Some results relating to the spectral decomposition of a linearized Vlasov equation in a uniform magnetic field

A. Rege[1], F. Charles[1], B. Despres[1], R. Weder[2]

[1]Laboratoire Jacques-Louis Lions
Sorbonne Université

[2]Universita Nacional Autonoma de Mexico

08 October 2019

Outline

1. Motivation : the Landau-Bernstein paradox
2. Spectral decomposition of a linearized Vlasov-Ampère system
3. Numerical study with a Semi-Lagrangian scheme : construction of reference solutions

Outline

The paradox

The Landau-Bernstein paradox

"In unmagnetized plasmas, waves exhibit Landau Damping, while in magnetized plasmas, waves perpendicular to the magnetic field are exactly undamped".[1]

- Several older physical papers[2] [1] and more recent mathematical papers[3] have studied the behaviour of magnetized plasmas.
- We want to better understand the transition between magnetized and unmagnetized frameworks.

---

1. A. I. Sukhorukov and P. Stubbe, On the Landau-Bernstein paradox, Phy. of Plasmas, 1997.
2. I. Bernstein, Waves in a Plasma in a Magnetic Field, Phy. Review, 1958.
3. J. Bedrossian and F. Wang, The linearized Vlasov and Vlasov-Fokker-Planck equations in a uniform magnetic field, preprint, 2018.

Numerical illustration : the model

1d-2v Vlasov-Poisson system with magnetic field

$$\begin{cases} \partial_t f + v_1 \partial_x f - E \partial_{v_1} f + \omega_c \left(-v_2 \partial_{v_1} + v_1 \partial_{v_2} \right) f = 0, \\ \partial_x E = 2\pi - \int f dv_1 dv_2. \end{cases} \tag{1}$$

Here $\omega_c > 0$ is the constant cyclotron frequency for electrons. The unknowns are the density of electrons $f(t, x, v_1, v_2) \geq 0$ and the electric field $E(t, x)$. The domain is $\Omega = \mathbb{T} \times \mathbb{R}^2, \quad \mathbb{T} = [0, 2\pi]_{\mathrm{per}}$ is the 1D-torus.

Ions are considered as a motionless background of neutralizing positive charge.

Numerical illustration : magnetic recurrence

Initialization : $f_0(x, v_1, v_2) = (1 + \varepsilon \cos kx) \exp\left(-\frac{v_1^2 + v_2^2}{2}\right)$ with $\varepsilon = 0.001$.

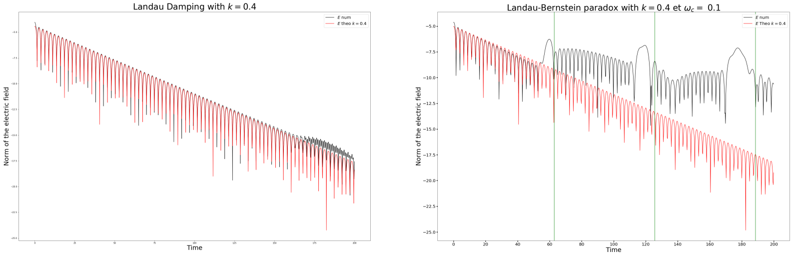


Figure – Damped and undamped electric field

Magnetic recurrence different from the numerical recurrence [4].

4. Recurrence phenomenon for Vlasov-Poisson simulations on regular finite element mesh, M. Mehrenberger, L. Navoret, N. Pham, 2018.

Outline

1. Motivation : the Landau-Bernstein paradox
2. Spectral decomposition of a linearized Vlasov-Ampère system
3. Numerical study with a Semi-Lagrangian scheme : construction of reference solutions

Linearized system

We linearize (1) by writing

$$f = f_0 + \varepsilon\sqrt{f_0}u + O(\varepsilon^2) \text{ and } E = \varepsilon F + O(\varepsilon^2).$$

where $(f_0, E_0) = (\exp(-\frac{v_1^2+v_2^2}{2}), 0)$ is a stationary solution of (1).

Linearized Vlasov-Poisson

$$\begin{cases} \partial_t u + v_1\partial_x u + Fv_1 e^{-\frac{v_1^2+v_2^2}{4}} + \omega_c\left(-v_2\partial_{v_1} + v_1\partial_{v_2}\right)u = 0, \\ \partial_x F = -\int u e^{-\frac{v_1^2+v_2^2}{4}} dv_1 dv_2. \end{cases} \quad (2)$$

- $\int u e^{-\frac{v_1^2+v_2^2}{4}} dx dv_1 dv_2 = 0$. (total mass equal zero)
- $\int F dx = 0$. (To solve the Poisson equation)

Scattering theory

Scattering theory : consider two self-adjoint operators H_0 and H on a Hilbert space $\mathcal{H}$ that are "close" in some sense, then we expect the spectral properties of H to also be close to those of H_0 $\Rightarrow$ the dynamics of $U'(t) = iHU$ similar to the dynamics of $U'(t) = iH_0U$.

- For the self-adjoint operator H, we have the following decomposition of the Hilbert space $\mathcal{H} = \mathcal{H}^{ac} \oplus \mathcal{H}^{sc} \oplus \mathcal{H}^{pp}$. [5] [6]
- One can show that for Vlasov-Poisson without magnetic field, there is only the absolutely continuous part $\mathcal{H}^{ac}$ $\Rightarrow$ Linear Landau damping.
- Scattering structures of inhomogeneous linear Vlasov equations are studied in [7] [8].

5. T. Kato, Perturbation theory for linear operators, 1966.
6. D.R. Yafaev, Scattering theory : Some old and new problems, 2000.
7. B. Despres, Scattering structure and Landau damping for linearized Vlasov eq. with inhomogeneous Boltzmannian states, Ann. IHP, 2019.
8. B. Despres, Trace class properties of the linear Vlasov-Poisson equation, preprint.

No self-adjointness

- We write the linearized system in the framework of scattering theory :

$$\partial_t u = i\mathcal{H}u, \quad i^2 = -1$$

with $\mathcal{H}u = i(v_1\partial_x + \omega_c(-v_2\partial_{v_1} + v_1\partial_{v_2}))u + iFv_1 e^{-\frac{v_1^2+v_2^2}{4}}$
and $\partial_x F = -\int u e^{-\frac{v_1^2+v_2^2}{4}} dv_1 dv_2$.

- Unfortunately,

$$\mathcal{H}^* \neq \mathcal{H}.$$

- Solution : rewrite the system with the Ampère equation (both systems are equivalent).

$$\begin{cases} \partial_t u + v_1 \partial_x u + F v_1 e^{-\frac{v_1^2+v_2^2}{4}} + \omega_c(-v_2 \partial_{v_1} + v_1 \partial_{v_2}) u = 0, \\ \partial_t F = 1^* \int u e^{-\frac{v_1^2+v_2^2}{4}} v_1 dv_1 dv_2. \end{cases} \tag{3}$$

$$\text{with } 1^* g(x) = g(x) - \frac{1}{2\pi} \int_{\mathbb{T}} g(x) dx.$$

Final formulation

$$\partial_t \begin{pmatrix} u \\ F \end{pmatrix} = iH \begin{pmatrix} u \\ F \end{pmatrix}, H = i \left(\begin{array}{c|c} v_1 \partial_x + \omega_c (v_2 \partial_{v_1} - v_1 \partial_{v_2}) & v_1 e^{-\frac{v_1^2+v_2^2}{4}} \\ \hline -1^* \int v_1 e^{-\frac{v_1^2+v_2^2}{4}} \cdot dv_1 dv_2 & 0 \end{array} \right).$$

$$V = \underbrace{\left(L^2(\mathbb{T} \times \mathbb{R}^2) \cap \left\{ \int u \sqrt{f_0} dx dv_1 dv_2 = 0 \right\}\right)}_{=L_0^2(\mathbb{T}\times\mathbb{R}^2)} \times \underbrace{\left(L^2(\mathbb{T}) \cap \left\{ \int F dx = 0 \right\}\right)}_{=L_0^2(\mathbb{T})}$$

Spectral study : eigenvalues and eigenvectors

- To conduct our study, we consider functions proportional to e^{nix}.
- For a non-zero Fourier mode $n \neq 0$, the eigenspaces are as follows :

Space	λ	m
$W_n^1 := \oplus_{m\in\mathbb{Z}^*} \left[e^{mi\varphi - in\frac{v^2}{\omega_c}} V_{n,m} \times \{0\} \right]$	$-m\omega_c$	$m \neq 0$
$W_n^2 := \oplus_{m\in\mathbb{Z}^*} \left\{ (e^{-in\frac{v^2}{\omega_c}} w_{n,m}, -ni) \right\}$	λ_m	$m \neq 0$
$W_n^3 := \text{Span}_\tau \left\{ (e^{-in\frac{v^2}{\omega_c}} \tau(r), 0) \right\} + \left\{ (e^{-\frac{r^2}{4}}, -in) \right\}$	0	

- The eigenspaces corresponding to $n = 0$ are :

Space	λ	m
$W_0^1 := \oplus_{m\in\mathbb{Z}^*} \left[e^{mi\varphi} L^2(\mathbb{R}^+) \times \{0\} \right]$	$-m\omega_c$	$m \neq 0$
$W_0^3 := \oplus_{p\in\mathbb{N}^*} \left[\{\tau_p\} \times \{0\} \right]$	0	$p > 0$

Spectral study : discrete spectrum

Theorem

We have completeness of the eigenspaces ($n \neq 0$).

$$L^2_0(\mathbb{T}\times\mathbb{R}^2)\times L^2_0(\mathbb{T}) = \oplus_{n\neq 0}\left[e^{inx}\left(W^1_n \oplus W^2_n \oplus W^3_n\right)\right] \oplus \left[L^2_0(\mathbb{R}^2)\times 0\right]$$

and so the eigenvalues of H are 0, $-m\lambda_c$ and λ_m, $m \neq 0$.

- This shows that H can be fully diagonalized $\Rightarrow$ their is only discrete spectrum $V = \mathcal{H}^{pp}$.
- New result for this kind of system [9].

9. J. Bedrossian and F. Wang, The linearized Vlasov and Vlasov-Fokker-Planck equations in a uniform magnetic field, preprint, 2018.

Back to the Landau-Bernstein paradox

Spectral explanation for the Landau-Bernstein paradox

1. The Vlasov-Ampère operator H is self-adjoint and it has a complete set of eigenfunctions $\Rightarrow$ electric field is undamped. Expression of electric field with the eigenvectors and eigenvalues :

$$F_n(t) = -nie^{nix} \sum_{m \neq 0} \frac{\left\langle u_0, e^{-in\frac{v_2}{\omega_c}} w_{n,m} \right\rangle + niF_0}{\left\| e^{-in\frac{v_2}{\omega_c}} w_{n,m} \right\|^2 + n^2} e^{i\lambda_m t}$$

2. The Vlasov system without magnetic field has only absolutely continuous spectrum $\Rightarrow$ electric field is damped.

Outline

Initialization : back to the spectral study

Objective : compare the numerical and theoretical solutions of Vlasov-Ampère when initializing with an eigenvector.

- We consider an eigenvector $\begin{pmatrix} w_{n,m} \\ F_n \end{pmatrix}$ associated to the Fourier mode $n \neq 0$ and the eigenvalue λ_m.
- $w_{n,m}$ and F_n are given by

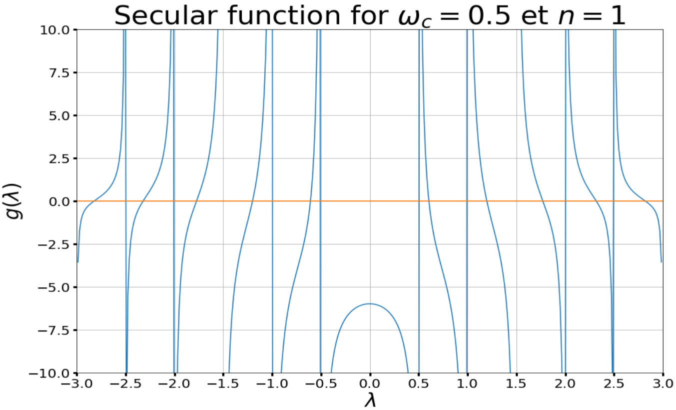
$$w_{n,m} = e^{in(x - \frac{v_2}{\omega_c})} e^{-\frac{r^2}{4}} \sum_{p \in \mathbb{Z}^*} \frac{p\omega_c}{p\omega_c + \lambda_m} e^{pi\varphi} J_p\left(\frac{nr}{\omega_c}\right) \text{ and } F_n = -ine^{inx}$$

- λ_m is one of the roots of a secular equation given by :

$$g(\lambda) = -1 - \frac{2\pi}{n^2} \sum_{m \in \mathbb{Z}^*} \frac{m\omega_c}{m\omega_c + \lambda} \int_0^\infty e^{-\frac{r^2}{2}} J_m\left(\frac{nr}{\omega_c}\right)^2 rdr = 0. \quad (4)$$

Secular equation

g has a unique root in $]m\omega_c\,;(m+1)\omega_c[$ for $m \geq 1$ and $](m-1)\omega_c\,;m\omega_c[$ for $m \leq -1$.



For $(n, m) = (1, 2)$, we compute $\lambda_2 \approx 1.19928$ with a numerical procedure.

Semi-Lagrangian scheme with splitting

Principle of the classical (backward) semi-lagrangian method (Cheng and Knorr JCP76)

The aim is to find an approximation f_n of the solution of $\partial_t f + E(x,t)\partial_x f = 0$ at all discrete time t_n.

- For every point x_i of the grid, we compute the foot at time t_n of the characteristic which is equal to x_i at time t_{n+1}.
- We compute f_{n+1} by interpolation using these values and f_n.

Semi-Lagrangian scheme with splitting

Principle of the classical (backward) semi-lagrangian method (Cheng and Knorr JCP76)

The aim is to find an approximation f_n of the solution of $\partial_t f + E(x,t)\partial_x f = 0$ at all discrete time t_n.

- For every point x_i of the grid, we compute the foot at time t_n of the characteristic which is equal to x_i at time t_{n+1}.
- We compute f_{n+1} by interpolation using these values and f_n.

Splitting

We approximate the solution of $\partial_t f + (\mathcal{A}+\mathcal{B})f = 0$ by solving $\partial_t f + \mathcal{A}f = 0$ and $\partial_t f + \mathcal{B}f = 0$ one after the other.

Splitting for Vlasov-Ampère

We split the Vlasov-Ampère so as to only solve transport equations with constant advection terms

$$\partial_t \begin{pmatrix} u \\ F \end{pmatrix} + (\mathcal{A} + \mathcal{B} + \mathcal{C} + \mathcal{D}) \begin{pmatrix} u \\ F \end{pmatrix} = 0$$

with

$$\mathcal{A} = \begin{pmatrix} v_1 \partial_x \\ 0 \end{pmatrix}, \mathcal{B} = \begin{pmatrix} F v_1 e^{-\frac{v_1^2+v_2^2}{4}} \\ 1^* \int u e^{-\frac{v_1^2+v_2^2}{4}} v_1 dv_1 dv_2 \end{pmatrix}$$

$$\mathcal{C} = \begin{pmatrix} -\omega_c v_2 \partial_{v_1} \\ 0 \end{pmatrix}, \mathcal{D} = \begin{pmatrix} -\omega_c v_1 \partial_{v_2} \\ 0 \end{pmatrix}$$

Algorithm to solve linearized Vlasov-Ampère

1. **Initialization** $U_0 = \begin{pmatrix} w_{n,m} \\ F_n \end{pmatrix}$ given.
2. **Going from t_n to t_{n+1}**
   Assume we know U_n, the approximation of U at time t_n.
   - We compute U^* by solving $\partial_t U + \mathcal{A}U = 0$ with a SL scheme during one time step Δt with initial condition U^n.
   - We compute $\hat{U}$ by solving $\partial_t U + \mathcal{B}U = 0$ with a Runge-Kutta 2 scheme during one time step Δt with initial condition U^*.
   - We compute U^{**} by solving $\partial_t U + \mathcal{C}U = 0$ with a SL scheme during one time step Δt with initial condition $\hat{U}$.
   - We compute U^{n+1} by solving $\partial_t U + \mathcal{D}U = 0$ with a SL scheme during one time step Δt with initial condition U^{**}.

Numerical results for u with $T_{end} = \frac{\pi}{2\lambda_m}$

For all of the simulations, $N_x = 33$, $N_{v_1} = N_{v_2} = 63$, $L_x = 2\pi$, $L_{v_1} = L_{v_2} = 10$ and we have taken $\omega_c = 0.5$ and $n = 1$.

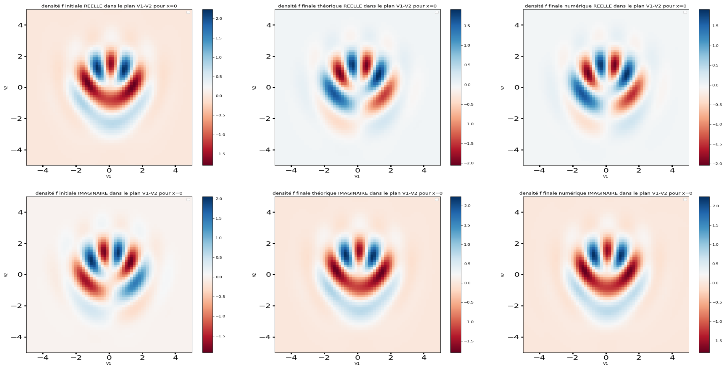


Figure – Real and imaginary parts of u in V1-V2 plane for $x = 0$.

Numerical results for u and F for $T_{end} = \frac{\pi}{2\lambda_m}$

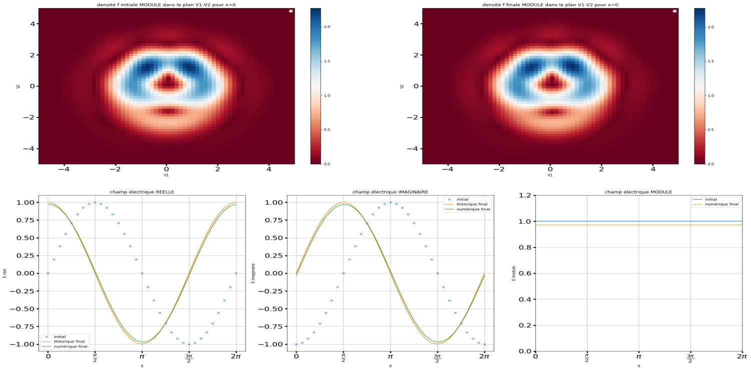


Figure – Module of u in V1-V2 plane for $x = 0$ and real and imaginary parts of F.

Algorithm to solve linearized Vlasov-Poisson

$U_0 = (w_{n,m}, F_n)$ is also a solution of the linearized Vlasov-Poisson system.

1. **Initialization** $u_0 = w_{n,m}$ and $F_0 = F_n$ given.
2. **Going from t_n to t_{n+1}**
   Assume we know u_n and F_n, the approximations of u and F at time t_n.
   - We compute u^* by solving $\partial_t u + v_1 \partial_x u = 0$ with a SL scheme during one time step Δt with initial condition u^n.
   - We compute F_{n+1} by solving the Poisson equation with u^*.
   - We compute u^{**} by solving $\partial_t u + F v_1 e^{-\frac{v_1^2+v_2^2}{4}} = 0$ with an Euler explicit scheme during one time step Δt with initial condition u^*.
   - We compute $\hat{u}$ by solving $\partial_t u - \omega_c v_2 \partial_{v_1} u = 0$ with a SL scheme during one time step Δt with initial condition u^{**}.
   - We compute u^{n+1} by solving $\partial_t u + \omega_c v_1 \partial_{v_2} u = 0$ with a SL scheme during one time step Δt with initial condition $\hat{u}$.

Numerical results for u with $T_{end} = \frac{\pi}{2\lambda_m}$

For all of the simulations, we have taken $\omega_c = 0.5$ and $n = 1$.

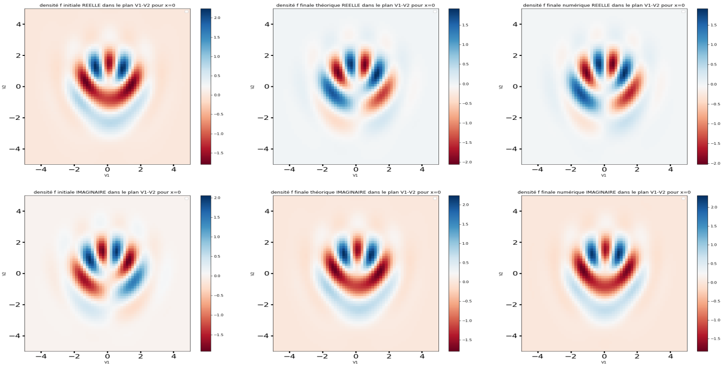


Figure – Real and imaginary parts of u in V1-V2 plane for $x = 0$.

Numerical results for u and F for $T_{end} = \frac{\pi}{2\lambda_m}$

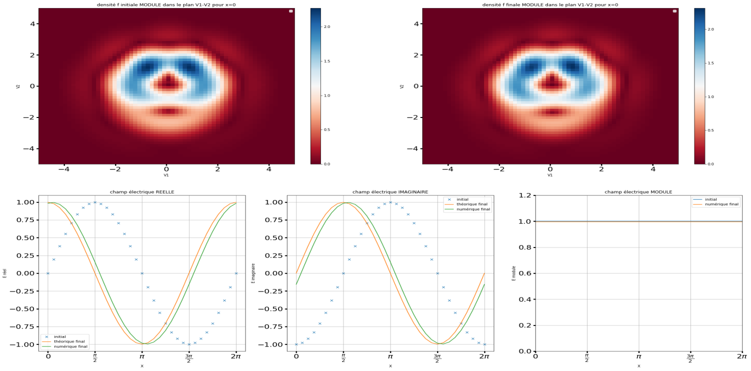


Figure – Module of u in V1-V2 plane for $x = 0$ and real and imaginary parts of F.

Algorithm for Vlasov-Poisson (non linear code)

We can also test the eigenvectors in the non-linear code for Vlasov-Poisson.

1. **Initialization** $f_{ini} = f_0 + \varepsilon\sqrt{f_0}\,\mathrm{Re}(w_{n,m})$ and $E_{ini} = \varepsilon\,\mathrm{Re}(F_n)$ given.
2. **Going from t_n to t_{n+1}**
   Assume we know f_n and E_n, the approximations of u and F at time t_n.
   - We compute f^* by solving $\partial_t f + v_1 \partial_x f = 0$ with a SL scheme during one time step Δt with initial condition f^n.
   - We compute E_{n+1} by solving the Poisson equation with f^*.
   - We compute $\hat{f}$ by solving $\partial_t f - (E_{n+1} + \omega_c v_2)\partial_{v_1} f = 0$ with a SL scheme during one time step Δt with initial condition f^*.
   - We compute f^{n+1} by solving $\partial_t f + \omega_c v_1 \partial_{v_2} f = 0$ with a SL scheme during one time step Δt with initial condition $\hat{f}$.

Numerical results for u and F with $T_{end} = \frac{\pi}{2\lambda_m}$

For all of the simulations, we have taken $\omega_c = 0.5$ and $n = 1$.

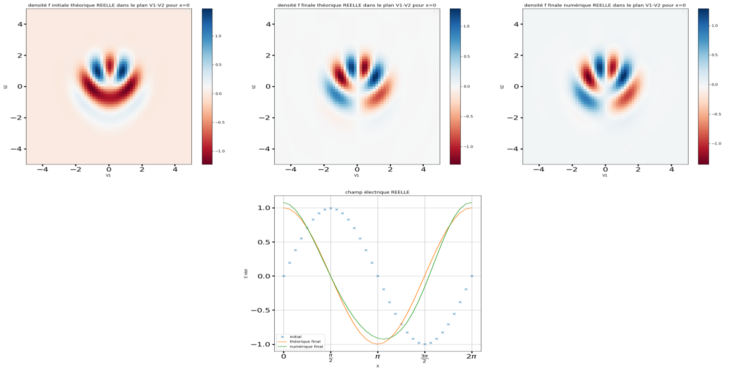


Figure – $\frac{f-f_0}{\varepsilon\sqrt{f_0}} \approx u$ in V1-V2 plane for $x = 0$ and $\frac{E}{\varepsilon} \approx F$.

Summary and perspectives

- Spectral decomposition of the Vlasov-Ampère system
- Reinterpretation of the Landau-Bernstein paradox as a AC spectrum versus PP spectrum.
- Constructed new reference solutions that can be tested on linear and non-linear schemes.

Summary and perspectives

- Spectral decomposition of the Vlasov-Ampère system
- Reinterpretation of the Landau-Bernstein paradox as a AC spectrum versus PP spectrum.
- Constructed new reference solutions that can be tested on linear and non-linear schemes.

Perspective

Limit $\omega_c \to 0$.
Mathematical difficulty :

$$w_{n,m} = e^{in(x - \frac{v_2}{\omega_c})} e^{-\frac{r^2}{4}} \sum_{p \in \mathbb{Z}^*} \frac{p\omega_c}{p\omega_c + \lambda_m} e^{pi\varphi} J_p\left(\frac{nr}{\omega_c}\right)$$

There is a singularity at $\omega_c = 0$.