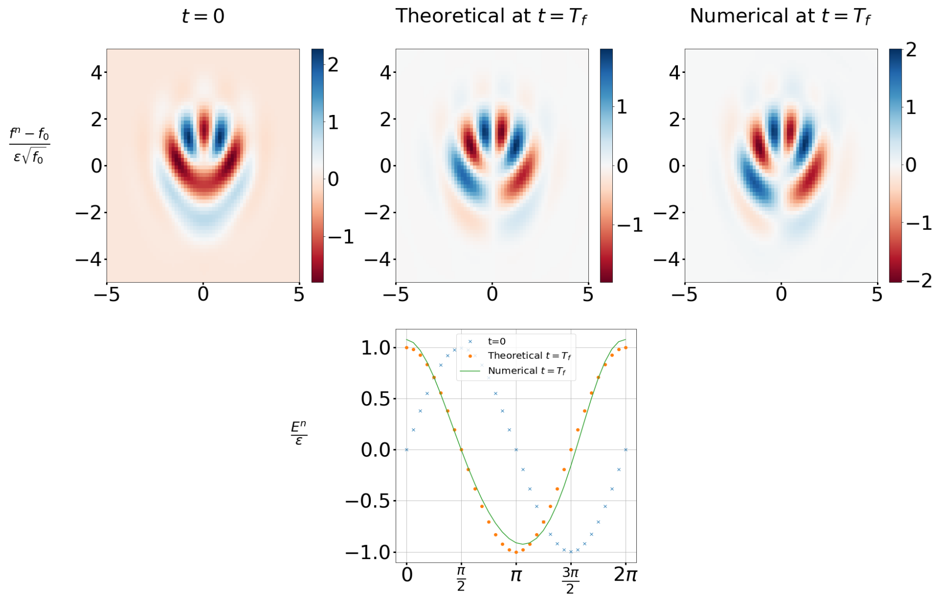

Figure 2.4 – u in $v_1 - v_2$ plane for $x = 0$ and electric field F.

The figures show that we can approximate the solution of the non-linear magnetized Vlasov–Poisson system using solutions of the linear magnetized Vlasov–Poisson system, initialized with the eigenfunctions of the Hamiltonian, $\mathbf{H}$ of the magnetized Vlasov–Ampère system.

2.8.5 The Bernstein–Landau paradox

In this subsection we illustrate the Bernstein–Landau paradox numerically, and we compare it with the Landau damping, using the above algorithm (similarly to [52]). In order to compare the numerical solutions to the non-linear Vlasov–Poisson system with the approximate analytical solution found in [117] in the case $\omega_c = 0$, we take below the charge of the ions equal to one. With this convention the non-linear Vlasov–Poisson system is written as,

$$\begin{cases} \partial_t f + v_1 \partial_x f - E \partial_{v_1} f + \omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2} \right) f = 0. \\ \partial_x E(t,x) = 1 - \displaystyle\int_{\mathbb{R}^2} f dv. \end{cases} \tag{2.8.8}$$

Furthermore, also with the purpose of comparing with the approximate analytical solution of [117], we initialize with the density function f_{LD} given by,

$$f_{LD}(x, v_1, v_2) = \frac{1}{2\pi} \left(1 + \varepsilon \cos kx\right) e^{\frac{-v^2}{2}}, \quad \varepsilon = 0.001, k = 0.4. \tag{2.8.9}$$

In this simulation the position interval is $[0, \frac{2\pi}{k}]$, since we keep periodic solutions. To introduce the approximate analytical solution of [117] let us consider the Vlasov–Poisson system (2.8.8) with $\omega_c = 0$,

$$\begin{cases} \partial_t f + v_1 \partial_x f - E \partial_{v_1} f = 0, \\ \partial_x E(t,x) = 1 - \int_{\mathbb{R}^2} f dv, \end{cases} \tag{2.8.10}$$

and initialized with (2.8.9).

Let us look for a solution of the form,

$$f(t,x,v) = f_1(t,x,v_1) \frac{1}{\sqrt{2\pi}} e^{\frac{-v_2^2}{2}}. \tag{2.8.11}$$

Then, $f(t,x,v)$ satisfies the (2.8.10) and it is initialized with (2.8.9) if and only if $f_1(t,x,v_1)$ is a solution of the following Vlasov–Poisson system in one dimension in space and velocity,

$$\begin{cases} \partial_t f_1 + v_1 \partial_x f_1 - E_1 \partial_{v_1} f_1 = 0, \\ \partial_x E_1(t,x) = 1 - \int_{\mathbb{R}} f_1 dv_1, \end{cases} \tag{2.8.12}$$

initialized with,

$$f_1(0,x,v_1) = \frac{1}{\sqrt{2\pi}} (1 + \varepsilon \cos kx) e^{\frac{-v_1^2}{2}}, \quad \varepsilon = 0.001, k = 0.4. \tag{2.8.13}$$

Furthermore, note that

$$E(t,x) = E_1(t,x). \tag{2.8.14}$$

Then, we can compute an approximate $E(t,x)$ using the approximate solution to (2.8.12), (2.8.13) given in page 58 of [117]. Namely,

$$E(x,t) \approx 4\varepsilon \times 0,424666 \exp(-0,0661t) \sin(0,4x) \cos(1,2850t - 0,3357725). \tag{2.8.15}$$

We have taken the values given in the second line of the table in page 58 of [117]. This approximate solution is a good approximation of the exact solution for large times.
Further, (2.8.15) is a classical test function to highlight Landau damping, more precisely the damping of the electric energy. In the figures below we report (2.8.15) in the black curves. Moreover, the figure below illustrates how when $\omega_c \neq 0$, the damping is replaced by a recurrence phenomenon of period $T_c = \frac{2\pi}{\omega_c}$, which follows the behaviour observed in [6, 121]. We take $\omega_c = 0.1$, and as in section 2.8.3, we use, $N_x = 33$ (number of points of discretization in position), $N_{v_1} = N_{v_2} = 63$ (number of points of discretization in both velocity variables), $L_{v_1} = L_{v_2} = 10$ (numerical truncation in both velocity variables).

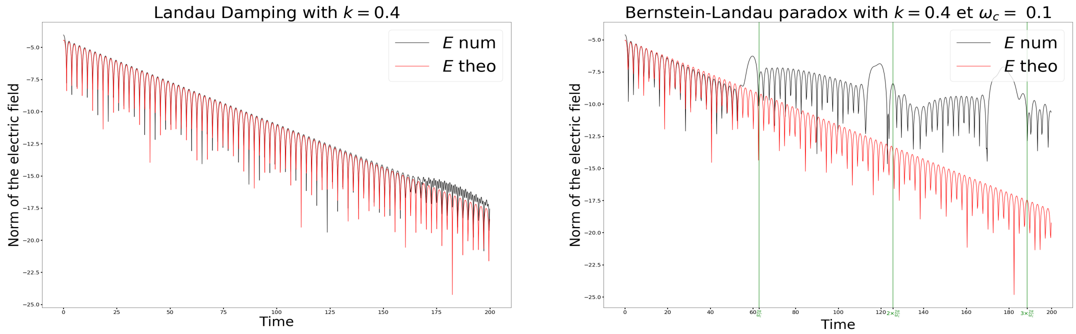

Figure 2.5 – Damped and undamped electric field

The recurrence visible on the right-hand side figure, i.e. the Bernstein paradox, is a fully "physical" phenomenon originating from the non-zero magnetic field and is to be distinguished from the recurrence in semi-Lagrangian schemes studied in [89], which deals with a purely numerical phenomenon. Let us show that this recurrence is a consequence of our series based on the eigenvectors expansion in the regime of non zero magnetic field. For this purpose, we take the charge of the ions equal to 2π, and solutions with period 2π, to be able to use our results of the previous sections. We consider the initial data.

$$\mathbf{G}_0 := (u_0(x,v), F_0(x)) = \left(e^{\frac{-v^2}{4}} \cos lx, -\frac{2\pi}{l} \sin lx \right), l \in \mathcal{Z}^*, \tag{2.8.16}$$

which satisfies the Gauss law. To compute the electric field with the expansion of the solution to the magnetic Vlasov–Ampère system given in (2.6.2), (2.6.4), (2.6.7) we only need to consider the eigenfunctions with non zero electric field, namely, $\mathbf{F}_n^{(0}$ and $\mathbf{Z}_{n,m}$. Recalling that the electric field is the second component of (2.6.2) we obtain,

$$\begin{aligned} F(x,t) = \sum_{n \in \mathbb{Z}^*} \left(\mathbf{G}_0, \mathbf{F}_n^{(0} \right) \mathbf{F}_n^{(0,2)}(x) + \sum_{n,m \in \mathbb{Z}^*} e^{-i\lambda_{n,m} t} \left(\mathbf{G}_0, \mathbf{Z}_{n,m} \right)_{\mathcal{H}} \mathbf{Z}_{n,m}^{(2)}(x) = \\ \left(\mathbf{G}_0, \mathbf{F}_l^{(0} \right) \mathbf{F}_l^{(0,2)}(x) + \sum_{m \in \mathbb{Z}^*} e^{-i\lambda_{l,m} t} \left(\mathbf{G}_0, \mathbf{Z}_{l,m} \right)_{\mathcal{H}} \mathbf{Z}_{l,m}^{(2)}(x), \end{aligned} \tag{2.8.17}$$

where we denote by $\mathbf{F}_n^{(0,2)}$, respectively, $\mathbf{Z}_{n,m}^{(2)}$ the second component of $\mathbf{F}_n^{(0)}$, and of $\mathbf{Z}_{n,m}$. Then, using (2.2.10) with $E_0 = 0$, and (2.8.17) we get the approximate formula for the electric field,

$$E(x,t) \approx \varepsilon \Re F(t) = \varepsilon \Re \left(\mathbf{G}_0, \mathbf{F}_l^{(0} \right) \mathbf{F}_l^{(0,2)}(x) + \varepsilon \, \Re \sum_{m \in \mathbb{Z}^*} e^{-i\lambda_{l,m} t} \left(\mathbf{G}_0, \mathbf{Z}_{l,m} \right)_{\mathcal{H}} \mathbf{Z}_{l,m}^{(2)}(x). \tag{2.8.18}$$

This approximate formula for the electric field shows the recurrence observed in the right-hand side of fig. 2.5. For clarity, we indicate the real part in (2.8.18), but note that since the initial data $\mathbf{G}_0$ in (2.8.16) is real valued, and the solution to the magnetic Vlasov-Ampère system (2.2.17) is unique, actually the electric field given by (2.8.17) is real valued.

2.9 Appendix A

In this appendix we further study the properties of the secular equation (2.5.50), (2.5.46), and (2.5.51). For later use we prepare the following result.

Proposition 2.9.1. *Let $a_{n,m}, n \in \mathbb{Z}^*, m = 1, \ldots,$ be the quantity defined in (2.5.46). Then, there is a constant, C, that depends on n, such that,*

$$a_{n,m} \leq C \frac{1}{\sqrt{m}} \left[\frac{en^2}{2\omega_c^2 m}\right]^m, \qquad m = 1, \ldots, \tag{2.9.1}$$

where e is Euler's number. In particular, for any $p > 0$ there is a constant C, that depends on n and p, such that,

$$a_{n,m} \leq C \frac{1}{m^p}. \tag{2.9.2}$$

Proof. By equation (10.22.67) in page 245 of [53]

$$a_{n,m} = e^{\frac{-n^2}{\omega_c^2}} I_m\left(\frac{n^2}{\omega_c^2}\right), \tag{2.9.3}$$

with $I_n(z)$ a modified Bessel function. Furthermore, by equation (10.41.1) in page 256 of [53],

$$I_m\left(\frac{n^2}{\omega_c^2}\right) = \frac{1}{\sqrt{2\pi m}} \left(\frac{en^2}{2\omega_c^2 m}\right)^m (1 + o(1)), \qquad m \to \infty. \tag{2.9.4}$$

Equation (2.9.1) follows from (2.9.3) and (2.9.4). Finally, (2.9.2) follows from (2.9.1). $\square$

We continue the analysis of the secular equation. Let $\lambda_{n,m}, m \geq 2$ be the root given in (2.5.6). Recall that $\lambda_{n,m} \in]m\omega_c, (m+1)\omega_c[$. Then, to isolate terms that can be large as $\lambda_{n,m}$ is close to $m\omega_c$ or to $(m+1)\omega_c$, we decompose $g(\lambda_{n,m})$ as follows,

$$g(\lambda_{n,m}) = g^{(1)}(\lambda_{n,m}) + g^{(2)}(\lambda_{n,m}) + g^{(3)}(\lambda_{n,m}) + g^{(4)}(\lambda_{n,m}), \tag{2.9.5}$$

where,

$$g^{(1)}(\lambda_{n,m}) := 4\pi \sum_{1 \leq q \leq m-1} \frac{q^2\, \omega_c^2}{q^2\omega_c^2 - \lambda_{n,m}^2} a_{n,q}, \tag{2.9.6}$$

$$g^{(2)}(\lambda_{n,m}) := 4\pi \frac{m^2\, \omega_c^2}{m^2\omega_c^2 - \lambda_{n,m}^2} a_{n,m}, \tag{2.9.7}$$

$$g^{(3)}(\lambda_{n,m}) := 4\pi \frac{(m+1)^2\, \omega_c^2}{(m+1)^2\omega_c^2 - \lambda_{n,m}^2} a_{n,m+1}, \tag{2.9.8}$$

$$g^{(4)}(\lambda_{n,m}) := 4\pi \sum_{q \geq m+2} \frac{q^2\, \omega_c^2}{q^2\omega_c^2 - \lambda_{n,m}} a_{n,q}. \tag{2.9.9}$$

Lemma 2.9.2. *Let $g^{(1)}(\lambda_{n,m})$ be the quantity defined in (2.9.6). Then, there is a constant C_n such that,*

$$\left|g^{(1)}(\lambda_{n,m})\right| \leq C_n \frac{1}{m^2}, \qquad m \geq 2. \tag{2.9.10}$$

Proof. First suppose that m is even. Then, $m/2$ is an integer, and we can decompose $g^{(1)}(\lambda_{n,m})$ as follows,

$$g^{(1)}(\lambda_{n,m}) = g^{(1,1)}(\lambda_{n,m}) + g^{(1,2)}(\lambda_{n,m}), \tag{2.9.11}$$

where,

$$g^{(1,1)}(\lambda_{n,m}) := 4\pi \sum_{1\le q\le m/2} \frac{q^2\,\omega_{\mathrm{c}}^2}{q^2\omega_{\mathrm{c}}^2 - \lambda_{n,m}^2} a_{n,q}, \tag{2.9.12}$$

and

$$g^{(1,2)}(\lambda_{n,m}) := 4\pi \sum_{m/2<q\le m-1} \frac{q^2\,\omega_{\mathrm{c}}^2}{q^2\omega_{\mathrm{c}}^2 - \lambda_{n,m}^2} a_{n,q}. \tag{2.9.13}$$

Note that,

$$\left|\frac{1}{q^2\omega_{\mathrm{c}}^2 - \lambda_{n,m}^2}\right| \le \frac{2}{m^2\omega_{\mathrm{c}}^2}, \qquad q = 1, \dots, \frac{m}{2}. \tag{2.9.14}$$

Then, by (2.9.2), (2.9.12) and, (2.9.14)

$$\left|g^{(1,1)}(\lambda_{n,m})\right| \le 4\pi\,\frac{2}{m^2\omega_{\mathrm{c}}^2}\sum_{1}^{m/2} q^2\,\omega_{\mathrm{c}}^2\,a_{n,q} \le C\frac{1}{m^2}. \tag{2.9.15}$$

Furthermore, we have

$$\left|\frac{1}{q^2\omega_{\mathrm{c}}^2 - \lambda_{n,m}^2}\right| \le \frac{1}{\omega_{\mathrm{c}}}\,\frac{1}{m\omega_{\mathrm{c}}}, \qquad q = \frac{m}{2}, \dots, m-1. \tag{2.9.16}$$

Then, by (2.9.2), (2.9.13) and, (2.9.16),

$$\left|g^{(1,2)}(\lambda_{n,m})\right| \le 4\pi\,\frac{1}{\omega_{\mathrm{c}}}\,\frac{1}{m\omega_{\mathrm{c}}} \sum_{m/2<q\le m-1} q^2\,\omega_{\mathrm{c}}^2 a_{n,q} \le C_p\frac{1}{m^p},\ p = 1, \dots. \tag{2.9.17}$$

Equation (2.9.10) follows from (2.9.11), (2.9.15) and, (2.9.17). In the case where m is odd, $(m-1)/2$ is an integer, and we decompose $g^{(1)}(\lambda_{n,m})$ as in (2.9.11) with,

$$g^{(1,1)}(\lambda_{n,m}) := 4\pi \sum_{1\le q\le (m-1)/2} \frac{q^2\,\omega_{\mathrm{c}}^2}{q^2\omega_{\mathrm{c}}^2 - \lambda_{n,m}} a_{n,q}, \tag{2.9.18}$$

and

$$g^{(1,2)}(\lambda_{n,m}) := 4\pi \sum_{(m-1)/2<q\le m-1} \frac{q^2\,\omega_{\mathrm{c}}^2}{q^2\omega_{\mathrm{c}}^2 - \lambda_{n,m}^2} a_{n,q}, \tag{2.9.19}$$

and we proceed as in the case of m even. □

In the following lemma we estimate $g^{(4)}(\lambda_{n,m})$.

Lemma 2.9.3. *Let $g^{(4)}(\lambda_{n,m})$ be the quantity defined in (2.9.9). Then, for every $p > 0$ there is a constant C_p such that,*

$$\left|g^{(4)}(\lambda_{n,m})\right| \le C_p\,\frac{1}{m^p}, \qquad m \ge 2. \tag{2.9.20}$$

Proof. Note that,

$$\left|\frac{1}{q^2\omega_{\rm c}^2-\lambda_{n,m}^2}\right| \leq \frac{1}{\omega_{\rm c}}\,\frac{1}{(m+3)\omega_{\rm c}}, \qquad q \geq m+2. \tag{2.9.21}$$

Equation (2.9.20) follows from (2.9.2), (2.9.9) and, (2.9.21). □

In the following lemma we estimate how $\lambda_{n,m}$ approaches $m\omega_{\rm c}$ as $m \to \pm\infty$.

Lemma 2.9.4. *We have,*

$$\lambda_{n,m} = m\omega_{\rm c} + 2\pi m\,\omega_{\rm c}\,\frac{a_{n,|m|}}{n^2} + a_{n,|m|}\,O\left(\frac{1}{|m|}\right), \qquad m \to \pm\infty. \tag{2.9.22}$$

Proof. Note that since $\lambda_{n,-m} = -\lambda_{n,m}$ it is enough to prove equation (2.9.22) when $m \to \infty$.

Using (2.9.10) and (2.9.20) we write (2.5.51) as follows

$$4\pi\,\frac{m^2\,\omega_{\rm c}^2}{\lambda_{n,m}^2 - m^2\omega_{\rm c}^2}a_{n,m} = n^2 + g^{(3)}(\lambda_{n,m}) + O\left(\frac{1}{m^2}\right), \qquad m \to \infty. \tag{2.9.23}$$

Moreover, as $g^{(3)}(\lambda_{n,m}) \geq 0$, we get,

$$4\pi\,\frac{m^2\,\omega_{\rm c}^2}{\lambda_{n,m}^2 - m^2\omega_{\rm c}^2}a_{n,m} \geq n^2 + O\left(\frac{1}{m^2}\right), \qquad m \to \infty.$$

Then, there is an m_0 such that $4\pi\,\frac{m^2\,\omega_{\rm c}^2}{\lambda_{n,m}^2-m^2\omega_{\rm c}^2}a_{n,m} \geq \frac{\pi}{4}$, $m \geq m_0$, and then, $\lambda_{n,m}^2 \leq m^2\omega_{\rm c}^2 + 16m^2\,\omega_{\rm c}^2\,a_{n,m}$, $m \geq m_0$, and taking the square root we obtain

$$m\omega_{\rm c} \leq \lambda_{n,m} \leq m\omega_{\rm c}\sqrt{1+16\,a_{n,m}}, \qquad m \geq m_0. \tag{2.9.24}$$

This already shows that $\lambda_{n,m}$ is asymptotic to $\omega_{\rm c}$ for large m. However, we can improve this estimate to obtain (2.9.22). By (2.9.2) and (2.9.24) for every $p > 0$,

$$((m+1)\omega_{\rm c} - \lambda_{n,m})^{-1} = \frac{1}{\omega_{\rm c}}\left(1 + O\left(\frac{1}{m^p}\right)\right), \qquad m \to \infty. \tag{2.9.25}$$

Further, introducing (2.9.8) and (2.9.25) into (2.9.23), and using (2.9.2) we obtain,

$$4\pi\,\frac{m^2\,\omega_{\rm c}^2}{\lambda_{n,m}^2 - m^2\omega_{\rm c}^2}a_{n,m} = n^2 + O\left(\frac{1}{m^2}\right), \qquad m \to \infty. \tag{2.9.26}$$

We rearrange (2.9.26) as follows,

$$\lambda_{n,m} - m\omega_{\rm c} = \frac{4\pi}{n^2}\,\frac{m^2\,\omega_{\rm c}^2}{\lambda_{n,m}+m\omega_{\rm c}}a_{n,m} + \frac{1}{n^2}\,(\lambda_{n,m}-m\omega_{\rm c})\,O\left(\frac{1}{m^2}\right), \qquad m \to \infty. \tag{2.9.27}$$

By (2.9.24)

$$\lambda_{n,m} - m\omega_{\rm c} \leq m\omega_{\rm c}\,O\left(a_{n,m}\right), \qquad m \to \infty. \tag{2.9.28}$$

Further,

$$(\lambda_{n,m}+m\omega_{\rm c})^{-1} = (2m\omega_{\rm c} + \lambda_{n,m} - m\omega_{\rm c})^{-1} = \frac{1}{2m\omega_{\rm c}}\,(1 + O\left(a_{n,m}\right)), \qquad m \to \infty. \tag{2.9.29}$$

Expansion (2.9.22) follows from (2.9.28) and, (2.9.29). □

2.10 Appendix B

2.10.1 A family of stationary solutions

In this appendix we explicitly construct a family of time-independent solutions to the linearized magnetized Vlasov–Poisson system. We first construct the family in dimension 1+2 (one dimension in space, two dimensions in velocity), which corresponds to the setting of our work. Then, we generalize our family of solutions to the case dimension 3+3 (three dimensions in space, three dimensions in velocity) that is the case considered by Bedrossian and Wang [13].

2.10.2 Dimension 1+2

For the purpose of making the comparison with [13] more transparent we consider the Vlasov equation,

$$\partial_t f + v_1 \partial_x f + \mathbf{F} \cdot \nabla_v f = 0, \tag{2.10.1}$$

with the electromagnetic Lorenz force,

$$\mathbf{F}(t,x) = \frac{q}{m} \left(\mathbf{E}(t,x) + v \times \mathbf{B}_0(t,x) \right). \tag{2.10.2}$$

Taking $\mathbf{B}_0 \neq 0$ the plasma is magnetized. The variable x is in the periodic torus $x \in \mathcal{T} = [0, 2\pi]_{\text{per}}$. The velocity variables are $(v_1, v_2) \in \mathbb{R}^2$. We take the charge $q > 0$ (remark 2.2.1) the mass $m > 0$, and we assume, as before, that the magnetic field $\mathbf{B}(t, \mathbf{x}) = \mathbf{B}_0$ is constant in space-time. We suppose again that the two-dimensional velocity v is perpendicular to the constant magnetic field, i.e., $\mathbf{B}_0 = (0, 0, B_0), B_0 > 0$. Moreover, we assume that the electric field is directed along the first coordinate axis, $\mathbf{E}(t,x) = (E(t,x), 0, 0)$, that it has mean zero,

$$\int_{\mathcal{T}} E(t,x)\, dx = 0,$$

and that it satisfies the Gauss law,

$$\partial_x E(t,x) = \frac{1}{4\pi} \left(-1 + q \int_{\mathbb{R}^2} f dv \right), \tag{2.10.3}$$

where as in [13] we introduced the factor $\frac{1}{4\pi}$ in the right-hand side of the Gauss law, and we have taken the charge of the heavy particles equal to minus one.

We linearize the equations around a homogeneous Maxwellian equilibrium state $f_0(v)$, where,

$$f_0(v) := \frac{1}{2\pi} e^{\frac{-v^2}{2}}.$$

This corresponds to the expansion,

$$f(t,x,v) = f_0(v) + \varepsilon h(t,x,v) + O(\varepsilon^2), \tag{2.10.4}$$

and

$$E(t,x) = E_0 + \varepsilon F(t,x) + O(\varepsilon^2), \tag{2.10.5}$$

with a null reference electric field $E_0 = 0$. Inserting (2.10.4) and (2.10.5) into (2.10.1)-(2.10.3), and keeping the terms up to linear in ε, we obtain the linearized magnetized Vlasov–Poisson system

$$\begin{cases} \partial_t h + v_1 \partial_x h - \dfrac{q}{m} F v_1 f_0 - \dfrac{q}{m} B_0 \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) h = 0, \\ \partial_x F = \dfrac{q}{4\pi} \displaystyle\int_{\mathbb{R}^2} h \, dv_1 dv_2, \int_{\mathcal{T}} F(t,x)\, dx = 0. \end{cases} \tag{2.10.6}$$

As we look for time-independent solutions, we have to solve,

$$\begin{cases} v_1 \partial_x h - \dfrac{q}{m} F v_1 f_0 - \dfrac{q}{m} B_0 \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) h = 0, \\ \partial_x F = \dfrac{q}{4\pi} \displaystyle\int_{\mathbb{R}^2} h \, dv_1 \, dv_2, \qquad \int_{\mathcal{T}} F(t,x)\, dx = 0. \end{cases} \tag{2.10.7}$$

Note that,

$$\left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) f_0(v_1, v_2) = 0. \tag{2.10.8}$$

Our objective hereafter is to construct a family of non trivial smooth solutions to (2.10.7) that have fast decay in velocity.

Lemma 2.10.1. *There exists an explicit family of non trivial smooth solutions (h, F) to the time-independent linearized magnetized Vlasov–Poisson system (2.10.7), where $F = -\varphi'(x)$, with $\varphi \in C^\infty(\mathcal{T})$, and where the function h can be taken either with l continuous derivatives with respect to $v, l = 1, 2, \dots,$ or infinitely differentiable with respect to v. Moreover, for each fixed $x \in \mathcal{T}, h \in L^1(\mathbb{R}^2)$. Further, the absolute value of h and of all its derivatives can be taken bounded by Gaussian functions of v, uniformly in $x \in \mathcal{T}$. Moreover, $h + \frac{q}{m}\varphi f_0$ can be taken with compact support in $\mathbf{v}$, uniformly in $x \in \mathcal{T}$.*

Proof. We introduce an electric potential $\varphi \in C^1(\mathcal{T})$ as

$$F(x) = -\varphi'(x),$$

with $\varphi(2\pi) = \varphi(0)$ and $\varphi'(2\pi) = \varphi'(0)$. Plugging in the first equation in (2.10.7), we obtain

$$v_1 \partial_x \left[h(x, v_1, v_2) + \frac{q}{m} \varphi(x) f_0(v_1, v_2) \right] - \frac{q}{m} B_0 \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) h = 0.$$

Let us define

$$G(x, v_1, v_2) = h(x, v_1, v_2) + \frac{q}{m} \varphi(x) f_0(v_1, v_2). \tag{2.10.9}$$

Since we have (2.10.8), G satisfies the equation

$$v_1 \partial_x G(x, v_1, v_2) - \frac{q}{m} B_0 \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) G(x, v_1, v_2) = 0. \tag{2.10.10}$$

Let us make another change of function which is valid since $B_0 \neq 0$,

$$H(x, v_1, v_2) = G(x - \frac{m\, v_2}{q B_0}, v_1, v_2) \Longleftrightarrow G(x, v_1, v_2) = H(x + \frac{m\, v_2}{q\, B_0}, v_1, v_2). \tag{2.10.11}$$

The equation (2.10.10) is rewritten as

$$\frac{q}{m} \omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2}\right) H(x, v_1, v_2) = 0. \tag{2.10.12}$$

Under this form, it is easy to find a general solution which writes

$$H(x, v_1, v_2) = K(x, v_1^2 + v_2^2)$$

where K is an arbitrary smooth function which decreases sufficiently fast at infinity with respect to its second variable. For example, we can take,

$$K(x, v_1^2 + v_2^2) = e^{inx} g(v_1^2 + v_2^2), \qquad n \in \mathcal{Z} \setminus \{0\}, \tag{2.10.13}$$

where $g(v_1^2 + v_2^2) \in C^l(\mathbb{R}^2), l = 1, 2, \ldots,$ or $g(v_1^2 + v_2^2) \in C^\infty(\mathbb{R}^2)$, and $g(v_1^2 + v_2^2) \in L^1(\mathbb{R}^2)$. For example, g can be taken with compact support, or a Gaussian. Going back to the perturbation h, one obtains the representation formula

$$h(x, v_1, v_2) = -\frac{q}{m}\varphi(x) f_0(v_1, v_2) + K(x + \frac{m\, v_2}{q\, B_0}, v_1^2 + v_2^2),$$

where the electric potential φ remains to be determined. All functions h of this form satisfy the first equation of (2.10.7).

It remains to verify the Gauss law, that is the second equation of (2.10.7). The right hand side of the Gauss law is

$$\frac{q}{4\pi}\int h\, dv_1\, dv_2 = \frac{q}{4\pi}\left(-\frac{q}{m}\,\varphi(x) + \int K(x + \frac{m\, v_2}{q\, B_0}, v_1^2 + v_2^2)\, dv_1\, dv_2\right).$$

So the Gauss law is rewritten as

$$-\varphi''(x) + \frac{q^2}{4\pi m}\,\varphi(x) = \frac{q}{4\pi}\int K(x + \frac{m\, v_2}{q\, B_0}, v_1^2 + v_2^2)\, dv_1\, dv_2.$$

Using (2.10.13), one gets

$$-\varphi''(x) + \frac{q^2}{4\pi m}\varphi(x) = \frac{q}{4\pi} e^{inx}\int e^{in\frac{mv_2}{qB_0}} g(v_1^2 + v_2^2)\, dv_1\, dv_2.$$

This is an equation for the electric potential. The periodic solution is explicit,

$$\varphi(x) = \frac{1}{n^2 + \frac{q^2}{4\pi m}}\,\frac{q}{4\pi}\, e^{inx}\int e^{in\frac{mv_2}{qB_0}} g(v_1^2 + v_2^2) dv_1 dv_2. \tag{2.10.14}$$

The remaining properties of the solution (h, F) follow immediately from the explicit representation of (h, F). □

Remark that the solutions given by lemma 2.10.1 satisfy,

$$\int_{\mathcal{T}} h(x, v_1, v_2)\, dx = 0, \int_{\mathcal{T}} F(x)\, dx = 0,$$

and in particular,

$$\int_{\mathcal{T}\times\mathbb{R}^2} h(x, v_1, v_2)\, dx\, dv_1\, dv_2 = 0.$$

The solutions given by lemma 2.10.1 are in agreement with (2.6.1), (2.6.2), (2.6.4), (2.6.7), and also with (2.6.14), (2.6.15), (2.6.16).

2.10.3 Dimension 3+3

We now consider solutions to the magnetized Vlasov–Poisson system in $\mathcal{T}^3 \times \mathbb{R}^3$, where $\mathcal{T}^3$ is the three-dimensional torus, $\mathcal{T}^3 := [0, 2\pi]^3_{\text{per}}$. We denote $\mathbf{x} := (x, y, z) \in \mathcal{T}^3$, and $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. The Vlasov equation is given by,

$$\partial_t f(t, \mathbf{x}, \mathbf{v}) + \mathbf{v} \cdot \nabla_{\mathbf{x}} f(t, \mathbf{x}, \mathbf{v}) + \mathbf{F}(t, \mathbf{x}) \cdot \nabla_{\mathbf{v}} f(t, \mathbf{x}, \mathbf{v}) = 0, \tag{2.10.15}$$

with the electromagnetic Lorenz force,

$$\mathbf{F}(t, \mathbf{x}) = \frac{q}{m} \left(\mathbf{E}(t, \mathbf{x}) + \mathbf{v} \times \mathbf{B}_0(t, \mathbf{x}) \right), \tag{2.10.16}$$

where $q > 0, m > 0$, and we assume, as before, that the magnetic field $\mathbf{B}(t, \mathbf{x}) = \mathbf{B}_0$ is constant in space-time, and that it is directed along the third coordinate, i.e., $\mathbf{B}_0 = (0, 0, B_0), B_0 > 0$. Moreover, we assume that the electric field satisfies the Gauss law,

$$\nabla_{\mathbf{x}} \cdot \mathbf{E}(t, \mathbf{x}) = \frac{1}{4\pi} \rho(t, \mathbf{x}), \qquad \rho(t, \mathbf{x}) := [-1 + q \int_{\mathbb{R}^3} f(t, \mathbf{x}, \mathbf{v})\, d^3\mathbf{v}]. \tag{2.10.17}$$

Further, we assume that the electric field has mean zero,

$$\int_{\mathcal{T}^3} \mathbf{E}(t, \mathbf{x})\, d^3\mathbf{x} = 0. \tag{2.10.18}$$

We linearize equations (2.10.15)-(2.10.18) around a homogeneous Maxwellian equilibrium state $f^0(v)$, where,

$$f^0(\mathbf{v}) := f_0(v_1, v_2)\, \frac{1}{\sqrt{2\pi T_\parallel}}\, e^{\frac{-v_3^2}{2T_\parallel}},$$

where $T_\parallel > 0$ is the temperature along the magnetic field. It corresponds to (2.1.2) by taking $\tilde{f} = 0$. This corresponds to the expansion,

$$f(t, \mathbf{x}, \mathbf{v}) = f^0(\mathbf{v}) + \varepsilon \mathcal{G}(t, \mathbf{x}, \mathbf{v}) + O(\varepsilon^2), \tag{2.10.19}$$

and

$$\mathbf{E}(t, \mathbf{x}) = \mathbf{E}_0 + \varepsilon \mathcal{F}(t, \mathbf{x}) + O(\varepsilon^2), \tag{2.10.20}$$

with a null reference electric field $\mathbf{E}_0 = 0$. Inserting (2.10.19) and (2.10.20) into (2.10.15)-(2.10.17), and keeping the terms up to linear in ε, we obtain the linearized magnetized Vlasov–Poisson system,

$$\begin{cases} \partial_t \mathcal{G}(t, \mathbf{x}, \mathbf{v}) + \mathbf{v} \cdot \nabla_{\mathbf{x}} \mathcal{G}(t, \mathbf{x}, \mathbf{v}) + \dfrac{q}{m} \mathcal{F}(t, \mathbf{x}) \cdot \nabla_{\mathbf{v}} f^0(\mathbf{v}) + \dfrac{q}{m} \mathbf{v} \times \mathbf{B}_0 \cdot \nabla_{\mathbf{v}} \mathcal{G}(t, \mathbf{x}, \mathbf{v}) = 0, \\ \nabla_{\mathbf{x}} \cdot \mathcal{F}(t, \mathbf{x}) = \dfrac{q}{4\pi}\, \rho(t, \mathbf{x}), \qquad \rho(t, \mathbf{x}) := \displaystyle\int_{\mathbb{R}^3} \mathcal{G}(t, \mathbf{x}, \mathbf{v})\, d^3\mathbf{v}, \qquad \int_{\mathcal{T}^3} \mathcal{F}(t, \mathbf{x})\, d^3\mathbf{x} = 0. \end{cases} \tag{2.10.21}$$

We look for solutions to (2.10.21) that satisfy,

$$\int_{\mathcal{T}^3 \times \mathbb{R}^3} \mathcal{G}(t, \mathbf{x}, \mathbf{v})\, d^3\mathbf{x}\, d^3\mathbf{v} = 0. \tag{2.10.22}$$

Under the condition (2.10.22) the Gauss law, that is the second equation in (2.10.21), is equivalent to the following equation,

$$\mathcal{F}(t,\mathbf{x}) = -\nabla_{\mathbf{x}} \int_{\mathcal{T}^3} W(\mathbf{x}-\mathbf{y})\,\rho(t,\mathbf{y})\,d^3\mathbf{y} \text{ with } W(\mathbf{x}) := \frac{q}{4\pi}\frac{1}{(2\pi)^3}\sum_{\mathbf{k}\in\mathbb{Z}^3\setminus\{0\}} \frac{1}{|\mathbf{k}|^2} e^{ik\cdot\mathbf{x}}. \tag{2.10.23}$$

Since we are looking for time-independent solutions we have to solve,

$$\begin{cases} \mathbf{v}\cdot\nabla_{\mathbf{x}}\mathcal{G}(t,\mathbf{x},\mathbf{v}) + \frac{q}{m}\mathcal{F}(t,\mathbf{x})\cdot\nabla_{\mathbf{v}}f^0(\mathbf{v}) + \frac{q}{m}\mathbf{v}\times\mathbf{B}_0\cdot\nabla_{\mathbf{v}}\mathcal{G}(t,\mathbf{x},\mathbf{v}) = 0, \\ \nabla_{\mathbf{x}}\cdot\mathcal{F}(t,\mathbf{x}) = \frac{q}{4\pi}\,\rho(t,\mathbf{x}), \qquad \rho(t,\mathbf{x}) := \int_{\mathbb{R}^3}\mathcal{G}(t,\mathbf{x},\mathbf{v})\,d^3\mathbf{v}, \qquad \int_{\mathcal{T}^3}\mathcal{F}(t,\mathbf{x})\,d^3\mathbf{x} = 0. \end{cases} \tag{2.10.24}$$

Let (h,F) be one of the solutions to (2.10.7) given by lemma 2.10.1. We define,

$$\mathcal{G}(\mathbf{x},\mathbf{v}) = h(x,v_1,v_2)\,\frac{1}{\sqrt{2\pi T_\|}}\,e^{\frac{-v_3^2}{2T_\|}}, \text{and } \mathcal{F}(\mathbf{x}) = \begin{pmatrix} F(x) \\ 0 \\ 0 \end{pmatrix}. \tag{2.10.25}$$

Lemma 2.10.2. *Let $(h(x,v_1,v_2),F(x))$ be one of the solutions to the time-independent linearized magnetized Vlasov–Poisson system (2.10.7) given by lemma 2.10.1. Then, the pair $(\mathcal{G}(\mathbf{x},\mathbf{v}),\mathcal{F}(\mathbf{x}))$ defined in (2.10.25) is a solution to the time-independent linearized magnetized Vlasov–Poisson system (2.10.24) in $\mathcal{T}^3\times\mathbb{R}^3$, with $\mathcal{F}\in C^\infty(\mathcal{T}^3)$, and where the function h can be taken either with l continuous derivatives with respect to $\mathbf{v}, l=1,2,\ldots,$ or infinitely differentiable with respect to $\mathbf{v}$. Moreover, for each fixed $x\in\mathcal{T}^3, h\in L^1(\mathbb{R}^3)$. Further, the absolute value of h and of all its derivatives can be taken bounded by Gaussian functions of $\mathbf{v}$, uniformly in $\mathbf{x}\in\mathcal{T}^3$. Further, the solution $(\mathcal{G}(\mathbf{x},\mathbf{v}),\mathcal{F}(\mathbf{x}))$ satisfies (2.10.22).*

Proof. We detail the calculations for the convenience of the reader. One has

$$\mathbf{v}\cdot\nabla_{\mathbf{x}}\mathcal{G}(x,y,z,v_1,v_2,v_3) = v_1\partial_x h(x,v_1,v_2)\,\frac{1}{\sqrt{2\pi T_\|}}\,e^{\frac{-v_3^2}{2T_\|}},$$

$$\nabla_{\mathbf{v}}\mathcal{G}(x,y,z,v_1,v_2,v_3) = \begin{pmatrix} \partial_{v_1}h(x,v_1,v_2)\,\frac{1}{\sqrt{2\pi T_\|}}\,e^{\frac{-v_3^2}{2T_\|}} \\ \partial_{v_2}h(x,v_1,v_2)\,\frac{1}{\sqrt{2\pi T_\|}}\,e^{\frac{-v_3^2}{2T_\|}} \\ h(x,v_1,v_2)\,\partial_{v_3}\frac{1}{\sqrt{2\pi T_\|}}\,e^{\frac{-v_3^2}{2T_\|}} \end{pmatrix}, \tag{2.10.26}$$

$$\mathcal{F}(x,y,z)\cdot\nabla_{\mathbf{v}}f^0 = -F(x)\,v_1\,f^0,$$

$$\mathbf{v}\times\mathbf{B}_0 = \begin{pmatrix} B_0v_2 \\ -B_0v_1 \\ 0 \end{pmatrix},$$

$$(\mathbf{v}\times\mathbf{B}_0)\cdot\nabla_{\mathbf{v}}\mathcal{G}(x,y,z,v_1,v_2,v_3) = B_0\,(v_2\partial_{v_1} - v_1\partial_{v_2})\,h(x,v_1,v_2)\frac{1}{\sqrt{2\pi T_\|}}\,e^{\frac{-v_3^2}{2T_\|}}.$$

Therefore, by (2.10.26) one gets the first equation in the linearized magnetized Vlasov–Poisson system (2.10.24),

$$\mathbf{v}\cdot\nabla_{\mathbf{x}}\mathcal{G}(t,\mathbf{x},\mathbf{v})+\frac{q}{m}\mathcal{F}(t,\mathbf{x})\cdot\nabla_{\mathbf{v}}f^0(v)+\frac{q}{m}\mathbf{v}\times\mathbf{B}_0\cdot\nabla_{\mathbf{v}}\mathcal{G}(t,\mathbf{x},\mathbf{v})=0. \tag{2.10.27}$$

Moreover, one has $\nabla_{\mathbf{x}}\cdot\mathcal{F}(\mathbf{x})=\partial_x F(x)$, and

$$\begin{aligned}\textstyle\int_{\mathbb{R}^3}\mathcal{G}(\mathbf{x},\mathbf{v})\,d^3\mathbf{v} &= \textstyle\int_{\mathbb{R}^2}h(x,v_1,v_2)dv_1dv_2\times\int_{\mathbb{R}}\frac{1}{\sqrt{2\pi_{\|}}}\,e^{\frac{-v_3^2}{2T_{\|}}}\,dv_3\\ &= \textstyle\int_{\mathbb{R}^2}h(x,v_1,v_2)dv_1dv_2.\end{aligned}$$

So one obtains immediately the Gauss law

$$\nabla_{\mathbf{x}}\cdot\mathcal{F}(\mathbf{x})=\frac{q}{4\pi}\,\rho(t,\mathbf{x}). \tag{2.10.28}$$

The fact that (2.10.22) holds, and the properties of the solution $(\mathcal{G},\mathcal{F})$ stated in the lemma hold, follow immediately from the definition of the pair (h,F). $\square$

Then the pair $(\mathcal{G},\mathcal{F})$ is a time independent solution to the linearized magnetized Vlasov equation,

$$\mathbf{v}\cdot\nabla_x\mathcal{G}-\mathcal{F}\cdot\nabla_v(f_0\,\tau(v_3))-(\mathbf{v}\times\mathcal{B}_0)\cdot\nabla_v\mathcal{G}=0, \tag{2.10.29}$$

and of the Gauss law,

$$\nabla_x\cdot\mathcal{F}(x,y,z)=-\int_{\mathbb{R}^3}\mathcal{G}dv. \tag{2.10.30}$$

In the notation of Theorem 1 of [13]

$$\hat{\rho}(t,k_\perp,0)=\mathcal{G}(x,y,z,v_1,v_2,v_3),$$

with $k_\perp=(n,0,0)$. Moreover, $f_3^0=\tau$, and $\tilde{f}^3=0$. Further, in [13] the charge q is taken positive and we take it equal to -1. However, the charge can be changed from one to minus one by the change of coordinate $x\to -x$. Note, moreover, that since we take $f_0=f_0(v_1,v_2)=e^{-(v_1^2+v_2^2)/2}$ and in [13] $f^0=\frac{1}{2\pi}\,e^{-(v_1^2+v_2^2)/2}$, we have to multiply our solution by $\frac{1}{2\pi}$, i.e. $(\frac{1}{2\pi}\mathcal{G},\frac{1}{2\pi}\,\mathcal{F})$.

2.10.4 The unmagnetized case $\omega_c=0$

It is instructing to calculate stationary solutions of the unmagnetized Vlasov–Poisson equations

$$\begin{cases} v_1\partial_x g+Fv_1f_0=0,\\ \partial_x F=-\displaystyle\int_{\mathbb{R}^2}gdv_1dv_2,\\ \displaystyle\int_{[0,2\pi]}F=0,\end{cases} \tag{2.10.31}$$

One obtains

$$v_1\partial_x\left[g(x,v_1,v_2)-\varphi(x)f_0(v_1,v_2)\right]=0.$$

It yields the representation

$$g(x,v_1,v_2)-\varphi(x)f_0(v_1,v_2)=M(v_1,v_2)$$

where M is arbitrary. Plugging in the 2nd equation of (2.10.31), one gets the Poisson equation

$$-\varphi''(x) + 2\pi\varphi(x) = -\int M(v_1, v_2)dv_1dv_2.$$

Since the right hand side is independent of x, the solution is also independent of x

$$\varphi(x) = -\frac{1}{2\pi}\int M(v_1, v_2)dv_1dv_2.$$

Therefore the electric field vanishes

$$F(x) = -\varphi'(x) = 0.$$

Also the mass is zero

$$\int_{\mathbb{R}^2} g dv_1 dv_2 = 0.$$

Hence, for the unmagnetized case, stationary solutions are trivial.

For the solutions $(\mathcal{G}, \mathcal{F})$ to the time-independent linearized magnetized Vlasov–Poisson system (2.10.24), that fulfills (2.10.22), the Fourier coefficient $\hat{\rho}(n, 0, 0)$ is, in general, not zero, and, moreover, they satisfy the assumption of Theorem 1 of Bedrossian and Wang, [13] (see theorem 2.1.1 above). For example, we can take $g(v_1^2 + v_2^2) = e^{-v_1^2+v_2^2}$.

2.10.5 Limit $B_0 \to 0$

An interesting question is passing to the limit $B_0 \to 0$ in the right-hand side of (2.10.14). One has weak convergence to zero of the right-hand side under standard integrability conditions on g since

$$\lim_{B_0\to 0}\int_{\mathbb{R}^2} e^{in\frac{mv_2}{qB_0}} h(v_1^2 + v_2^2)dv_1dv_2 = 0$$

because of the oscillating term $e^{in\frac{mv_2}{qB_0}}$. Therefore, by (2.10.14) the solutions of (2.10.7) given by lemma 2.10.1 satisfy

$$\varphi_{B_0\to 0} = 0.$$

The weak limit recovers the classical results in the non magnetized case.

II Mathematical properties of the magnetized Vlasov–Poisson system

3 Weak solutions to the magnetized Vlasov–Poisson system

Contents

Summary

In this chapter, we present the main mathematical properties of the magnetized Vlasov–Poisson system, detailing the most important a priori estimates and listing the different types of solutions to this system. Then we give a result on the existence of weak solutions to the magnetized Vlasov–Poisson system, analogous to the result in the unmagnetized framework. Finally, we detail the approximation procedure, first introduced by Arsenev in [4], that is central in the proof of both results. This approximation procedure is also used implicitly in the proof of our results regarding propagation of velocity moments in chapter 4 and chapter 5.

3.1 Introduction

We study the Vlasov–Poisson system with an external magnetic field, which we will call the magnetized Vlasov–Poisson system for the rest of this chapter, in a 3-dimensional framework. This system describes the behavior of a cloud of charged particles subject to an external magnetic field and is given by the following set of equations:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E + v \wedge B) \cdot \nabla_v f = 0, \\ E(t,x) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} f(t,y,v) dv dy, \end{cases} \tag{3.1.1}$$

with

$$\begin{cases} f_{|t=0}(x,v) := f^{in}(x,v) \geq 0 \\ E_{|t=0}(x) := E^{in}(x) := \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} f^{in}(y,v) dv dy. \end{cases} \tag{3.1.2}$$

Throughout this chapter, we will assume that the magnetic field B verifies

$$B \in W^{1,\infty}(\mathbb{R}^3). \tag{3.1.3}$$

We also define some macroscopic quantities, namely the macroscopic particle density ρ and the electric current j:

$$\begin{cases} \rho(t,x) := \displaystyle\int_{\mathbb{R}^3} f(t,x,v)dv \text{ with } \rho_{|t=0}(x) := \rho^{in}, \\ j(t,x) := \displaystyle\int_{\mathbb{R}^3} vf(t,x,v)dv \text{ with } j_{|t=0}(x) := j^{in}. \end{cases} \tag{3.1.4}$$

3.1.1 A priori estimates

In (3.1.1), the distribution function f is determined by the Vlasov equation which is a transport equation with a divergence free vector field. Furthermore, the electric potential Φ (E derives from the electric potential $E = -\nabla_x \Phi$) satisfies a Poisson equation. Thanks to these properties, we can derive many a priori estimates for the magnetized Vlasov–Poisson system, which we list below.

1. (**Maximum principle**) If the characteristics of the system are well-defined, then we can express the solution f using f^{in} and the characteristics. In this context for all $t \geq 0$ and for a.e. $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ we have

$$0 \leq f(t,x,v) \leq \sup_{(x,v)\in\mathbb{R}^3\times\mathbb{R}^3} f^{in}(x,v). \tag{3.1.5}$$

2. (**Conservation of L^p norms**) Since $(x,v) \to (v, E + v \wedge B)$ is a divergence free vector field, we deduce that for all smooth functions $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ we have for all $t \geq 0$

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f(t,x,v))dxdv = \int_{\mathbb{R}^3\times\mathbb{R}^3} \beta(f^{in}(x,v))dxdv. \tag{3.1.6}$$

   In particular, all the L^p norms ($1 \leq p < \infty$) are conserved and letting $p \to \infty$ we also find that the L^∞ norm is conserved:

$$\|f(t)\|_p = \left\|f^{in}\right\|_p \tag{3.1.7}$$

   for all $1 \leq p \leq \infty$.

3. (**Local conservation of mass**) Taking the macroscopic density ρ, we can write the following formal calculations:

$$\partial_t \rho = -\int_{\mathbb{R}^3} v \cdot \nabla_x f + (E + v \wedge B) \cdot \nabla_v f dv = -\mathrm{div}_x\, j - \int_{\mathbb{R}^3} \mathrm{div}_v \left((E + v \wedge B)f\right)dv.$$

   the last term is the second inequality is zero which means the macroscopic quantities ρ, j verify the following continuity equation:

$$\partial_t \rho(t,x) + \mathrm{div}_x\, j(t,x) = 0. \tag{3.1.8}$$

4. (**Conservation of energy**) We multiply the Vlasov equation by $|v|^2$, integrate over $\mathbb{R}^3 \times \mathbb{R}^3$ and integrate by parts to obtain formally

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 \frac{1}{2} f dxdv = -\int_{\mathbb{R}^3\times\mathbb{R}^3} (E + v \wedge B)\cdot v f dxdv &= -\int_{\mathbb{R}^3\times\mathbb{R}^3} E\cdot v f dxdv \\
&= -\int_{\mathbb{R}^3} E\cdot\left(\int_{\mathbb{R}^3} v f dv\right) dx \\
&= -\int_{\mathbb{R}^3} E\cdot j dx \\
&= \int_{\mathbb{R}^3} \Phi(t,x)\mathrm{div}_x\, j(t,x) dx \\
&= -\int_{\mathbb{R}^3} \Phi(t,x)\partial_t \rho(t,x) dx \\
&= \int_{\mathbb{R}^3} \Phi(t,x)\partial_t \Delta_x \Phi(t,x) dx \\
&= -\int_{\mathbb{R}^3} \nabla_x\Phi(t,x)\cdot\partial_t\nabla_x\Phi(t,x) dx \\
&= -\frac{d}{dt}\int_{\mathbb{R}^3} \frac{1}{2}|E(t,x)|^2 dx.
\end{aligned}$$

Finally, we can write the conservation of energy

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(t,x,v)dxdv + \int_{\mathbb{R}^3} |E(t,x)|^2 dx = \int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f^{in} dxdv + \int_{\mathbb{R}^3} \left|E^{in}(x)\right|^2 dx. \tag{3.1.9}$$

5. (**Uniform estimates**) We start by presenting an interpolation inequality for velocity moments. Let $g := g(x,v) \geq 0$ such that $g \in L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and $\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k g(x,v)dxdv < +\infty$. Then we have for $0 \leq k' \leq k$ and $r = \frac{k+3}{k'+3}$

$$\left\| \int_{\mathbb{R}^3} |v|^{k'} g(\cdot,v)dv \right\|_r \leq c(k,k') \|g\|_\infty^{\frac{k-k'}{k+3}} \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k g(x,v)dxdv\right)^{\frac{k'+3}{k+3}}. \tag{3.1.10}$$

The proof of this inequality can be found in the appendix of chapter 4.

Now we recall the weak Young inequality, whose proof can be found in [85].

Let $1 < p,q,r < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, then for all functions $f \in L^p(\mathbb{R}^n)$, $g \in L^{q,w}(\mathbb{R}^n)$ the convolution product $f \star g = \int_{\mathbb{R}^n} f(y)g(\cdot - y)dy \in L^r(\mathbb{R}^n)$ and satisfies the weak Young inequality

$$\|f\star g\|_r \leq c\,\|f\|_p\,\|g\|_{q,w} \tag{3.1.11}$$

with $c = c(p,q,n)$ and by definition $g \in L^{q,w}(\mathbb{R}^n)$ iff g is measurable and

$$\sup_{\tau>0}\left(\tau\left(\mathrm{vol}\{x\in\mathbb{R}^n \mid |g(x)| > \tau\}\right)^{\frac{1}{q}}\right) < \infty. \tag{3.1.12}$$

Furthermore, we can define a norm on $L^{q,w}(\mathbb{R}^n)$ given by

$$\|g\|_{q,w} = \sup_{|A|<\infty} |A|^{-\frac{1}{q'}} \int_A |g(x)|\,dx. \tag{3.1.13}$$

These inequalities yield uniform estimates under certain assumptions. Indeed if we assume that $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ such that $|v|^2 f \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ then we have uniform estimates on ρ, j and E, which we now list:

$$\sup_{t \geq 0} \|\rho(t,\cdot)\|_q \leq C \tag{3.1.14}$$

with $1 \leq q \leq \frac{5}{3}$ and C independent of q. This estimate is a consequence of the conservation of L^p norms, the conservation of energy and the interpolation inequality.

$$\sup_{t \geq 0} \|j(t,\cdot)\|_q \leq C \tag{3.1.15}$$

with $1 \leq q \leq \frac{5}{4}$ and C independent of q. Just like for the previous estimate on ρ, this is a consequence of the conservation of L^p norms, the conservation of energy and the interpolation inequality.

$$\sup_{t \geq 0} \|E(t,\cdot)\|_q \leq C \tag{3.1.16}$$

with $\frac{3}{2} < q \leq \frac{15}{4}$ and C independent of q. This estimate is a consequence of the uniform estimate on ρ and the weak Young inequality.

Most of these estimates can be proved rigorously by assuming sufficient regularity on the unknowns f, E and by using the following lemma:

Lemma 3.1.1 (Conservation law, [60])**.** *Let $a \in C(\mathbb{R}_+, L^1(\mathbb{R}^d)), b \in L^1((0,T) \times \mathbb{R}^d, \mathbb{R}^d), c \in L^1((0,T) \times \mathbb{R}^d)$ such that*

$$\partial_t a + \operatorname{div}_x b = c \quad \text{in } \mathcal{D}'(\mathbb{R}_+^* \times \mathbb{R}^d), \tag{3.1.17}$$

then

$$\frac{d}{dt}\int_{\mathbb{R}^d} a(t,x)dx = \int_{\mathbb{R}^d} c(t,x)dx \quad \text{in } \mathcal{D}'(\mathbb{R}_+^*). \tag{3.1.18}$$

3.1.2 Different types of solutions

We present the different notions of solutions for the magnetized Vlasov–Poisson system.

Definition 3.1.2 (Classical solutions)**.** Let $T > 0$ and $f := f(t,x,v) \geq 0$. Then f is a classical solution to the magnetized Vlasov–Poisson system (3.1.1) if $f \in C^1([0\,,T]\times\mathbb{R}^3\times\mathbb{R}^3)$, if $E := E(t,x)$ is Lipschitz with respect to x and if f verifies (3.1.1) in the strong sense.

In [103], Pfaffelmoser proved the existence of classical solutions to the Vlasov–Poisson system ($B = 0$). Using very similar ideas, Schäffer subsequently improved this result [114]. Both these proofs relied on estimating the support in velocity of the solution to Vlasov–Poisson (while initializing the system with f^{in} smooth and with compact support). The support in velocity of the solution is controlled by $Q := Q(t)$ defined by

$$Q(t) = 1 + \sup\left\{|v| : \exists x \in \mathbb{R}^3, \tau \in [0\,,t] \text{ such that } f(\tau,x,v) \neq 0\right\}. \tag{3.1.19}$$

We give Schäffer's existence theorem [114].

Theorem 3.1.3. *Let $f^{in} \in C_c^1(\mathbb{R}^3 \times \mathbb{R}^3)$. Then the Cauchy problem for the Vlasov–Poisson system ((3.1.1) with $B = 0$) has a unique solution C^1 solution, and for any $p > \frac{33}{17}$, there exists a constant c_p such that*

$$Q(t) \leq c_p(1+t)^p. \tag{3.1.20}$$

For the magnetized Vlasov–Poisson system, we proved the existence of classical solutions in chapter 4 for a constant magnetic field and in chapter 5 for a general magnetic field by showing propagation of velocity moments, which also yields propagation of regularity.

Now we present the notion of weak solutions.

Definition 3.1.4. Let $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^2(\mathbb{R}^3 \times \mathbb{R}^3))$, f is a weak solution of the magnetized Vlasov–Poisson system if $f\,|v|^2 \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ and we have

$$\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v((E + v \wedge B)f) = 0 \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3). \tag{3.1.21}$$

The initial condition is verified in the sense of distributions, i.e. for all $\phi \in C_c^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$, the function

$$t \mapsto \iint_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)\phi(t,x)dxdv \tag{3.1.22}$$

is continuous on $\mathbb{R}_+$ and satisfies

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(0,x,v)\phi(t,x)dxdv = \iint_{\mathbb{R}^3\times\mathbb{R}^3} f^{in}(x,v)\phi(t,x)dxdv. \tag{3.1.23}$$

In [4], Arsenev proved the existence of weak solutions to the Vlasov–Poisson system with an approximation procedure. This result can easily be adapted to (3.1.1), and in the following section we present an analogous result for the magnetized Vlasov–Poisson system with B verifying (3.1.3).

Finally we present the notion of renormalized solutions to (3.1.1) developed in [42]. It was notably shown in [42] that one requires weaker conditions to have existence of such renormalized solutions compared to the assumptions demanded for the existence of weak solutions that we will present in theorem 3.2.1. This notion, which was developed to study the Vlasov–Poisson system, can easily be adapted to the magnetized Vlasov–Poisson system.

Definition 3.1.5. Let $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$, f is a renormalized solution of the magnetized Vlasov–Poisson system if $E \in L^\infty(\mathbb{R}_+, L^{\frac{3}{2},w}(\mathbb{R}^3))$ and if we have

$$\partial_t \beta(f) + \mathrm{div}_x(v\beta(f)) + \mathrm{div}_v((E + v \wedge B)\beta(f)) = 0 \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3), \tag{3.1.24}$$

for all $\beta \in C(\mathbb{R}_+)$ such that β is bounded.

In this thesis, we haven't explored the existence of such solutions for the magnetized Vlasov–Poisson system. However, in [42] DiPerna and Lions proved a result where they detailed necessary and sufficient conditions to have equivalence between weak solutions and renormalized solutions.

Theorem 3.1.6. *Let $f \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3 \times \mathbb{R}^3))$.*

1. *If f is a renormalized solution to the Vlasov–Poisson system and if $f \in L^\infty(\mathbb{R}_+, L^{p_0}(\mathbb{R}^3 \times \mathbb{R}^3))$ with $p_0 = \frac{12+3\sqrt{5}}{11}$, $F \in L^\infty(\mathbb{R}_+, L^{p_0'}(\mathbb{R}^3))$ then f is a weak solution of the Vlasov–Poisson system.*

2. *If f is a weak solution to the Vlasov–Poisson system and if $f \in L^\infty(\mathbb{R}_+, L^{p_1}(\mathbb{R}^3 \times \mathbb{R}^3))$ with $p_1 = \frac{7+\sqrt{13}}{4}$, $|v|^2 f \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ then f is a renormalized solution of the Vlasov–Poisson system.*

This theorem can be adapted to the magnetized Vlasov–Poisson system with B verifying (3.1.3), which means the results on existence of weak solutions in chapter 4 and chapter 5 imply the existence of renormalized solutions to (3.1.1).

3.2 Global existence of weak solutions

As already announced above, we now present a theorem of existence of weak solutions to the magnetized Vlasov–Poisson system (3.1.1). This result is analogous to the theorem proved by Arsenev in [4] for the Vlasov–Poisson system and we show it by adapting the method from [4].

Theorem 3.2.1. *Let $f^{in} := f^{in}(x,v)$ with $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3), f^{in} \geq 0$ a.e. and assume that*

$$\frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f^{in} dx dv + \frac{1}{2} \int_{\mathbb{R}^3} \left|E^{in}\right|^2 dx := \mathcal{E}^{in} < \infty. \tag{3.2.1}$$

Let $B := B(t,x)$ with B verifying (3.1.3).

Then there exists a global weak solution $f \in L^\infty\left(\mathbb{R}_+, L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)\right)$ to the Cauchy problem for the magnetized Vlasov–Poisson system (3.1.1). This solution satisfies

$$0 \leq f(t,x) \leq \left\|f^{in}\right\|_\infty, \text{ for a.e. } (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \text{ and } \forall t \geq 0 \tag{3.2.2}$$

together with the L^p norm conservation for all $t \geq 0$ and $1 \leq p \leq \infty$

$$\|f(t)\|_p = \left\|f^{in}\right\|_p < \infty, \tag{3.2.3}$$

and the energy bound for a.e. $t \geq 0$

$$\frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f(t,x,v) dx dv + \frac{1}{2} \int_{\mathbb{R}^3} |E(t,x)|^2 dx \leq \mathcal{E}^{in} < \infty. \tag{3.2.4}$$

Just like in [4], this result is shown by regularizing the initial data and considering an approximate magnetized Vlasov–Poisson system, which is constructed with the following procedure:

Let $\zeta \in C^\infty(\mathbb{R}^3)$ which satisfies

$$\zeta(x) = \zeta(-x) \geq 0 \text{ for } x \in \mathbb{R}^3, \quad \operatorname{supp}(\zeta) \subset B(0,1) \quad \text{and} \quad \int_{\mathbb{R}^3} \zeta(x) dx = 1. \tag{3.2.5}$$

For $\varepsilon > 0$, we define

$$\zeta_\varepsilon(x) := \varepsilon^{-3} \zeta\left(\frac{x}{\varepsilon}\right) \quad \text{and} \quad \xi_\varepsilon(x) := \varepsilon^{-6} \zeta\left(\frac{x}{\varepsilon}\right) \zeta\left(\frac{v}{\varepsilon}\right). \tag{3.2.6}$$

We also define the Green function for the Laplacian

$$\mathcal{G}_3(x) = \frac{1}{4\pi}\frac{1}{|x|}. \tag{3.2.7}$$

This allows us to write the approximate magnetized Vlasov–Poisson system

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon + v \wedge B \cdot \nabla_v f_\varepsilon = 0, \\ E_\varepsilon = (\zeta_\varepsilon \star \zeta_\varepsilon \star \mathcal{G}_3) \star \rho_\varepsilon(t,\cdot), \\ \rho_\varepsilon(t,x) = \displaystyle\int_{\mathbb{R}^3} f_\varepsilon(t,x,v)dv, \\ f_{\varepsilon|t=0} = \xi_\varepsilon \star (\mathbf{1}_{\varepsilon|\mathbf{x}|<\mathbf{1}}\mathbf{1}_{\varepsilon|\mathbf{v}|<\mathbf{1}} f^{in}) := f_\varepsilon^{in}. \end{cases} \tag{3.2.8}$$

To prove the existence of weak solutions to (3.1.1), we require an intermediate existence result for the approximate magnetized Vlasov–Poisson system.

Proposition 3.2.2. *Let $f^{in} := f^{in}(x,v)$ with $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3), f^{in} \geq 0$ a.e. and assume that*

$$\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f^{in} dxdv + \frac{1}{2}\int_{\mathbb{R}^3} \left|E^{in}\right|^2 dx := \mathcal{E}^{in} < \infty. \tag{3.2.9}$$

Let $B := B(t,x)$ with B verifying (3.1.3).

Then we have the following estimates for the initial data of (3.2.8) f_ε^{in}:

$$\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f_\varepsilon^{in} dxdv + \frac{1}{2}\int_{\mathbb{R}^3} \left|\tilde{E}_\varepsilon^{in}\right|^2 dx := \mathcal{E}_\varepsilon^{in} < \infty \tag{3.2.10}$$

with

$$\tilde{E}_\varepsilon^{in} = -\zeta_\varepsilon \star \nabla_x K_3 \star \rho_\varepsilon^{in}, \tag{3.2.11}$$

and

$$\rho_\varepsilon^{in} := \int_{\mathbb{R}^3} f_\varepsilon^{in}(x,v)dv, \tag{3.2.12}$$

there exists a unique weak solution $f_\varepsilon \in C(R_+, L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3))$ to the approximate magnetized Vlasov–Poisson system (3.2.8).

This solution satisfies

$$0 \leq f_\varepsilon(t,x) \leq \left\|f^{in}\right\|_\infty, \text{ for a.e. } (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \text{ and } \forall t \geq 0, \tag{3.2.13}$$

together with L^p norm conservation for all $t \geq 0$,

$$\|f_\varepsilon(t)\|_p = \left\|f_\varepsilon^{in}\right\|_p < \infty, \tag{3.2.14}$$

and the energy bound for all $t \geq 0$,

$$\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f_\varepsilon(t,x,v)dxdv + \frac{1}{2}\int_{\mathbb{R}^3} \left|\tilde{E}_\varepsilon(t,x)\right|^2 dx = \mathcal{E}_\varepsilon^{in} < \infty, \tag{3.2.15}$$

where

$$\tilde{E}_\varepsilon(t,\cdot) := (\zeta_\varepsilon \star \nabla_x K_3) \star \rho_\varepsilon(t,\cdot). \tag{3.2.16}$$

Thanks to this result, we can show that $f_\varepsilon \underset{\varepsilon\to 0}{\to} f$, with f solution to (3.1.1) like in theorem 3.2.1, using the various estimates presented in section 3.1 to obtain compactness.

4 The Vlasov–Poisson system with a uniform magnetic field: propagation of moments and regularity

Contents

Summary

This chapter corresponds to a work that has been published in the SIAM Journal on Mathematical Analysis [110].

We show propagation of moments in velocity for the three-dimensional Vlasov–Poisson system with a uniform magnetic field $B = (0, 0, \omega)$ by adapting the work of Lions and Perthame. The added magnetic field also produces singularities at times which are the multiples of the cyclotron period $t = \frac{2\pi k}{\omega}$, $k \in \mathbb{N}$. This result also allows us to show propagation of regularity for the solution. For uniqueness, we extend Loeper's result by showing that the set of solutions with bounded macroscopic density is a uniqueness class.

4.1 Introduction

We consider the Cauchy problem for the Vlasov–Poisson system with an external magnetic field, which is given by

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f + v \wedge B \cdot \nabla_v f = 0, \\ E(t,x) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(t,y) dy \text{ with } \rho(t,x) := \int_{\mathbb{R}^3} f(t,x,v) dv, \\ f(0,x,v) = f^{in}(x,v) \geq 0. \end{cases} \tag{4.1.1}$$

This set of equations governs the evolution of a cloud of charged particles, where $f(t,x,v)$ is the distribution function at time $t \geq 0$, position $x \in \mathbb{R}^3$, and velocity $v \in \mathbb{R}^3$. E corresponds to the self-consistent electric field and B is an external, constant, and uniform magnetic field given by

$$B = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}, \tag{4.1.2}$$

where $\omega > 0$ is the cyclotron frequency.

The unmagnetized Vlasov–Poisson system has been extensively studied with the works of Arsenev [4] for weak solutions, Okabe and Ukai in dimension 2 [97], and Bardos and Degond for small initial data [8]. In the case of general initial data in dimension 3, two main approaches have been developed. The first one is based on the study of the charateristic curves with the papers from Pfaffelmoser [103] and Schäffer [114]. The second approach, first introduced for Vlasov type equations by Lions and Perthame [87], is based on the propagation of moments of the distribution function. This has resulted in several works where similar propagation properties are shown in the case of more general systems [58] and also in the case of more general assumptions [24, 29, 99, 100].

As for the Vlasov–Poisson system with an external magnetic field, it is a system of considerable importance for the modeling of tokamak plasmas. For this reason, there exists an abundant literature on the case with a strong magnetic field, where the aim is to derive asymptotic models [18, 35, 56, 57, 63, 64] and devise numerical methods that capture this asymptotic behavior [31, 55]. The Vlasov–Poisson system with an external and homogeneous magnetic field has also been studied in the half-space and in an infinite cylinder in [115, 116].

With the external magnetic field, the first difficulty is finding an appropriate representation formula for the macroscopic density, since the characteristics are a lot more complex than in the case without a magnetic field. The second and most arduous difficulty is the existence of singularities at times $t = 0, \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \ldots$, which correspond to the cyclotron periods, when we try to control the electric field. We manage to avoid these singularities because our estimates are valid for $t \in [0\,, T_\omega]$ with $T_\omega = \frac{\pi}{\omega}$ which is independent of f^{in}. This allows us to reiterate our analysis on $[T_\omega\,, 2T_\omega]$ and so on.

Hence, in this chapter, we succeed in extending the results of [87] to the case of Vlasov–Poisson with a homogeneous external magnetic field. This is a first step to proving propagation of moments in the case of a nonhomogeneous magnetic field.

First, we detail our main result and several additional results in section 4.2. Then, in section 4.3, we continue by presenting the basic definitions and lemmas that will be necessary for the proof of our main result in section 4.4, which is the core of this work. More precisely, we will give the new representation formula for the macroscopic density in section 4.4.1 and show how we control the electric field with the magnetized characteristics in section 4.4.2. To treat the singularities that appear, we establish a Grönwall inequality on $[0\,, T_\omega]$ in section 4.4.3 and show how this leads to propagation of moments for all time in section 4.4.4. In section 4.4.5, we explore a method where we place the magnetic part of the Lorentz force in the source term, which doesn't work, but is so simple that it's still interesting to mention. We will give the proofs of our additional results in section 4.5. In particular, we will explicate a new condition on the initial data so as to obtain the boundedness of the macroscopic density. Finally, in section 4.6 we will present a few questions and extensions related to the results in this chapter.

4.2 Results

First we give some notation and definitions.

For $k \geq 0$ we denote the kth order moment density and the kth order moment in velocity of a non-negative, measurable function $f\colon \mathbb{R}^6 \to [0\,,\infty[$ by

$$m_k(f)(x) := \int |v|^k f dv \quad \text{and} \quad M_k(f) := \int m_k(f)(x) dx = \iint |v|^k f dv dx.$$

We write $\mathcal{E}(t)$ for the energy of system (4.1.1), which is given by

$$\mathcal{E}(t) := \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 f(t,x,v) dx dv + \frac{1}{2} \int_{\mathbb{R}^3} |E(t,x)|^2 dx, \tag{4.2.1}$$

and we also write $\mathcal{E}_{in} := \mathcal{E}(0)$.

Last we define the notion of solutions for the magnetized Vlasov–Poisson system (4.1.1), which is analogous to the notion of weak solutions used in Arsenev [4].

Definition 4.2.1. Let $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^2(\mathbb{R}^3 \times \mathbb{R}^3))$, f is a weak solution of the magnetized Vlasov–Poisson system if $f\,|v|^2 \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ and we have

$$\partial_t f + \operatorname{div}_x(vf) + \operatorname{div}_v((E + v \wedge B)f) = 0 \text{ in } \mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3). \tag{4.2.2}$$

4.2.1 Main result

First we present this chapter's main result: propagation of velocity moments for the Vlasov–Poisson system with an external magnetic field.

Theorem 4.2.2 (Propagation of moments)**.** *Let $k_0 > 3, T > 0, f^{in} = f^{in}(x,v) \geq 0$ a.e. with $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and assume that*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{k_0} f^{in} dx dv < \infty. \tag{4.2.3}$$

Then there exists a weak solution

$$f \in C(\mathbb{R}_+; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \cap L^\infty(\mathbb{R}_+; L^p(\mathbb{R}^3 \times \mathbb{R}^3)) \tag{4.2.4}$$

$(1 \le p < +\infty)$ *to the Cauchy problem for the Vlasov–Poisson system with magnetic field* (4.1.1) *with* B *given by* (4.1.2) *in* $\mathbb{R}^3 \times \mathbb{R}^3$ *such that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f(t,x,v)dxdv \le C < +\infty, \quad 0 \le t \le T, \tag{4.2.5}$$

for all k *such that* $0 \le k \le k_0$*, and with* $C = C(T, k, \omega, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in}, M_k(f^{in})) > 0$.

Remark 4.2.3. As said in [87], the assumptions in theorem 4.2.2 guarantee that the initial energy $\mathcal{E}_{in}$ is finite.

Remark 4.2.4. Like in the original paper, all the a priori estimates that will be presented in the proof are true for smooth solutions (C^∞ with compact support). Since our estimates depend only on $T, k, \omega, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in}, M_k(f^{in})$, it is sufficient to pass to the limit in the approximate magnetized Vlasov–Poisson system. This approximate system is obtained by applying the regularization procedure introduced by Arsenev [4] to prove the existence of weak solutions.

Let's first mention that to prove the existence of weak solutions to (4.1.1) is relatively straightforward by adapting Arsenev's work [4], even when the external magnetic field isn't homogeneous. The only requirement is to have $B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3)$.

As said above, theorem 4.2.2 is an extension of the main result in [87]. To obtain (4.2.5), we follow approximately the same strategy, which is to establish a linear Grönwall inequality on the velocity moment. First, by writing a differential inequality on the velocity moment, we realize that to obtain a Grönwall inequality on the moments, we need to control a certain norm of the electric field. To do this, we require the information gained from the Vlasov equation. Hence, by using the characteristics, we can express the macroscopic charge density with a representation formula, which will in turn allow us to control the norm of the electric field. In our case, the added magnetic field significantly complicates the characteristics and the initial proof by extension.

4.2.2 Additional results

Now we state a result regarding propagation of regularity for solutions to (4.1.1), where the initial condition is sufficiently regular. This is also an extension of a result stated in [87] to the case with a magnetic field. However, here we present this result and its proof with much more detail than in [87] by adapting section 4.5 of [60].

Theorem 4.2.5 (Propagation of regularity)**.** *Let* $h \in C^1(\mathbb{R})$ *such that*

$$h \ge 0, h' \le 0 \text{ and } h(r) = \mathcal{O}(r^{-\alpha}) \text{ with } \alpha > 3,$$

and let $f^{in} \in C^1(\mathbb{R}^3)$ *be a probability density on* $\mathbb{R}^3 \times \mathbb{R}^3$ *such that* $f^{in}(x,v) \le h(|v|)$ *for all* x, v *and which verifies*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} (1 + |v|^{k_0}) f^{in}(x,v)dxdv < \infty \tag{4.2.6}$$

with $k_0 > 6$.

Then there exists a weak solution of the Cauchy problem for the Vlasov–Poisson system with magnetic field (4.1.1) $(f, E) \in C^1(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3) \times C^1(\mathbb{R}_+ \times \mathbb{R}^3)$ *satisfying the decay estimate*

$$\sup_{(t,x)\in[0\,,T]\times\mathbb{R}^3} f(t,x,v) + |D_x f(t,x,v)| + |D_v f(t,x,v)| = \mathcal{O}(|v|^{-\alpha}) \tag{4.2.7}$$

for all $T > 0$.

Next, we state a result on the uniqueness of solutions to (4.1.1) which is a direct adaptation of Loeper's paper [88].

Theorem 4.2.6 (Uniqueness)**.** *Let* $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *be a probability density such that for all* $T > 0$

$$\|\rho\|_{L^\infty([0\,,T]\times\mathbb{R}^3)} < +\infty \tag{4.2.8}$$

then there exists at most one solution to the Cauchy problem for the Vlasov–Poisson system with magnetic field (4.1.1).

Finally, we give a proposition that allows us to build solutions with bounded macroscopic density, which is analogous to the condition given in Corollary 3 in [87].

Proposition 4.2.7. *Let* f^{in} *verify the assumptions of theorem 4.2.2 with* $k_0 > 6$ *and assume that* f^{in} *also satisfies*

$$\begin{aligned} ess\ sup\{f^{in}(y+vt,w), |y-x| \leq (R+\omega\,|v|)t^2e^{\omega t}, |w-v| \leq (R+\omega\,|v|)te^{\omega t}\} \\ \in L^\infty([0\,,T[\,\times\mathbb{R}^3_x, L^1(\mathbb{R}^3_v)) \end{aligned} \tag{4.2.9}$$

for all $R > 0$ *and* $T > 0$.

Then, the solution of (4.1.1) *verifies*

$$\rho \in L^\infty([0\,,T] \times \mathbb{R}^3_x) \tag{4.2.10}$$

for all $T > 0$.

4.3 Preliminaries

As said above, we now present some basic results necessary for the proofs. First we recall the weak Young inequality. The proof of this basic inequality can be found in [85].

Lemma 4.3.1 (Weak Young inequality)**.** *Let* $1 < p, q, r < \infty$ *with* $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$*, then for all functions* $f \in L^p(\mathbb{R}^n)$*,* $g \in L^{q,w}(\mathbb{R}^n)$ *the convolution product* $f \star g = \int_{\mathbb{R}^n} f(y)g(\cdot - y)dy \in L^r(\mathbb{R}^n)$ *and satisfies*

$$\|f \star g\|_r \leq c\,\|f\|_p\,\|g\|_{q,w} \tag{4.3.1}$$

with $c = c(p,q,n)$ *and by definition* $g \in L^{q,w}(\mathbb{R}^n)$ *iff* g *is measurable and*

$$\sup_{\tau>0}\left(\tau\left(\mathrm{vol}\,\{x\in\mathbb{R}^n \mid |g(x)| > \tau\}\right)^{\frac{1}{q}}\right) < \infty. \tag{4.3.2}$$

Furthermore, we can define a norm on $L^{q,w}(\mathbb{R}^n)$ given by

$$\|g\|_{q,w} = \sup_{|A|<\infty} |A|^{-\frac{1}{q'}} \int_A |g(x)|\, dx. \tag{4.3.3}$$

Now we present three lemmas that will be used repeatedly to prove propagation of moments in section 4.4. We begin with a fundamental velocity moment inequality detailed in lemma 4.3.2, which yields uniform estimates on the electric field that we present in lemma 4.3.3. Then we give lemma 4.3.4 which is a basic functional inequality. To lighten the presentation, we place the proofs of these lemmas in the appendix 4.7.

Lemma 4.3.2. *Let $1 \le p, q \le \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$, $0 \le k' \le k < \infty$, and $r = \frac{k+\frac{3}{q}}{k'+\frac{3}{q}+\frac{k-k'}{p}}$. If $f \in L^p_+(\mathbb{R}^6)$ with $M_k(f) < \infty$, then $m_{k'}(f) \in L^r(\mathbb{R}^3)$ and*

$$\|m_{k'}(f)\|_r \le c\, \|f\|_p^{\frac{k-k'}{k+\frac{3}{q}}} M_k(f)^{\frac{k'+\frac{3}{q}}{k+\frac{3}{q}}}, \tag{4.3.4}$$

where $c = c(k, k', p) > 0$.

Lemma 4.3.3. *The estimate*

$$\|E(t)\|_p \le C, t \in [0\,, T[\tag{4.3.5}$$

holds for $p \in]\frac{3}{2}\,, \frac{15}{4}]$ with the constant $C = C(\|f^{in}\|_1\,, \|f^{in}\|_\infty\,, \mathcal{E}_{in})$ independent of p, so that we also have the estimate

$$\|E(t)\|_{\frac{3}{2},w} \le C, t \in [0\,, T[. \tag{4.3.6}$$

Lemma 4.3.4. *For all functions $g \in L^1 \cap L^\infty(\mathbb{R}^3)$ and $h \in L^{\frac{3}{2},w}(\mathbb{R}^3)$,*

$$\int_{\mathbb{R}^3} |gh|\, dx \le 3\left(\frac{3}{2}\right)^{\frac{2}{3}} \|g\|_1^{\frac{1}{3}} \|g\|_\infty^{\frac{2}{3}} \|h\|_{\frac{3}{2},w} \tag{4.3.7}$$

Last we give a Calderón–Zygmund inequality, whose proof one can find in [50].

Lemma 4.3.5 (Calderón–Zygmund)**.** *If $\Omega \in L^q(\mathcal{S}^{d-1})$, $q > 1$ so that $\int_{\mathcal{S}^{d-1}} \Omega(\omega) dS(\omega) = 0$, we consider the tempered distribution $T = \mathrm{vp}\frac{\Omega(\frac{x}{|x|})}{|x|^d} \in \mathcal{S}'(\mathbb{R}^d)$. The operator $\phi \in \mathcal{D}(\mathbb{R}^d) \mapsto T \star \phi$ can be uniquely extended into a bounded operator on $L^p(\mathbb{R}^d)$ for $p \in]1\,, \infty[$.*

4.4 Proof of propagation of moments

As said above, we extend the main result of [87] to the case of Vlasov–Poisson with a homogeneous magnetic field. However, here, we use the same steps for the proof as in [111], where the ideas of [87] are presented.

We begin by considering k_0, T, and f^{in} that follow the assumptions of theorem 4.2.2. Then, as in [111], we can write a differential inequality on M_k with $0 \le k \le k_0$.

We differentiate M_k, and by integration by parts, a Hölder inequality, and (4.3.2) with $p = \infty, q = 1, k' = k - 1$, we obtain

$$\begin{aligned}
\left|\frac{d}{dt}M_k(t)\right| &= \left|\iint |v|^k \left(-v\cdot\nabla_x f - (E+v\wedge B)\cdot\nabla_v f\right) dvdx\right| \\
&= \left|\iint |v|^k \operatorname{div_v}\left((E+v\wedge B)f\right) dvdx\right| \\
&= \left|\iint k\,|v|^{k-2}\, v\cdot E f dvdx\right| \\
&\le \iint k\,|v|^{k-1}\, f dv\,|E|\,dx \\
&\le k\,\|E(t)\|_{k+3}\,\|m_{k-1}(f)\|_{\frac{k+3}{k+2}}
\end{aligned}$$

and finally

$$\left|\frac{d}{dt}M_k(t)\right| \le C\,\|E(t)\|_{k+3}\,M_k(t)^{\frac{k+2}{k+3}} \tag{4.4.1}$$

with $C = c(k)\,\|f(t)\|_\infty^{\frac{1}{k+3}} = C(k, \left\|f^{in}\right\|_\infty)$. The computations above are almost the same as in the original case because the magnetic part vanishes. This means that, like in the unmagnetized case, we need to control $\|E(t)\|_{k+3}$ to obtain a Grönwall inequality on M_k.

4.4.1 A representation formula for ρ

Now we turn to the next step of the proof. Following [111], we write a representation formula for the macroscopic density using the characteristics associated to the Vlasov equation. With the added magnetic field, the characteristics are much more complicated than in the unmagnetized case. This translates to a generalized representation formula for the macroscopic density.

Lemma 4.4.1. *We have the representation formula for ρ,*

$$\rho(t,x) = \underbrace{\int_v f^{in}(X^0(t),V^0(t))dv}_{=:\rho_0(t,x)} + \operatorname{div}_x \int_0^t \int_v (fH_t)\,(s, X(s;t,x,v), V(s;t,x,v))\,dvds \tag{4.4.2}$$

with $(X(s;t,x,v),V(s;t,x,v))$ the characteristics associated to the transport operator $\partial_t + v\cdot\nabla_x + v\wedge B\cdot\nabla_v$ (here $E\cdot\nabla_v f$ is considered as a source term in the Vlasov equation) which are the solution to

$$\begin{cases} \dfrac{d}{ds}\left(X(s;t,x,v),V(s;t,x,v)\right) = \left(V(s;t,x,v), \omega V_2(s;t,x,v), -\omega V_1(s;t,x,v), 0\right), \\ \left(X(t;t,x,v),V(t;t,x,v)\right) = (x,v), \end{cases} \tag{4.4.3}$$

and after resolving the system are given by

$$\begin{cases} V(s;t,x,v) = \begin{pmatrix} v_1\cos(\omega(s-t)) + v_2\sin(\omega(s-t)) \\ -v_1\sin(\omega(s-t)) + v_2\cos(\omega(s-t)) \\ v_3 \end{pmatrix}, \\ X(s;t,x,v) = \begin{pmatrix} x_1 + \frac{v_1}{\omega}\sin(\omega(s-t)) + \frac{v_2}{\omega}(1-\cos(\omega(s-t))) \\ x_2 + \frac{v_1}{\omega}(\cos(\omega(s-t))-1) + \frac{v_2}{\omega}\sin(\omega(s-t)) \\ x_3 + v_3(s-t)) \end{pmatrix} \end{cases} \tag{4.4.4}$$

with $(X^0(t), V^0(t)) = (X(0;t,x,v), V(0;t,x,v))$ *and*

$$H_t(s,x) = \begin{pmatrix} \frac{\sin(\omega(s-t))}{\omega} E_1(s,x) + \frac{\cos(\omega(s-t))-1}{\omega} E_2(s,x) \\ \frac{1-\cos(\omega(s-t))}{\omega} E_1(s,x) + \frac{\sin(\omega(s-t))}{\omega} E_2(s,x) \\ (s-t)E_3(s,x) \end{pmatrix} \tag{4.4.5}$$

with E_i *the coordinates of the electric field* E.

Proof. First, thanks to the Vlasov equation, which we see as a transport equation in x and v with source term $-E \cdot \partial_v f$, we can express f by solving the characteristics and by applying the Duhamel formula

$$f(t,x,v) = f^{in}(X^0(t), V^0(t)) - \int_0^t \mathrm{div}_v(fE)(s, X(s;t,x,v), V(s;t,x,v))ds,$$

where $(X(\cdot,t,x,v), V(\cdot,t,x,v))$ is the solution to (4.4.3), hence the expressions in (4.4.4). Now if we consider

$$G_t(s,x) = \begin{pmatrix} \cos(\omega(s-t))E_1(s,x) - \sin(\omega(s-t))E_2(s,x) \\ \sin(\omega(s-t))E_1(s,x) + \cos(\omega(s-t))E_2(s,x) \\ E_3(s,x) \end{pmatrix},$$

then

$$\begin{aligned}
&\mathrm{div}_v \int_0^t fG_t(s, X(s;t,x,v), V(s;t,x,v))ds \\
&= \int_0^t \cos(\omega(s-t))\partial_{v_1}\left(fE_1\left(s, X(s;t,x,v), V(s;t,x,v)\right)\right) \\
&- \int_0^t \sin(\omega(s-t))\partial_{v_1}\left(fE_2\left(s, X(s;t,x,v), V(s;t,x,v)\right)\right) \\
&+ \int_0^t \sin(\omega(s-t))\partial_{v_2}\left(fE_1\left(s, X(s;t,x,v), V(s;t,x,v)\right)\right) \\
&+ \int_0^t \cos(\omega(s-t))\partial_{v_2}\left(fE_2\left(s, X(s;t,x,v), V(s;t,x,v)\right)\right) \\
&+ \int_0^t \partial_{v_3}\left(fE_3\left(s, X(s;t,x,v), V(s;t,x,v)\right)\right) \\
&= \int_0^t \frac{\cos\sin}{\omega}\partial_{x_1}(fE_1) + \frac{\cos(\cos-1)}{\omega}\partial_{x_2}(fE_1) + \cos^2\partial_{v_1}(fE_1) - \cos\sin\partial_{v_2}(fE_1) \\
&+ \int_0^t -\frac{\sin^2}{\omega}\partial_{x_1}(fE_2) + \frac{\sin(1-\cos)}{\omega}\partial_{x_2}(fE_2) - \cos\sin\partial_{v_1}(fE_2) + \sin^2\partial_{v_2}(fE_2) \\
&+ \int_0^t \frac{(1-\cos)\sin}{\omega}\partial_{x_1}(fE_1) + \frac{\sin^2}{\omega}\partial_{x_2}(fE_1) + \sin^2\partial_{v_1}(fE_1) + \cos\sin\partial_{v_2}(fE_1) \\
&+ \int_0^t \frac{\cos(1-\cos)}{\omega}\partial_{x_1}(fE_2) + \frac{\cos\sin}{\omega}\partial_{x_2}(fE_2) + \cos\sin\partial_{v_1}(fE_2) + \cos^2\partial_{v_2}(fE_2) \\
&+ \int_0^t (s-t)\partial_{x_3}(fE_3) + \partial_{v_3}(fE_3) \\
&= \int_0^t \mathrm{div}_v(fE)(s, X(s), V(s))ds + \mathrm{div}_x \int_0^t (fH_t)(s, X(s), V(s))ds,
\end{aligned}$$

where in the second to last equality, $\cos = \cos(\omega(s-t))$ (same for sin), $\partial_{x_i}(fE_i)$ is always evaluated at $(s, X(s;t,x,v), V(s;t,x,v))$ (same for $\partial_{v_i}(fE_i)$), and where in the last equality we write $(X(s), V(s))$ for the characteristics instead of $(X(s;t,x,v),$
$V(s;t,x,v))$. Then we integrate with respect to v, which gives us (4.4.2). □

Remark 4.4.2. The expression of H_t and the characteristics are coherent because $H_t \underset{\omega\to 0}{\longrightarrow} -tE$ and $(X^0, V^0) \underset{\omega\to 0}{\longrightarrow} (x-tv, v)$. These expressions obtained when $\omega \to 0$ correspond to the representation formula for ρ in the unmagnetized case.

4.4.2 Control of the electric field with the characteristics

Thanks to lemma 4.4.1 which gives us a new representation formula for ρ, we can start to write the estimates to control the electric field, still following the steps from [111]. A first difficulty here is adapting the estimates to this new context. We also see the appearance of the singularities mentioned above at (4.4.17), which will be a major difficulty.

First estimates

Thanks to the representation formula (4.4.2) for ρ, $E(t,\cdot)$ is given by

$$E(t,x) = -\left(\nabla K_3 \star \rho\right)(t,x) = E^0(t,x) + \tilde{E}(t,x), \tag{4.4.6}$$

where K_3 is Green's function for the Laplacian in dimension 3 given by

$$K_3(x) = \frac{1}{4\pi}\frac{1}{|x|}, \tag{4.4.7}$$

and

$$\begin{cases} E^0(t,x) = -\left(\nabla K_3 \star \rho_0\right)(t,x), \\ \tilde{E}(t,x) = -\nabla K_3 \star \left(\operatorname{div}_x \displaystyle\int_0^t \int_v (fH_t)(s, X(s;t,x,v), V(s;t,x,v))\, dv ds\right). \end{cases} \tag{4.4.8}$$

The first term E^0 is easier to control.

Lemma 4.4.3. *We have the following estimate for E^0:*

$$\left\|E^0(t,\cdot)\right\|_{k+3} \le C(k, \left\|f^{in}\right\|_1, M_k(f^{in})). \tag{4.4.9}$$

Proof. Thanks to the weak Young inequality, we can write

$$\left\|E^0(t,\cdot)\right\|_{k+3} \le \left\|\nabla K_3\right\|_{\frac{3}{2},w} \left\|\rho_0(t,\cdot)\right\|_p \tag{4.4.10}$$

with $p = \frac{3k+9}{k+6}$. And the $\frac{3k+9}{k+6}$ norm of $\rho_0(t,\cdot)$ can in turn be controlled using lemma 4.3.2, where $k' = 0, r = \frac{3k+9}{k+6}, p = \infty, q = 1$, and with a simple change of variables

$$\left\|\rho_0(t,\cdot)\right\|_{\frac{3k+9}{k+6}} \le c\left\|f\right\|_\infty^{\frac{l}{l+3}} \left(\iint |v|^l f^{in}(X^0(t), V^0(t)) dx dv\right)^{\frac{3}{l+3}} = CM_l(0)^{\frac{3}{l+3}}$$

with $\frac{l+3}{3} = \frac{3k+9}{k+6}$.

Since $k > 3$, $\frac{l}{3} = \frac{2k+3}{k+6} \leq \frac{2k+k}{6} = \frac{k}{3}$. Hence $l \leq k$, and thanks to lemma 4.3.2 with $p = \infty, q = 1, k' = l$ we obtain $M_l(0) \leq c \left\|f^{in}\right\|_1^{\frac{k-l}{k}} M_k(0)^{\frac{l}{k}}$.

This gives us a bound on $\rho_0(t, \cdot)$,

$$\|\rho_0(t,\cdot)\|_{\frac{3k+9}{k+6}} \leq \left(c \left\|f^{in}\right\|_1^{\frac{k-l}{k}} M_k(0)^{\frac{l}{k}}\right)^{\frac{3}{l+3}} = C(k, \left\|f^{in}\right\|_1, M_k(f^{in})) \tag{4.4.11}$$

with $\frac{l+3}{3} = \frac{3k+9}{k+6}$. □

To estimate the second term $\tilde{E}$, we first notice that it can be written as

$$\sum_{q,p=1}^{3} \partial_q \partial_p K_3 \star \int_0^t \int_v f H_t dv ds$$

so that we can apply the Calderón–Zygmund inequality (lemma 4.3.5)

$$\left\|\tilde{E}(t,\cdot)\right\|_{k+3} \leq \left\| \underbrace{\int_0^t \int_v (f H_t)\,(s, X(s;t,x,v), V(s;t,x,v))\, dv ds}_{\Sigma(t,x)} \right\|_{k+3}. \tag{4.4.12}$$

To simplify the expression of Σ, we consider the classical change of variables

$$\begin{aligned} \phi(v_1, v_2, v_3) &= \begin{pmatrix} v_1 \cos(\omega(s-t)) + v_2 \sin(\omega(s-t)) \\ -v_1 \sin(\omega(s-t)) + v_2 \cos(\omega(s-t)) \\ v_3 \end{pmatrix} \\ &= V(s;t,x,v) \end{aligned}$$

as well as the change of variable in time $\alpha(s) = t - s$, so that Σ can now be written

$$\Sigma(t,x) = \int_0^t \int_v f(t-s, X^*(s,x,v), v) D(t-s, s, X^*(s,x,v)) dv ds \tag{4.4.13}$$

with

$$D\,(t,s,x) = \begin{pmatrix} -\frac{\sin(\omega s)}{\omega} E_1(t,x) + \frac{\cos(\omega s)-1}{\omega} E_2(t,x) \\ \frac{1-\cos(\omega s)}{\omega} E_1(t,x) - \frac{\sin(\omega s)}{\omega} E_2(t,x) \\ -s E_3(t,x) \end{pmatrix} \tag{4.4.14}$$

and

$$X^*(s,x,v) = \begin{pmatrix} x_1 - \frac{v_1}{\omega} \sin(\omega s) + \frac{v_2}{\omega}(\cos(\omega s) - 1) \\ x_2 + \frac{v_1}{\omega}(1 - \cos(\omega s)) - \frac{v_2}{\omega} \sin(\omega s) \\ x_3 - v_3 s \end{pmatrix}. \tag{4.4.15}$$

We first study $\sigma(s,t,x)$ defined by

$$\sigma(s,t,x) = \int_v f(t-s, X^*(s,x,v), v) D(t-s, s, X^*(s,x,v)) dv ds. \tag{4.4.16}$$

Lemma 4.4.4. *We have the following estimate for σ:*

$$\|\sigma(s,t,\cdot)\|_{k+3} \leq C\frac{\sqrt{2}}{s}\left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))}\right)^{\frac{2}{3}} M_k(t-s)^{\frac{1}{k+3}}. \tag{4.4.17}$$

Proof. Thanks to lemma 4.3.4 we obtain

$$|\sigma(s,t,x)| \leq c\,\|D(t-s,s,X^*(s,x,\cdot))\|_{\frac{3}{2},w}\,\|f\|_\infty^{\frac{2}{3}}\,\|f(t-s,X^*(s,x,\cdot),\cdot)\|_1^{\frac{1}{3}}. \tag{4.4.18}$$

Let's first look at the weak $\frac{3}{2}$-norm of $D(t-s,s,X^*(s,x,\cdot))$ in (4.4.18). In the following computations D (respectively, E) and its coordinates D_i (respectively, E_i) are always evaluated at $(t-s,s,X^*(s,x,\cdot))$ (respectively, $(t-s,X^*(s,x,\cdot))$) and $\cos = \cos(\omega s)$ (respectively, $\sin = \sin(\omega s)$).

By definition,

$$\|D\|_{\frac{3}{2},w}^2 = \sum_{i=1}^{3}\|D_i\|_{\frac{3}{2},w}^2$$

so first we estimate $\|D_1\|_{\frac{3}{2},w}^2$:

$$\begin{aligned}
\|D_1\|_{\frac{3}{2},w}^2 &\leq \frac{\sin^2}{\omega^2}\|E_1\|_{\frac{3}{2},w}^2 + \frac{(1-\cos)^2}{\omega^2}\|E_2\|_{\frac{3}{2},w}^2 + 2\frac{|\sin|\,|(1-\cos)|}{\omega^2}\|E_1\|_{\frac{3}{2},w}\|E_2\|_{\frac{3}{2},w}\\
&\leq \frac{\sin^2}{\omega^2}\|E_1\|_{\frac{3}{2},w}^2 + \frac{(1-\cos)^2}{\omega^2}\|E_2\|_{\frac{3}{2},w}^2 + \frac{(1-\cos)^2}{\omega^2}\|E_1\|_{\frac{3}{2},w}^2 + \frac{\sin^2}{\omega^2}\|E_2\|_{\frac{3}{2},w}^2\\
&= \frac{2(1-\cos)}{\omega^2}\left(\|E_1\|_{\frac{3}{2},w}^2 + \|E_2\|_{\frac{3}{2},w}^2\right).
\end{aligned}$$

The computations are the same for $\|D_2\|_{\frac{3}{2},w}^2$ so that we can write

$$\|D\|_{\frac{3}{2},w}^2 \leq \frac{4(1-\cos(\omega s))}{\omega^2}\left(\|E_1\|_{\frac{3}{2},w}^2 + \|E_2\|_{\frac{3}{2},w}^2\right) + s^2\|E_3\|_{\frac{3}{2},w}^2 \tag{4.4.19}$$

and since for all $x \in \mathbb{R}$, $2(1-\cos(x)) \leq x^2$

$$\|D\|_{\frac{3}{2},w}^2 \leq 2s^2\left(\|E_1\|_{\frac{3}{2},w}^2 + \|E_2\|_{\frac{3}{2},w}^2\right) + s^2\|E_3\|_{\frac{3}{2},w}^2 \leq 2s^2\|E\|_{\frac{3}{2},w}^2. \tag{4.4.20}$$

Now let's try to express $\|E_1(t-s,X^*(s,x,\cdot))\|_{\frac{3}{2},w}$, by definition

$$\|E_1(t-s,X^*(s,x,\cdot))\|_{\frac{3}{2},w} = \sup_{|A|<\infty}|A|^{-\frac{1}{3}}\int_A |E_1(t-s,X^*(s,x,v))|\,dv, \tag{4.4.21}$$

and if we consider the change of variables $\psi(v) = X^*(s,x,v)$, for $s>0$, whose Jacobian matrix is given by

$$\mathrm{Jac}(\psi) = \begin{pmatrix} \frac{-\sin(\omega s)}{\omega} & \frac{\cos(\omega s)-1}{\omega} & 0\\ \frac{1-\cos(\omega s)}{\omega} & \frac{-\sin(\omega s)}{\omega} & 0\\ 0 & 0 & -s \end{pmatrix}, \tag{4.4.22}$$

we can write

$$\int_A |E_1(t-s,X^*(s,x,v))|\,dv = \int_{\psi(A)} |E_1(t-s,u))|\,|\mathrm{Jac}(\psi)|^{-1}\,du.$$

So finally

$$\begin{aligned}
&\|E_1(t-s, X^*(s,x,\cdot))\|_{\frac{3}{2},w} \\
&= \sup_{|A|<\infty} |A|^{-\frac{1}{3}} \int_{\psi(A)} |E_1(t-s,u))| \, |\mathrm{Jac}(\psi)|^{-1} du \\
&= \sup_{|A|<\infty} |\psi(A)|^{-\frac{1}{3}} \Bigg(\underbrace{\frac{|A|}{|\psi(A)|}}_{=|\mathrm{Jac}(\psi)|^{-1}}\Bigg)^{-\frac{1}{3}} |\mathrm{Jac}(\psi)|^{-1} \int_{\psi(A)} |E_1(t-s,u))| \, du \\
&= \sup_{|A|<\infty} |\psi(A)|^{-\frac{1}{3}} |\mathrm{Jac}(\psi)|^{-\frac{2}{3}} \int_{\psi(A)} |E_1(t-s,u))| \, du \\
&= |\mathrm{Jac}(\psi)|^{-\frac{2}{3}} \|E_1(t-s,\cdot)\|_{\frac{3}{2},w}.
\end{aligned}$$

The computations are the same for $\|E_2(t-s, X^*(s,x,\cdot))\|_{\frac{3}{2},w}$ and $\|E_3(t-s, X^*(s,x,\cdot))\|_{\frac{3}{2},w}$ so that

$$\begin{aligned}
\|E(t-s, X^*(s,x,\cdot))\|_{\frac{3}{2},w} &= |\mathrm{Jac}(\psi)|^{-\frac{2}{3}} \|E(t-s,\cdot)\|_{\frac{3}{2},w} \\
&= \left(\frac{\omega^2}{2s(1-\cos(\omega s))}\right)^{\frac{2}{3}} \|E(t-s,\cdot)\|_{\frac{3}{2},w}.
\end{aligned} \tag{4.4.23}$$

Combining (4.4.20) and (4.4.23) we obtain the following estimate

$$\|D(t-s,s,X^*(s,x,\cdot))\|_{\frac{3}{2},w} \leq \frac{\sqrt{2}}{s} \left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))}\right)^{\frac{2}{3}} \underbrace{\|E(t-s,\cdot)\|_{\frac{3}{2},w}}_{\leq C}. \tag{4.4.24}$$

and since $\|f\|_\infty \leq C$ we have

$$|\sigma(s,t,x)| \leq C \frac{\sqrt{2}}{s} \left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))}\right)^{\frac{2}{3}} \|f(t-s, X^*(s,x,\cdot),\cdot)\|_1^{\frac{1}{3}} \tag{4.4.25}$$

so that

$$\|\sigma(s,t,\cdot)\|_{k+3} \leq C \frac{\sqrt{2}}{s} \left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))}\right)^{\frac{2}{3}} \left\| \left(\int f(t-s, X^*(s,\cdot,v), v) dv \right)^{\frac{1}{3}} \right\|_{k+3}. \tag{4.4.26}$$

Furthermore, for any function ψ we have

$$\|\psi^\alpha\|_p = \|\psi\|_{\alpha p}^\alpha \tag{4.4.27}$$

so that

$$\left\| \left(\int f(t-s, X^*(s,\cdot,v), v) dv \right)^{\frac{1}{3}} \right\|_{k+3} = \left\| \int f(t-s, X^*(s,\cdot,v), v) dv \right\|_{\frac{k+3}{3}}^{\frac{1}{3}}, \tag{4.4.28}$$

and thanks to lemma 4.3.2 with $p=\infty, q=1, k'=0, r=\frac{k+3}{3}$ we obtain the desired estimate

$$\|\sigma(s,t,\cdot)\|_{k+3} \leq C \frac{\sqrt{2}}{s} \left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))}\right)^{\frac{2}{3}} M_k(t-s)^{\frac{1}{k+3}}$$

with $C = C(k, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in})$. □

Like in the unmagnetized case, we exactly obtain the desired exponent $\frac{1}{k+3}$ on M_k in our estimate. However, as mentioned above, we also see the singularities at times $\frac{2\pi q}{\omega}, q \in \mathbb{N}$.

To deal with the singularities that stem from the added magnetic field, we notice that all our estimates depend only on k, ω, and f^{in}, which means that if we can show propagation of moments on an interval $[0\,, T_\omega]$, then we can reiterate our analysis with the new initial condition $f_1^{in} = f(T_\omega)$ and so on.

Since the singularities depend on ω, it is logical to take T_ω that also depends on ω (this also justifies the notation). As said above, we choose to take $T_\omega = \frac{\pi}{\omega}$ (in fact, we could have taken any $t \in \,]0\,, \frac{2\pi}{\omega}[$).

Now to control $\|\Sigma(t,\cdot)\|_{k+3}$ with $M_k(t)^{\frac{1}{k+3}}$ we write

$$\Sigma(t,x) := \int_0^{t_0} \cdots + \int_{t_0}^{t} \ldots, \tag{4.4.29}$$

where $t_0 \in \,]0\,, T_\omega[$. This is an idea from the original paper [87]. The interval $[0\,, t_0]$ is considered small and thus we control the large t contribution ($\int_{t_0}^{t}$) precisely (with $M_k(t)^\beta$, $\beta \leq \frac{1}{k+3}$) and the small t contribution ($\int_0^{t_0}$) less precisely (with $M_k(t)^\gamma$, $\gamma > 0$). This last imprecise estimate is compensated by the fact that we integrate on a short length segment. However, the main difference with the unmagnetized case is that now we need t_0 to be small compared to $T_\omega = \frac{\pi}{\omega}$ to deal with the singularities.

Small time estimates

First we estimate the small contribution in time, as in [111], but with the added difficulty of the singularities.

Proposition 4.4.5. *We have the following estimate for the small contribution in time:*

$$\left\| \int_0^{t_0} \sigma(s,t,\cdot)ds \right\|_{k+3} \leq C t_0^{2-\frac{3}{d}} (1+t)^{\frac{l+3}{k+3}} \left(1 + \sup_{0\leq s\leq t} M_k(s)\right)^{\frac{3(l+3)}{(k+3)^2}} \tag{4.4.30}$$

with $C = C(k, \omega, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in})$ and l is an exponent defined in the proof.

Proof. Thanks to the Hölder inequality with $\frac{1}{d} + \frac{1}{d'} = 1$, we can write

$$\begin{aligned}
&|\sigma(s,t,x)| \\
&\leq \left(\int_{\mathbb{R}^3} |D\,(t-s,s,X^*(s,x,v))|^d\, dv\right)^{\frac{1}{d}} \left(\int_{\mathbb{R}^3} f\,(t-s,X^*(s,x,v),v)^{d'}\, dv\right)^{\frac{1}{d'}} \\
&\leq \sqrt{2}s \left(\frac{\omega^2}{2s(1-\cos(\omega s))}\right)^{\frac{1}{d}} \|E(t-s,\cdot)\|_d \, \|f\|_\infty^{\frac{1}{d}} \left(\int_{\mathbb{R}^3} f\,(t-s,X^*(s,x,v),v)\, dv\right)^{\frac{1}{d'}}.
\end{aligned}$$

This last inequality is obtained by using the same method as for the estimate (4.4.20) where we can show that

$$|D\,(t-s,s,X^*(s,x,v))| \leq \sqrt{2}s\,|E\,(t-s,X^*(s,x,v))|$$

and by applying the change of variable (4.4.22). Using (4.4.27) with $\alpha = \frac{1}{d'}, p = k+3$ and lemma 4.3.2 with $p = \infty, q = 1, k' = 0, r = \frac{k+3}{d'}$, this implies

$$\begin{aligned}
&\left\| \int_0^{t_0} \sigma(s,t,\cdot) ds \right\|_{k+3} \\
&\leq C \sup_{0 \leq s \leq t} \|E(t-s,\cdot)\|_d \\
&\times \sup_{0 \leq s \leq t} \left\| \left(\int_{\mathbb{R}^3} f\left(t-s, X^*(s,x,v), v\right) dv \right) \right\|_{\frac{k+3}{d'}}^{\frac{1}{d'}} \int_0^{t_0} s \left(\frac{\omega^2}{s(1-\cos(\omega s))} \right)^{\frac{1}{d}} ds \\
&\leq C \sup_{0 \leq s \leq t} \|E(t-s,\cdot)\|_d \sup_{0 \leq s \leq t} M_l(t-s)^{\frac{1}{k+3}} \int_0^{t_0} s \left(\frac{\omega^2}{s(1-\cos(\omega s))} \right)^{\frac{1}{d}} ds,
\end{aligned}$$

where thanks to lemma 4.3.2, the new exponent l verifies $\frac{k+3}{d'} = \frac{l+3}{3}$. Furthermore, we saw in lemma 4.3.3 that the electric field is uniformly bounded in $L^d(\mathbb{R}^3)$ for $\frac{3}{2} < d \leq \frac{15}{4}$ (so $\frac{15}{11} \leq d' < 3$). This implies the following estimate, with $\frac{k+3}{d'} = \frac{l+3}{3}$ and $\frac{15}{11} \leq d' < 3$:

$$\left\| \int_0^{t_0} \sigma(s,t,\cdot) ds \right\|_{k+3} \leq C \left(\int_0^{t_0} s \left(\frac{\omega^2}{s(1-\cos(\omega s))} \right)^{\frac{1}{d}} ds \right) \sup_{0 \leq s \leq t} M_l(s)^{\frac{1}{k+3}}. \tag{4.4.31}$$

Thanks to lemma 4.7.1 we have that

$$\sup_{0 \leq s \leq t} M_l(s) \leq C(1+t)^{l+3} \left(1 + \sup_{0 \leq s \leq t} M_k(s) \right)^{\frac{3(l+3)}{k+3}}$$

so that finally we obtain

$$\begin{aligned}
&\left\| \int_0^{t_0} \sigma(s,t,\cdot) ds \right\|_{k+3} \\
&\leq C \left(\int_0^{t_0} \underbrace{s \left(\frac{\omega^2}{s(1-\cos(\omega s))} \right)^{\frac{1}{d}}}_{\zeta(s)} ds \right) (1+t)^{\frac{l+3}{k+3}} \left(1 + \sup_{0 \leq s \leq t} M_k(s) \right)^{\frac{3(l+3)}{(k+3)^2}}
\end{aligned} \tag{4.4.32}$$

with $C = C(k, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in})$.

Now we must study $\int_0^{t_0} \zeta(s) ds$ (in the case without magnetic field $I = [0, t_0]$ and $\zeta(s) = s^{1-\frac{3}{d}}$).

We have

$$\begin{aligned}
\int_0^{t_0} \zeta(s) ds &= \omega^{\frac{3}{d}-2} \int_0^{\omega t_0} s \left(\frac{1}{s(1-\cos(s))} \right)^{\frac{1}{d}} ds \\
&= \omega^{\frac{3}{d}-2} \int_0^{\omega t_0} s^{1-\frac{3}{d}} \left(\frac{s^2}{(1-\cos(s))} \right)^{\frac{1}{d}} ds.
\end{aligned}$$

Since $\omega t_0 \leq \omega t \leq \pi$, the function $s \mapsto \left(\frac{s^2}{(1-\cos(s))} \right)^{\frac{1}{d}}$ is bounded on $[0, \omega t_0]$ (independently of t_0) so that finally

$$\int_0^{t_0} \zeta(s) ds \leq C \omega^{\frac{3}{d}-2} \int_0^{\omega t_0} s^{1-\frac{3}{d}} ds \leq C \omega^{\frac{3}{d}-2} (\omega t_0)^{2-\frac{3}{d}}. \tag{4.4.33}$$

□

Large time estimates

Now we look at the large t contribution, where our hope is to get a logarithmic dependence in t_0 just like in [87, 111].

Proposition 4.4.6. *We have the following estimate for the large contribution in time:*

$$\left\| \int_{t_0}^{t} \sigma(s,t,\cdot) ds \right\|_{k+3} \leq C \ln\left(\frac{t}{t_0}\right) \sup_{0\leq s\leq t} M_k(s)^{\frac{1}{k+3}} \tag{4.4.34}$$

with $C = C(k, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in})$.

Proof. Using (4.4.17), we can write

$$\begin{aligned}\left\| \int_{t_0}^{t} \sigma(s,t,\cdot) ds \right\|_{k+3} &\leq C \sup_{0\leq s\leq t} M_k(s)^{\frac{1}{k+3}} \int_{\omega t_0}^{\omega t} \frac{1}{s}\left(\frac{s^2}{(1-\cos(s))}\right)^{\frac{2}{3}} ds \\ &\leq C \sup_{0\leq s\leq t} M_k(s)^{\frac{1}{k+3}} \int_{\omega t_0}^{\omega t} \frac{1}{s} ds\end{aligned}$$

because in the same way as above the function $s \mapsto \left(\frac{s^2}{(1-\cos(s))}\right)^{\frac{1}{d}}$ is bounded on $[\omega t_0\,, \omega t]$ (independently of t_0, t, or ω) so that finally

$$\left\| \int_{t_0}^{t} \sigma(s,t,\cdot) ds \right\|_{k+3} \leq C \ln\left(\frac{t}{t_0}\right) \sup_{0\leq s\leq t} M_k(s)^{\frac{1}{k+3}}$$

with $C = C(k, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in})$. □

4.4.3 A Grönwall inequality for $t \in [0\,, T_\omega]$

Now we try to show propagation of moments on $[0\,, T_\omega]$ by establishing a Grönwall inequality like in [87, 111]

Proposition 4.4.7. *Theorem 4.2.2 is true for* $T = T_\omega$.

Proof. First, we define

$$\mu_k(t) := \sup_{0\leq s\leq t} M_k(s). \tag{4.4.35}$$

Next combining (4.4.11), (4.4.30), and (4.4.34), we obtain the following estimate for all $t \in [0\,, T]$:

$$\begin{aligned}\|E(t,\cdot)\|_{k+3} \leq \|\rho_0(t,\cdot)\|_{\frac{3k+9}{k+6}} &+ C t_0^{2-\frac{3}{d}} (1+t)^{\frac{l+3}{k+3}} (1+\mu_k(t))^{\frac{3(l+3)}{(k+3)^2}} \\ &+ C \ln\left(\frac{t}{t_0}\right) \mu_k(t)^{\frac{1}{k+3}}.\end{aligned} \tag{4.4.36}$$

Now, as was previously announced, we can absorb the term $(1+\mu_k(t))^{\frac{3(l+3)}{(k+3)^2}}$ by choosing a small t_0 such that $t_0 < t \leq T_\omega$. We choose t_0 in a different way than what was done in [87] and [111] by using the natural variable $\frac{t}{t_0}$. Hence t_0 is defined by the following relation:

$$(\frac{t_0}{t})^{2-\frac{3}{d}}\,(1+\mu_k(t))^{\frac{3(l+3)}{(k+3)^2}} = 1 \tag{4.4.37}$$

(the exponent $2-\frac{3}{d}$ is non-negative). Thus, we automatically have the inequality $t_0 < t \leq T_\omega$.

Then we can bound the three terms in (4.4.36) so as to obtain

$$\begin{aligned} \|E(t,\cdot)\|_{k+3} &\leq C_1 + C_2 t^{2-\frac{3}{d}}(1+t)^{\frac{l+3}{k+3}} + C_3 \frac{3(l+3)}{(2-\frac{3}{d})(k+3)^2}\mu_k(t)^{\frac{1}{k+3}} \ln(1+\mu_k(t)) \\ &\leq C\,(1+\mu_k(t))^{\frac{1}{k+3}}\,(1+\ln(1+\mu_k(t))) \end{aligned} \tag{4.4.38}$$

with $C = C(T,k,\omega,\left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

So now thanks to the inequality (4.4.1) we can write

$$\frac{d}{dt}M_k(t) \leq C\,(1+\mu_k(t))\,(1+\ln(1+\mu_k(t))) \tag{4.4.39}$$

and integrating the inequality on $[0\,,t]$ we conclude that

$$M_k(t) \leq M_k(0) + C\int_0^t (1+\mu_k(s))\,(1+\ln(1+\mu_k(s)))\,ds$$

for all $t \in [0\,,T]$.

Setting $y(t) = 1+\mu_k(t)$, we have

$$0 < y(t) \leq y(0) + C\int_0^t y(s)(1+\ln y(s))ds, \tag{4.4.40}$$

thus

$$\frac{Cy(t)(1+\ln y(t))}{y(0)+C\int_0^t y(s)(1+\ln y(s))ds} \leq C(1+\ln y(t))ds, \tag{4.4.41}$$

and integrating in time gives

$$\ln\left(\frac{y(t)}{y(0)}\right) \leq \ln\left(\frac{y(0)+C\int_0^t y(s)(1+\ln y(s))ds}{y(0)}\right) \leq C\int_0^t (1+\ln y(s))ds. \tag{4.4.42}$$

Hence $t \mapsto \ln y(t)$ verifies a classical Grönwall inequality,

$$\ln y(t) \leq \ln y(0) + Ct + C\int_0^t \ln y(s)ds \leq \ln y(0) + CT + C\int_0^t \ln y(s)ds, \tag{4.4.43}$$

which implies

$$\ln y(t) \leq (\ln y(0) + CT)\exp(Ct) \Leftrightarrow y(t) \leq \exp(CT\exp(Ct))\,y(0)^{\exp(Ct)} \tag{4.4.44}$$

for all $t \in [0\,,T]$ with $C = C(T,k,\omega,\left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$. □

4.4.4 Propagation of moments for all time

We conclude the proof of theorem 4.2.2 by showing propagation of moments for all time. Since the constant C in our estimate in proposition 4.4.7 depends only on $T, k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}$, and $M_k(f^{in})$, we can reiterate the procedure on any time interval $I_p = [pT_\omega\,, (p+1)T_\omega]$. Indeed, T, k and ω are constant $\|f(t)\|_1$, and $\|f(t)\|_\infty$ are conserved in time, the energy is bounded, and $M_k(f)$ is exactly the quantity we are studying.

Proposition 4.4.8. *Theorem 4.2.2 is true for all $T > T_\omega$.*

Proof. First, we show by induction on n that for all $n \in \mathbb{N}^*$

$$y(nT_\omega) \leq \beta_{n-1}\beta_{n-2}^{\alpha_{n-1}}\beta_{n-3}^{\alpha_{n-1}\alpha_{n-2}}\dots\beta_0^{\alpha_{n-1}\alpha_{n-2}\dots\alpha_1}y(0)^{\alpha_{n-1}\dots\alpha_0} \tag{4.4.45}$$

with $\beta_p = \exp\left(C_p T \exp\left(C_p T_\omega\right)\right)$ and $\alpha_p = \exp\left(C_p T_\omega\right)$ with

$$\begin{aligned}
C_p &= C_p(T, k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f(pT_\omega))) \\
&= C_p(T, k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in})).
\end{aligned}$$

The initial case is simply a consequence of proposition 4.4.7. Proving the induction step is also easy because thanks to the induction hypothesis, $f(nT_\omega)$ verifies the assumptions of theorem 4.2.2. This means we can apply the same analysis as in the previous subsections while initializing system (4.1.1) with $f(nT_\omega)$.

Hence we obtain

$$y((n+1)T_\omega) \leq \exp\left(C_n T \exp\left(C_n t\right)\right) y(nT_\omega)^{\exp(C_n t)} \tag{4.4.46}$$

with $C_n = C_n(T, k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f(nT_\omega)))$. The induction step is completed by writing $\beta_n = \exp\left(C_n T \exp\left(C_n t\right)\right)$ and $\alpha_n = \exp\left(C_n t\right)$ and by applying the induction hypothesis (4.4.45).

To conclude we consider $t \in [0\,, T]$ with $T > T_\omega$ and we write $t = (n+r)T_\omega$ with $n \in \mathbb{N}$ and $0 \leq r < 1$. Like in the induction proof above, we can apply the same analysis as in the previous section while initializing with $f(nT_\omega)$ to obtain

$$y(t) \leq \exp\left(C_n T \exp\left(C_n\left(t - \frac{n\pi}{\omega}\right)\right)\right) y\left(\frac{n\pi}{\omega}\right)^{\exp\left(C_n\left(t-\frac{n\pi}{\omega}\right)\right)}. \tag{4.4.47}$$

The proof is complete since we showed just before that we can bound $y(\frac{n\pi}{\omega})$. ☐

4.4.5 Difficulty of controlling the electric field with the magnetic field in the source term

In this section, we present a strategy for the proof of theorem 4.2.2 that does not permit us to conclude, but which is still interesting to detail because of its simplicity.

The idea is to consider the magnetic term $v \wedge B \cdot \nabla_v$ not as an added transport term in the Vlasov equation but as a source term. This allows us to write a new representation formula for the macroscopic density using the characteristics of the unmagnetized Vlasov–Poisson system.

Lemma 4.4.9. *We have a representation formula for ρ,*

$$\rho(t,x) = \rho_0(t,x) - \mathrm{div}_x \int_0^t s \int_v \left(f\left(E + v \wedge B\right)\right)(t-s, x-sv, v)\, dvds. \tag{4.4.48}$$

Proof. We use the methods of characteristics and the Duhamel formula but this time with the magnetic term in the source term, which allows us to write

$$\begin{aligned} f(t,x,v) &= f^{in}(x-tv, v) \\ &\quad - \int_0^t \left(E + v\wedge B\right)(s, x+(s-t)v) \cdot \nabla_v f(s, x+(s-t)v, v) ds \\ &= f^{in}(x-tv,v) - \int_0^t \mathrm{div}_v \left(\left(E + v\wedge B\right) f\right)(t-s, x-sv, v) ds, \end{aligned}$$

where we used the change of variable $s = t-s$ and because $\mathrm{div}_v \left(E + v\wedge B\right) = 0$.

Now we notice that

$$\begin{aligned} \mathrm{div}_v \left(\left(E + v\wedge B\right) f(t-s, x-sv, v)\right) = &-s\mathrm{div}_x \left(\left(E + v\wedge B\right) f(t-s, x-sv, v)\right) \\ &+ \mathrm{div}_v \left(\left(E + v\wedge B\right) f\right)(t-s, x-sv, v). \end{aligned}$$

Using this equality and integrating in v we obtain (4.4.48). □

Now we define

$$\begin{cases} \Sigma_E(t,x) = \displaystyle\int_0^t s \int_v E(t-s, x-sv) f\left(t-s, x-sv, v\right) dvds, \\ \Sigma_B(t,x) = \displaystyle\int_0^t s \int_v v\wedge B(t-s, x-sv) f\left(t-s, x-sv, v\right) dvds, \\ \Sigma(t,x) = \Sigma_E(t,x) + \Sigma_B(t,x). \end{cases} \tag{4.4.49}$$

Thanks to the Calderón–Zygmund inequality, to estimate the $k+3$-norm of $E(t,\cdot)$, we only need to estimate the $k+3$-norms of $\Sigma_E(t,\cdot)$ and $\Sigma_B(t,\cdot)$.

Using the exact same analysis as in [87, 111], we obtain the estimate for $\Sigma_E(t,\cdot)$ with $\mu(t)$ defined as in (4.4.35),

$$\left\|\Sigma_E(t,\cdot)\right\|_{k+3} \leq C t_0^{2-\frac{3}{d}} (1+t)^{\frac{l+3}{k+3}} \left(1+\mu_k(t)\right)^{\frac{3(l+3)}{(k+3)^2}} + C \ln\left(\frac{t}{t_0}\right) \mu_k(t)^{\frac{1}{k+3}}, \tag{4.4.50}$$

and then we choose t_0 like at (4.4.37) to obtain

$$\left\|\Sigma_E(t,\cdot)\right\|_{k+3} \leq C \left(1+\mu_k(t)\right)^{\frac{1}{k+3}} \left(1 + \ln\left(1+\mu_k(t)\right)\right), \tag{4.4.51}$$

which is a good estimate, analogous to (4.4.38).

Next we try to estimate $\left\|\Sigma_B(t,\cdot)\right\|_{k+3}$,

$$\begin{aligned} \left|\Sigma_B(t,x)\right| &= \omega \left| \int_0^t s \int_v \begin{pmatrix} v_2 \\ -v_1 \\ 0 \end{pmatrix} f\left(t-s, x-sv, v\right) dvds \right| \\ &\leq \omega \int_0^t s \int_v |v| \, f\left(t-s, x-sv, v\right) dvds = \omega \int_0^t s m_1(f\left(t-s, x-s\cdot, \cdot\right)) ds, \end{aligned}$$

so that

$$\|\Sigma_B(t,\cdot)\|_{k+3} \leq \omega \int_0^t sds \sup_{0\leq s\leq t} \|m_1(f(t-s,x-s\cdot,\cdot))\|_{k+3}$$
$$= \omega t^2 \sup_{0\leq s\leq t} \|m_1(f(t-s,x-s\cdot,\cdot))\|_{k+3}\,.$$

Unfortunately, $\|m_1(t)\|_{k+3}$ can't be controlled by $M_k(t)^\alpha$ because when we apply lemma 4.3.2 with $p=\infty, q=1, k'=1$ (which is the optimal case) we obtain

$$\|m_1(t)\|_{k+3} \leq c\,\|f\|_\infty^{\frac{l-1}{l+3}} M_l(t)^{\frac{4}{l+3}} \tag{4.4.52}$$

with $k+3 = \frac{l+3}{4}$, which implies $l > k$.

Indeed, its seems logical that with the added v in the magnetic part of the Lorentz force, controlling Σ_B requires a velocity moment of higher order than with Σ_E. Thus $\|\Sigma_B(t,\cdot)\|_{k+3}$ can't be controlled with $M_k(t)$, which means we can't deduce a Grönwall inequality on $M_k(t)$ with this method.

4.5 Proof of additional results

4.5.1 Proof of propagation of regularity

First we begin by presenting the proof of the propagation of regularity. Here we directly adapt subsection 4.5 of [60].

Remark 4.5.1. The mass conservation and the energy bound can be directly deduced from the assumptions of theorem 4.2.5

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} f(t,x,v)dxdv = \mathcal{M}^{in} = \iint_{\mathbb{R}^3\times\mathbb{R}^3} f^{in}dxdv < \infty, \tag{4.5.1}$$

$$\frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(t,x,v)dxdv + \frac{1}{2}\int_{\mathbb{R}^3} |E(t,x)|^2 dx \leq \mathcal{E}_{in} < \infty \tag{4.5.2}$$

for a.e. $t \geq 0$.

Proof. First step: L^∞ bound for E. Even though this step is the same in both magnetized and unmagnetized cases, in an effort to be clear, we detail the proof here.

Thanks to theorem 4.2.2, we have

$$M_k(t) \leq C_T \tag{4.5.3}$$

for all $0 \leq k \leq k_0$ and all $t \in [0\,,T]$. By lemma 4.3.2 with $p=\infty, q=1, k'=0, k=k_0$, we obtain

$$\|\rho(t)\|_{\frac{k_0+3}{3}} \leq C_T \tag{4.5.4}$$

for all $t \in [0\,,T]$.
Then we write

$$E(t) = -\nabla_x K_3 \star \rho(t) = -\left(\mathbb{1}_{B(0,1)}\nabla_x K_3\right)\star\rho(t) - \left(\mathbb{1}_{B(0,1)^c}\nabla_x K_3\right)\star\rho(t) := I + J \tag{4.5.5}$$

where

$$\mathbb{1}_{B(0,1)}\nabla_x K_3 = \mathcal{O}(|x|^{-2}\,\mathbb{1}_{|x|<1}) \in L^m(\mathbb{R}^3) \text{ for } 1 \leq m \leq \frac{3}{2} \tag{4.5.6}$$

and

$$\mathbb{1}_{B(0,1)^c}\nabla_x K_3 = \mathcal{O}(|x|^{-2}\,\mathbb{1}_{|x|\geq 1}) \in L^\infty(\mathbb{R}^3) \tag{4.5.7}$$

so that we conclude (this time thanks to the classical Young inequality)

$$\|I\|_\infty \leq \|\rho(t)\|_{\frac{k_0+3}{3}} \left\|\mathbb{1}_{B(0,1)}\nabla_x K_3\right\|_{\frac{k_0+3}{k_0}} \leq C_1 \|\rho(t)\|_{\frac{k_0+3}{3}} \tag{4.5.8}$$

and

$$\|J\|_\infty \leq \|\rho(t)\|_1 \left\|\mathbb{1}_{B(0,1)^c}\nabla_x K_3\right\|_\infty \leq C_2 \|\rho(t)\|_1 \,. \tag{4.5.9}$$

This finally implies

$$\|E(t)\|_\infty \leq C_1 C_T + C_2 \mathcal{M}^{in}. \tag{4.5.10}$$

Second step: L^∞ bound for ρ. We seek to show an inequality of the type

$$f(t,x,v) \leq h(|v| - A_T t) \tag{4.5.11}$$

for all $t \in [0\,,T]$.

And so we compute

$$\frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3} (f(t,x,v) - h(|v| - A_T t))_+ dxdv. \tag{4.5.12}$$

First we can write

$$\begin{aligned}
&\partial_t (f(t,x,v) - h(|v| - A_T t))_+ \\
&= (\partial_t f + h'(|v| - A_T t)A_T)\mathbb{1}_{f(t,x,v)\geq h(|v|-A_T t)} \\
&= (-v\cdot\nabla_x f - (E + v\wedge B)\cdot\nabla_v f + h'(|v| - A_T t)A_T)\mathbb{1}_{f(t,x,v)\geq h(|v|-A_T t)} \\
&= -v\cdot\nabla_x (f(t,x,v) - h(|v| - A_T t))_+ \\
&- (E + v\wedge B)\cdot\nabla_v (f(t,x,v) - h(|v| - A_T t))_+ \\
&+ h'(|v| - A_T t)\left(A_T - \underbrace{(E + v\wedge B)\cdot\frac{v}{|v|}}_{=E\cdot\frac{v}{|v|}}\right)\mathbb{1}_{f(t,x,v)\geq h(|v|-A_T t)}
\end{aligned}$$

so finally we obtain

$$\begin{aligned}
&\frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3} (f(t,x,v) - h(|v| - A_T t))_+ dxdv \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3} h'(|v| - A_T t)\left(A_T - E\cdot\frac{v}{|v|}\right)\mathbb{1}_{f(t,x,v)\geq h(|v|-A_T t)} dxdv.
\end{aligned} \tag{4.5.13}$$

We now choose $A_T = \|E\|_\infty \Rightarrow A_T - E(t,x)\cdot\frac{v}{|v|} \leq 0$ a.e., and since $h' \leq 0$ then we have

$$\frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3} (f(t,x,v) - h(|v| - A_T t))_+ dxdv \leq 0. \tag{4.5.14}$$

So the condition

$$f^{in}(x,v) \leq h(|v|) \tag{4.5.15}$$

implies that

$$f(t,x,v) \leq h(|v| - A_T t). \tag{4.5.16}$$

Since w in non-increasing, this gives us the L^∞ bound on ρ

$$\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3)} \leq R_T \tag{4.5.17}$$

Third step: Bound for $D_{x,v}f$. We set

$$L(t) := \|D_x f(t)\|_\infty + \|D_v f(t)\|_\infty \tag{4.5.18}$$

and differentiate the Vlasov equation in x and v to obtain

$$(\partial_t + v\cdot\nabla_x + (E + v\wedge B)\cdot\nabla_v)\begin{pmatrix} D_x f \\ D_v f\end{pmatrix} = \begin{pmatrix} 0 & D_x E(t,x)^T \\ I & D_v(v\wedge B(t,x))\end{pmatrix}\begin{pmatrix} D_x f \\ D_v f\end{pmatrix}$$

with

$$D_v(v\wedge B(t,x)) = \begin{pmatrix} 0 & -B_3(t,x) & B_2(t,x) \\ B_3(t,x) & 0 & -B_1(t,x) \\ -B_2(t,x) & B_1(t,x) & 0 \end{pmatrix} =: A(t,x) \tag{4.5.19}$$

so that

$$(\partial_t + v\cdot\nabla_x + (E+v\wedge B)\cdot\nabla_v)(|D_x f| + |D_v f|) \leq (1 + |D_x E(t,x)| + |A(t,x)|)(|D_x f| + |D_v f|). \tag{4.5.20}$$

Then setting

$$J(t) := \int_0^t (1 + \|D_x E(s)\|_\infty + \|A(s)\|_\infty) ds \tag{4.5.21}$$

we have

$$\begin{aligned}&(\partial_t + v\cdot\nabla_x + (E + v\wedge B)\cdot\nabla_v)\Big((|D_x f| + |D_v f|)e^{-J(t)}\Big) \\ &\leq (|D_x f| + |D_v f|)e^{-J(t)}(|D_x E(t,x)| + |A(t,x)| - \|D_x E(t)\|_\infty - \|A(t)\|_\infty) \leq 0.\end{aligned}$$

By the maximum principle we thus have

$$(|D_x f| + |D_v f|)e^{-J(t)} \leq (\|D_x f(0)\|_\infty + |D_v f(0)|)e^{-J(0)} = L(0) \tag{4.5.22}$$

and finally

$$L(t) \leq L(0)e^{J(t)}. \tag{4.5.23}$$

Fourth step: Bound for $D_x E$. Like in the unmagnetized case, thanks to an extension of the Calderón-Zygmund inequality, we can bound $D_x E(t)$.

First we write

$$D_x E(t) = \nabla_x^2 K_3 \star \rho(t), \tag{4.5.24}$$

and estimate $D_x E$ thanks to the following lemma, which extends the Calderón-Zygmund inequality to the case $p = \infty$.

Lemma 4.5.2. *Let $\Omega \in C(\mathcal{S}^2)$ such that $\int_{\mathcal{S}^2} \Omega(y) d\sigma(y=0)$ and let*

$$G = \mathrm{vp}\ \frac{\Omega(\frac{x}{|x|})}{|x|^3}. \tag{4.5.25}$$

Then for all $\varphi \in L^\infty(\mathbb{R}^3)$

$$\|G \star \varphi\| \leq \|\Omega\|_\infty \left(\left|\mathcal{S}^2\right| + \|\varphi\|_1 + \left|\mathcal{S}^2\right| \|\varphi\|_\infty \ln\left(1 + \|D_x\varphi\|_\infty\right)\right). \tag{4.5.26}$$

Applying this lemma, we obtain

$$\|D_x E(t)\|_\infty \leq C\left(1 + \ln\left(1 + \|D_x\rho(t)\|_\infty\right)\right). \tag{4.5.27}$$

Fifth step: Bound for $D_x\rho$. Recall that

$$\left(\partial_t + v\cdot\nabla_x + (E + v\wedge B)\cdot\nabla_v\right)\left((|D_x f| + |D_v f|)e^{-J(t)}\right) \leq 0 \tag{4.5.28}$$

Setting $g(t,x,v) := |D_x f| + |D_v f|)e^{-J(t)}$ we have

$$\begin{aligned}
&\frac{d}{dt}\iint_{\mathbb{R}^3\times\mathbb{R}^3} (g(t,x,v) - h(|v| - A_T t))_+ dxdv \\
&\leq \iint_{\mathbb{R}^3\times\mathbb{R}^3} w'(|v| - A_T t)\left(A_T - E\cdot\frac{v}{|v|}\right)\mathbb{1}_{g(t,x,v)\geq h(|v|-A_T t)} dxdv \leq 0.
\end{aligned}$$

so that

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} (g(t,x,v) - h(|v| - A_T t))_+ dxdv \leq 0 \tag{4.5.29}$$

and since $(f^{in} + \left|D_x f^{in}\right| + \left|D_v f^{in}\right|)(x,v) \leq h(|v|)$ for all $(x,v) \in \mathbb{R}^3\times\mathbb{R}^3$, we deduce that

$$(|D_x f| + |D_v f|)(t,x,v) \leq e^{J(t)} h(|v| - A_T t). \tag{4.5.30}$$

Integrating in v we obtain

$$|D_x\rho(t,x)| \leq e^{J(t)} \int_{\mathbb{R}^3} h(|v| - A_T t) dv \leq R_T e^{J(t)}. \tag{4.5.31}$$

So, like in the unmagnetized case, we have the following bound

$$|D_x\rho(t,x)| \leq R_T e^{J(t)} \tag{4.5.32}$$

for all $t \in [0\,,T]$ and a.e. $x \in \mathbb{R}^3$.

Sixth step: Last estimate. First, let's mention that $A \in L^\infty([0\,,T]\times\mathbb{R}^3)$ because $B \in L^\infty([0\,,T]\times\mathbb{R}^3)$.

$$\begin{aligned}
J(t) &= \int_0^t (1 + \|D_x E(s)\|_\infty + \|A(s)\|_\infty) ds \\
&\leq T + \int_0^t C\left(1 + \ln\left(1 + \|D_x\rho(s)\|_\infty\right)\right) + \|A(s)\|_\infty\, ds \\
&\leq T(1 + C + \|A\|_\infty) + \int_0^t C \underbrace{\ln\left(1 + R_T e^{J(s)}\right)}_{\leq \ln\left((1+R_T)e^{J(s)}\right)} ds \\
&\leq T(1 + C + \|A\|_\infty) + C_T T\ln(1 + R_T) + C_T \int_0^t J(s) ds.
\end{aligned}$$

Thanks to the Grönwall inequality

$$J(t) \leq T(1 + C + \|A\|_\infty + C_T \ln(1 + R_T))e^{TC_T}. \tag{4.5.33}$$

Thus we obtain the three following estimates:

$$\|D_x\rho(t)\|_\infty \leq R_T \exp(T(1 + C + \|A\|_\infty + C_T \ln(1 + R_T))e^{TC_T}) = R'_T, \tag{4.5.34}$$

$$\|D_x E(t)\|_\infty \leq C_T(1 + \ln(1 + R'_T)), \tag{4.5.35}$$

and

$$L(t) \leq L(0)\exp(T(1 + C + \|A\|_\infty + C_T \ln(1 + R_T))e^{TC_T}). \tag{4.5.36}$$

□

4.5.2 Proofs regarding uniqueness

Now we turn to the proof of theorem 4.2.6, which is a direct adaptation of Loeper's paper [88].

Proof. To prove our theorem, we only need to adapt subsection 3.2 from [88]. Thus we consider two solutions of (4.1.1) f_1, f_2 with initial datum f_0. We write the corresponding densities, electric fields, and characteristics ρ_1, ρ_2, E_1, E_2, and Y_1, Y_2. We define the following quantity Q:

$$Q(t) = \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} f_0(x,v)\,|Y_1(t,x,v) - Y_2(t,x,v)|^2\,dxdv. \tag{4.5.37}$$

Now we only need to differentiate Q

$$\begin{aligned}
\dot{Q}(t) &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} f_0(x,v)(Y_1(t,x,v) - Y_2(t,x,v))\cdot\partial_t(Y_1(t,x,v) - Y_2(t,x,v))dxdv \\
&= \iint_{\mathbb{R}^3\times\mathbb{R}^3} f_0(x,v)(X_1(t,x,v) - X_2(t,x,v))\cdot(V_1(t,x,v) - V_2(t,x,v))dxdv \\
&+ \iint_{\mathbb{R}^3\times\mathbb{R}^3} f_0(x,v)(V_1(t,x,v) - V_2(t,x,v))\cdot(E_1(t,X_1) - E_2(t,X_2))dxdv \\
&+ \iint_{\mathbb{R}^3\times\mathbb{R}^3} f_0(x,v)(V_1(t,x,v) - V_2(t,x,v))\cdot((V_1(t,x,v) - V_2(t,x,v))\wedge B)dxdv
\end{aligned}$$

We notice that the last term is null.

Using the analysis from [88], we conclude that

$$\frac{d}{dt}Q(t) \leq CQ(t)\left(1 + \ln\frac{1}{Q(t)}\right) \tag{4.5.38}$$

and thus $Q(0) = 0 \Rightarrow Q(t) = 0$ for all $t \geq 0$. □

Last, we detail the proof of proposition 4.2.7.

Proof. Like in Corollary 3 of [87], with $k_0 > 6$, we have sufficient regularity on E to consider the weak characteristics associated to system (4.1.1). Hence the solution to (4.1.1) is given by

$$f(t,x,v) = f^{in}(X^0(t), V^0(t)),$$

where $X^0(t), V^0(t) = (X(0;t,x,v), V(0;t,x,v))$, and we have

$$\dot{X}(s;t,x,v) = V(s;t,x,v), \quad \dot{V}(s;t,x,v) = E(s, X(s;t,x,v)) + V(s;t,x,v) \wedge B$$

with $X(t;t,x,v) = x$ and $V(t;t,x,v) = v$. To simplify things, we write $X(s)$ and $V(s)$ for the characteristics. Since $k_0 > 6$, we can show that E is bounded on $[0\,,T] \times \mathbb{R}^3$ so that we can write for $s \in [0\,,t]$ (using the same notation as in [87])

$$\begin{aligned}
|v - V(s)| &\le R(t-s) + \omega \int_s^t |V(u)|\, du \\
&\le R(t-s) + \omega \int_s^t |V(u) - v|\, du + \omega \int_s^t |v|\, du \\
&\le (R + \omega\,|v|)(t-s)\exp((t-s)\omega) \\
&\le (R + \omega\,|v|)t\exp(t\omega),
\end{aligned}$$

where the inequality between lines 2 and 3 is obtained thanks to the basic Grönwall inequality. Hence we can now write

$$|x + vt - X(0)| \le (R + \omega\,|v|)t^2 \exp(t\omega)$$

so that we obtain

$$f(t,x,v) \le \sup\{f^{in}(y+vt, w), |y-x| \le (R+\omega\,|v|)t^2 e^{\omega t}, |w-v| \le (R+\omega\,|v|)te^{\omega t}\}. \quad (4.5.39)$$

The condition (4.2.9) is deduced from this inequality in the same way as in [87] and implies that ρ is bounded. □

4.6 Remarks

There are several extensions and variants that would be interesting to explore in relation to this work.

The first extension that was already mentioned is to consider a nonhomogeneous magnetic field. For example, if we take a smooth B, this means that the characteristics associated to the Vlasov equation without the nonlinearity $\nabla_v f \cdot E$ are well defined. However, these characteristics cannot be written explicitly like (4.4.4), which makes it impossible to obtain a representation formula for ρ like (4.4.2). A solution would be to explore the method developed in [58]. In this paper, with an added space-moment hypothesis, the authors proved propagation of velocity moments of order $k > 2$ by splitting the Lorentz force into two parts, a long range part and a short range part. In our framework, we would of course split the magnetic part of the Lorentz force $v \wedge B$ into two parts and we would expect it to work for any bounded external magnetic field $B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3)$.

A second extension would be to adapt Castella's paper [24] on space moments to the magnetized Vlasov–Poisson system in order to consider solutions of (4.1.1) with infinite kinetic energy.

Finally, in the context of magnetic confinement, following [56], it seems reasonable to think that the propagation of velocity moments is verified when $\omega \to +\infty$. One would need to show that the condition (4.2.6) is propagated to the homogenized system obtained when $\omega \to +\infty$ and given by

$$\begin{cases} \partial_t f + v_\parallel \cdot \nabla_x f + E_\parallel \cdot \nabla_v f = 0, \\ E(t,x) = \dfrac{1}{4\pi} \displaystyle\int_{\mathbb{R}^3} \frac{x-y}{|x-y|^3} \rho(t,y) dy, \\ f_{|t=0} = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f^{in}(x, u(v,\tau)) d\tau, \end{cases} \tag{4.6.1}$$

where for any constant vector $\mathcal{M}$, $v_\parallel = (v \cdot \mathcal{M})\mathcal{M}$ (same for $E_\parallel$) and $u(v,\tau)$ is the rotation of angle τ around $\mathcal{M}$ applied to v. In our framework, we have of course $\mathcal{M} = (0,0,1)^T$.

4.7 Appendix

We begin by detailing the proofs of the three preliminary lemmas in section 4.3.

First we present the fundamental moment inequality.

Proof of lemma 4.3.2. The proof of lemma 4.3.2 is conducted using a very classical procedure, where we split the v-integral defining $m_{k'}(f)$ into small and large v's and optimize with respect to the splitting parameter.

$$\begin{aligned} m_{k'}(f)(x) &\le \int_{|v|\le R} |v|^{k'} f dv + \int_{|v|>R} |v|^{k'} f dv \\ &\le \|f(x,\cdot)\|_p \left(\int_{|v|\le R} |v|^{k'q} dv \right)^{\frac{1}{q}} + R^{k'-k} \int_{|v|>R} |v|^k f dv \\ &\le c \|f(x,\cdot)\|_p R^{k'+\frac{3}{q}} + R^{k'-k} m_k(f)(x) \end{aligned}$$

because $\int_{|v|\le R} |v|^{k'q} dv = \left(\frac{4\pi}{k'q+3}\right)^{\frac{1}{q}} R^{k'+\frac{3}{q}}$. Then we minimize the above quantity with respect to R which gives us,

$$R_{\min} = c \left(\frac{m_k(f)(x)}{\|f(x,\cdot)\|_p} \right)^{\frac{1}{k+\frac{3}{q}}}$$

The proof is finished by elevating the last inequality to the power r, integrating over x and applying a Hölder inequality once again. $\square$

This gives us the uniform estimates on the electric field.

Proof of lemma 4.3.3. First, since C in (4.3.5) is independent of p, the estimate

$$\tau^p \mathrm{vol}\left\{ x \in \mathbb{R}^3 \mid |E(t,x)| > \tau \right\} \le C^p, \tau > 0$$

holds for $p \in]\frac{3}{2}, \frac{15}{4}]$ and so also in the limiting case $p = \frac{3}{2}$, so that

$$\|E(t)\|_{\frac{3}{2},w} \le C, t \in [0, T[.$$

The estimate (4.3.5) is obtained by observing that $E(t) = -\nabla_x K_3 \star \rho$ is a convolution product in x, by the weak Young inequality (4.3.1) and by the fact $\rho \in L^{\frac{5}{3}}(\mathbb{R}^3) \cap L^1(\mathbb{R}^3)$ ($\rho \in L^{\frac{5}{3}}$ follows from lemma 4.3.2 with $k' = 0, k = 2, p = \infty, q = 1$). Let's detail this to be clear, thanks to (4.3.1) with $-\nabla_x K_3 \in \left(L^{\frac{3}{2},w}(\mathbb{R}^3)\right)^3$ and $\rho \in L^p(\mathbb{R}^3)$ for all $p \in [1, \frac{5}{3}]$, we get

$$\|E(t)\|_r \le \|\nabla_x K_3\|_{\frac{3}{2},w} \|\rho(t)\|_p \tag{4.7.1}$$

so that $E(t) \in L^r(\mathbb{R}^3)$ for all $r = \frac{3p}{3-p}$ with $p \in]1, \frac{5}{3}]$, hence for all $r \in]\frac{3}{2}, \frac{15}{4}]$. Finally, the constant in (4.3.5) is independent of time because for $0 < \alpha \le 1$ such that $p = \frac{1-\alpha}{1} + \frac{\alpha}{\frac{5}{3}}$

$$\|\rho(t)\|_p \le \|\rho(t)\|_1^{1-\alpha} \|\rho(t)\|_{\frac{5}{3}}^{\alpha} \le c \|\rho(t)\|_1^{1-\alpha} \|f(t)\|_\infty^{\frac{2\alpha}{5}} M_2(t)^{\frac{3\alpha}{5}} = C(\|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in}) \tag{4.7.2}$$

□

And last we give the proof of the basic functional inequality (4.3.7).

Proof of lemma 4.3.4. When we consider the layer cake representation of g (see [85, theorem 1.13]), we can write

$$\begin{aligned}
\int_{|h|>\tau} |h|\, dx &= \int_{|h|>\tau} \int_{t=0}^{\infty} \mathbb{1}_{\{|h|>t\}}(x) dt dx \\
&= \int_{t=0}^{\tau} \int_{|h|>\tau} \mathbb{1}_{\{|h|>t\}}(x) dx dt + \int_{t=\tau}^{\infty} \int_{|h|>\tau} \mathbb{1}_{\{|h|>t\}}(x) dx dt \\
&= \tau \mathrm{vol}\,(|h| > \tau) + \int_{t=\tau}^{\infty} \mathrm{vol}\,(|h| > t)\, dt \\
&= \tau^{\frac{3}{2}} \frac{\mathrm{vol}\,(|h| > \tau)}{\tau^{\frac{1}{2}}} + \int_{t=\tau}^{\infty} t^{\frac{3}{2}} \frac{\mathrm{vol}\,(|h| > t)}{t^{\frac{3}{2}}} dt \\
&\le \|h\|_{\frac{3}{2},w}^{\frac{3}{2}} \tau^{-\frac{1}{2}} + 2 \|h\|_{\frac{3}{2},w}^{\frac{3}{2}} \tau^{-\frac{1}{2}} = 3 \|h\|_{\frac{3}{2},w}^{\frac{3}{2}} \tau^{-\frac{1}{2}}
\end{aligned}$$

and hence

$$\int_{\mathbb{R}^3} |gh|\, dx = \int_{|g|\le\tau} |gh|\, dx + \int_{|g|>\tau} |gh|\, dx \le \tau \|g\|_1 + 3 \|h\|_{\frac{3}{2},w}^{\frac{3}{2}} \tau^{-\frac{1}{2}} \|g\|_\infty .$$

If we choose $\tau = \|h\|_{\frac{3}{2},w} \left(\frac{3}{2}\right)^{\frac{2}{3}} \left(\frac{\|g\|_\infty}{\|g\|_1}\right)^{\frac{2}{3}}$ the assertion follows. □

Finally, we present a technical estimate on the moments that was used to control the small time contribution (4.4.30). We separate this result from the main proof to lighten the presentation, but also because the proofs are identical in both magnetized and unmagnetized cases. One can find the proof of this lemma in [111, pp. 43–44]. To clarify our work, we present a more detailed version of the proof below.

Lemma 4.7.1. *Let $k > 3$ and $d' \in]\frac{3}{2}, \frac{15}{4}]$, then for l such that $\frac{k+3}{d'} = \frac{l+3}{3}$ we have the following estimate on $M_l(t)$:*

$$\sup_{0\le s\le t} M_l(s) \le C(1+t)^{l+3}\left(1+\sup_{0\le s\le t} M_k(s)\right)^{\frac{3(l+3)}{k+3}} \tag{4.7.3}$$

with $C = C(k, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in})$.

Proof. We first use the differential inequality (4.4.1)

$$\frac{d}{dt}M_l(t) \le C\|E(t)\|_{l+3} M_l(t)^{\frac{k+2}{k+3}}$$

so

$$(l+3)\frac{d}{dt}M_l(t)^{\frac{1}{l+3}} \le C\|E(t)\|_{l+3},$$

which implies

$$\begin{aligned} M_l(t) &\le \left(M_l(0)^{\frac{1}{k+3}} + \frac{C}{m+3}\int_0^t \|E(s,\cdot)\|_{l+3}\,ds\right)^{m+3} \\ &\le \left(M_l(0)^{\frac{1}{k+3}} + \frac{Ct}{m+3}\sup_{0\le s\le t}\|E(s,\cdot)\|_{l+3}\right)^{m+3}. \end{aligned}$$

This last inequality indicates that we need to control the q-norm of $E(t,\cdot)$ for any $q \ge l+3$ with $M_k(t)$, and this can be done by simply using the weak Young inequality and lemma 4.3.2 with $p=\infty, q=1, k'=0, r=\frac{k+3}{3}$,

$$\|E(s,\cdot)\|_q = \|\nabla K_3 \star \rho(t,\cdot)\|_q \le C\|\rho(t,\cdot)\|_{\frac{k+3}{3}} \le CM_k(t)^{\frac{3}{k+3}} \tag{4.7.4}$$

with $1+\frac{1}{q} = \frac{2}{3}+\frac{3}{k+3} \Rightarrow q = \frac{3k+9}{6-k}$, which implies that $k<6$. Furthermore, we want $q \ge l+3 \Leftrightarrow 6-k \le d' \in [\frac{15}{11}, 3[$ so this implies $k>3$.

Finally, with $3<k<6$, we can choose $d' \in [\frac{15}{11}, 3[$ so that l defined by $\frac{k+3}{d'} = \frac{l+3}{3}$ verifies $q \ge l+3$ ($q = \frac{3k+9}{6-k}$). With all this, the interpolation inequalities on L^p spaces allow us to write

$$\|E(s,\cdot)\|_{l+3} \le \|E(s,\cdot)\|_2^{\theta}\|E(s,\cdot)\|_q^{\theta} \tag{4.7.5}$$

$\theta \in [0,1]$.

Using this estimate and Young's classical inequality implies (4.7.3).

If $k \ge 6$, then for all $q \in]6, +\infty[$ there exists $3 < \bar{k} < 6$ such that

$$q = \frac{3\bar{k}+9}{6-\bar{k}} \quad \text{and} \quad \|E(s,\cdot)\|_q \le C\|\rho(t,\cdot)\|_{\frac{\bar{k}+3}{3}} \le CM_{\bar{k}}(t)^{\frac{3}{k+3}}. \tag{4.7.6}$$

$M_{\bar{k}}(t) < \infty$ because thanks to lemma 4.3.2 with $p=1, q=\infty, k'=\bar{k}, r = \frac{k+\frac{3}{q}}{k'+\frac{3}{q}+\frac{k-k'}{p}} = 1$ we have

$$\|m_{\bar{k}}(t)\|_r = M_{\bar{k}}(t) \le c\|f\|_1^{\frac{k-\bar{k}}{k}} M_k(t)^{\frac{\bar{k}}{k}} \le CM_k(t)^{\frac{\bar{k}}{k}} < \infty$$

for all $3 < \bar{k} < 6$.

Thus we choose $3 < \bar{k} < 6$ such that $q \geq l + 3$ with $\frac{k+3}{d'} = \frac{l+3}{3}$ as before, and now we try to estimate $\|E(s,\cdot)\|_q$ with $M_k(t)$,

$$\begin{aligned}\|E(s,\cdot)\|_q \leq C\,\|\rho(s,\cdot)\|_{\frac{\bar{k}+3}{3}} \leq C\,\|\rho(s,\cdot)\|_1^{1-\alpha}\,\|\rho(s,\cdot)\|_{\frac{k+3}{3}}^{\alpha} &\leq C\left(1 + \|\rho(s,\cdot)\|_{\frac{k+3}{3}}\right)\\ &\leq C(1+M_k(t))^{\frac{3}{k+3}}.\end{aligned}$$

This last estimate combined with the interpolation inequality (4.7.5) results in (4.7.3). $\square$

5 Propagation of velocity moments and uniqueness for the magnetized Vlasov–Poisson system

Contents

Summary

This chapter corresponds to an ongoing work that has yet to be published [109].

We study certain mathematical properties of the three-dimensional Vlasov–Poisson system in the full space with a general bounded magnetic field. First, we study the propagation of velocity moments for solutions to the system. We note that we only manage to prove propagation of velocity moments for the magnetized Vlasov–Poisson system if we assume that $B := B(t)$ is independent of the position variable x. However, we are confident that our proof can be adapted to more general magnetic fields that depend on x. We rely on Pallard's optimal result regarding the unmagnetized Vlasov–Poisson system and we combine it with an induction procedure depending on the cyclotron frequency $T_c = \frac{1}{\|B\|_\infty}$. This induction procedure, similar to the one used by the author in the case of a constant magnetic field, is necessary because we can only get satisfactory estimates on a small time scale compared to the cyclotron period. Second, we manage to extend a result by Miot regarding uniqueness for Vlasov–Poisson to the magnetized case. This result

relied heavily on the second-order structure of the Cauchy problem for the characteristics. The main difficulty in the magnetized case is that we lose this second-order structure.

5.1 Introduction

We study the Cauchy problem for the three-dimensional Vlasov–Poisson system with a general bounded magnetic field, which we will call magnetized Vlasov–Poisson system for the rest of the chapter, given by the following set of equations:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + (E + v \wedge B) \cdot \nabla_v f = 0, \\ f(0, x, v) = f^{in}(x, v) \geq 0. \end{cases} \tag{5.1.1}$$

where f^{in} is a positive measurable function and $f := f(t, x, v)$ is the distribution function of particles at time $t \in \mathbb{R}_+$, position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$. The self-consistent electric field $E := E(t, x)$ is given by:

$$E = -\nabla_x \mathcal{G}_3 * \rho, \tag{5.1.2}$$

with $\mathcal{G}_3 = \frac{1}{4\pi}\frac{1}{|x|}$ the Green function for the Laplacian and $\rho(t, x) := \int_{\mathbb{R}^3} f(t, x, v)dv$ the macroscopic particle density.

$B := B(t, x)$ is an magnetic field which will be considered bounded and lipschitz throughout this work:

$$B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3). \tag{5.1.3}$$

This system models the evolution of a set of charged particles that interact through the Coulomb force, and thus it is relevant for the study of various physical systems, most notably plasmas.

The mathematical theory for the unmagnetized ($B = 0$) Vlasov–Poisson system has been studied and developed in a great number of different works. In the three-dimensional framework, Arsenev [4] was the first to prove the existence of global weak solutions through a regularization procedure that preserves the main a priori estimates. The existence of global classical solutions for general initial data was established at the beginning of nineties in two separate works by Pfaffelmoser [103] and Lions, Perthame [87]. The first approach, extended and developed in [69, 114], relies on a very sharp study of the characteristics of the Vlasov–Poisson system while the latter approach, extended and developed in [24, 58, 100, 102, 113], is based upon the propagation of velocity moments. Even if these two approaches differ greatly, in both cases a key condition is to limit the influence of high velocities on the dynamics, by either considering initial data with compact support [103] or having a finite velocity moment of sufficiently high order ($k > 3$) [87] . More recently, Pallard combined the two approaches in [99], where he showed how to exploit the first approach by Pfaffelmoser to prove propagation of velocity moments, extending the main result of [87] by showing that this propagation property is true for moments of order $2 < k \leq 3$.

Going back to the magnetized Vlasov–Poisson system (5.1.1), the propagation of velocity moments for (5.1.1) with constant magnetic field $B = (0, 0, \omega)$ (with $\omega > 0$ the cyclotron frequency) was proven by the author in [110] by extending the method of propagation of velocity moments from [87] and adapting the uniqueness condition explicated in [87]. In order to extend the moment method, an important point in [110] was to establish a representation formula for the macroscopic density ρ. This was carried out by explicitly computing the characteristics of the transport equation

$$\partial_t f + v \cdot \nabla_x f + v \wedge B \cdot \nabla_v f = 0 \tag{5.1.4}$$

and then using the Duhamel formula, which just meant considering the Vlasov equation as a transport equation with the non-linear term $E \cdot \nabla_v f$ seen as a source term. Furthermore, in this analysis singularities at times $t = 0, \frac{2\pi}{\omega}, \frac{4\pi}{\omega}, \ldots$, which are just multiples of the cyclotron period $T_\omega = \frac{2\pi}{\omega}$, appeared in the velocity moment estimates because of the added magnetic field. This was remedied by the fact that all the estimates depended only on quantities conserved for all time and on the initial velocity moment, allowing for an induction argument to prove propagation of moments for all time. Unfortunately, in our configuration with a general magnetic field, this analysis breaks down at the first hurdle, simply because we can't explicitly compute the characteristics of (5.1.4), even with a smooth B.

The first main result of this chapter, which is the continuation of [110], is to prove the propagation of velocity moments for (5.1.1) with a general magnetic field $B := B(t)$ that is independent of the position variable. As said in the summary, we are confident that our proof can be adapted to the case of a more general magnetic field $B := B(t, x)$. For this reason, we present this conjecture and precise that we only succeed in proving it in the case of a magnetic field independent of x. We manage to obtain this result by combining Pallard's method [99] with an induction argument using the cyclotron period similar to the one in [110]. However, in this chapter we don't obtain explicit singularities like in [110] because we use the "Lagrangian" point of view (study of the characteristics) from [103] instead of using the "Eulerian" point of view (study of the distribution function) from [87, 110]. This means we need to work on a small time scale compared to the cyclotron period $T_c = \frac{1}{\|B\|_\infty}$ to obtain estimates analogous to those in [99]. This is due to the fact that on time scales comparable to T_c or greater than T_c, the variations of the characteristics are large and so the method in [99] fails. The second difficulty is to show that the estimates from [99] are also valid in our framework up to a certain time depending on T_c, and that these estimates depend only on quantities that don't prevent us from using the induction argument. Finally, propagation of velocity moments also implies propagation of the regularity of the initial data which means we have existence of classical solutions to (5.1.1). This result is detailed in [110] (theorem 2.5) for a constant magnetic field under additional conditions on f^{in} (f^{in} decays faster in velocity) but can be easily extended to the case of a general B.

Now we turn to results regarding uniqueness. First let's mention the major contribution by Loeper [88] who proved, for the unmagnetized Vlasov–Poisson system, that the set of solutions with bounded macroscopic density was a uniqueness class. This result was also extended to (5.1.1) for a constant B in [110] and we discuss how to prove a similar result for a general B below. Then in [95], Miot improved the result by showing uniqueness under the condition that the L^p norms of the macroscopic density grow at most linearly with respect to p. This allows for solutions with unbounded macroscopic density, more precisely with logarithmic blow-up.

This chapter's second main result is to prove that this uniqueness condition is also valid for (5.1.1), but only with added assumptions on the velocity moments of the initial data. Contrary to our result on propagation of velocity moments, we succeed in finding new uniqueness conditions even in the case of a $B := B(t, x)$ that depends on x. In [95], a key point was exploiting the second-order structure of the characteristics of the Vlasov–Poisson system. This explains why the uniqueness condition from [95] doesn't apply to the two-dimensional Euler model for incompressible fluids, which presents many similarities with Vlasov–Poisson, whereas the condition from [88] works for both models. It is simply due to the fact that the characteristics of the Euler model only verify a first-order ODE. In our case, the main difficulty is that the added magnetic field breaks the second-order structure of the Cauchy problem for the characteristics. We manage to get around this by proving that the characteristics in the magnetized case can be controlled by assuming more regularity on B and with the additional assumptions on the moments of the initial data mentioned above. With these additional assumptions, we deduce a new uniqueness condition which is actually the same as the sufficient condition imposed on the initial data to verify the uniqueness criterion in [95, theorem 1.2]

Outline of the chapter: This chapter will be organized as follows. We will finish this section by giving some notations and the classical a priori estimates satisfied by (5.1.1). In section 5.2 the main results of the chapter will be presented. Then section 5.3 will be devoted to the proof of propagation of velocity moments (5.1.1). More precisely, we will explain how we find estimates that are equivalent to those in [99] up to a time T_B depending on the cyclotron frequency and show how we can then use the same induction argument as in [110] to conclude. We finish with section 5.4 where we detail our proof of uniqueness for solutions to (5.1.1), highlighting how additional assumptions on the moments of the initial data allow us to control the added terms due to the added magnetic field in the analysis.

5.1.1 Preliminaries

Let's first detail the two main a priori bounds that we can deduce from system (5.1.1). The first bound is a direct consequence of the Vlasov equation where the coefficients are divergence-free.

$$\|f(t)\|_p = \left\|f^{in}\right\|_p \tag{5.1.5}$$

for all time t and exponents $p \in [0\,, +\infty]$.

The second bound is the conservation of the energy $\mathcal{E}(t)$ of the system, with

$$\mathcal{E}(t) := \frac{1}{2}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^2 f(t,x,v)dxdv + \frac{1}{2}\int_{\mathbb{R}^3} |E(t,x)|^2\, dx = \mathcal{E}(0) < +\infty. \tag{5.1.6}$$

Furthermore, thanks to the conservation of the energy $\mathcal{E}(t)$, we have the following bounds.

Lemma 5.1.1. *For all $t \geq 0$, we have $M_2(t) \leq C_1$ and $\|\rho(t)\|_{\frac{5}{3}} \leq C_2$ with the constants C_1, C_2 depending only on $\mathcal{E}(0), \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty$.*

We also present the standard notation for velocity moments: for any $k > 2$ and $t \geq 0$ we define:

$$M_k(t) = \sup_{0\leq s\leq t} \iint |v|^k f(s,x,v)dvdx. \tag{5.1.7}$$

As said before we will use the Lagrangian formulation detailed in [103], so we define the characteristics (X, V) of (5.1.1) which are solutions to the following Cauchy problem:

$$\begin{cases} \dfrac{d}{ds}X(s;t,x,v) = V(s;t,x,v), \\ \dfrac{d}{ds}V(s;t,x,v) = E(s,X(s;t,x,v)) + V(s;t,x,v)\wedge B(s,X(s;t,x,v)), \end{cases} \tag{5.1.8}$$

with
$$(X(t;t,x,v),V(t;t,x,v))=(x,v). \tag{5.1.9}$$

Then like in [99], we define for any $t>0$ and $\delta\in\,]0\,,t[$.

$$Q(t,\delta):=\sup\left\{\int_{t-\delta}^{t}|E(s,X(s;0,x,v))|\,ds,(x,v)\in\mathbb{R}^3\times\mathbb{R}^3\right\}. \tag{5.1.10}$$

For the unmagnetized Vlasov–Poisson system, $Q(t,\delta)$ quantifies the evolution of the characteristics on the interval $[t-\delta\,,t]$. However, in our context with the added magnetic field, $Q(t,\delta)$ will only quantify a part of the characteristic evolution.

5.2 Results

5.2.1 Propagation of moments

We now give the first main result of this section, which is the propagation of velocity moments of order $k>2$, extending theorem 1 in [99] to the magnetized Vlasov–Poisson system. A said above, our proof requires that $B:=B(t)$. However, since we are confident that this result is also true for $B:=B(t,x)$, we present this conjecture and precise that it is only true in the case $B:=B(t)$.

Conjecture 5.2.1 (Propagation of moments). *Let $k_0>2,T>0,f^{in}=f^{in}(x,v)\geq 0$ a.e. with $f^{in}\in L^1\cap L^\infty(\mathbb{R}^3\times\mathbb{R}^3)$ and assume that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3}|v|^{k_0}f^{in}dxdv<\infty. \tag{5.2.1}$$

Then there exists a weak solution

$$f\in C(\mathbb{R}_+;L^p(\mathbb{R}^3\times\mathbb{R}^3))\cap L^\infty(\mathbb{R}_+;L^p(\mathbb{R}^3\times\mathbb{R}^3)) \tag{5.2.2}$$

$(1\leq p<+\infty)$ *to the Cauchy problem for the Vlasov–Poisson system with magnetic field* (5.1.1) *in* $\mathbb{R}^3\times\mathbb{R}^3$ *such that*

$$\sup_{0\leq t\leq T}\iint_{\mathbb{R}^3\times\mathbb{R}^3}|v|^{k_0}f(t,x,v)dvdx\leq C \tag{5.2.3}$$

with C that depends only on

$$T,k_0,\|B\|_\infty,\mathcal{E}(0),\left\|f^{in}\right\|_1,\left\|f^{in}\right\|_\infty,\iint_{\mathbb{R}^3\times\mathbb{R}^3}|v|^{k_0}f^{in}dxdv. \tag{5.2.4}$$

In section 5.3, we will highlight exactly where our proof fails if we assume that B depends on x.

Remark 5.2.2. If f^{in} satisfies the assumptions of the previous theorem, then all the moments of order k such that $0\leq k<k_0$ are also propagated for the solution f, simply because of the following Hölder inequality

$$\iint|v|^k f(t,x,v)dvdx\leq\|f\|_1^{\frac{k_0-k}{k_0}}\left(\iint|v|^{k_0}f(t,x,v)dvdx\right)^{\frac{k}{k_0}} \tag{5.2.5}$$

where we use the decomposition $|v|^k f=f^{\frac{k_0-k}{k_0}}|v|^k f^{\frac{k}{k_0}}$ and the exponents $p=\frac{k_0}{k_0-k},q=\frac{k_0}{k}$.

Like in [87, 110], we assume we have smooth solutions to conduct the proof in section 5.3, and since the a priori estimates depend only on (5.2.4) we can pass to the limit in the approximate Vlasov–Poisson system first introduced in [4]. In fact, conjecture 5.2.1 will be a consequence of the main estimate in this chapter which will only hold for these smooth solutions because it is an estimate on Q (given in (5.1.10)), which isn't necessarily well-defined for functions in Lebesgue spaces. We now give this estimate on Q.

Main estimate on Q:

For all $T > 0$ we have

$$N(T) := \sup_{0\leq t\leq T} Q(t,t) \leq C, \tag{5.2.6}$$

with C that depends on the constants in (5.2.4). In the following remark, we explain how conjecture 5.2.1 is a consequence of (5.3.4).

Remark 5.2.3. The estimate on propagation of velocity moments (5.2.3) in conjecture 5.2.1 follows from the estimate on $N(T)$ (5.2.6) because we have:

$$\begin{aligned}\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f(t,x,v)dvdx &= \iint_{\mathbb{R}^3\times\mathbb{R}^3} |V(t;0,x,v)|^k f^{in}(x,v)dvdx \\ &\leq \iint_{\mathbb{R}^3\times\mathbb{R}^3} (|v| + N(T))^k \exp(kt\,\|B\|_\infty) f^{in}(x,v)dvdx \\ &\leq 2^{k-1}\exp(kt\,\|B\|_\infty)\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f^{in}(x,v)dvdx + N(T)^k \left\|f^{in}\right\|_1\right)\end{aligned}$$

The first inequality above is obtained through a Grönwall inequality on $|V(t;0,x,v)|$, indeed thanks to (5.1.8) we can write

$$V(t;0,x,v) = v + \int_0^t E(s,X(s;0,x,v))ds + \int_0^t V(s;0,x,v)\wedge B(s,X(s;0,x,v))ds \tag{5.2.7}$$

which implies

$$\begin{aligned}|V(t;0,x,v)| &\leq |v| + Q(t,t) + \|B\|_\infty \int_0^t |V(s;0,x,v)|\,ds \\ &\leq |v| + N(T) + \|B\|_\infty \int_0^t |V(s;0,x,v)|\,ds\end{aligned}$$

This is the classical Grönwall inequality which allows us to conclude that

$$|V(t;0,x,v)| \leq (|v| + N(T))\exp(t\,\|B\|_\infty). \tag{5.2.8}$$

The second inequality is just due to the fact that $2^{k-1}(1+x^k) \geq (1+x)^k$ for $x \geq 0$.

We finish this section by mentioning that, contrary to what is done in [99], we don't state any results on to the periodic case. This is simply explained by the fact that there are no results related to the existence of weak solutions to (5.1.1) in the periodic case. One possibility would be to adapt [12] to the magnetized case, and then combine the results from [99] for the periodic case and the proof of conjecture 5.2.1 to show propagation of moments in the periodic case.

5.2.2 Uniqueness

For all the uniqueness results and so also in section 5.4, we assume the same regularity on B (5.1.3).

We first mention a very important result by Loeper [88] where it was shown that the boundedness of the macroscopic density ρ was a sufficient condition for uniqueness in the Vlasov–Poisson system. This result was extended to the Vlasov–Poisson system with constant magnetic field in [110] and the proof can be adapted to the case of a general magnetic field verifying (5.1.3). However, in the magnetized case, we require extra conditions on the velocity moments and space moments of the initial data f^{in}, as shown in the following theorem:

Theorem 5.2.4. *Let B verify (5.1.3), let $f^{in} = f^{in}(x, v) \geq 0$ a.e. with $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and assume that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^6 f^{in} dxdv < \infty \text{ and } \iint_{\mathbb{R}^3\times\mathbb{R}^3} |x|^4 f^{in} dxdv < \infty. \tag{5.2.9}$$

Then there exists at most one weak solution f to (5.1.1) with initial data f^{in} such that

$$\rho \in L^\infty([0\,, T] \times \mathbb{R}^3_x) \tag{5.2.10}$$

for all $T > 0$.

However, to exploit this result, one needs to build solutions to (5.1.1) with bounded macroscopic density. Hence, in the next proposition, we give an explicit condition that guarantees the existence of solutions to (5.1.1) with a bounded ρ.

Proposition 5.2.5. *Let B verify (5.1.3) and be independent of x, let f^{in} satisfy the assumptions of conjecture 5.2.1 with $k_0 > 6$. We also assume that f^{in} is such that for all $R > 0$ and $T > 0$*

$$g_R(t, x, v) \in L^\infty([0\,, T] \times \mathbb{R}^3_x, L^1(\mathbb{R}^3_v)), \tag{5.2.11}$$

where

$$g_R(t, x, v) = \sup_{(y,w)\in S_{t,x,v,R}} f^{in}(y + vt, w) \tag{5.2.12}$$

with

$$S_{t,x,v,R} = \left\{(y, w) \colon\ |y - x| \leq (R + \|B\|_\infty |v|)t^2 e^{\|B\|_\infty t}, |w - v| \leq (R + \|B\|_\infty |v|)te^{\|B\|_\infty t}\right\}. \tag{5.2.13}$$

Then the weak solution f of (5.1.1) (provided by our result on propagation of velocity moments from the last subsection) verifies

$$\rho \in L^\infty([0\,, T] \times \mathbb{R}^3_x)$$

for all $T > 0$.

This proposition was shown in [110, proposition 2.7] in the case of a constant magnetic field and remains unchanged when we take a general B.

Now we present a theorem which is the second main result of this chapter, where we show that the uniqueness criterion proved in [95] (theorems 1.1 and 1.2) also applies to (5.1.1) with B verifying (5.1.3), improving theorem 5.2.4 because it allows for solutions with unbounded macroscopic density.

Theorem 5.2.6. *Let $T > 0$ and B verify (5.1.3). Furthermore, let $f^{in} \geq 0$ a.e. with $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ and such that*

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f^{in}(x,v)dxdv < +\infty, \tag{5.2.14}$$

for some $m > 6$.

If f^{in} also satisfies

$$\forall k \geq 1, \quad \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f^{in}(x,v)dxdv \leq (C_0 k)^{\frac{k}{3}}, \tag{5.2.15}$$

for some constant C_0 independent of k, then there exists at most one solution f to (5.1.1). If such a solution f exists then it will verify the following bound on the L^p norms of ρ:

$$\sup_{[0\,,T]} \sup_{p\geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty. \tag{5.2.16}$$

Remark 5.2.7. In our framework, an important difference with [95] is that the uniqueness criterion isn't given by the inequality on the macroscopic density (5.2.16) but rather the stronger assumption on the moments of the solution (5.2.15).

As mentioned above, the assumptions of theorem 5.2.6 are less restrictive than the condition (5.2.10) and thus allow us to consider initial data with unbounded macroscopic density. This result is illustrated by the following theorem ([95, theorem 1.3]):

Theorem 5.2.8 (Miot, [95])**.** *There exists $f^{in} \geq 0$ a.e. such that $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ satisfying the assumptions of theorem 5.2.6 and such that*

$$\rho_0(x) = \frac{4\pi}{3} \ln_-(|x|), \quad \forall x \in \mathbb{R}^3, \tag{5.2.17}$$

where $\ln_- = \max(-\ln(x), 0)$ is the negative part of the function $\ln$.

In section 5.4, we will detail the proof of theorem 5.2.6 first because it is the main result of this section. Then we will present the proof of theorem 5.2.4 where ingredients from the proof of theorem 5.2.6 are used, notably the boundedness of the velocity characteristic. However in theorem 5.2.4 we require a condition on the space moment of the initial data (5.2.9) which isn't the case in theorem 5.2.6.

Finally, with regards to uniqueness for Vlasov–Poisson, a major open problem is finding a uniqueness condition in the periodic case. Indeed, it would be very interesting to see if the conditions found in [88, 95] could be adapted to the periodic case.

5.3 Proofs regarding propagation of velocity moments

In this section, we shall denote by C a constant that can change from one line to another but that only depends on

$$\mathcal{E}(0), \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty. \tag{5.3.1}$$

As mentioned above, the whole proof is conducted using smooth functions.

We consider $k > 2$ and $\varepsilon > 0$ small enough, say $\varepsilon \in]0\,,\varepsilon_0[$ with $\varepsilon_0 \leq \frac{(k-2)}{2k}$. As said in the introduction, the main difference with the analysis in [99] is that we're going to show propagation of moments for all time by using an induction argument using the cyclotron period $T_c = \frac{1}{\|B\|_\infty}$. We begin with the initialization, so we're first going consider $T > 0$ with $T \leq T_B$, where T_B is the unique real number such that

$$T_B \in \mathbb{R}_+^* \text{ and } \|B\|_\infty T_B \exp(T_B \|B\|_\infty) = a, \tag{5.3.2}$$

with $a > 0$. In our method, since we can only obtain estimates on Q for $T_B \ll T_c$, we just need a small enough so we set $a = 2^{-10}$.

Thus, we show propagation of velocity moments on $[0\,, T_B]$ using the following result.

Proposition 5.3.1. *For all $T > 0$ such that $T \leq T_B$, (5.2.6) is verified. More precisely we have the following estimate on $Q(t,t)$*

$$Q(t,t) \leq C \exp(T \|B\|_\infty)^{\frac{2}{5}} (T^{\frac{1}{2}} + T^{\frac{7}{5}}) \tag{5.3.3}$$

with C that only depends on

$$k, \mathcal{E}(0), \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, M_k(0).$$

Remark 5.3.2. This estimate is the analogous of the estimate (13) in [99]. In our magnetized framework, we only manage to generalize this result up to the time T_B.

The following section will be devoted to the proof of this proposition, and just like in [99] the proof is done in three steps which correspond to proposition 5.3.3, proposition 5.3.6 and proposition 5.3.7.

5.3.1 The case $T \leq T_B$

Proposition 5.3.3. *For any $0 \leq \delta \leq t \leq T \leq T_B$ we have:*

$$Q(t,\delta) \leq C(\delta Q(t,\delta)^{\frac{4}{3}} + \delta^{\frac{1}{2}}(1 + M_{2+\varepsilon}(T))^{\frac{1}{2}}) \tag{5.3.4}$$

Proof. Let $(t, x_*, v_*) \in [0\,,T] \times \mathbb{R}^3 \times \mathbb{R}^3$ and set $(X_*, V_*)(s) = (X, V)(s; t, x_*, v_*)$. For any $\delta \in [0\,,t]$ we have by definition of E

$$\int_{t-\delta}^{t} |E(s, X(s; t, x_*, v_*))|\, ds \leq \int_{t-\delta}^{t} \int \frac{\rho(s,x)dx}{4\pi\, |x - X_*(s)|^2} ds$$

Our objective in the rest of this section will be to estimate the integral:

$$I_*(t,\delta) := \int_{t-\delta}^{t} \int \frac{\rho(s,x)dx}{|x - X_*(s)|^2} ds = \int_{t-\delta}^{t} \iint \frac{f(s,x,v)dvdx}{|x - X_*(s)|^2} ds \tag{5.3.5}$$

Now we will use a procedure that is inspired from [114] which consists in splitting $[t-\delta\,,t]\times\mathbb{R}^3\times\mathbb{R}^3$ into three parts. Here the partition is slightly different because, following [99], we introduce $\varepsilon > 0$.

$$
\begin{aligned}
G &= \{(s,x,v) : \min(|v|\,,|v-V_*(s)|) < P\}\,,\\
B &= \{(s,x,v) : |x-X_*(s)| \leq \Lambda_\varepsilon(s,v)\}\setminus G,\\
U &= [t-\delta\,,t]\times\mathbb{R}^3\times\mathbb{R}^3\setminus(G\cup B),
\end{aligned}
$$

with

$$
P = 2^{10}Q(t,\delta)\exp(\delta\,\|B\|_\infty) \text{ and } \Lambda_\varepsilon(s,v) = L(1+|v|^{2+\varepsilon})^{-1}\,|v-V_*(s)|^{-1} \tag{5.3.6}
$$

and $L > 0$ to be fixed later. The main difference here with [99] is the definition of P, because the added magnetic modifies the evolution of the characteristic in velocity $V(s)$. Furthermore, we take the same numerical constant 2^{10} in the definition of P as in [99] is (in truth this constant just needs to be large enough). Using obvious notations, we write $I_* = I_*^G + I_*^B + I_*^U$. The first two integrals will be more straightforward to estimate than I_*^U, which involves the set U considered as the "ugly set" according to [59].

The first two contributions I_*^G, I_*^B are treated the same in both magnetized and unmagnetized cases, simply because the modifications made to the sets G, B to take into account the added magnetic field don't change the computations required to estimate I_*^G and I_*^B. We succinctly present how to control both integrals following the calculations from [99]. The first bound is obtained by using a standard functional inequality.

For $\kappa \in L^\infty(\mathbb{R}^3)\cap L^{\frac{5}{3}}(\mathbb{R}^3)$ we have

$$
\left\|\kappa * |\cdot|^{-2}\right\|_\infty \leq c\,\|\kappa\|_{\frac{5}{3}}^{\frac{5}{9}}\,\|\kappa\|_\infty^{\frac{4}{9}}\,, \tag{5.3.7}
$$

with c a numerical constant.

We apply (5.3.7) to the quantity:

$$
\rho_G(s,x) = \int_{B(0,P)\cup B(V_*(s),P)} f(s,x,v)dv \leq \rho(s,x), \tag{5.3.8}
$$

which implies the following control on $I_*^G(t,\delta)$:

$$
I_*^G(t,\delta) \leq C(\delta P^{\frac{4}{3}}). \tag{5.3.9}
$$

To estimate the contribution on B, we first integrate in the space variable using a spherical change of variable.

$$\begin{aligned}
I_*^B(t,\delta) &\le \int_{t-\delta}^{t}\int_v\int_{|x-X_*(s)|\le\Lambda_\varepsilon(s,v)} \frac{f(s,x,v)dx}{|x-X_*(s)|^2}dvds,\\
&\le \int_{t-\delta}^{t}\int_v \left(4\pi\int_0^{\Lambda_\varepsilon(s,v)} \frac{\left\|f^{in}\right\|_\infty}{r^2}r^2dr\right)dvds,\\
&\le C\int_{t-\delta}^{t}\int_v \frac{L}{\left(1+|v|^{2+\varepsilon}\right)|v-V_*(s)|}dvds,\\
&\le C\int_{t-\delta}^{t}\left(\int_{|v|\le|v-V_*(s)|}\frac{L}{\left(1+|v|^{2+\varepsilon}\right)|v|}dv+\int_{|v|>|v-V_*(s)|}\frac{L}{\left(1+|v-V_*(s)|^{2+\varepsilon}\right)|v-V_*(s)|}dv\right)ds,\\
&\le C\int_{t-\delta}^{t}\int_v \frac{L}{\left(1+|v|^{2+\varepsilon}\right)|v|}dvds,\\
&\le C\int_{t-\delta}^{t}\int_{\mathbb{R}^3} \frac{Lr}{\left(1+r^{2+\varepsilon}\right)}drds,\\
&\le C\delta L.
\end{aligned}$$

The last contribution $I_*^U(t,\delta)$ can be written

$$I_*^U(t,\delta)=\int_{t-\delta}^{t}\iint \frac{f(s,x,v)\mathbf{1}_U(s,x,v)}{|x-X_*(s)|^2}dvdxds=\iint\int_{t-\delta}^{t}\frac{\mathbf{1}_U(s,X(s),V(s))}{|X(s)-X_*(s)|^2}dsf(t,x,v)dvdx \tag{5.3.10}$$

where we have the obvious notation $(X,V)(s)=(X,V)(s;t,x,v)$. Estimating this quantity is difficult and will occupy us for the rest of the proof of proposition 5.3.3.

The following lemma is very important in our proof because it highlights why we need to use the induction procedure mentioned above. In the unmagnetized case , we estimate I_*^U by noticing that because of the definition of U, the characteristic $V(s)$ stays close to v on $[t-\delta\,,t]$ because v is large compared to P and P is much larger $Q(t,\delta)$ which quantifies the total variation of $V(s)$ on $[t-\delta\,,t]$. However in the magnetized case, this stays true only under the condition (5.3.2) because if the magnetic field is large than the variations of $V(s)$ on $[t-\delta\,,t]$ can also be very large compared to P.

Lemma 5.3.4. *Let $s_1\in[t-\delta\,,t]$ such that $(s_1,X(s_1),V(s_1))\in U$, then for all $s\in[t-\delta\,,t]$ we have*

$$2^{-1}|v|\le|V(s)|\le 2|v|, \tag{5.3.11}$$

and

$$2^{-1}|v-v_*|\le|V(s)-V_*(s)|\le 2|v-v_*|. \tag{5.3.12}$$

Proof. First, because of the definition of U, we can write

$$\min(|V(s_1)|\,,|V(s_1)-V_*(s_1)|)\ge P. \tag{5.3.13}$$

Let's start by proving the first bound (5.3.11), thanks to (5.1.8) we have for all $s\in[t-\delta\,,t]$

$$V(s)-V(s_1)=\int_{s_1}^{s}E(\tau,X(\tau))d\tau+\int_{s_1}^{s}V(\tau)\wedge B(\tau,X(\tau))d\tau$$

Furthermore, one of the properties of the characteristics is that we have for all $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $\tau, t \in \mathbb{R}_+$

$$X(\tau;t,x,v) = X(\tau;0,X(0;t,x,v),V(0;t,x,v))$$

and also that the function $(x,v) \mapsto X(\tau;t,x,v)$ is a C^1-diffeomorphism which means that

$$\begin{aligned}\sup\left\{\int_{t-\delta}^{t} |E(s,X(s;t,x,v))|\,ds,(x,v)\in\mathbb{R}^3\times\mathbb{R}^3\right\} &= \sup\left\{\int_{t-\delta}^{t} |E(s,X(s;0,x,v))|\,ds,(x,v)\in\mathbb{R}^3\times\mathbb{R}^3\right\}\\ &= Q(t,\delta).\end{aligned}$$

Thus we can write

$$|V(s)-V(s_1)| \le Q(t,\delta) + \|B\|_\infty\left(\delta\,\|V(s_1)\| + \left|\int_{s_1}^{s} |V(\tau)-V(s_1)|\,d\tau\right|\right)$$

which is a Grönwall inequality, so we finally have for all $s \in [t-\delta\,,t]$

$$||V(s)|-|V(s_1)|| \le |V(s)-V(s_1)| \le (Q(t,\delta)+\|B\|_\infty\,\delta\,|V(s_1)|)\exp(\delta\,\|B\|_\infty). \tag{5.3.14}$$

This last inequality highlights the main difference with the unmagnetized case, indeed when we use $V(s_1)$ as a reference point to quantity the variation of $V(s)$, we see that the added term $\|B\|_\infty\,\delta\,|V(s_1)|\exp(\delta\,\|B\|_\infty)$, which is just the added variation of the velocity characteristic resulting from the magnetic field, is potentially unbounded. This is due to the fact that even if $0 \le \delta \le T$, $\|B\|_\infty$ is potentially large. This is the reason we introduce the time T_B which depends on the cyclotron frequency $\|B\|_\infty$.

Now using this last inequality, thanks to the relation between P and $Q(t,\delta)$ given in (5.3.6), to (5.3.13), and to (5.3.2) (because $t \le T_B$) we have

$$\begin{aligned}|V(s)| &\le |V(s_1)|\,(1+\|B\|_\infty\,\delta\exp(\delta\,\|B\|_\infty)) + 2^{-10}P\\ &\le |V(s_1)|\left(1+2^{-10}+2^{-10}\right) = |V(s_1)|\left(1+2^{-9}\right)\end{aligned}$$

and using the same relations but this time for $-|V(s_1)|$ we can write

$$\begin{aligned}|V(s_1)|\left(1-2^{-10}-2^{-10}\right) &\le |V(s_1)|\,(1-\|B\|_\infty\,\delta\exp(\delta\,\|B\|_\infty)) - 2^{-10}P\\ &\le |V(s_1)|\,(1-\|B\|_\infty\,\delta\exp(\delta\,\|B\|_\infty)) - Q(t,\delta)\exp(\delta\,\|B\|_\infty)\\ &\le |V(s)|\end{aligned}$$

These inequalities are valid for all $s \in [t-\delta\,,t]$ and so in particular for $s=t$. And so we can write

$$2^{-1}\,|V(s)| \le |V(s)|\,\frac{1-2^{-9}}{1+2^{-9}} \le |v| \le |V(s)|\,\frac{1+2^{-9}}{1-2^{-9}} \le 2\,|V(s)| \tag{5.3.15}$$

which is equivalent to (5.3.11).

Now let's look at the inequality (5.3.12). Like for (5.3.11), we try to write a Grönwall inequality but this time on $Z(s) = |(V(s)-V_*(s))-(V(s_1)-V_*(s_1))|$.

$$\begin{aligned}(V(s)-V_*(s))-(V(s_1)-V_*(s_1)) &= \int_{s_1}^{s} E(\tau,X(\tau))d\tau + \int_{s_1}^{s} V(\tau)\wedge B(\tau,X(\tau))d\tau\\ &\quad - \int_{s_1}^{s} E(\tau,X_*(\tau))d\tau - \int_{s_1}^{s} V_*(\tau)\wedge B(\tau,X_*(\tau))d\tau.\end{aligned}$$

This allows us to write

$$
\begin{aligned}
Z(s) &\leq 2Q(t,\delta) + \left| \int_{s_1}^{s} \left(V(\tau) \wedge B(\tau, X(\tau)) - V_*(\tau) \wedge B(\tau, X_*(\tau))\right) d\tau \right| \\
&\leq 2Q(t,\delta) + \int_{s_1}^{s} \left|V(\tau) \wedge (B(\tau, X(\tau)) - B(\tau, X_*(\tau)))\right| + \left|(V_*(\tau) - V(\tau)) \wedge B(\tau, X_*(\tau))\right| d\tau \\
&\leq 2Q(t,\delta) + 2\delta \left\|B\right\|_\infty 2\left|V(s_1)\right| \\
&+ \left\|B\right\|_\infty \left(\delta \left|(V_*(s_1) - V(s_1))\right| + \int_{s_1}^{s} \left|(V(\tau) - V_*(\tau)) - (V(s_1) - V_*(s_1))\right| d\tau \right)
\end{aligned}
$$

where in the second term in the last inequality we used the bound $|V(s)| \leq |V(s_1)|(1+2^{-9}) \leq 2|V(s_1)|$ that we established just before.

This is the point where we require that $B := B(t)$.

Thus we have our Grönwall inequality on $Z(s)$ which gives us

$$
Z(s) \leq (2Q(t,\delta) + 4\delta \left\|B\right\|_\infty |V(s_1)| + \left\|B\right\|_\infty \delta \left|(V_*(s_1) - V(s_1))\right|) \exp(\delta \left\|B\right\|_\infty). \tag{5.3.16}
$$

If we decompose the magnetic part of the Lorentz forces using $V_*(s)$ then we get an analogous inequality but this time with $|V_*(s_1)|$:

$$
Z(s) \leq (2Q(t,\delta) + 4\delta \left\|B\right\|_\infty |V_*(s_1)| + \left\|B\right\|_\infty \delta \left|(V_*(s_1) - V(s_1))\right|) \exp(\delta \left\|B\right\|_\infty). \tag{5.3.17}
$$

Of course, we can take the mean of the two inequalities to obtain

$$
Z(s) \leq (2Q(t,\delta) + 2\delta \left\|B\right\|_\infty \frac{|V_*(s_1)| + |V(s_1)|}{2} + \left\|B\right\|_\infty \delta \left|(V_*(s_1) - V(s_1))\right|) \exp(\delta \left\|B\right\|_\infty). \tag{5.3.18}
$$

In each of these inequalities, the second term in the inequality depends on $|V_*(s_1)|, |V(s_1)|$ which means that we can't get a lower bound on $|V(s) - V_*(s)|$ because we can't compare $|V(s_1) - V_*(s_1)|$ with $|V(s_1)|, |V_*(s_1)|$. However, if $B := B(t)$ then we don't have this extra term because we can just write

$$
Z(s) \leq 2Q(t,\delta) + \int_{s_1}^{s} \left|(V(\tau) - V_*(\tau)) \wedge B(\tau)\right|, \tag{5.3.19}
$$

and we obtain (5.3.12) by following exactly the same procedure used above to prove (5.3.11) but this time by exploiting the bound $|(V_*(s_1) - V(s_1))| > P$.

Going back to the case $B := B(t,x)$, we might be able to obtain a better estimate on $Z(s)$ by controlling the term $B(\tau, X(\tau)) - B(\tau, X_*(\tau))$ more precisely. However, since such an estimate would involve the quantity $|x - x_*|$, we would certainly have to modify the sets G, B, U in order to bound $|x - x_*|$. This is the lead that we want to explore to prove the conjecture 5.2.1 with $B := B(t,x)$.

$\square$

Lemma 5.3.5. *For any $(x,v) \in \mathbb{R}^6$ we have*

$$
\int_{t-\delta}^{t} \frac{\mathbf{1}_U(s, X(s), V(s))}{\left|X(s) - X_*(s)\right|^2} ds \leq C \left(\frac{1 + |v|^{2+\varepsilon}}{L} \right). \tag{5.3.20}
$$

Proof. If $(s, X(s), V(s)) \notin U$ for all $s \in [t-\delta\,, t]$ then the estimate (5.3.20) is verified. Now we assume that there exists $s_1 \in [t-\delta\,, t]$ such that $(s_1, X(s_1), V(s_1)) \in U$, then thanks to lemma 5.3.4 we can write

$$\Lambda_\varepsilon(s, V(s)) \geq L(1+(2\,|v|)^{2+\varepsilon})^{-1}(2\,|v-v_*|)^{-1} \geq 2^{-3-\varepsilon}\Lambda_\varepsilon(t,v) \tag{5.3.21}$$

and hence

$$\frac{\mathbf{1}_U(s, X(s), V(s))}{|X(s)-X_*(s)|^2} \leq \frac{\mathbf{1}_{\mathbb{R}^3\setminus B(X_*(s), 2^{-3-\varepsilon}\Lambda_\varepsilon(t,v))}(X(s))}{|X(s)-X_*(s)|^2} \leq h(|Y(s)|), \tag{5.3.22}$$

where $Y(s) = X(s) - X_*(s)$ and $h(u) = \min(|u|^{-2}\,, 4^{3+\varepsilon}\Lambda_\varepsilon(t,v)^{-2})$. Since h is a non-increasing function, we look for a lower bound on $|Y(s)|$.

For any $s_0 \in [t-\delta\,, t]$ we have, thanks to (5.3.12)

$$\begin{aligned} |Y(s)| &\geq |Y(s_0) + (s-s_0)Y'(s_0)| - \left|\int_{s_0}^{s}(s-u)Y''(u)du\right| \\ &\geq |Y(s_0) + (s-s_0)Y'(s_0)| - 2\,|s-s_0|\,(Q(t,\delta) + \delta\,\|B\|_\infty\,|v-v_*|). \end{aligned}$$

Now we consider $s = s_0$ that minimizes $|Y(s)|^2$ when $s \in [t-\delta\,, t]$, then this implies $(s-s_0)Y(s_0)\cdot Y'(s_0) \geq 0$ and so

$$|Y(s_0) + (s-s_0)Y'(s_0)|^2 \geq |Y'(s_0)|^2\,|s-s_0|^2 \tag{5.3.23}$$

and thanks to (5.3.12) we get $|Y'(s_0)| \geq 2^{-1}\,|v-v_*|$ and when we evaluate (5.3.12) in s_1 this also yields $Q(t,\delta) \leq 2^{-9}\,|v-v_*|\exp(-\delta\,\|B\|_\infty)$ so we have

$$\begin{aligned} |Y'(s_0)| - 2(Q(t,\delta) + \delta\,\|B\|_\infty\,|v-v_*|) &\geq |v-v_*|\,(2^{-1} - 2^{-8}\,|v-v_*|\exp(-\delta\,\|B\|_\infty) - 2\delta\,\|B\|_\infty) \\ &\geq |v-v_*|\,(2^{-1} - 2^{-8} - 2\delta\,\|B\|_\infty). \end{aligned}$$

We need the quantity $(2^{-1} - 2^{-8} - 2\delta\,\|B\|_\infty)$ to be strictly positive and so once again we need the condition (5.3.2) for $\delta\,\|B\|_\infty$ to be small and this inequality to be verified. Now we have $|Y(s)| \geq \alpha\,|v-v_*|\,|s-s_0|$ with $\alpha > 0$. Just as in [99], we bring this inequality into (5.3.22), integrate with respect to the time variable and estimate the integral as follows to obtain (5.3.20):

$$\begin{aligned} \int_{t-\delta}^{t} h(|Y(s)|)ds &\leq \int_{t-\delta}^{t} h(\alpha\,|v-v_*|\,|s-s_0|)ds, \\ &\leq \int_0^{+\infty} h(\alpha\,|v-v_*|\,r)dr, \\ &= (\alpha\,|v-v_*|)^{-1}\int_0^{+\infty} h(r)dr, \\ &= (\alpha\,|v-v_*|)^{-1}\left(\int_0^{2^{-3-\varepsilon}\Lambda_\varepsilon(t,v)} 4^{3+\varepsilon}\Lambda_\varepsilon(t,v)^{-2}dr + \int_{2^{-3-\varepsilon}\Lambda_\varepsilon(t,v)}^{+\infty}\frac{1}{r^2}dr\right), \\ &= (\alpha\,|v-v_*|)^{-1}(2^{3+\varepsilon}\Lambda_\varepsilon(t,v)^{-1} + 2^{3+\varepsilon}\Lambda_\varepsilon(t,v)^{-1}), \\ &\leq C\left(\frac{1+|v|^{2+\varepsilon}}{L}\right). \end{aligned}$$

□

Now integrating in x, v and using the mass conservation, we finally obtain

$$I_*^U(t,\delta) \leq CL^{-1}(1 + M_{2+\varepsilon}(T)). \tag{5.3.24}$$

We gather all the above estimates to conclude

$$\begin{aligned} I_*(t,\delta) &\leq C(\delta\,(Q(t,\delta)\exp(\delta\,\|B\|_\infty))^{\frac{4}{3}} + \delta L + L^{-1}(1 + M_{2+\varepsilon}(T))) \\ &\leq C(\delta Q(t,\delta)^{\frac{4}{3}} + \delta L + L^{-1}(1 + M_{2+\varepsilon}(T))) \end{aligned}$$

where the last inequality is justified by the fact that thanks to (5.3.2) we have $\exp(\delta\,\|B\|_\infty) \leq \exp(g(2^{-10}))$ where g is the inverse of the function $x \mapsto x\exp(x)$ on $\mathbb{R}_+$. We conclude in the same way as in [99], firstly by optimizing the parameter L and then by noticing that the pair (x_*, v_*) is arbitrary so that we have

$$\sup\left\{I_*(t,\delta), (x_*, v_*) \in \mathbb{R}^3 \times \mathbb{R}^3\right\} \geq Q(t,\delta). \tag{5.3.25}$$

Finally, we obtain (5.3.4). □

The next two propositions allow us to conclude. The proof of proposition 5.3.6 is identical to the one in [99] because it doesn't rely on the characteristics of the system, but rather on real analysis arguments. In an effort of clarity, and also because some arguments of the proof are more detailed in this chapter than in [99], we place the proof of proposition 5.3.6 in the appendix.

Proposition 5.3.6. *For any $t \in [0\,, T]$ with $T \leq T_B$, we have*

$$Q(t,t) \leq C(t^{\frac{1}{2}} + t)(1 + M_{2+\varepsilon}(T))^{\frac{4}{7}}. \tag{5.3.26}$$

Now we state the last result necessary in the proof of proposition 5.3.1.

Proposition 5.3.7. *There exists $\tau(\varepsilon, k) > 0$ such that for any $t \in [0\,, T]$ we have*

$$Q(t,t) \leq C(1 + M_k(0))^{\tau(\varepsilon,k)} \exp(T\,\|B\|_\infty)^{\frac{2}{5}}(T^{\frac{1}{2}} + T^{\frac{7}{5}}). \tag{5.3.27}$$

Remark 5.3.8. We notice that we obtain the same estimate (5.3.27) as in [99] when the magnetic field B is zero.

Proof. Using the same argument as in (5.2.3), we can write

$$M_k(t) \leq 2^{k-1}\exp(kt\,\|B\|_\infty)\left(M_k(0) + N(T)^k\,\|f^{in}\|_1\right) \tag{5.3.28}$$

with $N(T)$ defined in conjecture 5.2.1.

Now to obtain the desired estimate we have to bound $M_{2+\varepsilon}(t)$, which we manage with the Hölder inequality. Thus for any $t \in [0\,, T]$ we have

$$\iint |v|^{2+\varepsilon} f(t,x,v)dxdv \leq \left(\iint |v|^2 f(t,x,v)dxdv\right)^{\frac{2+\varepsilon-k}{2-k}} \left(\iint |v|^k f(t,x,v)dxdv\right)^{\frac{\varepsilon}{k-2}}. \tag{5.3.29}$$

With the conservation of the energy $E(t)$, this Hölder inequality implies that

$$M_{2+\varepsilon}(T) \leq CM_k(T)^{\frac{\varepsilon}{k-2}} \tag{5.3.30}$$

and bringing this inequality into (5.3.28) it yields

$$M_{2+\varepsilon}(T) \leq C \exp(\frac{k\varepsilon}{k-2} T \|B\|_{\infty}) \left(M_k(0) + N(T)^k \|f^{in}\|_1\right)^{\frac{\varepsilon}{k-2}}. \tag{5.3.31}$$

Now thanks to (5.3.26) we can deduce

$$\begin{aligned} M_{2+\varepsilon}(T) &\leq C \exp(\frac{k\varepsilon}{k-2} T \|B\|_{\infty}) \left(M_k(0) + N(T)^k \|f^{in}\|_1\right)^{\frac{\varepsilon}{k-2}} \\ &\leq C \exp(\frac{k\varepsilon}{k-2} T \|B\|_{\infty}) \left(M_k(0) + (T^{\frac{1}{2}} + T)^k (1 + M_{2+\varepsilon}(T))^{\frac{4k}{7}} \|f^{in}\|_1\right)^{\frac{\varepsilon}{k-2}} \\ &\leq C \exp(\frac{k\varepsilon}{k-2} T \|B\|_{\infty})(1 + M_k(0))^{\frac{\varepsilon}{k-2}} (T^{\frac{1}{2}} + T)^{\frac{k\varepsilon}{k-2}} (1 + M_{2+\varepsilon}(T))^{\frac{4k\varepsilon}{7(k-2)}} \end{aligned}$$

Like in [99], we write $\sigma(\epsilon, k) = \frac{4k\varepsilon}{7(k-2)}$ and notice that if we take ε small enough we have $\sigma(\epsilon, k) \in]0\,, 1[$. More precisely, if $\varepsilon < \varepsilon_0$ then $\sigma(\epsilon, k) \leq \frac{2}{7}$, $\frac{k\varepsilon}{k-2} \leq \frac{1}{2}$ and $\frac{\varepsilon}{k-2} \leq \frac{1}{2k}$ and we find

$$M_{2+\varepsilon}(T) \leq C(1 + M_k(0))^{\frac{1}{2k}} \exp(T \|B\|_{\infty})^{\frac{1}{2}} (1 + T)^{\frac{1}{2}} (1 + M_{2+\varepsilon}(T))^{\frac{2}{7}}, \tag{5.3.32}$$

where we used $T^{\frac{1}{2}} + T \leq 2(1 + T)$. Now since the right term in the last inequality is larger than 1 up to a constant C, we deduce that

$$(1 + M_{2+\varepsilon}(T))^{\frac{5}{7}} \leq C(1 + M_k(0))^{\frac{1}{2k}} \exp(T \|B\|_{\infty})^{\frac{1}{2}} (1 + T)^{\frac{1}{2}} \tag{5.3.33}$$

which finally yields

$$1 + M_{2+\varepsilon}(T) \leq C(1 + M_k(0))^{\frac{7}{10k}} \exp(T \|B\|_{\infty})^{\frac{7}{10}} (1 + T)^{\frac{7}{10}}. \tag{5.3.34}$$

Then, using (5.3.26) again we deduce

$$\begin{aligned} Q(t, t) &\leq C(T^{\frac{1}{2}} + T)(1 + M_k(0))^{\frac{2}{5k}} \exp(T \|B\|_{\infty})^{\frac{2}{5}} (1 + T)^{\frac{2}{5}} \\ &\leq C(1 + M_k(0))^{\frac{2}{5k}} \exp(T \|B\|_{\infty})^{\frac{2}{5}} (T^{\frac{1}{2}} + T^{\frac{7}{5}}). \end{aligned}$$

This concludes the proof of proposition 5.3.7 and proposition 5.3.1. □

5.3.2 The case $T \geq T_B$

We conclude the proof of (5.2.6) by showing that $Q(t, t)$ is bounded for all time.

Proposition 5.3.9. *The inequality* (5.2.6) *is valid for all* $T \geq T_B$.

Proof. For all $t \in [0\,, T]$, we write $t = nT_B + t_r$ with $n \in \mathbb{N}$ and $t_r \in [0\,, T_B[$. Since the constant C in proposition 5.3.1 depends only on $T, k, \|B\|_{\infty}, \|f^{in}\|_1, \|f^{in}\|_{\infty}, \mathcal{E}(0)$ and $M_k(0)$, we can reiterate the procedure on any time interval $I_p = [pT_B\,, (p+1)T_B]$. Indeed, T, k and $\|B\|_{\infty}$ are constants $\|f(t)\|_1$ and $\|f(t)\|_{\infty}$ are conserved in time and the energy $E(t)$ is bounded. This means we can write

$$\begin{aligned} Q(t, t) &\leq \sum_{p=0}^{n-1} Q((p+1)T_B, T_B) + Q(t, t_r) \\ &\leq C \sum_{p=0}^{n} (1 + M_k(pT_B))^{\frac{2}{5k}} \exp(T_B \|B\|_{\infty})^{\frac{2}{5}} (T_B^{\frac{1}{2}} + T_B^{\frac{7}{5}}). \end{aligned}$$

Furthermore, we can show by an immediate induction that for all $p \in \mathbb{N}$ with $p \leq n$, $M_k(pT_B)$ is bounded such that

$$M_k(pT_B) \leq C_p(k, \|B\|_\infty, \mathcal{E}(0), \|f^{in}\|_1, \|f^{in}\|_\infty, M_k(0)). \tag{5.3.35}$$

This is just because $M_k(pT_B) \leq C_1 \Rightarrow Q((p+1)T_B, T_B) \leq C_2 \Rightarrow M_k((p+1)T_B) \leq C_3$ with C_1, C_2, C_3 depending on (5.2.4). This concludes the proof of proposition 5.3.9 and conjecture 5.2.1.

□

5.4 Proofs regarding uniqueness

A said above, we manage to prove uniqueness using the conditions presented in section 5.2.2 with a magnetic field $B := B(t,x)$ that depends on time and position. The subsections 5.4.1, 5.4.2 and 5.4.3 will be devoted to the proof of theorem 5.2.6 and subsection 5.4.4 will be devoted to the proof of theorem 5.2.4. In this section, we shall denote by C a constant that can change from one line to another but that only depends on

$$\mathcal{E}(0), \|f^{in}\|_1, \|f^{in}\|_\infty, T, \iint |v|^m f^{in}. \tag{5.4.1}$$

5.4.1 Proof of the estimate on the L^p norms of ρ (5.2.16)

We consider f^{in} that satisfies the assumptions of theorem 5.2.6 and let f be the solution given by conjecture 5.2.1 with initial data f^{in}. By construction, we have propagation of moments:

$$\sup_{t\in[0,T]} \iint_{\mathbb{R}^3\times\mathbb{R}^3} |v|^m f(t,x,v)dxdv < +\infty. \tag{5.4.2}$$

Now thanks to a classical velocity moment inequality, we show how to control the L^p norms of the macroscopic density with velocity moments, this inequality is given by

$$\|\rho(t)\|_{\frac{k+3}{3}} \leq C \|f(t)\|_\infty^{\frac{k}{k+3}} M_k(t)^{\frac{3}{k+3}}. \tag{5.4.3}$$

with C independent of k. Since we want ρ to verify (5.2.16), this means that we need to prove

$$\forall k \geq 1, \sup_{t\in[0,T]} (\|f(t)\|_\infty^{\frac{k}{k+3}} M_k(t)^{\frac{3}{k+3}}) \leq Ck. \tag{5.4.4}$$

Since the solution $f \in L^\infty([0,T], L^\infty(\mathbb{R}^3\times\mathbb{R}^3))$, we finally need to show

$$\forall k \geq 1, \sup_{t\in[0,T]} M_k(t)^{\frac{3}{k+3}} \leq Ck. \tag{5.4.5}$$

First, we recall that thanks to (5.4.2) where $m > 6$ we can infer that $\rho \in L^\infty([0,T], L^p(\mathbb{R}^3))$ with $p = \frac{m+3}{3} > 3$ and following (5.4.25) we have $E \in L^\infty([0,T], L^\infty(\mathbb{R}^3))$. Then we write

$$\frac{d}{dt}|V(t,x,v)|^k = k|V(t,x,v)|^{k-1}\frac{\dot{V}(t,x,v)\cdot V(t,x,v)}{|V(t,x,v)|},$$

and thanks to the bound on E and the definition of the characteristics (5.1.8) we can infer that for all $k > m$

$$|V(t,x,v)|^k \leq |v|^k + k\int_0^t |V(s,x,v)|^{k-1}\,\frac{(E(s,X(s,x,v)) + V(s,x,v)\wedge B(s,X(s,x,v)))\cdot V(s,x,v)}{|V(s,x,v)|}ds$$
$$\leq |v|^k + k\,\|E\|_\infty \int_0^t |V(s,x,v)|^{k-1}\,ds.$$

Since the contribution of magnetic field B vanishes, the following computations are the same as in the unmagnetized case [95]. In an effort to be clear, we explicit these computations nonetheless.

Integrating this last inequality with respect to $f^{in}(x,v)dxdv$ we get

$$M_k(t) \leq M_k(0) + k\,\|E\|_\infty \int_0^t M_{k-1}(s)ds. \tag{5.4.6}$$

Thus by induction we deduce that $\sup\limits_{t\in[0\,,T]} M_k(t)$ is finite for all $k > m$. furthermore, by another classical velocity moment inequality we obtain that

$$M_{k-1}(s) \leq \|f(s)\|_1^{\frac{1}{k}}\, M_k(s)^{\frac{k-1}{k}}. \tag{5.4.7}$$

Since $\|f(t)\|_1$ is conserved, we get

$$M_k(t) \leq M_k(0) + Ck\int_0^t M_k(s)^{\frac{k-1}{k}}ds,. \tag{5.4.8}$$

Differentiating this inequality allows us to write

$$M_k'(t) \leq CkM_k(t)^{\frac{k-1}{k}} \Leftrightarrow \frac{d}{dt}(M_k(t)^{\frac{1}{k}}) \leq C \Rightarrow \sup_{t\in[0\,,T]} M_k(t)^{\frac{1}{k}} \leq M_k(0)^{\frac{1}{k}} + C.$$

By assumption on $M_k(0)$ we find for all $t \in [0\,,T]$

$$M_k(t)^{\frac{1}{k}} \leq (C_0k)^{\frac{1}{3}} + C \leq (Ck)^{\frac{1}{3}} \leq (Ck)^{\frac{1}{3}+\frac{1}{k}}, \tag{5.4.9}$$

which finally implies that

$$\sup_{t\in[0\,,T]} M_k(t)^{\frac{3}{k+3}} \leq Ck. \tag{5.4.10}$$

5.4.2 Estimate on the characteristics

We consider two solutions $f_1, f_2 \in L^\infty([0\,,T], L^1\cap L^\infty(\mathbb{R}^3\times\mathbb{R}^3))$ such that ρ_1,ρ_2 verify

$$\rho_1,\rho_2 \in L^\infty([0\,,T], L^p(\mathbb{R}^3)) \tag{5.4.11}$$

for some $p > 3$. This regularity on $\rho_{1,2}$ is guaranteed by the condition (5.2.14) thanks to the estimate (5.4.3). Then we write $Y_1 = (X_1, V_1)$ and $Y_2 = (X_2, V_2)$ for the corresponding characteristics, which are both solutions to (5.1.8) with $t = 0$. This means we can simplify the notation and will write $Y_i(t;0,x,v) = Y_i(t,x,v)$, $i = 1,2$. Regarding the existence of such characteristics, the condition (5.4.11) yields sufficient regularity on the electric field E_i, $i = 1,2$, so that with the added regularity assumption on the magnetic field (5.1.3) we can define weak characteristics thanks to theorem III.2 (section III.2) in [45].

Now we introduce the distance

$$D(t) = \iint_{\mathbb{R}^3\times\mathbb{R}^3} |X_1(t,x,v) - X_2(t,x,v)|\, f^{in}(x,v)dxdv. \tag{5.4.12}$$

From (5.1.8) we can infer that

$$\begin{aligned} X_1(t,x,v) - X_2(t,x,v) &= \int_0^t\int_0^s E_1(\tau, X_1(\tau,x,v)) - E_2(\tau, X_2(\tau,x,v)) \\ &+ V_1(\tau,x,v)\wedge B(\tau, X_1(\tau,x,v)) - V_2(\tau,x,v)\wedge B(\tau,X_2(\tau,x,v))d\tau ds \end{aligned} \tag{5.4.13}$$

which yields that

$$\begin{aligned} D(t) &\le \int_0^t\int_0^s\int_{\mathbb{R}^6} |E_1(\tau, X_1(\tau,x,v)) - E_2(\tau, X_2(\tau,x,v))| \\ &+ |V_1(\tau,x,v)\wedge B(\tau, X_1(\tau,x,v)) - V_2(\tau,x,v)\wedge B(\tau,X_2(\tau,x,v))|\, f^{in}(x,v)dxdvd\tau ds \\ &\le \int_0^t\int_0^s\int_{\mathbb{R}^6} |E_1(\tau, X_1(\tau,x,v)) - E_2(\tau, X_2(\tau,x,v))|\, f^{in}(x,v)dxdvd\tau ds \\ &+ \|B\|_\infty \int_0^t\int_0^s\int_{\mathbb{R}^6} |V_1(\tau,x,v) - V_2(\tau,x,v)|\, f^{in}(x,v)dxdvd\tau ds \\ &+ \int_0^t\int_0^s\int_{\mathbb{R}^6} |V_2(\tau,x,v)|\, |B(\tau, X_1(\tau,x,v)) - B(\tau,X_2(\tau,x,v))|\, f^{in}(x,v)dxdvd\tau ds \\ &= I(t) + J(t) + K(t) \end{aligned}$$

The term $I(t)$ is the quantity estimated thanks to the method in [95]. As for the other two terms $J(t)$ and $K(t)$, since we want to use the same method as in [95] which is to exploit the fact that the characteristics of the Vlasov equation verify an ODE of order 2, we need them to be controlled by $\int_0^t\int_0^s D(\tau)^{1-\frac{3}{p}}d\tau ds$. This is true and the estimates are given in the following proposition.

Proposition 5.4.1. *For all $t\in[0\,,T]$ and for all $p>3$, we have the following estimates:*

$$I(t) \le CpC_{\rho_1,\rho_2}\int_0^t\int_0^s D(\tau)^{1-\frac{3}{p}}d\tau ds \tag{5.4.14}$$

$$K(t) \le (CpK_B + K_{B,p})\int_0^t\int_0^s D(\tau)^{1-\frac{3}{p}}d\tau ds \tag{5.4.15}$$

$$\begin{aligned} J(t) &\le \|B\|_\infty \int_0^t\int_0^s\int_0^\tau (CpC_{\rho_1,\rho_2} + (CpK_B + K_{B,p}))\, D(u)^{1-\frac{3}{p}}dud\tau ds \\ &+ \|B\|_\infty^2 \exp(T\|B\|_\infty)\int_0^t\int_0^s\int_0^\tau\int_0^u (CpC_{\rho_1,\rho_2} + (CpK_B + K_{B,p}))\, D(w)^{1-\frac{3}{p}}dwdud\tau ds. \end{aligned} \tag{5.4.16}$$

with

$$\begin{aligned} C_{\rho_1,\rho_2} &= \max\left(1+\|\rho_1\|_{L^\infty([0\,,T],L^p)}\,, 1+\|\rho_2\|_{L^\infty([0\,,T],L^p)}\right), \\ K_{B,p} &= 2\|B\|_{W^{1,\infty}}\|E_2\|_\infty \exp(T\|B\|_\infty), \\ K_B &= 2\|B\|_{W^{1,\infty}}\exp(T\|B\|_\infty), \end{aligned}$$

and where C denotes a constant that depends only on T, $\left\|f^{in}\right\|_\infty$, $\left\|f^{in}\right\|_1$.

Proof of proposition 5.4.1. As said above, the term $I(t)$ is the quantity estimated thanks to the method in [95], so we treat it identically to find the estimate (5.4.14).

Let's first look at the term $K(t)$. Since $B \in W^{1,\infty}(\mathbb{R}_+ \times \mathbb{R}^3)$ then for all $t \in [0\,, T]$ and $\alpha \in]0\,, 1]$

$$B(t) \in C^{0,\alpha}(\mathbb{R}^3) \tag{5.4.17}$$

with Hölder coefficient $C_{B(t)}$ verifying $C_{B(t)} \leq \max(2\,\|B\|_\infty\,, \|\nabla B\|_\infty) \leq 2\,\|B\|_{W^{1,\infty}}$.

Then we simply have for all $p > 3$

$$K(t) \leq 2\,\|B\|_{W^{1,\infty}} \int_0^t \int_0^s \int_{\mathbb{R}^6} |V_2(\tau, x, v)|\,|X_1(\tau, x, v) - X_2(\tau, x, v)|^{1-\frac{3}{p}} f^{in}(x, v) dx dv d\tau ds \tag{5.4.18}$$

Now we need to estimate the velocity characteristic V_2, and using (5.1.8) we can write once again

$$\begin{aligned} |V(t, x, v)| &\leq |v| + \int_0^t |E(s, X(s, x, v))|\,ds + \|B\|_\infty \int_0^t |V(s, x, v)|\,ds \\ &\leq |v| + T\,\|E\|_\infty + \|B\|_\infty \int_0^t |V(s, x, v)|\,ds. \end{aligned}$$

This classical Grönwall inequality yields for all $t \in [0\,, T]$

$$|V(t, x, v)| \leq (|v| + T\,\|E\|_\infty) \exp(t\,\|B\|_\infty). \tag{5.4.19}$$

So that we can write

$$\begin{aligned} &K(t) \leq 2\,\|B\|_{W^{1,\infty}} \exp(T\,\|B\|_\infty), \\ &\quad \int_0^t \int_0^s \int_{\mathbb{R}^6} (|v| + T\,\|E_2\|_\infty)\,|X_1(\tau, x, v) - X_2(\tau, x, v)|^{1-\frac{3}{p}} f^{in}(x, v) dx dv d\tau ds, \\ &\quad = K_1(t) + K_2(t). \end{aligned} \tag{5.4.20}$$

By applying Jensen's inequality for concave functions to $x \mapsto x^{1-\frac{3}{p}}$ we obtain

$$K_2(t) \leq K_{B,p} \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds. \tag{5.4.21}$$

Then we estimate $K_1(t)$ by writing $f^{in} = (f^{in})^{\frac{1}{p}} (f^{in})^{\frac{1}{p'}}$ where $\frac{1}{p} + \frac{1}{p'} = 1$, so that with the Hölder inequality applied to $|v|\,(f^{in})^{\frac{1}{p}}$ and $|X_1(\tau, x, v) - X_2(\tau, x, v)|^{1-\frac{3}{p}} (f^{in})^{\frac{1}{p'}}$ with the exponents p and p' we have

$$K_1(t) \leq K_B \left(\int_{\mathbb{R}^6} |v|^p f^{in}(x, v) dx dv \right)^{\frac{1}{p}} \int_0^t \int_0^s \left(\int_{\mathbb{R}^6} |X_1(\tau, x, v) - X_2(\tau, x, v)|^{(1-\frac{3}{p})p'} f^{in}(x, v) dx dv \right)^{\frac{1}{p'}} d\tau ds.$$

Using (5.2.15) we have $\left(\int_{\mathbb{R}^6} |v|^p f^{in}(x, v) dx dv\right)^{\frac{1}{p}} \leq (C_0 p)^{\frac{1}{3}} \leq Cp$. Furthermore, we can once again use the Jensen inequality because $(1 - \frac{3}{p})p' = \frac{p-3}{p} \frac{p}{p-1} = \frac{p-3}{p-1} < 1$, which gives us

$$K_1(t) \leq Cp K_B \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds. \tag{5.4.22}$$

This concludes the proof of (5.4.16).

To estimate the last term $J(t)$, we also use (5.1.8) to obtain a Grönwall inequality on $|V_1(t) - V_2(t)|$, and since the computations are complicated we write $V_{1,2}(s), X_{1,2}(s)$ for the characteristics. First we write

$$\begin{aligned}|V_1(t) - V_2(t)| \leq \int_0^t |E_1(s, X_1(s)) - E_2(s, X_2(s))|\, ds + \|B\|_\infty \int_0^t |V_1(s) - V_2(s)|\, ds \\ + \int_0^t |V_2(s)|\, |B(s, X_1(s)) - B(s, X_2(s))|\, ds.\end{aligned}$$

Now using (5.4.17) and (5.4.19) we deduce

$$\begin{aligned}|V_1(t) - V_2(t)| \leq \int_0^t |E_1(s, X_1(s)) - E_2(s, X_2(s))|\, ds + \|B\|_\infty \int_0^t |V_1(s) - V_2(s)|\, ds \\ + (K_B\, |v| + K_{B,p}) \int_0^t |X_1(s) - X_2(s)|^{1-\frac{3}{p}}\, ds\end{aligned}$$

which is just the Grönwall inequality on $|V_1(t) - V_2(t)|$ we were looking for and which yields

$$\begin{aligned}&|V_1(t) - V_2(t)| \leq \int_0^t |E_1(s, X_1(s)) - E_2(s, X_2(s))| + (K_B\, |v| + K_{B,p})\, |X_1(s) - X_2(s)|^{1-\frac{3}{p}}\, ds \\ &+ \int_0^t \left(\int_0^s |E_1(\tau, X_1(\tau)) - E_2(\tau, X_2(\tau))| + (K_B\, |v| + K_{B,p})\, |X_1(\tau) - X_2(\tau)|^{1-\frac{3}{p}}\, d\tau \right) \\ &\times \|B\|_\infty \exp((t-s)\, \|B\|_\infty) ds.\end{aligned}$$

Now we insert this inequality in the definition of $J(t)$ to obtain

$$\begin{aligned}&J(t) \leq \|B\|_\infty \int_0^t \int_0^s \int_0^\tau \int_{\mathbb{R}^6} \left(|E_1(u, X_1(u)) - E_2(u, X_2(u))| + (K_B\, |v| + K_{B,p})\, |X_1(u) - X_2(u)|^{1-\frac{3}{p}} \right) \\ &\times f^{in}(x,v) dx dv du d\tau ds \\ &+ \|B\|_\infty^2 \exp(T\, \|B\|_\infty) \times \\ &\int_0^t \int_0^s \int_0^\tau \int_0^u \int_{\mathbb{R}^6} \left(|E_1(w, X_1(w)) - E_2(w, X_2(w))| + (K_B\, |v| + K_{B,p})\, |X_1(w) - X_2(w)|^{1-\frac{3}{p}} \right) \\ &\times f^{in}(x,v) dx dv dw du d\tau ds.\end{aligned} \tag{5.4.23}$$

Like previously, we can use the Jensen inequality to bound the terms $K_{B,p}\, |X_1 - X_2|^{1-\frac{3}{p}}$, the relation (5.4.14) to bound the terms $|E_1 - E_2|$ and the Hölder inequality used to estimate $K_1(t)$ to bound $K_B\, |v|\, |X_1 - X_2|^{1-\frac{3}{p}}$.

This gives the desired estimate (5.4.15) on $J(t)$:

$$\begin{aligned}J(t) \leq \|B\|_\infty \int_0^t \int_0^s \int_0^\tau (CpC_{\rho_1,\rho_2} + (CpK_B + K_{B,p}))\, D(u)^{1-\frac{3}{p}}\, du d\tau ds \\ + \|B\|_\infty^2 \exp(T\, \|B\|_\infty) \int_0^t \int_0^s \int_0^\tau \int_0^u (CpC_{\rho_1,\rho_2} + (CpK_B + K_{B,p}))\, D(w)^{1-\frac{3}{p}} dw du d\tau ds.\end{aligned}$$

□

5.4.3 A second order inequality on $D(t)$

We begin by looking at the dependence of $K_{B,p}$ with respect to p. The only term in $K_{B,p}$ which depends on p is $\|E_2\|_\infty$, and since $\rho_2 \in L^\infty([0,T], L^p(\mathbb{R}^3))$ with $p > 3$, then we can deduce the desired L^∞ bound on E_2 because for all $t \in [0,T]$

$$\|E_2(t)\|_\infty \leq \left\|\mathbf{1}_{|x|\geq 1}\nabla\mathcal{G}_3\right\|_\infty \|\rho_2(t)\|_1 + \left\|\mathbf{1}_{|x|<1}\nabla\mathcal{G}_3\right\|_q \|\rho_2(t)\|_p \tag{5.4.24}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

From this last inequality we can finally deduce

$$\|E_2\|_\infty \leq C(1 + \|\rho_2\|_{L^\infty([0,T],L^p)}), \tag{5.4.25}$$

where C depends only on $\left\|f^{in}\right\|_1$.

Now we consider that the solutions f_1, f_2 verify the assumptions of theorem 5.2.6. This means that $\max(\|\rho_1\|_{L^\infty([0,T],L^p)}, \|\rho_2\|_{L^\infty([0,T],L^p)}) \leq Cp$ for all $p \geq 1$, and so thanks to (5.4.25) and proposition 5.4.1 we have for all $p > 3$

$$\begin{aligned} D(t) &\leq C_1 p^2 \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds \\ &+ C_2(p^2+p) \int_0^t \int_0^s \int_0^\tau D(u)^{1-\frac{3}{p}} du d\tau ds \\ &+ C_3(p^2+p) \int_0^t \int_0^s \int_0^\tau \int_0^u D(w)^{1-\frac{3}{p}} dw du d\tau ds, \end{aligned} \tag{5.4.26}$$

where C_1, C_2, C_3 are constants that depend on T, $\left\|f^{in}\right\|_\infty, \left\|f^{in}\right\|_1, \|B\|_\infty, \|\nabla B\|_\infty$.

Let $\mathcal{F}(t) = \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds$. Since $\mathcal{F}$ is increasing by construction and since $p > 3$ we can finally conclude that

$$D(t) \leq Cp^2 \int_0^t \int_0^s D(\tau)^{1-\frac{3}{p}} d\tau ds, \tag{5.4.27}$$

with C that depends on T, $\left\|f^{in}\right\|_\infty, \left\|f^{in}\right\|_1, \|B\|_\infty, \|\nabla B\|_\infty$.

Finally, we obtain the same second order differential inequality as in [95], for all $t \in [0,T]$ we have:

$$\mathcal{F}''(t) \leq Cp^2 \mathcal{F}(t). \tag{5.4.28}$$

From this inequality, we use the same method as in [95] to conclude that for all $t \in [0,T]$ we have $f_1(t) = f_2(t)$ a.e. on $\mathbb{R}^3 \times \mathbb{R}^3$. This concludes the proof of theorem 5.2.6.

5.4.4 Proof of theorem 5.2.4

We finish this section with the proof of theorem 5.2.4, which is the extension of Loeper's result [88] to the magnetized Vlasov–Poisson system. As said above, this proof was already done for

B constant in [110]. In fact in [110], it was already proved that under the assumptions (5.2.11), (5.2.12), and (5.2.13), the macroscopic density ρ is bounded (5.2.10), and this proof doesn't change in the case of a general magnetic field.

Like in the theorem 5.2.6, we require additional assumptions on the moments of f^{in} to obtain uniqueness (compared to the unmagnetized case). However, these assumptions on the moments aren't as strong as in theorem 5.2.6 because the boundedness of ρ is already a strong assumption.

To prove our theorem, we only need to adapt subsection 3.2 from [88]. Thus we consider two solutions of (5.1.1) f_1, f_2 with initial datum f^{in} that verifies the assumptions of theorem 5.2.4. Like in the previous proof, we write the corresponding densities, electric fields, and characteristics ρ_1, ρ_2, E_1, E_2, and $Y_1(t,x,v), Y_2(t,x,v) = (X_1(t,x,v), V_1(t,x,v)), (X_2(t,x,v), V_2(t,x,v))$. To simplify the presentation, we will write $Y_i(t)$ for the characteristics. We define the following quantity Q:

$$Q(t) = \frac{1}{2}\int_{\mathbb{R}^6} f^{in}(x,v)\left|Y_1(t,x,v) - Y_2(t,x,v)\right|^2 dxdv. \tag{5.4.29}$$

Now we differentiate Q (which we couldn't do with the distance D (5.4.12)) splitting the magnetic part of the Lorentz force $V \wedge B$ like in the previous section:

$$\begin{aligned}
\dot{Q}(t) &= \int_{\mathbb{R}^6} f^{in}(x,v)(Y_1(t) - Y_2(t)) \cdot \partial_t(Y_1(t) - Y_2(t))dxdv,\\
&= \int_{\mathbb{R}^6} f^{in}(x,v)(X_1(t) - X_2(t)) \cdot (V_1(t) - V_2(t))dxdv\\
&+ \int_{\mathbb{R}^6} f^{in}(x,v)(V_1(t) - V_2(t)) \cdot (E_1(t, X_1(t)) - E_2(t, X_2(t)))dxdv\\
&+ \int_{\mathbb{R}^6} f^{in}(x,v)(V_1(t) - V_2(t)) \cdot \left[V_2(t) \wedge (B_1(t, X_1(t)) - B_2(t, X_2(t))\right] dxdv\\
&+ \int_{\mathbb{R}^6} f^{in}(x,v)(V_1(t) - V_2(t)) \cdot \left[(V_1(t) - V_2(t)) \wedge B(t, X_1(t))\right] dxdv.
\end{aligned}$$

First, we notice that the last term is null, which means we only need to control the second to last term (due to the added magnetic field) which we denote $P(t)$. The first term is bounded by $Q(t)$ and the second term can be estimated using the analysis from [88] and is bounded by $Q(t)\ln(\frac{1}{Q(t)})$. To control $P(t)$ we first use the bound on the velocity characteristic (5.4.19).

$$\begin{aligned}
P(t) &\leq \|B\|_{W^{1,\infty}} \int_{\mathbb{R}^6} f^{in}(x,v)\left|V_1(t) - V_2(t)\right| \left|V_2(t)\right| \left|X_1(t) - X_2(t)\right| dxdv,\\
&\leq \|B\|_{W^{1,\infty}} \int_{\mathbb{R}^6} f^{in}(x,v)\left|V_1(t) - V_2(t)\right| (|v| + T\left\|E_2\right\|_\infty)e^{T\|B\|_\infty} \left|X_1(t) - X_2(t)\right| dxdv,\\
&= R(t) + S(t).
\end{aligned}$$

We recall that since $\|\rho_{1,2}\|_\infty \leq +\infty$ we can bound $\|E_{1,2}\|_\infty$ thanks to (5.4.25) and the interpolation inequality:

$$\|E_i\|_\infty \leq C(\|\rho\|_1, \|\rho\|_\infty) := C_\rho, \tag{5.4.30}$$

with $i = 1, 2$.

This means we can simply estimate $S(t)$ with the Cauchy–Schwarz inequality applied on the functions $(f^{in})^{\frac{1}{2}}|V_1(t)-V_2(t)|$ and $(f^{in})^{\frac{1}{2}}|X_1(t)-X_2(t)|$.

$$\begin{aligned} S(t) &\leq TC_\rho C_{B,T}\left(\int_{\mathbb{R}^6} f^{in}(x,v)\,|V_1(t)-V_2(t)|^2\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^6} f^{in}(x,v)\,|X_1(t)-X_2(t)|^2\right)^{\frac{1}{2}} \\ &\leq TC_\rho C_{B,T}Q(t), \end{aligned}$$

with $C_{B,T}=\|B\|_{W^{1,\infty}}\,e^{T\|B\|_\infty}$.

To control $R(t)$ we first use the Cauchy–Schwarz inequality and then the bound on the velocity characteristic (5.4.19), which also gives us a bound on the position characteristic.

$$\begin{aligned} R(t) &\leq C_{B,T}\int_{\mathbb{R}^6} f^{in}(x,v)\,|v|\,|Y_1(t)-Y_2(t)|^2\,dxdv \\ &\leq C_{B,T}\int_{\mathbb{R}^6} f^{in}(x,v)\,|v|\,|Y_1(t)-Y_2(t)|\left(|V_1|^2+|V_2|^2+|X_1|^2+|X_2|^2\right)^{\frac{1}{2}}dxdv \\ &\leq C_{B,T}Q(t)\int_{\mathbb{R}^6} f^{in}(x,v)\,|v|^2\left(|V_1|^2+|V_2|^2+|X_1|^2+|X_2|^2\right)dxdv \\ &\leq C_{B,T}Q(t)\underbrace{\int_{\mathbb{R}^6} f^{in}(x,v)\,|v|^2\,2\left((|v|+TC_\rho)^2e^{2T\|B\|_\infty}+\left(|x|+T(|v|+TC_\rho)e^{T\|B\|_\infty}\right)^2\right)dxdv}_{=I}. \end{aligned}$$

Thanks to the assumption (5.2.9) of theorem 5.2.4, I is bounded because we have

$$I\leq C\left(\int f^{in}\,|v|^6,\int f^{in}\,|x|^4\right). \tag{5.4.31}$$

From these estimates, we conclude that

$$\frac{d}{dt}Q(t)\leq CQ(t)\left(1+\ln\frac{1}{Q(t)}\right) \tag{5.4.32}$$

with $C:=C\left(T,\|B\|_{W^{1,\infty}},\|\rho\|_1,\|\rho\|_\infty,\int f^{in}\,|v|^6,\int f^{in}\,|x|^4\right)$.

With this inequality we can show, using standard Grönwall type arguments, that $Q(0)=0\Rightarrow Q(t)=0$ for all $t\geq 0$, which concludes the proof of theorem 5.2.4.

5.5 Appendix

As said above, we present a slightly more detailed version of the proof of proposition 5.3.6 compared to the one found in [99].

Proof of proposition 5.3.6. Let $t\in[0,T]$. We note here $H=1+M_{2+\varepsilon}(T)$ and for any $\delta\in[0,t]$ we define $N_1(t,\delta)=\delta Q(t,\delta)^{\frac{4}{3}}$ and $N_2(t,\delta)=(\delta H)^{\frac{1}{2}}$ as in the left hand side of inequality (5.3.4). We set:

$$I=\{\delta\in[0,t]:N_1(t,\delta)\geq N_2(t,\delta)\}. \tag{5.5.1}$$

First let's suppose that I is empty. Then $Q(t,\delta) \lesssim N_2(t,\delta)$ thanks to (5.3.4) for any $\delta \in [0\,,t]$, which means that

$$Q(t,\delta) \lesssim (\delta H)^{\frac{1}{2}} \leq t^{\frac{1}{2}}(1+M_{2+\varepsilon}(T))^{\frac{4}{7}}, \tag{5.5.2}$$

so that (5.3.26) is automatically verified. Now we suppose that there exists $\delta_*(t) \in]0\,,t]$ such that $N_1(t,\delta_*(t)) = N_2(t,\delta_*(t))$. It comes:

$$Q(t,\delta_*(t)) = (\delta_*(t)^{-1}H)^{\frac{3}{8}}. \tag{5.5.3}$$

Then we use the inequality (5.3.4) again so $Q(t,\delta_*(t)) \lesssim N_1(t,\delta_*(t))+N_2(t,\delta_*(t)) = 2N_2(t,\delta_*(t)) \lesssim (\delta H)^{\frac{1}{2}}$, which implies that

$$H^{-\frac{1}{7}} \lesssim \delta_*(t) \tag{5.5.4}$$

and again using (5.5.3) we obtain

$$Q(t,\delta_*(t)) \lesssim H^{\frac{3}{7}}.$$

Now let c_*^{-1} be the implicit constant in (5.5.4), which depends only on the constants in (5.3.1), thanks to (5.5.4) we can write for any $t \in [c_* H^{-\frac{1}{7}}\,,T]$

$$Q(t,c_* H^{-\frac{1}{7}}) \lesssim H^{\frac{3}{7}} \tag{5.5.5}$$

Then for any such t, we can write $t = nc_* H^{-\frac{1}{7}} + r$ with $n \in \mathbb{N}^*$ and $r < c_* H^{-\frac{1}{7}}$ and thanks to the last inequality we obtain

$$\begin{aligned}
Q(t,t) &\leq Q(r,r) + \sum_{p=1}^{n} Q(pc_* H^{-\frac{1}{7}} + r, c_* H^{-\frac{1}{7}}) \\
&\lesssim (rH)^{\frac{1}{2}} + nH^{\frac{3}{7}} \\
&\lesssim c_*(rH)^{\frac{1}{2}} + nc_* H^{-\frac{1}{7}} H^{\frac{4}{7}} \\
&\lesssim c_* t^{\frac{1}{2}} H^{\frac{1}{2}} + tH^{\frac{4}{7}}
\end{aligned}$$

So that finally for all $t \in [c_* H^{-\frac{1}{7}}\,,T]$ we have

$$Q(t,t) \lesssim (t^{\frac{1}{2}} + t)H^{\frac{4}{7}}. \tag{5.5.6}$$

Lastly, if $t \leq c_* H^{-\frac{1}{7}}$ then thanks to (5.5.2) and (5.5.4) we can write

$$Q(t,t) \lesssim (tH)^{\frac{1}{2}}. \tag{5.5.7}$$

This concludes the proof of proposition 5.3.6 because $H > 1$. □

Bibliography

[1] M. Abramovitz and I.A. Stegun. *Handbook of Mathematical Functions with Formulas Graphs and Mathematical Tables.* National Bureau of Standards, USA, 1972.

[2] R. A. Adams and J. J. F. Fournier. *Sobolev Spaces.* Elsevier Science, Oxford, 2003.

[3] A. V. Antonov. Remarks on the problem of stability in stellar dynamics. *Soviet Astronomy*, 4:859–867, 1961.

[4] A. A. Arsenev. Global existence of a weak solution of Vlasov's system of equations. *USSR Computational Mathematics and Mathematical Physics*, 15:136–147, 1975.

[5] G. Backus. Linearized plasma oscillations in arbitrary electron Velocity distributions. *Journal of Mathematical Physics*, 1:178–191, 1960.

[6] D.E. Baldwin and G. Rowlands. Plasma oscillations perpendicular to a weak magnetic field. *The Physics of Fluids*, 9:2444–2453, 1966.

[7] C. Bardos and P. Degond. Existence globale et comportement asymptotique de la solution de l'équation de Vlasov– Poisson. *Comptes Rendus de l'Académie des sciences Série 1 - Mathématiques*, 297:321–324, 1983.

[8] C. Bardos and P. Degond. Global existence for the Vlasov–Poisson equation in 3 space variables with small initial data. *Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire*, 2:101–118, 1985.

[9] J. Batt. Ein Existenzbeweis für die Vlasov-Gleichung der Stellardynamik bei gemittelter Dichte. *Archive for Rational Mechanics and Analysis*, 13:296–308, 1963.

[10] J. Batt. Global symmetric solutions of the initial value problem of stellar dynamics. *Journal of differential equations*, 25:342–364, 1977.

[11] J. Batt and G. Rein. Global classical solutions of the periodic Vlasov–Poisson system in three dimensions. *Comptes Rendus de l'Académie des sciences Série 1 - Mathématiques*, 313:411–416, 1991.

[12] J. Batt and G. Rein. A Rigorous Stability Result for the Vlasov–Poisson System in Three Dimensions. *Annali di Matematica pura ed applicata*, 164:133–154, 1993.

[13] J. Bedrossian and F. Wang. The linearized Vlasov and Vlasov–Fokker–Planck equations in a uniform magnetic field. *Journal of Statistical Physics*, 178:552–594, 2020.

[14] I. B. Bernstein. Waves in a plasma in a magnetic field. *Physical Review*, 109:10–21, 1958.

[15] P. Bertrand, A.Ghizzo, J. Roche, and E. Sonnendrücker. The semi-lagrangian method for the numerical resolution of vlasov equations. *Rapport de recherche n^o 3393, Inria Lorraine*, 1998.

[16] M. S. Birman and M. Z. Solomjak. *Spectral Theory of Self-Adjoint Operators in Hilbert Space*. D. Reidel, Dordrecht, 1987.

[17] L. Boltzmann. Weitere studien über das Wärmegleichgewicht unter Gasmolekülen. *Sitzungsberichte der Kaiserlichen Akademie der Wissenschaften in Wien, mathematisch-naturwissenschaftliche Classe*, 66:275–370, 1872.

[18] M. Bostan. The Vlasov–Poisson system with strong external magnetic field. Finite Larmor radius regime. *Asymptotic Analysis*, 61:91–123, 2009.

[19] W. Braun and K. Hepp. The Vlasov dynamics and its fluctuations in the $\frac{1}{N}$ limit of interacting classical particles. *Communications in Mathematical Physics*, 56:101–113, 1977.

[20] W. Braun and K. Hepp. Vlasov equations. *Functional Analysis and Its Applications*, 13:115–123, 1979.

[21] E. Caglioti and C. Maffei. Time asymptotics for solutions of Vlasov–Poisson equation in a circle. *Journal of Statistical Physics*, 92:301–323, 1998.

[22] Y. Cao, C. Kim, and D. Lee. Global Strong Solutions of the Vlasov–Poisson–Boltzmann System in Bounded Domains. *Archive for Rational Mechanics and Analysis*, 233:1027–1130, 2019.

[23] K.M. Case. Plasma oscillations. *Annals of Physics*, 7:349–364, 1959.

[24] F. Castella. Propagation of space moments in the Vlasov–Poisson equation and further results. *Annales de l'Institut Henri Poincaré C, Analyse Non Linéaire*, 16:503–533, 1999.

[25] F. Charles, B. Després, B. Perthame, and R. Sentis. Nonlinear stability of a Vlasov equation for magnetic plasma. *Kinetic and Related Models*, 6:269–290, 2013.

[26] F. Charles, B. Després, A. Rege, and R. Weder. The magnetized Vlasov–Ampère system and the Bernstein–Landau paradox. *Journal of statistical physics*, 183:23, 2021.

[27] F. Chen. *Introduction to Plasma Physics and Controlled Fusion*. Springer, New York, 1984.

[28] Z. Chen and J. Chen. Moments propagation for weak solutions of the Vlasov–Poisson system in the three-dimensional torus. *Journal of Mathematical Analysis and Applications*, 472:728–737, 2019.

[29] Z. Chen and X. Zhang. Global existence to the Vlasov–Poisson system and propagation of moments without assumption of finite kinetic energy. *Communications in Mathematical Physics*, 343:851–879, 2016.

[30] M. Z. Cheng and G. Knorr. The integration of the Vlasov equation in configuration space. *Journal of Computational Physics*, 22:330–351, 1976.

[31] N. Crouseilles, M. Lemou, F. Méhats, and X. Zhao. Uniformly accurate particle-in-cell method for the long time solution of the two-dimensional Vlasov–Poisson equation with uniform strong magnetic field. *Journal of Computational Physics*, 346:172–190, 2017.

[32] J. M. Dawson. Particle simulation of plasmas. *Reviews of Modern Physics*, 55:403–447, 1983.

[33] A. Decoster, P. A. Markowich, and B. Perthame. *Modeling of Collisions*. Gauthier-Villars, Paris, 1998.

[34] P. Degond. Spectral theory of the linearized Vlasov–Poisson equation. *Transactions of the American Mathematical Society*, 294:435–453, 1986.

[35] P. Degond and F. Filbet. On the asymptotic limit of the three dimensional Vlasov–Poisson system for large magnetic field: Formal derivation. *Journal of Statistical Physics*, 165:765–784, 2016.

[36] B. Després. Trace class properties of the non homogeneous linear Vlasov–Poisson equation in dimension 1+1. *to appear in Journal of Spectral Theory*.

[37] B. Després. Scattering structure and Landau damping for linearized Vlasov equations with inhomogeneous boltzmannian states. *Annales Henri Poincaré*, 20:2767–2818, 2019.

[38] B. Després. Symmetrization of Vlasov–Poisson equations. *SIAM Journal on Mathematical Analysis*, 46:2554–2580, 2014.

[39] L. Desvillettes. On the convergence of splitting algorithms for some kinetic equations. *Asymptotic analysis*, 6:315–333, 1993.

[40] L. Desvillettes and S. Mischler. About the splitting algorithm for Boltzmann and B.G.K. equations. *Mathematical Models and Methods in Applied Sciences*, 6:1079–1101, 1996.

[41] G. Dimarco and L. Pareschi. Numerical methods for kinetic equations. *Acta Numerica*, 23:369–520, 2014.

[42] R. J. DiPerna and P.-L. Lions. Solutions globales d'équations du type Vlasov- Poisson. *Comptes Rendus de l'Académie des sciences Série 1 - Mathématiques*, 307:655–658, 1988.

[43] R. J. Diperna and P.-L. Lions. Global weak solutions of Vlasov–Maxwell systems. *Communications on Pure and Applied Mathematics*, 42:729–757, 1989.

[44] R. J. DiPerna and P.-L. Lions. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Annals of Mathematics*, 130:321–366, 1989.

[45] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and sobolev spaces. *Inventiones mathematicae*, 98:511–547, 1989.

[46] R. J. DiPerna and P.-L. Lions. Global weak solutions of kinetic equations. *Rendiconti del Seminario Matematico Università e Politecnico di Torino*, 46:259–288, 1990.

[47] R. J. DiPerna and P.-L. Lions. Global solutions of Boltzmann's equation and the entropy inequality. *Archive for Rational Mechanics and Analysis*, 114:47–55, 1991.

[48] R. J. DiPerna and P.-L. Lions. Unicité de la solution faible à support compact de l'équation de Vlasov–Poisson. *Comptes Rendus de l'Académie des sciences Série 1 - Mathématiques*, 324:873–877, 1997.

[49] G. Dubus. *From Plain Visualisation to Vibration Sensing: Using a Camera to Control the Flexibilities in the ITER Remote Handling Equipment*. PhD thesis, 10 2014.

[50] J. Duoandikoetxea. *Fourier Analysis.* American Mathematical Society: Graduate Studies in Mathematics Volume 29, Providence, Rhode Island, 2001.

[51] EcosimPro and PROOSIS. www.ecosimpro.com/cryogenic-simulation-applications-in-iter-project-with-cryo-l

[52] B. Eliasson. Numerical simulations of the Fourier transformed Vlasov–Maxwell system in higher dimensions: theory and applications. *Transport Theory and Statistical Physics*, 39:387–465, 2010.

[53] R. F. Boisvert F. W. J. Olver, D.l W. Lozier and C. W. Clark. *Scattering Theory.* National Institute of Standards and Technology and Cambridge University Press, Cambridge, 2010.

[54] J. Freidberg. *Plasma physics and Fusion Energy.* Cambridge University Press, Cambridge, 2007.

[55] E. Frenod, S. Hirstoaga, M. Lutz, and E. Sonnendrücker. Long time behaviour of an exponential integrator for a Vlasov–Poisson system with strong magnetic field. *Communications in Computational Physics*, 18:263–296, 2015.

[56] E. Frenod and E. Sonnendrücker. Homogenization of the Vlasov equation and of the Vlasov–Poisson system with a strong external magnetic field. *Asymptotic Analysis*, 18:193–213, 1998.

[57] E. Frenod and E. Sonnendrücker. Long time behavior of the two-dimensional Vlasov equation with a strong external magnetic field. *Mathematical Models and Methods in Applied Sciences*, 10:539–553, 2000.

[58] I. Gasser, P.-E. Jabin, and B. Perthame. Regularity and propagation of moments in some nonlinear Vlasov systems. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 130:1259–1273, 2000.

[59] R. Glassey. *The Cauchy Problem in Kinetic Theory.* Philadelphia: SIAM, 1996.

[60] F. Golse. Mean field kinetic equations, 2013.

[61] F. Golse, P-L. Lions, B. Perthame, and R. Sentis. Regularity of the moments of the solution of a transport equation. *Journal of Functional Analysis*, 76:110–125, 1988.

[62] F. Golse and T. Paul. Empirical measures and quantum mechanics: applications to the mean-field limit. *Communications in Mathematical Physics*, 369:1021–1053, 2009.

[63] F. Golse and L. Saint-Raymond. The Vlasov–Poisson system with strong magnetic field. *Journal de Mathématiques Pures et Appliquées*, 78:791–817, 1999.

[64] F. Golse and L. Saint-Raymond. The Vlasov–Poisson system with strong magnetic field in quasineutral regime. *Mathematical Models and Methods in Applied Sciences*, 13:661–714, 2003.

[65] Y. Guo. The Vlasov–Maxwell–Boltzmann system near Maxwellians. *Inventiones mathematicae*, 153:593–630, 2003.

[66] Y. Guo. The Vlasov–Poisson–Landau system in a periodic box. *Journal of the American Mathematical Society*, 25:759–812, 2012.

[67] M. Herda. On massless electron limit for a multispecieskinetic system with external magnetic field. *Journal of Differential Equations*, 260:7861–7891, 2016.

[68] E. Horst. Global strong solutions of Vlasov's Equation. Necessary and sufficient conditions for their existence. *Banach Center Publications*, 19:143–153, 1987.

[69] E. Horst. On the asymptotic growth of the solutions of the Vlasov–Poisson system. *Mathematical Methods in the Applied Sciences*, 16:75–85, 1993.

[70] E. Horst and R. Hunze. Weak solutions of the initial value problem for the unmodified nonlinear Vlasov equation. *Mathematical Methods in the Applied Sciences*, 6:262–279, 1984.

[71] H. J. Hwang and Velasquez J. J. L. On the Existence of Exponentially Decreasing Solutions of the Nonlinear Landau Damping Problem. *Indiana University Mathematics Journal*, 68:2623–2660, 2009.

[72] R. Illner and G. Rein. The time decay of the solutions of the Vlasov–Poisson system in the plasma physical case. *Mathematical Methods in the Applied Sciences*, 19:1409–1413, 1996.

[73] IRFM. irfm.cea.fr/fusion/grands-principes/introduction/.

[74] ITER. www.iter.org.

[75] J. H. Jeans. On the theory of star-streaming and the structure of the universe. *Monthly Notices of the Royal Astronomical Society*, 76:70–84, 1915.

[76] T. Kato. *Perturbation Theory of Linear Operators.* Springer, Berlin, 1995.

[77] Y. L. Klimontovitch. *The Statistical Theory of non–Equilibrium Processes in a Plasma.* International Series of Monographs in Natural Philosophy, Vol. 9, Pergamon press, Oxford, 1967.

[78] M. D. Kruskal and C. R. Oberman. On the stability of plasma in static equilibrium. *Physics of Fluids*, 1:275–280, 1958.

[79] R. Kurth. Das Anfangswertproblem der Stellardynamik. *Zeitschrift für Astrophysik*, 30:213–229, 1952.

[80] L. D. Landau. On the vibration of the electronic plasma. *Journal of physics of the Academy of Sciences of the USSR*, 10:25–34, 1946.

[81] L.D. Landau. Die kinetische Gleichung für den Fall Coulombscher Wechselwirkung. *Phys.Z. Sowjet*, 154, 1936.

[82] L.D. Landau. The transport equation in the case of the Coulomb interaction. *Collected papers of L.D. Landau, Pergamon press, Oxford*, pages 163–170, 1965.

[83] I. Langmuir. Oscillations in Ionized Gases. *Proceedings of the National Academy of Science*, 14:627–637, 1928.

[84] P. Lax and R. S. Phillips. *Scattering Theory.* Academic Press, New York, 1967.

[85] E. Lieb and M. Loss. *Analysis.* American Mathematical Society: Graduate Studies in Mathematics Volume 14, Providence, Rhode Island, 1997.

[86] P.-L. Lions. Global solutions of kinetic models and related questions. *Nonequilibrium Problems in Many-Particle Systems. Lecture Notes in Mathematics*, 1551:58–86, 1993.

[87] P. L. Lions and B. Perthame. Propagation of moments and regularity for the 3-dimensional Vlasov–Poisson system. *Inventiones mathematicae*, 105:415–430, 1991.

[88] G. Loeper. Uniqueness of the solution to the Vlasov–Poisson system with bounded density. *Journal de Mathématiques Pures et Appliquées*, 86:68–79, 2006.

[89] L. Navoret M. Mehrenberger and N. Pham. Recurrence phenomenon for Vlasov–Poisson simulations on regular finite element mesh. *Communications in Computational Physics*, 28:877–901, 2020.

[90] J. H. Malmberg and C. B. Wharton. Collisionless Damping of Electrostatic Plasma Waves. *Physical Review Letters*, 13:184–186, 1964.

[91] A. Mangeney, F. Valentini, and P. Veltri. Magnetic-field effects on nonlinear electrostatic-wave Landau damping. *Physical review E*, 71:016402, 2005.

[92] J. C. Maxwell. Illustrations of the dynamical theory of gases. Part I. On the motions and collisions of perfectly elastic spheres. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 19:19–32, 1860.

[93] J. C. Maxwell. Illustrations of the dynamical theory of gases. Part I. On the motions and collisions of perfectly elastic spheres. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 20:21–37, 1860.

[94] J. C. Maxwell. A Dynamical Theory of the Electromagnetic Field. *Philosophical Transactions of the Royal Society of London*, 155:459–512, 1865.

[95] E. Miot. A uniqueness criterion for unbounded solutions to the Vlasov–Poisson system. *Communications in Mathematical Physics*, 346:469–482, 2016.

[96] C. Mouhot and C. Villani. On Landau damping. *Acta Mathematica*, 207:29–201, 2011.

[97] T. Okabe and S. Ukai. On classical solutions in the large in time of two-dimensional Vlasov's equation. *Osaka Journal of Mathematics*, 15:245–261, 1978.

[98] C. Pallard. Large velocities in a collisionless plasma. *Journal of Differential Equations*, 252:2864–2876, 2012.

[99] C. Pallard. Moment propagation for weak solutions to the Vlasov–Poisson system. *Communications in Partial Differential Equations*, 37:1273–1285, 2012.

[100] C. Pallard. Space moments of the Vlasov–Poisson system: propagation and regularity. *SIAM Journal on Mathematical Analysis*, 46:1754–1770, 2014.

[101] B. Perthame. Higher moments for kinetic equations: application to Vlasov–Poisson and Fokker-Planck equations. *Mathematical Methods in the Applied Sciences*, 13:441–452, 1987.

[102] B. Perthame. Time decay, propagation of low moments and dispersive effects for kinetic equations. *Communications in Partial Differential Equations*, 21:659–686, 1996.

[103] K. Pfaffelmoser. Global classical solutions of the Vlasov–Poisson system in three dimensions for general initial data. *Journal of Differential Equations*, 95:281–303, 1992.

[104] M. Reed and B. Simon. *Methods of Modern Mathematical Physics: II Fourier Analysis Self-Adjointness*. Academic Press, San Diego, 1975.

[105] M. Reed and B. Simon. *Methods of Modern Mathematical Physics: IV Analysis of Operators*. Academic Press, San Diego, 1978.

[106] M. Reed and B. Simon. *Methods of Modern Mathematical Physics: III Scattering Theory*. Academic Press, San Diego, 1979.

[107] M. Reed and B. Simon. *Methods of Modern Mathematical Physics: I Functional Analysis*. Academic Press, San Diego, 1980.

[108] A. Rege. github.com/Alex-Rege/VA-P_wB.

[109] A. Rege. Propagation of velocity moments and uniqueness for the magnetized Vlasov–Poisson system. *preprint*.

[110] A. Rege. The Vlasov–Poisson system with a uniform magnetic field: propagation of moments and regularity. *SIAM Journal on Mathematical Analysis*, 53:2452–2475, 2021.

[111] G. Rhein. *Collisionless Kinetic Equations from Astrophysics - The Vlasov–Poisson System*. Handbook of Differential Equations: Evolutionary Equations, Volume 3, Elsevier/North–Holland, Amsterdam, 2007.

[112] D. Ryutov. Landau damping: half a century with the great discovery. *Plasma physics and controlled fusion*, 41:A1–A11, 1999.

[113] D. Salort. Transport equations with unbounded force fields and application to the Vlasov–Poisson equation. *Mathematical Models and Methods in Applied Sciences*, 19:199–228, 2009.

[114] J. Schäffer. Global existence of smooth solutions to the Vlasov–Poisson system in three dimensions. *Communications in Partial Differential Equations*, 16:1313–1335, 1991.

[115] A. L. Skubachevskii. Vlasov–Poisson equations for a two-component plasma in a homogeneous magnetic field. *Russian Mathematical Surveys*, 69:291–330, 2014.

[116] A. L. Skubachevskii and Y. Tsuzuki. Classical solutions of the Vlasov–Poisson equations with external magnetic field in a half-space. *Computational Mathematics and Mathematical Physics*, 57:541–557, 2017.

[117] E. Sonnendrücker. *Modèles Cinétiques pour la Fusion.* Notes du cours de M2, IRMA, Université Louis Pasteur, Strasbourg, 2008.

[118] T. H. Stix. *The theory of Plasma Waves.* McGraw-Hill, New York, 1962.

[119] R. M. Strain and K. Zhu. The Vlasov–Poisson–Landau system in $\mathbb{R}^3$. *Archive for Rational Mechanics and Analysis*, 210:615–671, 2013.

[120] R.M. Strain. The Vlasov–Maxwell–Boltzmann System in the Whole Space. *Communications in Mathematical Physics*, 268:543–567, 2006.

[121] A. I. Sukhorukov and P. Stubbe. On the Bernstein–Landau paradox. *Physics of Plasmas*, 4:2497–2507, 1997.

[122] D. G. Swanson. *Plasma Waves (2nd Edition).* Institute of Physics publishing, Bristol, 2003.

[123] C. Villani. On a new class of weak solutions to the spatially homogeneous Boltzmann and Landau equations. *Archive for Rational Mechanics and Analysis*, 143:273–307, 1998.

[124] Cédric Villani. *Review of mathematical topics in collisional kinetic theory.* Elsevier Science, Amsterdam, 2002.

[125] A. A. Vlasov. The vibrational properties of an electron gas. *Soviet Physics Uspekhi*, 8:291, 1938.

[126] A. A. Vlasov. Vlasov equation and plasma dispersion relation. *Journal of physics of the Academy of Sciences of the USSR.*, 9:25, 1945.

[127] R. Weder. *Scattering Theory for Wave Propagation in Perturbed Stratified Media.* Springer Verlag, New York, 1991.

[128] D. R. Yafaev. *Scattering Theory: Some Old and New Problems.* Springer-Verlag, Berlin, Heidelberg, 2000.

[129] T. Yang, H. Yu, and H. Zhao. Cauchy Problem for the Vlasov–Poisson–Boltzmann System. *Archive for Rational Mechanics and Analysis*, 182:415–470, 2006.

[130] T. Yang and H. Zhao. Global Existence of Classical Solutions to the Vlasov–Poisson–Boltzmann System. *Communications in Mathematical Physics*, 268:569–605, 2006.

Résumé Cette thèse est consacrée à l'étude de modèles cinétiques qui décrivent des plasmas sans collisions soumis à un champ magnétique externe.
Dans une première partie, nous nous intéressons aux plasmas électrostatiques soumis à un champ magnétique externe constant, où l'on constate que certaines ondes ne sont pas amorties. Ce phénomène, qui ne dépend pas de l'intensité du champ magnétique, est connu sous le nom de paradoxe de Bernstein–Landau, car il met en évidence une discontinuité avec la théorie des plasmas non magnétisés, où les ondes longitudinales de charge d'espace, dont le champ électrique, sont amorties. Habituellement, ces propriétés physiques sont étudiées en considérant le système de Vlasov–Poisson linéarisé. Dans ce travail, nous parvenons à réinterpréter le paradoxe de Bernstein–Landau en reformulant ce système comme une équation de type Schrödinger, ce qui nous amène à considérer le système de Vlasov–Ampère magnétisé. Grâce à cette reformulation, nous expliquons le paradoxe de Bernstein–Landau par rapport au spectre d'un certain opérateur auto-adjoint associé à Vlasov–Ampère magnétisé. Enfin, nous construisons des schémas semi-Lagrangiens pour tester les vecteurs propres de l'opérateur Vlasov–Ampère magnétisé.
Dans une deuxième partie, nous étudions certaines propriétés mathématiques du système de Vlasov–Poisson magnétisé, en nous concentrant d'abord sur la propagation des moments en vitesse pour les solutions faibles. Dans le cas d'un champ magnétique constant, nous utilisons une approche eulérienne afin de contrôler directement les moments en vitesse. La présence d'un champ magnétique génère des singularités aux multiples de la période cyclotron. Dans le cas d'un champ magnétique quelconque, nous utilisons une méthode lagrangienne en étudiant cette fois les caractéristiques du système pour obtenir des estimations a priori qui impliquent la propagation des moments. Une contribution originale de ce travail est que, afin de démontrer la propagation de moments pour tout temps dans les deux cas, on utilise un raisonnement par récurrence impliquant la période cyclotron. Enfin, nous mettons en évidence de nouvelles conditions d'unicité pour les solutions du système de Vlasov–Poisson magnétisé avec des hypothèses supplémentaires sur les moments en vitesse.

Mots clefs: Vlasov–Poisson, Champ magnétique, Paradoxe de Bernstein–Landau, propagation de moments en vitesse, période cyclotron, unicité

Abstract This thesis is devoted to the study of kinetic models that describe collisionless plasmas subject to an external magnetic field.
In a first part, we explore how in electrostatic plasmas subject to a constant external magnetic field certain waves are undamped, independently of the strength of the magnetic field. This phenomenon, known as the Bernstein–Landau paradox, highlights a discontinuity with the theory of unmagnetized plasmas, where the longitudinal space charge waves, which include the electric field, are damped. Usually, we study these physical properties by considering the linearized Vlasov–Poisson system. In this work, we manage to reinterpret the Bernstein–Landau paradox by reformulating this system as a Schrödinger type equation which leads us to consider the magnetized Vlasov–Ampère system. With this reformulation, we explain the Bernstein–Landau paradox with respect to the spectrum of a certain self-adjoint magnetized Vlasov–Ampère operator. Finally, we build semi-Lagrangian schemes to test the eigenfunctions of the magnetized Vlasov–Ampère operator.
In a second part, we study certain mathematical properties of the magnetized Vlasov–Poisson system, focusing first on the propagation of velocity of moments for weak solutions. In the case of a constant magnetic field, we use a Eulerian approach in order to control the velocity moments directly. Because of the added magnetic field, we find singularities at multiples of the cyclotron period. In the case of a general magnetic field, we use a Lagrangian method, focusing on the characteristics of the system to obtain a priori estimates that imply propagation of moments. An original feature of this work is that, in both cases, we rely on an induction procedure involving the cyclotron period to prove propagation of velocity moments for all time. Finally, we find new uniqueness conditions for solutions to the magnetized Vlasov–Poisson system that include assumptions on the velocity moments.

Keywords: Vlasov–Poisson, Magnetic field, Bernstein–Landau paradox, propagation of velocity moments, cyclotron period, uniqueness