VARIANTS OF THE DE JONG FUNDAMENTAL GROUP

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Abstract. For a rigid space X, we answer two questions of de Jong about the category $\mathbf{Cov}_X^{\mathrm{adm}}$ of coverings which are locally in the admissible topology on X the disjoint union of finite étale coverings: we show that this class is different from the one used by de Jong, but still gives a tame infinite Galois category. In addition, we prove that the objects of $\mathbf{Cov}_X^{\mathrm{et}}$ (with the analogous definition) correspond precisely to locally constant sheaves for the pro-étale topology defined by Scholze.

1. Introduction

In [ALY21a], we have defined a new notion of covering space in rigid-analytic geometry, called *geometric coverings*, and showed that the category $\mathbf{Cov}_X$ of geometric coverings of a rigid K-space X gives rise to a fundamental group $\pi_1^{\mathrm{ga}}(X, \overline{X})$, called the *geometric arc fundamental group*. These notions generalized those previously introduced by de Jong [dJ95]. In this paper, we use these ideas to answer some questions pertaining to previously studied notions: de Jong's fundamental group and Scholze's pro-étale topology.

Extensions of de Jong covering spaces. The notion of 'covering space' of a rigid K-space has been a notoriously difficult concept to fully encapsulate. It was realized early on that, unlike the case of schemes, even relatively well-behaved rigid spaces admit interesting connected covers of infinite degree. The most classical examples of this phenomenon are the Tate uniformization map $\mathbb{G}_{m,K}^{\mathrm{an}} \to E^{\mathrm{an}}$, for an elliptic curve E over K with split multiplicative reduction, and the Gross–Hopkins period map

$$\pi_{\mathrm{GH}} \colon \{x \in \mathbb{A}_K^{1,\mathrm{an}} : |x| < 1\} \to \mathbb{P}_K^{1,\mathrm{an}}.$$

Building off ideas of Berkovich, in [dJ95] de Jong sought to define a notion of covering space which could encompass these examples but was robust enough to support a notion of fundamental group.

To make this precise, let us consider the full subcategories

$$\mathbf{Cov}_X^{\tau} (\subseteq \mathbf{Cov}_X) \subseteq \mathbf{\acute{E}t}_X, \quad \tau \in \{\mathrm{adm}, \text{ét}, \mathrm{oc}\}$$

consisting of étale maps $Y \to X$ for which there exists a τ-cover $U \to X$ such that $Y_U \to U$ is the disjoint union of finite étale coverings of U. Here adm, ét, or oc denotes the usual (i.e. 'admissible'), étale, or overconvergent (i.e. partially proper open) Grothendieck topology on X, respectively. These

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categories come with a natural fiber functor given by the restriction of the fiber functor

$$F_{\overline{x}}\colon \acute{\mathbf{E}}\mathbf{t}_X \to \mathbf{Set}, \qquad F_{\overline{x}}(Y) = \mathrm{Hom}_X(\overline{x}, Y)$$

for every geometric point $\overline{x} \to X$. In particular, the category $\mathbf{Cov}_X^{\mathrm{oc}}$, called the category of *de Jong covering spaces*, is precisely the category considered in op. cit. Intuitively, one may think about the category $\mathbf{Cov}_X^{\mathrm{oc}}$ as being a synthesis of the notion of finite étale covering and topological covering (of the Berkovich space associated to X).

The category $\mathbf{Cov}_X^{\mathrm{oc}}$ is shown in op. cit. to have favorable properties. First, it is shown that the natural infinite degree covering spaces mentioned above are examples of de Jong covering spaces. Significantly deeper, it is shown that if one sets $\mathbf{UCov}_X^{\mathrm{oc}}$ to be the category of arbitrary disjoint unions of de Jong covering spaces, then the pair $(\mathbf{UCov}_X^{\mathrm{oc}}, F_{\overline{x}})$ is a *tame infinite Galois category* when X is connected. The notion of a pair $(\mathcal{C}, F)$ being a tame infinite Galois category was developed in [BS15] as a generalization of the classical theory of Galois categories. In particular, there is a topological group $\pi_1(\mathcal{C}, F)$, called the *fundamental group* of $(\mathcal{C}, F)$, such that

$$F\colon \mathcal{C} \xrightarrow{\sim} \pi_1(\mathcal{C}, F)\text{-}\,\mathbf{Set}$$

is an equivalence where $\pi_1(\mathcal{C}, F)\text{-}\,\mathbf{Set}$ is the category of sets endowed with a continuous action of $\pi_1(\mathcal{C}, F)$. We call the fundamental group of the pair $(\mathbf{UCov}_X, F_{\overline{x}})$ the *de Jong fundamental group* and denote it $\pi_1^{\mathrm{dJ}}(X, \overline{x})$.

Despite these positive aspects of the category $\mathbf{Cov}_X^{\mathrm{oc}}$, there is an obvious downside. Namely, it is not obvious whether the notion of a de Jong covering space is local on the target for the admissible topology. For this reason, in [dJ95] the following two questions are posed (using different language):

- Does the equality $\mathbf{Cov}_X^{\mathrm{oc}} = \mathbf{Cov}_X^{\mathrm{adm}}$ hold?
- If not, is $(\mathbf{UCov}_X^{\mathrm{adm}}, F_{\overline{x}})$ a tame infinite Galois category?

We give a negative answer to the first question using an explicit construction relying on a careful analysis of Artin–Schreier coverings.

Theorem 1 (See Proposition 2.1.4)**.** *Let K be a non-archimedean field of characteristic p and let X be an affinoid annulus over K. Then, the containment $\mathbf{Cov}_X^{\mathrm{oc}} \subsetneq \mathbf{Cov}_X^{\mathrm{adm}}$ is strict.*

One may think the existence of such an example is a subtlety related to characteristic p geometry. But, in [Gau] Gaulhiac has cleverly adapted our construction to produce an analogous example in mixed characteristic.

Remark 2. Note that we do not expect such examples to exist when K is of equicharacteristic 0. In fact, in [ALY21b, Corollary 4.17], we show that the equality $\mathbf{Cov}_X^{\mathrm{oc}} = \mathbf{Cov}_X^{\mathrm{\acute{e}t}}$ often holds in equicharacteristic 0.

In [ALY21a] we show that the pair $(\mathbf{Cov}_X, F_{\overline{x}})$ is a tame infinite Galois category. As the notion of geometric covering is étale local on the target, is closed under disjoint unions, and contains finite étale coverings we see that $\mathbf{UCov}_X^{\tau}$, with the obvious definition, is contained in $\mathbf{Cov}_X$. Combining these two results we answer de Jong's second question.

Theorem 3 (See Theorem 3.3.1)**.** *Let X be a connected rigid K-space with geometric point $\overline{x}$. For every $\tau \in \{\mathrm{adm}, \text{ét}, \mathrm{oc}\}$, the pair $(\mathbf{UCov}^{\tau}_X, F_{\overline{x}})$ is a tame infinite Galois category.*

Consequently, there is a topological group $\pi_1^{\mathrm{dJ},\tau}(X, \overline{x})$, which we call the *$\tau$-adapted de Jong fundamental group*, and an equivalence of categories

$$F_{\overline{x}}\colon\ \mathbf{UCov}^{\tau}_X \xrightarrow{\sim} \pi_1^{\mathrm{dJ},\tau}(X, \overline{x})\text{-}\,\mathbf{Set}.$$

Local systems for the pro-étale topology. In the second part of this paper, we show the largest of these three categories $\mathbf{Cov}^{\text{ét}}_X$ is not just of purely theoretical interest and connects to previously studied objects. Recall that in [Sch13], Scholze introduced a site $X_{\text{proét}}$ for a rigid K-space X, called there the *pro-étale topology*. Its covers are roughly an étale cover of X followed by an inverse limit of finite étale covers.

As local systems are usually analyzed using fundamental group techniques, it natural to ask the following question: if X is connected, is the pair $(\mathbf{Loc}(X_{\text{proét}}), F_{\overline{x}})$ a tame infinite Galois category?

Theorem 4 (See Theorem 4.4.1)**.** *The functor associating to a geometric covering $Y \to X$ the corresponding sheaf on the pro-étale site $X_{\text{proét}}$ induces an equivalence of categories*

$$\mathbf{Cov}^{\text{ét}}_X \xrightarrow{\sim} \mathbf{Loc}(X_{\text{proét}}).$$

Considering Theorem 3, we may give a precise answer to our question. Set $\mathbf{ULoc}(X_{\text{proét}})$ to be the category of disjoint unions of objects of $\mathbf{Loc}(X_{\text{proét}})$. Then the pair $(\mathbf{ULoc}(X_{\text{proét}}), F_{\overline{x}})$ is a tame infinite Galois category. Moreover, the fundamental group of $(\mathbf{ULoc}(X_{\text{proét}}), F_{\overline{x}})$ is identified with $\pi_1^{\mathrm{dJ},\text{ét}}(X, \overline{x})$ and so one has an equivalence

$$F_{\overline{x}}\colon\ \mathbf{ULoc}(X_{\text{proét}}) \xrightarrow{\sim} \pi_1^{\mathrm{dJ},\text{ét}}(X, \overline{x})\text{-}\,\mathbf{Set}$$

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Notation and conventions. We shall use freely the material from [Hub96] and [FK18]. We follow the conventions and notations concerning valuative spaces, adic spaces, and rigid K-spaces set out in [ALY21a, §2-3].

2. A non-overconvergent covering space

In this section we construct an example of a morphism $Y \to X$ that belongs to $\mathbf{Cov}^{\mathrm{adm}}_X$ but not $\mathbf{Cov}^{\mathrm{oc}}_X$. For X we take an annulus over a characteristic p non-archimedean field, and $Y \to X$ is constructed by carefully gluing together finite étale covers of a neighborhood of the Guass point which extend over shrinking overconvergent neighborhoods. After that, we establish a result that, in some sense, implies that every element of $\mathbf{Cov}_X^{\mathrm{adm}}$ not in $\mathbf{Cov}_X^{\mathrm{oc}}$ must come from such a gluing construction.

2.1. **The example.** Let K be a non-archimedean field of characteristic $p > 0$ and let $\varpi \in K$ be a pseudouniformizer. Set

$$X = \{|\varpi| \leq |x| \leq |\varpi|^{-1}\}.$$

Our example $Y \to X$ is obtained by gluing two families $Y_n^{\pm}$ of Artin–Schreier coverings of two annuli

$$U^- = \{|\varpi| \leq |x| \leq 1\}, \qquad U^+ = \{1 \leq |x| \leq |\varpi|^{-1}\}$$

which are split over shrinking overconvergent neighborhoods of the intersection

$$C = U^- \cap U^+ = \{|x| = 1\}.$$

We begin with an analysis of Artin–Schreier coverings of $\mathbf{A}_K^{1,\mathrm{an}}$. For a rational number α, we define the affinoid opens

$$D(\alpha) = \{|x| \leq |\varpi|^{-\alpha}\} \quad \supseteq \quad S(\alpha) = \{|x| = |\varpi|^{-\alpha}\}.$$

For integers a, b with $b > 0$, we denote by $Y_{a,b}$ the following Artin–Schreier covering of $\mathbf{A}_K^{1,\mathrm{an}}$:

$$Y_{a,b} = \Big(\operatorname{Spec} K[x,y]/(y^p - y - \varpi^a x^b)\Big)^{\mathrm{an}}.$$

It is a $\mathbf{Z}/p\mathbf{Z}$-torsor over $\mathbf{A}_K^{1,\mathrm{an}}$, and since $\mathbf{Z}/p\mathbf{Z}$ does not have any non-trivial subgroups, the restriction of $Y_{a,b}$ to an affinoid subdomain $U \subseteq \mathbf{A}_K^{1,\mathrm{an}}$ is disconnected if and only if it splits completely over U and if and only if the equation $y^p - y = \varpi^a x^b$ has a solution in $\mathcal{O}(U)$.

Lemma 2.1.1. *Let a, b be integers with $b > 0$ and let α be a rational number. The following are equivalent:*

(a) The covering $Y_{a,b}$ splits over $D(\alpha)$.
(b) The covering $Y_{a,b}$ splits over $S(\alpha)$.
(c) We have $\alpha < a/b$.

Proof. Write $g = \varpi^a x^b$. The unique solution to $y^p - y = -g$ in $K[\![x]\!]$ satisfying $y(0) = 0$ is the power series

$$f = g + g^p + g^{p^2} + \cdots = \sum_{s \geq 1} \varpi^{p^s a} x^{p^s b}.$$

It converges on $D(\alpha)$ if and only if $\alpha < a/b$, as $\mathcal{O}(D(\alpha))$ consists of power series $\sum a_n x^n$ with $|a_n| \cdot |\varpi|^{-n\alpha} \to 0$. Therefore, (a) is equivalent to (c).

Since clearly (a) implies (b), it remains to show (b) implies (a). The ring $\mathcal{O}(S(\alpha))$ consists of Laurent series $f = \sum_{n \in \mathbf{Z}} a_n x^n$ with $|a_n| \cdot |\varpi|^{-n\alpha} \to 0$ as $|n| \to \infty$. It suffices to show that if $f^p - f \in \mathcal{O}(D(\alpha))$, then $f \in \mathcal{O}(D(\alpha))$ (i.e. $a_n = 0$ for $n < 0$). We have $f^p - f$ is equal to $\sum_{n \in \mathbf{Z}} (a_{n/p}^p - a_n) x^n$ where we set $a_{n/p} = 0$ if p does not divide n. Since $f^p - f \in \mathcal{O}(D(\alpha))$, we have $a_{n/p}^p - a_n = 0$ for $n < 0$. By induction we see that $a_n = 0$ for all $n < 0$. $\square$

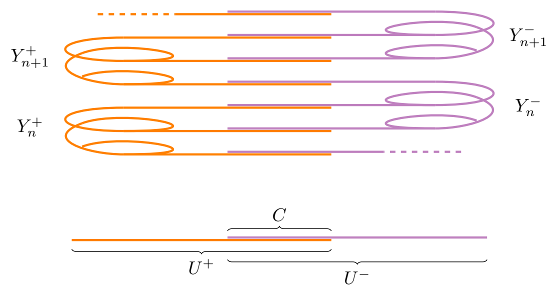

FIGURE 1. Construction of the covering $Y \to X$ (for $p = 3$).

Remark 2.1.2. The equivalence of (a) and (b) holds more generally for every finite étale cover of $D(\alpha)$, by [dJ95, proof of Proposition 7.5].

Construction 2.1.3. Fix two sequences of positive integers $(a_n)_{n\in\mathbf{Z}}$ and $(b_n)_{n\in\mathbf{Z}}$ such that $a_n/b_n > 1$ for all $n \in \mathbf{Z}$ and $\lim_{|n|\to\infty} a_n/b_n = 1$. For $n \in \mathbf{Z}$, we set Y_n to be the restriction of the Artin–Schreier covering Y_{a_n,b_n} to U^+. By Lemma 2.1.1, every Y_n splits completely over $C = S(1)$, while for every $m > 0$ its restriction to $D(1+\frac{1}{m}) \cap U^+$ and $S(1+\frac{1}{m})$ is connected for $|n| \gg 0$. We set $Y^+ = \coprod_{n\in\mathbf{Z}} Y_n^+$, which is an object of $\mathbf{UF\acute{E}t}_{U^+}$.

The automorphism $x \mapsto x^{-1}$ of X induces an isomorphism $i\colon U^- \xrightarrow{\sim} U^+$. We let $Y^- = \coprod_{n\in\mathbf{Z}} Y_n^-$ be the pullback of $Y^+ \to U^+$ under the map i. The restriction of Y_n^- to $S(1-\frac{1}{m})$ is thus connected for $|n| \gg 0$, while Y_n^- splits completely over C for all n.

Label the irreducible components of $Y_n^\pm \times_{U^\pm} C$ by

$$Z^\pm_{np}, Z^\pm_{np+1}, \ldots, Z^\pm_{np+p-1}$$

(in any order); every $Z^\pm_m$ maps isomorphically onto C. Identify

$$Y_n^+ \times_{U^+} C = \coprod_{m\in\mathbf{Z}} Z_m^+ \quad \text{with} \quad Y_n^- \times_{U^-} C = \coprod_{m\in\mathbf{Z}} Z_m^-$$

by identifying Z_m^+ with Z_{m-1}^- for all $m \in \mathbf{Z}$ (see Figure 1). This defines an étale morphism $Y \to X$ whose restriction to U^+ (resp. U^-) is Y^+ (resp. Y^-), in particular $Y \to X$ it is an object of $\mathbf{Cov}_X^{\mathrm{adm}}$.

Proposition 2.1.4. *The map $Y \to X$ does not belong to $\mathbf{Cov}_X^{\mathrm{oc}}$.*

Proof. Let V be a connected overconvergent open subset of X, which by shrinking we may assume satisfies that $V \cap U^\pm$ is connected containing the Gauss point η of the unit disk. It suffices to show that $Y_V \to V$ contains a connected open subset with infinite fiber size.

By Lemma 2.1.5 below, V contains $S(1+\frac{1}{m}) \cup S(1-\frac{1}{m})$ for some $m > 0$. Since for $n \gg 0$, say $n > n_0$, the restriction of $Y_n^\pm$ to $S(1 \pm \frac{1}{m})$ is connected, the restriction $Y_n^\pm \cap Y_V$ of $Y_n^\pm$ to $V \cap U^\pm$ is connected as well.

Let Y' be the image in Y of the union of Y_n^+ and Y_n^- for all $n > n_0$, which is an open subset of Y. We claim that $Y'_V = Y' \cap Y_V$ is connected. This will give the required assertion since $Y'_V \to V$ has infinite fiber size. To see the claim, it suffices to note that in the infinite sequence

$$Y^{+}_{n_0+1} \cap Y_V, \quad Y^{-}_{n_0+1} \cap Y_V, \quad Y^{+}_{n_0+2} \cap Y_V, \quad Y^{-}_{n_0+2} \cap Y_V, \quad Y^{+}_{n_0+3} \cap Y_V, \quad \ldots$$

each set is a connected open subset of $Y' \cap Y_V$ with a non-empty intersection with the subsequent one (e.g. by considering the fiber over a point of C). Consulting Figure 1 again might help the reader with the last step. □

Lemma 2.1.5. *Every overconvergent neighborhood of η inside X contains $S(1+\frac{1}{m}) \cup S(1-\frac{1}{m})$ for some $m > 0$.*

Proof. We can replace X with the ambient disc $D = D(|\varpi|^{-1})$. As the topological space $[D]$ agrees with the Berkovich disk D^{Berk} (cf. [FK18, Theorem C.6.12]), it suffices to prove the analogous claim for D^{Berk}. By [BR10, Proposition 1.6], a basis of the topology for D^{Berk} is given by sets of the form $D(a,r)^- \setminus \bigcup_{i=1}^n D(a_i, r_i)$ where $D(a,r)$ (resp. $D(a,r)^-$) denotes the open (resp. closed) disc with center a and radius r, and where we allow $r > 0$ or $n = 0$. We note that for every a, r we have either $D(a,r) = D(0,r)$ (if $|a| \le |r|$) or $D(a,r) \subseteq S(0,|a|)$ where $S(0,|a|) = \{|x| = |a|\}$ (if $|a| > |r|$), and similarly for $D(a,r)^-$.

Let V be an open subset of D^{Berk} containing η which is of the above form. By the above observation, we may assume that $a = 0$, $r > \rho$, $a_1 = 0$, $r_1 < \rho$, and $D(a_i, r_i) \subseteq S(0, |a_i|)$ for $i \ge 2$. Let $\rho' \in (\rho, r)$ be such that the interval (ρ, ρ') does not contain any of the r_i, then the annulus $\{\rho < |x| < \rho'\}$ is contained in V. Taking preimages in D we obtain the desired claim. □

2.2. **Extending finite étale components.** The map $Y \to X$ was obtained by gluing together finite étale coverings Y_n of an admissible open neighborhood U of the point η with the property that the maximal overconvergent open neighborhood V_n of U over which Y_n extends shrinks to U as n tends towards infinity. We now show that this is a general phenomemon.

Proposition 2.2.1. *Let X be a taut rigid K-space and let $f\colon Y \to X$ be étale and partially proper. Let U be a quasi-compact open subset of X and let W be an open subset of Y_U which is finite étale over U. Then, there exists an overconvergent open subset V of X containing U and an open subset W' of Y_V which is finite étale over V and such that $W' \cap Y_U = W$.*

To prove Proposition 2.2.1 we first establish Lemma 2.2.2 which essentially gives us the desired result at the topological level of the universal separated quotients. We then upgrade this to our desired result about rigid geometry using Proposition 2.2.3.

Lemma 2.2.2. *Let $Y \to X$ be a map of Hausdorff topological spaces with Y locally compact. Let $Z \subset X$ be a subspace and let $W \subset Y_Z$ be a clopen and compact subspace. Then there exist an open subspace V of X containing Z and a subspace W' of Y_V such that:*

(1) $W' \cap Y_Z = W$;
(2) W' is open in Y,
(3) $W' \to V$ is a proper map of topological spaces.

Proof. Let $\mathfrak{B}$ be a collection of compact subspaces of Y containing a neighborhood basis of every point of Y. It suffices to find finitely many $B_1, \ldots, B_m$ in $\mathfrak{B}$ and an open V containing Z such that if $B = \bigcup_i B_i$ and $g\colon B \to X$ is the induced map, then:

(1) $B \cap Y_Z = W$;
(2) $g^{-1}(V) \subset \mathrm{int}_Y(B)$.
(3) $g\colon g^{-1}(V) \to V$ is a proper map of topological spaces.

as one may then take $W' = g^{-1}(V)$.

To prove the existence of such $B_1, \ldots, B_m$ observe that as $Y_Z \setminus W$ is closed we may find a closed subspace C of Y such that $C \cap Y_Z = Y_Z \setminus W$. As $Y \setminus C$ is an open containing the compact set W, we may find finitely many $B_1, \ldots, B_n$ in $\mathcal{B}$ contained in $Y \setminus C$ such that $W \subseteq \mathrm{int}_Y(B)$ where $B = \bigcup_i B_i$. Moreover, we see that $B \cap Y_Z = W$. Note that $g^{-1}(Z) = W$. Since B is compact, g is proper and so we may produce an open neighborhood V of Z such that $g^{-1}(V) \subseteq \mathrm{int}_Y(B)$. Clearly V satisfies the first two desired conditions, and the last property follows as g is proper. $\square$

Proposition 2.2.3. *Let $f\colon Y \to X$ be a taut and valuative morphism of taut valuative spaces. Then $[f]$ is universally closed if and only if f is quasi-compact.*

Proof. If f is quasi-compact then $[f]$ is universally closed by [FK18, Proposition 2.3.27, Chapter 0]. Suppose that $[f]$ is universally closed and let U be a quasi-compact open subset of X. Note that as X is quasi-separated U is retrocompact and taut. By [ALY21a, Proposition 2.2.5] we know that $[f^{-1}(U)]$ is homeomorphic to $[f]^{-1}(\mathrm{sep}_X(U))$. But, since $[f]$ is universally closed and $\mathrm{sep}_X(U)$ is quasi-compact, we know that $[f]^{-1}(\mathrm{sep}_X(U))$ is quasi-compact by [Sta21, Tag 005R]. So, in particular, $[f^{-1}(U)]$ is quasi-compact. But, since $f^{-1}(U)$ is taut we know that the map

$$\mathrm{sep}_{f^{-1}(U)} : f^{-1}(U) \to [f^{-1}(U)]$$

is universally closed by [FK18, Chapter 0, Theorem 2.5.7], and so we deduce that $f^{-1}(U)$ is quasi-compact again by [Sta21, Tag 005R]. $\square$

Proof of Proposition 2.2.1. By our assumptions, together with [FK18, Theorem 2.5.7, Chapter 0] and [ALY21a, Proposition 2.2.3 and Proposition 2.2.5]

- $[f]\colon [Y] \to [X]$ is a map of locally compact Hausdorff spaces,
- $\mathrm{sep}_X(U)$ is a subspace of $[X]$ homeomorphic to $[U]$ (and similarly for W),
- $[Y_U] \to [f]^{-1}([U])$ is a homeomorphism,
- and $\mathrm{sep}_X(W)$ a compact clopen subset of $\mathrm{sep}_X(Y_U)$.

Thus, by Lemma 2.2.2, one may find an open subspace V_0 of $[X]$ containing $[U]$ and a subspace W_0' of $[Y]_{V_0}$ such that W_0' is open in $[Y]$, $W_0' \to V_0$ is proper, and $W_0' \cap [f]^{-1}([U]) = [W]$. Set

$$V = \mathrm{sep}_X^{-1}(V_0) \subseteq X, \qquad W' = \mathrm{sep}_Y^{-1}(W_0') \subseteq Y,$$

which are overconvergent open subsets of their ambient spaces. Note then that by construction $[W'] \to [V]$ is proper. By Proposition 2.2.3 and the fact that $Y \to X$ is étale and partially proper, the morphism $W' \to V$ is étale, partially proper, and quasi-compact and so finite étale by [Hub96, Proposition 1.5.5]. We finally claim that $W' \cap Y_U = W$. But, since $W' \cap Y_U$ and W are overconvergent open subsets of Y_U, this can be checked in $[Y]$ from where it's true by construction. □

3. Variants of the de Jong fundamental group

In this section we examine the categories $\mathbf{Cov}_X^\tau$ extending de Jong's category $\mathbf{Cov}_X^{\mathrm{oc}}$, and show they have good geometric properties.

3.1. The categories $\mathbf{Cov}_X$ and $\mathbf{Cov}_X^\tau$. Throughout this subsection let us fix a rigid K-space X. We will consider below the following three Grothendieck topologies $\tau = \mathrm{oc}, \mathrm{adm}, \text{ét}$ on $\mathbf{\acute{E}t}_X$, whose covers are:

- ($\tau = \mathrm{oc}$, for 'overconvergent') open covers by overconvergent opens,
- ($\tau = \mathrm{adm}$, for 'admissible') open covers,
- ($\tau = \text{ét}$) jointly surjective étale morphisms.

which are in increasing order of fineness.

Definition 3.1.1. Define the category $\mathbf{Cov}_X^\tau$ to be the full subcategory of $\mathbf{\acute{E}t}_X$ consisting of those morphisms $Y \to X$ for which there exists a cover $U \to X$ in the τ-topology so that $Y_U \to U$ is an object of $\mathbf{UF\acute{E}t}_U$.

As the category of geometric coverings plays a pivotal role in our proof that $\mathbf{UCov}_X^\tau$ is a tame infinite Galois category, we presently recall its definition.

Definition 3.1.2 ([ALY21a, Definition 5.2.2])**.** An étale and partially proper morphism $f\colon Y \to X$ is a *geometric covering* if for any embedding $i\colon [0,1] \to [X]$, any lift of $[0,1) \to [X]$ along $[f]$ can uniquely extended to a lift of i.

By [ALY21a, Propositions 5.2.4, 5.2.6, and 5.2.7] being a geometric covering is étale local on the target, is closed under disjoint unions, and contains finite étale coverings. Therefore, the following proposition follows .

Proposition 3.1.3. *The containment* $\mathbf{UCov}_X^\tau \subseteq \mathbf{Cov}_X$ *holds.*

3.2. A brief recollection of tame infinite Galois categories. In the this section we briefly recall the setup of the theory of tame infinite Galois categories in the sense of [BS15, §7]. [1]

Definition 3.2.1 ([BS15, Definition 7.2.1])**.** Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to \mathbf{Set}$ be a functor (called the *fiber functor*). We then call the pair $(\mathcal{C}, F)$ an *infinite Galois category* if the following properties hold:

(1) the category $\mathcal{C}$ is cocomplete and finitely complete,
(2) Each object X of $\mathcal{C}$ is a coproduct of categorically connected objects of $\mathcal{C}$.
(3) There exists a set S of connected objects of $\mathcal{C}$ which generates $\mathcal{C}$ under colimits.
(4) The functor F is faithful, conservative, cocontinuous, and finitely continuous.

[1] A similar formalism, under different names, appears also in [Lep10].

We say that $(\mathcal{C}, F)$ is *tame* if for every categorically connected object X of $\mathcal{C}$ the action of $\pi_1(\mathcal{C}, F)$ on $F(X)$ is transitive. The *fundamental group of* $(\mathcal{C}, F)$, denoted $\pi_1(\mathcal{C}, F)$ is the group $\mathrm{Aut}(F)$ endowed with the compact-open topology.[2]

In the above we used the terminology that an object Y of a category $\mathcal{C}$ is *categorically connected*. This, by definition, means that if Y' is a non-initial object of $\mathcal{C}$ then every monomorphism $Y' \to Y$ is an isomorphism.

The upshot of the theory of (tame) infinite Galois categories is as follows.

Proposition 3.2.2 ([BS15, Theorem 7.2.5.(d)])**.** *Let $(\mathcal{C}, F)$ be a tame infinite Galois category. Then, the functor*

$$F\colon \mathcal{C} \xrightarrow{\sim} \pi_1(\mathcal{C}, F)\text{-}\mathbf{Set}$$

is an equivalence.

From Proposition 3.2.2 we see that for a tame infinite Galois category $(\mathcal{C}, F)$ the isomorphism classes of connected objects form a set. It is moreover clear that every object Y isomorphic to a disjoint union of its *categorically connected components* which, by definition, are the connected subobjects, excluding the initial object.

Using this, it is easy to deduce the following criterion for when a subcategory of a tame infinite Galois category is itself tame infinite Galois.

Lemma 3.2.3. *Let $(\mathcal{C}, F)$ be a tame infinite Galois category. Let $\mathcal{C}'$ be a strictly*[3] *full subcategory of $\mathcal{C}$ satisfying the following three properties:*

- *(a) an object X of $\mathcal{C}'$ is categorically connected as an object of $\mathcal{C}'$ if and only if it is categorically connected as an object of $\mathcal{C}'$,*
- *(b) an object X of $\mathcal{C}$ belongs to $\mathcal{C}'$ if and only if its categorically connected components belong to $\mathcal{C}'$,*
- *(c) the subcategory $\mathcal{C}'$ is closed under (small) colimits and finite limits.*

Then, $(\mathcal{C}', F)$ is a tame infinite Galois category.

3.3. **The categories $\mathbf{UCov}_X^\tau$ are tame infinite Galois categories.** Consider the categories $\mathbf{UCov}_X^\tau$ of disjoint unions of objects of $\mathbf{Cov}_X^\tau$.

Theorem 3.3.1. *Let X be a connected rigid K-space and $\overline{x}$ a geometric point in X. Then, $(\mathbf{UCov}_X^\tau, F_{\overline{x}})$ is a tame infinite Galois category.*

Proof. By Lemma 3.2.3 it suffices to verify that conditions (a), (b), and (c) hold when $\mathcal{C} = \mathbf{Cov}_X$ and $\mathcal{C}' = \mathbf{UCov}_X^\tau$.

By [ALY21a, Proposition 5.4.5] and its proof, to show that (a) holds it suffices to prove that for $Y_i \to X$ for $i = 1, 2$ and $Z \to X$ objects of $\mathbf{UCov}_X^\tau$ that the fiber product $Y_1 \times_Z Y_2$ as adic spaces is an object of $\mathbf{UCov}_X^\tau$. To

[2]More precisely, for each s in S, where S is as in **(IGC3)**, we endow $\mathrm{Aut}(s)$ with the compact-open topology. We then endow $\mathrm{Aut}(F)$ with the subspace topology inherited from the natural map $\mathrm{Aut}(F) \to \prod_{s\in S} \mathrm{Aut}(s)$.

[3]I.e. an object of $\mathcal{C}$ isomorphic to an object of $\mathcal{C}'$ is itself an object of $\mathcal{C}'$.

show this let $U \to X$ be a τ-cover such that $(Y_i)_U$ for $i = 1, 2$ and Z_U are objects of $\mathbf{UF\acute{E}t}_U$. Let us write

$$(Y_1)_U = \coprod_{i \in I} Y_{1,i}, \quad Z_U = \coprod_{j \in J} Z_j, \quad (Y_2)_U = \coprod_{k \in K} Y_{2,k}$$

where each piece is connected, and thus finite étale, over U. We have functions $f\colon I \to J$ and $g\colon K \to J$ as $(Y_1)_U \to Z_U$ and $(Y_2)_U \to Z_U$ must send connected components into connected components. So then,

$$(Y_1 \times_Z Y_2)_U \simeq (Y_1)_U \times_{Z_U} (Y_2)_U = \coprod_{\substack{(i,j,k) \in I \times J \times K \\ f(i)=k=g(j)}} Y_{1,i} \times_{Z_j} Y_{2,k}$$

from where the claim follows.

Claim (b) is clear. Indeed, $\mathbf{UCov}_X^\tau$ is closed under disjoint unions, and therefore for an object Y of $\mathbf{Cov}_X$ its connected components belonging to $\mathbf{UCov}_X^\tau$ implies it itself belongs to $\mathbf{UCov}_X^\tau$. Conversely, if Y belongs to $\mathbf{UCov}_X^\tau$ and C is any connected component of Y then C_U is a connected component of Y_U for any morphism of rigid K-spaces $U \to X$. In particular, if Y_U is in $\mathbf{UF\acute{E}t}_U$ then C_U is in $\mathbf{UF\acute{E}t}_U$. Taking $U \to X$ to be a τ-cover such that Y_U is in $\mathbf{UF\acute{E}t}_U$ thus shows that C is an object of $\mathbf{UCov}_X^\tau$ as desired.

To verify condition (c) it suffices to consider the case of either a coproduct, coequalizer, or fibered product diagram. The claim in the case of fibered products and disjoint unions has already been addressed earlier in this proof. Thus, it suffices to prove this result in the case of a coequalizer diagram. But, the formation of coequalizers commutes with base change (cf. [Sta21, Tag 03I4]). In particular, by considering the base change to an appropriate τ-cover we reduce ourselves to showing the coequalizer of a diagram in $\mathbf{UF\acute{E}t}_X$ taken in $\mathbf{Cov}_X$ is in $\mathbf{UF\acute{E}t}_X$. But, this is clear (cf. [Sta21, Tag 0BN9]). □

Definition 3.3.2. Let $\overline{x}$ be a geometric point of X. We define the *τ-adapted de Jong fundamental group* of the pair $(X, \overline{x})$, denoted $\pi_1^{\mathrm{dJ},\tau}(X, \overline{x})$, to be the fundamental group of the tame infinite Galois category $(\mathbf{UCov}_X^\tau, F_{\overline{x}})$.

Corollary 3.3.3. *Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then, the functor*

$$F_{\overline{x}}\colon \mathbf{UCov}_X^\tau \to \pi_1^{\mathrm{dJ},\tau}(X, \overline{x})\text{-}\mathbf{Set}$$

is an equivalence of categories.

4. The categories $\mathbf{Cov}_X^{\text{ét}}$ and $\mathbf{Loc}(X_{\text{proét}})$

In this section we show that if X is an adic space then there is a natural equivalence between $\mathbf{Cov}_X^{\text{ét}}$ (whose definition is the same as in Definition 3.1.1) and the category $\mathbf{Loc}(X_{\text{proét}})$ of locally constant sheaves (see [AGV73, Tome III, Exposé IX, §2.0]) on the pro-étale topology as in [Sch13].

4.1. **The pro-étale site of a rigid space and classical sheaves.** We begin by recalling the pro-étale site $X_{\text{proét}}$ of an adic space X. Moreover, we single out a class of sheaves in $\mathbf{Sh}(X_{\text{proét}})$, the analogue of classical sheaves from [BS15], which play an important role in the proof of our main result.

The pro-étale site. For a category $\mathcal{C}$, denote by $\mathbf{Pro}(\mathcal{C})$ the pro-completion of $\mathcal{C}$ as in [KS06, §6]. We write an object of $\mathbf{Pro}(\mathcal{C})$ as $\{U_i\}$. Identify $\mathcal{C}$ as a full subcategory of $\mathbf{Pro}(\mathcal{C})$ by sending U to the constant system $\{U\}$. To distinguish constant objects from general objects we write the latter as U.

Denote by $\mathbf{Pro\acute{E}t}_X$ the full subcategory of $\mathbf{Pro}(\mathbf{\acute{E}t}_X)$ consisting of those objects U such that $\mathsf{U} \to X$ is pro-étale in the sense of [Sch13, Definition 3.9]. Every object of $\mathbf{Pro\acute{E}t}_X$ has a presentation of the form $\{U_i\}_{i\in\mathcal{I}}$ where

- $\mathcal{I}$ has a final object 0 such that $U_0 \to X$ is étale,
- the maps $U_j \to U_i$ for $i \geqslant j \geqslant 0$ are finite étale and surjective.

When speaking of a presentation of an object of $\mathbf{Pro\acute{E}t}_X$ we shall assume it is of this form. By [Sch13, Lemma 3.10 vii)] the category $\mathbf{Pro\acute{E}t}_X$ admits all finite limits, a fact we use without further comment.

As in [Sch13, §3], for an object $\mathsf{U} = \{U_i\}$ we denote by $|\mathsf{U}|$ the topological space $\varprojlim |U_i|$ and call it the *underlying topological space* of U. We call an object U of $\mathbf{Pro\acute{E}t}_X$ *quasi-compact and quasi-separated* if its underlying space $|\mathsf{U}|$ is. Denote the subcategory of quasi-compact and quasi-separated objects of $\mathbf{Pro\acute{E}t}_X$ by $\mathbf{Pro\acute{E}t}_X^{\text{qcqs}}$. By [Sch13, Proposition 3.12 i), ii), and v)], $\mathbf{Pro\acute{E}t}_X^{\text{qcqs}}$ is closed under fiber products and every object of $\mathbf{Pro\acute{E}t}_X$ has a cover in $X_{\text{proét}}$ by objects of $\mathbf{Pro\acute{E}t}_X^{\text{qcqs}}$ which moreover can be assumed to be open embeddings on the underlying topological spaces. Denote by $X_{\text{proét}}^{\text{qcqs}}$ the induced site structure on $\mathbf{Pro\acute{E}t}_X^{\text{qcqs}}$ from $X_{\text{proét}}$. By [AGV73, Tome I, Exposé III, Thèoremé 4.1] the natural morphism of topoi $\mathbf{Sh}(X_{\text{proét}}) \to \mathbf{Sh}(X_{\text{proét}}^{\text{qcqs}})$ is an equivalence.

Define the *pro-étale site* of X, denoted $X_{\text{proét}}$, to be the site whose underlying category is $\mathbf{Pro\acute{E}t}_X$ and whose topology is as in [BMS19, §5.1]. The inclusion functor $\mathbf{\acute{E}t}_X \to \mathbf{Pro\acute{E}t}_X$ preserves finite limits, and is a continuous morphism of categories with Grothendieck topologies. So, we obtain an induced morphism of sites $X_{\text{proét}} \to X_{\text{ét}}$. We denote by ν_X, or ν when X is clear from context, the induced morphism of topoi $\mathbf{Sh}(X_{\text{proét}}) \to \mathbf{Sh}(X_{\text{ét}})$.

Classical sheaves. Using the morphism of topoi ν_X, we can define the appropriate notion of 'classical sheaf' as in [BS15, Definition 5.1.3].

Definition 4.1.1. A sheaf $\mathcal{G}$ in $\mathbf{Sh}(X_{\text{proét}})$ is called *classical* if it is in the essential image of the pullback functor $\nu^*\colon \mathbf{Sh}(X_{\text{ét}}) \to \mathbf{Sh}(X_{\text{proét}})$.

Denote by $\mathcal{F}^{(\to)}$ the association

$$\mathcal{F}^{(\to)}(\{U_i\}) = \varinjlim \mathcal{F}(U_i)$$

which is an element of $\mathbf{PSh}(X_{\text{proét}})$. In fact, $\mathcal{F}^{(\to)}$ is the presheaf pullback of $\mathcal{F}$, and thus there is a natural map $\mathcal{F}^{(\to)} \to \nu_X^*(\mathcal{F})$ of objects of $\mathbf{PSh}(X_{\text{proét}})$.

Proposition 4.1.2 (cf. [Sch13, Lemma 3.16])**.** *For any sheaf $\mathcal{F}$ in $\mathbf{Sh}(X_{\text{ét}})$ the map $\mathcal{F}^{(\to)} \to \nu_X^*(\mathcal{F})$ is a bijection when evaluated on any object of $X_{\text{proét}}^{\text{qcqs}}$.*

We thus obtain a structure theorem for classical sheaves.

Proposition 4.1.3 (cf. [BS15, Lemma 5.1.2])**.** *For any sheaf $\mathcal{F}$ in $\mathbf{Sh}(X_{\text{ét}})$, the unit map $\mathcal{F} \to \nu_*\nu^*\mathcal{F}$ for the adjunction $\nu^* \dashv \nu_*$ is an isomorphism. Therefore, the functor $\nu^* \colon \mathbf{Sh}(X_{\text{ét}}) \to \mathbf{Sh}(X_{\text{proét}})$ is fully faithful with essential image those $\mathcal{F}$ such that the counit map $\nu^*\nu_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.*

Proof. The latter statements follow from general category theory (see [Mac71, §IV.1, Theorem 1] and [Sta21, Tag 07RB]). To see the first statement we note that by [AGV73, Tome I, Exposé III, Thèoremé 4.1] it suffices to show that the map $\mathcal{F} \to \nu_*\nu^*\mathcal{F}$ is an isomorphism when evaluated on any quasi-compact and quasi-separated object of $X_{\text{ét}}$. This follows from Proposition 4.1.2. □

Arguing as in [BS15, Lemma 5.1.4], we obtain the following corollary.

Lemma 4.1.4. *Let $\mathcal{F}$ be an object of $\mathbf{Sh}(X_{\text{proét}})$. If there exists a covering $\{\mathsf{U}_i \to X\}$, and for each i a classical sheaf $\mathcal{F}_i$ such that $\mathcal{F}|_{\mathsf{U}_i} \simeq \mathcal{F}_i|_{\mathsf{U}_i}$, then $\mathcal{F}$ is classical. In particular, locally constant sheaves are classical.*

From this and Lemma 4.1.6 below we obtain the following.

Proposition 4.1.5. *For any set S the natural map of sheaves $\underline{S}_{X_{\text{proét}}} \to \mathrm{Hom}_{\text{cnts}}(\pi_0(|-|), S)$ is a bijection when applied to any object of $X^{\text{qcqs}}_{\text{proét}}$.*

Lemma 4.1.6. *Let $\{X_i\}$ be a projective system of spectral spaces with quasi-compact transition maps. Set $X = \varprojlim_i X_i$. Then, X is spectral and the natural map $\pi_0(X) \to \varprojlim_i \pi_0(X_i)$ is a homeomorphism of profinite spaces.*

Proof. Under the given assumptions, X is spectral by [FK18, Chapter 0, Theorem 2.2.10]. Suppose first that the X_i are connected and non-empty. By [FK18, Chapter 0, Proposition 3.1.10], the induced map $\varinjlim \Gamma(X_i, \underline{\mathbf{F}_2}) \to \Gamma(\varprojlim X_i, \underline{\mathbf{F}_2})$ is a bijection. Thus $\Gamma(X, \underline{\mathbf{F}_2}) = \mathbf{F}_2$, and so X is connected and non-empty, showing the assertion.

By [Sta21, Tag 0906], $\pi_0(X)$ and each $\pi_0(X_i)$ are profinite, and thus also $\varprojlim_i \pi_0(X_i)$ is profinite. Since profinite spaces are compact, it suffices to prove that the map in question is a bijection. To this end, let (C_i) be in $\varprojlim_i \pi_0(X_i)$. As C_i is closed in X_i, the C_i form an inverse system of non-empty connected spectral spaces with quasi-compact transition maps, and hence by the first paragraph the space $C = \varprojlim_i C_i$ is a non-empty and connected spectral space.

By [FK18, Chapter 0, Lemma 2.2.19] we have $C = \bigcap_i p_i^{-1}(C_i)$, where $p_i : X \to X_i$ is the natural map. Let x be a point in C and let C_x be the component of X containing it. Since $p_i(C_x)$ is connected and intersects C_i at $p_i(x)$ we see that $p_i(C_x) \subseteq C_i$ for all i, so that C_x maps to (C_i). Conversely, if the connected components of C_x and C_y for $x, y \in X$ both map to (C_i), then $x, y \in \bigcap_i p_i^{-1}(C_i) = C$, which is connected, and so $C_x = C_y$. □

For an object Y of $\mathbf{\acute{E}t}_X$ we denote by $h_{Y,\text{ét}}$ its corresponding sheaf. For an object Y of $\mathbf{Pro\acute{E}t}_X$ we denote by $h_{\mathsf{Y},\text{proét}}$ its corresponding presheaf, and by $h^\sharp_{\mathsf{Y},\text{proét}}$ its sheafification. This sheafification is necessary.

Example 4.1.7. Let X be the disjoint union of copies X_n of $\mathrm{Spa}(K)$ indexed by $n \geq 0$, and let L be a non-trivial finite separable extension of K. We set Y to be the disjoint union of $Y_n = \mathrm{Spa}(L)$, with the natural étale map $Y \to X$. Let $\mathsf{U} = \{U_i\}$ be the following object of $X_{\mathrm{proét}}$ indexed by $i \geq 0$: the space U_i is the disjoint union of $U_{i,n}$ indexed by $n \geq 0$, where $U_{i,n} = X_n$ for $n > i$ and $U_{i,n} = Y_n$ for $n \leq i$. Note that for every i, we have $\mathrm{Hom}_X(U_i, Y) = \emptyset$, and hence $h_{Y,\mathrm{proét}}(\mathsf{U})$ is empty. Then $\{Y_n \to \mathsf{U}\}_{n\geq 0}$ forms a pro-étale cover which violates the sheaf condition for $h_{Y,\mathrm{proét}}$.

While $X_{\mathrm{proét}}$ is not subcanonical, we can use Proposition 4.1.2 to show that $h_{\mathsf{Y},\mathrm{proét}}$ is a sheaf when restricted to $X^{\mathrm{qcqs}}_{\mathrm{proét}}$.

Proposition 4.1.8. *Let Y be an object of $\mathbf{Pro\acute{E}t}_X$. Then, the natural map $h_{\mathsf{Y},\mathrm{proét}} \to h^{\sharp}_{\mathsf{Y},\mathrm{proét}}$ a bijection when evaluated on any element of $\mathbf{Pro\acute{E}t}_X^{\mathrm{qcqs}}$. In particular, the site $X^{\mathrm{qcqs}}_{\mathrm{proét}}$ is subcanonical.*

Proof. By Lemma 4.1.9 below, it suffices to show that $h_{\mathsf{Y},\mathrm{proét}}$ is a sheaf when restricted to $X^{\mathrm{qcqs}}_{\mathrm{proét}}$. But, if $\mathsf{Y} = \{Y_i\}$ then $h_{\mathsf{Y},\mathrm{proét}} = \varprojlim h_{Y_i,\mathrm{proét}}$ where this inverse limit is taken in $\mathbf{PSh}(X_{\mathrm{proét}})$. Since the inverse limit of sheaves, taken in the category of presheaves, is a sheaf, we're restricted to showing that for all i the restriction of $h_{Y_i,\mathrm{proét}}$ to $X^{\mathrm{qcqs}}_{\mathrm{proét}}$ is a sheaf. But, since $h_{Y_i,\mathrm{proét}} = h^{(\to)}_{Y_i,\mathrm{ét}}$ this follows from Proposition 4.1.2. $\square$

Lemma 4.1.9. *Let $\mathcal{C}$ be a site and let $\mathcal{B} \subseteq \mathcal{C}$ be a full subcategory closed under fiber products and such that every object V of $\mathcal{C}$ admits a covering family $\{U_\alpha \to V\}_{\alpha\in I}$ with each U_α an object of $\mathcal{B}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$ whose restriction to $\mathcal{B}$ is a sheaf, i.e. which satisfies the sheaf condition for coverings of $\{U_\alpha \to U\}_{\alpha\in I}$ with U and all U_α objects of $\mathcal{B}$, and let $\mathcal{F}^{\#}$ be its sheafification. Then, for every object U of $\mathcal{B}$, the natural map $\mathcal{F}(U) \to \mathcal{F}^{\#}(U)$ is an isomorphism.*

Proof. Let us recall the construction of $\mathcal{F}^{\#}$ from [Sta21, Tag 00W1]. For a covering family $\mathcal{V} = \{V_\alpha \to V\}_{\alpha\in I}$ in $\mathcal{C}$, we denote by $H^0(\mathcal{V}, \mathcal{F})$ the corresponding equalizer. For an object V of $\mathcal{C}$, we denote by $\mathcal{J}_V$ the category of coverings of V: its objects are covering families $\{V_\alpha \to V\}_{\alpha\in I}$ and morphisms are refinements. We note that by [Sta21, Tag 00W7] and [Sta21, Tag 00W6] that the diagram $H^0(-, \mathcal{F})\colon \mathcal{J}_V \to \mathbf{Set}$ is filtered (see [Sta21, Tag 002V]). We define $\mathcal{F}^+(V) = \varinjlim_{\mathcal{J}_V} H^0(\mathcal{V}, \mathcal{F})$. This is a presheaf endowed with a natural transformation $\mathcal{F} \to \mathcal{F}^+$. The sheafification of $\mathcal{F}$ is identified with the composition $\mathcal{F} \to \mathcal{F}^+ \to (\mathcal{F}^+)^+$.

For V an object of $\mathcal{C}$, consider the full subcategory $\mathcal{J}'_V$ of $\mathcal{J}_V$ consisting of coverings whose elements are in $\mathcal{B}$. One can check that the diagram $H^0(-, \mathcal{F})\colon \mathcal{J}'_V \to \mathbf{Set}$ is cofinal in the diagram $H^0(-, \mathcal{F})\colon \mathcal{J}_V \to \mathbf{Set}$. Therefore $\mathcal{F}^+(V) = \varinjlim_{\mathcal{J}'_V} H^0(\mathcal{V}, \mathcal{F})$. This colimit depends only on the values of $\mathcal{F}$ on V, and if V is itself an object of $\mathcal{B}$, then $\mathcal{F}^+(V) = \mathcal{F}(V)$. Therefore $\mathcal{F}^{\#}(V) = (\mathcal{F}^+)^+(V) = \mathcal{F}^+(V) = \mathcal{F}(V)$. $\square$

4.2. **Sheaves on G-pFSet and $X_{\mathrm{profét}}$.** In this subsection we establish a profinite version of our main result.

The site G- pFSet. For a profinite group G, denote by G- **FSet** the category of finite sets with a continuous action of G. One has a natural identification $\mathbf{Pro}(G\text{-}\mathbf{FSet})$ with G- **pFSet** where the latter category is the category of profinite topological spaces with a continuous action of G.

We endow G- **pFSet** with a Grothendieck topology where the coverings $\{S_i \to S\}$ are jointly surjective maps such that each map $S_i \to S$ satisfies a condition similar to that in the definition of the pro-étale site as in [BMS19, §5.1] (with '$\mathsf{U}_0 \to \mathsf{U}$ étale' replaced by '$\mathsf{U}_0 \to \mathsf{U}$ is the pullback of a morphism in G- **FSet**' and '$\mathsf{U}_\mu \to \mathsf{U}_{<\mu}$ is finite étale surjective' is replaced by '$\mathsf{U}_\mu \to \mathsf{U}_{<\mu}$ is the pullback of a finite surjective map in G- **FSet**').

Proposition 4.2.1 ([Sch16, (1)])**.** *The collection $\{G \to *\}$ is a cover.*

Proposition 4.2.2. *The site G- **pFSet** is finite cocomplete and subcanonical.*

Proof. Since one has an equivalence $G\text{-}\mathbf{pFSet} \simeq \mathbf{Pro}(G\text{-}\mathbf{FSet})$ and G- **FSet** has all finite colimits, the existence of all finite colimits in G- **pFSet** follows from [KS06, Corollary 6.1.17 i)]. To show G- **pFSet** is subcanonical, let S be an element of G- **pFSet**. We can write $S = \varprojlim S_i$ where S_i are finite objects of G- **pFSet**. Since $h_S = \varprojlim_i h_{S_i}$ where the right hand side is a limit in the category of presheaves, we are reduced to showing that h_S is a sheaf when S is finite. This is handled in the proof of Proposition 4.2.3 below. □

Locally constant sheaves on G- pFSet. We now describe the category $\mathbf{Loc}(G\text{-}\mathbf{pFSet})$ in terms of the category G- **Set**.

Proposition 4.2.3. *Let G be a profinite topological group. Then, the functor*

$$G\text{-}\mathbf{Set} \to \mathbf{Loc}\,(G\text{-}\mathbf{pFSet})\,, \qquad T \mapsto (\mathcal{F}_T : S \mapsto \mathrm{Hom}_{\mathrm{cnts},G}(S,T))$$

is an equivalence of categories.

Proof. We first show that $\mathcal{F}_T$ is a sheaf. As covers in G- **pFSet** are jointly surjective, it follows that the sheaf condition is satisfied for the presheaf sending S to $\mathrm{Hom}_G(S,T)$ on G- **pFSet**. One needs to check that continuity condition is local, which can be checked as follows. Any cover $\{U_i \to U\}$ in G- **pFSet** has a finite subcover. Passing to such subcover and considering $\coprod_i U_i \to U$, we can assume the map to be a surjective map of compact spaces. It is then automatically a quotient map of topological spaces, thus the continuity of functions can be tested after pullback to $\coprod_i U_i$.

For S (resp. T) an object of G- **pFSet** (resp. G- **Set**), denote the underlying set of S (resp. T) with trivial G-action by S_{ta} (resp. T_{ta}). We claim that $\mathcal{F}_T$ is trivialized after restricting to G seen as an element of G- **pFSet**. Indeed, the slice category $G\text{-}\mathbf{pFSet}/G$ is equivalent to the category of profinite sets $*$- **pFSet** via $(S \xrightarrow{f} G) \mapsto (f^{-1}(1_G))$. Under this equivalence, the pushforward of $\mathcal{F}_T$ is the sheaf on $*$- **pFSet** given by $U \mapsto \mathrm{Maps}_{\mathrm{cnts}}(U,T)$, which is precisely a constant sheaf in $*$- **pFSet**.

Suppose now that $\mathcal{F}$ is a locally constant sheaf on G- **pFSet**. We now show there exists T in G- **Set** such that $\mathcal{F} = \mathcal{F}_T$. We claim that, the obvious map $\operatorname{colim} \mathcal{F}(S) \to \mathcal{F}(S_i)$, where $S = \varprojlim S_i$ where S_i are finite G-sets, is a bijection. To show such an equality it suffices to check it after a cover (cf. [BS15, Lemma 5.1.4]). This reduces us to the constant case, which is trivial.

We define our candidate for T as follows. As a discrete set, we set $T = \mathcal{F}(G)$. The G-action on $T = \mathcal{F}(G)$ is defined using the map $G \to \mathrm{Aut}_{G\text{-}\mathbf{pFSet}}(G)^{\mathrm{op}}$ given by $h \mapsto (g \mapsto g \cdot h)$ (where we view G as an element of G-$\mathbf{pFSet}$ by acting on the left). To see that this action is continuous we must show the equality $\mathcal{F}(G) = \operatorname{colim} \mathcal{F}(G/U)$, as U travels over the open normal subgroups of G, but this follows from the classicality condition.

We now verify that $\mathcal{F}_T$ is isomorphic to $\mathcal{F}$. By Lemma 4.2.1 the map $G \to *$ is a cover, and thus $G \times S \to S$ is a cover for any S an object of G-$\mathbf{pFSet}$. Observe that $G \times S_{\mathrm{ta}} \to S$ defined by $(g, s) \mapsto gs$ is isomorphic to $G \times S \overset{\mathrm{prs}}{\to} S$ in G-$\mathbf{pFSet}/S$ via the map $(g, s) \mapsto (g, gs)$. Similarly, $G \times G_{\mathrm{ta}} \times S_{\mathrm{ta}}$ is isomorphic to $G \times G \times S$ via the map $(g, h, s) \mapsto (g, gh, gs)$. We thus have the following isomorphism of diagrams.

$$
\begin{tikzcd}
S \arrow[d, equal] & G \times S_{\mathrm{ta}} \arrow[l] \arrow[d, "\wr"] & G \times G_{\mathrm{ta}} \times S_{\mathrm{ta}} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "\wr"] \\
S & G \times S \arrow[l] & G \times G \times S. \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd}
\tag{4.2.1}
$$

By the classicality of $\mathcal{F}$ we have a canonical identification

$$\mathcal{F}(G \times S_{\mathrm{ta}}) = \operatorname*{colim}_i \mathcal{F}(G \times S_i) = \mathrm{Maps}(S_i, T) = \mathrm{Hom}_{\mathrm{cnts}}(S_{\mathrm{ta}}, T)$$

after presenting $S_{\mathrm{ta}} = \lim_i S_i$ for finite (discrete) S_i, and the middle equality follows from the canonical identifications $\mathcal{F}(G \times S_i) = \mathcal{F}(\coprod_{s \in S_i} G) = \prod_{s \in S_i} \mathcal{F}(G) = \mathrm{Maps}(S_i, T)$. Consider the exact sequence of sets

$$\mathcal{F}(S) \to \mathcal{F}(G \times S_{\mathrm{ta}}) \rightrightarrows \mathcal{F}(G \times G_{\mathrm{ta}} \times S_{\mathrm{ta}})$$

obtained by using the identification given in Equation (4.2.1), the observation that as an object of G-$\mathbf{Set}$ we have that $(G \times S) \times_S (G \times S)$ is isomorphic to $G \times G \times S$, and the sheaf sequence for the cover $G \times S \to S$. We then make the identifications

$$\mathcal{F}(G \times S_{\mathrm{ta}}) = \mathrm{Hom}_{\mathrm{cnts}}(S_{\mathrm{ta}}, T), \qquad \mathcal{F}(G \times G_{\mathrm{ta}} \times S_{\mathrm{ta}}) = \mathrm{Hom}_{\mathrm{cnts}}(G_{\mathrm{ta}} \times S_{\mathrm{ta}}, T) \tag{4.2.2}$$

as above. As the maps $G \times G_{\mathrm{ta}} \times S_{\mathrm{ta}} \rightrightarrows G \times S_{\mathrm{ta}}$ are explicitly given by $(g, h, s) \mapsto (g, s)$ and $(g, h, s) \mapsto (gh, h^{-1}s)$, and by the definition of the action of G on T, we see that the corresponding maps $\mathrm{Hom}_{\mathrm{cnts}}(S_{\mathrm{ta}}, T) \rightrightarrows \mathrm{Hom}_{\mathrm{cnts}}(G_{\mathrm{ta}} \times S_{\mathrm{ta}}, T)$ are given by $f \mapsto f \circ \mathrm{pr}_{S_{\mathrm{ta}}}$ and $f \mapsto ((h, s) \mapsto h \cdot f(h^{-1} \cdot s))$. Using this, and the sequence given in Equation (4.2.2) we get the canonical identification of $\mathcal{F}(S)$ and $\mathrm{Hom}_{\mathrm{cnts},G}(S, T) = \mathcal{F}_T(S)$, as desired.

We have shown that our functor is essentially surjective, and showing that it is fully faithful is routine. ☐

The site $X_{\mathrm{profét}}$. Consider $\mathbf{Pro}(\mathbf{FÉt}_X)$ as a full subcategory of $\mathbf{ProÉt}_X$ and endow it with a Grothendieck topology as in [BMS19, §5.1] (with '$\mathsf{U}_0 \to \mathsf{U}$ étale' replaced by '$\mathsf{U}_0 \to \mathsf{U}$ finite étale').

Proposition 4.2.4 ([Sch13, Proposition 3.5],[Sch16])**.** *Let X be a connected adic space and let $\overline{x}$ be a geometric point of X. The functor*

$$X_{\text{profét}} \to \pi_1^{\text{alg}}(X, \overline{x})\text{-}\mathbf{pFSet}, \qquad \{U_i\} \mapsto \varprojlim F_{\overline{x}}(U_i)$$

is an equivalence of sites.

From this and Proposition 4.2.2 we immediately deduce the following.

Corollary 4.2.5. *The site $X_{\text{profét}}$ has all finite colimits and is subcanonical.*

Localy constant sheaves on $X_{\text{profét}}$. We end this section by describing all the objects of $\mathbf{Loc}(X_{\text{profét}})$ using Proposition 4.2.3. Define for any object $\mathsf{Y} = \{Y_j\}$ of $\mathbf{Pro}(\mathbf{\acute{E}t}_X)$ the presheaf $h_{\mathsf{Y},\text{profét}}$ on $X_{\text{profét}}$ obtained by restricting $h_{\mathsf{Y},\text{proét}}$ to $X_{\text{profét}}$. More explicitly, we have the following formula

$$h_{\mathsf{Y},\text{profét}}(\{U_i\}) = \varprojlim_j \varinjlim_i \operatorname{Hom}_{\mathbf{\acute{E}t}_X}(U_i, Y).$$

We define $h^\sharp_{\mathsf{Y},\text{profét}}$ to be the sheafification of this presheaf. The following result then follows from Proposition 4.2.4, Proposition 4.2.3, and the fact that $\pi_1^{\text{alg}}(X, \overline{x})\text{-}\mathbf{Set} \simeq \mathbf{UF\acute{E}t}_X$.

Proposition 4.2.6. *For all Y in $\mathbf{UF\acute{E}t}_X$, the presheaf $h_{Y,\text{profét}}$ is a sheaf and the functor*

$$\mathbf{UF\acute{E}t}_X \to \mathbf{Loc}(X_{\text{profét}}), \qquad Y \mapsto h_{Y,\text{profét}}$$

is an equivalence of categories.

Proof. It suffices to verify the first claim. We may assume that X is connected. The presheaf $h_{Y,\text{profét}}$ corresponds via the equivalence in Proposition 4.2.4 to the sheaf $\mathcal{F}_T$ from Proposition 4.2.3 with $T = Y_{\overline{x}}$, and thus is a sheaf. □

4.3. **Sheaves on $X_{\text{proét}}$ which are pro-finite étale locally constant.** We now to upgrade Proposition 4.2.6 to a result about sheaves on $X_{\text{proét}}$ which are pro-finite étale locally constant.

Comparing $\mathbf{Sh}(X_{\text{proét}})$ and $\mathbf{Sh}(X_{\text{profét}})$. The inclusion $\mathbf{Pro}(\mathbf{F\acute{E}t}_X) \to \mathbf{Pro\acute{E}t}_X$ is a continuous functor of categories with Grothendieck topologies. Since this morphism preserves fiber products we get an induced map of sites $X_{\text{proét}} \to X_{\text{profét}}$. Denote the induced morphism of topoi $\mathbf{Sh}(X_{\text{proét}}) \to \mathbf{Sh}(X_{\text{profét}})$ by θ_X or just θ when X is clear from context.

Proposition 4.3.1 (cf. [AG16, Proposition VI.9.18])**.** *Let X be a quasi-compact and quasi-separated adic space. Then, the unit map $\mathbf{1} \to \theta_*\theta^*$ for the adjunction $\theta^* \dashv \theta_*$ is an isomorphism. Thus, $\theta^* \colon \mathbf{Sh}(X_{\text{profét}}) \to \mathbf{Sh}(X_{\text{proét}})$ is fully faithful with essential image those sheaves $\mathcal{F}$ for which the counit map $\theta^*\theta_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.*

Proof. Again, the latter statements follow from general category theory. To show the first statement begin by noting that θ^*, being a left adjoint, commutes with colimits. We claim that θ_* also commutes with filtered colimits. By [AGV73, Tome II, Exposé VI, Thèoremé 5.1] it suffices to prove that θ

is a coherent morphism between coherent topoi. The fact that $\mathbf{Sh}(X_{\text{proét}})$ is coherent is verified in [Sch13, Proposition 3.12 (vii)], and the fact that $\mathbf{Sh}(X_{\text{profét}})$ is coherent follows easily from this. To show that θ is coherent let $\mathsf{V} = \{V_j\}$ be an object of $X_{\text{profét}}$. This system consists of spectral spaces and the transition maps are quasi-compact. We deduce from Proposition 4.1.6 that $|\mathsf{V}|$ is quasi-compact and quasi-separated from where we are finished by [Sch13, Proposition 3.12 (iv)].

Fix a sheaf $\mathcal{F}$ in $\mathbf{Sh}(X_{\text{proét}})$. We now show that the unit map $\mathcal{F} \to \theta^*\theta_*\mathcal{F}$ is an isomorphism. By Corollary 4.2.5 the category $X_{\text{profét}}/\mathcal{F}$ is filtered, where this category means the subcategory of the slice category $\mathbf{PSh}(X_{\text{profét}})/\mathcal{F}$ consisting of representable presheaves (see [AGV73, Tome I, Exposé I, 3.4.0]). By [AGV73, Tome I, Exposé II, Corollaire 4.1.1] the natural map

$$\varinjlim_{\mathsf{Y}\in\mathbf{Pro}(\mathbf{F\acute{E}t}_X)/\mathcal{F}} h_{\mathsf{Y},\text{profét}} \to \mathcal{F}$$

is an isomorphism where we do not need to sheafify $h_{\mathsf{Y},\text{profét}}$ by Corollary 4.2.5. Since θ^* and θ_* both commute with filtered colimits we've thus reduced to showing that the unit map is an isomorphism when evaluated on representable sheaves $h_{\mathsf{Y},\text{profét}}$. A computation using the adjunction property and [Sta21, Tag 04D3] shows this is equivalent to the map $h_{\mathsf{Y},\text{profét}}(\mathsf{V}) \to h^\sharp_{\mathsf{Y},\text{proét}}(\mathsf{V})$ being a bijection. This follows from Proposition 4.1.8. $\square$

Proposition 4.3.2. *Let X be a quasi-compact and quasi-separated adic space and let Y be an object of $\mathbf{UF\acute{E}t}_X$. Then, $\theta^* h_{Y,\text{profét}} \simeq h^\sharp_{Y,\text{proét}}$.*

Proof. Write $Y = \varinjlim_j Y_j$ with Y_j objects of $\mathbf{F\acute{E}t}_X$ and the transition maps are open embeddings. Since θ^* commutes with colimits, we're reduced to showing that $h_{Y,\text{profét}} \simeq \varinjlim_j h_{Y_j,\text{profét}}$ and $h^\sharp_{Y,\text{proét}} \simeq \varinjlim_j h^\sharp_{Y_j,\text{proét}}$. The former is clear since every object of $X_{\text{profét}}$ is quasi-compact and quasi-separated, and the latter is clear by combining Proposition 4.1.8 and [AGV73, Tome I, Exposé III, Thèoremé 4.1]. $\square$

Locally constant sheaves on $X_{\text{proét}}$ trivialized on a pro-finite étale cover. We now establish the aforementioned special case of our main theorem, classifying locally constant sheaves on $X_{\text{proét}}$ which become constant on a pro-finite étale cover of X.

Proposition 4.3.3. *Let X be a quasi-compact and quasi-separated adic space. Then, the functor*

$$\mathbf{UF\acute{E}t}_X \to \mathbf{Loc}(X_{\text{proét}}), \qquad Y \mapsto h^\sharp_{Y,\text{proét}}$$

is fully faithful with essential image those objects $\mathcal{F}$ of $\mathbf{Loc}(X_{\text{proét}})$ which become constant on a pro-finite étale cover of X.

Proof. We may assume that X is connected. To see this functor has image contained in the correct subcategory of $\mathbf{Loc}(X_{\text{proét}})$ let Y be an object of $\mathbf{UF\acute{E}t}_X$. As observed in the proof of Proposition 4.2.6 the sheaf $h_{Y,\text{profét}}$ corresponds, under Proposition 4.2.4 to $\mathcal{F}_T$ where $T = Y_{\overline{x}}$. From loc. cit. we see that $h_{Y,\text{profét}}$ is an object of $\mathbf{Loc}(X_{\text{profét}})$. In particular, $\theta^* h_{Y,\text{profét}}$ is an object of $\mathbf{Loc}(X_{\text{proét}})$ trivialized on a pro-finite étale cover of X and so we're done by Proposition 4.3.2. Moreover, our functor is fully faithful since it can be described, by Proposition 4.1.8, as ν^* restricted to $\mathbf{UF\acute{E}t}_X$ from where we deduce fully faithfulness by Proposition 4.1.3.

To show our functor is essentially surjective let $\mathcal{F}$ an object of $\mathbf{Sh}(X_{\text{proét}})$ trivialized on a pro-finite étale cover of X. In particular, it is trivialized on a fixed universal pro-finite Galois cover $\tilde{X}$ of X which, under the equivalence in Proposition 4.2.4, corresponds to $\pi_1^{\text{alg}}(X,\overline{x})$ acting on itself by left multiplication. By combining Proposition 4.2.4 and Proposition 4.2.3, there exists some Y in $\mathbf{UF\acute{E}t}_X$ and an isomorphism $\psi : h_{Y,\text{profét}} \xrightarrow{\sim} \theta_*\mathcal{F}$ of objects of $\mathbf{Sh}(X_{\text{profét}})$. One then obtains a morphism $h^\sharp_{Y,\text{proét}} \xrightarrow{\sim} \theta^* h_{Y,\text{profét}} \xrightarrow{\sim} \theta^*\theta_*\mathcal{F} \to \mathcal{F}$ where the first isomorphism comes from Proposition 4.3.2, the second map is $\theta^*\psi$, and the last map is the counit map. We claim that this is an isomorphism. It remains to check that the counit map $\phi : \theta^*\theta_*\mathcal{F} \to \mathcal{F}$ is an isomorphism. To see this, first observe that since this is a morphism of sheaves and $\tilde{X} \to X$ is a cover in $X_{\text{proét}}$, it is enough to check that $\phi_{|\tilde{X}}$ is an isomorphism. As both sheaves are locally constant and trivialized when restricted to the connected cover $\tilde{X}$, it is enough to check that $\phi(\tilde{X})\colon \theta^*\theta_*\mathcal{F}(\tilde{X}) \to \mathcal{F}(\tilde{X})$ is a bijection. But this can be checked after applying θ_*. The map obtained from unit and counit $\theta_* \to \theta_*\theta^*\theta_* \to \theta_*$ is the identity morphism (see [Mac71, §IV.1, Theorem 1]). Moreover, by Proposition 4.3.1, we know that the first morphism in this composition is an isomorphism. It follows that the second morphism is an isomorphism too. Thus, $\theta_*\theta^*\theta_*\mathcal{F}(\tilde{X}) = \theta_*\mathcal{F}(\tilde{X})$, as desired. $\square$

4.4. Main result.

We now arrive at the main result of this section.

Theorem 4.4.1. *Let X be an adic space. Then, the functor*

$$\mathbf{Cov}^{\text{ét}}_X \to \mathbf{Loc}(X_{\text{proét}}), \quad Y \mapsto h^\sharp_{Y,\text{proét}}$$

is an equivalence of categories. Therefore, if X is a rigid K-space the stalk functor

$$F_{\overline{x}}\colon \mathbf{ULoc}(X_{\text{proét}}) \to \pi_1^{\text{dJ,ét}}(X,\overline{x})\text{-}\mathbf{Set}$$

is an equivalence of categories.

Proof. The final claim follows immediately Corollary 3.3.3 so we focus on the first claim. Suppose first that $Y \to X$ is an object of $\mathbf{Cov}^{\text{ét}}_X$. We claim that the object $h^\sharp_{Y,\text{proét}}$ of $\mathbf{Sh}(X_{\text{proét}})$ lies in the subcategory $\mathbf{Loc}(X_{\text{proét}})$. But, there exists an étale cover $\{U_i \to X\}$ such that U_i is affinoid for all i, and such that Y_{U_i} is an object of $\mathbf{UF\acute{E}t}_{U_i}$ for all i. Since $h^\sharp_{Y,\text{proét}}$ restricted to U_i is $h^\sharp_{Y_{U_i},\text{proét}}$ in $\mathbf{Sh}(X_{\text{proét}}/U_i) = \mathbf{Sh}((U_i)_{\text{proét}})$ we know from Proposition 4.3.3 that $h^\sharp_{Y,\text{proét}}$ restricted to each U_i is locally constant. Thus, $h^\sharp_{Y,\text{proét}}$ itself is locally constant.

As our our functor is nothing but ν^*_X by Proposition 4.1.8, it is fully faithful by Proposition 4.1.3. Given this, and the fact both the source $\mathbf{Cov}^{\text{ét}}_X$ and the target $\mathbf{Loc}(X_{\text{proét}})$ naturally form stacks on X for the étale topology (the former by [War17, Corollary 3.1.9]), it is enough to show essential surjectivity étale locally on X. Thus, it is enough to assume X is affinoid and show that every sheaf of sets $\mathcal{F}$ of $X_{\text{proét}}$ which becomes constant on a pro-finite étale cover of X comes from an object of $\mathbf{UF\acute{E}t}_X$. This is Proposition 4.3.3. $\square$

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