Asymmetric wave functions from tiny perturbations

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(Received 12 November 2014; accepted 18 June 2015)

The quantum mechanical behavior of a particle in a double well defies our intuition based on classical reasoning. Not surprisingly, an asymmetry in the double well will restore results more consistent with the classical picture. What is surprising, however, is how a very small asymmetry can lead to essentially classical behavior. In this paper, we use the simplest version of a double-well potential to demonstrate these statements. We also show how this system accurately maps onto a two-state system, which we refer to as a "toy model."
[http://dx.doi.org/10.1119/1.4923249]

I. WHERE IS THE PARTICLE?

Given a double-well potential, which of the two wave functions shown in Figs. 1(a) and 1(b) is the correct ground state?

The actual correct answer is that we do not have enough information. We do not have enough information because we are given only a picture of the double-well potential and not an explicit definition. An explicit definition of the potential used in Fig. 1 will be provided below, and this definition will reveal an asymmetry not apparent in the figure: the potential well on the right-hand side is slightly lower than that on the left-hand side. The potential is actually drawn with this asymmetry, but the asymmetry is too small to be visible to the eye on this scale (by many orders of magnitude). The result, however, is that the actual ground state is that pictured in Fig. 1(b), with the wave function located (almost) entirely in the right-side well. Hence, a tiny perturbation results in a state very different from the familiar symmetric superposition of "left" and "right" well occupancy shown in Fig. 1(a).

II. THE ASYMMETRIC DOUBLE-WELL POTENTIAL

A. Introduction

The double-well potential is often used in quantum mechanics to illustrate situations in which more than one state is accessible in a system, with a coupling from one to the other through tunneling.[1,2] For example, in the *Feynman Lectures* the ammonia molecule is used to illustrate a physical system that has a double-well potential for the nitrogen atom.[3] Feynman used an effective two-state model to illustrate these ideas, and in most undergraduate textbooks, a two-state system is used for a similar purpose. Here, instead we will first focus on a full solution to a double-well potential; the features inherent in a two-state system will emerge from our calculations. Indeed, we will also present a refined two-state model to capture the essence of the asymmetry in a microscopic double-well potential, as we are using in Fig. 1.

The notion that a slight asymmetry can result in a drastic change in the wave function is not new. It was first discussed in Refs. 4–6 but in a manner and context inaccessible to undergraduate students. More recently the topic has been revisited[7,8] to illustrate an emerging phenomenon in the semiclassical limit ($\hbar \to 0$). These authors refer to the very minor perturbation in the potential (as in our example above) as the "flea," and to the very deep double-well potential as the "elephant."[6] Schrödinger's cat has crept into the discussion[7,8] because the "flea" disrupts the entangled character of the usual Schrödinger cat-like double-well wave function of Fig. 1(a).

The purpose of this paper is to use a very simple model of an asymmetric double-well potential, solvable either analytically or through an application of matrix mechanics,[2,9] to demonstrate the potent effect of a rather tiny imperfection in the otherwise symmetric double-well potential. Contrary to the impression one might get from the references on this subject, there is nothing "semiclassical" about the asymmetry of the wave function illustrated in Fig. 1(b). We will show, using a slight modification of Feynman's ammonia example,[3,8] that the important parameter to which the asymmetry should be compared is the tunneling probability; this latter parameter can be arbitrarily small. This correspondence

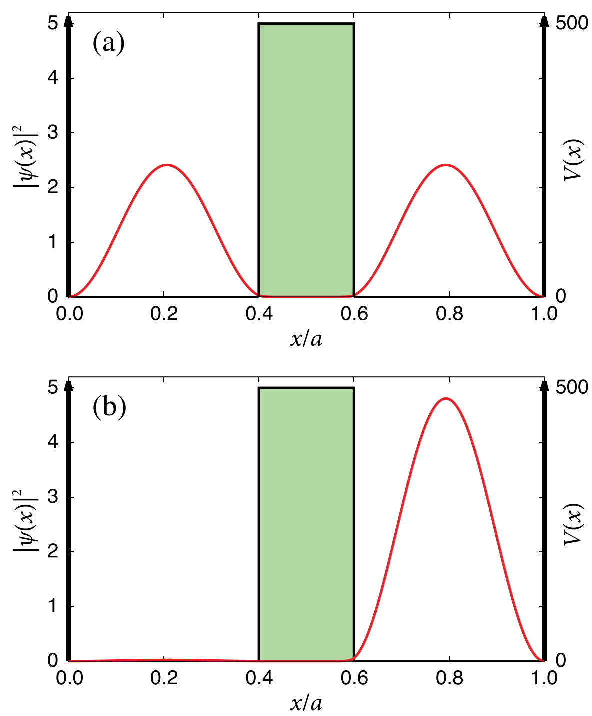

Fig. 1. (a) A symmetric "left *and* right" wave function, with equal amplitude in either of the two potential wells. (b) A "just right" wave function, entirely in the right-side well. A third possibility would be a "just left" wave function, entirely in the left-side well. Which is the ground state for the double-well potential shown?.

applies for excited states as well, along with other asymmetric double-well shapes.

B. Square double well with asymmetry

A variety of symmetric double-well potentials were used in Ref. 2 to illustrate the universality of the energy splitting described there. Here instead we will use perhaps the simplest model to exhibit the impact of asymmetry. We have checked other versions and the same physics applies universally. Our model uses two square wells, with left and right wells having base levels V_L and V_R, respectively, separated by a barrier of width b and height V_0 and enclosed within an infinite square well extending from $x=0$ to $x=a$. For our purposes, we assign the two wells equal width $w=(a-b)/2$. Mathematically, the well is described by

$$V(x)=\begin{cases}\infty & \text{if } x<0 \text{ or } x>a\\ V_0 & \text{if } (a-b)/2<x<(a+b)/2\\ V_L & \text{if } 0<x<(a-b)/2\\ V_R & \text{if } (a+b)/2<x<a,\end{cases}\tag{1}$$

and is shown in Fig. 2. We can readily recover the symmetric double well by using $V_L=V_R$. Units of energy are those of the ground state for the infinite square well of width a, $E_1^0=\pi^2\hbar^2/2ma^2$, where m is mass of the particle, and units of length are the width a of the infinite square well. We will typically use a barrier width $b=a/5$ so that the individual wells have widths $w=2a/5$. The height of the barrier then controls the degree to which tunneling from one well to the other occurs, and $\delta\equiv(V_L-V_R)/2$ controls the asymmetry. For a symmetric potential, we will use $V_L=V_R=0$; asymmetry will typically be implemented with $V_R<0$, and therefore $\delta>0$. Actual values are provided below.

In the following subsection, we will work through detailed solutions to this problem, first analytically and then numerically. It is important to realize that these solutions retain the full Hilbert space in the problem. A "toy" model is introduced in a later subsection, and reduces this complex problem to a two-state problem. We then proceed to illustrate how the two-state problem reproduces remarkable features of the full problem, as a function of the asymmetry in the two wells.

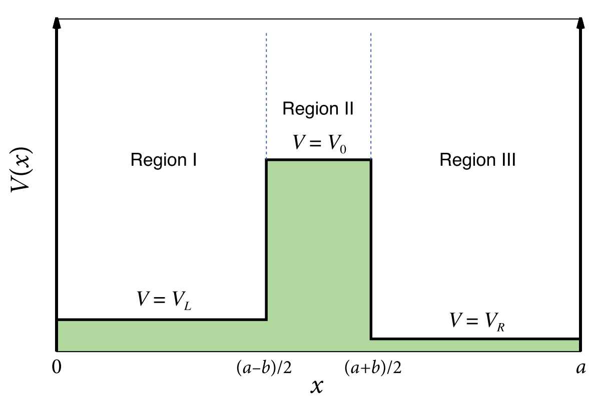

Fig. 2. A schematic of the generic asymmetric square double-well potential. The well widths are the same but the left- and right-side levels can be independently adjusted.

C. Preliminary analysis

Figure 1 was produced using $v_0\equiv V_0/E_1^0=500$, $b/a=0.2$, $v_L\equiv V_L/E_1^0=0$, and $v_R\equiv V_R/E_1^0=0$ in panel (a) while making $v_R=-0.00001$ in panel (b). The change in potential strength compared with the barrier between these two cases is one part in 50,000,000. These parameters result in ground-state energies of $e_1\equiv E_1/E_1^0=5.827034$ and $e_1\equiv E_1/E_1^0=5.827025$ for the symmetric and asymmetric cases, respectively. Needless to say, both the difference in potentials and the difference in ground state energies are minute changes compared to the tremendous qualitative change in the wave function between panels (a) and (b) of Fig. 1.

The potential used in Eq. (1) is simple enough that an "analytical" solution to this problem is possible. The word "analytical" is in quotations here because, in reality, the solution to the equation for the energy for each state must be obtained graphically (i.e., numerically). While this process poses no significant difficulty, it is sufficient work that essentially all textbooks stop here and do not examine the wave function.[10–12]

Assuming that $E<V_0$, the analytical solution is

$$\begin{aligned}\psi_{\mathrm{I}}(x)&=A\sin kx \quad k=[2m(E-V_L)/\hbar^2]^{1/2},\\ \psi_{\mathrm{II}}(x)&=Be^{\kappa x}+Ce^{-\kappa x} \quad \kappa=[2m(V_0-E)/\hbar^2]^{1/2},\\ \psi_{\mathrm{III}}(x)&=D\sin q(a-x) \quad q=[2m(E-V_R)/\hbar^2]^{1/2},\end{aligned}\tag{2}$$

where the regions I, II, and III are depicted in Fig. 2. Applying the matching conditions at $x=(a\pm b)/2$ leads to an equation for the allowed energies:

$$\begin{aligned}&\left[\sin(kw)+\frac{k}{\kappa}\cos(kw)\right]\left[\sin(qw)+\frac{q}{\kappa}\cos(qw)\right]\\ &\quad=e^{-2\kappa b}\left[\sin(kw)-\frac{k}{\kappa}\cos(kw)\right]\left[\sin(qw)-\frac{q}{\kappa}\cos(qw)\right],\end{aligned}\tag{3}$$

where $w\equiv(a-b)/2$ is the width of the individual wells. The matching conditions also provide expressions for the relative amplitude D/A:

$$\left|\frac{D}{A}\right|=e^{-\kappa b}\left|\frac{\kappa\sin(kw)-k\cos(kw)}{\kappa\sin(qw)+q\cos(qw)}\right|,\tag{4}$$

or, alternatively,

$$\left|\frac{D}{A}\right|=e^{+\kappa b}\left|\frac{\kappa\sin(kw)+k\cos(kw)}{\kappa\sin(qw)-q\cos(qw)}\right|.\tag{5}$$

To proceed further, Eq. (3) is solved numerically for the allowed energy values. As might be expected, the low-energy solutions come in pairs. Once E is known, then so are k, q, and κ, and the D/A ratio can be determined through either Eq. (4) or (5). In addition, through similar relations and normalization, all other coefficients in Eq. (2) can be determined. Notice that at first glance Eq. (4) suggests that $|D|\ll|A|$, while Eq. (5) suggests the opposite. In reality, both equations provide the correct answer, though with limited numerical precision one is generally more accurate than the other. Which is more accurate depends on whether $V_R<V_L$ or vice-versa. For $V_R\lessgtr V_L$, we expect $|D|\gtrless|A|$.

Alternatively, we can solve the original Schrödinger equation numerically right from the start by expressing it as a matrix in the infinite-square-well basis, $\phi_n(x) = \sqrt{2/a}\sin(n\pi x/a)$, for $n = 1, 2, 3\ldots$. More explicitly, we write

$$|\psi\rangle = \sum_n c_n|\phi_n\rangle, \tag{6}$$

and insert this into the Schrödinger equation to obtain the matrix equation

$$\sum_m H_{nm}c_m = E_n c_n, \tag{7}$$

where

$$\begin{aligned} H_{nm} = &\delta_{nm}[n^2E_1^0 + (V_L + V_R)w/a + V_0 b/a] \\ &+\delta_{nm}[2V_0 - V_L - V_R]\,\mathrm{sinc}(2n) \\ &+(1-\delta_{nm})D_{nm}[V_L - V_0 + (V_R - V_0)(-1)^{n+m}], \end{aligned} \tag{8}$$

with

$$D_{nm} \equiv \mathrm{sinc}(n-m) - \mathrm{sinc}(n+m), \tag{9}$$

and

$$\mathrm{sinc}(n) \equiv \frac{\sin(\pi n w)}{\pi n}. \tag{10}$$

As before, $w \equiv (a-b)/2$ is the width of either well. This matrix procedure is explained in detail in Refs. 9 and 2. Using either the analytical expressions or the numerical matrix diagonalization, the results are identical. The advantage of the latter method is that the study is not limited to simple well geometries consisting of boxes, so students can easily explore a variety of double-well potential shapes. The matrix must be truncated, and the solutions presented in this paper were all converged as a function of matrix size, with a typical size being 2000×2000. While a much smaller matrix size achieves accurate results for the energies and even achieves accurate results for the wave function for the symmetric double well, this rather large size is required for accurate results for the wave function when the well is asymmetric.

D. Results

Figure 3 shows the resulting wave function for a variety of asymmetries. The results are remarkable. As V_R/E_1^0 decreases from 0 to a value of -0.00001, the probability density changes from a symmetric profile (equal probability in left and right wells) to an entirely asymmetric profile (entire probability localized to the right well). The other "obvious" energy scales in the problem are the barrier height ($V_0/E_1^0 = 500$) and the ground-state energies ($E_{GS}/E_1^0 \approx 5.827$), so these changes in the potential are minute in comparison. Even more remarkable is that as far as the energies are concerned, these minute changes give rise to equally minute changes in the ground-state energy: E_{GS}/E_1^0 shifts from 5.827034 down to 5.827025 as V_R/E_1^0 is reduced from zero to -0.00001, while the changes in the wave functions are qualitatively spectacular.

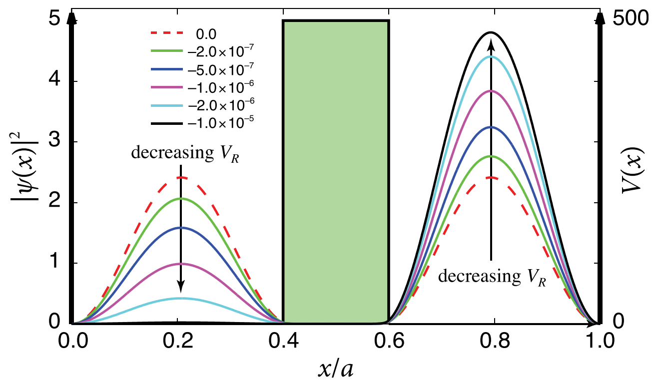

Fig. 3. Progression of the wave function as the right well level is lowered from 0 (same as the left well level) to -10^{-5} (in units of $E_1^0 = \hbar^2\pi^2/2ma^2$). When the double well is symmetric the probability density ($|\psi(x)|^2$) is symmetric (dashed curve); the degree of asymmetry in the probability density increases monotonically as V_R/E_1^0 decreases from 0 to -10^{-5}. The actual values of V_R used in this plot, in units of E_1^0, are indicated in the legend. Any of these values is absolutely indistinguishable on the energy scale of the barrier ($V_0/E_1^0 = 500$) or the ground state energies ($E_{GS}/E_1^0 \approx 5.827$).

For anyone familiar with the symmetric double well, these results are perhaps less surprising, because, as we discuss further below, the splitting between the two lowest energy eigenvalues provides the relevant energy scale. This is an energy scale that *emerges from the solution* to the symmetric double well; it is not an input parameter of the Hamiltonian.

E. Discussion

That such enormous qualitative changes can result from such minute asymmetries in the double-well potential is of course important for experiments in this area, where it would be very difficult to control deviations from perfect symmetry in a typical double-well potential at the 10^{-5} level. Why is this phenomenon not widely disseminated in textbooks? And what precisely controls the energy scale for the "flea-like" perturbation that eventually results in a completely asymmetric wave function situated in only one of the two wells [as in Fig. 1(b)]? The answer to the first question is undoubtedly connected to the lack of a straightforward analytical demonstration of the strong asymmetry in the wave function. It is difficult to coax from Eqs. (4) and (5) an explicit demonstration of the resulting asymmetry apparent in Fig. 3 as a function of lowering (or raising) the level of the potential well on the right. To shed more light on this phenomenon and to elaborate on an answer to the second question, we resort to a "toy model" slightly modified from the one used by Feynman[3] to explain tunneling in a symmetric double-well system and introduced more recently by Landsman and Reuvers[7,8] in a perturbative way. Such a model is also used in standard textbooks to discuss "fictitious" spins interacting with a magnetic field[11] and two-level systems subject to an electric field.[12]

III. A TOY MODEL FOR THE ASYMMETRIC DOUBLE WELL

An aid towards understanding the results of our calculations is provided by a simplified "effective" model that strips the system of its complexity and focuses on the essentials. In this instance, the key features are whether the particle is in the right well, the left well, or a combination thereof.

Following Feynman, we begin with two isolated wells, each with a particular energy level, and with each coupled to the other through some matrix element t:

$$\begin{aligned} H\psi_L &= E_L\psi_L - t\psi_R, \\ H\psi_R &= E_R\psi_R - t\psi_L, \end{aligned} \tag{11}$$

where, in the absence of coupling, the left (right) well would have a ground-state energy E_L (E_R), and ψ_L (ψ_R) represents a wave function localized in the left-side (right-side) well. A straightforward solution of this two-state system results in an energy splitting, as in the symmetric case:

$$E_{\pm} = \frac{E_L + E_R}{2} \pm \sqrt{\left(\frac{E_L - E_R}{2}\right)^2 + t^2}. \tag{12}$$

For typical barriers ($t \ll (E_L + E_R)/2$) and small asymmetries ($E_L - E_R \ll (E_L + E_R)/2$), very little difference occurs in the energies, in agreement with the results from our more complete calculations above. If we define $\delta \equiv (E_L - E_R)/2$ (that is, $(V_L - V_R)/2$ for the square double-well potential), then the ground-state wave function becomes

$$\psi = \frac{1}{\sqrt{2}}\sqrt{1 - \frac{\delta}{\sqrt{\delta^2 + t^2}}}\psi_L + \frac{1}{\sqrt{2}}\sqrt{1 + \frac{\delta}{\sqrt{\delta^2 + t^2}}}\psi_R. \tag{13}$$

In the symmetric case, we recover the (symmetric) linear superposition of the state with the particle in the left well along with the state with the particle in the right well (see the remark in Ref. 10). However, with increasing asymmetry, say with $V_R < V_L$ (so that $\delta > 0$), the amplitude for the particle being in the right well rises to unity, while that for the particle in the left well decreases to zero. Our toy model illustrates that the energy scale for this cross-over is the tunneling matrix element t. This energy scale must be clearly present in the microscopic model defined in Eq. (1), but it is not there explicitly.

A. Comparison of the toy model to the microscopic model

To see how well the toy model defined by the two-state system in Eq. (11) reproduces properties of the microscopic calculations of Sec. II, we make an attempt to compare the results from the two approaches. This is most readily accomplished by the following procedure. First, the solid curves displayed in Fig. 4 are readily obtained by plotting the two amplitudes in Eq. (13),

$$\begin{aligned} |c_L|^2 &\equiv \frac{1}{2}\left(1 - \frac{\delta}{\sqrt{\delta^2 + t^2}}\right), \\ |c_R|^2 &\equiv \frac{1}{2}\left(1 + \frac{\delta}{\sqrt{\delta^2 + t^2}}\right), \end{aligned} \tag{14}$$

as a function of δ/t. Then, for one of the results shown in Fig. 3, we compute the area under the curve on the left; it will correspond to an amplitude $|c_L|^2$ in Fig. 4. By placing this value on the appropriate curve in Fig. 4, we are able to extract a value of δ/t and hence an effective value of t (since $\delta \equiv (V_L - V_R)/2$ is known). This value is marked by a large circle in Fig. 4. We have thus identified a value of t, strictly only defined for the toy model, with a specific barrier height and width in the more microscopic calculations connected with Eqs. (2)–(5) or their numerical counterparts. We can then vary the value of δ (as was done to generate the curves shown in Fig. 3) and plot the values of the total probability density in the left and right wells as a function of δ/t. The smaller circles in Fig. 4 are the results of these calculations, and they almost perfectly lie on the curves generated from the toy model, thus showing that the asymmetric double well system indeed behaves like a two-state system described phenomenologically by Eq. (11). We have done this for other barrier heights and widths and similar very accurate agreement between the two approaches is achieved. We have also carried out such comparisons for excited states as well as for other kinds of double wells (e.g., so-called Gaussian wells) with similar agreement.

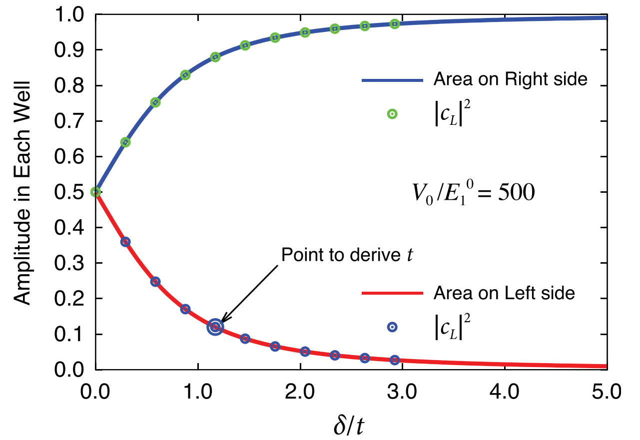

Fig. 4. Plot of the amplitude in each well vs the asymmetry parameter δ/t. The circles indicate the areas integrated from Fig. 3 on each side of the barrier. The large circle shows the point used to establish a value of $t \approx 6.84 \times 10^{-7}E_1^0$, so that the expression in Eq. (14) matches precisely the value determined by the more microscopic calculation of the previous subsection. With this value of t the curves corresponding to the expressions in Eq. (14) are also plotted, and the agreement is excellent over the entire range of δ/t. This indicates that the "toy model" phenomenology is very accurate.

As an example, in Figs. 5 and 6, we show results analogous to those of Figs. 3 and 4 for a double well with a barrier of the same width but with a significantly increased height. The sequence of probability densities in Fig. 5 is similar to those shown in Fig. 3 except that the changes in the potential asymmetry are orders of magnitude smaller. Figure 6 then confirms that the significantly enhanced sensitivity is due to the significantly reduced effective "hopping" amplitude t between the two wells, such that the asymmetry in probability densities as a function of potential asymmetry in Fig. 6 is as it is in Fig. 4 as a function of δ/t, with both of these parameters greatly reduced.

B. Origin of the coupling *t*

The origin of the coupling parameter t in the two-state toy model is clearly the possibility of tunneling that exists from one well into the other. It is rather involved to "derive" this parameter t from the parameters of the original double-well potential specified in Eq. (1). In fact, it suffices to provide an estimate based only on the symmetric case ($V_L = V_R = 0$),

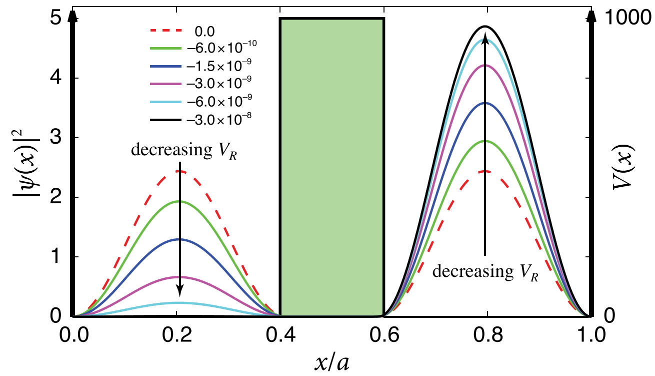

Fig. 5. As in Fig. 3, progression of the wave function as the right well level is lowered from 0 (same as the left well level) to -3.0×10^{-8} (in units of $E_1^0 = \hbar^2\pi^2/2ma^2$), but now with a barrier height of $V_0 = 1000E_1^0$. When the double well is symmetric the probability density ($|\psi(x)|^2$) is symmetric (dashed curve); the degree of asymmetry in the probability density increases monotonically as V_R/E_1^0 decreases from 0 to -3.0×10^{-8}. The actual values of V_R used in this plot, in units of E_1^0, are indicated in the legend. Even more so than before, any of these values is absolutely indistinguishable on the energy scale of the barrier ($V_0/E_1^0 = 1000$) or the ground state energies ($E_{GS}/E_1^0 \approx 5.947$).

and we provide a brief exposition here, following Merzbacher.[13]

One starts with a variational wave function of the form

$$\psi_\pm(x) = \frac{N_\pm}{\sqrt{2}}[\psi_L(x) \pm \psi_R(x)], \tag{15}$$

where the $\pm$ refers to the ground state (+) or first excited state (−), respectively, and the subscripts L and R refer to the left and right wells, respectively. First taking the two wells in isolation, we obtain $\psi_L(x)$, for example, with a solution similar to that in Eq. (2),

$$\psi_L(x) = \begin{cases} A\sin(kx) & 0 < x < w, \\ A\sin(kw)\,e^{-\kappa(x-w)} & x > w. \end{cases} \tag{16}$$

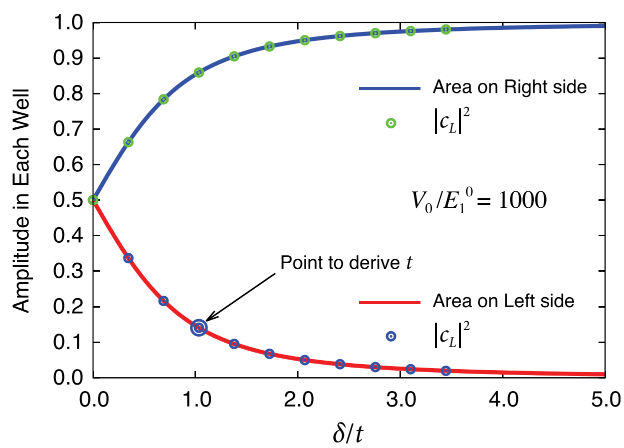

Fig. 6. As in Fig. 4, plot of the amplitude in each well vs the asymmetry parameter δ/t, but now for a higher barrier potential $V_0/E_1^0 = 1000$. The circles indicate the areas integrated from Fig. 5 on each side of the barrier. The large circle shows the point used to establish a value of $t \approx 1.45 \times 10^{-9}E_1^0$, so that the expression in Eq. (14) matches precisely the value determined by the more microscopic calculation of the previous subsection. With this value of t the curves corresponding to the expressions in Eq. (14) are also plotted, and the agreement is excellent over the entire range of δ/t. As long as the barrier is sufficiently high to delineate two very distinct states ("left" and "right"), the two-state model works very well.

A similar formula is used for the well on the right; with all coordinates displaced a distance b to the right, it forms a mirror image of the one on the left. If we consider this distance to be very large, we can introduce the coupling between the two wells as a perturbation. Disregarding the unimportant normalization constants, $N_\pm$ in Eq. (15) and A in Eq. (16), the energy splitting between the two states is determined by the "overlap" between the two wells. A straightforward calculation gives

$$E_{\rm split} = \int dx\,\psi_L(x)H\psi_R(x) \propto e^{-\kappa b} \approx e^{-b\sqrt{V_0}}, \tag{17}$$

so as V_0 (the height of the barrier) increases for fixed width b, the splitting becomes less and less. Note, however, that the parameter t, first introduced in Eq. (11), is proportional to this same quantity:

$$t \propto e^{-b\sqrt{V_0}}. \tag{18}$$

The actual values of $e^{-\kappa b}$, with $\kappa \equiv \sqrt{2m(V_0 - E)/\hbar^2}$ and $b/a = 0.2$, are 8.6×10^{-7} and 2.5×10^{-9} for $V_0/E_1^0 = 500$ and 1000, respectively (note that $E/E_1^0 \approx 5.827$). The actual values of t obtained phenomenologically through the fitting procedure described above are 6.8×10^{-7} and 1.5×10^{-9}, respectively, which tracks very closely these exponentially decaying factors.[14]

IV. SUMMARY

We have examined the simplest asymmetric double-well potential and explored the behavior of the wave function as a function of asymmetry. In the symmetric case, the ground state is a linear superposition of the particle in the left well and the particle in the right well. As the floor level of the potential on the right side (V_R) decreases, the probability for the particle to be on the right side "slowly" increases. The remarkable result of our calculations is that the energy scale for the transition from symmetric ground state to completely asymmetric ground state can be made arbitrarily small. As our "toy model" calculation demonstrated, this energy scale is controlled by the tunneling probability between the two wells, which is an energy scale that is not obviously present in the microscopic parameters (height and width of the barrier). In fact, the weaker the coupling between the wells in the two-well system, the smaller this energy scale. Figure 3 (or Fig. 5) demonstrates this quite dramatically, where imperceptibly small asymmetries in the potential give rise to a completely asymmetric wave function. There is very little indication of this resulting asymmetry in the ground-state energy; instead, a calculation of the wave function is required to demonstrate this.

These calculations serve to demonstrate a number of important principles for the novice. First, the numerical calculation is "simpler" than the analytical calculation, and less likely to lead to error. By this we mean that solving for the wave function through Eqs. (3)–(5) is a little subtle and, for example, the wrong choice of using either Eq. (4) or Eq. (5) can lead to inaccuracies. In contrast the numerical matrix solution is straightforward. Then, the solution obtained here can be tied to perturbation theory. An accurate calculation of the energy can be achieved with perturbation theory, but not so with the wave function! This outcome ties into the variational principle and teaches the important lesson that a (very) accurate estimate of the energy does not imply an even qualitatively correct wave function.

ACKNOWLEDGMENTS

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada (NSERC), by the Alberta iCiNano program, and by a University of Alberta Teaching and Learning Enhancement Fund (TLEF) grant.

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[9]F. Marsiglio, "The harmonic oscillator in quantum mechanics: A third way," Am. J. Phys. **77**, 253–258 (2009).

[10]To be more precise, many textbooks work through in some form or other the eigenvalues of a *symmetric* double well potential. They will also at least sketch the wave functions corresponding, for example, to the two lowest eigenvalues. As is well known, the eigenvalues are *nearly* degenerate, and the ground state is as shown in Fig. 1(a) while the first excited state is simply the antisymmetric version of this. The discussion of the asymmetric double well potential is absent from almost all undergraduate textbooks, so of course not even a prescription for determining the eigenvalues is provided. The tacit assumption is that there is very little change in the eigenvalues from the symmetric case, and this is correct, as indicated by the numbers we have given in the text. A further assumption is that the eigenstates also suffer very little change from the symmetric case, and this is incorrect, and the main point of this paper. Also note that almost all textbooks provide a discussion of two-state systems as an illustration of what happens in the symmetric double-well potential, and some even discuss the asymmetric case (see, for example, the next two references), but these tend to fixate on the energies and not the wave functions.

[11]C. Cohen-Tannoudji, B. Diu, and F. Laloë, *Quantum Mechanics* (Wiley, Toronto, 1977). See especially Complements B_{IV} and C_{IV}.

[12]J. S. Townsend, *A Modern Approach to Quantum Mechanics* (University Science Books, Sausilito, CA, 2000). See especially the discussion of the two-state depiction of the ammonia molecule (following Feynman in Ref. 3) in the presence of an electric field, though here he resorts to perturbation theory when discussing the wave function.

[13]E. Merzbacher, *Quantum Mechanics*, 3rd ed. (Wiley, Hoboken, NJ, 1998). See in particular Section 8.5 and Eqs. (8.69)–(8.78), where the author provides an estimate for the energy splitting of the parabolic double-well potential.

[14]Note that the ratio of $e^{-\kappa b}/t$ remains of order unity, going from 1.3 to 2.1 as V_0 changes from 500 to 1000.

Student Potentiometer

This small student potentiometer, currently in the Greenslade Collection, was sold by Leeds & Northrup of Philadelphia for $80.00 in 1924. The range is from 0 to 1.6 V with an uncertainty of 0.00005 V. An external galvanometer is connected to the circuit through the binding posts at the back. The step switch is used to select voltages in units of 0.1 V, and the single turn slide wire is used to spread 0.1 V into 1000 parts. Since the instrument is essentially a slide wire with switch-selected resistances in series, it can also be used as part of a Wheatstone bridge. (Notes and picture by Thomas B. Greenslade, Jr., Kenyon College)