On electrostatic plasmas and their magnetization

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1. The Vlasov–Poisson system
2. Local solutions to the Vlasov–Poisson system
3. Global solutions to the Vlasov–Poisson system
4. Propagation of moments for weak solutions to the magnetized Vlasov–Poisson system
5. The case of a non-constant magnetic field

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Kinetic formalism and the Vlasov equation

Trajectory $(X(t), V(t))$ of one particle of mass m subject to a force $F(t)$ given by:

$$\begin{cases} \dot{X}(t) = V(t), \\ \dot{V}(t) = \frac{1}{m}F(t). \end{cases}$$

Large number of identical particles allows for a kinetic description of the system. $f(t, x, v)$ is the number density of particles which are located at the position x and have velocity v at time t and satisfies:

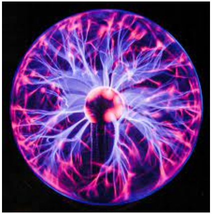
$$\partial_t f(t, x, v) + v \cdot \partial_x f(t, x, v) + \frac{1}{m} F(t, x) \cdot \partial_v f(t, x, v) = 0$$

with $(t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3$.

Long-range interactions

The force F(t) models the long-range interaction between particles.

Coulomb interaction
(repulsive) (Vlasov, 1938)



$$F(t,x) = \frac{q^2}{4\pi\epsilon_0} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{x-y}{|x-y|^3} f(t,y,v)dvdy$$

Gravitational interaction
(attractive) (Jeans, 1915)

$$F(t,x) = -\Gamma m^2 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{x-y}{|x-y|^3} f(t,y,v)dvdy$$

Collisions: Boltzmann type operator in the r.h.s. of the Vlasov equation.

The Vlasov–Poisson system for electrons

- At the time scale of electrons: ions are static $\Longrightarrow f_{ion}$ is constant.
- At the time scale of ions: electrons are at thermal equilibrium $\Longrightarrow f_{electron} =$ Maxwell–Boltzmann type distribution.

At the time scale of electrons:

$$\begin{cases} \partial_t f + v \cdot \partial_x f + \frac{q_e}{m_e} E \cdot \partial_v f = 0, \\ \mathrm{div}_x E(t,x) = \underbrace{\frac{q_{ion}}{\epsilon_0} \int_{\mathbb{R}^3} f_{ion}(x,v) dv}_{=:\rho_{ion}} + \underbrace{\frac{q_e}{\epsilon_0} \int_{\mathbb{R}^3} f(t,x,v) dv}_{=:\rho(t,x)}, \\ E(t,x) = -\partial_x \phi(t,x), \\ \left(-\Delta_x \phi = \frac{q_e}{\epsilon_0} \rho + \frac{q_{ion}}{\epsilon_0} \rho_{ion} \right). \end{cases} \tag{VP}$$

with $f \equiv f(t,x,v)$ the distribution function of electrons, $f_{ion}(x,v)$ the constant ion distribution, $E \equiv E(t,x)$ the self-consistent electric field and ϕ the electrostatic potential.

A priori estimates

- Conservation of L^p norms of f: $\partial_t f + \mathrm{div}_x(vf) + \mathrm{div}_v(Ef) = 0$, so for $1 \leq p \leq \infty$ and $t \geq 0$,
$$\|f(t)\|_p = \left\|f^{in}\right\|_p .$$
- Local conservation of charge:
$$\partial_t \rho(t,x) + \mathrm{div}_x \underbrace{\left(\int_{\mathbb{R}^3} vf(t,x,v)dv \right)}_{=:j(t,x)} = 0.$$
- Conservation of energy:
$$\frac{d}{dt}\left(\iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{1}{2}|v|^2 f(t,x,v)dxdv + \int_{\mathbb{R}^3} \frac{1}{2}|E(t,x)|^2 dx \right) = 0.$$

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Local solution to (VP) for compactly supported initial data

A function $f\colon I \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ is a classical solution to the Vlasov–Poisson system (VP) on the interval $I \in \mathbb{R}$ if

- $f \in C^1(I \times \mathbb{R}^3 \times \mathbb{R}^3)$ and induces a charge density $\rho \in C^1(I \times \mathbb{R}^3)$ and electrostatic potential $\phi \in C^1(I; C^2(\mathbb{R}^3))$.
- For every compact subinterval $J \in I$, the electric field $E = -\partial_x \phi$ is bounded on $J \times \mathbb{R}^3$.
- The functions f, ρ, ϕ satisfy the Vlasov–Poisson system (VP).

Theorem (Existence of local solutions to (VP))

Every $f^{in} \in C_c^1(\mathbb{R}^6), f^{in} \geq 0$ launches a unique classical solution f of (VP) *on some time interval $[0, T[$ with $f(0) = f^{in}$. For all $t \in [0, T[$, $f(t)$ is compactly supported and $f(t) \geq 0$.*

Solving the Vlasov equation (I)

We consider the Vlasov equation

$$\partial_t f + v \cdot \partial_x f + F \cdot \partial_v f = 0.$$

where F is given. Its characteristic system is given by

$$\dot{X}(s) = V(s), \dot{V}(s) = F(s, X(s)).$$

Lemma

Let $I \in \mathbb{R}$ an interval and $F \in C^1(I \times \mathbb{R}^3)$. Then, for $(t, x, v) \in I \times \mathbb{R}^3 \times \mathbb{R}^3$ there exists a unique characteristic flow $Z(s) := (X, V)(s; t, x, v)$ solution to (10) on I, which also verifies

- $Z \in C^1(I \times I \times \mathbb{R}^3 \times \mathbb{R}^3)$.
- *For all $s, t \in I$, the mapping $Z(s; t, \cdot) \colon \mathbb{R}^6 \to \mathbb{R}^6$ is a measure preserving C^1-diffeomorphism.*

Solving the Vlasov equation (II)

A solution f to the Vlasov equation is constant along the characteristic flow $Z(s)$:

$$\frac{d}{ds} f(s, Z(s)) = (\partial_t f + v \cdot \partial_x f + F \cdot \partial_v f)(s, Z(s)) = 0.$$

Lemma

If F verifies the assumptions of the previous lemma, then for $f^{in} \in C^1(\mathbb{R}^6)$ the function

$$f(t, x, v) := f^{in}(Z(0; t; x, v)), t \in I, (x, v) \in \mathbb{R}^6$$

is the unique solution to the Vlasov equation (10).

Solutions to the Poisson equation and interpolation inequalities

Lemma

Let $\rho \in C_c^1(\mathbb{R}^3)$, then ϕ_ρ defined by

$$\phi_\rho(t,x) = \int_{\mathbb{R}^3} \frac{\rho(y)}{4\pi|x-y|} dy,$$

is the unique solution of the Poisson equation $-\Delta\phi = \rho, \lim_{|x|\to\infty} \phi(x) =$ in $C^2(\mathbb{R}^3)$. Furthermore, we have

$$\|\partial_x \phi_\rho\|_\infty \leq c \|\rho\|_1^{\frac{1}{3}} \|\rho\|_\infty^{\frac{2}{3}}, c = 3(2\pi)^{\frac{2}{3}},$$
$$\left\|\partial_x^2 \phi_\rho\right\|_\infty \leq c\left((1 + \|\rho\|_\infty)(1 + \ln_+ \|\partial_x \rho\|_\infty) + \|\rho\|_1\right).$$

Iterative scheme

We consider R_{max}, P_{max} such that

$$f^{in}(x,v) = 0 \text{ for } |v| \geq P_{max} \text{ or } |x| \geq R_{max}.$$

We construct a solution to (VP) using the following iterative scheme:

- $\mathbf{n} = \mathbf{0}$: $f_0(t,x,v) = f^{in}(x,v)$ for $t \geq 0, (x,v) \in \mathbb{R}^6$.
- $\mathbf{n} \to \mathbf{n} + \mathbf{1}$: We define $\rho_n := \int_{\mathbb{R}^3} f_n dv$, $\phi_n := \phi_{\rho_n}$, and $Z_n(s) = (X_n, V_n)(s;t;x;v)$ the solution of the characteristic system

$$\begin{cases} \dot{X} = V, \dot{V} = -\partial_x \phi_n(s, X(s)), \\ (X,V)(t) = (x,v). \end{cases}$$

  Then

$$f_{n+1}(t,x,v) := f^{in}(Z_n(0;t,x,v)).$$

Bounded support for f_n

We have $f_n(t) \in C_c^1(\mathbb{R}^6)$

$$f_n(t,x,v) = 0 \text{ for } |v| \geq P_n(t) \text{ or } |x| \geq R_{max} + \int_0^t P_n(s)ds,$$

where

$$P_0(t) := P_{max}, P_n(t) := \sup\left\{|V_{n-1}(s,0,z)|, \text{ for } z \in \operatorname{supp} f^{in}, 0 \leq s \leq t\right\}.$$

$\implies \rho_n(t) \in C_c^1(\mathbb{R}^3)$,

$$\rho_n(t,x) = 0 \text{ for } |x| \geq R_{max} + \int_0^t P_n(s)ds.$$

Existence time

We have the following estimates on $\rho_n, \partial_x \phi_n$:

$$\|\rho_n\|_\infty \leq c \left\|f^{in}\right\|_\infty P_n^3(t) \underset{\|\partial_x \phi\|_\infty \leq c\|\rho\|_1^{\frac{1}{3}} \|\rho\|_\infty^{\frac{2}{3}}}{\Longrightarrow} \|\partial_x \phi_n\|_\infty \leq C(\left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty) P_n^2(t).$$

By the characteristic system we deduce

$$P_n(t) \leq P_n(0) + C(\left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty) \int_0^t P_n^2(s) ds.$$

Let δ be the maximum existence time of the integral equation

$$P(t) = P_{max} + C(\left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty) \int_0^t P^2(s) ds.$$

We have

$$P(t) = P_{max} \left(1 - P_{max} C(\left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty) t\right)^{-1}, \; \delta = \frac{1}{P_{max} C(\left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty)}.$$

Uniform bound on P_n

For all $n \in \mathbb{N}$ and $t \in [0, \delta[$,

$$P_n(t) \leq P(t).$$

- $\mathbf{n = 0}$: $P_{max} \leq P(t)$.
- $\mathbf{n \to n+1}$:

$$\begin{aligned}|V_n(s; 0, x, v)| &\leq |v| + \int_0^s \|\partial_x \phi_n(\tau)\|_\infty \, d\tau \\ &\leq P_{max} + C(\|f^{in}\|_1, \|f^{in}\|_\infty) \int_0^s P_n^2(\tau) d\tau \\ &\leq P_{max} + C(\|f^{in}\|_1, \|f^{in}\|_\infty) \int_0^t P^2(\tau) d\tau = P(t).\end{aligned}$$

For $n \in \mathbb{N}$ and $t \in [0, \delta[$, we have

$$\|\rho_n\|_\infty \leq c \|f^{in}\|_\infty P^3(t), \;\; \|\partial_x \phi_n\|_\infty \leq C(\|f^{in}\|_1, \|f^{in}\|_\infty) P^2(t).$$

Estimates on the characteristics

We have

$$\begin{aligned}|f_{n+1}(t,x,v)-f_n(t,x,v)| &= |f^{in}(Z_n(0;t,x,v))-f^{in}(Z_{n-1}(0;t,x,v))|\\ &\leq C|Z_n(0;t,x,v)-Z_{n-1}(0;t,x,v)|,\end{aligned}$$

For $0\leq s\leq t$ we have

$$\begin{aligned}|X_n(s)-X_{n-1}(s)| &\leq \int_s^t |V_n(\tau)-V_{n-1}(\tau)|d\tau,\\ |V_n(s)-V_{n-1}(s)| &\leq \int_s^t (|\partial_x\phi_n(\tau,X_n(\tau))-\partial_x\phi_{n-1}(\tau,X_n(\tau))|\\ &\quad +|\partial_x\phi_{n-1}(\tau,X_n(\tau))-\partial_x\phi_{n-1}(\tau,X_{n-1}(\tau))|)d\tau\\ &\leq \int_s^t (\|\partial_x\phi_n(\tau,X_n(\tau))-\partial_x\phi_{n-1}(\tau,X_n(\tau))\|_\infty\\ &\quad +\left\|\partial_x^2\phi_{n-1}(t)\right\|_\infty |X_n(\tau)-X_{n-1}(\tau)|)d\tau.\end{aligned}$$

Estimates on $\partial_x \rho_n, \partial_x^2 \phi_n$

We have

$$|\partial_x \rho_{n+1}(t,x)| \leq \int_{|v|\leq P(t)} |\partial_x (f^{in}(Z_n(0;t,x,v)))| dv \leq C \left\| \partial_x Z_n(0;t,\cdot) \right\|_\infty ,$$

$$|\partial_x Z_n(s)| = |\partial_x X_n(s)| + |\partial_x V_n(s)| \leq \exp \int_0^t (1 + \left\| \partial_x^2 \phi_n(\tau) \right\|_\infty) d\tau.$$

So

$$\left\| \partial_x \rho_{n+1} \right\|_\infty \leq C \exp \int_0^t \left\| \partial_x^2 \phi_n(\tau) \right\|_\infty d\tau,$$

and since $\left\| \partial_x^2 \phi_\rho \right\|_\infty \leq c\left((1 + \|\rho\|_\infty)(1 + \ln_+ \|\partial_x \rho\|_\infty) + \|\rho\|_1\right)$, we finally have for $t \in [0, \delta_0]$ $(0 < \delta_0 < \delta)$ and $n \in \mathbb{N}$

$$\left\| \partial_x^2 \phi_{n+1}(t) \right\|_\infty \leq C \left(1 + \int_0^t \left\| \partial_x^2 \phi_n(\tau) \right\|_\infty d\tau \right) \implies \left\| \partial_x^2 \phi_n(t) \right\|_\infty \leq C \exp Ct.$$

Final estimate on the characteristics

Thanks to Grönwall's inequality we obtain

$$
\begin{aligned}
|Z_n(s) - Z_{n-1}(s)| &\leq C \int_s^t \|\partial_x \phi_n(\tau) - \partial_x \phi_{n-1}(\tau)\|_\infty \, d\tau, \\
&\leq C \int_s^t \|\rho_n(\tau) - \rho_{n-1}(\tau)\|_\infty^{\frac{2}{3}} \|\rho_n(\tau) - \rho_{n-1}(\tau)\|_1^{\frac{1}{3}} \, d\tau, \\
&\leq C \int_s^t \|\rho_n(\tau) - \rho_{n-1}(\tau)\|_\infty \, d\tau \leq C \int_s^t \|f_n(\tau) - f_{n-1}(\tau)\|_\infty \, d\tau.
\end{aligned}
$$

Convergence of the iterative scheme

We finally obtain

$$\|f_{n+1}(t) - f_n(t)\|_\infty \leq C \int_0^t \|f_n(\tau) - f_{n-1}(\tau)\|_\infty \, d\tau,$$

and by induction,

$$\|f_{n+1}(t) - f_n(t)\|_\infty \leq C_* \frac{C^n}{n!}.$$

$\implies f_n \to f$ uniformly on $[0, \delta_0] \times \mathbb{R}^6$ $(0 < \delta_0 < \delta)$, with f verifying

$$f(t, x, v) = 0 \text{ for } |v| \geq P(t) \text{ or } |x| \geq R_{max} + \int_0^t P(s) ds,$$

and $\rho_n \to \rho_f$, $\phi_n \to \phi_{\rho_f}$ uniformly on $[0, \delta_0] \times \mathbb{R}^3$.

Uniqueness

Consider two solutions f, g on some interval $[0, \delta]$ with $f^{in} = g^{in}$, we have

$$\begin{aligned} |f(t,x,v) - g(t,x,v)| &= |f^{in}(Z_f(0;t,x,v)) - g^{in}(Z_g(0;t,x,v))| \\ &\leq C|Z_f(0;t,x,v) - Z_g(0;t,x,v)|. \end{aligned}$$

$\implies$ We can reproduce the same estimates on $f - g$ as what was done for $f_{n+1} - f_n$ to obtain

$$\|f(t) - g(t)\|_\infty \leq C \int_0^t \|f(s) - g(s)\|_\infty \, ds.$$

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Continuation criterion

Theorem

Let $f^{in} \in C_c^1(\mathbb{R}^6), f^{in} \geq 0$ and f be the associated solution on some time interval $[0, T[$. If $T > 0$ is chosen maximal and if

$$P^* = \sup\{|v|, \text{ for } (x, v) \in \text{supp}\, f(t), 0 \leq t < T\} < \infty$$

or

$$R^* = \sup\left\{\rho(t, x), \text{ for } x \in \mathbb{R}^3, 0 \leq t < T\right\} < \infty,$$

then the solution is global.

Proof:
For t_0 close to T, we consider the integral equation

$$P(t) = P^* + \underbrace{C(\|f(t_0)\|_1, \|f(t_0)\|_\infty)}_{=C(\|f^{in}\|_1, \|f^{in}\|_\infty)} \int_{t_0}^{t} P^2(s) ds.$$

The size δ' of the existence interval $[t_0, t_0 + \delta'[$ is independent of t_0.

Estimating the variation of $V(s)$

We define

$$P(t) := 1 + \max\left\{|v| \text{ for } \in \operatorname{supp} f(s), 0 \le s \le t\right\}.$$

Objective: Show that $P(t)$ is bounded on bounded time intervals.
Set $(X^*, V^*)(s) = (X, V)(s; t, x^*, v^*)$, we have

$$\begin{aligned}
|V^*(t) - V^*(t-\Delta)| &\le \int_{t-\Delta}^{t} |E(s, X^*(s))| ds \\
&\le \int_{t-\Delta}^{t} \int_{\mathbb{R}^3} \frac{\rho(s,x)}{|x - X^*(s)|^2} dxds \\
&= \underbrace{\int_{t-\Delta}^{t} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{f(s,x,v)}{|x - X^*(s)|^2} dvdxds}_{:= I^*(t,\Delta)}.
\end{aligned}$$

The good, the bad, and the ugly

We split the phase space in three parts (Pfaffelmoser 1989, Schaeffer 1991):

$$G := \{ \, (s,x,v) : \min(|v|, |v - V^*(s)|) \le p \, \} ,$$

$$\begin{aligned} B := \{ \, (s,x,v) : \min(|v|, |v - V^*(s)|) > p \\ \text{and } \; |x - X^*(s)| \le \max(r|v|^{-3}, r|v - V^*(s)|^{-3}) \big\} , \end{aligned}$$

$$\begin{aligned} U := \{ \, (s,x,v) : \min(|v|, |v - V^*(s)|) > p \\ \text{and } \; |x - X^*(s)| > \max(r|v|^{-3}, r|v - V^*(s)|^{-3}) \big\} , \end{aligned}$$

with $p, r > 0$ parameters to fix later.

$$I^*(t,\Delta) = I^*_G(t,\Delta) + I^*_B(t,\Delta) + I^*_U(t,\Delta).$$

Estimate on each set

For Δ small enough

$$I_G^*(t,\Delta) \leq C p^{\frac{4}{3}} \Delta$$

$$I_B^*(t,\Delta) \leq C r \ln \frac{4P(t)}{p} \Delta$$

$$I_U^*(t,\Delta) \leq C r^{-1}$$

Optimize in p, r so to have the same power of $P(t)$ in order to obtain

$$|V^*(t) - V^*(t-\Delta)| \leq C P(t)^{\frac{16}{33}} \ln P(t) \Delta,$$

This implies

$$P(t) \leq C(1+t) P(t)^{\frac{16}{33}+\varepsilon} \implies P(t) \leq C(1+t)^{\frac{33}{17}+\varepsilon}.$$

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The magnetized Vlasov–Poisson system for electrons

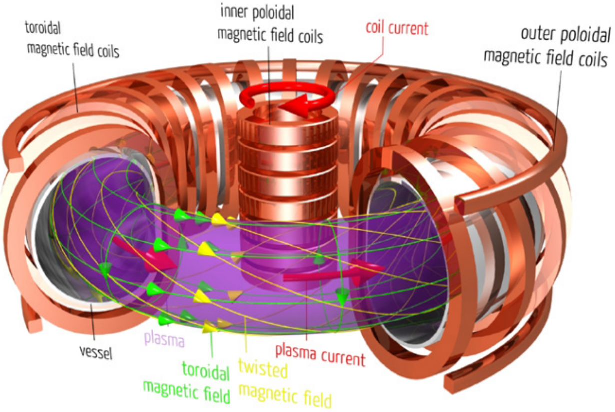
At the time scale of electrons, we have:

$$\begin{cases} \partial_t f + v \cdot \partial_x f + \frac{q_e}{m_e}(E + v \wedge B) \cdot \partial_v f = 0, \\ \mathrm{div}_x E(t,x) = \frac{q_{ion}}{\epsilon_0} \int_{\mathbb{R}^3} f_{ion}(x,v) dx dv + \frac{q_e}{\epsilon_0} \int_{\mathbb{R}^3} f(t,x,v) dx dv, \\ E(t,x) = -\partial_x \phi(t,x), \end{cases} \tag{VPwB}$$

with $f \equiv f(t,x,v)$ the distribution function of electrons, $f_{ion}(x,v)$ the constant ion distribution, $E \equiv E(t,x)$ the self-consistent electric field, ϕ the electrostatic field, and $B \equiv B(t,x)$ the **given** external magnetic field.

Magnetic confinement fusion

- Use an intense external (not self-induced) magnetic field B to confine the hot plasma.
- Feasibility of controlled nuclear fusion: ITER tokamak under construction in Cadarache, France.



Constant B

Results on existence of solutions in the unmagnetized case:

- Existence of weak solutions [Arsenev, 1975]
- Small initial data [Bardos, Degond, 1985]
- Existence of smooth solutions [Pfaffelmoser, 1989]
- Propagation of velocity moments [Lions, Perthame, 1991]

We will first consider a constant B

$$B = \begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}.$$

Weak solutions

A function $f : \mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}_+$ is a weak solution of (VPwB) if we have

- $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3) \cap L^2(\mathbb{R}^3 \times \mathbb{R}^3))$.
- $|v|^2 f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$.
- $\partial_t f + v \cdot \partial_x f + (E + v \wedge B) \cdot \partial_v f = 0$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$.

We will consider solutions such that

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{k_0} f^{in} dxdv < +\infty,$$

$$\implies \mathcal{E}^{in} = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{2} |v|^2 f^{in} dxdv + \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{2} |E^{in}|^2 dxdv < +\infty.$$

Propagation of velocity moments

Theorem (Lions, Perthame 1991)

Let $k_0 > 3, T > 0, f^{in} = f^{in}(x,v) \geq 0$ *a.e. with* $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ *and assume that*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{k_0} f^{in} dxdv < \infty.$$

Then there exists a weak solution f *to* (VP) *and*

$$C = C\left(T, k_0, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}^{in}, \iint |v|^{k_0} f^{in}\right)$$

such that

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^{k_0} f(t,x,v) dxdv \leq C < +\infty, \quad 0 \leq t \leq T.$$

Differential inequality on M_k

For $k \geq 0$, we write

$$M_k(t) := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t,x,v) dvdx.$$

$$\begin{aligned}
|\frac{d}{dt} M_k(t)| &= |\iint |v|^k (-v \cdot \partial_x f - (E + v \wedge B) \cdot \partial_v f) dvdx|, \\
&= |\iint |v|^k \operatorname{div_v}((E + v \wedge B) f)\, dvdx|, \\
&= |\iint k|v|^{k-2} v \cdot (E + v \wedge B) f dvdx|, \\
&\leq \int \left(\int k |v|^{k-1} f dv \right) |E| dx \\
&\leq C \left\| E(t) \right\|_{k+3} M_k(t)^{\frac{k+2}{k+3}}.
\end{aligned}$$

Next step: we need to control of $\|E(t)\|_{k+3}$ with $M_k(t)^\alpha$ with $\alpha \leq \frac{1}{k+3}$.

A representation formula for ρ (I)

We rewrite the Vlasov equation

$$\partial_t f + v \cdot \partial_x f + (v \wedge B) \cdot \partial_v f = -E \cdot \partial_v f.$$

The associated characteristic system is given by

$$\begin{cases} \dot{X}(s) = V(s), \dot{V}(s) = V(s) \wedge B = (\omega V_2(s), -\omega V_1(s), 0), \\ (X(t), V(t)) = (x, v). \end{cases}$$

$$\begin{cases} V(s) = \begin{pmatrix} v_1 \cos(\omega(s-t)) + v_2 \sin(\omega(s-t)) \\ -v_1 \sin(\omega(s-t)) + v_2 \cos(\omega(s-t)) \\ v_3 \end{pmatrix}, \\ X(s) = \begin{pmatrix} x_1 + \frac{v_1}{\omega} \sin(\omega(s-t)) + \frac{v_2}{\omega}(1 - \cos(\omega(s-t))) \\ x_2 + \frac{v_1}{\omega}(\cos(\omega(s-t)) - 1) + \frac{v_2}{\omega} \sin(\omega(s-t)) \\ x_3 + v_3(s-t)) \end{pmatrix}. \end{cases}$$

A representation formula for ρ (II)

We apply the Duhamel formula

$$f(t,x,v) = f^{in}(X(0)), V(0)) - \int_0^t \mathrm{div}_v (fE)(s, X(s), V(s)) ds,$$

We also have

$$\begin{aligned}
&\mathrm{div}_v \int_0^t (fG_t)(s, X(s), V(s)) ds \\
&= \int_0^t \mathrm{div}_v (fE)(s, X(s), V(s)) ds + \mathrm{div}_x \int_0^t (fH_t)(s, X(s), V(s)) ds,
\end{aligned}$$

so that

$$\begin{aligned}
f(t,x,v) = f^{in}(X(0), V(0)) &+ \mathrm{div}_x \int_0^t (fH_t)(s, X(s), V(s)) ds \\
&- \mathrm{div}_v \int_0^t (fG_t)(s, X(s), V(s)) ds.
\end{aligned}$$

$$\rho(t,x) = \underbrace{\int_v f^{in}(X(0), V(0)) dv}_{=:\rho_0(t,x)} + \mathrm{div}_x \int_0^t \underbrace{\int_v (fH_t)(s, X(s), V(s)) \, dv}_{=:\sigma(s,t,x)} ds.$$

Controlling E with σ

$$
\begin{aligned}
E(t,x) &= -\left(\partial_x \frac{1}{|x|} \star \rho\right)(t,x) \\
&= -\left(\partial_x \frac{1}{|x|} \star \rho_0\right)(t,x) - \left(\underbrace{\partial_x \frac{1}{|x|} \star \operatorname{div}_x \int_0^t \sigma(s,t,x) ds}_{=\sum_{q,p=1}^{3} \partial_q \partial_p K_3 \star \int_0^t \int_v f H_t dvds}\right)(t,x) \\
&= E^0(t,x) + \tilde{E}(t,x).
\end{aligned}
$$

Thanks to the Calderón–Zygmund inequality we have

$$
\left\|\tilde{E}(t)\right\|_{k+3} \leq \int_0^t \|\sigma(s,t,x)\|_{k+3} \, ds.
$$

And we can bound E^0 uniformly

$$
\left\|E^0(t)\right\|_{k+3} \leq C(k, \left\|f_1^{in}\right\|, M_k(f^{in})).
$$

Singularities at multiples of the cyclotron frequency

We have

$$|\sigma(s,t,x)| \leq c \left\| H_t(s, X^*(s,x,\cdot), V^*(s,x,\cdot)) \right\|_{L_v^{\frac{3}{2},w}} \|f\|_\infty^{\frac{2}{3}} \left\| f(t-s, X^*(s,x,\cdot),\cdot) \right\|_{L_v^1}^{\frac{1}{3}},$$

which yields

$$\|\sigma(s,t,\cdot)\|_{k+3} \leq C \frac{\sqrt{2}}{s} \left(\frac{\omega^2 s^2}{2(1-\cos(\omega s))} \right)^{\frac{2}{3}} M_k(t-s)^{\frac{1}{k+3}}.$$

Proposition (Propagation of moments on a finite interval)

For all $0 \leq t \leq T_\omega := \frac{\pi}{\omega}$ *we have*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f(t,x,v) dx dv \leq C < +\infty,$$

with $C = C(k, \omega, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

Dealing with the singularity in 0 (I)

We deal with the singularity at 0 by writing

$$\left\| \int_0^t \sigma(s,t,x)ds \right\|_{k+3} = \left\| \int_0^{t_0} \dots \right\|_{k+3} + \left\| \int_{t_0}^{t} \dots \right\|_{k+3}$$

with t_0 a "small time".
Rough estimate in small time

$$\left\| \int_0^{t_0} \sigma(s,t,x)ds \right\|_{k+3} \leq (1+t)^{\delta} t_0^{\beta} \left(1 + \sup_{0\leq s\leq t} M_k(s)\right)^{\alpha} \text{ with } \alpha > \frac{1}{k+3}, \beta > 0.$$

Dealing with the singularity in 0 (II)

Precise estimate in large time

$$\left\| \int_{t_0}^{t} \sigma(s,t,\cdot)ds \right\|_{k+3} \leq C \ln\left(\frac{t}{t_0}\right) \sup_{0\leq s\leq t} M_k(s)^{\frac{1}{k+3}}.$$

Optimizing in t_0, we obtain

$$\|E(t,\cdot)\|_{k+3} \leq C \left(1 + \sup_{0\leq s\leq t} M_k(s)\right)^{\frac{1}{k+3}} \left(1 + \ln\left(1 + \sup_{0\leq s\leq t} M_k(s)\right)\right).$$

Propagation of moments for all time

We have that

- $\left\|f^{in}\right\|_1 = \|f(T_\omega)\|_1$ and $\left\|f^{in}\right\|_\infty = \|f(T_\omega)\|_\infty$,
- $\mathcal{E}(T_\omega) \leq \mathcal{E}_{in}$,
- $M_k(f(T_\omega)) \leq C(k, \omega, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

This means $f(T_\omega)$ verifies the same assumptions as $f^{in} \implies$ we can show propagation of moments for all time by induction.

Uniqueness for bounded ρ

Uniqueness with bounded charge density (Loeper, 2006)

Let $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ be a probability density such that for all $T > 0$

$$\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3)} < +\infty$$

then there exists at most one solution to (VPwB).

Example: $f^{in}(x,v) = \frac{\phi(|x|)}{1+|v|^7}$ with $\phi \in L^1 \cap L^\infty(\mathbb{R}, \mathbb{R}_+)$.

Uniqueness for unbounded ρ

Uniqueness with unbounded charge density (Miot, 2016)

Let $f^{in} \in L^1 \cap L^\infty(\mathbb{R}^3 \times \mathbb{R}^3)$ be a probability density such that for all $T > 0$

$$\sup_{[0,T]} \sup_{p \geq 1} \frac{\|\rho(t)\|_p}{p} < +\infty$$

then there exists at most one solution to (VPwB).

Example: $f^{in}(x,v) = \phi\left(|v|^2 - (\ln_-|x|)^{2/3}\right)$ with $\phi \in L^\infty(\mathbb{R}, \mathbb{R}_+)$.

Propagation of regularity

Theorem (Propagation of regularity)

Let $h \in C^1(\mathbb{R})$ *such that*

$$h \geq 0, h' \leq 0 \text{ and } h(r) = \mathcal{O}(r^{-\alpha}) \text{ with } \alpha > 3,$$

and let $f^{in} \in C^1(\mathbb{R}^3)$ *be a probability density on* $\mathbb{R}^3 \times \mathbb{R}^3$ *such that* $f^{in}(x, v) \leq h(|v|)$ *for all* x, v *and which verifies*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} (1 + |v|^{k_0}) f^{in}(x, v) dxdv < \infty$$

with $k_0 > 6$.
Then there exists a unique classical solution f *to* (VPwB) *with* $f(0) = f^{in}$.

1 The Vlasov–Poisson system

2 Local solutions to the Vlasov–Poisson system

3 Global solutions to the Vlasov–Poisson system

4 Propagation of moments for weak solutions to the magnetized Vlasov–Poisson system

5 The case of a non-constant magnetic field

Lagrangian formulation for propagation of velocity moments

Characteristic system of the Vlasov–Poisson system

$$\dot{X}(s) = V(s), \dot{V}(s) = E(s, X(s)).$$

Consider the quantity

$$Q(t,\delta) := \sup \left\{ \int_{t-\delta}^{t} |E(s, X(s; 0, x, v))| ds, (x, v) \in \mathbb{R}^3 \times \mathbb{R}^3 \right\},$$

then

$$\begin{aligned} M_k(t) &= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |V(t; 0, x, v)|^k f^{in}(x, v) dv dx, \\ &\leq \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(|v| + \left(\sup_{0 \leq t \leq T} Q(t,t) \right) \right)^k f^{in}(x, v) dv dx \\ &\leq 2^{k-1} \left(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^k f^{in}(x, v) dv dx + \left(\sup_{0 \leq t \leq T} Q(t,t) \right)^k \left\| f^{in} \right\|_1 \right) \end{aligned}$$

Propagation of moments of order $k > 2$

Theorem (Pallard, 2012)

Let $T > 0, k > 2$, and f^{in} such that $M_k(f^{in}) < +\infty$, then for all $0 \leq t \leq T$,

$$Q(t) \leq C(T^{\frac{1}{2}} + T^{\frac{7}{5}}),$$

with $C = C(k, \left\|f^{in}\right\|_1, \left\|f^{in}\right\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

The good, the bad, and the ugly 2

$$\int_{t-\delta}^{t} |E(s, X(s; t, x_*, v_*))| ds \leq \int_{t-\delta}^{t} \int \frac{\rho(s,x)dx}{|x - X_*(s)|^2} ds = \int_{t-\delta}^{t} \iint \frac{f(s,x,v)dvdx}{|x - X_*(s)|^2} ds.$$

$$\begin{aligned} G &= \{(s,x,v) : \min(|v|, |v - V_*(s)|) < P\}, \\ B &= \{(s,x,v) : |x - X_*(s)| \leq \Lambda_\varepsilon(s,v)\} \setminus G, \\ U &= [t-\delta, t] \times \mathbb{R}^3 \times \mathbb{R}^3 \setminus (G \cup B), \end{aligned}$$

with $P = 2^{10} Q(t,\delta)$ and $\Lambda_\varepsilon(s,v) = L(1 + |v|^{2+\varepsilon})^{-1} |v - V_*(s)|^{-1}$. Estimate on U:

$$I_U^*(t,\delta) \leq L^{-1}(1 + M_{2+\varepsilon}(T)).$$

Adding a general magnetic field

Now we have $B := B(t,x)$ such that

$$B \in L^\infty(\mathbb{R}_+; W^{1,\infty}(\mathbb{R}^3)),$$

$$\dot{X}(s) = V(s), \dot{V}(s) = E(s, X(s)) + V(s) \wedge B(s, X(s)).$$

So

$$V(t;0,x,v) = v + \int_0^t E(s, X(s;0,x,v))ds + \int_0^t V(s;0,x,v) \wedge B(s, X(s;0,x,v))ds$$

$\implies |V(t;0,x,v)| \leq (|v| + \sup_{0\leq t\leq T} Q(t,t)) \exp(t\|B\|_\infty)$, which yields

$$M_k(t) \leq 2^{k-1} \exp(kt\|B\|_\infty) \left(\int_{\mathbb{R}^3\times\mathbb{R}^3} |v|^k f^{in} x + \left(\sup_{0\leq t\leq T} Q(t,t) \right)^k \left\| f^{in} \right\|_1 \right).$$

Non-constant uniform B

Assume

$$B = B(t).$$

Theorem

For all time t such that $0 \leq t \leq T_B$,

$$Q(t) \leq C \exp(T_B \|B\|_\infty)^{\frac{2}{5}} (T_B^{\frac{1}{2}} + T_B^{\frac{7}{5}}),$$

with $C = C(k, \|B\|_\infty, \|f^{in}\|_1, \|f^{in}\|_\infty, \mathcal{E}_{in}, M_k(f^{in}))$.

$$\begin{aligned}
G &= \{(s,x,v) : \min(|v|, |v - V_*(s)|) < P\}, \\
B &= \{(s,x,v) : |x - X_*(s)| \leq \Lambda_\varepsilon(s,v)\} \setminus G, \\
U &= [t-\delta, t] \times \mathbb{R}^3 \times \mathbb{R}^3 \backslash (G \cup B),
\end{aligned}$$

with $P = 2^{10} Q(t,\delta) \exp(\delta \|B\|_\infty)$ and $\Lambda_\varepsilon(s,v) = L(1 + |v|^{2+\varepsilon})^{-1} |v - V_*(s)|^{-1}$.

Non constant, non-uniform B

Assume

$$B = B(t, x)$$

Difficulty to control the difference between velocity characteristics $|V(s) - V_*(s)|$ in terms of $Q(t)$ and $|V(s) - V_*(s)|$:

$$\begin{aligned}|V(s) - V_*(s)| \leq\ & |v - v_*| + 2Q(t) \\ & + \int_s^t |V(s) \wedge B(s, X(s)) - V_*(s) \wedge B(s, X_*(s))| ds.\end{aligned}$$

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