Quasineutral limit of Vlasov-Poisson to Yudovich solutions of Incompressible Euler

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Kinetic formalism and the Vlasov equation

Trajectory $(X(t), V(t))$ of one particle of mass m subject to a force $F(t)$ given by:

$$\begin{cases} \dot{X}(t) = V(t), \\ \dot{V}(t) = \frac{1}{m}F(t). \end{cases}$$

Large number of identical particles allows for a kinetic description of the system. $f(t, x, v)$ is the number density of particles which are located at the position x and have velocity v at time t and satisfies:

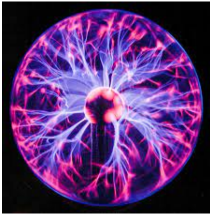
$$\partial_t f(t, x, v) + v \cdot \partial_x f(t, x, v) + \frac{1}{m} F(t, x) \cdot \partial_v f(t, x, v) = 0$$

with $(t, x, v) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$.

Long-range interactions

The force F(t) models the long-range interaction between particles.

Coulomb interaction (repulsive) (Vlasov, 1938)



$$F(t,x) = \frac{q^2}{4\pi\epsilon_0} \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{x-y}{|x-y|^3} f(t,y,v)dvdy$$

Gravitational interaction (attractive) (Jeans, 1915)

$$F(t,x) = -\Gamma m^2 \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{x-y}{|x-y|^3} f(t,y,v)dvdy$$

Collisions: Boltzmann type operator in the r.h.s. of the Vlasov equation.

The Vlasov–Poisson system for electrons

- At the time scale of electrons: ions are static $\Longrightarrow f_{ion}$ is constant.
- At the time scale of ions: electrons are at thermal equilibrium $\Longrightarrow f_{electron} =$ Maxwell–Boltzmann type distribution.

At the time scale of electrons:

$$\begin{cases} \partial_t f + v \cdot \partial_x f + \frac{q_e}{m_e} E \cdot \partial_v f = 0, \\ \mathrm{div}_x E(t,x) = \frac{q_{ion}}{\epsilon_0} \underbrace{\int_{\mathbb{R}^3} f_{ion}(x,v) dv}_{=: \rho_{ion}} + \frac{q_e}{\epsilon_0} \underbrace{\int_{\mathbb{R}^3} f(t,x,v) dv}_{=: \rho(t,x)}, \\ E(t,x) = -\partial_x \phi(t,x), \\ -\Delta_x \phi = \frac{q_e}{\epsilon_0} \rho + \frac{q_{ion}}{\epsilon_0} \rho_{ion}. \end{cases} \tag{VP}$$

with $f \equiv f(t,x,v)$ the distribution function of electrons, $f_{ion}(x,v)$ the constant ion distribution, $E \equiv E(t,x)$ the self-consistent electric field, and ϕ the electrostatic potential.

Well-posedness results

Existence

- Existence of weak solutions in dimension 3 [Arsenev, 1975],
- Existence of smooth solutions in dimension 2 [Okabe, Ukai, 1978],
- Existence of smooth solutions in dimension 3 [Pfaffelmoser, 1992],
- Propagation of velocity moments with Eulerian approach in dimension 3 [Lions, Perthame, 1991],
- Propagation of velocity moments with Lagrangian approach in dimension 3 [Pallard, 2012].

Uniqueness

- Uniqueness of solutions with bounded density, [Loeper, 2006],
- Uniqueness for certain solutions with unbounded density, [Miot, 2016].

Quasineutrality

Charge disturbances in a plasma are screened, as particles within the plasma move to counteract the imbalance.
The **Debye length** describes the scale of **charge separation**,

$$\lambda_D := \left(\frac{\epsilon_0 k_B T}{nq^2} \right)^{1/2}.$$

Plasmas are typically **quasineutral**
$\rightarrow$ the Debye length is much shorter than the scale of observation:

$$\varepsilon := \frac{\lambda_D}{L}$$

Plasma type	Tokamak	Solar loop	Magnetosphere	Solar wind
Debye length	10^{-4}m	10^{-3}m	10^{-1}m	10m
Observation length	$\sim$ 1m	$\geq 10^3$m	$\geq 10^3$m	$\geq 10^3$m

The **quasineutral limit** is the limit $\varepsilon \rightarrow 0$.

Quasineutral limit

In appropriate units, (VP) becomes:

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \partial_x f_\varepsilon + E_\varepsilon \cdot \partial_v f_\varepsilon = 0, \\ E_\varepsilon(t,x) = -\partial_x \phi_\varepsilon(t,x), \\ -\varepsilon^2 \Delta_x \phi_\varepsilon = 1 - \rho_\varepsilon. \end{cases} \tag{VP_ε}$$

It's formal limit is the **kinetic incompressible Euler** system:

$$\begin{cases} \partial_t f + v \cdot \partial_x f + E \cdot \partial_v f = 0, \\ E(t,x) = -\partial_x \phi(t,x), \\ \rho = 1. \end{cases} \tag{kIE}$$

Why kinetic Euler?

A monokinetic profile

$$f(t,x,v) = \rho(t,x)\delta(v - u(t,x))$$

is a solution to (kIE) iff (ρ, u) solves the incompressible Euler equations

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho u) = 0, \\ \partial_t u + (u \cdot \nabla_x) u = -\nabla p, \quad \mathrm{div}_x u = 0. \end{cases} \tag{E}$$

Some results on the quasineutral limit

- Convergence of (VP_ε) to (kIE) using defect measures [Brenier, Grenier, 1994, 1995],
- Convergence of (VP_ε) to (kIE) for analytic solutions [Grenier, 1996]
- Convergence for monokinetic target solutions (convergence to Euler) [Brenier, Masmoudi, Han-Kwan, 2000, 2001, 2011]
- Convergence in 1d for solutions with high Sobolev regularity [Grenier, Han-Kwan, Hauray, 1999, 2015]

Convergence to dissipative solutions of Euler

Brenier shows[1] the convergence of weak solutions of $(\mathrm{VP}_\varepsilon)$ to **dissipative** solutions of (E).

Dissipative solutions of (E)

We say that u is a *dissipative solution* to (E) with initial data $u(0) = u_0$ if

- $u \in L^\infty((0,\infty); L^2(\mathbb{T}^2)) \cap C((0,\infty); L^2_w(\mathbb{T}^2))$,
- $\mathrm{div}(u) = 0$ in $\mathcal{D}'((0,T) \times \mathbb{T}^2)$,
- It holds that

$$\begin{aligned}\int_{\mathbb{T}^2} |u-v|^2(t,x)dx \leq \exp\left(2\int_0^t \|d(u-v)(s,\cdot)\|_\infty ds\right)\int_{\mathbb{T}^2} |u_0 - v_0|^2(x)dx \\ +2\int_0^t\int_{\mathbb{T}^2} \exp\left(2\int_s^t \|d(u-v)(\tau,\cdot)\|_\infty d\tau\right) A(v)\cdot(u-v)dxds\end{aligned}$$

for all smooth v.

[1] Y. Brenier, *Convergence of the Vlasov–Poisson system to the incompressible Euler equations*, Comm. Partial Differential Equations, 2000.

2D case: Yudovich solutions

$$\partial_t \omega + \mathrm{div}(\omega u) = 0,\ \omega = \Delta\psi,\ u = (\nabla\psi)^{\perp}, \tag{E}$$

with u the fluid velocity and ω the fluid vorticity.

Definition of Yudovich solutions to 2D Euler

We say that $\omega : [0, T] \times \mathbb{T}^2 \to \mathbb{R}$ is a weak solution to (E) if $\omega \in L^\infty([0, T]; L^1(\mathbb{T}^2) \cap L^p(\mathbb{T}^2))$ for some $p > 2$ and for all $\varphi \in W^{1,\infty}([0, T]; C_0^1(\mathbb{R}^2))$ and all $t \in [0, T]$ it holds that

$$\begin{aligned}\int_{\mathbb{T}^2} \omega(t,x)\varphi(t,x)dx = \int_{\mathbb{T}^2} \omega_0(x)\varphi(0,x)dx + \int_0^t \int_{\mathbb{T}^2} \omega(s,x)\partial_s\varphi(s,x)dxds \\ + \int_0^t \int_{\mathbb{T}^2} \nabla\varphi(s,x) \cdot \nabla^{\perp} V \star \omega(s,x)\omega(s,x)dxds.\end{aligned}$$

- Their exists a unique Yudovich solution if $\omega_0 \in L^\infty(\mathbb{T}^2)^2$.
- For these solutions, we have $\nabla_x u \in \mathrm{BMO}(\mathbb{T}^2)$.

[2] V. Yudovich. *Non-stationary flow of an ideal incompressible liquid*. USSR Computational Mathematics and Mathematical Physics, 1963.

Modulated energy

We use the modulated energy of Brenier:

$$E_\varepsilon(t) \coloneqq \frac{1}{2}\int |\xi - u(t,x)|^2 f^\varepsilon(t,x,\xi)dxd\xi + \frac{\varepsilon}{2}\int |\nabla\phi^\varepsilon|^2(t,x)dx.$$

We have that

$$\begin{aligned}\dot{E}_\varepsilon(t) = &-\int_{\mathbb{T}^2}\int_{\mathbb{R}^2} d(u)(t,x) : (\xi - u(t,x)) \otimes (\xi - u(t,x)) f^\varepsilon(t,x,\xi)dxd\xi \\ &+ \varepsilon\int_{\mathbb{T}^2} d(u)(t,x) : \nabla\phi^\varepsilon \otimes \nabla\phi^\varepsilon dx \\ &+ \int_{\mathbb{T}^2} A(u)(t,x)\cdot u(t,x)\left(\rho^\varepsilon(t,x) - 1\right)dx.\end{aligned}$$

Harmonic analysis

Definition of BMO

Let $f \in L^1(\mathbb{R}^d)$. We shall say that f is of *bounded mean oscillation* or in brief $\mathrm{BMO}(\mathbb{R}^d)$, if

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^d)} := \sup_{Q \subset \mathbb{R}^d} \frac{1}{|Q|} \int_Q \left| f(x) - \frac{1}{|Q|} \int_Q f \right| dx < \infty.$$

Definition of $\mathcal{H}^1$

For each $f \in L^1(\mathbb{T}^d)$ *the maximal function of* f is denoted by $\mathcal{M}f$ and is defined for all $x \in \mathbb{T}^d$ by

$$\mathcal{M}f(x) := \sup_{x \in Q \subset \mathbb{T}^d} \frac{1}{|Q|} \int_Q |f|(y) dy.$$

$$\mathcal{H}^1(\mathbb{T}^d) := \left\{ f \in L^1(\mathbb{T}^d) \,\middle|\, \mathcal{M}f \in L^1(\mathbb{T}^d) \right\},$$

Tools

- Fefferman showed[3] that BMO is the dual of the Hardy space $\mathcal{H}^1$, supplemented with the inequality

$$\left| \int_{\mathbb{R}^d} fg \right| \leq C \left\| g \right\|_{\mathcal{H}^1(\mathbb{R}^d)} \left\| f \right\|_{\mathrm{BMO}(\mathbb{R}^d)} .$$

- Novel quantitative estimates on $\left\| \rho_\varepsilon \right\|_{L^\infty(\mathbb{T}^2)}$.

[3] C. Fefferman, *"Characterizations of bounded mean oscillation*, Bulletin of the American Mathematical Society, 1971.

Convergence of (VP_ε) to Yudovich solutions of (E)

Theorem (Ben-Porat, Iacobelli, R. 2024)

Let the following assumptions on the initial data hold
H1. $0 \leq f_0^\varepsilon \in L^\infty \cap L^1(\mathbb{T}^2 \times \mathbb{R}^2)$ *is a probability density.*
H2. $\left\|(1+|\xi|^{k_0})f_0^\varepsilon\right\|_{L^\infty \cap L^1} \leq \overline{M}_\varepsilon$ *where* $k_0 > 2$ *and* $\overline{M}_\varepsilon = O(\varepsilon^{-\alpha})$ *for some* $\alpha > 0$.
H3. *There is some* $\beta > 0$ *such that* $E_\varepsilon(0) = O(\varepsilon^\beta)$.
H4. $\omega_0 \in L^\infty(\mathbb{T}^2)$.
Let f^ε *be the unique solution to* (VP_ε) *with initial data* f_0^ε, *and let* ω *be the Yudovich solution to* (E) *with initial data* ω_0. *Then, there is some* $T_* = T_*(\|\omega_0\|_{L^\infty}) > 0$ *such that*

$$\rho^\varepsilon(t,\cdot) \underset{\varepsilon\to 0}{\rightharpoonup} 1 \quad \text{and} \quad J^\varepsilon(t,\cdot) \underset{\varepsilon\to 0}{\rightharpoonup} u(t,\cdot)$$

weakly in the sense of measures.

Open questions

- Non-local in time?
- $\mathbb{R}^2$?
- 3d case?
- Gyrokinetic limit?

Weak solutions

A function $f: \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}_+$ is a weak solution of (VP) if we have

- $f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^d \times \mathbb{R}^d) \cap L^2(\mathbb{R}^d \times \mathbb{R}^d))$.
- $|v|^2 f \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}^d \times \mathbb{R}^d))$.
- $\partial_t f + v \cdot \partial_x f + E \cdot \partial_v f = 0$ in $\mathcal{D}'(\mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{R}^d)$.

We will consider solutions such that, for $k_0 > 2$

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} |v|^{k_0} f^{in} dxdv < +\infty,$$

$$\implies \mathcal{E}^{in} = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}|v|^2 f^{in} dxdv + \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{2}|E^{in}|^2 dxdv < +\infty.$$