Prediction-Correction Method of N-Body Gravitational Physics Using Linear Multistep Methods

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1 Overview of the method using fourth order Adams–Bashforth prediction and fourth order Adams–Moulton correction

Consider the second order ordinary differential equation (ODE):

$$\ddot{\mathbf{r}} = g(t, \dot{\mathbf{r}}) \tag{1}$$

We have the initial conditions:

$$\mathbf{r}(0) = \mathbf{r}_0 \tag{2}$$
$$\dot{\mathbf{r}}(0) = \dot{\mathbf{r}}_0 \tag{3}$$

This can be re-written as a pair of coupled ODEs:

$$\dot{\mathbf{v}} = g(t, \mathbf{r}) \tag{4}$$
$$\dot{\mathbf{r}} = f(t, \mathbf{v}) \tag{5}$$

And the initial conditions become

$$\mathbf{r}(0) = \mathbf{r}_0 \tag{6}$$
$$\mathbf{v}(0) = \mathbf{v}_0 \tag{7}$$

2 Generating the inital points

In order to use the fourth order Adams methods, we need to generate three points extra from the initial conditions, this is because multistep methods retain knowledge of previous points which helps gain efficiency. We can choose any method to generate these points, the fourth order Runge-Kutta method is a good choice and we will implement it in §4, but for this overview we will use

Adams methods of increasing order.

The first prediction step is performed using the Euler method and the first correction step is done using either the backward Euler method or the trapezoidal rule. For this case we choose the trapezoidal rule, for a justification see §4. With step size h and $t_{n+1} = t_n + h$, the first step is:

$$\begin{aligned}
\hat{\mathbf{v}}_1 &= \mathbf{v}_0 + h \cdot g(t_0, \mathbf{r_0}) \\
\hat{\mathbf{r}}_1 &= \mathbf{r}_0 + \frac{h}{2}\left(f(t_1, \hat{\mathbf{v}}_1) + f(t_0, \mathbf{v}_0)\right) \\
\mathbf{v}_1 &= \mathbf{v}_0 + \frac{h}{2}\left(g(t_1, \hat{\mathbf{r}}_1) + g(t_0, \mathbf{r_0})\right) \\
\mathbf{r}_1 &= \mathbf{r}_0 + \frac{h}{2}\left(f(t_1, \mathbf{v}_1) + f(t_0, \mathbf{v_0})\right)
\end{aligned} \tag{1}$$

Note that in the calculating $\hat{\mathbf{r}}_1$—the prediction of $\mathbf{r}_1$—we make use of the trapezoidal rule to gain some extra accuracy over the backward Euler method. We could also ignore the last correction step, $\mathbf{r}_1$ does not give a great improvement over $\hat{\mathbf{r}}_1$.There is also the option to change the order in which we calculate each term, the second term calculated ought to be more accurate. We can then use the two step Adams methods:

$$\begin{aligned}
\hat{\mathbf{v}}_2 &= \mathbf{v}_1 + \frac{h}{2}\left(3g(t_1, \mathbf{r}_1) - g(t_0, \mathbf{r}_0)\right) \\
\hat{\mathbf{r}}_2 &= \mathbf{r}_1 + \frac{h}{12}\left(5f(t_2, \hat{\mathbf{v}}_2) + 8f(t_1, \mathbf{v}_1) - f(t_0, \mathbf{v}_0)\right) \\
\mathbf{v}_2 &= \mathbf{v}_1 + \frac{h}{12}\left(5g(t_2, \hat{\mathbf{r}}_2) + 8g(t_1, \mathbf{r}_1) - g(t_0, \mathbf{r}_0)\right) \\
\mathbf{r}_2 &= \mathbf{r}_1 + \frac{h}{12}\left(5f(t_2, \mathbf{v}_2) + 8f(t_1, \mathbf{v}_1) - f(t_0, \mathbf{v}_0)\right)
\end{aligned} \tag{2}$$

It is clear by step 2 that only four new function evaluations are necessary for each step and only two are remembered, we will see later that the result of this is that we can increase the accuracy for a relatively small cost in computation and memory. Because Adams–Moulton methods are of order $s + 1$ for step s, this is a third order method.

$$\begin{aligned}
\hat{\mathbf{v}}_3 &= \mathbf{v}_2 + \frac{h}{12}\left(23g(t_2, \mathbf{r}_2) - 16g(t_1, \mathbf{r}_1) + 5g(t_0, \mathbf{r}_0)\right) \\
\hat{\mathbf{r}}_3 &= \mathbf{r}_2 + \frac{h}{24}\left(9f(t_3, \hat{\mathbf{v}}_3) + 19f(t_2, \mathbf{v}_2) - 5f(t_1, \mathbf{v}_1) + f(t_0, \mathbf{v}_0)\right) \\
\mathbf{v}_3 &= \mathbf{v}_2 + \frac{h}{24}\left(9g(t_3, \hat{\mathbf{r}}_3) + 19g(t_2, \mathbf{r}_2) - 5g(t_1, \mathbf{r}_1) + g(t_0, \mathbf{r}_0)\right) \\
\mathbf{r}_3 &= \mathbf{r}_2 + \frac{h}{24}\left(9f(t_3, \mathbf{v}_3) + 19f(t_2, \mathbf{v}_2) - 5f(t_1, \mathbf{v}_1) + f(t_0, \mathbf{v}_0)\right)
\end{aligned} \tag{3}$$

We can then finally find the four step methods:

$$
\begin{aligned}
\hat{\mathbf{v}}_4 &= \mathbf{v}_3 + \frac{h}{24}\left(55g(t_3,\mathbf{r}_3) - 59g(t_2,\mathbf{r}_2) + 37g(t_1,\mathbf{r}_1) - 9g(t_0,\mathbf{r}_0)\right) \\
\hat{\mathbf{r}}_4 &= \mathbf{r}_3 + \frac{h}{720}\left(251f(t_4,\hat{\mathbf{v}}_4) + 646f(t_3,\mathbf{v}_3) - 264f(t_2,\mathbf{v}_2) + 106f(t_1,\mathbf{v}_1) - 19f(t_0,\mathbf{v}_0)\right) \\
\mathbf{v}_4 &= \mathbf{v}_3 + \frac{h}{720}\left(251g(t_4,\hat{\mathbf{r}}_4) + 646g(t_3,\mathbf{r}_3) - 264g(t_2,\mathbf{r}_2) + 106g(t_1,\mathbf{r}_1) - 19g(t_0,\mathbf{r}_0)\right) \\
\mathbf{r}_4 &= \mathbf{r}_3 + \frac{h}{720}\left(251f(t_4,\mathbf{v}_4) + 646f(t_3,\mathbf{v}_3) - 264f(t_2,\mathbf{v}_2) + 106f(t_1,\mathbf{v}_1) - 19f(t_0,\mathbf{v}_0)\right)
\end{aligned}
\tag{4}
$$

With these initial points created we can use the above four step predictor and five step corrector to compute future points. Define:

$$f_n := f(t_n, \mathbf{v}_n) \tag{5}$$

$$g_n := g(t_n, \mathbf{r}_n) \tag{6}$$

$$\hat{f}_n := f(t_n, \hat{\mathbf{v}}_n) \tag{7}$$

$$\hat{g}_n := g(t_n, \hat{\mathbf{r}}_n) \tag{8}$$

$$
\begin{aligned}
\hat{\mathbf{v}}_{n+1} &= \mathbf{v}_n + \frac{h}{24}\left(55g_n - 59g_{n-1} + 37g_{n-2} - 9g_{n-3}\right) \\
\hat{\mathbf{r}}_{n+1} &= \mathbf{r}_n + \frac{h}{720}\left(251\hat{f}_{n+1} + 646f_n - 264f_{n-1} + 106f_{n-2} - 19f_{n-3}\right) \\
\mathbf{v}_{n+1} &= \mathbf{v}_n + \frac{h}{720}\left(251\hat{g}_{n+1} + 646g_n - 264g_{n-1} + 106g_1n - 2 - 19g_{n-3}\right) \\
\mathbf{r}_{n+1} &= \mathbf{r}_n + \frac{h}{720}\left(251f_{n+1} + 646f_n - 264f_{n-1} + 106f_{n-2} - 19f_{n-3}\right)
\end{aligned}
\tag{9}
$$

Less computational work is performed by storing function evaluations as well as the previous four points. Thus in memory we will have:

- $\mathbf{r}_n$
- $\mathbf{v}_n$
- g_{n-1}, g_{n-2}, g_{n-3}
- f_n, f_{n-1}, f_{n-2}, f_{n-3}

Every iteration the only function evaluations are, g_n, f_{n+1}, $\hat{f}_{n+1}$ and $\hat{g}_{n+1}$, only g_n, f_{n+1} are stored.

3 The general prediction correction method

4 Newtonian Gravitation