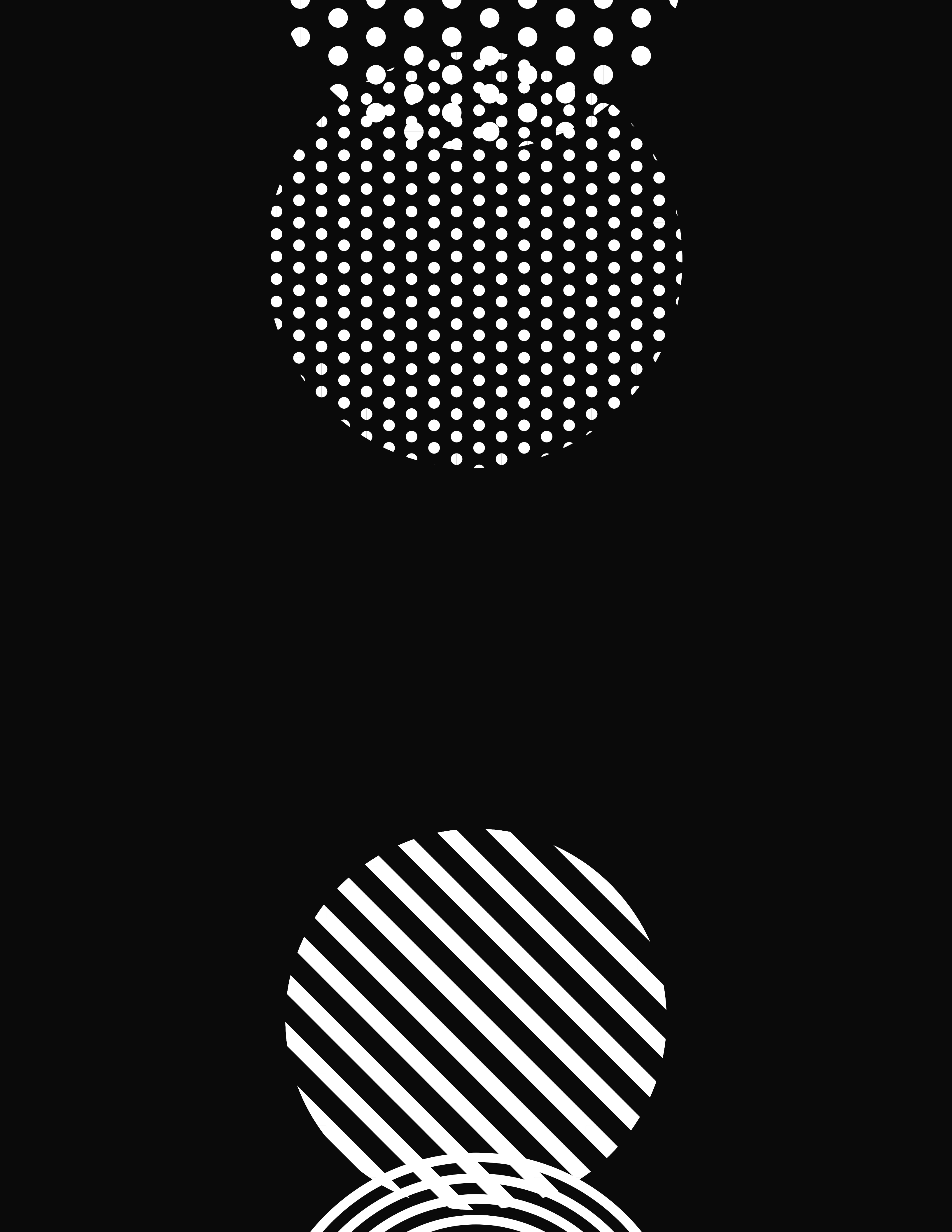


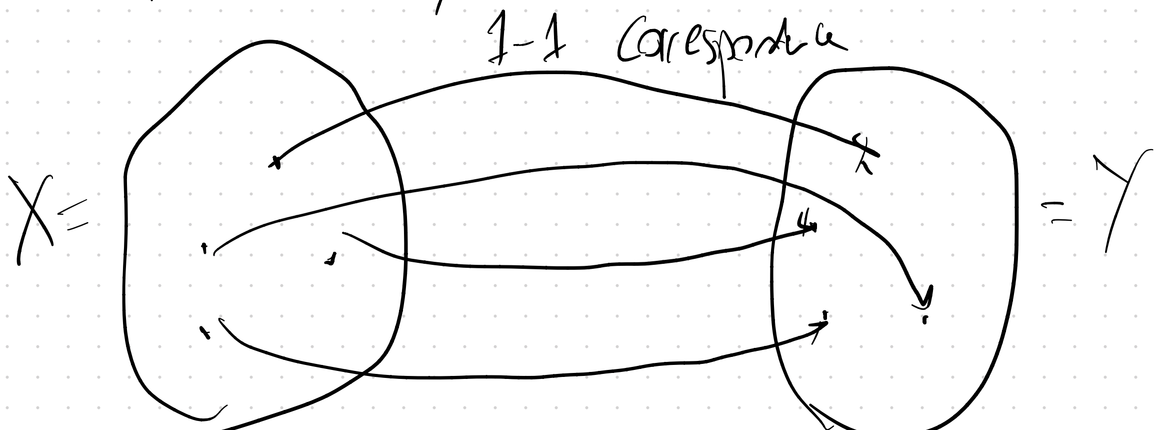
§0 Themes for this week

- Getting more comfortable w/ abstract functions
- Getting more Comfortable w/ Structural thinking
- Understanding ∞

§1 In / Sur / bi-jections

Goal: Want to understand when two sets X and Y have the same "size".

Idea:



Definition: Let $f: X \to Y$ be a function

- f is an injection if $\forall y \in Y$

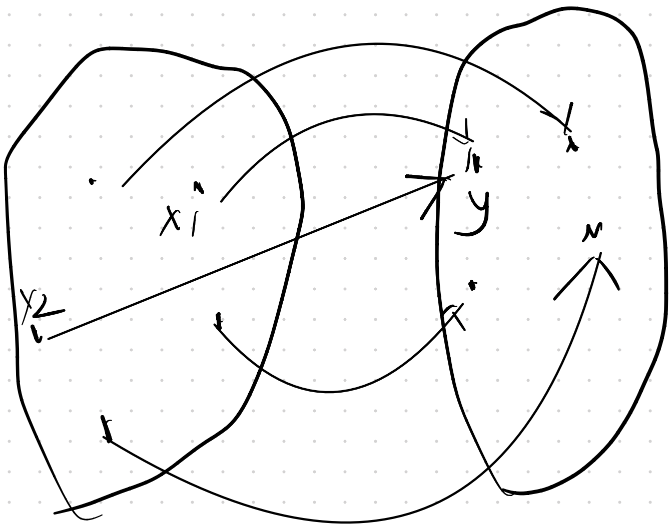
$$f^{-1}(y) := \{x \in X : f(x) = y\}$$

has at most 1 element,

- a surjection if $\forall y$, $f^{-1}(y)$ has at least 1 element.

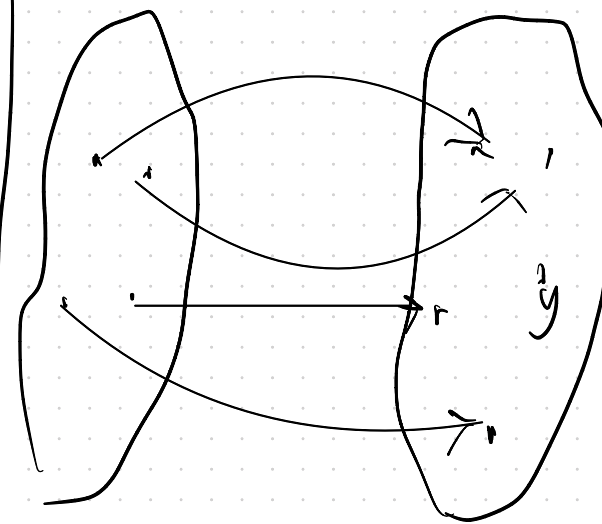
- a bijection if $\forall y$, $f^{-1}(y)$ has exactly 1 element.

Visually:



Surjection but not injection

$f^{-1}(y) = \{x_1, x_2\}$



Injection but not Surjection: $f^{-1}(y) = \emptyset$

<u>Intuition:</u> $f: X \to Y$ a function

- f is a surj. $\Rightarrow$ $\text{Size}(X) \geq \text{Size}(Y)$
- f is an inj. $\Rightarrow$ $\text{Size}(X) \leq \text{Size}(Y)$
- f is a bij $\Rightarrow$ $\text{Size}(X) = \text{Size}(Y)$

<u>e.g.</u> $\mathbb{N} \to \mathbb{N} - \{0\}$, $n \mapsto n+1$ is a bij. so
$\text{Size}(\mathbb{N}) = \text{Size}(\mathbb{N} - \{0\})$ — like $\infty = \infty - 1$.

Exercise: (a) Give an example of a surj. $f: \mathbb{R} \to \mathbb{R}$ which is not an inj. but $f(\mathbb{Q}) \subseteq \mathbb{Q}$ and $f: \mathbb{Q} \to \mathbb{Q}$ is not a surj.

(b) Is this possible if "inj." and "surj." are switched?

Obs: $f: X \to Y$ function

(a) f is inj. $\iff$ $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ then $x_1 = x_2$

$\iff$ $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$

(b) f is surj. $\iff$ $f(X) = Y$

- f is bij. $\Longleftrightarrow$ f is inj. <u>and</u> surj.

<u>Note</u>: A function has domain/codomain as part of data! Changing this can change the properties of f.

<u>e.g.</u> $f : [0,\infty) \to [0,\infty)$, $x \longmapsto x^2$ surj.

$f : [0,\infty) \to \mathbb{R}$, $x \longmapsto x^2$, <u>not</u> surj.

Prop: Let $f: X \to Y$ and $g: Y \to Z$ be functions.

- If f and g are both in/sur/bi-jections then $g \circ f$ is as well.
- $g \circ f$ is surj. $\Rightarrow$ g surj.
- $g \circ f$ inj $\Rightarrow$ f inj.

Pf:

(1, inj): Assume f and g inj and let $x_1, x_2 \in X$. If $(g \circ f)(x_1) = (g \circ f)(x_2) \Rightarrow g(f(x_1)) = g(f(x_2))$. So as g inj. $\Rightarrow f(x_1) = f(x_2)$. As f is inj. $\Rightarrow x_1 = x_2$, as desired.

(3) Assume $g \circ f$ is surj., then $\forall z \in Z$ $\exists$ $x \in X$ s.t. $(g \circ f)(x) = z$. But, then $y = f(x) \in Y$ is s.t. $g(y) = z$. As z was arb., g is surj. $\blacksquare$

Definition: For a function $f: X \to Y$, an **inverse function** is a function $g: Y \to Z$ s.t.

$g(f(x)) = x \ \forall x \in X$ and $f(g(y)) = y \ \forall y \in Y$.

Thm: For a function $f: X \to Y$, TFAE:

(1) f is a bij.

(2) $\exists$ an inverse function $g : Y \to Z$ of f

Moreover, an inverse of f is unique if it exists.