# Weil-Deligne representations and $p$-adic Hodge theory: motivation

## Prologue

I'd quickly like to explain what this short note is about. I do this mostly to orient the reader, since (upon rereading) it's somewhat non-obvious what the 'goal' of this note is.

So, this is an extension of some notes for a seminar at Berkeley whose overall goal was to understand the statement of the global Langlands conjecture (and related topics). This specific talk was aimed at trying to motivate $p$-adic Hodge theory, deferring the basic definitions and technical results we'd need (i.e. de Rham representations, Hodge-Tate representations, and a bit about semi-stable representations) to a later talk.

Now, there are undoubtedly many ways in which one can attempt to motivate $p$-adic Hodge theory. For example, a somewhat glaringly obvious one being the titular theorem: namely, an analogue of the usual Hodge decomposition over $p$-adic fields. Considering our goal (in the overall seminar) it seemed more relevant to, instead, discuss why $p$-adic Hodge theory might/should come up in our discussion of Langlands. I wanted to talk about *families* of Galois representations.

Specifically, the goal of this talk was to:

1. Explain what $p$-adic Hodge theory is at large (this is the content of Section 1).

2. Motivate, from a somewhat naive perspective, why one might want/hope to define what it means to have a 'family' of $\ell$-adic representations of $G_{\mathbb{Q}}$ and explain why the technical push needed to make such a notion rigorous is the elimination of topological notions in the definition of $\ell$-adic representations. (This is carried out in Section 2).

3. Explain through a specific example (Tate modules of elliptic curves) why naturally occurring $p$-adic of $G_{\mathbb{Q}_p}$ representations are much richer (and consequently much *harder*) than $\ell$-adic (where $\ell \neq p$) representations of $G_{\mathbb{Q}_p}$. (This is section 4).

4. Explain how to carry out our desire from 2. (i.e. the elimination of topology) in the case of $\ell$-adic representations of $G_{\mathbb{Q}_p}$ (with $\ell \neq p$)—in other words, talking about *Weil-Deligne representations*. The emphasis being not the proofs but, really, that the operative thing was the incongruity between the $p$-adic and $\ell$-adic topologies—the operative thing being Grothendieck's $\ell$-adic monodromy theorem.

5. Outline how one might make such a thing work for $p$-adic representations of $G_{\mathbb{Q}_p}$ (i.e. the association of Weil-Deligne representations) using $p$-adic Hodge theory. Specifically, explaining that the procedure of associating Weil-Deligne representations to $p$-adic representations is formally similar to $\ell$-adic case and, again, the operative thing lies on the extremely deep *$p$-adic monodromy theorem*. (This is section 5.)

6. Apply this 'developed' theory to define compatible systems and give a glib idea of how this should relate to the global Langlands correspondence (and even give an equally glib statement of the global Langlands conjecture for $\mathrm{GL}_n$). We also apply the association of Weil-Deligne representations to give the 'correct' definition of the $L$-function of an $\ell$-adic representation (or rather, the non-archimedean factors of the $L$-function). (This is section 6).

To summarize, the point of this note (the upshot, as it were) is that really the global Langlands program should be about associations between (cuspidal) automorphic representations and *families* of Galois representations. The rigorous notion of family requires the elimination of continuity in the definition of

$\ell$-adic representation—more correctly, one wants to associate Weil-Deligne representations to these $\ell$-adic representations. One achieves this for $\ell$-adic (for $\ell \neq p$) representations fairly easily because the clash of $\ell$-adic topology and $p$-adic topology make the representations (as a whole) 'simpler'—this is codified by Grothendieck's $\ell$-adic monodromy theorem. One cannot hope to achieve the association of Weil-Deligne representations to $p$-adic representations of $G_{\mathbb{Q}_p}$ since their topologies *don't* clash, and they are not at all simple. So, one needs to complicated machinery of $p$-adic Hodge theory to obtain the same result (the association of Weil-Deligne representations) by a spiritually identical argument—everything relying again on a monodromy theorem the *p-adic monodromy* theorem.

# 1 A matter of taxonomy

In this section we would like to give a very, very broad overview of what $p$-adic Hodge theory 'is' before we dive into some problems whose solution requires it.

The subject of '$p$-adic Hodge theory' can (to an outsider like myself) sometimes seem like it suffers from a crisis of identity. Namely, when one Googles '$p$-adic Hodge theory' one will find many, many sources, and these sources will be of two, ostensibly unrelated, types. This is because, $p$-adic Hodge theory has 'two sides' (of the same coin!) that we would now like to roughly explain:

**Representation theoretic:** From this perspective $p$-adic Hodge theory is the study of $p$-adic representations of $G_K$ where $K$ is a $p$-adic field. More specifically, one might say that the study of continuous representations $G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ is what $p$-adic Hodge theory 'does' or 'is about'. More to the point, the study of such representations (as we will see concretely later) is much more difficult than the study of $\ell$-adic representations of $G_{\mathbb{Q}_p}$ (where $\ell \neq p$) and so one of $p$-adic Hodge theory's main functions is to sort the veritable wilderness of continuous representations $G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ in to various "nice" classes of representations, and attempt to describe these "nice" classes by (semi)linear algebra not directly involving the Galois action (for example, one might associate to a Galois representation a *filtered* vector space).

**Geometric:** The other side of $p$-adic Hodge theory is one of geometry. And, even here, there are two sub-perspectives (which are really different only in motivation—the results they aim to prove are the same):

1. [**Motivic**] Try and compare various cohomology theories for varieties over $\mathbb{Q}_p$: étale, de Rham, Hodge, crystalline, etc. and understand what these comparisons tells us geometrically.

2. [**Rep theory**] Prove that for every class of "nice" representations (as mentioned above) there is a "correspondingly nice" class of varieties so that the geometric $p$-adic representations coming from this class of varieties (i.e. $H^m_{\text{ét}}(X, \mathbb{Q}_p)$ for $X$ "correspondingly nice") are, in fact, "nice".

   Here are some examples of what this might mean:

   | Nice | Correspondingly nice |
   |---|---|
   | de Rham | Smooth proper |
   | Semi-stable | Smooth proper with semi-stable reduction |
   | Crystalline | Smooth proper with good reduction |

   where it should be pointed out that each of these results is incredibly difficult.

I mention these differences in an attempt to dispel the type of confusion that I had when first learning $p$-adic Hodge theory. Namely, if one looks at the notes of, say, Conrad-Brinon they look *totally* different than the papers of, say, Faltings, Beilinson, Scholze, or Niziol. This is because the former is focused on the representation theoretic aspects of $p$-adic Hodge theory whereas the latter papers are focused on the geometric perspectives.

## 2 Compatible systems

Now, the real main technical goal of this note is to explain how one can formalize what it means for a system of $\ell$-adic Galois representations (say of $G_\mathbb{Q}$) to form a 'family'. We'll see that this shall (despite first guesses) be technically daunting and require, ultimately, $p$-adic Hodge theory.

So, let us begin by trying to explain what, intuitively, it means for an $\ell$-adic representation to belong to a 'family' by looking at some natural examples.

Namely, in this seminar so far we have only talked about a few large classes of Galois representations. These are as follows:

1. To any abelian variety $A/\mathbb{Q}$ and prime $\ell$ we obtain the $\ell$-adic representation $T_\ell A$ of $G_\mathbb{Q}$, the *$\ell$-adic Tate module* of $A$.
2. For any prime $\ell$ we have the $\ell$-adic *cyclotomic character* $\chi_\ell : G_\mathbb{Q} \to \mathbb{Q}_\ell^\times$.
3. For any prime $\ell$ and any normalized Hecke eigen-newform of some level $\Gamma_0(N)$ we obtained an $\ell$-adic representation $\rho_{f,\ell} : G_\mathbb{Q} \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$.

Now, one thing that one should immediately notice, as is extremely notationally clear, is that these representations come in 'families'. Namely, for each of the above three examples we got an $\ell$-adic representation for each choice of a prime $\ell$.

**Remark 2.1:** It is somewhat humorous to note that, in some sense, examples 2. and 3. above are really *derived from* 1. in some sense. Namely, $\chi_\ell = \det T_\ell E$ (for $E$ an elliptic curve) and $\rho_{f,\ell}$ was constructed, roughly, as a quotient of $T_\ell \operatorname{Jac}(X_0(N))$. ♦

More generally, if $X/\mathbb{Q}$ is a variety, then one cannot help in creating a *family* of representations when trying to create just *one* representation. Namely, the only way that I personally know how to get representations of $G_\mathbb{Q}$ from $X$ is by considering $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$ but, of course, by design this lives in a family as $\ell$-varies. This also is, when interpreted correctly, why all of the above representations came in families—they were all built from geometric origins—even 2. which were, secretly, built from Kugo-Sato varieties: see the nice paper of Tony Scholl on the subject).

So, one may begin to wonder whether or not this is some general phenomenon or if it is a coincidence of how we made étale cohomology—we had a cohomology group for *every* prime $\ell$. But, of course, before we start wondering about such generalities it's helpful first to even explain what we mean by a 'family' of representations.

Namely, let us begin by remarking that the representation $\rho : G_\mathbb{Q} \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ is determined by its local factors $\rho\mid_{G_{\mathbb{Q}_p}}$ (certainly this is the case in the representations we are interested in: those unramified almost everywhere) and, for technical reasons, it's really easier to try and phrase what a compatible family means by turning an eye towards these local representations $\rho\mid_{G_{\mathbb{Q}_p}}$.

So, one way in which one might optimistically try and define what it means for a sequence of representations $\rho_\ell : G_\mathbb{Q} \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ to form a *family* is to say that each $\rho_\ell$, or more correctly $\rho_\ell\mid_{G_{\mathbb{Q}_p}}$, comes from a single object. Namely, perhaps there is some sort of algebraic objects $V_p$ over $\overline{\mathbb{Q}}$, something like a representation of $G_{\mathbb{Q}_p}$ into $\mathrm{GL}_n(\overline{\mathbb{Q}})$, such that each $\rho_\ell\mid_{G_{\mathbb{Q}_p}}$ is something like $V_p \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}_\ell}$ (of course this depends on the choice of embedding $\iota : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$ but we ignore this important technicality). Namely, this would say that the representations $\rho_\ell$ form a 'family' since they are just the '$\ell$-adic realization' of some rationally defined objects (or, again, more correctly the local factors of $\rho_\ell$ are realizations of local factors of some rationally defined objects).

But, this is bound to have issues. Namely, this posits a fairly direct comparison between $\ell$-adic and $\ell'$-adic representations of $G_{\mathbb{Q}_p}$ for $\ell \neq \ell'$ and such comparisons are bound to be an issue. Why? Because $\mathbb{Q}_\ell$ and $\mathbb{Q}_{\ell'}$ have *incommensurate topologies*—one may of trying to make this precise is to note that there are no continuous homomorphisms $\mathbb{Q}_\ell \to \mathbb{Q}_{\ell'}$ if $\ell \neq \ell'$. The reason this is bound to be an issue is that whatever the action $G_{\mathbb{Q}_p} \to \mathrm{GL}(V_p)$ is we need to have that its composition with each $\mathrm{GL}(V_p) \to \mathrm{GL}(V_p \otimes_{\overline{\mathbb{Q}}} \overline{\mathbb{Q}_\ell})$ is continuous <u>for all $\ell$</u> and this is just not going to happen. Thus, we'd like to figure out how to actually talk about comparision $\ell$-adic and $\ell'$-adic representations of $G_{\mathbb{Q}_p}$ correctly.

An obvious idea presents itself: eliminate the topology. Namely, the only thing that made the above comparisions impossible was the fact that the $\ell$-adic and $\ell'$-adic topologies don't play nicely together if $\ell \neq \ell'$. If we're able to eliminate the dependence on the topologies then we're golden. Indeed, once the topologies are removed only algebra remains (this sounds like some ancient prophecy...) and $\overline{\mathbb{Q}_\ell}$ and $\overline{\mathbb{Q}_{\ell'}}$ are very comparable algebraically—they're isomorphic! In fact, we might as well replace every instance of $\overline{\mathbb{Q}_\ell}$ with $\mathbb{C}$ if we can ignore topologies.

So, the way forward towards defining what it means for a sequence $\rho_\ell$ of $\ell$-adic representations to be a 'family' is now clear: we need to figure out how to eliminate topologies. But we will see that this elimination will be broken into two very different steps:

1. Elimating the dependence on the topology of an $\ell$-adic representation of $G_{\mathbb{Q}_p}$ with $\ell \neq p$.
2. Eliminating the dependence on the topology of a $p$-adic representation of $G_{\mathbb{Q}_p}$.

What we'll see is that 1. is fairly easy and mostly formal—this, as we'll see, is largely a function of the incommensurability of the topologies on $G_{\mathbb{Q}_p}$ and $\overline{\mathbb{Q}_\ell}$ if $\ell \neq p$ (which might be expected since, after all, $G_{\mathbb{Q}_p}$ is heavily related to $\mathbb{Q}_p$ itself—its (Hausdorff) abelianization contains $\mathbb{Q}_p^\times$ as a dense subset). Intuitively, since the topologies don't play well together to begin with, there probably aren't too many interesting things happening topologically between $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ and so it's easy to imagine that it won't be too difficult to disentangle these representations from the topology.

In contrast 2. rests on some very deep mathematics and, really, is a huge non-trivial part of $p$-adic Hodge theory. This is because, unlike the $\ell$-adic case, the topologies of $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_n(\overline{\mathbb{Q}_p})$ <u>do</u> interact in highly interesting, non-trivial ways, and so it will be much more difficult to extract a non-topological object from such things. What is also at play is the fact that just, in general, $p$-adic representations of $G_{\mathbb{Q}_p}$ are much more numerous and complicated, and so harder to really deal with (this is a direct consequence of the last sentence though: the numerosity is a function of the fact that the topologies mesh well allowing for more interesting constructions).

To hammer into our heads how large of a difficulty gap there is between 1. and 2. let us remark on the following. The carrying out of 1. is classical due (I believe) to Grothendieck, and done in the '60s. The process of 2. (in the form in which we will interpret it) was not completed until 2004 (by Kedlaya, Berger et al.)—quite a difference.

# 3 A concrete example

Before we charge headfirst into eliminating the topologies, and thus making rigorous the notion of compatible systems, we'd like to pause and try to justify something we've said multiple times now. Namely, in this section we'd like to understand, through a concrete example, the statement which I have made several times in this note: "$p$-adic representations are 'harder' than $\ell$-adic ones".

In particular, I'd like to think about the Tate module $T_\ell E$ (with, possibly, $\ell = p$) of an elliptic curve $E/\mathbb{Q}_p$ with good reduction, and see how the case $\ell \neq p$ differs from the case $\ell = p$.

But, before we dive into the two cases, let us consider what we know in general. Namely, by definition of $E/\mathbb{Q}_p$ having good reduction, we know that there is a unique elliptic scheme $\mathcal{E}/\mathbb{Z}_p$ with $\mathcal{E}_{\mathbb{Q}_p} \cong E$. Let us denote by $f$ the map $\mathcal{E} \to \mathrm{Spec}(\mathbb{Z}_p)$.

<u>**Case 1:** $\ell \neq p$</u> We know from the smooth proper base change theorem that we have isomorphisms

$$T_\ell E \cong (R^1 f_* \mathbb{Q}_\ell)^\vee_{\overline{\mathbb{Q}_p}} \cong (R^1 f_* \mathbb{Q}_\ell)^\vee_{\overline{\mathbb{F}_p}} \cong T_\ell \mathcal{E}_{\overline{\mathbb{F}_p}} \tag{1}$$

which 'intertwine the $\pi_1^{\text{ét}}$ actions'. Or, less cryptically, we have that the isomorphism in (1) satisfies

$$\begin{tikzcd} T_\ell E \arrow[r, "\approx"] & T_\ell \mathcal{E}_{\overline{\mathbb{F}_p}} \\ \circlearrowright & \circlearrowright \\ G_{\mathbb{Q}_p} \arrow[r, two heads] & G_{\mathbb{F}_p} \end{tikzcd} \tag{2}$$

where the surjection $G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p}$ is the usual one.

From this we conclude several things. First and foremost, we see that $T_\ell E$ must be unramified since $I_{\mathbb{Q}_p}$ acts trivially on $T_\ell \mathcal{E}_{\mathbb{F}_p}$ (since it acts through $G_{\mathbb{F}_p}$). We also see that, really, $T_\ell E$ only depends on $\mathcal{E}_{\mathbb{F}_p}$ and not at all on the deformation $\mathcal{E}$ to $\mathbb{Z}_p$ or, equivalently, on $E$. In particular, we see that we have lost quite a bit of information for in the following diagram

$$\begin{tikzcd} E \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] & \mathcal{E}_{\mathbb{F}_p} \arrow[l] \arrow[d] \\ \mathrm{Spec}(\mathbb{Q}_p) \arrow[r] & \mathrm{Spec}(\mathbb{Z}_p) & \mathrm{Spec}(\mathbb{F}_p) \arrow[l] \end{tikzcd} \tag{3}$$

the left Cartesian square is not at all 'lossy' (i.e. $E \rightsquigarrow \mathcal{E}$ is an equivalence) but the right Cartesian square is extremly 'lossy' (the map $\mathcal{E} \rightsquigarrow \mathcal{E}_{\mathbb{F}_p}$ is far from being fully faithful). Thus, we see that we have lost quite a bit of information about $E$ when looking at $T_\ell E$, beyond what we might expect.

If this is not convincing enough, let us note that the above observation tells us that the data of the Galois representation $T_\ell E$ is equivalent to the data of the Frobenius eigenvalues $\alpha, \beta \in \overline{\mathbb{Z}}$ which, in turn, is equivalent to the data of the cardinality $\#\mathcal{E}_{\mathbb{F}_p}(\mathbb{F}_p)$. In particular, the isomorphism type of $T_\ell E$ is *equivalent* to the isogeny type of $\mathcal{E}_{\mathbb{F}_p}$.

**Case 2:** $\ell = p$ In this case we still have a map $T_p E \to T_p \mathcal{E}_{\mathbb{F}_p}$ but unlike the map in (1) it is far from being an isomorphism. A silly reason this must be the case, without thinking very much, is that $T_p E$ is always a rank 2 $\mathbb{Z}_p$-module (since $p$ is invertible in $\mathbb{Q}_p$!) but $T_p \mathcal{E}_{\mathbb{F}_p}$ is *at most* rank 1 (being rank 1 precisely when $\mathcal{E}_{\mathbb{F}_p}$ is ordinary). Thus, we see that $T_p E$ contains much more information than just $\mathcal{E}_{\mathbb{F}_p}$.

Let us note what the kernel of the map $T_p E \to T_p \mathcal{E}_{\mathbb{F}_p}$ is. There are several ways of describing it, all important in their own little way. First, we can think of the kernel as being precisely $T_p \mathcal{E}[p^\infty]^c_{\mathbb{Q}_p}$ which, while notationally intimidating, is not overly complicated to define. Namely, $\mathcal{E}$ has associated to it a $p$-divisible group $\mathcal{E}[p^\infty]$ over $\mathbb{Z}_p$. But, since $\mathbb{Z}_p$ is a complete Noetherian local ring, we know that $\mathcal{E}[p^\infty]$ sits in a connected-étale sequence

$$0 \to \mathcal{E}[p^\infty]^c \to \mathcal{E}[p^\infty] \to \mathcal{E}[p^\infty]^{\text{ét}} \to 0 \tag{4}$$

By a theorem of Tate the generic fiber functor from $p$-divisible groups over $\mathbb{Z}_p$ to those over $\mathbb{Q}_p$ is fully faithful, and thus no information is lost when passing from $\mathcal{E}[p^\infty]^c$ to its generic fiber $\mathcal{E}[p^\infty]^c_{\mathbb{Q}_p}$. But, once one is over a field in which $p$ is invertible, $p$-divisible groups are entirely captured by the Galois representation that is their Tate module—in our case, this means that $\mathcal{E}[p^\infty]^c_{\mathbb{Q}_p}$ is, essentially, the same thing as the $G_{\mathbb{Q}_p}$-module $T_p \mathcal{E}[p^\infty]^c_{\mathbb{Q}_p}$.

**Remark 3.1:** I think that two remarks are in order here that are not strictly important for this note at large.

First, the above really well illustrates how the important object is *not* the Tate module, but the $p$-divisible group. Indeed, when one tries to make connections between $\mathbb{Q}_p$ geometry and $\mathbb{F}_p$ geometry of abelian varieties, the somehow universal constant is the notion of $p$-divisible groups whereas the Tate module is somehow a shadow—dealing with $\mathbb{Q}_p$ geometry just spoils one since there the Tate module captures precisely the same information the $p$-divisible group does (whereas this is not the case over $\mathbb{F}_p$).

Secondly, since it's so well-known, I think it's worth unmasking another identity of $T_p \mathcal{E}[p^\infty]^c_{\mathbb{Q}_p}$ that will not strictly be important to us. Namely, by the work of Tate we know that the connected $p$-divisible group $\mathcal{E}[p^\infty]^c$ is 'equivalent to' (i.e. represents the same fppf sheaf) as some formal Lie group over $\mathbb{Z}_p$. In particular, for our case know that $\mathcal{E}[p^\infty]^c$ is 'the same thing' (in the sense of sheaves mentioned above) to the usual formal group $\widehat{\mathcal{E}}$ over $\mathrm{Spf}(\mathbb{Z}_p)$ associated to $\mathcal{E}/\mathbb{Z}_p$. Thus, $T_p \mathcal{E}[p^\infty]^c_{\mathbb{Q}_p}$ can be seen to be the same as $T_p \widehat{\mathcal{E}}$ (i.e. $\varprojlim \widehat{\mathcal{E}}(\mathfrak{m}_{\overline{\mathbb{Q}_p}})[p^n]$). If this still doesn't look overly familiar, it's the spiritual (and essentially literal) Tate module analogue of the well-known result that there is a short exact sequence

$$0 \to \widehat{\mathcal{E}}(p\mathbb{Z}_p) \to E(\mathbb{Q}_p) \to \mathcal{E}_{\mathbb{F}_p}(\mathbb{F}_p) \to 0 \tag{5}$$

(one obtains the claim about the kernel of the reduction map by replacing $\mathbb{Q}_p$ by $\overline{\mathbb{Q}_p}$ in (5) and taking inverse limit of $p^n$-torsion). ♦

But, we can describe the kernel of this reduction map $T_pE \to T_p\mathcal{E}_{\mathbb{F}_p}$ in another way which will be enlightening for us. Namely, let us define for any $G_{\mathbb{Q}_p}$-module $M$ the subrepresentation $C(M)$ given by

$$C(M) = \operatorname{Span}_{\mathbb{Q}_p}\left\{\sigma m - m : \sigma \in I_{\mathbb{Q}_p}\right\} \tag{6}$$

Then, let us recall that the $I_{\mathbb{Q}_p}$-coinvariants of $M$, denoted $M_{I_{\mathbb{Q}_p}}$, are given as the $G_{\mathbb{Q}_p}$-quotient $M/C(M)$. Then, as one might guess, one can check that the kernel of the map $T_pE \to T_p\mathcal{E}_{\mathbb{F}_p}$ is precisely $(T_pE)_{I_{\mathbb{Q}_p}}$ showing that the act of passing from $T_pE$ is *precisely* killing the inertia action on $T_pE$. Of course, for this to be of interest it's worth noting that $I_{\mathbb{Q}_p}$ *always* acts non-trivially on $T_pE$ (opposed to the case $T_\ell E$ with $\ell \neq p$). Indeed, one way to see this is to note that if it were not true then the map $T_pE \to T_p\mathcal{E}_{\mathbb{F}_p}$ would be an isomorphism which we've already decided it's not. Alternatively, and perhaps more illustratively, is the fact that if $T_pE$ had trivial inertia action (i.e. was unramified) then so would be $\det T_pE$ but this is nothing other than $\mathbb{Q}_p(1)$—the $p$-adic cyclotomic character $\chi_p$ which is certainly not unramified.

**Remark 3.2:** One might be slightly perturbed by the advent of the inertia *co*invariants in the above since, usually, one thinks about inertia invariants of Galois representations. This is really just an a phenomenon of functoriality. Namely, if we were considering étale cohomology instead of étale homology (which is the Tate module) then indeed we'd have a map $H^1_{\text{ét}}(\mathcal{E}_{\mathbb{F}_p}, \mathbb{Q}_p) \to H^1_{\text{ét}}(E, \mathbb{Q}_p)$ identifying $H^1_{\text{ét}}(\mathcal{E}_{\mathbb{F}_p}, \mathbb{Q}_p)$ precisely with the $I_{\mathbb{Q}_p}$-invariants of $H^1_{\text{ét}}(E, \mathbb{Q}_p)$. ♦

We can roughly summarize the differences in the above two cases as follows. If $\ell \neq p$ then $T_\ell E$ is unramified meaning that it contains no interesting information about $G_{\mathbb{Q}_p}$ (since the complicated part of $G_{\mathbb{Q}_p}$ is $I_{\mathbb{Q}_p}$) whereas $T_pE$ is *always* ramified and so *always* contains some interesting information about $G_{\mathbb{Q}_p}$. We can phrase this geometrically as follows. If $\ell \neq p$ then the information of $T_\ell E$ is, essentially, just the information of (the isogeny class) of $\mathcal{E}_{\mathbb{F}_p}$ which tells you that $T_\ell E$ can't contain that much information about $E$ (since $E \rightsquigarrow \mathcal{E}_{\mathbb{F}_p}$ is a very 'lossy' process) whereas $T_pE$ is always a finer invariant of $E$ than just the isogeny class of $\mathcal{E}_{\mathbb{F}_p}$.

**Remark 3.3:** The above might not be overly convincing that this was a defecit of $\ell$-adic representations and, perhaps, just a defecit of these *specific* $\ell$-adic representations. We'll see later that this is somewhat representative though—while not every $\ell$-adic representations is unramified it's 'almost tamely ramified'.

That said, it is also somewhat nice to think that in the family $\{T_\ell E\}$ of representations of $G_{\mathbb{Q}_p}$ (family used here in an intuitive sense) we expect, in a very intuitive non-rigorous way, to have that the relative complicatedness of $T_\ell E$ amongst all $\ell$-adic representations is somehow 'uniform' accross all $\ell$. Thus, the fact that $T_pE$ is more complicated than any of the other $T_\ell E$ should somehow indicate that $p$-adic representations are actually more complicated than $\ell$-adic ones. ♦

We can try and justify this last claim as follows. Namely, the above analysis, and Tate's isogeny theorem, tells us that

$$\operatorname{Hom}_{\mathbb{Z}_\ell[G_{\mathbb{Q}_p}]}(T_\ell E, T_\ell E') \cong \operatorname{Hom}(\mathcal{E}_{\mathbb{F}_p}, \mathcal{E}'_{\mathbb{F}_p}) \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \tag{7}$$

whereas one has that

$$\operatorname{Hom}_{\mathbb{Z}_p[G_{\mathbb{Q}_p}]}(T_pE, T_pE')\left[\frac{1}{p}\right] \cong \left\{f \in \operatorname{Hom}(\mathcal{E}_{\mathbb{F}_p}, \mathcal{E}'_{\mathbb{F}_p}) \otimes_{\mathbb{Z}} \mathbb{Q}_p : f \text{ preserves the Hodge-Messing filtration}\right\} \tag{8}$$

where the Hodge-Messing flitration on $D(\mathcal{E}_{\mathbb{F}_p}[p^\infty])$ (the Dieudonné module of $\mathcal{E}_{\mathbb{F}_p}[p^\infty]$) is obtained either by:

1. Messing theory noting that $\mathcal{E}[p^\infty]$ is a deformation of $\mathcal{E}_{\mathbb{F}_p}[p^\infty]$ and thus gives a filtration on the Dieudonné module.
2. Hodge theory by identifying $D(\mathcal{E}_{\mathbb{F}_p}[p^\infty])$ with $H^1_{\text{crys}}(\mathcal{E}_{\mathbb{F}_p}/\mathbb{Z}_p)$ which, in turn, is identified with $H^1_{\text{dR}}(\mathcal{E}/\mathbb{Z}_p)$ providing $D(\mathcal{E}_{\mathbb{F}_p}[p^\infty])$ with a Hodge filtration.

and moreover this condition on the Hodge-Messing filtration is *never* trivial (i.e. there are always maps between the special fibers that don't preserves the Hodge filtration—this is clear from the $p$-divisible group version of Tate's isogeny theorem). Thus, we see that maps between the $p$-adic Tate module really are more than just maps between the reductions—they keep track of the filtration which, by Serre-Tate and Messing theory, is equivalent to respecting the deformation $\mathcal{E}$.

# 4 Eliminating topology when $\ell \neq p$

We made the claim above that eliminating the dependence on the topology of $\ell$-adic representations (with $\ell \neq p$) is not too difficult and is really a consequence of the incommensurability of a $p$-adic and $\ell$-adic topology. We'd like to now justify this.

Now this discussion will be put on much more sure footing if we establish some terminology first. Namely, we want to make the claim that any $\ell$-adic representation of $G_{\mathbb{Q}_p}$ (with $\ell \neq p$) is in a class of representations of $G_{\mathbb{Q}_p}$ which are 'not too complicated' (I want to say 'nice' but that's already been used in this note!). Of course, the least complicated type of representations one can think of are those which are unramified. Alas, this is not to be expected in general. Namely, any ellpitic curve $E/\mathbb{Q}_p$ with bad reduction will have $T_\ell E$ ramified (by Neron-Ogg-Shafarevich)—see the appendix of this note for a more down-to-earth example. So, we need to enlarge the class of representations beyond unramified but to a class which is still managable.

The class that will turn out to be relevant for us will be related to the notion of semi-stable representations. First, let us recall that we call a representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(V)$ (for any finite-dimensional vector space $V$) *semi-stable* if $\rho\mid_{I_{\mathbb{Q}_p}}$ acts unipotently—recall that this means that for all $g \in I_{\mathbb{Q}_p}$ we have that $\rho(g)$ has characteristic polynomial $(T-1)^{\dim(V)}$. Equivalently, if $V^{\mathrm{ss}}$ denotes the semi-simplification of $V$ then $\rho$ is semi-stable if and only if $I_{\mathbb{Q}_p}$ acts trivially on $V^{\mathrm{ss}}$. The terminology comes from the theory of abelian varieties, where $A/\mathbb{Q}_p$ has what's called semi-stable reduction if and only if $T_\ell A$ is semi-stable—there the semi-stable makes sense since such reduction type is stable under base change.

Now, the class of semi-stable $\ell$-adic representations of $G_{\mathbb{Q}_p}$ is still not large enough to contain all $\ell$-adic representations. For example, if $E/\mathbb{Q}_p$ has additive bad reduction then, as mentioned at the end of the last paragraph, it cannot be the case that $T_\ell E$ is semi-stable since this would imply that $E$ has semi-stable reduction, which it doesn't (for an elliptic curve semi-stable reduction is the same thing as good or split/non-split multiplicative reduction). That said, this example also tips us off as to how we might enlarge our class further. Namely, it's a famous theorem of Grothendieck that every abelian variety has *potentially* semi-stable reduction: that after some (possibly highly ramified) base extension it obtains semi-stable reduction. Thus, one might imagine that we could replace semi-stable representations with *potentially semi-stable* representations: those which become semi-stable after finite base change.

More rigorously, let us define an $\ell$-adic representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(V)$ to be *potentially semi-stable* if there exists an open subgroup $H \subseteq I_{\mathbb{Q}_p}$ such that $\rho\mid_H$ acts unipotently on $V$. Equivalently, $\rho$ is potentially semi-stable if there exists a finite extension $K/\mathbb{Q}_p$ such that $\rho\mid_{G_K}$ is semi-stable (thus the name—potentially in the same sense of potential good reduction—it obtains the property after finite base change).

**Remark 4.1:** The above is a bit anachronistic. Namely, Grothendieck used the notion of semi-stable representations in his proof of the semi-stable reduction theorem—for our purposes though this is not important.⧫

So, let us define the category $\mathsf{Rep}_{\mathrm{pst},\mathbb{Q}_\ell}(G_{\mathbb{Q}_p})$ to be the category of potentially semi-stable $\ell$-adic representations of $G_{\mathbb{Q}_p}$. We then have the following beautiful theorem of Grothendieck:

**Theorem 4.2 (Grothendieck's $\ell$-adic Monodromy theorem):** *The two categories of representations $\mathsf{Rep}_{\mathrm{pst},\mathbb{Q}_\ell}(G_{\mathbb{Q}_p})$ and $\mathsf{Rep}_{\mathbb{Q}_\ell}(G_{\mathbb{Q}_p})$ are equal. In other words, every $\ell$-adic representations of $G_{\mathbb{Q}_p}$ is potentially semi-stable.*

I will not go through the proof here (one can see a nice exposition of it here), but let me point out that a pivotal lemma in the proof is the following:

**Lemma 4.3:** *Let $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ be a continuous representation with $\ell \neq p$. Then, $\rho(P_{p,\ell})$ is finite.*

Let me explain what $P_{p,\ell}$ is here. Namely, it's well-known that if $P_{\mathbb{Q}_p} \subseteq I_{\mathbb{Q}_p}$ denotes the wild inertia group then there is a canonical isomorphism

$$I_{\mathbb{Q}_p}/P_{\mathbb{Q}_p} \cong \widehat{\mathbb{Z}}^{(p)} \tag{9}$$

(where we denote $\prod_{\ell \neq p} \mathbb{Z}_\ell$ by $\widehat{\mathbb{Z}}^{(p)}$) which is done by showing that $\mathbb{Q}_p^{\mathrm{tr}}$ (the maximal tamely ramified extension of $\mathbb{Q}_p$) is $\mathbb{Q}_p^{\mathrm{ur}}(\{\sqrt[n]{p} : (n,p)=1\})$ which is elementary (it follows from the fact that $\mathbb{Q}_p^{\mathrm{ur}}$ has strictly Henselian integer ring). So, then we define $P_{p,\ell}$ to be the preimage in $I_{\mathbb{Q}_p}$ of $\prod_{\ell' \neq \ell, p} \mathbb{Z}_\ell$ under the surjection in (9).

Before we sketch the proof of Lemma 4.3 let us explain why it is already an incredibly powerful statement. Let us say that a representation $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}(V)$ is *tamely ramified* if $\rho(P_{\mathbb{Q}_p})$ is trivial. Such representations should be regarded as extremely simple since it will then factor through $G_{\mathbb{Q}_p}/I_{\mathbb{Q}_p}$ and, from (9), we know precisely what this group is, it's a split extension

$$1 \to \widehat{\mathbb{Z}}^{(p)} \to G_{\mathbb{Q}_p}/P_{\mathbb{Q}_p} \to \widehat{\mathbb{Z}} \to 1 \tag{10}$$

where the splitting can be (non-canonically) described as

$$G_{\mathbb{Q}_p}/P_{\mathbb{Q}_p} \cong \widehat{\mathbb{Z}}^{(p)} \rtimes \widehat{\mathbb{Z}} \tag{11}$$

where once we identify $\mathbb{Q}_p^{\mathrm{tr}} = \mathbb{Q}_p^{\mathrm{ur}}(\{\sqrt[n]{p} : (n,p) = 1\})$ the action of $\widehat{\mathbb{Z}}$ on $\widehat{\mathbb{Z}}^{(p)}$ is by $\chi^{(p)} := \prod_{\ell \neq p} \chi_\ell$. All of this is by way of saying that tamely ramified extensions should be regarded as simple since they will then factor through a very explicit, simple topological group. One can then see Lemma 4.3 as saying that <u>every</u> $\ell$-adic representation of $G_{\mathbb{Q}_p}$ is *almost tamely ramified*—in fact, it says that, up to a finite group, $\rho$ will factor through $\mathbb{Z}_\ell \rtimes \widehat{\mathbb{Z}}$ where $\widehat{\mathbb{Z}}$ acts on $\mathbb{Z}_\ell$ by $\chi_\ell$—this is what the finiteness of $\rho(P_{p,\ell})$ (opposed to just $\rho(P_{\mathbb{Q}_p})$) gives us.

So now that we (hopefully) appreciate the power of what Lemma 4.3 says, let us explain, roughly, why it's true. The reason we choose to do so is to point out that its veracity is <u>precisely</u> due to the incongruity of the $\ell$-adic and $p$-adic topologies.

So, let's just show that $\rho(P_{\mathbb{Q}_p})$ is finite (although the full statement of Lemma 4.3 follows similarly). The key point is the following. We may as well assume that $\rho$ takes values in $\mathrm{GL}_n(\mathbb{Z}_\ell)$ (since we can always find a $G_{\mathbb{Q}_p}$-stable lattice by compactness) and $\mathrm{GL}_n(\mathbb{Z}_\ell)$ has a natural decomposition:

$$1 \to 1 + \ell\,\mathrm{Mat}_n(\mathbb{Z}_\ell) \to \mathrm{GL}_n(\mathbb{Z}_\ell) \to \mathrm{GL}_n(\mathbb{F}_\ell) \to 1 \tag{12}$$

We then claim that, from (12), if $g, g' \in P_{\mathbb{Q}_p}$ have the same image in $\mathrm{GL}_n(\mathbb{F}_\ell)$ they must be equal which evidently proves the claim that $\rho(P_{\mathbb{Q}_p})$ is finite. Why is this true? Well, let $K$ be the kernel of the composition $P_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathbb{Z}_\ell) \to \mathrm{GL}_n(\mathbb{F}_\ell)$. Then, by definition, we see that $\rho$ provides a continuous homomorphism $K \to 1 + \ell\,\mathrm{Mat}_n(\mathbb{Z}_\ell)$. But, $K$ is pro-$p$ (being a closed subgroup of $P_{\mathbb{Q}_p}$) and $1 + \ell\,\mathrm{Mat}_n(\mathbb{Z}_\ell)$ is pro-$\ell$. Since $\ell \neq p$ this implies that the map $K \to 1 + \ell\,\mathrm{Mat}_n(\mathbb{Z}_\ell)$ is trivial, which proves the claim.

To emphasize, once again, note that the key point here was that $P_{\mathbb{Q}_p}$ was pro-$p$ and $1 + \ell\,\mathrm{Mat}_n(\mathbb{Z}_\ell)$ is pro-$\ell$ and that these were only incompatible *topologically*. Namely, there <u>are</u> non-continuous non-trivial group maps from pro-$p$ groups to pro-$\ell$ groups. Thus, the above was really a function of the clashing topologies of $G_{\mathbb{Q}_p}$ and $\mathrm{GL}_n(\mathbb{Q}_\ell)$ (the former has a large closed pro-$p$ subgroup and the latter is locally pro-$\ell$).

OK, great. We see that Lemma 4.3 says something deep about how $\ell$-adic represenations of $G_{\mathbb{Q}_p}$ are all 'somewhat simple' but it's non-obvious how this relates to Theorem 4.2 and how either of these relates to eliminating the topology in $\ell$-adic representations of $G_{\mathbb{Q}_p}$.

Let us speak about the latter since, again, the former is well-known and exposited quite well in other sources (see loc. cit.). The key is the following definition.

Recall that the *Weil group* $W_{\mathbb{Q}_p}$ of $\mathbb{Q}_p$ is a topological group constructed as follows. Consider the usual inertia sequence for $G_{\mathbb{Q}_p}$:

$$1 \to I_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p} \to G_{\mathbb{F}_p} \to 1 \tag{13}$$

Let us denote by $W_{\mathbb{Q}_p}$ the group obtained by the preimage of $\mathrm{Frob}_p^{\mathbb{Z}} \subseteq G_{\mathbb{F}_p}$. Thus, by definition, we have a short exact sequence

$$1 \to I_{\mathbb{Q}_p} \to W_{\mathbb{Q}_p} \xrightarrow{v} \mathrm{Frob}_p^{\mathbb{Z}} \to 1 \tag{14}$$

We then topologize $W_{\mathbb{Q}_p}$ in a non-obvious way. Namely, we do *not* give it the subspace topology from $G_{\mathbb{Q}_p}$. We topologize it so that $I_{\mathbb{Q}_p} \subseteq W_{\mathbb{Q}_p}$ is open and <u>does</u> have the subspace topology from $G_{\mathbb{Q}_p}$ and we also require that $v$ is continuous when $\mathrm{Frob}_p^{\mathbb{Z}} \cong \mathbb{Z}$ is given the discrete topology. Thus, topologically, $W_{\mathbb{Q}_p}$ is a countable disjoint union of $I_{\mathbb{Q}_p}$ with the standard topology. In particular, note that $W_{\mathbb{Q}_p}$ is not compact (it surjects onto the non-compact group $\mathbb{Z}$).

**Remark 4.4:** The reason that $W_{\mathbb{Q}_p}$ comes up is that it, opposed to $G_{\mathbb{Q}_p}$, is really the object related to the local Langlands program. For example, if one thinks about local class field theory (i.e. $\mathrm{GL}_1$ local Langlands) then one usually sees the statement that $G_{\mathbb{Q}_p}^{\mathrm{ab}} \cong \widehat{\mathbb{Q}_p^\times}$ where, here, the hat does denote profinite completion (for function fields one is completing with respect to the norm subgroups—it's a coincidence of the $p$-adic setting that such a norm completion agrees with profinite completion). But, we're really interested not in $\mathbb{Q}_p^\times$'s profinite completion (since it's not clear how to relate this directly to smooth representations of $\mathbb{Q}_p^\times$—one needs to add finiteness conditions I suppose) but $\mathbb{Q}_p^\times$ itself.

The fix is to realize that under the isomorphism $G_{\mathbb{Q}_p}^{\mathrm{ab}} \cong \widehat{\mathbb{Q}_p^\times}$ that $W_{\mathbb{Q}_p}^{\mathrm{ab}}$ maps isomorphically onto $\mathbb{Q}_p^\times$. This is to be expected. Namely, non-canonically we know that $\mathbb{Q}_p^\times \cong \mathbb{Z} \times \mathbb{Z}_p^\times$ and the latter group is already profinitely complete, thus completing merely replaces $\mathbb{Z}$ by $\widehat{\mathbb{Z}}$. So, if we want to not have to complete $\mathbb{Q}_p^\times$ (in thinking about some isomorphism of $G_{\mathbb{Q}_p}^{\mathrm{ab}}$) we really want to find the analogue of $\mathbb{Z} \subseteq \widehat{\mathbb{Z}}$ in $G_{\mathbb{Q}_p}^{\mathrm{ab}}$ and a little thought shows that it's precisely $W_{\mathbb{Q}_p}^{\mathrm{ab}}$. ♦

Now, note that $W_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p}$ (which is continuous, just not a topological embedding) has dense image (since it differs from $G_{\mathbb{Q}_p}$ essentially by $\mathbb{Z} \subseteq \widehat{\mathbb{Z}}$ which is a dense embedding) and thus the natural map $\mathsf{Rep}_{\mathbb{Q}_\ell}(G_{\mathbb{Q}_p}) \to \mathsf{Rep}_{\mathbb{Q}_\ell}(W_{\mathbb{Q}_p})$ is fully faithful (the fullness statement requires more than just denseness, but we ignore that here) where, here $\mathsf{Rep}$ does mean continuous representations. Thus, if we can find a way of eliminating the topology for continuous $\ell$-adic representations of $W_{\mathbb{Q}_p}$ we'll have also figured out how to eliminate the topology for $\ell$-adic representations of $G_{\mathbb{Q}_p}$. But, somewhat miraculously, one can actually elimiante the topology from $\ell$-adic $W_{\mathbb{Q}_p}$-representations.

Namely, let us define *Weil-Deligne* representation of $W_{\mathbb{Q}_p}$ to be a continuous representation $\rho : W_{\mathbb{Q}_p} \to \mathrm{GL}(V)$ (where $V$ is a finite-dimensional vector space over some field), where $\mathrm{GL}(V)$ is given the discrete topology, together with a *monodromy operator* $N \in \mathrm{End}(V)$ such that for all $g \in W_{\mathbb{Q}_p}$ one has the relation

$$\rho(g) N \rho(g)^{-1} = p^{-v(g)} N \tag{15}$$

where, here, $v(g)$ is the integer described by the map in (14). Note that while this still seems topological (because we've used the word 'continuous') is it not. Namely, the topology we're putting on $\mathrm{GL}(V)$ is a non-interesting one—a discrete one. If one wants to eliminate all topological language whatsoever the continuity of $\rho$ is equivalent to the fact that $(\ker \rho) \cap I_{\mathbb{Q}_p} \subseteq I_{\mathbb{Q}_p}$ is finite index. We will often denote a Weil-Deligne representation by the pair $(\rho, N)$. We will call a Weil-Deligne representation *$\ell$-adic* if $V$ is a $\mathbb{Q}_\ell$ space. Let us denote the category of $\ell$-adic Weil-Deligne representations of $\mathbb{Q}_p$ by $\mathrm{WD}_\ell(\mathbb{Q}_p)$.

Let us say that Weil-Deligne representation $(\rho, N)$ is *unramified* if $N = 0$ and $\rho(I_{\mathbb{Q}_p})$ is trivial. Let us say that $(\rho, N)$ is *irreducible* if $\rho$ is. Similarly, let us say that $(\rho, N)$ is *semi-simple* (or sometimes called *Frobenius semi-simple*) if $\rho$ is semi-simple. Finally, if $(\rho, N)$ is $\ell$-adic we call it *integral* if for all $g \in W_{\mathbb{Q}_p}$ all eigenvalues of $\rho(g)$ in $\overline{\mathbb{Q}_\ell}$ are units.

We then have the following remarkable theorem:

**Theorem 4.5:** *Let $\ell \neq p$. Then, there is a natural equivalence of categories $\mathsf{Rep}_{\mathbb{Q}_\ell}(W_{\mathbb{Q}_p}) \cong \mathrm{WD}_\ell(\mathbb{Q}_p)$. Moreover, under this equivalence the fully faithful embedding $\mathsf{Rep}_{\mathbb{Q}_\ell}(G_{\mathbb{Q}_p}) \hookrightarrow \mathrm{WD}_\ell(\mathbb{Q}_p)$ has essential image the integral $\ell$-adic Weil-Deligne representation. Moreover, if $R$ in $\mathsf{Rep}_{\mathbb{Q}_\ell}(G_{\mathbb{Q}_p})$ maps to $\mathrm{WD}(R)$ in $\mathrm{WD}_\ell(\mathbb{Q}_p)$ then $R$ is semi-simple, irreducible, or unramified if and only if $\mathrm{WD}(R)$ possessess the same property.*

Again, we will not prove this result (see loc. cit. for a nice exposition) but let us remark where the $\ell$-adic monodromy theorem comes into play. Namely, what one first shows (without too much difficulty) is that the category $\mathsf{Rep}_{\mathrm{pst},\mathbb{Q}_\ell}(G_{\mathbb{Q}_p})$ is equivalent to the category of integral $\ell$-adic Weil-Deligne representations of $\mathbb{Q}_p$. It is then only by the $\ell$-adic monodromy theorem that one obtains Theorem 4.5.

Thus we see that, by mostly simple theorems (I didn't prove Theorem 4.2 or Theorem 4.5 but they are relatively simple) we have been able to eliminate the reference to topologies in our study of $\ell$-adic representations of $G_{\mathbb{Q}_p}$ by replacing such represenations with integral $\ell$-adic Weil-Deligne representations.

# 5 Eliminating topology when $\ell = p$

We now seek to do for $\ell = p$ what we did for $\ell \neq p$ in the last section: eliminate the dependence on topology for $p$-adic representations of $G_{\mathbb{Q}_p}$. Somewhat surprisingly, this will also be done by associating a $p$-adic Weil-Deligne representation of $\mathbb{Q}_p$ to a $p$-adic representation of $G_{\mathbb{Q}_p}$ but, of course, it will have to be by an entirely different (and entirely more complicated) method.

Why is this so? Well, the crux of Theorem 4.5 was the $\ell$-adic monodromy theorem and nothing like that holds for $p$-adic representations. As a simple example consider the $p$-adic cyclotomic character $\chi_p : G_{\mathbb{Q}_p} \to \mathbb{Q}_p^\times$. Note that since $\chi_p$ is a charcter being potentially semi-stable is the same thing as being potentially unramified (i.e. killing an open subgroup of $I_{\mathbb{Q}_p}$) since it's automatically semi-simple. But, $\chi_p$ is certainly not potentially unramified: for any finite extension $K/\mathbb{Q}_p$ we will certainly have all of the infinite group $\mathrm{Gal}(K(\mu_{p^\infty})/K)$ acting non-trivially on $\mathbb{Q}_p$ via $\chi_p$.

So, how are we going to associate to a $p$-adic representation of $G_{\mathbb{Q}_p}$ a $p$-adic Weil-Deligne representation? We're not! Now, while I can't state a rigorous theorem to this effect, the only known (known to me, that is) case where we can attach a Weil-Deligne representation to a $p$-adic representation of $G_{\mathbb{Q}_p}$ is when this Galois representation is a certain type of "nice" represenation alluded to in the very first section of this note.

Since the goal of this note is <u>not</u> to be hyper-rigorous, I will not define rigorously the type of "nice" representation we'll need (this will be deferred to our next talk). But, I will at least say the name and say the operative property of these representations which make them amenable to attaching Weil-Deligne representations. Namely, the class of representations we'll be dealing with are called *de Rham* representation. The name, as one might guess, is related to the fact that proving $H^m_{\text{ét}}(X, \mathbb{Q}_p)$ is de Rham (if $X/\mathbb{Q}_p$ is smooth proper), as was done by Faltings-Tsuji, basically equates to showing a comparision theorem between étale cohomology and (algebraic) de Rham cohomology of $X/\mathbb{Q}_p$.

OK, so while it's not really helpful for me to say these words (specifically 'de Rham'), let me somewhat counter-intuitively say even more words which, again while not helpful in their own right, fit together to make the picture look somewhat like the last section. Namely, there is also a notion of a *semi-stable* $p$-adic representation of $G_{\mathbb{Q}_p}$. Be careful though—it is <u>not</u> the same as the naive guess (i.e. that after some finite extension our representation becomes semi-stable in the sense listed above). It's defined, as are de Rham representations, using a certain *period ring* that we'll not get into here. The key is that, much like semi-stable represetnations in the $\ell$-adic setting, one can attach to semi-stable representations $p$-adic representations of $G_{\mathbb{Q}_p}$ Weil-Deligne representations. Thus, essentially everything would fall into place like the last section (at least formally, in terms of the words used) if we had an analogue of the $\ell$-adic monodromy theorem.

The extremely deep result is that we do:

**Theorem 5.1 ($p$-adic monodromy theorem):** *A $p$-adic representation of $G_{\mathbb{Q}_p}$ is de Rham if and only if it's potentially semi-stable.*

This was not proven until, I believe, 2004 by combining deep results of L. Berger, K. Kedlaya, and others.

Thus, just as in Theorem 4.5 we can associate to each de Rham representaion $R : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\overline{\mathbb{Q}_p})$ a $p$-adic Weil-Deligne representation $\mathrm{WD}(R)$ but, be warned, this is not fully faithful (it's not even fully faithful for semi-stable representations—one has to keep track of extra structure—the Hodge filtration coming from being de Rham).

Regardless, we see that we can attach to a <u>de Rham</u> $p$-adic representation $R$ of $G_{\mathbb{Q}_p}$ a $p$-adic Weil-Deligne representation $\mathrm{WD}(R)$ thus, again, eliminating the dependence on topology (if by an altogether deeper, more sophisticated method).

# 6 Compatible systems (for real) and $L$-functions

## 6.1 Compatible systems

Now that we have figured out (roughly) how to eliminate topological terminology from our discussion of representations of $G_{\mathbb{Q}_p}$ we can come back and rigorously formulate the meaning of a 'family' of $\ell$-adic representations. But, before we do this we should probably exlain *why* we wish to do so besides a) it's a natural question to ask (whether representations come in families) and b) to eliminate the artificial dependence on a prime $\ell$.

Namely, the real reason that we should be interested in compatible families of $\ell$-adic representations is, perhaps not shockingly, due to our desire to understand what the Langlands conjecture says (as is the goal of the seminar). Now, one usually thinks about the Langlands conjecture as being a correspondence between Galois representations and automorphic representations. But, more correctly, Langlands posits a connection between Galois representations and *motives*.

It would be foolish to attempt to describe precisely what I mean by a motive here (for those in the know: I mean pure Chow motive with numerical equivalence—do you just call these 'numerical motives'?) but suffice it to say that this is some sort of 'linear algebraic avatar' for a variety $X$. Namely, associated to $X$ is this linear algebra gadget $M(X)$ (its 'motive'). And, part of the deal of these motives is that they have 'realizations' for every Weil cohomology theory. In particular, for every prime $\ell$ there is an $\ell$-adic realization (corresponding to the usual $\ell$-adic cohomology theory) denote $H_\ell(M(X))$. Now one can, very roughly, think that that the motive $M(X)$ is captured by its $\ell$-adic realizations (really one should consider its de Rham realization but this is part of the Archimedean side of the Langlands program that we sweep under the rug here). Thus, if one thinks that Langlands posits some sort of correspondence between automorphic cuspidal representations of $\mathrm{GL}_n$ and motives $M(X)$ then, really, associated to an automorphic representation $\rho$ should be a *family* of $\ell$-adic representations $H_\ell(M(X))$. So, in short, Langlands is more appropriately a correspondence between automorphic representations and *families* of $\ell$-adic representations.

**Remark 6.1:** This should not be shocking even if one ignores all this motivic language. Namely, on the automorphic side there is no distinguished $\ell$ and so, really, one should think that the Langlands conjecture shouldn't have to deal with automorphic representations and $\ell$-adic representations for a fixed $\ell$—it should deal with all equally—it should correspond to *families*. Of course, one may then ask why are excluding the infinite place in this setup and, as mentioned above, this Archimedean theory corresponds to the de Rham realization of the motive which we ignore here (mostly for simplicity). ♦

So, all of this motivation aside we can formally define what we mean by a family of $\ell$-adic representations. Namely, a collection of semi-simple $\ell$-adic representations $R = \{R_{\iota,\ell}\}$, as $\ell$ ranges over primes, $\iota$ ranges over embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}_\ell}$, and where $R_{\iota,\ell} : G_\mathbb{Q} \to \mathrm{GL}(V \otimes_\mathbb{Q} \overline{\mathbb{Q}_\ell})$ (where $V$ is some fixed $\overline{\mathbb{Q}}$ space and the tensor product is with respect to $\iota$), is called a *compatible family* if

1. For almost all $p$ one has that $\mathrm{WD}_p(R_{\iota,\ell} \mid_{G_{\mathbb{Q}_p}})$ is unramified for all pairs $(\iota, \ell)$ (to be clear: there is a finite set $S$ of primes such that if $p \notin S$ then for all $\ell$ we have that $\mathrm{WD}_p(R_{\iota,\ell} \mid_{G_{\mathbb{Q}_p}})$ is unramified—i.e. that $R_{\iota,\ell}$ is unramified at $p$ for all pairs $(\iota, \ell)$).

2. There exists a multi-set of integers $\mathrm{HT}(R)$ such that for all $(\iota, \ell)$ one has that $R_{\iota,\ell} \mid_{G_{\mathbb{Q}_\ell}}$ is de Rham and $\mathrm{HT}(R) = \mathrm{HT}(R_{\iota,\ell} \mid_{G_{\mathbb{Q}_\ell}})$.

3. For all primes $p$ there exists a Weil-Deligne representation of $G_{\mathbb{Q}_p}$ on a $\overline{\mathbb{Q}}$ vector space, denote it $\mathrm{WD}_p(R)$, such that for all $(\iota, \ell)$ one has that $\mathrm{WD}_p(R)^{\mathrm{ss}} \cong \mathrm{WD}_p(R_{\iota,\ell} \mid_{G_{\mathbb{Q}_p}})$.

(note that what we have written here is what is called a *strongly compatible family* in Taylor's article—the two notions are conjectured to coincide). This seems like a lot of conditions, so let me break them down. But first let me note that while the $\iota$ dependence <u>is</u> important, it's of somewhat secondary importance, so let me just ignore the $\iota$ in the following informal discussion.

Condition 1. basically says that not only is each $R_\ell$ unramified almost everywhere, but that they are unramified almost everywhere *uniformly* in $\ell$. Why should this make sense? Well, intuitively we imagine that $R_\ell$ is something like $H^i_{\acute{e}t}(X, \mathbb{Q}_\ell)$ for $X/\mathbb{Q}$ smooth proper. But, we know that $X/\mathbb{Q}$ will lift to a smooth proper model $\mathfrak{X}/U$ for some $U = \mathrm{Spec}(\mathbb{Z}) - S$ (for some finite set $S$). Note then that by abstract nonsense (i.e. the smooth proper base change theorem) one has that $H^i_{\acute{e}t}(X, \mathbb{Q}_\ell)$ will be unramified at $p$ if $p \notin S$ which is a condition *independent* of $\ell$.

Condition 2. says that not only are all the constituents of our family (uniformly) unramified almost everywhere, but that they are all de Rham at the relevant prime for all $\ell$ (i.e. $R_\ell$ is de Rham at $\ell$ or, more correctly, $R_\ell \mid_{G_{\mathbb{Q}_\ell}}$ is de Rham). Again, this is true for the family $H^i_{\acute{e}t}(X, \mathbb{Q}_\ell)$ due to work of Tsuji-Faltings. The condition on Hodge-Tate weights (this HT thing) is important—it has to do with the notion of a $p$-adic representation of $G_{\mathbb{Q}_p}$ being Hodge-Tate (which is *weaker* than being de Rham) and they (the Hodge-Tate weights) are a multi-set of integers describing precisely how this representation decomposes after drastic base change (specifically base change to $\mathbb{C}_p$)—they are, roughly, keeping track of the Hodge diamond of $X$ if, again, $R_\ell$ is $H^i_{\text{ét}}(X, \mathbb{Q}_\ell)$. Since we haven't discussed these at any length, I won't say more about them here.

Finally, 3. is really the thing that we wanted all along (the first two conditions were really just 'reasonability conditions' we'd expect any nice family to have). Namely, 3. says that there is some primordial object over $\overline{\mathbb{Q}}$ for which all the $\ell$-adic representations $R_\ell$ are just *realizations* of. Of course, thinking motivically one might imagine that these $\overline{\mathbb{Q}}$ objects are, somehow, the motive and the representations $R_\ell$ their $\ell$-adic realizations. Of course, note that we were really only to make rigorous sense of this using our notion of Weil-Deligne representations which made comparisions between various $\ell$ possible—in particular, making good sense of these representations coming from something over $\overline{\mathbb{Q}}$.

One upshot of all of this work is that we can rigorously formulate one version of the global Langlands conjecture for $\mathrm{GL}_n$—of course, defining none of the important terms on the automorphic side. Namely, let us fix a multi-set of integers $H$. Then, there is a unique bijection between cuspidal automorphic representations $\pi$ with infinitesimal character $H$ (the infinitesimal character is just specifying how $Z(U(\mathfrak{gl}_n))$ acts on our automorphic forms) and compatible systems of $\ell$-adic representations $R$ such that

1. We have that $H = \mathrm{HT}(R)$.
2. For all $p$ we have that $\mathsf{LL}(\pi_p) = \mathrm{WD}_p(R)$.

Here $\pi_p$ is the $p^{\text{th}}$ Flath component of $\pi$ (i.e. the admissible $\mathrm{GL}_n(\mathbb{Q}_p)$-representation showing up in the Flath decomposition of $\pi$) and $\mathsf{LL}$ is the local Langlands correspondence for $\mathrm{GL}_n(\mathbb{Q}_p)$ (which is actually now a theorem of Harris and Taylor!).

While it may seem like 'cheating' to use the local Langlands correspondence in the definition of the global one this is not at all surprising. Think for example, about class field theory. It's extremely common to phrase global class field theory as the claim that the kernel of the global Artin map $\mathrm{Art} : \mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times \to G_{\mathbb{Q}}^{\mathrm{ab}}$ is $(\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times)^\circ$ (thus realizing $G_{\mathbb{Q}}^{\mathrm{ab}}$ as $\pi_0(\mathbb{A}_{\mathbb{Q}}^\times/\mathbb{Q}^\times)$). But, of course, *what* is the global Artin map? Well, it's the product of the local Artin maps $\mathrm{Art}_p$! It's almost always the case that we think about global objects locally—it's nature of the subject—and this should be in particular true for the Langlands correspondence.

## 6.2 $L$-functions

Let us end this section with something much less heavy. Namely, there is a much more concrete application of our theory of Weil-Deligne representation (namely the $\mathrm{WD}(\rho)$ associated to a $p$-adic representation of $G_{\mathbb{Q}_p}$) than formulating a version of the global Langlands conjecture. Namely, it allows us to define $L$-functions correctly.

At this point it's conceivable the reader is exasperatedly perplexed: "I know how to define $L$-functions and I don't need *any* of this fancy machinery!" Well, that's kind of true. For example, let $\rho$ be an Artin representation of $G_{\mathbb{Q}}$—a continuous representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C})$ (or, what amounts to the same thing, a representation $\mathrm{Gal}(K/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{C})$ where $K/\mathbb{Q}$ is finite Galois). Then, the definition of the $L$-function of $\rho$ is classical:

$$L(\rho, s) := \prod_p L_p(\rho, s), \qquad L_p(\rho, s) := \chi_p(\rho, p^{-s})^{-1}, \qquad \chi_p(\rho, T) := \det(1 - \mathrm{Frob}_p T \mid V^{I_{\mathbb{Q}_p}}) \tag{16}$$

but this definition gives the *wrong answer* when applied to an $\ell$-adic representation opposed to an Artin one.

Indeed, let us consider the usual $\ell$-adic cyclotomic character $\chi_\ell : G_{\mathbb{Q}} \to \mathbb{Q}_\ell^\times$. The, one can compute that for all $p \neq \ell$ we have that $\chi_\ell$ is unramified at $p$ and $\chi_p(\chi_\ell, T) = 1 - pT$ so that $L_p(\chi_\ell, s) = (1 - p^{-(s-1)})^{-1}$. Thus, we see that the $p$-factor of the $L$-function $L(\chi_\ell, s)$ matches the $p$-factor of $\zeta(s-1)$. Thus, the 'correct' guess for what $L_\ell(\rho, s)$ is should be $(1 - \ell^{-(s-1)})^{-1}$. But this is *not* what we get. Namely, since $\chi_\ell$ is ramified we have the inertia invariants are trivial and so if we define $L_\ell(\chi_\ell, s)$ as in 16 we get that $L_\ell(\chi_\ell, s) = 1$ which is certainly 'incorrect'.

The issue is that the inertia action is just 'too big' for a $p$-adic representation of $G_{\mathbb{Q}_p}$ and thus taking $I_{\mathbb{Q}_p}$-invariants is too drastic of an operation. Thus, one needs a more sophisticated perspective to get the 'correct' factor at $\ell = p$.

Thankfully Weil-Deligne representations come to the rescue. Namely, for a an element $(\rho, N)$ in $\mathrm{WD}_p(G_{\mathbb{Q}_p})$ let us define $L((\rho,N),s)=\chi((\rho,N),p^{-s})^{-1}$ where $\chi((\rho,N),T)=\det(1-\mathrm{Frob}_p T \mid V^{I_{\mathbb{Q}_p},N=0})$. Then, for an $\ell$-adic representation $R : G_{\mathbb{Q}} \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})$ we can define the correct $L$-function as follows:

$$L(\rho, s) := \prod_p L(\mathrm{WD}_p(R\mid_{G_{\mathbb{Q}_p}}), s) \tag{17}$$

which, for example, does produce the "correct answer" $L(\chi_\ell, s) = \zeta(s-1)$.

# Appendix 1: Odds and ends

## Geometric motivation

Because it seems a crime to not even mention the geometric motivation for $p$-adic Hodge theory, I felt required to include this small section explaining it in embarrassingly little detail or clarity.

The very rough idea is that if one starts with a variety $X/\mathbb{Q}_p$ then there is an embarrassment of riches when it comes to cohomology theories one can use to study $X$. For example, one can certainly consider the etale cohomology $H^m_{\acute{e}t}(X,\mathbb{Q}_p)$. But, of course, one can also consider the algebraic de Rham cohomology $H^m_{\mathrm{dR}}(X/\mathbb{Q}_p)$. There's also the Hodge cohomology $H^m_{\mathrm{Hodge}}(X/\mathbb{Q}_p) := \displaystyle\bigoplus_{i+j=m} H^j(X, \Omega^i_{X/\mathbb{Q}_p})$—the list goes on.

Of course, one would hope that, just as in the classical setting of varieties over $\mathbb{C}$, we might be able to compare these various cohomology groups. For example, we know for abstract nonsesense reasons that $H^m_{\acute{e}t}(X,\mathbb{Q}_p)$, $H^m_{\mathrm{dR}}(X/\mathbb{Q}_p)$, and $H^m_{\mathrm{Hodge}}(X/\mathbb{Q}_p)$ are isomorphism as $\mathbb{Q}_p$ spaces—this follows by reduction to $\mathbb{C}$ where one can then consider analytic techniques on the analytification of these varieties. But, one can ask for something more. Namely, is there any sort of *canonicalness* between these isomorphisms. Moreover, can we make these isomorphisms preserve any sort of extra structure that the cohomology groups might have (étale has a Galois action, de Rham has a filtration, and Hodge has a grading)?

This is the goal of the geometric side of $p$-adic Hodge theory—to try and create meaningful (canonical) isomorphisms between these various cohomology groups that preserves the extra structure each have.

Of course, we don't expect to be able to do this rationally. For example, if $X/\mathbb{Q}$ is a smooth variety then we know that

$$H^*_{\mathrm{dR}}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^*_{\mathrm{dR}}(X_{\mathbb{C}}/\mathbb{C}) \cong H^*_{\mathrm{dR}}(X^{\mathrm{an}}/\mathbb{C}) \cong H^*_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C}) \tag{18}$$

but we don't expect these isomorphisms to happen rationally—we don't expect canonical isomorphisms between $H^*_{\mathrm{dR}}(X/\mathbb{Q})$ and $H^*_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{Q})$.

Why? Well, how are the isomorphsims in (18) made? Roughly, they come from a pairing on integral homology and de Rham cohomology—the integration pairing. For example, if $X = \mathbf{G}_m$ and if $S^1 \subseteq \pi_1(X^{\mathrm{an}}, 0)$ is the usual loop then for the form $\dfrac{dz}{z} \in H^1_{\mathrm{dR}}(X/\mathbb{Q})$ we can think about the integral

$$\int_{S^1} \frac{dz}{z} = 2\pi i \tag{19}$$

We see then that this will force us to have to work over $\mathbb{C}$, as in (18), to create our pairing—not over $\mathbb{Q}$.

Similarly, we'd expect any various isomorphisms between étale, de Rham, or Hodge cohomology of varieties over $\mathbb{Q}_p$ to only happen after some large base change—base change to some big field extension $K/\mathbb{Q}_p$. Since the integrals like in (19) are called *periods* it is tempting to call this field $K$ a *period ring*. The period ring will, *a priori*, (and actually!) depend on what sort of isomorphism we are trying to create.

So, what should our period rings be? For example, what should our period ring $K$ be so that we get a comparison between étale cohomology and Hodge cohomology:

$$H^m_{\acute{e}t}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K \cong H^m_{\mathrm{Hodge}}(X/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} K \tag{20}$$

What's the obvious guess? Well, what about the analogue of $\mathbb{C}$, $\mathbb{C}_p$? This, as it turns out, has issues. Namely, if we try and consider the isomorphism as in (20) with $K = \mathbb{C}_p$ then there are two obvious issues.

We want this isomorphism to preserve all structures existent on both sides. But, what structures do both sides have?

Well, $H^m_{\text{Hodge}}(X/K)$ has a grading and, thus, so does the entire right hand side of (20) with $K = \mathbb{C}_p$. The right hand side also has a Galois action since $\mathbb{C}_p$ does. The left hand side has a Galois action (where the Galois group acts *diagonally*—on both the cohomology factor and $\mathbb{C}_p$) but no grading. Thus, there are two issues at play here, one obvious, one not:

1. There is no grading on the left hand side.
2. It's not true that with the Galois actions on both sides (as described above) there is an isomorphism as in (20) with $K = \mathbb{C}_p$ that's Galois equivariant.

The second claim, as indicated above, is certainly not obvious at first glance.

**Remark 6.2:** In fact, a much stronger fact is true. If such a decomposition like (20) happens with $K = \mathbb{C}_p$ in a Galois-equivariant way then, by a theorem of Sen, one actually has that $H^m_{\text{ét}}(X, \mathbb{Q}_p)$ is potentially unramified—an extremely strong result. ♦

So, shockingly, the above allows us to conclude that $\mathbb{C}_p$ is *not big enough*! This is somewhat not too surprising considering the following clever observation of Tate. Namely, I claim that $2\pi i \notin \mathbb{C}_p$. Of course, this is meaningless—$\mathbb{C}_p \cong \mathbb{C}$ so any transcendental element of $\mathbb{C}_p$ 'looks like' (algebraically) $2\pi i$. So, what does this mean? The usual justification goes as follows. Suppose that there was an element $2\pi i \in \mathbb{C}_p$. Then, one would imagine that $e_n := \exp\left(\frac{2\pi i}{p^n}\right)$ would be $\zeta_{p^n}$. Thus, if we consider $\sigma(e_n)$ we should just get $e_n^{\chi(\sigma)}$. That said, since exp is continuous it's easy to see that one should have that $\sigma(e_n)$ is $\exp\left(\frac{\sigma(2\pi i)}{p^n}\right)$ and thus, comparing this with $e_n^{\chi_p(\sigma)}$, we see that $\sigma(2\pi i) = \chi_p(\sigma) 2\pi i$. What Tate showed is that there is no element $t \in \mathbb{C}_p$ such that $\sigma(t) = \chi_p(\sigma)t$ for all $\sigma$—there is no $2\pi i$.

In fact, one way of justifying the above 'argument' is that just as $2\pi i$ was the impediment to realizing an isomorphism between singular and de Rham cohomology for $\mathbf{G}_m$ one can check that the impediment to having a Galois-equivariant isomorphism as in (20) with $K = \mathbb{C}_p$ is precisely the non-existence of ane element $t \in \mathbb{C}_p$ satisfying $\sigma(t) = \chi_p(\sigma)t$ for all $\sigma$.

So, what's the next thing one might do? Well, let's add $2\pi i$ into $\mathbb{C}_p$! Namely, let's consider the ring $B_{\text{HT}} = \bigoplus_i \mathbb{C}_p t^i$ with the obvious multiplication/addition and with $G_{\mathbb{Q}_p}$-module structure defined so that $\sigma(t) = \chi_p(\sigma)t$ for all $\sigma$—i.e. make $t$ be $2\pi i$. This has created a larger period ring than $\mathbb{C}_p$ since, evidently, $\mathbb{C}_p \subseteq B_{\text{HT}}$. Moreover, one can note that by set-up we have that $B_{\text{HT}}$ is a graded ring.

So, now, one can ask if $B_{\text{HT}}$ is large enough. Namely, is it true that (20) holds with $K = B_{\text{HT}}$. Of course, note now that not only do both sides have Galois actions (same as above—left side has diagonal action, right side just acts on $B_{\text{HT}}$) but now both sides have gradings! Namely, the left hand side has a grading inherited just from $B_{\text{HT}}$ and the right hand side has a 'diagonal grading' (i.e. the usual grading one puts on the tensor product of graded spaces).

The astounding thing is that, in fact, it is actually true:

**Theorem 6.3 ($C_{\text{HT}}$):** *Let $X/\mathbb{Q}_p$ be smooth proper. Then, there is a Galois-equivariant isomorphism of filtered spaces*

$$H^m_{\acute{e}t}(X, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{HT}} \cong H^m_{Hodge}(X/\mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\text{HT}} \tag{21}$$

One can then ask what the period rings are other comparisons like, say, comparisions between étale cohomology and de Rham cohomology (in particular, ask if they *exist*). This study of the existence of/properties of period rings for comparisions of various cohomology theories is what comprises a large portion of the geometric side of $p$-adic Hodge theory.

## A taste of 'semi-linear algebra'

In this section I would like to give the smallest glimpse into what type of 'semi-linear algebra' one might encounter when doing $p$-adic Hodge theory. Since the stuff that will actually be important to us (i.e. the theory of period rings) will be discussed next time, let us instead give a more esoteric example (well, esoteric depending on what circles you run in).

The goal is that we want to, somehow, get rid of the dependence on $G_{\mathbb{Q}_p}$ in the definition of a representation of $G_{\mathbb{Q}_p}$. Well, this certainly sounds ridiculous, no? Well, as it turns out, it's not. Namely, we'd like to find some other sort of 'linear algebra category' $\mathscr{C}$ and some equivalence between $\mathsf{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p}) \cong \mathscr{C}$. The condition on $\mathscr{C}$ is that it shouldn't involve $G_{\mathbb{Q}_p}$ and, ultimately, should be simpler to deal with than just $\mathsf{Rep}_{\mathbb{Q}_p}(G_{\mathbb{Q}_p})$ itself. Specifically, we'll see that one can trade in the complicatedness of the group $G_{\mathbb{Q}_p}$ for an increase in the compliacatedness of the ring our linear algebra objects live over.

The avatar type theorem in this direction is the following:

**Theorem 6.4:** *Let $E$ be a perfect field of characteristic $p$. Then, the category $\mathsf{Rep}_{\mathbb{Z}_p}(G_E)$ is equivalent to the category of finite free $W(E)$-modules $M$ together with an isomorphism of $W(E)$-modules $\varphi_M : \varphi^* M \to M$ where, here, $\varphi : W(E) \to W(E)$ is the canonical lift of $\mathrm{Frob}_p : E \to E$.*

What this says is that we can trade in however complicated the group $G_E$ is for however complicated the ring $W(E)$ is. In particular, this seems like a marked improvement since we've gone from having to understanding something like a group algebra $\mathbb{Z}[G_E]$ (colloquially—$G_E$'s not finite) to understanding the ring $W(E)$ which, after all, is just a DVR. We've, somehow, traded in understanding a hard group to understanding 'easy' linear algebra (easy is certainly in square quotes because, as it turns out, even DVRs can get quite complicated!).

One can then think of $p$-adic Hodge theory as trying to replicate the success of Theorem 6.4 in other contexts. Specifically, to try and find analogues of this result in characteristic $p$ for, say, $\mathbb{Q}_p$ representations of $G_{\mathbb{Q}_p}$—linear algebra data equivalent to the group representation.

Again, it seems very difficult to imagine how to do this since, after all, Theorem 6.4 didn't just characteristic $p$ in its proof (which we didn't give…) but in its *very statement*, and moreover in a way which is not at all obvious how to generalize.

So, it seems, that if one wants to analogize Theorem 6.4 to characteristic $p$ one needs to first relate characteristic 0 representations to characteristic $p$ representations. Thus, perhaps not shockingly, almost all of $p$-adic Hodge theory (in one way or another) is foundationally dependent upon the notion of tilting due, originally, to Fontaine (and famously later developed in a geometric setting by Scholze). This is, almost by definition, a method of turning characteristic 0 things into characteristic $p$ things.

So, again, not wanting to steal the thunder of period rings for our next talk, let me give a brief indication of how one might adapt Theorem 6.4 to characteristic 0 in a crude way. The key ingredient is the following:

**Theorem 6.5 (Fontaine-Winterberger):** *There is a canonical isomorphism $G_{\mathbb{Q}_p(\mu_{p^\infty})} \cong G_{\mathbb{F}_p((T))^{\mathrm{perf}}}$.*

Then, one sees an immediate way to use this to bootstrap Theorem 6.4 to characteristic 0. Namely, it should be formally clear that $\mathsf{Rep}_{\mathbb{Z}_p}(G_{\mathbb{Q}_p})$ is the same as $\mathsf{Rep}_{\mathbb{Z}_p,\Gamma}(G_{\mathbb{Q}_p(\mu_{p^\infty})})$ where, here, $\Gamma$ is just the group $\mathbb{Z}_p^\times = \mathrm{Gal}(\mathbb{Q}_p(\mu_{p^\infty})/\mathbb{Q}_p)$, and $\mathsf{Rep}_{\mathbb{Z}_p,\Gamma}(G_{\mathbb{Q}_p(\mu_{p^\infty})})$ means representations of $G_{\mathbb{Q}_p(\mu_{p^\infty})}$ with a $\Gamma$-action—the $\Gamma$-action is, intuitively, descent datum from a representation $\rho : G_{\mathbb{Q}_p(\mu_{p^\infty})} \to \mathrm{GL}_n(\mathbb{Z}_p)$ to a representation $\rho' : G_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathbb{Z}_p)$. But, then, by Theorem 6.5 we know that $\mathsf{Rep}_{\mathbb{Z}_p,\Gamma}(G_{\mathbb{Q}_p(\mu_{p^\infty})})$ is the same as $\mathsf{Rep}_{\mathbb{Z}_p,\Gamma}(G_{\mathbb{F}_p((T))^{\mathrm{perf}}})$. But, by Theorem 6.4 this is the same thing as free $W(\mathbb{F}_p((T))^{\mathrm{perf}})$-modules $M$ together with a $\Gamma$-action and an isomorphism $\varphi_M : \varphi^* M \to M$. Thus, really, we *can* basically make Theorem 6.4 work in characteristic 0 as long as we are willing to keep track of some extra group action (but it's just a group action by $\mathbb{Z}_p^\times$!).

**Remark 6.6:** By the way, for those that know what this means (although if you know what this means it's probably obvious anyways) the above is a nascent form of étale $(\varphi, \Gamma)$-modules. Of course, to make things like slope theory work one needs to develop a slightly more nuanced presentation of these objects. ⧫

# Appendix 2: An example

As always, while it is certainly not necessary to give an example, it is certainly helpful, and I happened to write this one down while preparing this talk.

The goal of the following example is to illustrate a few things:

1. Give a concrete, computable $\ell$-adic representations of $G_{\mathbb{Q}_p}$ (with $\ell \neq p$) which is semi-stable but has huge inertia action.

2. Since semi-simple semi-stable representations are unramified, this also gives a concrete example of an $\ell$-adic (with $\ell \neq p$) representation which is not semi-simple whereas geometric representations, in the naive sense of being a cohomology group of a smooth proper thing, are semi-simple (conjecturally).

3. It gives a concrete non-trivial element of $\mathrm{Ext}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ by taking $\ell = p$. This seems somewhat silly but one of the founding results of $p$-adic Hodge theory is that $\mathrm{Ext}^1(\mathbb{C}_p, \mathbb{C}_p(1)) = 0$—thus it's nice to understand why this is a surprising result by showing how badly it fails with $\mathbb{C}_p$ replaced by $\mathbb{Q}_p$.

So, without further adieu, let us write down our slightly interesting example. Let's start by fixing an isomorphism $\varprojlim \mu_{\ell^n}(\overline{\mathbb{Q}_p}) \cong \mathbb{Z}_\ell$ (i.e. fix a primitive $\ell^{\text{th}}$ root of unity $\zeta$)—we let $\ell$ be arbitrary at this point with the possibility that $\ell = p$. We will then construct a map $\beta : G_{\mathbb{Q}_p} \to \mathbb{Z}_\ell(1)$ for which our desired representation $V$ will comes from the map

$$g \mapsto \begin{pmatrix} \chi_\ell(g) & \beta(g) \\ 0 & 1 \end{pmatrix} \tag{22}$$

where, as usual, $\chi_\ell$ is the $\ell$-adic cyclotomic character.

So, let us define elements $\alpha_n \in \mathbb{Q}_p^\times$ by setting $\alpha_n = p^{\frac{\ell^n - 1}{\ell - 1}}$ which is totally reasonable since, of course, $\dfrac{\ell^n - 1}{\ell - 1}$ is an integer. We then define our map $\beta$ by the rule that $\beta(g) \in \varprojlim \mu_{\ell^n}(\overline{\mathbb{Q}_p}) = \mathbb{Z}_\ell$ has constituents $\beta_n(g) \in \mu_{\ell^n}(\overline{\mathbb{Q}_p})$ given by

$$\beta_n(g) := \frac{g\left(\sqrt[\ell^n]{\alpha_n}\right)}{\sqrt[\ell^n]{\alpha_n}} \tag{23}$$

for any (it's independent) choice of $\sqrt[\ell^n]{\alpha_n} \in \overline{\mathbb{Q}_p}$. Note that, in fact, $\beta_n(g) \in \mu_{\ell^n}(\overline{\mathbb{Q}_p})$ since

$$\beta_n(g)^{\ell^n} = \frac{g(\alpha_n)}{\alpha_n} = 1 \tag{24}$$

since $\alpha_n \in \mathbb{Q}_p^\times$ so that $g(\alpha_n) = \alpha_n$. Finally, let us check that these sequence of $(\ell^n)^{\text{th}}$-roots are compatible. Namely, we need to check that $\beta_n(g)^\ell = \beta_{n-1}(g)$. But, this is equivalent to check that

$$g\left(\frac{\sqrt[\ell^n]{\alpha_n}}{\sqrt[\ell^{n-1}]{\alpha_{n-1}}}\right) = \frac{\sqrt[\ell^n]{\alpha_n}}{\sqrt[\ell^{n-1}]{\alpha_{n-1}}} \tag{25}$$

But, note that, by design, $\dfrac{\sqrt[\ell^n]{\alpha_n}}{\sqrt[\ell^{n-1}]{\alpha_{n-1}}} = p$ so that (25) holds automatically.

So, let us now claim that the map $\rho : G_{\mathbb{Q}_p} \to \mathrm{GL}_2(\overline{\mathbb{Q}_\ell})$ given by (22) is actually a group map. To see this we merely note that

$$\rho(gh) = \begin{pmatrix} \chi_\ell(gh) & \beta(gh) \\ 0 & 1 \end{pmatrix} \tag{26}$$

and

$$\rho(g)\rho(h) = \begin{pmatrix} \chi_\ell(g)\chi_\ell(h) & \chi(g)\beta(h) + \beta(g) \\ 0 & 1 \end{pmatrix} \tag{27}$$

and thus we need to check that (26) and (27) are equal which, of course, is the same as checking that $\beta(gh) = \chi(g)\beta(h) + \beta(g)$. But, this is written in the additive language of $\mathbb{Z}_\ell$ and it's really easier to verify it in the multiplicative language of $\varprojlim \mu_{\ell^n}(\overline{\mathbb{Q}_p})$ which really comes down to checking that $\beta(gh) = \beta(g)\beta(h)^{\chi_\ell(g)}$ which can also be checked on the $n^{\text{th}}$-level constituents. In other words we want to check that

$$\frac{g(h(\sqrt[\ell^n]{\alpha_n}))}{\sqrt[\ell^n]{\alpha_n}} = \frac{g(\sqrt[\ell^n]{\alpha_n})}{\sqrt[\ell^n]{\alpha_n}} \left(\frac{h(\sqrt[\ell^n]{\alpha_n})}{\sqrt[\ell^n]{\alpha_n}}\right)^{\chi_\ell(g)} \tag{28}$$

But, by definition of what $\chi_\ell(g)$ is, we know that

$$\left(\frac{h(\sqrt[\ell^n]{\alpha_n})}{\sqrt[\ell^n]{\alpha_n}}\right)^{\chi_\ell(g)} = \frac{g(h(\sqrt[\ell^n]{\alpha_n}))}{g(\sqrt[\ell^n]{\alpha_n})} \tag{29}$$

from where the claim follows.

Finally, let us claim that $\rho$ is continuous. This really only requires explaining why $\beta$ is continuous. But, this is clear since $\beta$ is built from an inverse limit on the finite quotients $\mu_{\ell^n}(\overline{\mathbb{Q}_p})$ with open stabilizers (namely $\mathrm{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p(\sqrt[\ell^n]{\alpha_n}, \zeta_{\ell^n}))$.

So, let us verify that our $\ell$-adic representation $\rho$ satisfies the properties 1., 2., and 3. claimed about if above.

In this paragraph we have that $\ell \neq p$. To see that $\rho$ is semi-stable is trivial. Namely, we see that for all $g \in I_{\mathbb{Q}_p}$ we have that $\rho(g) = \begin{pmatrix} 1 & \beta(g) \\ 0 & 1 \end{pmatrix}$ (since $\chi_\ell$ is unramified) and so evidently $\rho(g)$ has eigenvalues 1. To see that the inertia action is non-trivial note the following. Namely, consider that for all $n$ one has that $\mathrm{Gal}(\mathbb{Q}_p(\sqrt[\ell^n]{\alpha_n}, \zeta_{\ell^n})/\mathbb{Q}_p)$ has non-trivial image under $\beta$. But $\mathbb{Q}_p(\sqrt[\ell^n]{\alpha_n}, \zeta_{\ell^n})/\mathbb{Q}_p$ is ramified since $\mathbb{Q}_p(\sqrt[\ell^n]{\alpha_n})/\mathbb{Q}_p$ is totally ramified because it's residue field extension is trivial. Thus, considering this as $n$ increases, we see that $I_{\mathbb{Q}_p}$ has infinite image under $\rho$. Thus, these two observations imply that 1. holds.

In this pargraph let $\ell \neq p$. The proof 2. was given in its statement. Namely, since $\rho$ is semi-stable it can't be semi-simple, else $I_{\mathbb{Q}_p}$ would act trivially, but it doesn't.

Finally let us show that 3. holds by explaining why $V$, when $\ell = p$, is a non-trivial element of $\mathrm{Ext}^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$. It's evident that we have a short exact sequence

$$0 \to \mathbb{Q}_p(1) \to V \to \mathbb{Q}_p \to 0 \tag{30}$$

and we claim that this sequence doesn't split. Indeed, if it did split then the only part of $I_{\mathbb{Q}_p}$ which would act non-trivially on $V$ would be $\mathrm{Gal}(\mathbb{Q}_p(\zeta_{p^\infty})/\mathbb{Q}_p)$. But, as noted above, we have that $\mathrm{Gal}(\mathbb{Q}_p(\sqrt[p^n]{\alpha_n}, \zeta_{p^n})/\mathbb{Q}_p)$ acts non-trivially on $V$ for all $n$.

It is interesting to note that the above example, as $\ell$ varies, jives completely with our discussion of the behavior of $\ell$-adic (for $\ell \neq p$) represetnations in contrast to $p$-adic ones. Namely, $V$ is tamely ramified for $\ell \neq p$ (since the only part of $I_{\mathbb{Q}_p}$ which acts non-trivially is the $\mathrm{Gal}(\sqrt[\ell^n]{\alpha_n}, \zeta_{\ell^n})$ which are tamely ramified) but when $\ell = p$ one has that $V$ is very wildly ramified.

**Remark 6.7:** Depending on how one reads the above, it may be completely obvious or completely nebulous how I thought of creating $V$ as above. Namely, I wanted to create a non-trivial element of $\mathrm{Ext}^1(\mathbb{Q}_\ell, \mathbb{Q}_\ell(1))$ (possibly with $\ell = p$). But, by a simple computation one has that this group is just $H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_\ell(1)))$ (this is slightly less trivial than it initially looks—it doesn't follow immediately from the definition that $H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_\ell(1)) = \mathrm{Ext}^1_{\mathrm{cont.}, \mathbb{Z}[G_{\mathbb{Q}_p}]}(\mathbb{Z}, \mathbb{Q}_\ell(1)))$.

But, of course, sitting inside of $H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p(1))$ is $H^1(G_{\mathbb{Q}_p}, \mathbb{Z}_p(1))$ which is just $\varprojlim H^1(G_{\mathbb{Q}_p}, \mu_{\ell^n})$ which, by Kummer theory, is just $\varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{\ell^n}$. With the explicit isomorphsim coming from cocycles as in the formula (23). One can then see that one way of making an element of $\varprojlim \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{\ell^n}$ is to start with any element $u_0 \in \mathbb{Q}_p^\times$ and define $\alpha_n := u_0^{\frac{\ell^n - 1}{\ell - 1}}$ which gives you the compatible family of units one desires. I then chose $u_0$ to be $p$ precisely so that my extension in $\mathrm{Ext}^1(\mathbb{Q}_\ell, \mathbb{Q}_\ell(1))$ would be highly ramified. ♦