AN APPROACH TO THE CHARACTERIZATION OF THE LOCAL LANGLANDS CORRESPONDENCE

ALEXANDER BERTOLONI MELI AND ALEX YOUCIS

Abstract. In this paper, we explain how Scholze's characterization of the local Langlands correspondence for general linear groups can be made to work for a more general class of reductive groups. Our paper shows, in the case of discrete L-parameters, how to overcome the key representation-theoretic problem of understanding how Scholze's construction works in a setting with non-singleton L-packets.

1. Introduction

In [Sch13b], Scholze gave a new construction of the local Langlands correspondence for $\mathrm{GL}_n(F)$, for F a p-adic field. A key component of Scholze's analysis is that he was able to characterize his correspondence by an explicit equation relating his construction to certain functions $f_{\tau,h}$ defined in terms of the cohomology of certain Berthelot tubes inside of Rapoport–Zink spaces.

A major appeal of Scholze's characterization of the local Langlands correspondence for $\mathrm{GL}_n(F)$ is that it should be possible to generalize to the setting of a general reductive group G. In contrast, the standard characterization for the $\mathrm{GL}_n(F)$ case, as first described fully in [Hen00], is specialized to work for $\mathrm{GL}_n(F)$. Similarly, the characterizations for classical groups following from [Art13] use twisted endoscopy to reduce to the case of $\mathrm{GL}_n(F)$ where Henniart's characterization can be applied. Unfortunately, many groups cannot be related to $\mathrm{GL}_n(F)$ via endoscopy and for these cases one needs a more general approach. Moreover, even in cases where this is possible, having a characterization internal to the group G is desirable.

The two major complications of generalizing the results of [Sch13b] to arbitrary groups G are:

- **(Q1)** how to generalize the functions $f_{\tau,h}$ of [Sch13b] and prove they satisfy analogous equations,
- **(Q2)** decide whether two constructions of the local Langlands correspondence satisfying the generalized equations must coincide.

The question in **(Q1)** has been considered by several authors. Namely, the functions $f_{\tau,h}$ were generalized in [Sch13a] to PEL/EL type cases (and in [You21] to abelian type cases) and in [SS13], Scholze and Shin give a precise conjecture generalizing the main equation in [Sch13b]. We call the generalized equation in [SS13] the *Scholze–Shin equation*, which roughly is of the following form

$$S\Theta_{\psi}(f^{\mu}_{\tau,h}) = \mathrm{tr}(\tau \mid r_{-\mu} \circ \psi^{\mathrm{ss}}) S\Theta_{\psi}(h). \tag{1}$$

Here $r_{-\mu}$ is a certain representation of ${}^LG$, $S\Theta_{\psi}$ is the stable character of ψ, and ψ^{ss} is the 'semi-simplification' of ψ. In [SS13] it is proven that equations of this form hold in EL type cases and in [BMY19], the authors prove the Scholze–Shin equations hold for discrete parameters of unramified unitary groups (with the local Langlands conjecture as in [Mok15]).

That said, the question in **(Q2)** does not seem to have been studied at all in [SS13] or subsequent work. In the context of [Sch13b] this is not surprising, as **(Q2)** has a trivial solution there, but difficulties do arise for more general G due to the existence of non-singleton L-packets.

The goal of this paper is to resolve **(Q2)**, modulo some extra conditions, for a substantial class of groups in the case of *discrete L-parameters*. These are the L-parameters $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ that do not factor through a proper Levi subgroup of ${}^LG$. The discrete L-parameters are

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conjecturally those whose L-packet consists entirely of discrete series representations. This is a natural class of parameters to consider since their conjectured properties are well understood. We propose a list of axioms for an assignment of packets to discrete L-parameters and call such an assignment a *discrete local Langlands correspondence* (see §3 for a precise definition). Our axioms consist of standard desiderata except that we additionally impose that Equation (1) is satisfied.

Theorem 1.1 (Imprecise version of Theorem 3.4)**.** *Let G be a 'good' reductive group over F and suppose Π^i for $i = 1, 2$ are discrete local Langlands correspondences for G which satisfy the Scholze–Shin equations with respect to the same set of functions $\{f^{\mu}_{\tau,h}\}$. Then, $\Pi^1 = \Pi^2$.*

The phrase 'good' in the above theorem statement means, roughly, that the semi-simplified parameters ψ^{ss} of G may be recovered from the compositions $r_{-\mu} \circ \psi^{\mathrm{ss}}$ (see Definition 3.2 below). These include classes of groups such as general linear groups, odd special orthogonal groups, and unitary groups.

Remark 1.2. From the perspective of the work of Kaletha (for instance [Kal19]) on supercuspidal L-packets, it is perhaps natural to consider *supercuspidal* local Langlands correspondences, instead of discrete ones. These have as their domain the equivalence classes of supercuspidal L-parameters (i.e. discrete L-parameters with trivial restriction to SL_2) and their target finite subsets of supercuspidal representations. The results of this paper go through, verbatim, for this setup with one key difference. For technical reasons it becomes necessary to axiomatize which supercuspidal representations show up inside the L-packets of supercuspidal parameters. This question is subtle, in distinction to the discrete case, since there are supercuspidal representations which do not appear in the L-packet of any supercuspidal representation. In op. cit., Kaletha suggests that when p does not divide the order of the Weyl group of G, the answer is given by *non-singular supercuspidal* representations.

The proof of Theorem 1.1 proceeds by first reducing to the case where the L-packet of ψ is a singleton using elliptic endoscopy. For singleton packets, one deduces equality of $\Pi^1(\psi)$ and $\Pi^2(\psi)$ from the Scholze–Shin equations, the *atomic stability* property, and the fact that two discrete parameters with isomorphic semi-simplifications are isomorphic. Atomic stability is roughly the property that the stable distributions $S\Theta_\psi$ for ψ a discrete parameter are a basis for the stable distributions arising as linear combinations of discrete characters. We prove that our axiomatization of a discrete Langlands correspondence implies the atomic stability property.

Combining the above theorem with the aforementioned proof that the Scholze–Shin equations hold for the 'discrete' local Langlands conjecture as in [Mok15], we obtain the following.

Theorem 1.3 (See Theorem 4.2)**.** *The discrete local Langlands correspondence for unramified unitary groups as given by* [Mok15] *is characterized by* Theorem 1.1*.*

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2. Notation and terminology

The following notation will be used throughout the rest of the paper unless stated otherwise.

Let F be a p-adic local field. Fix an algebraic closure $\overline{F}$ and let F^{un} be the maximal unramified extension of F in $\overline{F}$. Let L be the completion of F^{un} and fix an algebraic closure $\overline{L}$.

Let G be a (connected) reductive group over F. We denote by $G(F)^{\mathrm{reg}}$ the regular semisimple elements in $G(F)$ and by $G(F)^{\mathrm{ell}}$ the subset of elliptic regular semisimple elements. We denote by

D, or D_G, the Harish-Chandra discriminant map on $G(F)$. If $\gamma, \gamma' \in G(F)$ are stably conjugate we denote this by $\gamma \sim_{\mathrm{st}} \gamma'$.

Let $\widehat{G}$ be the connected Langlands dual group of G and let ${}^LG$ be the Weil group version of the L-group of G as defined in [Kot84b, §1]. We denote by $\mathrm{Irr}(G(F))$ the set of irreducible smooth representations of $G(F)$, by $\mathrm{Irr}^{\mathrm{disc}}(G(F))$ the subset of essentially square-integrable representations, and by $\mathrm{Irr}^{\mathrm{sc}}(G(F))$ the subset of supercuspidal representations. For a finite group C the notation $\mathrm{Irr}(C)$ means all irreducible $\mathbb{C}$-valued representations of C. Let $\mathrm{Groth}(G(F))$ denote the Grothendieck group of admissible representations of $G(F)$.

A *discrete Langlands parameter* is an L-parameter (see [Bor79, §8.2]) $\psi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ such that the image of ψ is not contained in a proper Levi subgroup of ${}^LG$. A *supercuspidal Langlands parameter* is a discrete L-parameter that is also trivial on the $\mathrm{SL}_2(\mathbb{C})$ factor. We say that discrete parameters ψ and ψ' are equivalent if they are conjugate in $\widehat{G}$ and denote this by $\psi \sim \psi'$. Let C_ψ be the centralizer of $\psi(W_F \times \mathrm{SL}_2(\mathbb{C}))$ in $\widehat{G}$. Then by [Kot84b, §10.3.1] (given the remark at the start of page 648 of loc. cit.), ψ is discrete if and only if the identity component C_ψ° of C_ψ is contained in $Z(\widehat{G})^{\Gamma_F}$. We define the group $\overline{C_\psi} := C_\psi / Z(\widehat{G})^{\Gamma_F}$ which is finite by our assumptions on ψ. For the sake of comparison, in [Kal16a, Conj. F], Kaletha defines $S_\psi^\natural := C_\psi/(C_\psi \cap [\widehat{G}]_{\mathrm{der}})^\circ$. For ψ a discrete parameter, we have $S_\psi^\natural = C_\psi$ Indeed,

$$(C_\psi \cap [\widehat{G}]_{\mathrm{der}})^\circ = (C_\psi^\circ \cap [\widehat{G}]_{\mathrm{der}})^\circ \subset (Z(\widehat{G})^{\Gamma_F} \cap [\widehat{G}]_{\mathrm{der}})^\circ = \{1\},$$

from where the equality follows.

Define $Z^1(W_F, G(\overline{L}))$ to be the set of continuous cocycles of W_F valued in $G(\overline{L})$ and let $\mathbf{B}(G) := H^1(W_F, G(\overline{L}))$ be the corresponding cohomology group. Let $\kappa : \mathbf{B}(G) \to X^*(Z(\widehat{G})^{\Gamma_F})$ be the Kottwitz map as in [Kot97].

An *elliptic endoscopic datum* of G (cf. [Kot84b, 7.3-7.4]) is a triple (H, s, η) of a quasisplit reductive group H, an element $s \in Z(\widehat{H})^{\Gamma_F}$, and a homomorphism $\eta : \widehat{H} \to \widehat{G}$. We require that η gives an isomorphism

$$\eta : \widehat{H} \to Z_{\widehat{G}}(\eta(s))^\circ,$$

that the $\widehat{G}$-conjugacy class of η is stable under the action of Γ_F, and that $(Z(\widehat{H})^{\Gamma_F})^\circ \subset Z(\widehat{G})$.

An *extended elliptic endoscopic datum* of G is a triple $(H, s, {}^L\eta)$ such that ${}^L\eta : {}^LH \to {}^LG$ and $(H, s, {}^L\eta|_{\widehat{H}})$ gives an elliptic endoscopic datum of G.

An *extended elliptic hyperendoscopic datum* is a sequence of tuples of data $(H_1, s_1, {}^L\eta_1), \dots, (H_k, s_k, {}^L\eta_k)$ such that $(H_1, s_1, {}^L\eta_1)$ is an extended elliptic endoscopic datum of G, and for $i > 1$, the tuple $(H_i, s_i, {}^L\eta_i)$ is an extended elliptic endoscopic datum of H_{i-1}. An *elliptic hyperendoscopic group* of G is a quasisplit connected reductive group H_k appearing in an extended elliptic hyperendoscopic datum for G as above.

For simplicity we assume throughout the paper that each elliptic hyperendoscopic group H of G and each elliptic endoscopic datum (H', s', η') of H, one can extend (H', s', η') to an extended elliptic endoscopic datum $(H', s', {}^L\eta')$ such that ${}^L\eta' : {}^LH' \to {}^LH$.

Remark 2.1. The authors are not aware of any example for G a group over F where this property does not hold. If G and all its hyperendoscopic groups have simply connected derived subgroup, then this condition is automatic from [Lan79, Prop. 1]. In particular, unitary groups satisfy satisfy this condition. Similarly, all elliptic endoscopic data (H, s, η) for G a symplectic or special orthogonal group can also be extended to a datum $(H, s, {}^L\eta)$ ([Kal16a, pg.5]). Since the elliptic endoscopic groups of symplectic and special orthogonal groups are products of groups of this type ([Wal10, §1.8]), it follows that symplectic and special orthogonal groups also satisfy this condition. One could remove this assumption altogether at the cost of having to consider z-extensions of endoscopic groups (see [KS99a]).

3. Statement of main result

We now state the main result. Let us fix G^* to be a quasi-split reductive group over F. We define a *discrete local Langlands correspondence* for a group G^* to be an assignment

$$\Pi_H : \left\{ \begin{matrix} \text{Discrete } L\text{-parameters} \\ \text{for } H \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Subsets of} \\ \mathrm{Irr}^{\mathrm{disc}}(H(F)) \end{matrix} \right\},$$

for every elliptic hyperendoscopic group H of G^* satisfying the following properties.

(Dis) If $\Pi_H(\psi) \cap \Pi_H(\psi') \neq \emptyset$ then $\psi \sim \psi'$.

(Bij) For each Whittaker datum $\mathfrak{w}_H$ of H, a bijection

$$\iota_{\mathfrak{w}_H} : \Pi_H(\psi) \to \mathrm{Irr}(\overline{C_\psi}).$$

This bijection $\iota_{\mathfrak{w}_H}$ gives rise to a pairing

$$\langle -, - \rangle_{\mathfrak{w}_H} : \Pi_H(\psi) \times \overline{C_\psi} \to \mathbb{C},$$

defined as follows:

$$\langle \pi, s \rangle_{\mathfrak{w}_H} := \mathrm{tr}(s \mid \iota_{\mathfrak{w}_H}(\pi)).$$

(St) For all discrete L-parameters ψ of H, the distribution

$$S\Theta_\psi := \sum_{\pi \in \Pi_H(\psi)} \langle \pi, 1 \rangle_{\mathfrak{w}_H} \Theta_\pi,$$

is stable and does not depend on the choice of $\mathfrak{w}_H$.

(ECI) For all extended elliptic endoscopic data $(H', s', {}^L\eta')$ for H and all $h \in \mathscr{H}(H(F))$, suppose ψ is a discrete L-parameter of H that factors through ${}^L\eta$ by some parameter $\psi^{H'}$. Then we assume that $\psi^{H'}$ satisfies the *endoscopic character identity*: for any Whittaker datum $\mathfrak{w}_H$ the equality

$$S\Theta_{\psi^{H'}}(h^{H'}) = \Theta^s_\psi(h),$$

where we define $h^{H'}$ to be a transfer of h to H' (e.g. see [Kal16a, §1.3]) and we define

$$\Theta^s_{\psi^H} := \sum_{\pi \in \Pi_H(\psi)} \langle \pi, s \rangle_{\mathfrak{w}_H} \Theta_\pi,$$

the s-twisted character of ψ.

(Surj) The union of $\Pi_H(\psi)$ over all discrete ψ equals $\mathrm{Irr}^{\mathrm{disc}}(H(F))$.

Suppose now that $z_{\mathrm{iso}} \in Z^1(W_F, G(\overline{L}))$ projecting to an element of $\mathbf{B}(G)_{\mathrm{bas}}$. Let G be the inner form of G^* corresponding to the projection of z_{iso} to $Z^1(W_F, \mathrm{Aut}(G)(\overline{F}))$. We then define a *discrete local Langlands correspondence* for the *extended pure inner twist* (G, z_{iso}) (cf. [Kal16a, §2.5]) to be a discrete local Langlands correspondence for G^* as well as a correspondence

$$\Pi_{(G,z_{\mathrm{iso}})} : \left\{ \begin{matrix} \text{Discrete } L\text{-parameters} \\ \text{for } G \end{matrix} \right\} \to \left\{ \begin{matrix} \text{Subsets of} \\ \mathrm{Irr}^{\mathrm{disc}}(G(F)) \end{matrix} \right\},$$

satisfying

(Bij') For each Whittaker datum $\mathfrak{w}_G$ of G, a bijection

$$\iota_{\mathfrak{w}_G} : \Pi_G(\psi) \to \mathrm{Irr}(C_\psi, \chi_{z_{\mathrm{iso}}}),$$

where $\mathrm{Irr}(C_\psi, \chi_{z_{\mathrm{iso}}})$ denotes the set of equivalence classes of irreducible algebraic representations of C_ψ with central character on $Z(\widehat{G})^{\Gamma_F}$ equal to $\chi_{z_{\mathrm{iso}}} := \kappa(\overline{z_{\mathrm{iso}}})$. This gives rise to a pairing

$$\langle -, - \rangle_{\mathfrak{w}_G} : \Pi_G(\psi) \times C_\psi \to \mathbb{C},$$

defined as

$$\langle \pi, s \rangle_{\mathfrak{w}_G} := \mathrm{tr}(s \mid \iota_{\mathfrak{w}_G}(\pi)).$$

(ECI') For all discrete parameters ψ of G and all extended elliptic endoscopic data $(H, s, {}^L\eta)$ of G such that ψ factors as $\psi = {}^L\eta \circ \psi^H$, there is an equality

$$\Theta_\psi^s(h) = S\Theta_{\psi^H}(h^H),$$

where $h \in \mathscr{H}(G(F))$ and $S\Theta_\psi$ is independent of choice of Whittaker datum in **(Bij')**.

(Surj') The union of $\Pi_G(\psi)$ over all discrete ψ equals $\mathrm{Irr}^{\mathrm{disc}}(G(F))$.

The above axioms are not enough to uniquely specify a discrete local Langlands correspondence Π for G^*. The goal of our main theorem is to explain a sufficient extra condition which does uniquely specify a discrete local Langlands correspondence.

In the statement of this condition we need to assume an extra property of G. To state it, we first define a map

$$\iota : W_F \to W_F \times \mathrm{SL}_2(\mathbb{C}),$$

given by

$$w \mapsto \left(w, \begin{pmatrix} ||w||^{\frac{1}{2}} & 0 \\ 0 & ||w||^{-\frac{1}{2}} \end{pmatrix}\right).$$

We refer to $\psi \circ \iota$ as the *semi-simplification*[1] of ψ.

Definition 3.1. Then we say that G^* is *good* if for every elliptic hyperendoscopic group H of G^* we have that there exists a set S^H of dominant cocharacters of $H_{\overline{F}}$ with the following property: for any pair ψ_1^H and ψ_2^H of discrete parameters of H such that for all dominant cocharacters $\mu \in S^H$, we have an equivalence $r_{-\mu} \circ \psi_1^H \circ \iota \sim r_{-\mu} \circ \psi_2^H \circ \iota$, then $\psi_1^H \circ \iota \sim \psi_2^H \circ \iota$. Here $r_{-\mu}$ is the representation of ${}^LH$ as defined in [Kot84a, (2.1.1)]. We say that G is *good* if G^* is.

It is worth noting that many classes of groups satisfy this assumption and S^H can often be taken to be a single minuscule cocharacter (see §4).

Definition 3.2. Suppose G is a good group and fix S^H as above. A *Scholze–Shin datum* $\{f_{\tau,h}^\mu\}$ for G (and S^H) consists of the following data for each elliptic hyperendoscopic group H of G:

- A compact open subgroup $K^H \subset H(F)$,
- For each $\mu \in S^H$ with reflex field E_μ, each $\tau \in W_{E_\mu}$, and each $h \in \mathscr{H}(K^H)$, a function $f_{\tau,h}^\mu \in \mathscr{H}(H(F))$.

Let us say that a discrete local Langlands correspondence for G satisfies the *Scholze–Shin equations* relative to the Scholze–Shin datum $\{f_{\tau,h}^\mu\}$ if the following holds:

(SS) For all elliptic hyperendoscopic groups H, all $h \in \mathscr{H}(K^H)$, all $\mu \in S^H$, and all parameters ψ of H one has the equality

$$S\Theta_\psi(f_{\tau,h}^\mu) = \mathrm{tr}(\tau \mid (r_{-\mu} \circ \psi \circ \iota)(\chi_\mu))S\Theta_\psi(h),$$

where $\chi_\mu := |\cdot|^{-\langle \rho, \mu\rangle}$ and ρ is the half-sum of the positive roots of H (for a representation V and character χ we denote by $V(\chi)$ the character twist of V by χ).

Remark 3.3. As mentioned in the introduction, equations of this form were originally conjectured to hold in [SS13] (with the obvious extension to the abelian type case in the forthcoming work [You21]). These equations use Scholze–Shin data constructed from the cohomology of certain tubular neighborhoods inside of Rapoport–Zink spaces (cf. [Sch13a]). It seems conceivable to the authors that such Scholze–Shin data could also be constructed in a much greater generality using Scholze's theory of moduli spaces of mixed characteristic shtuka. For this

[1] Note that for any representation r of ${}^LG$ the representation $r \circ \psi \circ \iota$ is semi-simple. Indeed, by [BH06, Proposition 28.7] it suffices to show that

$$(r \circ \psi)\left(\mathrm{Frob}, \begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}\right) = (r \circ \psi)(\mathrm{Frob})\,(r \circ \psi)\left(\begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}\right)$$

is semi-simple. But, this is evidently a commuting product of semi-simple elements, so is semi-simple.

reason, and not knowing the precise form of these generalizations, we have chosen to work in the maximal generality for which our arguments work.

We then have the following result:

Theorem 3.4. *Let G be a good group and suppose Π^i for $i = 1, 2$ are discrete local Langlands correspondences for (G, z_{iso}) which satisfy the Scholze–Shin equations relative to the same Scholze–Shin data $\{f^{\mu}_{\tau,h}\}$. Then,*

- *we have $\Pi^1 = \Pi^2$,*
- *and for every choice of Whittaker datum $\mathfrak{w}_H$, the bijections $\iota^i_{\mathfrak{w}_H}$ for $i = 1, 2$ agree.*

Remark 3.5. In this paper we have considered only G that arise as extended pure inner twists of G^* (e.g. see [Kal16a]). In general, the map $\mathbf{B}(G^*)_{\mathrm{bas}} \to \mathrm{Inn}(G^*)$, where $\mathrm{Inn}(G^*)$ denotes the set of inner twists of G^*, need not be surjective. However, when G^* has connected center, this map will be surjective (see [Kal16a, pg.20]). In general, one can consider all inner twists by adapting the arguments of this paper to the language of rigid inner twists as in [Kal16b] (cf. [Kal16a]).

4. Examples

In this section we discuss some examples of where the conditions necessary to apply Theorem 3.4 are satisfied.

4.1. The characterization in the unitary case.

We start by discussing the case of unitary groups which was mentioned in the introduction. Namely, let F be an extension of $\mathbb{Q}_p$ and let E be a quadratic extension of F. Let us set $U^*_{E/F,n}$ to be the quasi-split unitary group associated to the extension E/F. We call a group G over F a *unitary group* $G^* = U^*_{E/F,n}$ for some n, F, and E. Note though that every inner form of $U^*_{E/F,n}$ can be upgraded to an extended pure inner twist, and we leave such choice implicit. We say that G is *unramified* if the extension $E/\mathbb{Q}_p$ is unramified.

Proposition 4.1. *Let G be a unitary group. Then, G is good.*

Proof. Without loss of generality we may assume that $G = U^*_{E/F,n}$. Note that every elliptic endoscopic group of G is of the form $U^*_{E/F,a} \times U^*_{E/F,b}$ for some $a, b \in \mathbb{N}$ such that $a + b = n$ (e.g. see [Rog90, Proposition 4.6.1]). From this we deduce that the elliptic hyperendoscopic groups of G are given by $U^*_{E/F,P}$ where $P = (n_1, \ldots, n_k)$ is a partition of n and

$$U^*_{E/F,P} := U^*_{E/F,n_1} \times \cdots \times U^*_{E/F,n_m}.$$

We note then that G is clearly good as we can take $S^{U^*_{E/F,P}}$ to be $\{\mu_{P,\mathrm{std}}\}$ where

$$\mu_{P,\mathrm{std}} = \mu_{n_1,\mathrm{std}} \times \cdots \times \mu_{n_m,\mathrm{std}}$$

where $\mu_{n_i,\mathrm{std}}$ is the cocharacter corresponding to the standard representation of $U^*_{E/F,n_i}$. □

In particular, let us define a Scholze–Shin datum $\{f^{\mu_{P,\mathrm{std}}}_{\tau,h}\}$ as in [You21]. Let us then set Π^{Mok} to be the discrete local Langlands correspondence associated to an unramified unitary group as in [Mok15]. Then, the results of [BMY19] and Theorem 3.4 show the following:

Theorem 4.2. *Let G be an unramified unitary group. Then, Π^{Mok} is characterized by the Scholze–Shin datum $\{f^{\mu_{P,\mathrm{std}}}_{\tau,h}\}$ and the set of singly Π^{Mok}-accessible representations.*

4.2. **The odd orthogonal case.** We now discuss the case of odd special orthogonal groups. Let F be a finite extension of $\mathbb{Q}_p$ and let $n \geqslant 1$ be an integer. By an *odd special orthogonal group* we mean a group G over F of the form $\mathrm{SO}(V, q)$ where (V, q) is a quadratic space over F of odd dimension. We denote by $\mathrm{SO}_{2n+1,F}$ the special odd orthogonal group of the split quadratic space of dimension $2n+1$ which is a split group. If G is any odd special orthogonal group then $G^* = \mathrm{SO}_{2n+1,F}$ for $2n+1 = \dim(V)$. We call an odd special orthogonal group *unramified* if F is unramified.

Proposition 4.3. *Let G be an odd special orthogonal group. Then, G is good.*

Proof. Since G being good depends only on G^*, and $G^* = \mathrm{SO}_{2n+1,F}$ for some n we may assume that $G = \mathrm{SO}_{2n+1,F}$. Let us denote by μ_n the unique non-trivial minuscule cocharacter of $\mathrm{SO}_{2n+1,F}$. By [GGP12, Theorem 8.1], we can recover ψ from $r \circ \psi$ where ψ any admissible homomorphism $\psi : W_F \to \mathrm{Sp}(2n)(\mathbb{C})$ and r is the standard representation. Let us also note that $\widehat{G} = \mathrm{Sp}_{2n}(\mathbb{C})$ and $r_{-\mu_n} = r$. Now, since every such elliptic hyperendoscopic group of G is a product of odd special orthogonal groups (e.g. by [Wal10, §1.8]), we are done as in the definition of good we may take $S^H = \{\mu_H\}$ where if

$$H = \mathrm{SO}_{2n_1+1,F} \times \cdots \times \mathrm{SO}_{2n_k+1,F},$$

we denote by μ_H the cocharacter $\mu_{n_1} \times \cdots \times \mu_{n_k}$. □

Remark 4.4. We end by remarking to what extent one might hope that the result Theorem 4.2 extends to the case of odd special orthogonal groups. A construction of the local Langlands correspondence for odd special orthogonal groups is complete by [Art13]. Moreover, the works of Arthur and [Tai19] prove the global multiplicity formula results. Such multiplicity results play a pivotal role in the proof in [BMY19] that Mok's Langlands correspondence for unramified unitary groups satisfies the Scholze–Shin equations for the data $\{f_{\tau,h}^{\mu_P,\mathrm{std}}\}$ from the last section.

Moreover, there are well-studied Shimura data associated to the odd special orthogonal groups over number fields (see [Zhu18]). The cocharacter associated to this Shimura datum is μ_n. In [You21] there are constructed functions $f_{\tau,h}^{\mu_H}$ which serve as candidate Scholze–Shin data. Combining this geometric input with the aforementioned results of Arthur and Taïbi it then seems conceivable to prove that Arthur's local Langlands correspondence for unramified odd special orthogonal group satisfies the Scholze–Shin equations relative to the functions in [You21]. This would then allow one to prove the analogue of Theorem 4.2 for unramified odd special orthogonal groups.

5. Atomic stability of L-packets

Before we begin the proof of Theorem 3.4 in earnest, we first discuss the following extra assumption one might make on a discrete local Langlands correspondence Π for the group G which, for this section, we assume is quasi-split.

Definition 5.1. Suppose that for any hyperelliptic endoscopic group H of G and for all finite subsets $S \subseteq \mathrm{Irr}^{\mathrm{disc}}(H(F))$ and complex numbers $\{a_\pi\}_{\pi \in S} \subseteq \mathbb{C}$ such that $\Theta := \sum a_\pi \pi$ is a stable distribution, there is a partition $S = \bigsqcup_\psi \Pi_H(\psi)$ such that $\Theta = \sum b_\psi S\Theta_\psi$ for some $\{b_\psi\} \subseteq \mathbb{C}$. We then say that Π satisfies *atomic stability.*

Somewhat surprisingly, this condition on Π is automatically implied by the axioms we have already imposed.

Proposition 5.2. *Let Π be a discrete local Langlands correspondence for a group G. Then, Π possesses atomic stability.*

For notational convenience, we may assume that $H = G$ in the proof of the above. For discrete L-parameters $\psi_1, \ldots, \psi_n$ we denote by $D(\psi_1, \ldots, \psi_n)$ the $\mathbb{C}$-span of the distributions Θ_π for $\pi \in \Pi_G(\psi_1) \cup \cdots \cup \Pi_G(\psi_n)$ and let $S(\psi_1, \ldots, \psi_n)$ be the subspace of stable distributions in $D(\psi_1,\ldots,\psi_n)$. To prove Proposition 5.2, it suffices to show that $\{S\Theta_{\psi_1},\ldots,S\Theta_{\psi_n}\}$ is a basis for $S(\psi_1,\ldots,\psi_n)$. Indeed, we can enlarge S to be a union $\Pi_{\psi_1}(G)\sqcup\cdots\sqcup\Pi_{\psi_n}(G)$ of L-packets. Proposition 5.2 is then clear since every stable distribution in the span of S is contained in $S(\psi_1,\ldots,\psi_n)$.

Before we proceed with the proof of Proposition 5.2 we establish some further notation and basic observations. For an element π of $\mathrm{Irr}^{\mathrm{disc}}(G(F))$ we denote by f_π the locally constant $\mathbb{C}$-valued function on $G(F)^{\mathrm{reg}}$ given by the Harish-Chandra regularity theorem. We then obtain a linear map

$$R : D(\mathrm{Irr}^{\mathrm{disc}}(G(F))) \to C^\infty(G(F)^{\mathrm{ell}},\mathbb{C})$$

given by linearly extending the association $\Theta_\pi \mapsto f_\pi\,|_{G(F)^{\mathrm{ell}}}$. Here we denote by $D(\mathrm{Irr}^{\mathrm{disc}}(G(F)))$ the $\mathbb{C}$-span of the distributions on $\mathcal{H}(G(F))$ of the form Θ_π for π in $\mathrm{Irr}^{\mathrm{disc}}(G(F))$.

We then have the following likely well-known lemma concerning R.

Lemma 5.3. *The linear map R is injective.*

Proof. Suppose that $R\left(\sum_i a_i\Theta_{\pi_i}\right)$ is zero. By [KHW21, Theorem C.1.1] this implies that we may write

$$\sum_i a_i\pi_i = \sum_j b_j \mathrm{Ind}_{P_i}^G(\rho_j)$$

for proper parabolic subgroups P_j of G, and ρ_i representations on $L_j(F)$ where L_j is a Levi factor of P_j. Up to replacing P_j by smaller parabolics, we may assume by the p-adic Langlands classification that each ρ_i is an essentially tempered representation (see [BW00, §IX.2]). By [Wal03, Proposition III.4.1] we may find a unique (up to conjugacy) parabolic subgroup $Q_j\subseteq L_j$ with Levi factor $M_j\subseteq Q_j$ and an essentially square integrable representation σ_j of $M_j(F)$ such that ρ_j is a direct summand of $\mathrm{Ind}_{Q_j}^{L_j}(\sigma_j)$. Let us write δ_j as the complement of ρ_j in $\mathrm{Ind}_{Q_j}^{L_j}(\sigma_j)$. So then, we see that

$$\sum_j b_j\mathrm{Ind}_{Q_j}^{L_j}(\sigma_j) = \sum_j b_j\delta_j + \sum_i a_i\pi_i.$$

From the independence theory of characters, and the fact that each $\mathrm{Ind}_{Q_j}^{L_j}(\sigma_j)$ is semi-simple (cf. loc. cit.) we deduce that each π_i must be a direct summand of some $\mathrm{Ind}_{Q_j}^{L_j}(\sigma_j)$. But, since Q_j is a proper parabolic subgroup of G, this contradicts the unicity part of loc. cit. □

We also have an averaging map

$$\mathrm{Avg} : C^\infty(G(F)^{\mathrm{ell}},\mathbb{C}) \to C^\infty(G(F)^{\mathrm{ell}},\mathbb{C})$$

given by

$$\mathrm{Avg}(f)(\gamma) := \frac{1}{n_\gamma}\sum_{\gamma'} f(\gamma')$$

where γ' runs over representatives of conjugacy classes of $G(F)$ whose stable class is equal to γ, and n_γ is the number of such classes.

The following lemma describes the relationship between this averaging map and the map R. The proof of this follows from the well-known fact that since Θ is a stable distribution, the function $R(\Theta)$ is constant on stable conjugacy classes in $G(F)^{\mathrm{ell}}$.

Lemma 5.4. *Let $\Theta\in D(\mathrm{Irr}^{\mathrm{disc}}(G(F)))$ be stable as a distribution. Then,* $\mathrm{Avg}(R(\Theta)) = R(\Theta)$.

We may now proceed to the proof of Proposition 5.2.

Proof of Proposition 5.2. In the following, we fix a Whittaker datum $\mathfrak{w}_G$.

Step 1. Note that by assumption **(Bij)**, the set of virtual characters $\Theta^s_{\psi_i}$, as s runs through representatives for the conjugacy classes in $\overline{C_\psi}$ and i runs through $\{1,\ldots,n\}$, is a basis of $D(\psi_1,\ldots,\psi_n)$. To see this, it suffices to show this in the case when $n=1$. Write ψ instead of

ψ_1. Since $\{\Theta^s_\psi\}$ is the result of applying the matrix $(a_{\pi,s})$ where $a_{\pi,s} = \langle \pi, s\rangle_{\mathfrak{w}_G}$ to the basis $\{\Theta_\pi\}$, we are done if we can show that this matrix is invertible. But, this matrix is in fact unitary by the orthogonality relations of characters for the finite group $\overline{C_\psi}$.

Step 2. We next show that for any discrete L-parameter ψ and any non-trivial s in $\overline{C_\psi}$, the function $\mathrm{Avg}(R(\Theta^s_\psi))$ is zero. Let $(H, s, {}^L\eta, \psi^H)$ be the quadruple corresponding to the pair (ψ, s) as in [BMY19, Proposition I.2.15]. To begin, a simple application of the Weyl integration formula to the equations in **(ECI)** yields the following (cf. [HS12, Lemma 6.20])

$$\mathrm{Avg}(R(\Theta^s_\psi))(\gamma) = \frac{1}{n_\gamma}\sum_{\gamma'}\sum_{\gamma_H\in X(\gamma')/\sim_{\mathrm{st}}}\Delta(\gamma_H,\gamma')\left|\frac{D_H(\gamma_H)}{D_G(\gamma')}\right| S\Theta_{\psi^H}(\gamma_H),$$

where here γ' travels over the set of conjugacy classes of $G(F)$ stably equal to the conjugacy class of γ and, as in loc. cit., $X(\gamma')$ is the set of conjugacy classes in $H(F)$ that transfer to γ, and $\Delta(\gamma_H, \gamma')$ is the $\mathfrak{w}_G$-normalized transfer factor of [LS87], and D_G (resp. D_H) denotes the Weyl discriminant function for G (resp. H). Let us note that we can rewrite this sum as

$$\frac{1}{n_\gamma}\sum_{\gamma_H\in X(\gamma)/\sim_{\mathrm{st}}}\left(\sum_{\gamma'}\Delta(\gamma_H,\gamma')\left|\frac{D_H(\gamma_H)}{D_G(\gamma')}\right|\right) S\Theta_{\psi^H}(\gamma_H),$$

because $X(\gamma')/\sim_{\mathrm{st}}$ is independent of the choice of γ'.

As $D_G(\gamma')$ is defined in terms of the characteristic polynomial for $\mathrm{Ad}(\gamma')$, we have that $D_G(\gamma') = D_G(\gamma)$ for all γ' stably conjugate to γ. Thus, we can further rewrite this as

$$\frac{1}{n_\gamma}\sum_{\gamma_H\in X(\gamma)/\sim_{\mathrm{st}}}\left|\frac{D_H(\gamma_H)}{D_G(\gamma)}\right|\left(\sum_{\gamma'}\Delta(\gamma_H,\gamma')\right) S\Theta_{\psi^H}(\gamma_H),$$

and so it suffices to show that this inner sum is zero.

Let $\mathfrak{K}(I_\gamma/F)^D$ be the Pontryagin dual of the Kottwitz group associated to γ (see [Kot86, §4.6]). As in [Kot84b, §5.6] the element s defines an element $\mathfrak{K}(I_\gamma/F)$ in a natural way, and associated to any $\gamma' \sim_{\mathrm{st}} \gamma$ is an element $\mathrm{inv}(\gamma, \gamma')$ of $\mathfrak{K}(I_\gamma/F)^D$. By [KS99b, Theorem 5.1.D], the equality

$$\Delta(\gamma_H,\gamma') = \langle \mathrm{inv}(\gamma,\gamma'), s\rangle\Delta(\gamma_H,\gamma)$$

holds, where $\langle -, -\rangle$ denotes the tautological pairing between a finite group and its Pontryagin dual. Since γ is elliptic, $\gamma' \mapsto \mathrm{inv}(\gamma, \gamma')$ gives a bijection between F-conjugacy classes in the stable conjugacy class of γ and $\mathfrak{K}(I_\gamma/F)^D$ (see [Kot86, §4]). Hence,

$$\sum_{\gamma'}\Delta(\gamma_H,\gamma') = \Delta(\gamma_H,\gamma)\sum_{\chi\in\mathfrak{K}(I_\gamma)^D}\chi(s).$$

In particular, it suffices to show that s gives a nontrivial element of $\mathfrak{K}(I_\gamma/F)$. Since (H, s, η) is a nontrivial elliptic endoscopic datum and γ is elliptic, this follows from [Shi10, Lemma 2.8] (again cf. [BMY19, Proposition I.2.15]).

Step 3. Note that to show $\{S\Theta_{\psi_1}, \ldots, S\Theta_{\psi_n}\}$ is a basis of $S(\psi_1, \ldots, \psi_n)$, we need only show it spans since $\{S\Theta_{\psi_1}, \ldots, S\Theta_{\psi_n}\}$ is independent by the independence of characters and assumption **(Dis)**. But, this is now clear since if $\Theta \in S(\psi_1, \ldots, \psi_n)$ then we know by Lemma 5.4 that $\mathrm{Avg}(R(\Theta)) = R(\Theta)$. On the other hand, since we have already observed that $\{\Theta^s_{\psi_i}\}$ is a basis of $D(\psi_1, \ldots, \psi_n)$ we may write

$$\Theta = \sum_{i=1}^n\sum_s a_{is}\Theta^s_{\psi_i}.$$

We see from the above discussion, as well as combining assumption **(St)** with Lemma 5.4, that

$$\mathrm{Avg}(R(\Theta)) = \sum_{i=1}^n R(a_{ie}S\Theta_{\psi_i}) = R\left(\sum_{i=1}^n a_{ie}S\Theta_{\psi_i}\right)$$

where we have identified $S\Theta_{\psi_i}$ with $\Theta^e_{\psi_i}$ (where e is the identity conjugacy class in $\overline{C_\psi}$). Thus, putting everything together we see that

$$R(\Theta) = R\left(\sum_{i=1}^n a_{ie} S\Theta_{\psi_i}\right),$$

and so the claim then follows from Lemma 5.3. $\square$

6. Proof of main result

We now begin the proof of our main result. Our notation throughout this section will be as in §3. For discrete local Langlands correspondences Π^i for $i = 1, 2$ for a group G, we shall use the superscript i to indicate which local Langlands correspondence the object refers to (e.g. $S\Theta^i_\psi$ denotes the stable character associated to ψ by Π^i).

We begin by establishing that it suffices to assume that G is quasi-split.

Lemma 6.1. *Suppose that the Theorem 3.4 holds for $(G^*, 1)$, then it holds for (G, z_{iso}).*

Proof. Let ψ be any discrete L-parameter for G. By assumption **(ECI')** we have that

$$S\Theta^1_\psi(h) = S\Theta^1_{\psi^{G^*}}(h^{G^*}) = S\Theta^2_{\psi^{G^*}}(h^{G^*}) = S\Theta^2_\psi(h)$$

for all $h \in \mathscr{H}(G(F))$. By the independence of characters, this implies that $\Pi^1_{(G,z_{\mathrm{iso}})}(\psi)$ is equal to $\Pi^2_{(G,z_{\mathrm{iso}})}(\psi)$. It remains to show that $\iota^1_{\mathfrak{w}_G}$ is equal to $\iota^2_{\mathfrak{w}_G}$, and hence that for all $\pi \in \Pi^1_{(G,z_{\mathrm{iso}})}(\psi) = \Pi^2_{(G,z_{\mathrm{iso}})}(\psi)$ and $s \in C_\psi$, one has that $\langle \pi, s\rangle^1_{\mathfrak{w}_G} = \langle \pi, s\rangle^2_{\mathfrak{w}_G}$. By independence of characters, it suffices to show that $\Theta^{1,t}_\psi = \Theta^{2,t}_\psi$ for all $t \in C_\psi$. But, there exists an elliptic endoscopic triple $(H, s, {}^L\eta)$ and a parameter ψ^H for H such that ${}^L\eta \circ \psi^H$ is equivalent to ψ, and such that s has the same image in $\overline{C_\psi}$ as t (cf. [BM21, Prop. 2.10]). The claim then easily follows from **(ECI')** since $\Theta^{i,t}_\psi(h)$ equals $S\Theta^i_\psi(h^H)$ and by assumption, we have that $S\Theta^1_\psi(h^H)$ is equal to $S\Theta^2_\psi(h^H)$. $\square$

Due to Lemma 6.1, it is sufficient to prove Theorem 3.4 in the quasi-split case. Thus, throughout the entire rest of this subsection we assume that G is quasi-split and that z_{iso} is trivial.

Before we begin the proof in the quasi-split case, we note that since the Scholze–Shin equations only involve the semi-simplified parameter $\psi \circ \iota$, we will certainly need the following fact which says that a discrete parameter ψ may be recovered from its semi-simplified parameter. This result is likely well-known to the experts, but an explicit proof will appear in the forthcoming work [BMIY21][2].

Proposition 6.2. *Let G be a reductive group over F and suppose that ψ_1 and ψ_2 are discrete parameters for G. If $\psi_1 \sim \psi_2$, then $\psi_1 \circ \iota \sim \psi_2 \circ \iota$.*

We now begin the proof of the quasi-split case of Theorem 3.4 in earnest. We wish to, in particular, show that $\Pi^1_H(\psi)$ is equal to $\Pi^2_H(\psi)$ for every hyperelliptic endoscopic group H, and every parameter ψ of H. Our proof will involve an inductive-type argument to reduce to the case when $\Pi^1_H(\psi)$ is a singleton. This case is then handled in the following lemma.

Lemma 6.3. *Suppose that H is an elliptic hyperendoscopic group of G and suppose that $\Pi^1_H(\psi)$ is a singleton set $\{\pi\}$. Then, in fact, $\{\pi\} = \Pi^2_H(\psi)$.*

[2]For the convenience of the reader, a sketch of the proof may be given as follows. Let σ_i for $i = 1, 2$ be the Weil–Deligne parameters associated to ψ_i (e.g. see [Ima20, Definition 1.3 and Proposition 1.13]). One may establish the surprisingly subtle fact that each σ_i must also be discrete in an appropriate sense. Since $\psi_1 \circ \iota \sim \psi_2 \circ \iota$, one sees that without loss of generality one may assume that the semi-simplified parameters $\sigma_i^{\mathrm{ss}} := \sigma_i|_{W_F}$ are equal. By [Vog93, Proposition 4.5], the moduli space of Weil–Deligne parameters with a given fixed semi-simplification has the structure of a vector space on which the conjugation action has finitely many orbits. A quick argument then shows that discrete Weil–Deligne parameters must form an open orbit. Since there is a unique such open orbit, the conclusion follows.

Proof. Since $\{\pi\}$ is a discrete packet for Π^1_H, we have by assumption **(St)** that Θ_π is stable. By Proposition 5.2 applied to Π^2_H, we have $\{\pi\} = \Pi^2_H(\psi')$ for some discrete L-parameter ψ' of H. Then, by the assumption of the theorem we have that

$$\operatorname{tr}(\tau \mid (r_{-\mu}\circ\psi\circ\iota)(\chi_\mu))\operatorname{tr}(h \mid \pi) = \operatorname{tr}(f^\mu_{\tau,h} \mid \pi) = \operatorname{tr}(\tau \mid (r_{-\mu}\circ\psi'\circ\iota)(\chi_\mu))\operatorname{tr}(h \mid \pi)$$

In particular, choosing $h \in \mathscr{H}(K^H)$ such that $\operatorname{tr}(h \mid \pi) \neq 0$ and letting τ vary we deduce that

$$\operatorname{tr}(\tau \mid (r_{-\mu}\circ\psi\circ\iota)(\chi_\mu)) = \operatorname{tr}(\tau \mid (r_{-\mu}\circ\psi'\circ\iota)(\chi_\mu)) \tag{2}$$

for all $\tau \in W_E$. Since the representations $r_{-\mu}\circ\psi$ and $r_{-\mu}\circ\psi'$ are semi-simple, by the Brauer–Nesbitt theorem we deduce that $r_{-\mu}\circ\psi\circ\iota$ is equivalent to $r_{-\mu}\circ\psi'\circ\iota$ for all $\mu \in S^H$. By our assumption on S^H, we deduce that $\psi\circ\iota$ is equivalent to $\psi'\circ\iota$. Then by Proposition 6.2, we deduce that ψ is equivalent to ψ'. In particular, $\{\pi\} = \Pi^2_\psi(H)$ as desired. □

We now proceed to the proof of Theorem 3.4 which, as mentioned before, is an induction-type argument to reduce to the singelton packet case.

Proof of Theorem 3.4. We prove this by inducting on the number of absolute roots k for elliptic hyperendoscopic groups H of G. If $k = 0$ then H is a torus. Since every distribution on H is stable, one deduces from assumption **(Dis)** and assumption **(St)** that $\Pi^1_H(\psi)$ is a singleton and thus we are done by Lemma 6.3. Suppose now that the result is true for elliptic hyperendoscopic groups of G with at most k roots. Let H be an elliptic hyperendoscopic group of G with $k+1$ roots and let ψ be a discrete parameter of H. We wish to show that $\Pi^1_H(\psi) = \Pi^2_H(\psi)$. If $\Pi^1_H(\psi)$ is a singleton, then we are done again by Lemma 6.3. Otherwise, we show that $\Pi^1_H(\psi) \subset \Pi^2_H(\psi)$, which by **(Bij)** will imply that $\Pi^1_H(\psi) = \Pi^2_H(\psi)$.

To show this containment, let π be an element of $\Pi^1_H(\psi)$ and fix a Whittaker datum $\mathfrak{w}_G$ of G. We may choose a non-trivial $\overline{s} \in \overline{C_\psi}$ and a lift $s \in C_\psi$ such that $\langle \pi, \overline{s}\rangle_{\mathfrak{w}_G} \neq 0$[3]. By definition of $\overline{C_\psi}$, we have that $s \notin Z(\widehat{G})$. Now, it suffices to show that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ for this particular s since then by independence of characters, we may deduce that $\pi \in \Pi^2_H(\psi)$ as desired. We in fact claim that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ for all non-trivial $s \in \overline{C_\psi}$. To show this, we begin by noting that by [BMY19, Proposition I.2.15], there exists a quadruple $(H', s, {}^L\eta, \psi^{H'})$ with $\psi^{H'}$ so that $\psi = {}^L\eta \circ \psi^{H'}$. The parameter $\psi^{H'}$ is discrete since if it factored through a Levi subgroup of ${}^LH$, its image would commute with a nontrivial torus of ${}^LH$ and hence also of ${}^LG$. This would contradict the discreteness of ψ. One then has from Assumption **(ECI)** that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ if and only if $S\Theta^1_{\psi^{H'}} = S\Theta^2_{\psi^{H'}}$. Moreover, since s is non-central, we know that H' has a smaller number of roots than H and thus $S\Theta^1_{\psi^{H'}} = S\Theta^2_{\psi^{H'}}$ by induction. The conclusion that $\Pi^1 = \Pi^2$ follows.

Let us now show that for any discrete L-parameter ψ one has that $\iota^1_{\mathfrak{w}_H} = \iota^2_{\mathfrak{w}_H}$ for all elliptic hyperendoscopic groups H of G and Whittaker data $\mathfrak{w}_H$ of H. By the orthogonality relations, it suffices to show that $\langle \pi, s\rangle^1_{\mathfrak{w}_H} = \langle \pi, s\rangle^2_{\mathfrak{w}_H}$ for all $\pi \in \Pi^1_\psi(H) = \Pi^2_\psi(H)$. By independence of characters, it suffices to show that $\Theta^{1,s}_\psi = \Theta^{2,s}_\psi$ for all $s \in \overline{C_\psi}$. As s is an element of $\overline{C_\psi}$, there again exists associated to the pair (ψ, s) a quadruple $(H', s, {}^L\eta, \psi^{H'})$, where H' is an elliptic endoscopic group of H and $\psi^{H'}$ is a parameter such that $\psi = {}^L\eta\circ\psi^{H'}$. By assumption **(ECI)** it suffices to show that $S\Theta^1_{\psi^{H'}} = S\Theta^2_{\psi^{H'}}$, but this follows from the previous part of the argument since we know that $\Pi^1_{H'}(\psi^{H'}) = \Pi^2_{H'}(\psi^{H'})$. □

[3] If M is a finite group and χ the character of the irreducible representation ρ of M, then $\chi(m) \neq 0$ for some non-trivial m in M. Indeed, evidently ρ is non-trivial, but also by the orthogonality relations we have that $\dim(\rho)^2 = |M|$. Since $|M| = 1 + \sum \dim(\rho')^2$, where ρ' travels over the non-trivial elements of $\operatorname{Irr}(M)$, this is a contradiction.

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University of Michigan, 530 Church St, Ann Arbor, MI 48109, United States
E-mail address: abertolo@umich.edu

University of Tokyo, Komaba Campus, 3 Chōme-8-1 Komaba, Meguro City, Tokyo 153-8914
E-mail address: ayoucis@g.ecc.u-tokyo.ac.jp