Special Topics in Physics

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Contents

1 Lecture: Introduction and Schedule **1**

2 Lecture: Path Integral Formalism **1**

2.1 Ising Model Example 1
2.2 Path Integral Derivation 3
2.3 Functional Derivatives 4
2.4 Generating Functional 5

1 Lecture: Introduction and Schedule

21/02/2023

- Week 2: Path Integral Methods in QFT - Statistical Field Theory (Euclidean QFT) - **James & Cian**
- Week 3- 6: Renormalisation in QFT (φ^4) - **Alex, Achintya, James & Cian**
- Assignment: Renormalisation of $\frac{\lambda}{3!}\varphi^3$ in 6D (1 loop).
- Week 7-9: QED Scattering amplitudes. **TBD.**
- Week 10-13: Renormalisation of QED, SM and QFT on curved spacetime. **TBD.**

2 Lecture: Path Integral Formalism

03/03/2023

This lecture and accompanying notes follow closely LeBellac Chapter 8. - Cian

2.1 Ising Model Example

Consider a single spin-$\frac{1}{2}$ system with Hamiltonian

$$\begin{aligned} H &= -J\sigma_x \\ &= -J\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

which has eigenstates

$$|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad\qquad |1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

with eigenvalues $E_0 = -J$ and $E_1 = J$.

Definition 2.1: We define the matrix element of the evolution operator $U(t) = \exp(-iHt)$ by

$$F(t, S_b \mid 0, S_a) = \langle S_b | e^{-iHt} | S_a \rangle ,$$

where $|S_i\rangle$ are eigenstates of $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, (with eigenvalues $S_i = \pm 1$) but in general could be arbitrary states. This matrix element represents the probability amplitude to observe S_b at time t given that the system was in state S_a at $t = 0$.

Next, note that as $\{|S_a\rangle, |S_b\rangle\}$ spans the Hilbert space (i.e. they form a complete basis), we can write

$$\mathbb{I} = \sum_{S_i = \pm 1} |S_i\rangle \langle S_i| .$$

Note. While this expression may be familiar, it is interesting to notice that the $\langle S_i|$ can be considered projection operators onto the state $|S_i\rangle$. For example, in 2D, one may project onto the x and y axes and rewrite the state as $\varphi = (\hat{x}P_x + \hat{y}P_y)\varphi$ where P_x is a projection operator and $\hat{x}$ here is analogous to a ket.

Supposing that $t = N \gg 1$, we can insert an identity operator at every integer t by splitting the evolution operator into N time steps such that

$$\begin{aligned} \langle S_b | e^{-iHt} | S_a \rangle &= \langle S_b | e^{-iH} \mathbb{I} \cdots \mathbb{I} e^{-iH} | S_a \rangle \\ &= \sum_{S_1 = \pm 1} \cdots \sum_{S_{N-1} = \pm 1} \langle S_b | e^{-iHt} | S_{N-1} \rangle \\ &\left(\prod_{i=N-1}^{2} \langle S_i | e^{-iH} | S_{i-1} \rangle \right) \langle S_1 | e^{-iH} | S_a \rangle . \end{aligned}$$

Then, we take

$$\langle S | e^{-iH} | S' \rangle = e^{-iV(S, S')},$$

which allows us to write

$$\langle S_b | e^{-iH} | S_a \rangle = \sum_{[S_i]} \exp\left(-i \left(V(S_b, S_{N-1}) + \sum_{i=N-1}^{2} V(S_i, S_{i-1}) + V(S_1, S_a) \right) \right) .$$

Now, if we take the time to be imaginary with $t = -i\tau$, then the matrix element becomes

$$\langle S_b | e^{-H\tau} | S_a \rangle = \sum_{[S_i]} \exp\left(V(S_b, S_{N-1}) + \sum_{i=N-1}^{2} V(S_i, S_{i-1}) + V(S_1, S_a) \right) .$$

Observe that

$$e^{-H} = e^{J\sigma_1}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n!}$$

$$= \begin{pmatrix} \cosh J & \sinh J \\ \sinh J & \cosh J \end{pmatrix},$$

where if we take

$$Z_N = \sum_{[S_i]} e^{-H}$$

we see that it equals

$$= \operatorname{tr} e^{-iH\tau}.$$

Namely, the partition function of the quantum spin system is equal to that of a classical N spin system.

2.2 Path Integral Derivation

Consider now a particle evolving according to general H with probability amplitude now given by

$$\begin{aligned} F\left(q', t' \mid q, t\right) &= \langle q', t' | q, t \rangle \\ &= \langle q' | \, e^{-iH\left(t'-t\right)} \, | q \rangle \, . \end{aligned}$$

As before we subdivide the time interval into n intervals of length $\varepsilon = \left(t' - t\right)/n$ and we specify

$$H\left(t' - t\right) = n\varepsilon \left(\frac{\hat{p}^2}{2m} + V\left(q\right) \right).$$

We can then insert complete sets of eigenstates $\{|q_i\rangle\}$ as identity operators yielding

$$F\left(q', t' \mid q, t\right) = \int \left(\prod_{l=1}^{n} \mathrm{d}q_i \right) \prod_{l=0}^{n} \langle q_{l+1} | \exp\left(-i\varepsilon \left(\frac{\hat{p}^2}{2m} \right) \right) \exp\left(-i\varepsilon V\left(\hat{q}\right) \right) | q_l \rangle \, .$$

The matrix element of interest simplifies to

$$\begin{aligned} &= \exp\left(-i\varepsilon V\left(q_l\right)\right) \langle q_{l+1} | \exp\left(-i\varepsilon \frac{\hat{p}^2}{2m} \right) | q_l \rangle \\ &= \exp\left(-i\varepsilon V\left(q_l\right)\right) \int \frac{\mathrm{d}p_l}{2\pi} \langle q_{l+1} | \exp\left(-i\varepsilon \frac{\hat{p}^2}{2m} \right) | p_l \rangle \langle p_l | q_l \rangle \\ &= \exp\left(-i\varepsilon V\left(q_l\right)\right) \int \frac{\mathrm{d}p_l}{2\pi} \exp\left(-i\varepsilon \frac{p_l^2}{2m} \right) \langle q_{l+1} | p_l \rangle \langle p_l | q_l \rangle \end{aligned}$$

where $\langle p_l | q_l \rangle = \frac{1}{2\pi} e^{-iq_l p_l}$ yields

$$= \exp\left(-i\varepsilon V\left(q_l\right)\right) \int \frac{\mathrm{d}p_l}{2\pi} \exp\left(-i\varepsilon \frac{p_l^2}{2m} \right) \exp\left(i \left(q_{l+1} - q_l \right) p_l \right).$$

Returning this matrix element to the probability amplitude expression, we see that

$$F\left(q', t' \mid q, t\right) = \int \left(\prod_{l=1}^{n} \mathrm{d}q_i\right) \prod_{l=0}^{n} \int \frac{\mathrm{d}p_l}{2\pi} \exp\left(-i\varepsilon\left(V\left(q_l\right) + \frac{p_l^2}{2m}\right)\right) \exp\left(i\left(q_{l+1} - q_l\right) p_l\right).$$

As V is independent of p_l, we can integrate over p_l and obtain

$$F\left(q', t' \mid q, t\right) = \lim_{\varepsilon \to 0} \left(-i\frac{m}{2\pi\varepsilon}\right)^{\frac{1}{2}} \int \left(\prod_{l=1}^{n} \left(-i\frac{m}{2\pi\varepsilon}\right)^{\frac{1}{2}} \mathrm{d}q_l\right) \exp\left(i\sum_{l=0}^{n} \varepsilon \frac{m\left(q_l - q_{l+1}\right)^2}{2\varepsilon^2} - \varepsilon V\left(q_l\right)\right)$$

Where in the limit of $\varepsilon \to 0$,

$$F\left(q', t' \mid q, t\right) = \int \mathcal{D}q \exp\left(i\int_t^{t'} \frac{1}{2}m\dot{q}^2 - V\left(q\right)\right)$$

$$F\left(q', t' \mid q, t\right) = \int \mathcal{D}q \exp\left(iS\right),$$

and we have denoted the integration measure by

$$\mathcal{D}q = \left(-i\frac{m}{2\pi\varepsilon}\right)^{\frac{1}{2}} \prod_{l=1}^{n} \left(-i\frac{m}{2\pi\varepsilon}\right)^{\frac{1}{2}} \mathrm{d}q_l .$$

we arrive at the desired path integral expression. Namely, this equation states that it is through summing over all possible paths the particle can take, weighted by their action, that we obtain the amplitude for a given process. The weighting by the action ensures that unlikely paths contribute negligibly (in fact the fast rotation of the action in the complex plane means they cancel themselves). On the other hand paths close to the classical equation of motion (which is stationary with respect to the action) will add constructively and thus contribute measurably to the probability amplitude of the process.

Note. We can then analogously obtain matrix elements of time ordered products such that

$$\langle q', t'| TQ\left(t_1\right) Q\left(t_2\right) |q, t\rangle = \int \mathcal{D}q q\left(t_1\right) q\left(t_2\right) e^{iS},$$

namely such that $t_1 > t_2$.

2.3 Functional Derivatives

Definition 2.2: Given a functional $F : M \to \mathbb{R}$ where M is some space (manifold), we have that the functional derivative with respect to some function ρ is given by

$$\begin{aligned}\int \frac{\delta F}{\delta \rho\left(x\right)} \varphi\left(x\right) \mathrm{d}x &= \lim_{\varepsilon \to 0} \frac{F\left[\rho + \varepsilon\varphi\right] - F\left[\rho\right]}{\varepsilon} \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}\varepsilon} F\left[\rho + \varepsilon\varphi\right]\right]_{\varepsilon=0}.\end{aligned}$$

One can consider the functional derivative as the gradient of F at the point ρ in function space in the direction of φ.

Examples. Given

$$F\left[\varphi\left(x\right)\right] = e^{\int \varphi(x) g(x) \mathrm{d}x},$$

the functional derivative with $\varphi = \delta\left(x-y\right)$ is

$$\begin{aligned}
\frac{\delta F\left[\varphi\left(x\right)\right]}{\delta\varphi\left(y\right)} &= \frac{\delta F\left[\varphi\left(x\right)+\varepsilon\delta\left(x-y\right)\right] - F\left[\varphi\left(x\right)\right]}{\varepsilon} \\
&= \lim_{\varepsilon\to 0} \frac{e^{\int(\varphi(x)+\varepsilon\delta(x-y))g(x)\mathrm{d}x} e^{\int \varphi(x)g(x)\mathrm{d}x}}{\varepsilon} \\
&= e^{\int \varphi(x)g(x)\mathrm{d}x} \lim_{\varepsilon\to 0} \frac{e^{\varepsilon\int \delta(x-y)g(x)\mathrm{d}x} - 1}{\varepsilon} \\
&= e^{\int \varphi(x)g(x)\mathrm{d}x} \lim_{\varepsilon\to 0} \frac{e^{\varepsilon g(y)} - 1}{\varepsilon} \\
&= g\left(y\right) F\left[\varphi\left(x\right)\right].
\end{aligned}$$

2.4 Generating Functional

We modify the Lagrangian to include a source term

$$L = \frac{1}{2} m\dot{q}^2 - V\left(q\right) + j\left(t\right) q\left(t\right),$$

which results in an external force $j\left(t\right)$ which we take to be nonzero on $[t, t']$.

We define the generating functional

$$\begin{aligned}
Z\left(y\right) &= \lim_{T\to i\infty, T'\to -i\infty} \frac{\langle Q', T' | Q, T\rangle}{e^{-iE_0(T'-T)}\varphi_0^*\varphi_0\left(Q'\right)} \\
Z\left(j\right) &= \int \mathrm{d}q\,\mathrm{d}q'\,\varphi_0^*\left(q', t'\right)\langle q', t' | q, t\rangle \varphi_0\left(q, t\right)
\end{aligned}$$

$$Z\left(j\right) = \langle 0 | e^{iHt'} U_j\left(t', t\right) e^{-iHt} | 0\rangle,$$

with U_j satisfying

$$i\frac{\mathrm{d}U_j}{\mathrm{d}t} = \left(H - j\left(t\right)Q\right) U_j\left(t\right).$$

Writing the matrix elements as a path integral we have

$$Z\left(j\right) = \mathcal{N} \lim_{T^{(\prime)}\to\pm i\infty} \int \mathcal{D}q \exp\left(i\int_T^{T'} \left(L\left(q,\dot{q}\right) + j\left(t\right)q\left(t\right)\right)\right),$$

where by using the result derived in the example above, the functional derivative gives

$$\langle 0 | T\left(Q\left(t_1\right)Q\left(t_2\right)\right) | 0\rangle = \frac{\left(-i\right)^2}{Z\left(0\right)} \frac{\mathrm{d}^2 Z\left(j\right)}{\mathrm{d}j\left(t_1\right)^2} j\left(t_2\right)\Bigg|_{j=0}$$

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