Splitting on a Vlasov equation for magnetic plasmas

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Outline

Context : Fusion plasmas

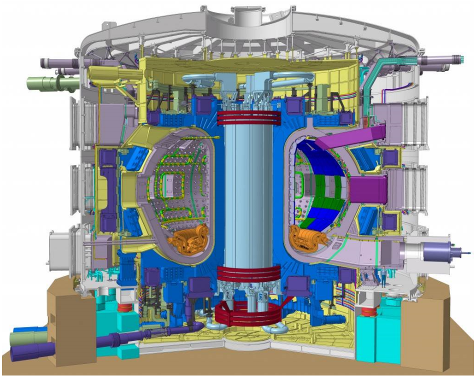

Figure – Artist's vue of the ITER Tokamak

- Laboratory fusion plasmas $\Rightarrow$ Kinetic models
- $f(t, x, \mathbf{v}) / f_e(t, x, \mathbf{v})$: Distribution functions for ions/electrons

Evolution equation on the magnetic field $\mathbf{B}$

- Mean electron temperature T_e = constant + weak mass approximation $m_e = 0$ + conservation of moments for electrons + Joules effect
  $\Rightarrow$ Relation for the electric field $\mathbf{E}$: Generalised Ohm law

$$n_e \mathbf{E} = -T_e \nabla n_e - n_I \mathbf{u}_I \wedge \mathbf{B} + \mathbf{J} \wedge \mathbf{B} + n_e \eta \, \mathbf{rot}\, \mathbf{B}$$

- Maxwell-Faraday equation $\partial_t \mathbf{B} + \mathbf{rot}\, \mathbf{E} = 0 \Rightarrow$ evolution equation on $\mathbf{B}$

$$\frac{\partial \mathbf{B}}{\partial t} - \mathbf{rot}\left(n_I \mathbf{u}_I \wedge \frac{\mathbf{B}}{n_e} \right) + \mathbf{rot}\left(\mathbf{rot}\, \mathbf{B} \wedge \frac{\mathbf{B}}{n_e} \right) + \mathbf{rot}(\eta \, \mathbf{rot}\, \mathbf{B}) = 0$$

Modelling

- Vlasov equation on the ion distribution function f

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{\partial}{\partial \mathbf{v}} \cdot \left[\mathbf{F} f \right] = 0$$

with $\mathbf{F}(t, x, \mathbf{v}) = \dfrac{1}{n_e}(-T_e \nabla n_e - n_I \mathbf{u}_I \wedge \mathbf{B} + \mathbf{J} \wedge \mathbf{B}) + \mathbf{v} \wedge \mathbf{B}$

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- Equation on the electron density $n_e(t, x) = \int_{\mathbb{R}^3} f_e(t, x, \mathbf{v}) d\mathbf{v}$: Gauss relation on the dominant term of the electric field $-T_e \nabla \ln n_e$

$$\mathrm{div}(-T_e \epsilon_0 \nabla \ln n_e(t, x)) = \underbrace{\int_{\mathbb{R}^3} f(t, x, \mathbf{v}) d\mathbf{v}}_{n_I(t,x)} - n_e(t, x)$$

$\Rightarrow$ Poisson equation

$$-\lambda^2 \Delta \ln n_e = n_e - n_I \quad \lambda^2 = T_e \epsilon_0$$

The Model

$$\begin{cases} -\lambda^2 \Delta \ln n_e = n_e - n_I \\ \dfrac{\partial \mathbf{B}}{\partial t} - \mathbf{rot}\left(n_I \mathbf{u}_I \wedge \dfrac{\mathbf{B}}{n_e} \right) + \mathbf{rot}\left(\mathbf{rot}\, \mathbf{B} \wedge \dfrac{\mathbf{B}}{n_e} \right) + \mathbf{rot}(\eta\, \mathbf{rot}\, \mathbf{B}) = 0 \\ \dfrac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \dfrac{\partial}{\partial \mathbf{v}} \cdot \left[\left(-\dfrac{T_e}{n_e} \nabla n_e + \mathbf{rot}\, \mathbf{B} \wedge \dfrac{\mathbf{B}}{n_e} + (\mathbf{v} - \dfrac{n_I \mathbf{u}_I}{n_e}) \wedge \mathbf{B} \right) f \right] = 0 \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$$

with

$$n_I(t,x)\mathbf{u}_I(t,x) = \int_{\mathbb{R}^3} f(t,x,\mathbf{v})\mathbf{v} d\mathbf{v} \qquad \text{(the macroscopic velocity of ions)}$$

Boundary conditions

$\Omega \subseteq \mathbb{R}^3$ bounded domain of class $\mathcal{C}^{1,1}$

- $\mathbf{n}_x \cdot \nabla n_e(t,x) = 0 \quad x \in \partial\Omega \Rightarrow$ global neutrality of the plasma $\int_\Omega n_e(t,x)dx = \int_\Omega n_I(t,x)dx$
- $\mathbf{n}_\Omega \wedge \mathbf{B}(t,x) = 0 \quad x \in \partial\Omega$
- $f(t,x,\mathbf{v} - 2(\mathbf{v}\cdot\mathbf{n}_x)\mathbf{n}_x) = f(t,x,\mathbf{v}), \quad \mathbf{v} \in \mathbb{R}^3, \quad x \in \partial\Omega$
  This is the specular reflection condition ; it implies the no-slip boundary condition

$$\mathbf{u}_I(t,x)\cdot\mathbf{n}_x = 0 \quad x \in \partial\Omega$$

Splitting strategy : The energy balance

$$\mathcal{E}_I(t) = \frac{1}{2}\int_\Omega \int_{\mathbb{R}^3} f(t,x,\mathbf{v})|\mathbf{v}|^2 d\mathbf{v} dx, \qquad \mathcal{E}_m(t) = \frac{1}{2}\int_\Omega |\mathbf{B}(t,x)|^2 dx$$

Classical solutions to the system have the following energy identity

$$\frac{d}{dt}\left[\mathcal{E}_I + \mathcal{E}_m + T_e \int_\Omega (n_e \ln n_e - n_e + 1) dx + T_e \frac{\lambda^2}{2}\int_\Omega |\nabla \ln n_e|^2 dx\right] = -\int_\Omega \eta |\mathbf{rot}\, \mathbf{B}|^2$$

- L.Desvillettes and S.Mischler : *Splitting for Boltzmann and B.G.K. equations*

$$\begin{cases} -\lambda^2 \Delta \ln n_e = n_e - n_I \\ \dfrac{\partial \mathbf{B}}{\partial t} - \mathbf{rot}\left(n_I \mathbf{u}_I \wedge \dfrac{\mathbf{B}}{n_e}\right) + \mathbf{rot}\left(\mathbf{rot}\,\mathbf{B} \wedge \dfrac{\mathbf{B}}{n_e}\right) + \mathbf{rot}(\eta\,\mathbf{rot}\,\mathbf{B}) = 0 \\ \dfrac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \dfrac{\partial}{\partial \mathbf{v}} \cdot \left[\left(-\dfrac{T_e}{n_e} \nabla n_e + \mathbf{rot}\,\mathbf{B} \wedge \dfrac{\mathbf{B}}{n_e} + \left(\mathbf{v} - \dfrac{n_I \mathbf{u}_I}{n_e}\right) \wedge \mathbf{B}\right) f\right] = 0 \\ \nabla \cdot \mathbf{B} = 0 \end{cases}$$

Second stage : First Magnetic part

- We freeze the magnetic field in certain terms$\Rightarrow$ linear equation on $\mathbf{B}$

During Δt, we solve

$$\begin{cases} \dfrac{\partial \mathbf{B}}{\partial t} - \mathbf{rot}\left(n_I \mathbf{u}_I \wedge \dfrac{\mathbf{B}_f}{n_e}\right) + \mathbf{rot}\left(\mathbf{rot}\,\mathbf{B} \wedge \dfrac{\mathbf{B}_f}{n_e}\right) + \mathbf{rot}(\eta\,\mathbf{rot}\,\mathbf{B}) = 0 \\ \dfrac{\partial f}{\partial t} + \dfrac{\partial}{\partial \mathbf{v}} \cdot \left[\left(\mathbf{rot}\,\mathbf{B} \wedge \dfrac{\mathbf{B}_f}{n_e}\right) f\right] = 0 \end{cases}$$

- $\partial_t n_I = 0 \Rightarrow \partial_t n_e = 0$
- $\partial_t \mathbf{u}_I = \mathbf{J} \wedge \mathbf{c}, \quad \mathbf{c} = \dfrac{\mathbf{B}_f}{n_e^f}$
- Energy identity : $\mathcal{E}'_{tot}(t) = -\int_\Omega \eta |\mathbf{rot}\,\mathbf{B}|^2$

Second stage : System on $\mathbf{B}$ and $\mathbf{u}_I$

$$\begin{cases} \dfrac{\partial \mathbf{B}}{\partial t} - \mathbf{rot}(n_I \mathbf{u}_I \wedge \mathbf{c}) + \mathbf{rot}(\mathbf{rot}\,\mathbf{B} \wedge \mathbf{c}) + \mathbf{rot}(\eta\, \mathbf{rot}\,\mathbf{B}) = 0 \\ \partial_t \mathbf{u}_I = \mathbf{rot}\,\mathbf{B} \wedge \mathbf{c} \end{cases}$$

1. $\mathbf{B}^{\square}(t_k) \in \underbrace{H^2_{\mathrm{div}=0}(\Omega)^3 \cap H^2_{t0}(\Omega)^3}_{=H^2_{t0,\mathrm{div}=0}}$
2. $\mathbf{u}_I^{\square}(t_k) \in H^1(\Omega)^3$
3. $n_I^f \in W^{1,\infty}(\Omega)$ and $0 < \min_{\Omega}(n_I^f)$
4. $\mathbf{c} \in W^{1,\infty}(\Omega)^3$
5. $C(\Omega, \eta) \max\left(\|\mathbf{c}\|_{W^{1,\infty}}, \|\mathbf{n}\|_{W^{1,\infty}} \|\mathbf{c}\|_{W^{1,\infty}}\right) < 1$ where $C(\Omega, \eta) > 0$
6. $\partial\Omega$ is $\mathcal{C}^2$

The Hille-Yosida theorem

Theorem (Hille-Yosida)

Let $\Phi\colon D(\Phi) \to H$ *be a maximal monotone operator* $(\Phi(D(\Phi)) = H$ *and* $\langle \Phi\mathbf{u}, \mathbf{u}\rangle_H \geq 0, \forall \mathbf{u} \in H)$*. Then, given any* $u_0 \in D(\Phi)$ *there exists a unique function*

$$u \in \mathcal{C}^1([0\,;\,T[;H) \cap \mathcal{C}([0\,;\,T[;D(\Phi))$$

satisfying

$$\begin{cases} \dfrac{du}{dt} + \Phi u = 0 \\ u(0) = u_0 \end{cases}$$

- From an evolution problem to a stationary equation
- Contraction semigroup $(S_A(t))_{t\geq 0}$ defined by $S_A(t)\colon u_0 \in D(A) \mapsto u(t) \in D(A)$

Φ is monotone

Framework

$$\frac{d}{dt}\begin{pmatrix} \mathbf{B} \\ \mathbf{u}_I \end{pmatrix} + \Phi\begin{pmatrix} \mathbf{B} \\ \mathbf{u}_I \end{pmatrix} = 0$$

with

$$\Phi\colon \underbrace{H^2_{t0,\mathrm{div}=0}(\Omega)^3 \times H^1_{n_I}(\Omega)^3}_{=D(\Phi)} \to \underbrace{L^2(\Omega)^3 \times H^1_{n_I}(\Omega)^3}_{=H}$$

$$\begin{pmatrix} \mathbf{B} \\ \mathbf{u}_I \end{pmatrix} \mapsto \begin{pmatrix} -\operatorname{rot}(n_I \mathbf{u}_I \wedge \mathbf{c}) + \operatorname{rot}(\operatorname{rot}\mathbf{B} \wedge \mathbf{c}) + \operatorname{rot}(\eta \operatorname{rot}\mathbf{B}) \\ -\operatorname{rot}\mathbf{B} \wedge \mathbf{c} \end{pmatrix}$$

$$\left\langle \Phi\begin{pmatrix} \mathbf{B} \\ \mathbf{u}_I \end{pmatrix}, \begin{pmatrix} \mathbf{B} \\ \mathbf{u}_I \end{pmatrix} \right\rangle_H = \int_\Omega \eta |\operatorname{rot}\mathbf{B}|^2 \geq 0$$

Φ maximal monotone ?

- Idea : symmetric part controls the anti-symmetric part

Construction of the sequence

$$(I + \Phi_{sym}) \begin{pmatrix} \mathbf{B}_{p+1} \\ \mathbf{u}_{p+1} \end{pmatrix} = \underbrace{-\Phi_{anti} \begin{pmatrix} \mathbf{B}_p \\ \mathbf{u}_p \end{pmatrix} + \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}}_{\in H} = \begin{pmatrix} \tilde{\mathbf{f}} \\ \tilde{\mathbf{g}} \end{pmatrix} \qquad (1)$$

with

$$\Phi_{sym} \begin{pmatrix} \mathbf{B} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{rot}(\eta \, \mathbf{rot}\, \mathbf{B}) \\ 0 \end{pmatrix}$$

and

$$\Phi_{anti} \begin{pmatrix} \mathbf{B} \\ \mathbf{u} \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ \mathbf{u} \end{pmatrix} \mapsto \begin{pmatrix} \mathbf{rot}(n_I \mathbf{u}_I \wedge \mathbf{c}) - \mathbf{rot}(\mathbf{rot}\, \mathbf{B} \wedge \mathbf{c}) \\ \mathbf{rot}\, \mathbf{B} \wedge \mathbf{c} \end{pmatrix}$$

Φ maximal monotone ?

Elliptic boundary-value problem

$$\begin{cases} (\mathbf{B}_{p+1} + \mathbf{rot}(\eta\, \mathbf{rot}\, \mathbf{B}_{p+1})) = \tilde{\mathbf{f}} \text{ in } \Omega \\ \mathbf{B}_{p+1} \wedge n_\Omega = 0 \text{ on } \partial\Omega \end{cases}$$

$\mathbf{B}_{p+1}$ is a weak solution

$$\Psi : H^1_{t0,\mathrm{div}=0}(\Omega)^3 \times H^1_{t0,\mathrm{div}=0}(\Omega)^3 \to \mathbb{R}$$

$$(\mathbf{B}, \mathbf{C}) \mapsto \int_\Omega \mathbf{B} \cdot \mathbf{C} + \int_\Omega \eta\, \mathbf{rot}\, \mathbf{B} \cdot \mathbf{rot}\, \mathbf{C}$$

Lax-Milgram on Ψ

- $\mathbf{B}_{p+1} \notin H^2(\Omega)^3$

Simplified model for the second stage : 2D

- Assumption : $\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ b(x,y) \end{pmatrix} \Rightarrow f(t,x,v) = f(t,(x_1,x_2),(v_1,v_2))$

Simplified second stage

$$\begin{cases} \partial_t b - \mathrm{div}(c n_I u_I) + (\partial_y c)(\partial_x b) - (\partial_x c)(\partial_y b) - \eta \Delta b = 0 \\ \partial_t \mathbf{u}_I = -c \begin{pmatrix} \partial_x b \\ \partial_y b \\ 0 \end{pmatrix} \end{cases}$$

- Boundary condition $\Rightarrow b = 0$ on $\partial\Omega$
- The sequence $(b_p, \mathbf{u}_p)_{p\geq 0} \in (H^2(\Omega) \times H^1(\Omega)^2)^{\mathbb{N}}$ is well-defined
- Estimates from H^2-regularity theorem $\Rightarrow$

$$(I + \Phi)\begin{pmatrix} \mathbf{B} \\ \mathbf{u}_I \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{g} \end{pmatrix}$$

Perspectives

- Look what happens when the Debye length $\lambda \to 0$
- Numerical resolution