GEOMETRIC ARCS AND FUNDAMENTAL GROUPS OF RIGID SPACES

PIOTR ACHINGER, MARCIN LARA, AND ALEX YOUCIS

Abstract. We develop the notion of a *geometric covering* of a rigid space X, which yields a larger class of covering spaces than that studied previously by de Jong. Geometric coverings are closed under disjoint unions and are étale local on X. If X is connected, its geometric coverings form a tame infinite Galois category, and hence are classified by a topological group. The definition is based on the property of lifting of "geometric arcs" .

We answer two questions of de Jong about the category $\mathbf{Cov}_X^{\mathrm{adm}}$ of coverings which are locally in the admissible topology on X the disjoint union of finite étale coverings: we show that this class is different from the one used by de Jong, but still gives a tame infinite Galois category. We prove that the objects of $\mathbf{Cov}_X^{\mathrm{et}}$ (with the analogous definition) corresponds precisely to locally constant sheaves for the pro-étale topology defined by Scholze.

Résumé. Nous développons la notion de *revêtement géométrique* d'un espace rigide X, qui donne une classe de revêtements plus large que celle étudiée précédemment par de Jong. Ses revêtements géométriques sont fermés sous des unions disjointes et sont locales sur X pour la topologie étale. Les revêtements géométriques forment une catégorie Galoisienne infinie modérée, et sont donc classifiés par un groupe topologique. La définition est basée sur la propriété de relèvement des «arcs géométriques».

Nous résolvons deux questions de de Jong sur la catégorie $\mathbf{Cov}_X^{\mathrm{adm}}$ des revêtements qui sont localement dans la topologie admissible sur X l'union disjointe de revêtements étales finis: nous montrons que cette classe est différente de celle utilisée par de Jong, mais donne quand même une catégorie Galoisienne infinie modérée. Nous montrons que les objets de $\mathbf{Cov}_X^{\mathrm{et}}$ (défini de manière analogue) correspondent précisément aux les faisceaux localement constants pour la topologie pro-étale sur X définie par Scholze.

Contents

1. Introduction

1.1. **Geometric coverings, geometric arcs, and our main theorem.** While the geometry of non-archimedean analytic spaces has become relatively robust in the 60 years since its inception, a definitive theory of covering spaces has remained elusive. Ideologically, the main reason for the difficulty is that non-archimedean analytic spaces are not locally simply connected in any meaningful way. Here we mean 'simply connected' in the rigid-geometric sense, not from a topological perspective (work of Berkovich [Ber99] shows that smooth p-adic Berkovich spaces are locally contractible as topological spaces). An important step in this direction, prompted by the theory of p-adic period mappings, was taken by de Jong in [dJ95b]. Following Berkovich

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[Ber93, 6.3.4 ii)], he blended the notions of topological covering and finite étale map by considering morphisms which locally in the Berkovich topology become the disjoint union of finite étale coverings. By doing so, he was able to develop a reasonable notion of the fundamental group of a rigid space.

Unfortunately, de Jong's theory of covering spaces lacks some of the properties one would expect such as being closed under compositions and closed under disjoint unions. Even more serious, the property of being a covering space in his sense is not local in the admissible topology on the target (see Theorem 5 below).

Similar issues arise in topology when one considers covering spaces of topological spaces which aren't locally simply connected. An alternative perspective on covering spaces, present in the recent work of Brazas [Bra12], is to overcome this difficulty by converting one of the key properties of covering spaces into a definition — the ability to lift paths: for any commutative square of solid arrows as below there exists a unique dotted arrow making the diagram commute.

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Y \arrow[d] \\ {[0,1]} \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

An analogue of this approach is also hidden in the work of Bhatt and Scholze on the pro-étale fundamental group in [BS15], where the corresponding notion of a covering space is called a *geometric covering*. Namely, if one defines a 'geometric path' between two geometric points of a scheme X as a sequence of specializations and generalizations of points in X where subsequent points are 'connected' by a strictly Henselian local ring, then geometric coverings are precisely étale maps for which one has unique lifting of all geometric paths.

The main goal of this article is the development of a good theory of covering spaces in terms of path lifting, thus continuing the above pattern. It enlarges de Jong's category[1] and is based on the notion of a geometric arc, to be explained (along with the shift from paths to arcs) in more detail below.

Definition 1 (See Definition 5.2.2)**.** Let X be an adic space locally of finite type over a non-archimedean field K. An étale morphism of adic spaces $Y \to X$ is a *geometric covering* if it is partially proper and if the following condition holds:

> For every test curve $C \to X$, the map $Y_C \to C$ of adic curves satisfies unique lifting of geometric arcs.

We denote by $\mathbf{Cov}_X$ the category of geometric coverings of X.

In addition to our definition being geometrically intuitive, we show in §5 that the notion of geometric covering is closed under composition, closed under disjoint unions, and is étale local on the target. Moreover, every finite étale covering is a geometric covering, and thus combining these results we see that every covering space in the sense of de Jong is a geometric covering.[2]

To explain 'geometric arcs' it is useful to note that, unlike schemes, for a geometric point $\overline{x}$ of a rigid space X, the étale localization $X_{(\overline{x})}$ (i.e. the inverse limit of pointed étale neighborhoods) is often nothing more than a point (see [Hub96, Proposition 2.5.13 i)]) and thus is not large enough to emulate the notion of 'path' in algebraic geometry. But, also unlike schemes, rigid spaces X (or more precisely their associated Berkovich space $[X]$) have an abundance of topological arcs, e.g. $[X]$ is arc connected if X is connected. Our notion of *geometric arcs* is a synthesis of the algebro-geometric and topological notions of paths: a geometric arc $\overline{\gamma}$ in a rigid space X is an arc γ in $[X]$ together with a choice of a geometric point above each of its points and a compatible family of étale paths along γ (see Definition 4.2.1).

[1] This is in contrast with the work of André [And03] and Lepage [Lep10b] on the tempered fundamental group, which aims at making the de Jong fundamental group smaller and therefore more manageable.

[2] It is interesting to note that our definition of geometric coverings matches quite nicely to that of a semi-covering as defined in [Bra12]. In particular, the condition is the replacement for the unique lifting of paths, and étale and partially proper constitute the replacement for local homeomorphism (cf. Proposition 3.7.5).

If X is one-dimensional, every two points of X are connected by a geometric arc. This statement is significantly more subtle than the analogue from algebraic geometry (see Section 4). In general, extending a theorem of de Jong [dJ95a, Proposition 6.1.1], we show in Appendix A that every two points of a connected X can be connected by a sequence of *test curves* (i.e. one-dimensional rigid spaces over some extension of K). Combining these two statements we see that X is "geometric path connected". We can then more precisely state the unique lifting property for a map $Y \to X$ as in Definition 1 by saying that for every geometric arc $\overline{\gamma}$ in a test curve C and every lifting $\overline{y}$ of the geometric left endpoint $\overline{x}$ of $\overline{\gamma}$ to Y_C, there exists a unique lifting of $\overline{\gamma}$ to a geometric arc in Y_C with geometric left endpoint $\overline{y}$.

Using this "geometric path connectedness", the path lifting property of our geometric coverings gives a concrete way of constructing an isomorphism of fiber functors $F_{\overline{x}_0} \simeq F_{\overline{x}_1}$ for any two geometric points of X in the same connected component. This quickly implies that the category $\mathbf{Cov}_X$, with the fiber functor $F_{\overline{x}}$ associated to a geometric point $\overline{x}$, is a tame infinite Galois category in the sense of [BS15, §7]. In words this means that the category $\mathbf{Cov}_X$ of geometric coverings has enough structure to support a notion of Galois theory. This is the main theorem of our paper.

Theorem 2 (See Theorem 5.4.1)**.** *For a connected adic space X locally of finite type over a non-archimedean field K and a geometric point $\overline{x}$ of X, the pair $(\mathbf{Cov}_X, F_{\overline{x}})$ forms a tame infinite Galois category.*

From the general yoga of tame infinite Galois categories one thus obtains from the pair $(\mathbf{Cov}_X, F_{\overline{x}})$ a topological group $\pi_1^{\mathrm{ga}}(X, \overline{x})$, which we call the *geometric arc fundamental group*[3]. Moreover, $\pi_1^{\mathrm{ga}}(X, \overline{x})$ is a Noohi group (in the sense of loc. cit.), and so $F_{\overline{x}}$ induces an equivalence of categories

$$F_{\overline{x}} \colon \mathbf{Cov}_X \xrightarrow{\sim} \pi_1^{\mathrm{ga}}(X, \overline{x})\text{-}\,\mathbf{Set}.$$

An analogue of this result was proven, in different language, by de Jong for the category of disjoint unions of his coverings (see [BS15, Remark 7.4.11]). His proof, however, does not generalize to our situation (see §1.5 below). Moreover, our main theorem directly implies tameness of the natural generalizations of de Jong's category (see §1.3 below).

1.2. **AVC and an alternative characterization of geometric coverings.** The proofs of several key facts about our geometric coverings, most notably étale descent, rely on a different, more hands-on, characterization in terms of a 'topological valuative criterion', called the *arcwise valuative criterion (AVC)*.

A similar duality is present in the theory of the pro-étale fundamental group. There, geometric coverings, despite the aforementioned characterization in terms of 'path lifting,' are defined as étale morphisms which satisfy the valuative criterion of properness (i.e. are partially proper). In rigid geometry, étale and partially proper maps do not form a suitably well-behaved notion of a 'covering space.' For instance, the inclusion of the open unit disk into the closed unit disk is a partially proper étale map, but should not be considered a covering space. At the heart of the issue is that Bhatt–Scholze work with locally topologically Noetherian schemes which disallow open partially proper embeddings which aren't the inclusion of a connected component. While this assumption is not unreasonable for schemes, essentially all rigid spaces are not locally topologically Noetherian (e.g. the disk).

To explain the condition needed in order to eliminate the gap between geometric coverings and partially proper étale maps, it is useful to recall the topological structure of a rigid space X. Namely, there is a natural map (the *separation map*)

$$\mathrm{sep}_X \colon X \to [X]$$

[3]Given the analogies with [BS15], it might seem reasonable to call this group the pro-étale fundamental group. However, we do not know if there is a suitable 'pro-étale topology' whose local systems are $\mathbf{Cov}_X$ — see the discussion below.

to the corresponding Berkovich analytic space which can be characterized as the quotient map to the universal Hausdorff quotient (this observation and the notation are due to Fujiwara [Fuj95, §4.6], see also [FK18]). The fibers of the separation map are, essentially, Riemann–Zariski type spaces, and one may see the valuative criterion for properness as a statement about the fibers of the separation map sep_X. Thus, to effect a situation similar to that in [BS15] we need, in addition to the usual valuative criterion, a 'valuative criterion relative to $[X]$'. To this end, we have the following simple looking definition.

Definition 3. A map of topological spaces $Y \to X$ satisfies the *arcwise valuative criterion (AVC)* if for every commutative square of solid arrows as below with the bottom map a topological embedding there exists a unique dotted arrow making the diagram commute.

$$\begin{tikzcd} {[0,1)} \arrow[r] \arrow[d] & Y \arrow[d] \\ {[0,1]} \arrow[r] \arrow[ur, dotted] & X \end{tikzcd}$$

It is easy to observe that that a partially proper open immersion $U \to X$ with X connected and such that $[U] \to [X]$ satisfies AVC is an isomorphism. Thus, AVC at least disallows the previously mentioned examples showing that partially proper and étale morphisms do not constitute a good theory of covering spaces. In fact, as previously hinted, we may characterize geometric coverings as those étale maps which not only are partially proper but satisfy AVC over every test curve[4].

Proposition 4 (Proposition 5.1.9)**.** *Let $Y \to X$ be an étale and partially proper morphism of rigid spaces. Then, $Y \to X$ is a geometric covering if and only if for all test curves $C \to X$ the map $[Y_C] \to [C]$ satisfies AVC.*

Again, at an intuitive level the valuative criterion included in the definition of a partially proper map works along the fibers of the separation map $\mathrm{sep}_X : X \to [X]$, whereas AVC works on the target $[X]$, and so the two are complementary, and jointly give rise to a situation much more representative of the valuative criterion from algebraic geometry. See Figure 1 for a graphical representation of this idea.

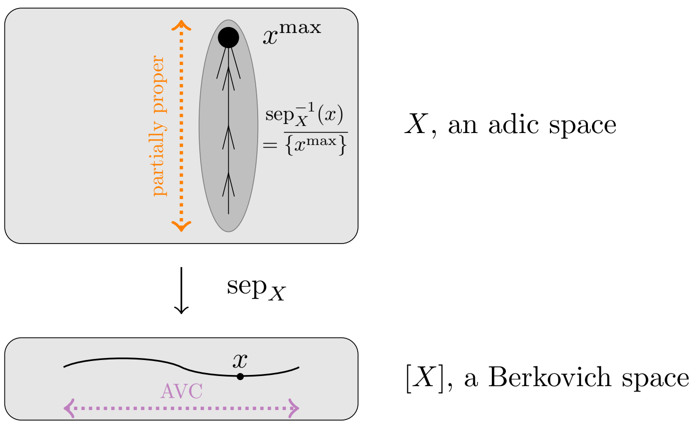

FIGURE 1. The separation map from the adic space to the Berkovich space [X] and an arc in [X]. Here, AVC allows us to connect different fibers, while the valuative criterion (partial properness) allows us to relate points in the fibers.

[4]One should view the prescence of test curves here as a positive since one would probably like to say that AVC happens universally (i.e. after base change to any rigid space), and we are demanding that it only happens on base change to test curves.

1.3. **Applications to previously studied objects.** Finally, we show that the theory of geometric coverings is useful in contexts previously studied by other authors.

We start with certain questions raised by de Jong in [dJ95b]. To this end, we consider the full subcategories

$$\mathbf{Cov}_X^{\tau} \subseteq \mathbf{Cov}_X, \quad \tau \in \{\mathrm{adm}, \text{ét}, \mathrm{oc}\}$$

consisting of étale maps $Y \to X$ for which there exists a τ-cover $U \to X$ such that $Y_U \to U$ is the disjoint union of finite étale coverings of U. Here adm, ét, or oc denotes the usual (i.e. 'admissible'), étale, or overconvergent (i.e. partially proper open) Grothendieck topology on X, respectively. In particular, $\mathbf{Cov}_X^{\mathrm{oc}}$ is the category of covering spaces of X (or rather the corresponding Berkovich analytic space) considered by de Jong, which we call here *de Jong covering spaces*. We denote by $\mathbf{UCov}_X^{\tau}$ the category of disjoint unions of objects of $\mathbf{Cov}_X^{\tau}$. As mentioned before, in [dJ95b], de Jong essentially showed that for X connected, $(\mathbf{UCov}_X^{\mathrm{oc}}, F_{\overline{x}})$ is a tame infinite Galois category, and posed the following two questions (using different language):

- Does the equality $\mathbf{Cov}_X^{\mathrm{oc}} = \mathbf{Cov}_X^{\mathrm{adm}}$ hold?
- If not, is $\mathbf{UCov}_X^{\mathrm{adm}}$ a tame infinite Galois category (for X connected)?

The first question is answered negatively using an explicit construction relying on a careful analysis of Artin–Schreier coverings.

Theorem 5 (See Proposition 6.3.4)**.** *Let K be a non-archimedean field of characteristic p and let X be an affinoid annulus over K. Then, the containment $\mathbf{Cov}_X^{\mathrm{oc}} \subseteq \mathbf{Cov}_X^{\mathrm{adm}}$ is strict.*

On the other hand, our result that $\mathbf{Cov}_X$ is a tame infinite Galois category allows us to obtain a positive answer to the second question as an immediate corollary.

Theorem 6 (See Theorem 6.2.1)**.** *Let X be a connected adic space locally of finite type over K and let $\overline{x}$ be a geometric point of X. For every $\tau \in \{\mathrm{adm}, \text{ét}, \mathrm{oc}\}$, the pair $(\mathbf{UCov}_X^{\tau}, F_{\overline{x}})$ is a tame infinite Galois category.*

Consequently, there is a Noohi group $\pi_1^{\mathrm{dJ},\tau}(X, \overline{x})$ and an equivalence of categories

$$F_{\overline{x}} \colon \mathbf{UCov}_X^{\tau} \xrightarrow{\sim} \pi_1^{\mathrm{dJ},\tau}(X, \overline{x})\text{-}\,\mathbf{Set}.$$

As a last application, we wish to show that the larger category $\mathbf{Cov}_X^{\text{ét}}$ is not just of purely theoretical interest and connects to previously studied objects. To this end, we recall that in [Sch13] (see also [Sch16]), Scholze introduced a topology on rigid analytic spaces, called there the *pro-étale topology* (though the term might now be outdated in view of subsequent developments). Its covers are roughly an étale cover of X followed by an inverse limit of finite étale covers. In Section 7, we show the following result.

Theorem 7 (See Theorem 7.5.1)**.** *The functor associating to a geometric covering $Y \to X$ the corresponding sheaf on the pro-étale site $X_{\text{proét}}$ induces an equivalence of categories*

$$\mathbf{Cov}_X^{\text{ét}} \xrightarrow{\sim} \mathbf{Loc}(X_{\text{proét}}),$$

where $\mathbf{Loc}(X_{\text{proét}})$ is the category of locally constant sheaves on $X_{\text{proét}}$. If X is connected and $\overline{x}$ is a geometric point of X, then $\mathbf{UCov}_X^{\text{ét}}$ and $\mathbf{ULoc}(X_{\text{proét}})$ are further equivalent to the category $\pi_1^{\mathrm{dJ},\text{ét}}(X, \overline{x})\text{-}\,\mathbf{Set}$.[5]

1.4. **Future directions.** We end with some directions of future research. We do not know if there exists a variant of the pro-étale topology on a rigid space whose locally constant sheaves of sets are naturally equivalent to $\mathbf{Cov}_X$. Such a result seems to be a natural next step to completing the analogy of our results with [BS15], but would likely be important in its own right. We also hope that geometric arcs and paths could also be used as exit paths in order to describe a certain class of constructible sheaves on rigid spaces.

[5] Here, $\mathbf{ULoc}(X_{\text{proét}})$ denotes the category of disjoint unions of locally constant sheaves of discrete sets on $X_{\text{proét}}$. The result implies, in particular, that $\mathbf{ULoc}(X_{\text{proét}})$ is a tame infinite Galois category. This property might be lost when trying to work with a version of 'the pro-étale topology' for rigid spaces that is too fine, as suggested by [BS15, Remark 7.3.12].

1.5. **A remark about the history of the paper.** Originally we sought only to answer the question posed by de Jong on the existence of a Galois theory of covering spaces which are almost split (i.e. the disjoint union of finite unions of finite étale coverings) on an admissible open cover. We succeeded in this goal by modifying the delicate proof of [dJ95b, Theorem 2.9] to this situation, and then decided to take the construction one step further and show that a category of maps which are almost split on an arbitrary étale cover has a suitable Galois theory. After many attempts at adapting de Jong's proof, we found the implicit 'theory of geometric arcs' lurking in the background.

1.6. **A remark on the usage of adic spaces.** For the reader mostly familiar with Berkovich spaces, let us note that even though in this paper the primary object is the adic space X, it is likely one could equally well work with Berkovich analytic spaces throughout. Doing so would have the advantage of making many of the definitions more literal (e.g. arcs would then be in X, not some space associated to X). On the other hand, we feel that theory of adic spaces does provide a useful framework to think about the notion of partially proper (boundaryless in the vernacular of Berkovich theory) in terms of a valuative criterion. As implied above, one of the main thematic takeaways of this paper should be that both perspectives are useful and that only by combining the two different valuative criteria a) partial properness in terms of the adic space X (Proposition 3.3.2), and b) AVC on the Berkovich space X^{Berk} (see Definition 4.1.4) does the picture become truly clear. But more practically, from the point of view of the Berkovich space X^{Berk}, the adic space X could be seen as a gadget on which the structure sheaf of X should live (e.g. affinoid domains are open subsets). Let us also lastly point out that our results are novel even when phrased entirely in the language of Berkovich spaces.

Notation and conventions:

- adic spaces are analytic and locally strongly noetherian (see §3.1),
- K denotes a non-archimedean field (complete with respect to a rank one valuation),
- a *rigid K-space* is an adic space locally of finite type over K (Definition 3.1.2),
- for a Huber pair of the form (A, A°) we abbreviate $\mathrm{Spa}(A, A^\circ)$ to $\mathrm{Spa}(A)$,
- Vakil's 'cancellation principle' [Vak17, 10.1.19] will be used freely, and says the following: given a category $\mathcal{C}$ with pullbacks and a property P of morphisms in $\mathcal{C}$ closed under pullbacks and composition, for every two morphisms $f\colon Y \to X$, $g\colon Z \to Y$ such that $f \circ g\colon Z \to X$ and the diagonal $\Delta_{Y/X}\colon Y \to Y \times_X Y$ have P, also g has P,
- a topological space is compact if it is quasi-compact and Hausdorff, and locally compact if every point has an open neighborhood basis of relatively compact open sets,
- a map of topological spaces $Y \to X$ is called *separated* if the diagonal map $\Delta_{Y/X}$ is a closed embedding, universally closed if it satisfies any of the equivalent properties in [Stacks, Tag 005R], and *proper* if it is universally closed and separated.

Glossary of notation:

- $\mathrm{sep}_X\colon X \to [X]$: the separation map to the universal separated quotient, §2.2
- $x^{\max}$: the unique maximal point specializing to x, §2.2
- $\mathrm{int}_X(U)$: overconvergent interior, §2.4
- open oc neighborhood, §2.4
- $\mathbf{\acute{E}t}_X, \mathbf{F\acute{E}t}_X, \mathbf{UF\acute{E}t}_X$: the category of étale/finite étale/disjoint unions of finite étale spaces over X
- $F_{\overline{x}}$: the fiber functor, §3.2
- $\pi_1^{\mathrm{alg}}(X; a, b)$: the space of algebraic étale paths from $\overline{a}$ to $\overline{b}$, §3.2

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2. Preliminaries on valuative spaces and separated quotients

In this section we collect various definitions and results of a purely topological nature that will be useful later on. We encourage the reader to skip this section upon first reading, returning only as necessary. Most of the material is based on [FK18, Chapter 0].

2.1. Locally spectral spaces.

In this subsection we review some of the topological properties of locally spectral spaces, a class of spaces encompassing the underlying spaces of both schemes and adic spaces.

Let us recall that a map of topological spaces $f\colon Y \to X$ is *quasi-compact* if $f^{-1}(U)$ is quasi-compact for every quasi-compact open subset U of X. We call an open subset U of a topological space X *retrocompact* if the inclusion map $U \hookrightarrow X$ is quasi-compact. Finally, we call a topological space X *quasi-separated* if every quasi-compact open subset is retrocompact. A map $f\colon Y \to X$ of topological spaces is *quasi-separated* if $f^{-1}(U)$ is quasi-separated for every quasi-separated open subset U of X.

Let us also recall that a topological space X is called *irreducible* if it cannot be written as the union of two proper closed subsets. A topological space X is *sober* if it is T_0 and for every non-empty irreducible closed subset Z of X there exists a unique point η of Z such that $Z = \overline{\{\eta\}}$.

Definition 2.1.1. A topological space X is called *spectral*[6] if it is quasi-compact, quasi-separated, sober, and has a basis of quasi-compact open subsets. A topological space X is called *locally spectral* if it admits an open cover by spectral subspaces.

It is easy to see that a locally spectral space is spectral if and only if it is quasi-compact and quasi-separated. We shall make use of this observation freely.

Let us call a map $f : Y \to X$ of locally spectral spaces *locally quasi-compact* if for all quasi-compact and quasi-separated open subsets $U \subseteq Y$ and $V \subseteq X$ with $f(U) \subseteq V$ the map $f|_U : U \to V$ is quasi-compact.[7] The following will often be used without comment.

Proposition 2.1.2 ([FK18, Chapter 0, Proposition 2.2.25])**.** *Let $f : Y \to X$ be a locally quasi-compact morphism of locally spectral spaces. Then, for any retrocompact open subset V of X, $f^{-1}(V)$ is a retrocompact open subset of Y.*

Another useful property of locally spectral spaces is that their retrocompact open subsets have closures which may be computed 'pointwise'. More precisely, we have the following.

Proposition 2.1.3 ([FK18, Chapter 0, Corollary 2.2.27])**.** *Let X be a locally spectral space and U a retrocompact open subset of X. Then,*

$$\overline{U} = \bigcup_{x \in U} \overline{\{x\}}$$

We finish this subsection by discussing projective systems of spectral spaces, their inverse limits, and the connected components of the inverse limit. Here the set of connected components $\pi_0(X)$ of a topological space X is given the quotient topology with respect to the projection $X \to \pi_0(X)$.

Proposition 2.1.4. *Let $\{X_i\}$ be a projective system of spectral spaces with quasi-compact transition maps. Set $X = \varprojlim_i X_i$. Then, X is spectral and the natural map*

$$\pi_0(X) \to \varprojlim_i \pi_0(X_i)$$

is a homeomorphism of profinite spaces.

[6] Called 'coherent' in [FK18].

[7] Note that the map $f : \mathrm{Spa}(\mathbf{Z}_p[\![T]\!]) \to \mathrm{Spa}(\mathbf{Z}_p)$ is not locally quasi-compact since $f^{-1}(\mathrm{Spa}(\mathbf{Q}_p))$ is the open unit disk over $\mathrm{Spa}(\mathbf{Q}_p)$. Such behavior does not occur for analytic adic spaces (see Theorem 3.1.1).

Proof. Under the given assumptions, X is spectral by [FK18, Chapter 0, Theorem 2.2.10]. Suppose first that the X_i are connected and non-empty. By [FK18, Chapter 0, Proposition 3.1.10], the induced map $\varinjlim \Gamma(X_i, \underline{\mathbf{F}_2}) \to \Gamma(\varprojlim X_i, \underline{\mathbf{F}_2})$ is a bijection. Thus $\Gamma(X, \underline{\mathbf{F}_2}) = \mathbf{F}_2$, and so X is connected and non-empty, showing the assertion.

In general, combining [Laz67, Corollaire 8.5] and [Stacks, Tag 0900], we see that $\pi_0(X)$ and each $\pi_0(X_i)$ are profinite, and thus also $\varprojlim_i \pi_0(X_i)$ is profinite. Since profinite spaces are compact, it suffices to prove that the map in question is a bijection. To this end, let (C_i) be in $\varprojlim_i \pi_0(X_i)$. As C_i is closed in X_i, the C_i form an inverse system of non-empty connected spectral spaces with quasi-compact transition maps, and hence by the first paragraph the space $C = \varprojlim_i C_i$ is a non-empty and connected spectral space.

By [FK18, Chapter 0, Lemma 2.2.19] we have $C = \bigcap_i p_i^{-1}(C_i)$, where $p_i : X \to X_i$ is the natural map. Let x be a point in C and let C_x be the component of X containing it. Since $p_i(C_x)$ is connected and intersects C_i at $p_i(x)$ we see that $p_i(C_x) \subseteq C_i$ for all i, so that C_x maps to (C_i). Conversely, if the connected components of C_x and C_y for $x, y \in X$ both map to (C_i), then $x, y \in \bigcap_i p_i^{-1}(C_i) = C$, which is connected, and so $C_x = C_y$. $\square$

2.2. **Valuative spaces and universal separated quotients.** In the last subsection we discussed locally spectral spaces, spaces that appear in algebraic geometry and rigid geometry. In this subsection we specialize even further to spaces that only appear in rigid geometry, and discuss the key property of these spaces: that they admit rich T_1 quotient spaces.

Definition 2.2.1 ([FK18, Chapter 0, Definition 2.3.1])**.** A topological space X is called *valuative* if it is locally spectral and if for each x in X the set of generizations of x is totally ordered by the generization relation.

We note that every point x of a valuative space X has a unique maximal generization, denoted $x^{\max}$ (see [FK18, Chapter 0, Remark 2.3.2 (1)]). We then call a point x of X *maximal* if $x = x^{\max}$. With this, we can define the best T_1-approximation to a valuative space.

Definition 2.2.2 ([FK18, Chapter 0, §2.3.(c)])**.** Let X be a valuative space. Then, the *universal separated quotient* of X, denoted $[X]$, is the quotient topological space $X/\sim$ where $x \sim y$ if $x^{\max} = y^{\max}$. The quotient map

$$\mathrm{sep}_X \colon X \to [X]$$

is called the *separation map*.

For a point x of $[X]$ we will sometimes write $x^{\max}$ for the unique maximal point of X in $\mathrm{sep}_X^{-1}(x)$. Let us note that for a point x in a valuative space X, one has the equality $\mathrm{sep}_X^{-1}(\mathrm{sep}_X(x)) = \overline{\{x^{\max}\}}$. Equivalently, for a point x in $[X]$ one has $\mathrm{sep}_X^{-1}(x) = \overline{\{x^{\max}\}}$. These simple equations will be useful quite often below.

By [FK18, Chapter 0, Proposition 2.3.9], the map $\mathrm{sep}_X : X \to [X]$ is initial amongst continuous maps from X to T_1-topological spaces and, in particular, $[X]$ is T_1. This implies that the universal separated quotient of X is functorial with respect to continuous maps. For a map $f \colon Y \to X$ of valuative spaces, we denote the induced map $[Y] \to [X]$ by $[f]$. One of the themes of this paper it the interplay of the geometry of X and the topology of $[X]$.

Another useful observation that follows from the universality of the association $X \mapsto [X]$, is that X is connected if and only if $[X]$ is connected. Indeed, this comes from observing that the natural map $\mathrm{Hom}([X], \{0,1\}) \to \mathrm{Hom}(X, \{0,1\})$ is a bijection where $\{0,1\}$ is the discrete two-point space. In fact, one can upgrade this observation to the statement that the map $\pi_0(X) \to \pi_0([X])$ is bijective.

Finally, we single out a property of valuative spaces that will be ubiquitous in the theory below. Namely, we call a map $f \colon Y \to X$ of valuative spaces *valuative* (see [FK18, Chapter 0, Definition 2.3.21]) if it has the property that $f(y)$ is maximal whenever y is maximal.

2.3. **Taut valuative spaces.** In this section we discuss a technical condition on a valuative space X, namely when such a space is 'taut'. Tautness will play an important role in our article since it will imply that $[X]$ and sep_X have reasonable topological properties.

Definition 2.3.1 ([Hub96, Definition 5.1.2 i)])**.** Let X be a valuative space. Then, X is called *taut*[8] if it is quasi-separated and for all quasi-compact open subsets U of X, the closure $\overline{U}$ is also quasi-compact.

This definition is relatively tame since [FK18, Chapter 0, Proposition 2.5.15] shows that if X is a valuative space which is quasi-separated and paracompact (in the sense of [FK18, Chapter 0, Definition 2.5.13]) then X is automatically taut.

As already mentioned, taut valuative spaces enjoy very useful topological properties with respect to their universal separated quotients. We now discuss these properties which, for the rest of the article, we will use without comment.

Proposition 2.3.2 ([FK18, Chapter 0, Theorem 2.5.7 and Corollary 2.5.9])**.** *Let X be a taut valuative space. Then, the following statements are true:*

(a) the universal separated quotient $[X]$ is locally compact and Hausdorff,
(b) the separation map $\mathrm{sep}_X : X \to [X]$ is a universally closed map of topological spaces.

Tautness is also useful for resolving a possible subtlety concerning the separation map and open subspaces of a valuative space. Namely, if U is an open subset of X, then one has a continuous injective map $[U] \to [X]$ with image $\mathrm{sep}_X(U)$. In general, this map will not be a homeomorphism. If X is taut and U retrocompact, this is a non-issue.

Proposition 2.3.3. *Let X be a taut valuative space and U a retrocompact open subspace. Then, the natural map $[U] \to [X]$ is a homeomorphism onto $\mathrm{sep}_X(U)$.*

Proof. The inclusion map $j\colon U \to X$ is taut (see Definition 2.3.4) since it is quasi-compact and quasi-separated. Thus, the natural map $[j]\colon [U] \to [X]$ is a map of Hausdorff spaces. It is clearly continuous and injective and has image $\mathrm{sep}_X(U)$. Thus, it suffices to prove that $[j]$ is closed, but this follows from [FK18, Chapter 0, Proposition 2.3.27]. $\square$

In particular, if U is a retrocompact open subspace of a taut valuative space X, then we shall often identify $\mathrm{sep}_X(U)$ and $[U]$ without comment. No confusion should arise in practice.

It is also useful to have the notion of when a morphism is taut, which is furnished by the following definition.

Definition 2.3.4 ([Hub96, Definition 5.1.2 ii)])**.** Let $f\colon Y \to X$ be a morphism of valuative spaces. Then, f is called *taut* if it is locally quasi-compact and for every taut open subspace U of X the space $f^{-1}(U)$ is taut.

Such maps are again reasonable as [Hub96, Lemma 5.1.3 and Lemma 5.1.4] show that taut morphims are closed under composition and pullback, as well as containing the class of quasi-compact and quasi-separated morphisms.

We end this subsection by recording a result we will often implicitly use which shows that, in good situations, the universal separated quotients commutes with pullbacks of maps along open embeddings.

Proposition 2.3.5. *Let $f\colon Y \to X$ be a valuative, locally quasi-compact map, and taut map between valuative spaces, and let $U \subseteq X$ be retrocompact open. Then the natural map*

$$[Y \times_X U] \to [Y] \times_{[X]} [U]$$

is a homeomorphism.

[8]In the terminology of Fujiwara–Kato, taut means quasi-separated and locally strongly compact (see [FK18, Chapter 0, Remark 2.5.6].

Proof. We first claim that $\operatorname{sep}_Y(f^{-1}(U)) = [f]^{-1}(\operatorname{sep}_X(U))$ (this only requires that f be a valuative map of valuative spaces, and $U \subseteq X$ to be open). To see this, we identify $[X]$ with maximal points of X, and therefore sep_X with the map $x \mapsto x^{\max}$, and similarly for Y. Since f is valuative, we have $f(y)^{\max} = f(y^{\max})$, and $[f]$ corresponds to the restriction of f to maximal points. The left hand side consists of $y = z^{\max}$ where $f(z) \in U$, but (since f is valuative and continuous and U is open) this is equivalent to $f(y) \in U$.

By Proposition 2.3.3, we have $[U] = \operatorname{sep}_X(U)$, and hence the target of the above map is $[f]^{-1}(\operatorname{sep}_X(U))$. By the previous paragraph, this equals $\operatorname{sep}_Y(f^{-1}(U))$. But $f^{-1}(U)$ is retrocompact in Y (Proposition 2.1.2), and again by Proposition 2.3.3 we have $\operatorname{sep}_Y(f^{-1}(U)) = [f^{-1}(U)] = Y \times_X U$. □

2.4. **Overconvergent open subsets, overconvergent interiors.** Another aspect of valuative spaces that will be important to us is the refinement of the notion of open subset of a valuative space X that the separation map $\operatorname{sep}_X : X \to [X]$ allows.

Proposition 2.4.1 ([FK18, Chapter 0, Proposition 2.3.13])**.** *Let X be a valuative space and let U be an open subset of X. Then, the following conditions are equivalent:*

- *(a) There exists an open subset $V \subseteq [X]$ such that $U = \operatorname{sep}_X^{-1}(V)$.*
- *(b) The equality $\operatorname{sep}_X^{-1}(\operatorname{sep}_X(U)) = U$ holds.*
- *(c) For all x in U one has that $\overline{\{x^{\max}\}} \subseteq U$.*

We can then use this to make the following definition.

Definition 2.4.2. Let X be a valuative space. An open subset U of X is called *overconvergent* if any of the equivalent conditions of Proposition 2.4.1 holds.

Overconvergent open subsets of X form a topology called the *oc topology*. For a subset $F \subseteq X$ we denote by $\operatorname{int}_X(F)$ its interior in this topology, and call it the *overconvergent interior* of F. The following proposition shows that the overconvergent interior is particularly well-behaved in the case of taut valuative spaces.

Proposition 2.4.3. *Let X be a taut valuative space, U an open subset of X, and S a subset of U. Consider the following conditions:*

- *(1) For all x in S one has that $\overline{\{x^{\max}\}} \subseteq U$,*
- *(2) $\operatorname{sep}_X^{-1}(\operatorname{sep}_X(S))$ is contained in U,*
- *(3) S is contained in $\operatorname{int}_X(U)$,*
- *(4) the topological interior of $\operatorname{sep}_X(U)$ in $[X]$ contains $\operatorname{sep}_X(S)$.*

Then,

$$(1) \iff (2) \iff (3) \implies (4)$$

Proof. To see that (1), (2), and (3) are equivalent it is clear that we may assume that $S = \{x\}$. To prove the equivalence of (1) and (2) we merely apply the equality $\overline{\{x^{\max}\}} = \operatorname{sep}_X^{-1}(\operatorname{sep}_X(x))$. We note that the equivalence of (1) and (3) is given in [FK18, Chapter 0, Proposition 2.3.29] when U is assumed quasi-compact (note that condition $(*)$ in loc. cit. is automatic since X is assumed taut). But, by Proposition 2.3.2 and the equality $\overline{\{x^{\max}\}} = \operatorname{sep}_X^{-1}(\operatorname{sep}_X(x))$ we see that the subspace $\overline{\{x^{\max}\}}$ is quasi-compact. It is then clear one can remove the quasi-compactness assumption in [FK18, Chapter 0, Proposition 2.3.29].

That (3) implies (4) is also simple since then $\operatorname{sep}_X(S) \subseteq \operatorname{sep}_X(\operatorname{int}_X(U))$. But, since $\operatorname{int}_X(U)$ is overconvergent we see that $\operatorname{sep}_X(\operatorname{int}_X(U))$ is open in $[X]$ and contained in $\operatorname{sep}_X(U)$. □

Remark 2.4.4. For a counterexample of the equivalence of (3) and (4) in Proposition 2.4.3 take X to $\mathbf{D}_K$, the closed unit disk over K, and $U = X - \{x\}$ where x is a type 5 point in the closure of the Gauss point η of X. Then, $\operatorname{sep}_X(U) = [X]$ and so $\operatorname{sep}_X(\eta)$ is contained in the interior of $\operatorname{sep}_X(U)$. But, since $x \in \overline{\{\eta\}}$ we see from (1) that η is not in $\operatorname{int}_X(U)$.

Let X be a taut valuative space, U an open subset, and S a subset of U. Below we shall often use the terminology that U is an *open oc neighborhood of* S to mean that S is contained in $\mathrm{int}_X(U)$. By the above this is equivalent to the claim that U contains $\mathrm{sep}_X^{-1}(\mathrm{sep}_X(S))$. So we harmlessly abuse notation and say that for an open subset $U \subseteq X$ and a subset S of $[X]$ that U is an *open oc neighborhood of* S if U contains $\mathrm{sep}_X^{-1}(S)$.

Implicit in the above proof, but useful to note in its own right, is the following (cf. [Hub96, Lemma 8.1.5]).

Proposition 2.4.5. *Let X be a taut valuative space, and Σ a quasi-compact subset of X. Then, the set of quasi-compact open oc neighborhoods of Σ constitute an open oc neighborhood basis of Σ.*

Proof. Assume first that $\Sigma = \{x\}$ is a point. Let U be an open oc neighborhood of x. Note then that necessarily U contains $x^{\max}$. We see then by Proposition 2.4.3 that $\overline{\{x\}}$ is contained in U. Since $\overline{\{x^{\max}\}}$ is quasi-compact (as observed in the proof of Proposition 2.4.3) it is clear then that there exists a quasi-compact open subset V of U such that $\overline{\{x^{\max}\}} \subseteq V$. So then, by Proposition 2.4.3 we're done. Now, for arbitrary Σ, we can apply the above to all $x \in \Sigma$ and employ quasi-compactness of Σ to find a finite subcover. $\square$

3. Preliminaries on rigid geometry

In this section we review some material from the theory of rigid geometry that will be important in the main part of this article. Again, we encourage the reader to skip this section upon first reading, returning only as necessary.

3.1. Adic spaces and rigid K-spaces.

In this subsection we set our terminology concerning adic spaces and rigid spaces that will be used for the rest of this paper.

General adic spaces. Most importantly, the following conventions shall be used throughout this article.

- All Huber pairs (A, A^+) are assumed complete, analytic (see [Ked19, Definition 1.1.2]), and strongly Noetherian (see [Ked19, Definition 1.2.10]),
- we only consider adic spaces X for which every affinoid open subspace is of the form $\mathrm{Spa}(A, A^+)$ where (A, A^+) is a Huber pair of the aforementioned type,
- for a Huber pair of the form (A, A°) we abbreviate $\mathrm{Spa}(A, A^\circ)$ to $\mathrm{Spa}(A)$.

In particular, we are in a situation where the contents of [Hub96] apply. We denote by $\mathbf{\acute{E}t}_X$ (resp. $\mathbf{F\acute{E}t}_X$) the full subcategory of the category of adic spaces over X consisting of étale (resp. finite étale) morphisms.

We freely use the terminology concerning finite type hypotheses as in [Hub96, Definition 1.2.1]. We note that by [Hub96, Proposition 1.2.2] the fiber product $Y_1 \times_X Y_2$ exists in the category of adic spaces as long as the morphism $Y_1 \to X$ is locally of weakly finite type. For a morphism $Y \to X$ locally of weakly finite type and a morphism $X' \to X$, we shall often abbreviate $Y \times_X X'$ to $Y_{X'}$.

By an *affinoid field* we mean a Huber pair (L, L^+) where L is a field, L^+ is a valuation ring, and L has the valuation topology. Note that for each point x of an adic space X one obtains, in the usual way, an affinoid field $(k(x), k(x)^+)$ called the *residue pair* of x. We call $k(x)$ the *residue field* of X at x. We shall denote by $\mathrm{Spa}(k(x), k(x)^+) \to X$ the canonical map. For a morphism $Y \to X$ locally of weakly finite type we shorten the notation $Y \times_X \mathrm{Spa}(k(x), k(x)^+)$ to Y_x.

We now observe that the material about valuative spaces from §2 applies our study of adic spaces.

Theorem 3.1.1 (Huber)**.** *The underlying topological space of any adic space is valuative. Moreover, any morphism $f\colon Y \to X$ of adic spaces is valuative and locally quasi-compact.*

Proof. The first claim follows by combining [Hub93, Theorem 3.5 (i)] and [Hub96, (1.1.9)]. The claim about maps being valuative follows from [Hub96, Lemma 1.1.10 (4)]. Finally, the claim about locally quasi-compact follows from the easy fact that any map of affinoid (analytic) adic spaces is quasi-compact. □

In particular, we shall freely use the material from §2 about valuative spaces for adic spaces. Whenever we talk about a property of an adic space (resp. map of adic spaces) defined for valuative spaces (resp. maps of valuative spaces) we mean that the underlying topological space of the adic space (resp. the map on underlying topological spaces) satisfies that property.

Lastly, we note that the underlying topological space of an adic space is locally connected, and therefore its connected components are open and hence are adic spaces themselves. We shall use this observation freely.

Rigid K-spaces. Let K be a non-archimedean field, by which we shall always mean a field complete with respect to a rank one valuation. We shall use the following definition of rigid spaces throughout the article.

Definition 3.1.2. By a *rigid K-space* we mean an adic space X locally of finite type over $\mathrm{Spa}(K)$. By a *morphism* of rigid K-spaces we mean a morphism of adic spaces over $\mathrm{Spa}(K)$. We denote the category of rigid K-spaces by $\mathbf{Rig}_K$.

Note that a morphism of $Y \to X$ rigid K-spaces is automatically locally of finite presentation, and so we shall almost always forego finite type hypotheses when talking about morphisms of rigid K-spaces. Note that, in particular though, if $Y \to X$ is a map of rigid K-spaces then for any adic space $X' \to X$ the fiber product $Y_{X'}$ exists. So, we see that $\mathbf{Rig}_K$ admits all fibered products and for any non-archimedean extension L of K there is a base extension functor $\mathbf{Rig}_K \to \mathbf{Rig}_L$ which we denote by $X \mapsto X_L$. We also note that if $f\colon Y \to X$ is a morphism locally of finite type between adic spaces where X is a rigid K-space, then Y is automatically a rigid K-space.

By the *dimension* of a rigid K-space X we mean the supremum of lengths of chains of specializations in X. Thanks to [Hub96, Lemma 1.8.6] this is equal to the supremum of Krull dimensions $\dim(A)$ as $\mathrm{Spa}(A)$ runs over the open affinoid subspaces of X. We say that a rigid K-space X is *equidimensional* if $\dim(X) = \dim(U)$ for all non-empty open subspaces of X.

Since 'curves' perform the role of workhorses throughout, we make the notion rigorous as follows.

Definition 3.1.3. A *rigid K-curve* is a rigid K-space which is equidimensional of dimension 1.

3.2. **Geometric points, fiber functors, and algebraic fundamental groups.** In this subsection we recall the notion of geometric points for an adic space. We then define the fiber functor associated to such a geometric point, and then use this to define the algebraic fundamental group of a rigid K-space.

Geometric points. By a *geometric point* of an adic space X we mean a morphism of adic spaces $\overline{x}\colon \mathrm{Spa}(L, L^+) \to X$ where (L, L^+) is an affinoid field and L is separably closed. By the *anchor point* of a geometric point $\overline{x}\colon \mathrm{Spa}(L, L^+) \to X$ we mean the image of the unique closed point of $\mathrm{Spa}(L, L^+)$. We shall often write x for the anchor point of $\overline{x}$. For such a geometric point we write $\overline{x}^{\max}$ for the associated *maximal geometric point* obtained as the composition $\mathrm{Spa}(L) \to \mathrm{Spa}(L, L^+) \to X$. We note that the anchor point of $\overline{x}^{\max}$ is the point $x^{\max}$ (see §2.2). For a geometric point $\overline{x}\colon \mathrm{Spa}(L, L^+) \to X$ and a morphism $Y \to X$ locally of finite type we abbreviate $Y \times_X \mathrm{Spa}(L, L^+)$ to $Y_{\overline{x}}$. Finally, for a geometric point $\overline{x}\colon \mathrm{Spa}(L, L^+) \to X$ we shall often denote L (resp. L^+) by $k(\overline{x})$ (resp. $k(\overline{x})^+$).

Suppose that $\overline{x}\colon \mathrm{Spa}(L, L^+) \to X$ is a geometric point with anchor point x. One then obtains a morphism of Huber pairs $(k(x), k(x)^+) \to (L, L^+)$. Let F_0 denote the separable closure of $k(x)$ in L, which is separably closed since L is. By [Bou98, §8.6, Chapter VI] there is a valuation ring F_0^+ of F_0, unique up to Galois conjugacy, such that $F_0^+ \cap k(x) = k(x)^+$. The completion (F, F^+) of (F_0, F_0^+) in (L, L^+) is a Huber pair and we call the choice of such a pair a *separable closure* of x in $\overline{x}$. Note that if (L, L^+) is a maximal geometric point then there is a unique separable closure of x in $\overline{x}$ given by (F, F°), the completion of (F_0, F_0°).

Using this notion we can define *equivalence* between geometric points. Fix geometric points $\overline{x}_i\colon \operatorname{Spa}(L_i, L_i^+) \to X$ for $i = 0, 1$. By an *equivalence* $\overline{x}_1 \to \overline{x}_2$, which only exists if the anchor points of these geometric points are the same point x of X, we mean an isomorphism $(F_1, F_1^+) \to (F_2, F_2^+)$ of chosen separable closures (F_i, F_i^+) of x in $\overline{x}_i$. It is obvious that an equivalence exists between any two geometric points of X with the same anchor point.

Lastly, we recall two basic operations for geometric points. Namely, let $f\colon Y \to X$ be a morphism of adic spaces. If $\overline{y}\colon \operatorname{Spa}(L, L^+) \to X$ is a geometric point, then the *image* of this geometric point under f is the geometric point $f(\overline{y}) = f \circ \overline{y}$. Similarly, by a *lifting* of a geometric point $\overline{x}\colon \operatorname{Spa}(L, L^+) \to X$ along f we mean a morphism $\overline{y}\colon \operatorname{Spa}(L, L^+) \to X$ such that $f(\overline{y}) = \overline{x}$.

Fiber functors and algebraic fundamental groups. Let $\overline{x}\colon \operatorname{Spa}(L, L^+) \to X$ be a geometric point of an adic space X. The following definition will play a central role in this article.

Definition 3.2.1. The *fiber functor* associated to $\overline{x}$ is the functor

$$F_{\overline{x}}\colon \mathbf{\acute{E}t}_X \to \mathbf{Set}, \qquad (Y \to X) \mapsto \operatorname{Hom}_X(\operatorname{Spa}(L, L^+), Y) = \pi_0(Y_{\overline{x}})$$

We shall also use the notation $F_{\overline{x}}$ to refer to the restriction to any subcategory of $\mathbf{\acute{E}t}_X$.

In the parlance of liftings we see that $F_{\overline{x}}(Y)$ nothing more than the set of all liftings of $\overline{x}$ along $Y \to X$. We note that there is a natural functorial association $F_{\overline{x}}(Y) \to Y_{\overline{x}}$, where here we consider $Y_{\overline{x}}$ as just a set, given by associating a lifting $\overline{y}$ in $\operatorname{Hom}_X(\operatorname{Spa}(L, L^+), Y)$ to the anchor point of the geometric point $\operatorname{Spa}(L, L^+) \to Y_{\overline{x}}$ induced by $\overline{y}$. We note that this map is not a bijection in general, unless $\overline{x}$ is a maximal geometric point, but does induce a functorial bijection $F_{\overline{x}}(Y) \to \pi_0(Y_{\overline{x}})$.

We would like to define the 'algebraic fundamental group' (the analogue in rigid geometry of the usual étale fundamental group from algebraic geometry), the crux of which is the following claim concerning Galois categories (see [SGA1, Exposé V, Definition 5.1]) whose proof is much the same as in the theory of schemes.

Proposition 3.2.2. *Let X be a connected adic space and let $\overline{x}$ and $\overline{y}$ be a geometric points of X. Then, the pair $(\mathbf{F\acute{E}t}_X, F_{\overline{x}})$ is a Galois category. Moreover, the fiber functors $F_{\overline{x}}$ and $F_{\overline{y}}$ are isomorphic.*

Using this we can define the algebraic fundamental group (following the terminology of [dJ95b]) as follows.

Definition 3.2.3. Let X be a connected adic space and let $\overline{x}$ and $\overline{y}$ be geometric points of X. Then, the set of *étale paths* from $\overline{x}$ to $\overline{y}$ is the set

$$\pi_1^{\mathrm{alg}}(X; \overline{x}, \overline{y}) = \operatorname{Isom}_{\mathbf{F\acute{E}t}_X}(F_{\overline{x}}, F_{\overline{y}})$$

When $\overline{x} = \overline{y}$ we shorten the notation to $\pi_1^{\mathrm{alg}}(X, \overline{x})$ and call the resulting profinite topological group the *algebraic fundamental group* of X.

We use the following well-known result freely (see [Hub96, Example 1.6.6 ii)]): for an affinoid adic space $X = \operatorname{Spa}(R, R^+)$, the category $\mathbf{F\acute{E}t}_X$ is equivalent to the category $\mathbf{F\acute{E}t}_{\operatorname{Spec}(R)}$, or equivalently to the category of finite étale R-algebras. In particular, for X connected we have $\pi_1^{\mathrm{alg}}(X, \overline{x}, \overline{y}) \simeq \pi_1^{\mathrm{\acute{e}t}}(\operatorname{Spec}(R), \overline{x}, \overline{y})$.

3.3. **Partially proper morphisms.**

In this section we recall the definition of a partially proper morphism, which will be ubiquitious in the theory below.[9]

We first recall that a morphism $f\colon Y \to X$ of topological spaces *specializing* if for any point y of Y and any specialization x' of $x = f(y)$ there exists a specialization y' of y such that

[9] One should note that partially proper morphisms appear in the theory of Berkovich spaces under the name 'boundaryless'.

$f(y') = x'$. If $f\colon Y \to X$ is a morphism of adic spaces, we say that f is *universally specializing* if for all morphisms of adic spaces $X' \to X$ the morphism of topological spaces $Y_{X'} \to X'$ is specializing.

Next recall that a morphism $f\colon Y \to X$ of adic spaces is called *separated* if it is locally of weakly finite type and the diagonal morphism $\Delta_{Y/X}\colon Y \to Y \times_X Y$ has closed image[10]. If X is a rigid K-space, and if the structure map $X \to \mathrm{Spa}(K)$ of a rigid K-space is separated, we say that X itself is *separated*.

We can then define partially proper morphism of adic spaces as follows.

Definition 3.3.1 ([Hub96, Definition 1.3.3])**.** A morphism $f\colon Y \to X$ of adic spaces is *partially proper* if it is locally of $^+$weakly finite type, universally specializing, and separated.

The reader only interested in rigid spaces should note that if $Y \to X$ is a morphism of rigid K-spaces, then it is automatically locally $^+$weakly of finite type, and so it is partially proper if and only if it is universally specializing and separated.

We note that the composition of partially proper morphisms is partially proper. Also, if $Y \to X$ is a partially proper morphism of adic spaces spaces, then so is $Y_{X'} \to X'$ for any map of adic spaces $X' \to X$.

Thematically it is important to note that partial properness (and separatedness) can be understood in terms of a certain valuative criterion akin to the valuative criterion of properness [Stacks, Tag 0A40].

Proposition 3.3.2 ([Hub96, Lemma 1.3.10], [Lud20, Proposition 6.13])**.** *Let $f\colon Y \to X$ be a morphism of adic spaces which is quasi-separated and locally of $^+$weakly finite type (resp. locally of weakly finite type). Then, the following properties are equivalent.*

(1) The morphism f is partially proper (resp. separated),
(2) for every affinoid field (L, L^+) and every morphism $\mathrm{Spa}(L, L^+) \to X$ the morphism $\mathrm{Hom}_X(\mathrm{Spa}(L, L^+), Y) \to \mathrm{Hom}_X(\mathrm{Spa}(L), Y)$ is a bijection (resp. injection),
(3) for every Huber pair (R, R^+) and every morphism $\mathrm{Spa}(R, R^+) \to X$ the morphism $\mathrm{Hom}_X(\mathrm{Spa}(R, R^+), Y) \to \mathrm{Hom}_X(\mathrm{Spa}(R), Y)$ is a bijection (resp. injection).

We record here two useful facts about the theory of valuative spaces which can only be phrased in the context of adic spaces and, in particular, both are in relation to the notion of partially proper morphisms.

The first is the following surprisingly non-trivial result.

Proposition 3.3.3 ([Hub96, Lemma 5.1.4.ii)])**.** *Let $f\colon Y \to X$ be a partially proper morphism of adic spaces. Then, f is taut.*

Since most of the morphisms that appear in this article are partially proper, this will effectively allow us to always be in a situation where our morphism is taut.

The second result gives a useful reinterpretation of overconvergent opens in terms of partially proper maps. This observation, whose proof follows from Proposition 3.3.2 (2) and part (2) of Proposition 2.4.1, will be used freely in the rest of this article.

Proposition 3.3.4. *Let X be an adic space and U an open subset of X. Then, the inclusion morphism $U \hookrightarrow X$ is partially proper if and only if U is an overconvergent open.*

3.4. **Huber's universal compactifications and tautness of curves.** As intimated before, tautness is a desirable property which forces the topology of rigid spaces to be especially well-behaved. A large portion of this article is devoted to the study of rigid K-curves and so it is comforting to know that, as it turns out, tautness is automatic for such spaces. To prove this though, we first need to recall a portion of Huber's theory of universal compactifications of (taut) morphisms of adic spaces.

[10] Equivalently that this morphism is a Zariski closed embedding. See [FK18, Chapter II, §7.5.(b)].

Definition 3.4.1 ([Hub96, Definition 5.1.1]). A *universal compactification* of a map of adic spaces $U \to X$ is a locally closed embedding $U \to U^{\mathrm{univ}}_{/X}$ over X with $U^{\mathrm{univ}}_{/X} \to X$ partially proper, and such that for every factorization $U \to U' \to X$ with $U' \to X$ partially proper there exists a unique morphism $U^{\mathrm{univ}}_{/X} \to U'$ under U and over X.

The following theorem shows the existence, with good properties, of universal compactifications of taut morphisms.

Theorem 3.4.2 (Huber, [Hub96, Theorem 5.1.5]). *Let $U \to X$ be a taut and separated morphism of rigid K-spaces. Then, a universal compactification $j : U \to U^{\mathrm{univ}}_{/X}$ exists, j is a quasi-compact open embedding, and every point of $U^{\mathrm{univ}}_{/X}$ is a specialization of a point of U.*

If U is a rigid K-space, taut and separated over $\mathrm{Spa}(K)$, then we shall shorten the notation $U^{\mathrm{univ}}_{/\mathrm{Spa}(K)}$ to U^{univ}. An example of such an object is given in the following.

Example 3.4.3. For $\mathbf{D}^1_K = \mathrm{Spa}(K\langle X\rangle)$, we have $(\mathbf{D}^1_K)^{\mathrm{univ}} = \mathbf{D}^1_K \cup \{\nu_{0,1^+}\}$ where $\nu_{0,1^+}$ is a certain rank two valuation in $\mathbf{A}^{1,\mathrm{an}}_K$ in the closure of the Gauss point

We would like to explicitly link Huber's definition of universal compactifications with the approach taken by Scholze in [Sch17, §18] in the context of v-sheaves.

In the lemma below, we denote by $\mathcal{A}$ the category of affinoid adic spaces $T = \mathrm{Spa}(R, R^+)$ such that R is topologically of finite type over some non-archimedean field. Note that the category of adic spaces locally of the form $\mathrm{Spa}(R, R^+)$, for $\mathrm{Spa}(R, R^+)$ an object of $\mathcal{A}$, admit a fully faithful embedding into presheaves on $\mathcal{A}$.

Lemma 3.4.4. *Let $U \to X$ be a separated and taut morphism of rigid K-spaces. Then, we have an isomorphism of presheaves on $\mathcal{A}$:*

$$\mathrm{Hom}(T, U^{\mathrm{univ}}_{/X}) \simeq \left\{ (f, \tilde{f}) \, : \begin{tikzcd} T^\circ \arrow[r, "\tilde{f}"] \arrow[d] & U \arrow[d] \\ T \arrow[r, "f"'] & X \end{tikzcd} \text{ commutes} \right\}$$

where $T = \mathrm{Spa}(R, R^+)$ and $T^\circ = \mathrm{Spa}(R)$.

Proof. It suffices to show that for $T = \mathrm{Spa}(R, R^+) \to X$ an object of $\mathcal{A}_{/X}$ there is a functorial identification $\mathrm{Hom}_X(T, U^{\mathrm{univ}}_{/X}) \simeq \mathrm{Hom}_X(T^\circ, U)$. Let us note that one has a natural injective map of sets $\mathrm{Hom}_X(T^\circ, U) \to \mathrm{Hom}_X(T^\circ, U^{\mathrm{univ}}_{/X})$. We also have a bijection

$$\mathrm{Hom}_X(T, U^{\mathrm{univ}}_{/X}) \to \mathrm{Hom}_X(T^\circ, U^{\mathrm{univ}}_{/X})$$

by Proposition 3.3.2 (3) applied to the partially proper map $U^{\mathrm{univ}}_{/X} \to X$. In particular, we get an injective map

$$\mathrm{Hom}_X(T^\circ, U) \to \mathrm{Hom}_X(T^\circ, U^{\mathrm{univ}}_{/X}) \simeq \mathrm{Hom}_X(T, U^{\mathrm{univ}}_{/X})$$

This map is clearly functorial, and thus it suffices to show that it is a bijection. To do this, it suffices to show that for every map $f \colon T \to U^{\mathrm{univ}}_{/X}$ over X, that $f(T^\circ) \subseteq U$.

To see this, note that by Theorem 3.4.2 that every maximal point of $U^{\mathrm{univ}}_{/X}$ is a point U. In particular, since f is valuative, we see that $f^{-1}(U)$ contains every maximal point of T. But, by Theorem 3.4.2 the map $U \to U^{\mathrm{univ}}_{/X}$ is quasi-compact. Thus, $f^{-1}(U) \to T$ is quasi-compact, and thus $V = f^{-1}(U)$ is quasi-compact.

It remains to show $T^\circ \subseteq V$. Since T is quasi-separated, $V \cap T^\circ$ is a quasi-compact open subset of T°. By the above, we see that $V \cap T^\circ$ contains all maximal and hence all classical points of T°, and hence is equal to it by [Hub93, Theorem 4.3]. ∎

Remark 3.4.5. The only place in the above proof that we used that R is an affinoid algebra over some non-archimedean field is in the last sentence. In this way, one sees that (R, R^+) can be any Huber pair where $\mathrm{Spa}(R)$ is reflexive in the sense of [FK18, Chapter 0, Definition 2.4.1]. This includes pairs (R, R^+) where R has a Noetherian ring of definition, and by using standard techniques to reduce to this situation, that all (analytic) Huber pairs satisfy this property.

Note that if X is a rigid K-space, then for any morphism of rigid K-spaces $U \to V$ which are separated and taut over X, then Lemma 3.4.4 implies that one obtains an induced morphism of adic spaces $f^{\mathrm{univ}}_{/X}\colon U^{\mathrm{univ}}_{/X} \to V^{\mathrm{univ}}_{/X}$. If $X = \mathrm{Spa}(K)$ we shall shorten this notation to just $f^{\mathrm{univ}} : U^{\mathrm{univ}} \to V^{\mathrm{univ}}$.

We record some further properties of the universal compactification that will be useful for us in showing that quasi-separated rigid K-curves are taut.

Proposition 3.4.6. *Let U and X be a rigid K-spaces. Let $U \to X$ be a separated and taut morphism. Then, the following statements hold true.*

- *(a) Suppose that $U \to X$ is an open immersion. Then, $U^{\mathrm{univ}}_{/X} \to X$ is a homeomorphism onto the closure of U in X.*
- *(b) If U is taut and separated over* $\mathrm{Spa}(K)$*, then we have an injective map $U^{\mathrm{univ}}_{/X} \to U^{\mathrm{univ}}$ under U.*
- *(c) For $U =$* $\mathrm{Spa}(A)$ *we have a, functorial in A, identification $U^{\mathrm{univ}} =$* $\mathrm{Spa}(A, A')$ *where $A' \subseteq A^\circ$ is the integral closure of the $\mathcal{O}_K$-subalgebra of A generated by $A^{\circ\circ}$.*
- *(d) If $U \to V$ is a finite morphism of affinoid K-spaces, then the induced map $U^{\mathrm{univ}} \to V^{\mathrm{univ}}$ is finite.*

Proof. To prove the first statement, note that by Lemma 3.4.4 the map

$$\mathrm{Hom}(\mathrm{Spa}(R, R^+), U^{\mathrm{univ}}_{/X}) \to \mathrm{Hom}(\mathrm{Spa}(R, R^+), X)$$

is injective for all objects $\mathrm{Spa}(R, R^+)$ of $\mathcal{A}$. Thus, $U^{\mathrm{univ}}_{/X} \to X$ is injective. To see the image is $\overline{U}$, note that by Lemma 3.4.4 applied to $T = \mathrm{Spa}(k(x), k(x)^+)$ for $x \in X$ shows that $x^{\max} \in U$ if and only if $T \to X$ lifts to $U^{\mathrm{univ}}_{/X}$, in which case that lifting is unique. Combining this with Proposition 2.1.3 shows that the image is indeed $\overline{U}$. The map $U^{\mathrm{univ}}_{/X} \to X$ is quasi-compact by [Hub96, Corollary 5.1.6], and thus it is a homeomorphism onto $\overline{U}$ by [Hub96, Lemma 1.3.15]

For the second statement, using the notation of Lemma 3.4.4, the map $U^{\mathrm{univ}}_{/X} \to U^{\mathrm{univ}}$ corresponds via the Yoneda lemma to the map $(f, \tilde{f}) \mapsto (\pi_X \circ f, \tilde{f})$ (where $\pi_X\colon X \to \mathrm{Spa}(K)$ is the structure map). This map is clearly under U, and is injective by Proposition 3.3.2 (3) since $U \to X$ is separated

For the third statement, note that every map $(A, A^\circ) \to (R, R^\circ)$ over $(K, \mathcal{O}_K)$ sends A' to R', and $R' \subseteq R^+$ because R' is the smallest subring of integral elements of R containing $\mathcal{O}_K$. This shows that (A, A') represents the correct presheaf as in Lemma 3.4.4.

To see the final claim, write $V = \mathrm{Spa}(B)$ and $U = \mathrm{Spa}(A)$. We know that $B \to A$ is finite and hence that $B^\circ \to A^\circ$ is integral by [BGR84, §6.3.4, Proposition 1]. We claim that this implies $B' \to A'$ is integral, which is sufficient to prove our claim. But, for this it suffices to show that every element of $A^{\circ\circ}$ satisfies a monic equation with non-leading coefficients in $B^{\circ\circ}$. Let $f \in A^{\circ\circ}$, then we can write $f^m = cg$ where $m \geq 1$ and $c \in K$ with $|c| < 1$ and $g \in A^\circ$ (e.g. apply the contents of [BGR84, §6.2.3]). Let $P = T^n + a_1 t^{n-1} + \ldots + a_n \in B^\circ[T]$ be a monic polynomial with $P(g) = 0$. Then f satisfies the monic polynomial $c^n P(T^m/c)$ whose non-leading coefficients lie in $B^{\circ\circ}$. ☐

Corollary 3.4.7. *Let U be a quasi-compact and quasi-separated rigid K-curve. Then $U^{\mathrm{univ}} \setminus U$ is finite.*

Proof. Let us note that by Proposition 2.1.3 and Theorem 3.4.2 that $\overline{U} = U^{\mathrm{univ}}$. Thus, we are trying to show that $\overline{U} \setminus U$ is finite. If $\{V_i\}$ is a finite open cover of U by affinoid open subsets, we note that it suffices to show that $\overline{V_i} \setminus V_i$ is finite or all i. But, by combining Proposition 3.4.6

(a) and Proposition (b) it suffices to show that $V_i^{\mathrm{univ}} \setminus V_i$ is finite. Thus, we may assume that U is affinoid.

Let $f\colon U \to D := \mathbf{D}^1_K$ be a finite surjective map, which exists by Noether normalization (see [BGR84, §6.1.2, Corollary 2]). By Proposition 3.4.6 (d), the induced map of compactifications $f^{\mathrm{univ}}\colon U^{\mathrm{univ}} \to D^{\mathrm{univ}}$ is finite. But since f is finite we know that $U \to (f^{\mathrm{univ}})^{-1}(D)$ is finite, and thus has closed image. This then implies that $\overline{U} \cap (f^{\mathrm{univ}})^{-1}(D) = U$. But, by combining Proposition 2.1.3 and Theorem 3.4.2 we know that $\overline{U} = U^{\mathrm{univ}}$. Thus, $(f^{\mathrm{univ}})^{-1}(D) = U$, and hence

$$U^{\mathrm{univ}} \setminus U = (f^{\mathrm{univ}})^{-1}(D^{\mathrm{univ}} \setminus D) = (f^{\mathrm{univ}})^{-1}(\nu_{0,1^+}) \quad \text{(see Example 3.4.3)}$$

which is a finite set because f^{univ}, being a finite map, has finite fibers by [Hub96, Lemma 1.5.2]. $\square$

We now prove that every quasi-separated rigid K-curve is taut, a fact which we shall use without comment in the sequel.

Proposition 3.4.8. *Let X be a quasi-separated rigid K-curve. Then, X is taut.*

Proof. Since X is quasi-separated, it suffices to show that for every quasi-compact $U \subseteq X$, the closure $\overline{U}$ is quasi-compact. By Proposition 3.4.6 (a), $\overline{U} \setminus U = U^{\mathrm{univ}}_{/X} \setminus U$, which by Proposition 3.4.6 (b) is contained in $U^{\mathrm{univ}} \setminus U$. The latter set is finite by Corollary 3.4.7. Therefore $\overline{U} = U \cup (\overline{U} \setminus U)$ is quasi-compact, as the union of a quasi-compact open and a finite set. $\square$

3.5. **Goodness of curves.** While Proposition 2.4.5 implies that every maximal point of a taut adic space is contained in a quasi-compact open oc neighborhood, it does not imply that one can take such a quasi-compact open oc neighborhood to be affinoid. Since this condition is sometimes useful, we codify it as follows.

Definition 3.5.1. A taut adic space X is called *good* if every point x of X has an open oc neighborhood basis of affinoid open oc neighborhoods.

It is not difficult to see that a taut adic space X is good if and only if every point x admits some affinoid oc open neighborhood. This notion of 'goodness' was borrowed from the theory of Berkovich spaces, see [Ber93, Remark 1.2.16].

It will be useful for our construction of geometric intervals on rigid K-curves (see Proposition 4.4.4) to recall the fact that any reasonably nice rigid K-curve is automatically good (cf. [dJ95b, Corollary 3.4]). In fact, we show the following strengthening of this claim.

Proposition 3.5.2. *Let X be a smooth and separated rigid K-curve and let Σ be a connected quasi-compact subset of X. Suppose that $\mathrm{sep}_X(\Sigma)$ is not a connected component of $[X]$, then the set of affinoid open oc neighborhoods of Σ forms an open oc neighborhood basis of Σ. In particular, X is good.*

Proof. Let U be an open oc neighborhood of Σ. By Proposition 2.4.5 we can find a quasi-compact open oc neighborhood W of Σ contained in U. By passing to a connected component of W containing Σ we may assume without loss of generality that W is connected. By [FM86, Théorème 2], W is either affinoid or projective.

In the former case we are done, so suppose that W is projective, and in particular a connected component of X. We claim that there exists a classical point p of W not contained in Σ. Otherwise $\mathrm{sep}_W(\Sigma)$ contains all maximal points of $[W]$, and since these are dense, we have $\mathrm{sep}_W(\Sigma) = [W]$, contradicting that $\mathrm{sep}_X(\Sigma)$ is not a connected component of $[X]$. Replace W again with a quasi-compact open oc neighborhood W' of Σ but this time in $W \setminus \{p\}$. Now W' cannot be projective (being contained in the affine $W \setminus \{p\}$), so it is affinoid. $\square$

3.6. **Étale monomorphisms.** In this subsection we recall the rigid geometric analogue of Grothendieck's fudamental property for étale morphisms (see [Stacks, Tag 025F]). While this result, essentially relying on the Gerritzen–Grauert theorem, is certainly well-known to experts (cf. [FK06, Theorem 6.21]) we are unaware of an explicit reference, so we have included a proof for the convenience of the reader.

Proposition 3.6.1. *Let $f\colon Y \to X$ be an étale morphism of rigid K-spaces. Then, f is an open embedding (resp. an isomorphism) if and only if its geometric fibers have at most one (resp. exactly one) point.*

Proof. The assertion about isomorphisms follows from the one about open embeddings, so we only show the latter. Since the assertion is local on X, we can reduce to X affinoid. We claim that we can also assume that Y is affinoid. Indeed, if $Y = \bigcup_{\alpha \in I} Y_\alpha$ with $Y_\alpha \subseteq Y$ affinoid opens, then the compositions $Y_\alpha \to Y$ satisfy the same assumptions, and then if the claim holds for maps between affinoids, then $Y \to X$ is a local isomorphism. But a local isomorphism whose fibers have at most one element is an open embedding.

We claim that for every classical point $y \in Y$, the map on stalks $f^*\colon \mathcal{O}_{X,f(y)} \to \mathcal{O}_{Y,y}$ is an isomorphism. Indeed, this map is finite by [Hub96, 1.5.4], and since both of these rings are Noetherian (see [BGR84, §7.3.2, Proposition 7]) this map is of finite presentation. We note that this map is also flat and and unramified (see [Hub96, Proposition 1.7.5]), and thus is finite étale. But, the local ring $\mathcal{O}_{Y,y}$ is Henselian (see [KL15, Lemma 2.4.17 (a)]). Moreover, by the assumption on geometric fibers of f, and the fact that the extension of generic fibers is finite separable since f is étale, the above map induces an isomorphism on residue fields. We conclude that f^* is an isomorphism.

The above implies that the map of rigid spaces $\operatorname{Sp}\mathcal{O}_Y(Y) \to \operatorname{Sp}\mathcal{O}_X(X)$ is an open immersion in the sense of [BGR84, 7.3.3]. By [Hub96, 1.1.11 (b)], the map $Y \to X$ is an open immersion of adic spaces. □

3.7. **Étale and partially proper maps.** In this subsection we discuss some special properties of étale and partially proper morphisms of rigid K-spaces.[11]

Étale localness of étale and partially proper maps. We start by showing that étale and partially proper are properties of maps which can be checked étale locally on the target. We start with the easier of these two statements.

Proposition 3.7.1. *Let $Y \to X$ be a morphism locally of finite type between adic spaces and let $X' \to X$ be an étale surjection. If $Y_{X'} \to X'$ is étale, then $Y \to X$ is étale.*

Proof. This follows from Lemma 3.7.2 by considering the étale surjection $Y_{X'} \to Y$ over X. □

Lemma 3.7.2. *Let $Y \to X$ be a flat surjective map of adic spaces locally of finite type over the adic space S. Suppose that $Y \to S$ is étale, then $X \to S$ is étale.*

Proof. We first show that $X \to S$ is flat. Let x be a point of X mapping to s in S and let y be a point of Y mapping to x. We then have a series of ring maps

$$\mathcal{O}_{S,s} \to \mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$$

By assumption we know the composition is flat, and that the second map is flat, and thus must be the first by [Stacks, Tag 02JZ] (note that $\operatorname{Spec}(\mathcal{O}_{Y,y}) \to \operatorname{Spec}(\mathcal{O}_{X,x})$ is surjective by [Stacks, Tag 00HR]). Since $\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}$ is flat so therefore is the map $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x} \to \mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$, where $\mathfrak{m}_s$ is the maximal ideal of $\mathcal{O}_{S,s}$, but since this is a map of local rings it is automatically injective (by loc. cit. and the fact that faithfully flat ring maps are injective). But, since $Y \to S$ is étale we know that $\mathcal{O}_{Y,y}/\mathfrak{m}_s\mathcal{O}_{Y,y}$ is a finite separable field extension of $\mathcal{O}_{S,s}/\mathfrak{m}_s$ and therefore so must be $\mathcal{O}_{X,x}/\mathfrak{m}_s\mathcal{O}_{X,x}$. So, $X \to S$ is étale at x by [Hub96, Proposition 1.7.5]. Since x was arbitrary the conclusion follows. □

The corresponding statement for partially proper morphisms is slightly more sophisticated, and requires Corollary 3.8.3 below.

Proposition 3.7.3. *Let $Y \to X$ be a locally of $^+$weakly finite type morphism of adic spaces. Suppose that $X' \to X$ is an étale surjection such that $Y_{X'} \to X'$ is partially proper. Then, $Y \to X$ is partially proper.*

[11] We note as a side remark that as proved in [Hub96, p. 427], étale and partially proper morphisms (of taut rigid K-spaces) correspond to étale morphisms of Berkovich K-analytic spaces.

Proof. Let us write $Y' = Y_{X'}$. Since partially proper morphisms are affinoid open local on the target, we may assume by Corollary 3.8.3 that $X' \to X$ is a finite étale Galois cover. Note that once we know $Y \to X$ is separated, the claim then follows from Lemma 3.7.4 since $Y \to X$ is separated, $Y' \to Y$ is surjective, and the composition $Y' \to X' \to X$ partially proper.

To see that $Y \to X$ is separated, set Y' to be $Y_{X'}$. Note that since $Y' \to Y$ is surjective, the image of $\Delta_{Y/X}\colon Y \to Y \times_X Y$ is the image of the closed subset $\mathrm{im}(\Delta_{Y'/X'})$ under the finite étale map $Y' \times_{X'} Y' \to Y \times_X Y$. Since this map is closed, $\mathrm{im}(\Delta_{Y/X})$ is closed. $\square$

Lemma 3.7.4. *Let $Y \to X$ be a surjective map of adic spaces over the adic space S. Suppose that $Y \to S$ is partially proper and that $X \to S$ is separated and locally of ${}^+$weakly finite type. Then, $X \to S$ is partially proper.*

Proof. Since $X \to S$ is assumed separated and locally of ${}^+$weakly finite type the only property to be checked is that $Y \to S$ is universally specializing. But, let $S' \to S$ be any map of adic spaces. Then, $Y_{S'} \to X_{S'}$ is a surjection. Let x be a point of $X_{S'}$ mapping to the point s of S', and let y be a point of $Y_{S'}$ mapping to x. Let s' be a point of S' specializing s. Since $Y_{S'} \to S'$ is specializing there exists some y' in Y specializing y which maps to s' under $Y_{S'} \to S'$. Since $Y_{S'} \to X_{S'}$ is continuous the image of y' specializes x and maps to s' under $X_{S'} \to S'$ from where the conclusion follows. $\square$

Local finiteness of étale and partially proper maps. We now state a result which will be an ever-present component of some of the more involved proofs below. In essence, this result says that an étale and partially proper map of adic spaces is 'locally finite'. More precisely, we have the following (cf. [Hub96, Proposition 1.5.6]).

Proposition 3.7.5. *Let $f\colon Y \to X$ be an étale partially proper morphism of adic spaces where X is taut. Then, for any point y of Y the set*

$$\left\{ W \subseteq Y : \begin{matrix} (1) & W \text{ is an open oc neighborhood of } y \\ (2) & f|_W : W \to f(W) \text{ is finite étale} \end{matrix} \right\}$$

forms an open oc neighborhood basis of y.

Proof of Proposition 3.7.5. Let U be an open oc neighborhood of y. Note then that by Proposition 3.3.4 the composition $\mathrm{int}_Y(U) \to Y \to X$ is étale and partially proper. In particular, by replacing Y with $\mathrm{int}_X(U)$ it suffices to show that for every point y of Y there exists an open oc neighborhood W of y such that $f|_W : W \to f(W)$ is finite étale.

By Proposition 2.4.5, there is a quasi-compact open oc neighborhood V of $x = f(y)$. Since X is quasi-separated so is V and since f is quasi-separated so then is Y_V. So we may apply [Hub96, Proposition 1.5.6] to the étale and partially proper map $Y_V \to V$ to produce an open oc neighborhood W of y in Y_V and an open oc neighborhood W' of x such that $f(W) \subseteq W'$ and $f|_W : W \to W'$ is finite, and thus finite étale. Note though that $f(W)$ must be a clopen subset of of W', and therefore $f|_W : W \to f(W)$ is finite étale. But, since V contains x in its oc interior, Y_V must contain y in its oc interior by [FK18, Chapter 0, Corollary 2.3.23]. Since W is an oc open neighborhood of y in Y_V, it is then also an open oc neighborhood of y in Y. $\square$

Properties of $[f]$ for étale and partially proper maps. We now show that the assumption that f is étale and partially proper implies many nice consequences for the map $[f]$.

Proposition 3.7.6. *Let $f\colon Y \to X$ be an étale and partially proper morphism of adic spaces, and suppose that X is taut. Then, the following properties hold true.*

- *(a) The map $[f]$ is open,*
- *(b) the map $[f]$ has discrete fibers,*
- *(c) the map $[f]$ satisfies the following condition:*
  - *$(*)$ For $y \in Y$ and every open neighborhood U of y there exists an open neighborhood V of $f(y)$ and a clopen subset W of $f^{-1}(V)$ containing y and contained in U.*

Proof. The final statement follows from Lemma 3.7.7 below, so we focus only on the first two.

To show the first statement, note that if $V \subseteq [Y]$ is open then $\mathrm{sep}_Y^{-1}(V)$ is an overconvergent subset of Y. So then, $f(\mathrm{sep}^{-1}(V))$ is open since f is étale, but it is also overconvergent since it is closed under specialization since f is specializing and $\mathrm{sep}^{-1}(V)$ is closed under specialization. Thus, $\mathrm{sep}_X(f(\mathrm{sep}_Y^{-1}(V)))$ is an open subset of $[X]$. But, since the separation maps are surjective this is equal to $[f](V)$ as desired.

We first address the second statement in the case f is finite étale. Since f is valuative it suffices to show that f itself is quasi-finite, but this follows from [Hub96, Lemma 1.5.2]. Let $y \in [Y]$ and let W be as in Proposition 3.7.5 (for the point $y^{\max} \in Y$). By the finite étale case, the assertion holds for $W \to f(W)$. Then, $\mathrm{sep}_X(W)$ is a neighborhood of y which intersects $[f]^{-1}(f(y))$ at finitely many points. So, we're done. $\square$

Lemma 3.7.7. *Let $f\colon Y \to X$ be a map with discrete fibers between Hausdorff spaces, with Y locally compact. Then, condition $(*)$ holds.*

Proof. If y is a point of Y and $Y_0 \subseteq Y$ a closed subset such that y is in the interior of Y_0 and $f|_{Y_0}$ satisfies the assertion $(*)$ at y, then so does f. Indeed, let U be an open neighborhood of y in Y and let U_0 be the intersection of U and the interior of Y_0. This is an open neighborhood of y in Y_0, and hence there exists an open neighborhood V of $f(y)$ in X and a clopen subset $W \subseteq f^{-1}(V) \cap Y_0$ containing y and contained in U_0 and hence in U. Then W is closed in $f^{-1}(V)$. It is also open in Y, and hence open in $f^{-1}(V)$.

So then, let $y \in Y$ and let $x = f(y)$. Since Y is locally compact, there exists an open neighborhood U_1 of y such that $\overline{U}_1$ is compact. In particular, $\overline{U}_1 \cap f^{-1}(x)$ is finite (being compact and discrete). Because Y is Hausdorff, we can find an open neighborhood $U_0 \subseteq U_1$ of y such that $\overline{U}_0 \cap f^{-1}(x) = \{y\}$. By the first paragraph of this proof applied to $Y_0 = \overline{U}_0$, it is enough to show that $Y_0 \to X$ satisfies $(*)$ at y. In other words, we may assume that Y is compact and $f^{-1}(x) = \{y\}$. Let U be an open neighborhood of y, and let $Z = Y \setminus U$. Then $f(Z)$ is closed and does not contain x, and hence $V = X \setminus f(Z)$ is an open neighborhood of x. Then $f^{-1}(V) \subseteq U$, and we can take $W = f^{-1}(V)$. $\square$

Bounding cardinality of étale and partially proper maps. The following strange-looking result will be used in order to prove that the category $\mathbf{Cov}_X$ of geometric coverings of a rigid K-space X is not too big (is generated under colimits by a *set* of connected objects):

Lemma 3.7.8. *Let X be a rigid K-space. Then, there exists a cardinal κ such that for every étale and partially proper map $Y \to X$ with Y connected, the cardinality of $|Y|$ is less than κ.*

Proof. The cardinality of any affinoid rigid K-space is bounded by some κ_0 which depends only on K. Indeed, by Noether normalization it suffices to take κ_0 bigger than the cardinality of $|\mathbf{D}_K^n|$ for all $n \geq 0$.

Let Γ be the set of all connected affinoid opens $V \subseteq Y$ such that $V \to f(V)$ is finite étale. By Proposition 3.7.5, such opens V cover Y. We make Γ into a graph where V and V' are adjacent if $V \cap V' \neq \emptyset$. The graph Γ is connected since Y is. Therefore to find a bound on the cardinality of Γ it is enough to find an absolute bound on the degree of any vertex V of Γ.

Let V_0' and V_1' be elements of Γ with $f(V_0') = f(V_1')$ (call this subset U') and such that $V_0' \cap V = V_1' \cap V \neq \emptyset$. Then $V_i' \to U'$ are connected finite étale coverings. Since $Y \to X$ is partially proper and in particular separated, the map $Y_{U'} \to U'$ is separated, and the cartesian square

$$\begin{tikzcd} V_0' \cap V_1' \arrow[r] \arrow[d] & V_0' \times_{U'} V_1' \arrow[d] \\ Y_{U'} \arrow[r] & Y_{U'} \times_{U'} Y_{U'} \end{tikzcd}$$

shows that the top map is a closed immersion. Since this top map is a map between étale spaces over U', it is étale and hence open. Therefore $V_0' \cap V_1' \to V_0' \times_{U'} V_1' \to U'$ is finite étale and non-empty. Since V_0' and V_1' are connected, we must have $V_0' = V_1'$. Indeed, since $V_0' \cap V_1'$ is finite étale over U' and V'_i for $i = 0, 1$ are finite étale over U', the inclusion map $V'_0 \cap V'_1 \to V'_i$ for $i = 0, 1$ is finite, thus the image is clopen and therefore everything.

The set of possible values of $V' \cap V$ for varying $V' \in \Gamma$ is bounded by 2^{κ_0}, and the set of possible values of $f(V')$ is bounded by 2^{κ_1} where κ_1 is the cardinality of $|X|$. By the previous paragraph, the degree of V as a node of Γ is bounded by $2^{\kappa_0+\kappa_1}$. □

3.8. **The étale bootstrap principle.** In this final subsection, we observe that all étale morphisms of adic spaces can be refined by covers of a particularly simple form. In practice, this implies that to show a property satisfies étale descent, it suffices to verify descent with respect to affinoid open covers and finite étale Galois covers.

The lynchpin needed to prove this result is the following general statement about factorizations of étale morphisms of affinoid adic spaces.

Lemma 3.8.1 ([Hub96, Lemma 2.2.8])**.** *Let $f\colon V \to X$ be an étale morphism of affinoid adic spaces. Then, there exists an affinoid open cover $X = \bigcup_{i\in I} U_i$, finite étale maps $Z_i \to U_i$, and open immersions $f^{-1}(U_i) \to Z_i$ over U_i.*

The desired refinement statement is then as follows.

Proposition 3.8.2. *Let X be an adic space and let $V \to X$ be an étale cover. Then $\{V \to X\}$ admits as a refinement an étale cover $\{V_{ij} \to X\}_{i\in I, j\in J_i}$ such that for all $i \in I$ and $j \in J_i$ one has a factorization*

$$V_{ij} \to W_i \to U_i \to X$$

such that the following holds

- *$\{U_i \to X\}_{i\in I}$ forms an open cover, with U_i connected,*
- *the maps $W_i \to U_i$ are connected finite étale Galois covers with Galois group G_i,*
- *for every $i \in I$, the family $\{V_{ij} \to W_i\}_{j\in I_j}$ is a finite affinoid open cover, stable under the action of G_i.*

Proof. If $X = \bigcup U_i$ is an affinoid open cover and the statement holds for $V \times_X U_i \to U_i$, then it also holds for X. We may therefore assume that X is affinoid. Similarly, we may replace $V \to X$ with any étale surjection $V' \to X$ which factors through V. We may therefore assume that V is the disjoint union of affinoids. Since $V \to X$ is open and X is quasi-compact, the images of finitely many of the affinoids covering V will cover X, and hence we may assume that V is the disjoint union of a finite number of affinoids and hence is affinoid.

Applying Lemma 3.8.1 to $V \to X$, we find an open cover $X = \bigcup U_i$ by connected affinoids and factorizations $V \times_X U_i \to Z_i \to U_i$ with Z_i finite étale. Applying again the previous step, we may assume that V and X are affinoid, with X connected, and that we have a factorization $V \to Z \to X$ with $Z \to X$ finite étale.

Let $V_1, \ldots, V_m$ be the connected components of V, and let $U_i \subseteq X$ be the image of V_i. For each i, let $Z_i \subseteq Z \times_X U_i$ be the connected component of $Z \times_X U_i$ containing the image of V_i. Then $Z_i \to U_i$ is a connected finite étale cover; let $W_i \to U_i$ be a Galois connected finite étale cover dominating $Z_i \to U_i$. If G_i denotes the automorphism group of W_i/U_i, then for $g \in G_i$ let $V_{ig} = g^{-1}(V_i \times_{Z_i} W_i)$. We set $J_i = G_i$.

By construction, V_i intersects every fiber of $Z_i \to U_i$, and hence $V_i \times_{Z_i} W_i$ intersects every fiber of $W_i \to U_i$. Since G_i acts transitively on the fibers of $W_i \to U_i$, we see that the translates $V_{ig} = g^{-1}(V_i \times_{Z_i} W_i)$ cover W_i. □

The above result can be interpreted as saying that the big étale topology of (analytic, locally strongly Noetherian) adic spaces is generated by the Zariski topology and the finite étale topology. While we do not wish to make this statement entirely precise, let us note the following concrete and useful corollary.

Corollary 3.8.3. *Let X be an adic space and let $\mathcal{C}$ be a fibered category over $\mathbf{\acute{E}t}_X$. Let P be a property of objects of $\mathcal{C}$ such that the following holds:*

(1) property P is stable under base change,

(2) property P can be checked on an affinoid open cover,

(3) property P can be checked on a finite étale Galois cover of connected affinoids,

Then, property P can be checked on an arbitrary étale cover.

4. Geometric arcs

In this section we discuss the theory of geometric arcs, and more generally the theory of geometric intervals.

4.1. **Intervals in rigid K-spaces and AVC.** We start by recalling some basic definitions concerning half-open and closed intervals which will play an important role in our definition of geometric coverings. Since intervals rarely exist in non-Hausdorff spaces, we define the notions of intervals in rigid K-spaces using the universal separated quotient.

Definition 4.1.1. Let X be a rigid K-space. By an *interval* in X we mean a subspace $\ell \subseteq [X]$ of its universal separated quotient, equipped with an order relation $\leqslant$ for which there is a monotonic homeomorphism to one of the following two model spaces: $[0, 1]$ (in which case we call ℓ an *arc*) or $[0, 1)$ (in which case we call ℓ an *ray*).

By a *parameterization* of an interval we mean a given monotonic homeomorphism with its model space. Given a parameterization $i\colon [0, 1] \xrightarrow{\sim} \gamma$ of an arc $\gamma \subseteq [X]$ the points $i(0), i(1) \in \gamma$ are referred to as the *left* and *right endpoint* of γ, respectively (they do not depend on the choice of i). In a similar vein, if $i : [0, 1) \xrightarrow{\sim} \rho$ is a parameterized ray in X, then we call $i(0)$ the *(left) endpoint* of ρ.

A *subinterval* (*subarc*, *subray*) ℓ' of an interval ℓ is an interval in X contained in ℓ whose order relation agrees with that induced by ℓ. For an interval ℓ and $a, b \in \ell$ with $a < b$, we write $[a, b]$ for the unique subarc of ℓ with left endpoint a and right endpoint b.

It will often be useful to allow *degenerate* arcs by which we mean points, and use the notation $[a, a] = \{a\}$. By convention, the left and right endpoint of a degenerate arc coincide. A *possibly degenerate interval (resp. arc)* is either an interval (resp. arc) or a degenerate interval.

Recall that a topological space is called *arc connected* if for any two distinct points x and y there exists an arc γ which has left endpoint equal to x and right endpoint equal to y. We say that it is *uniquely arc connected* if such an arc is always unique.

In §4.4 it will be useful to know that an arc in rigid K-curves have an oc neighborhood basis consisting of affinoids. To this end, we make the following observations.

- If X is taut then for any arc γ in X the subset $\mathrm{sep}_X^{-1}(\gamma)$ is closed and quasi-compact by Proposition 2.3.2 (b),
- $\mathrm{sep}_X^{-1}(\gamma)$ is connected (as any map $\mathrm{sep}_X^{-1}(\gamma) \to \{0, 1\}$ must factor through γ),
- γ cannot be a connected component of $[X]$ (since such a connected component has a dense set (of classical) points whose removal does not disconnect the space).

Combining these, one sees that Proposition 3.5.2 yields the following.

Proposition 4.1.2. *Let X be a smooth and separated rigid K-curve and let γ be an arc in X. Then, the set of affinoid open oc neighborhoods of γ forms an oc neighborhood basis of γ.*

We now state a fundamental theorem of Berkovich which will be one of the key topological underpinnings for our theory of geometric coverings.

Theorem 4.1.3 (Berkovich)**.** *Let X be a connected good rigid K-space. Then, the topological space $[X]$ is arc connected. Moreover, if X is a smooth and separated rigid K-curve then X has a basis $\{U_i\}$ where each U_i is affinoid and $[U_i] = \mathrm{sep}_X(U_i)$ is uniquely arc connected.*

Proof. Combining [FK18, Chapter II, Theorem A.5.2] with [FK18, Chapter II, Theorem C.6.12] we see that the topological space $[X]$ is the underlying space of a connected Hausdorff strictly K-analytic space X^{Berk} as in [Ber93]. Moreover, one sees from [FK18, Chapter II, Proposition C.6.13] that our assumption that X is good implies that X^{Berk} is also good. Thus, our desired result follows from [Ber90, Theorem 3.2.1]. The second result follows by applying similar reasoning, [Ber90, Corollary 4.3.3], and Proposition 2.3.3. ☐

One of the ways that arcs in rigid spaces will enter into our theory of geometric coverings is via the following 'topological valuative criterion'.

Definition 4.1.4. Let $f\colon Y \to X$ be a map of rigid K-spaces and let $i\colon [0,1] \to [X]$ be a parameterized arc. We say that f *satisfies the arcwise valuative criterion (AVC) with respect to* i if for every commutative square of solid arrows

$$\begin{tikzcd} {[0,1)} \arrow[r] \arrow[d] & {[Y]} \arrow[d, "{[f]}"] \\ {[0,1]} \arrow[r, "i"'] \arrow[ur, dotted] & {[X]} \end{tikzcd} \tag{4.1.1}$$

there exists a unique dotted arrow making the diagram commute. We say that f satisfies the *arcwise valuative criterion (AVC)* if it satisfies AVC with respect to every parameterized arc $i : [0,1] \to [X]$.

Remark 4.1.5. One can show that every proper map of topological spaces whose fibers are discrete satisfies AVC.

Remark 4.1.6. Suppose that $f\colon Y \to X$ is map of taut rigid K-spaces. Then in any given diagram as in (4.1.1), the dotted arrow is unique if it exists. Indeed, it suffices to note that since X and Y are taut, the spaces $[X]$ and $[Y]$ are Hausdorff, and hence $[f]\colon [Y] \to [X]$ is separated.

4.2. **Definition of geometric intervals and basic operations.** In this subsection we define the notion of a geometric structure on an interval, provide a practical way of building such geometric structures, and finally provide some simple operations one can use to produce new such structures from old ones.

Definition of geometric intervals. We start upgrading the notion of an interval to that of a 'geometric interval', which is the synthesis of the notion of an interval and an étale path in algebraic geometry (see §3.2 for a recollection of notation).

Definition 4.2.1. Let X be a rigid K-space. A *geometric interval* $\bar{\ell}$ in X consists of the following data:

- an interval ℓ in X (called the *underlying interval* of $\bar{\ell}$),
- for every point $z \in \ell$, a geometric point $\bar{z}$ of X anchored at $z^{\max}$,
- for every subarc $[a,b] \subseteq \ell$, and for every open oc neighborhood U of $[a,b]$ an étale path

$$\iota^U_{a,b} \in \pi_1^{\mathrm{alg}}(U; \bar{a}, \bar{b}),$$

such that the following conditions hold:

(1) for a subarc $[a,b] \subseteq \ell$ and two open oc neighborhoods U, U' of $[a,b]$ such that $U \subseteq U'$, the map

$$\pi_1^{\mathrm{alg}}(U; \bar{a}, \bar{b}) \to \pi_1^{\mathrm{alg}}(U'; \bar{a}, \bar{b})$$

induced by the inclusion $U \to U'$ maps $\iota^U_{a,b}$ to $\iota^{U'}_{a,b}$,

(2) for subarcs $[a,b]$ and $[b,c]$ of ℓ with common endpoint b, and for every $U \subseteq X$ open oc neighborhood of $[a,c] = [a,b] \cup [b,c]$, the composition map

$$\pi_1^{\mathrm{alg}}(U; \bar{a}, \bar{b}) \times \pi_1^{\mathrm{alg}}(U; \bar{b}, \bar{c}) \to \pi_1^{\mathrm{alg}}(U; \bar{a}, \bar{c})$$

maps $(\iota^U_{a,b}, \iota^U_{b,c})$ to $\iota^U_{a,c}$.

We sometimes write $\iota^{U,\bar{\ell}}_{a,b}$ if we want to emphasize the role of $\bar{\ell}$. We also extend the definition of $\iota^U_{a,b}$ by $\iota^U_{a,a} = \mathrm{id}$ and $\iota^U_{b,a} = (\iota^U_{a,b})^{-1}$.

A geometric structure on an interval ℓ is a suitably transitive system of étale paths along the arc. In practice what this means is that every point of ℓ comes equipped with a geometric structure (the structure of a geometric point anchored at the unique maximal point over it) as well as a means of performing 'parallel transport' in an algebro-geometric sense between these points with geometric structure (i.e. the étale paths). The role of the interval itself then, opposed to a general set, is to impose a suitably well-defined notion of 'continuity' in this procedure.

We would like to say what it means for two geometric intervals to be 'equivalent', which intuitively means that they give equivalent means of 'parallel transport'.

Definition 4.2.2. Two geometric arcs $\overline{\ell}_1$ and $\overline{\ell}_2$ in the rigid K-space X are *equivalent* if $\ell_1 = \ell_2 =: \ell$ and there exists equivalences $\overline{z}_1 \to \overline{z}_2$ between the geometric points lying over each points z of ℓ such that for all $a, b \in \ell$ the map

$$\pi_1^{\mathrm{alg}}(U, \overline{a}_1, \overline{b}_1) \to \pi_1^{\mathrm{alg}}(U, \overline{a}_2, \overline{b}_2)$$

carries $\iota_{a,b}^{U,\overline{\ell}_1}$ to $\iota_{a,b}^{U,\overline{\ell}_2}$. In this case, we call the data of such equivalences an *equivalence* between $\overline{\ell}_1$ and $\overline{\ell}_2$.

Geometric structures defined on a basis. In practice it is useful to be able to endow an interval with a geometric structure by only specifying the étale paths $\iota_{a,b}^U$ for 'sufficiently many U'. This is similar to the ability to define a sheaf by only specifying the values on a basis of open sets.

To make this precise, let X be a rigid K-space and let ℓ be an interval in X. Let us say that a set $\mathcal{U}$ of open subsets of X is an *ℓ-basis* if for each point z of ℓ the set $\mathcal{U}$ contains oc neighborhood basis of z.

Definition 4.2.3. Let X be a rigid K-space and let ℓ be an interval in X. Let $\mathcal{U}$ be an ℓ-basis. Then, a *geometric $\mathcal{U}$-structure* consists of the data of a geometric point $\overline{z}$ anchored at $z^{\max}$ for each point z of ℓ, and an étale path $\iota_{a,b}^U \in \pi_1^{\mathrm{alg}}(U; \overline{a}, \overline{b})$ for every subarc $[a, b] \subseteq \ell$ and every oc open neighborhood U of $[a, b]$ in $\mathcal{U}$, such that Axioms 1 and 2 of Definition 4.2.1 are satisfied under the assumption that $U, U' \in \mathcal{U}$.

We then have the following result.

Lemma 4.2.4. *Let X be a rigid K-space, ℓ an interval in X, and $\mathcal{U}$ an ℓ-basis. Then, every geometric $\mathcal{U}$-structure on ℓ extends uniquely to the structure of a geometric interval on ℓ (with the chosen geometric points).*

Proof. Let $[a, b] \subseteq \gamma$ be a subarc and let U be an open oc neighborhood of $[a, b]$. By definition of $\mathcal{U}$ and the fact that $[a, b]$ is compact, there exists a finite subdivision $[a, b] = \bigcup_{i=1}^m [z_{i-1}, z_i]$ (with $z_0 = a$ and $z_m = b$) and open oc neighborhoods U_i of $[z_{i-1}, z_i]$ which belong to $\mathcal{U}$ and are contained in U. We want to define $\iota_{a,b}^U$ as the composition inside the fundamental groupoid of U of the images of $\iota_{z_{i-1},z_i}^{U_i}$. To this end, we need to check that this composition does not depend on the choice of z_i and U_i. For every two such choices there exists a third one which refines both in the obvious way, and we reduce to the case $m = 2$ i.e. a subdivision $[a, b] = [a, z] \cup [z, b]$ and opens $U', U'_0, U'_1 \in \mathcal{U}$ which contain $[a, b]$, $[a, z]$, and $[z, b]$ respectively in their oc interiors. Then Axioms 1 and 2 imply that the composition of the images of $\iota_{a,z}^{U'_0}$ and $\iota_{z,b}^{U'_1}$ in the fundamental groupoid of U' equals $\iota_{a,b}^{U'}$. It is clear that the elements $\iota_{a,b}^U$ defined as above satisfy Axioms 1 and 2. ∎

Various operations on geometric intervals. We now would like to give some basic constructions of new geometric intervals from old ones. In what follows X will be a rigid K-space and $\overline{\ell}$ a geometric interval in X with underlying interval ℓ.

Construction 4.2.5 (Subinterval)**.** Let ℓ' be a subinterval of ℓ. By considering only the geometric points $\overline{z}$ for $z \in \ell'$ and the étale paths $\iota_{a,b}^U$ for $[a, b] \subseteq \ell'$ one endows ℓ' with an induced geometric structure, denoted by $\overline{\ell}'$.

Construction 4.2.6 (Concatenation)**.** Let ℓ be an interval in X and let $z \in \ell$ be an interior point (i.e. not and endpoint of ℓ). Thus $\ell = \ell' \cup \ell''$ is a union of two intervals with $\ell \cap \ell' = \{z\}$ and $x \leq y$ for $x \in \ell'$ and $y \in \ell''$. Suppose that we are given structures of geometric intervals $\overline{\ell}'$ and $\overline{\ell}''$ for which the chosen geometric points $\overline{z}$ above z agree. Then there exists a unique structure of a geometric interval $\overline{\ell}$ whose restrictions to ℓ' and ℓ'' are the given ones. Namely, for $a < z < b$ in ℓ and an open oc neighborhood U of $[a, b]$, we define

$$\iota_{a,b}^{U,\overline{\ell}} = \iota_{z,b}^{U,\overline{\ell}''} \circ \iota_{a,z}^{U,\overline{\ell}'}.$$

Construction 4.2.7 (Image)**.** Let $f\colon X \to X'$ be a map of rigid K-spaces. Suppose that $[f]$ maps ℓ homeomorphically onto its image ℓ'. One then obtains a geometric interval structure $\overline{\ell}'$ on ℓ' as follows. For each point z of ℓ we denote $[f](z)$ by z'. We define the geometric point $\overline{z}'$ to be the image of the geometric point $\overline{z}$ (see §3.2) under f. For each subarc $[a', b'] \subseteq \ell'$ and each open oc neighborhood U of $[a', b']$ we define $\iota_{a',b'}^{U,\overline{\ell}'}$ to be the image of $\iota_{a,b}^{f^{-1}(U),\ell}$ under the canonical map

$$\pi_1^{\mathrm{alg}}(f^{-1}(U); \overline{a}, \overline{b}) \to \pi_1^{\mathrm{alg}}(U; \overline{a'}, \overline{b'})$$

We call this the *image* of the geometric arc $\overline{\ell}$, and denote it by $f(\overline{\ell})$.

4.3. **$\mathcal{P}$-fields and their algebraic closures.** As of now we do not know that geometric intervals exist, let alone any sort of abundance result which says that 'many' intervals can be endowed with a geometric structure. To remedy this, we first need to develop some basic theory about $\mathcal{P}$-fields which, in short, are diagrams of fields indexed by intervals in a totally ordered set $\mathcal{P}$.

To see why this helpful, let us note the following example.

Example 4.3.1. Let us recall that for any connected ring R and any two points x_i for $i = 0, 1$ of $\mathrm{Spec}(R)$ the set $\pi_1^{\mathrm{alg}}(\mathrm{Spec}(R); \overline{x}_0, \overline{x}_1)$ of étale paths from $\overline{x}_0$ to $\overline{x}_1$ is non-empty for any geometric points $\overline{x}_i$ anchored at x_i. When R is a domain there is a simple proof of this fact.

Indeed, let $\overline{R}$ be the integral closure of R in an algebraic closure $\overline{K}$ of its field of fractions. Let $\tilde{x}_i$ for $i = 0, 1$ be lifts of x_i to $\mathrm{Spec}(\overline{R})$. The natural maps $\overline{y}_i\colon \mathrm{Spec}(k(\tilde{x}_i)) \to \mathrm{Spec}(R)$ are geometric points anchored at x_i, and there is a natural isomorphism $F_{\overline{y}_0} \simeq F_{\overline{y}_1}$ coming from the fact that $\overline{y}_i$ factorize through the connected object $\mathrm{Spec}(\overline{R})$ and that every connected finite étale cover of $\mathrm{Spec}(\overline{R})$ is split (which follows from [Stacks, Tag 0BQM]).

Since one can easily create isomorphisms $F_{\overline{x}_i} \simeq F_{\overline{y}_i}$ one then uses this to show that the set $\pi_1^{\mathrm{alg}}(\mathrm{Spec}(R), \overline{x}_0, \overline{x}_1)$ is non-empty as desired.

We would like to emulate this approach to create geometric structures on intervals. The notion of $\mathcal{P}$-fields plays the analogue of the generic point $\mathrm{Spec}(K) \to \mathrm{Spec}(R)$ in Example 4.3.1.

A (closed, possibly degenerate) interval in a totally ordered set $\mathcal{P}$ is a subset of the form

$$[a, b] = \{x \in \mathcal{P} \,:\, a \leq x \leq b\}$$

for some $a, b \in \mathcal{P}$ with $a \leq b$. We denote by $\mathbf{Int}(\mathcal{P})$ the poset of intervals in $\mathcal{P}$ ordered by inclusion. It is isomorphic as a poset to the set $\{(a, b) \in \mathcal{P}^2 \,:\, a \leq b\}$ with $(a, b) \leq (a', b')$ if $a' \leq a \leq b \leq b'$. If $\varphi\colon \mathcal{Q} \to \mathcal{P}$ is a monotone map of totally ordered sets, we obtain a monotone map $\varphi\colon \mathbf{Int}(\mathcal{Q}) \to \mathbf{Int}(\mathcal{P})$ sending $[a, b]$ to $[\varphi(a), \varphi(b)]$.

Definition 4.3.2. Let $\mathcal{P}$ be a totally ordered set. A *$\mathcal{P}$-field* is a functor

$$\mathcal{K}\colon \mathbf{Int}(\mathcal{P})^{\mathrm{opp}} \to \mathbf{Fields}, \quad [a, b] \mapsto \mathcal{K}_{[a,b]}.$$

A *morphism* or *extension* of $\mathcal{P}$-fields is a natural transformation $\mathcal{K} \to \mathcal{L}$. For $\mathcal{Q} \subseteq \mathcal{P}$, we denote by $\mathcal{K}|_Q$ the $\mathcal{Q}$-field obtained by restriction of $\mathcal{K}$ to $\mathbf{Int}(\mathcal{Q})^{\mathrm{opp}}$.

One of the key results in the basic theory of fields is the existence of algebraic and separable closures. We would like to formulate an analogue for $\mathcal{P}$-fields; such a fact is required to carry out our desire to emulate Example 4.3.1. Towards this end, we say that an extension $\mathcal{K} \to \mathcal{L}$ *algebraic* (resp. *separable*) if for all $[a, b] \in \mathbf{Int}(\mathcal{P})$, the extension $\mathcal{K}_{[a,b]} \to \mathcal{L}_{[a,b]}$ is algebraic (resp.

separable). We say that a $\mathcal{P}$-field $\mathcal{L}$ is *algebraically closed* (resp. *separably closed*) if all the fields $\mathcal{L}_{[a,b]}$ are algebraically (resp. separably) closed. We then have the following definition.

Definition 4.3.3. Let $\mathcal{P}$ be a totally ordered set and $\mathcal{K}$ a $\mathcal{P}$-field. Then an *algebraic (resp. separable) closure* of $\mathcal{K}$ is an extension of $\mathcal{P}$-fields $\mathcal{K} \to \mathcal{L}$ which is algebraic (resp. separable algebraic) and for which $\mathcal{L}$ is algebraically (resp. separably) closed.

We then have the following result.

Proposition 4.3.4. *Let $\mathcal{P}$ be a totally ordered set. Then, every $\mathcal{P}$-field has an algebraic (resp. separable) closure.*

Proof. By Lemma 4.3.5 below the ring $C = \operatorname{colim}(\iota \circ \mathcal{K})$ is non-zero. Therefore, it admits a homomorphism $C \to L$ into an algebraically closed field L. In particular, we see that one has compatible embeddings

$$\mathcal{K}_{[a,b]} \to C \to L$$

for all $[a,b] \in \mathbf{Int}(\mathcal{P})$. If one sets $\overline{\mathcal{K}}_{[a,b]}$ to be the algebraic (resp. separable) closure of $\mathcal{K}_{[a,b]}$ in L, one sees that such fields are functorial, and so form an algebraic (resp. separable) closure $\overline{\mathcal{K}}$ of $\mathcal{K}$. $\square$

Lemma 4.3.5. *Let $\mathcal{P}$ be a totally ordered set. Then, for every $\mathcal{P}$-field $\mathcal{K}$ the colimit $\operatorname{colim}(\iota \circ \mathcal{K})$ is a non-zero ring, where ι: **Fields** $\hookrightarrow$ **Rings** is the natural inclusion.*

Proof. Let us first observe that by [KS06, Lemma 3.2.8], the colimit can be rewritten as

$$\operatorname{colim}(\iota \circ \mathcal{K}) = \varinjlim_{J \in \mathcal{J}} \operatorname{colim}(\iota \circ \mathcal{K}|_J)$$

where $\mathcal{J}$ is any cofinal set of finite subposets of $\mathbf{Int}(\mathcal{P})^{\mathrm{op}}$. Since this outer colimit is filtered, and the filtered colimit of non-zero rings is non-zero (see [FK18, Chapter 0, Proposition 3.1.1]), it suffices to find a cofinal set of finite subposets J of $\mathbf{Int}(\mathcal{P})^{\mathrm{op}}$ such that $\operatorname{colim}(\iota \circ \mathcal{K}|_J)$ is non-zero.

To do this, for each finite subset T of $\mathcal{P}$, let

$$Q_T = \{[a,b] \in \mathbf{Int}(\mathcal{Q})^{\mathrm{op}} : a, b \in T\}.$$

Note that Q_T is finite, and that the set of Q_T is cofinal in the set of all finite subposets of $\mathbf{Int}(\mathcal{P})^{\mathrm{op}}$, since for any finite subposet J one has that $J \subseteq Q_T$ where T comprises of the finitely many endpoints of elements of J.

To show that $\operatorname{colim}(\iota \circ K|_{Q_T})$ is non-zero, let $Z_T \subseteq Q_T$ be the set of all intervals $[a,b] \in Q_T$ such that $[a,b] \cap T = \{a,b\}$. Since for every $q \in Q_T$, the set $\{x \in Z_T : x \geq q\}$ of its upper bounds in Z_T is a non-empty and connected poset, by [Mac71, Chapter IX, §3, Theorem 1] we have

$$\operatorname{colim}(\iota \circ \mathcal{K}|_{Q_T}) = \operatorname{colim}(\iota \circ \mathcal{K}|_{Z_T}).$$

If $T = \{x_0 < x_1 < \ldots < x_m\}$, then Z_T consists of the intervals $[x_i, x_i]$ $(i = 0, \ldots, m)$ and $[x_{i-1}, x_i]$ $(i = 1, \ldots, m)$. The colimit over Z_T is an iterated pushout (tensor product)

$$\operatorname{colim}(\iota \circ \mathcal{K}|_{Z_T}) = K_{[x_0,x_0]} \otimes_{K_{[x_0,x_1]}} K_{[x_1,x_1]} \otimes_{K_{[x_1,x_2]}} K_{[x_2,x_2]} \otimes \cdots \otimes_{K_{[x_{m-1},x_m]}} K_{[x_m,x_m]}.$$

Since the tensor product of two non-zero algebras over a field is non-zero, we deduce that $\operatorname{colim}(\iota \circ \mathcal{K}|_{Z_T}) \neq 0$. $\square$

The following example shows the importance of restricting our attention to diagrams indexed by posets of the type $\mathbf{Int}(\mathcal{P})$: not every diagram of fields indexed by a poset admits an "algebraic closure" in the sense similar to Definition 4.3.3.

Example 4.3.6. Consider the following commutative diagram indexed by a connected poset with five elements:

$$\begin{tikzcd}[column sep=small]
 & \mathbf{Q}(\sqrt{2}) & & \mathbf{Q}(\sqrt{2}) & \\
\mathbf{Q}(\sqrt{2}) \arrow[ur, "\sigma"] \arrow[urrr, "\mathrm{id}"'] & & & & \mathbf{Q}(\sqrt{2}) \arrow[ul, "\mathrm{id}"'] \arrow[ulll, "\mathrm{id}"] \\
 & & \mathbf{Q} \arrow[ull] \arrow[urr] & &
\end{tikzcd}$$

where σ is the nontrivial automorphism. Due to the "monodromy" around the figure eight, there is no compatible choice of a square root of 2. For this reason, this diagram does not have an algebraic closure, and its colimit in the category of rings is the zero ring.

The following lemma will be used later for extending geometric rays to geometric arcs.

Lemma 4.3.7. *Let $\mathcal{P}$ be a totally ordered set with largest element 1, and let $\mathcal{P}^\circ = \mathcal{P} \setminus \{1\}$. Let $\mathcal{K}$ be a $\mathcal{P}$-field, and let $\mathcal{K}^\circ$ be its restriction to $\mathcal{P}^\circ$. Let $\mathcal{K}^\circ \to \overline{\mathcal{K}}^\circ$ be an algebraic (resp. separable) closure of $\mathcal{K}^\circ$. Then there exists an algebraic (resp. separable) closure $\mathcal{K} \to \overline{\mathcal{K}}$ whose restriction to $\mathcal{P}^\circ$ coincides with $\mathcal{K}^\circ \to \overline{\mathcal{K}}^\circ$.*

Proof. We only treat algebraic closures, the proof for separable closures is the same. The intervals in $\mathbf{Int}(\mathcal{P}) \setminus \mathbf{Int}(\mathcal{P}^\circ)$ are of the form $[a, 1]$ with $a \in \mathcal{P}$. We first extend the definition of $\overline{\mathcal{K}}^\circ$ to such intervals with $a < 1$ by defining $\overline{\mathcal{K}}_{[a,1]}$ to be the algebraic closure of $\mathcal{K}_{[a,1]} \subseteq \mathcal{K}_{[a,a]} = \mathcal{K}^\circ_{[a,a]}$ in $\overline{\mathcal{K}}^\circ_{[a,a]}$.

We check that this naturally defines a functor on $\mathbf{Int}(\mathcal{P})^{\mathrm{opp}} \setminus \{[1,1]\}$. First, we note that $\overline{\mathcal{K}}_{[a,1]}$ coincides with the algebraic closure of $\mathcal{K}_{[a,1]} \subseteq \mathcal{K}_{[a,b]}$ in $\overline{\mathcal{K}}^\circ_{[a,b]}$ for any $a \leq b < 1$. If $a \leq b < 1$, then the commutative diagram

$$\begin{tikzcd}
\overline{\mathcal{K}}^\circ_{[a,b]} \arrow[r] & \overline{\mathcal{K}}^\circ_{[b,b]} \\
\mathcal{K}^\circ_{[a,b]} \arrow[r] \arrow[u] & \mathcal{K}^\circ_{[b,b]} \arrow[u] \\
\mathcal{K}_{[a,1]} \arrow[r] \arrow[u] & \mathcal{K}_{[b,1]} \arrow[u]
\end{tikzcd}$$

produces a map $\overline{\mathcal{K}}_{[a,1]} \to \overline{\mathcal{K}}_{[b,1]}$. These maps are clearly functorial for $a \leq b \leq c < 1$, and allow us to extend the functor $\overline{\mathcal{K}}^\circ$ to $\mathbf{Int}(\mathcal{P})^{\mathrm{opp}} \setminus \{[1,1]\}$.

To extend the functor to $[1,1]$, we let $\mathcal{K}_{[1-,1]}$ (resp. $\overline{\mathcal{K}}_{[1-,1]}$) be the colimit of $\mathcal{K}_{[a,1]}$ (resp. $\overline{\mathcal{K}}_{[a,1]}$) over all $a < 1$. Then $\overline{\mathcal{K}}_{[1-,1]}$ is an algebraic closure of $\mathcal{K}_{[1-,1]}$. We pick an algebraic closure $\overline{\mathcal{K}}_{[1,1]}$ of $\mathcal{K}_{[1,1]}$ fitting inside a commutative square

$$\begin{tikzcd}
\overline{\mathcal{K}}_{[1-,1]} \arrow[r] & \overline{\mathcal{K}}_{[1,1]} \\
\mathcal{K}_{[1-,1]} \arrow[r] \arrow[u] & \mathcal{K}_{[1,1]}. \arrow[u]
\end{tikzcd}$$

We then define the maps $\overline{\mathcal{K}}_{[a,1]} \to \overline{\mathcal{K}}_{[1,1]}$ by the composition $\overline{\mathcal{K}}_{[a,1]} \to \overline{\mathcal{K}}_{[1-,1]} \to \overline{\mathcal{K}}_{[1,1]}$. This extends the functor $\overline{\mathcal{K}}$ and the natural transformation $\mathcal{K} \to \overline{\mathcal{K}}$ to all of $\mathbf{Int}(\mathcal{P})^{\mathrm{opp}}$ as desired. □

4.4. **Abundance of geometric intervals on curves.** In this section we finally verify that, at least on reasonable curves, there is an abundance of geometric intervals. More precisely, we wish to show that any interval is the underlying interval of a geometric interval. We establish this by two constructions. One will notice a similarity with the construction from Example 4.3.1.

Let X be a smooth and separated rigid K-curve and let ℓ be an interval in X. Our first construction associates to ℓ a natural ℓ-field, a sort of 'meromorphic structure sheaf' for ℓ.

Construction 4.4.1. For any subarc $[a,b]$ of ℓ, let us define the ring

$$\mathcal{R}_{[a,b]} = \varinjlim_{\substack{\text{open oc}\\ \text{neighborhood of } [a,b]}} \mathcal{O}_X(U) = \varinjlim_{\substack{\text{affinoid open}\\ \text{oc neighborhood of } [a,b]}} \mathcal{O}_X(U)$$

where the latter equality follows from Proposition 4.1.2. We then have the following simple observation.

Lemma 4.4.2. *The rings $\mathcal{R}_{[a,b]}$ are integral domains, and for $[a,b] \subseteq [a',b']$ the natural restriction maps $\mathcal{R}_{[a',b']} \to \mathcal{R}_{[a,b]}$ are injections.*

Proof. Let us start by showing that $\mathcal{R}_{[a,b]}$ is a domain. By [FK18, Chapter 0, Proposition 3.1.1(2)] it suffices to show that cofinal in the set of open affinoid oc neighborhoods of γ are such open affinoid oc neighborhoods U with $\mathcal{O}_X(U)$ an integral domain. But, cofinal in such affinoid domains are those U which are (smooth and) connected, and for such U the ring $\mathcal{O}_X(U)$ is an integral domain (cf. [Ber90, Corollary 3.3.21]).

Let us now show that the map $\mathcal{R}_{[a',b']} \to \mathcal{R}_{[a,b]}$ for $[a,b] \subseteq [a',b']$ is injective. To see this let $f \in \mathcal{O}_X(U)$ be an element of $\mathcal{R}_{[a',b']}$ with zero image in $\mathcal{R}_{[a,b]}$, where U is a connected oc open neighborhood of $[a,b]$. Then, by definition there exists a connected oc open neighborhood U' of $[a,b]$ contained in U such that $f|_{U'} = 0$. But, this implies that $V(f) \subseteq U$ contains an oc open subset. Therefore (cf. loc. cit.) that $V(f) = U$ and so f equals zero in $\mathcal{O}_X(U)$ and so f is zero in $\mathcal{R}_{[a,b]}$ as desired. □

Let us denote by $\mathcal{K}_{[a,b]}$ the fraction field of $\mathcal{R}_{[a,b]}$. By the above injectivity statement, we see that if $[a,b] \subseteq [a',b']$ then one gets an induced map $\mathcal{K}_{[a',b']} \to \mathcal{K}_{[a,b]}$. Thus, we see that one has a functor

$$\mathcal{K}_\ell \colon \mathbf{Int}(\ell)^{\mathrm{opp}} \to \mathbf{Fields}, \quad [a,b] \mapsto \mathcal{K}_{[a,b]}$$

and thus we obtain an ℓ-field $\mathcal{K}_\ell$ which we call the ℓ-field *associated* to ℓ in X. This finishes our first construction. □ *(Construction 4.4.1)*

We next show that from an algebraic closure of the ℓ-field $\mathcal{K}_\ell$, one can naturally build a geometric structure on ℓ.

Construction 4.4.3. Let $\overline{\mathcal{K}}_\ell$ be an algebraic closure of $\mathcal{K}_\ell$. For a in ℓ, let us pick a geometric point $\overline{a}$ of X anchored at $a^{\max}$ and let us abbreviate $\mathcal{R}_{[a,a]}$ (resp. $\mathcal{K}_{[a,a]}$) to $\mathcal{R}_a$ (resp. $\mathcal{K}_a$). The point $\overline{a}$ defines a geometric point on $\mathrm{Spec}(\mathcal{R}_a)$, denoted $\overline{a}$ as well. As in Example 4.3.1, we pick an étale path $\iota_a \in \pi_1^{\mathrm{alg}}(\mathrm{Spec}(\mathcal{R}_a); \overline{a}, \tilde{a})$ where $\tilde{a}$ is the geometric point $\mathrm{Spec}(\overline{\mathcal{K}}_a) \to \mathrm{Spec}(\mathcal{R}_a)$.

Suppose now that $[a,b] \subseteq \ell$ and that $U = \mathrm{Spa}(A)$ is an affinoid open oc neighborhood of $[a,b]$ in X. We would like to give an isomorphism of fiber functors:

$$\iota^U_{a,b} \colon F_{\overline{a}} \to F_{\overline{b}}$$

where $F_{\overline{a}}$ and $F_{\overline{b}}$ are the natural fiber functors on $\mathbf{UF\acute{E}t}_U$. Note that we have a natural bijection (see the end of §3.2)

$$\pi_1^{\mathrm{alg}}(\mathrm{Spa}(A); \mathrm{Spa}(k(\overline{a})), \mathrm{Spa}(k(\overline{b}))) \xrightarrow{\sim} \pi_1^{\mathrm{alg}}(\mathrm{Spec}(A); \mathrm{Spec}(k(\overline{a})), \mathrm{Spec}(k(\overline{b}))) \tag{4.4.1}$$

But, we also have a diagram

$$\begin{tikzcd}
\mathrm{Spec}(\overline{\mathcal{K}}_a) \arrow[dr] & & & \\
& \mathrm{Spec}(\overline{\mathcal{K}}_{[a,b]}) \arrow[r] & \mathrm{Spec}(\mathcal{R}_{[a,b]}) \arrow[r] & \mathrm{Spec}(A) \\
\mathrm{Spec}(\overline{\mathcal{K}}_b) \arrow[ur] & & &
\end{tikzcd}$$

which gives isomorphisms of fiber functors $j^U_{a,b} : F_{\overline{\mathcal{K}}_a} \simeq F_{\overline{\mathcal{K}}_b}$.

We define $\iota^U_{a,b}$ to be the preimage under the map described in Equation (4.4.1) of the following composition

$$F_{\mathrm{Spec}(k(\overline{a}))} \xrightarrow{\iota_a} F_{\mathrm{Spec}(\overline{\mathcal{K}}_a)} \xrightarrow{j^U_{a,b}} F_{\mathrm{Spec}(\overline{\mathcal{K}}_b)} \xrightarrow{\iota_b^{-1}} F_{\mathrm{Spec}(k(\overline{b}))}$$

It is clear that $\iota^U_{a,b}$ doesn't depend on the affinoid open oc neighborhood U of $[a,b]$. So then, for an open oc neighborhood U of $[a,b]$ we can, as at the beginning of the proof, find an affinoid neighborhood U' of $[a,b]$ contained in U. We then set $\iota^U_{a,b}$ to be the image of $\iota^{U'}_{a,b}$. As above indicated above, this definition is independent of the choice of U'.

The set of isomorphisms $\{\iota^U_{a,b}\}$, where now $[a,b] \subseteq \ell$ is arbitrary and U is allowed to either be an affinoid or open neighborhood of $[a,b]$, is easily seen to satisfy the desired conditions. Thus, the $\iota^U_{a,b}$ together define a geometric interval $\overline{\ell}$ with underlying interval ℓ.

Combining these two constructions we obtain the following.

Proposition 4.4.4. *Let X be a smooth and separated rigid K-curve and let ℓ be an interval in X. Then, there exists a geometric interval $\overline{\ell}$ with underlying interval ℓ.*

Moreover, we can take this one step further and show that often all geometric structures an interval ℓ come from this construction.

Proposition 4.4.5. *Let X be a smooth and separated rigid K-curve and let ℓ be an interval in X whose endpoints are not classical points. Then, up to equivalence every geometric structure comes from Construction 4.4.3.*

Proof. Fix $[a,b] \subseteq \ell$ and let $\{U_i = \mathrm{Spa}(R_i)\}_{i\in I}$ be a cofinal family of affinoid oc neighborhoods of $[a,b]$. Then $\mathcal{R}_{[a,b]} = \varinjlim_{i\in I} R_i$, and hence

$$\pi_1^{\mathrm{alg}}(\mathrm{Spec}(\mathcal{R}_{[a,b]}); \overline{a}, \overline{b}) = \varprojlim_{i\in I} \pi_1^{\mathrm{alg}}(\mathrm{Spec}(R_i); \overline{a}, \overline{b}) = \varprojlim_{i\in I} \pi_1^{\mathrm{alg}}(U_i; \overline{a}, \overline{b}),$$

(see e.g. [KL15, Remark 1.2.9]). The elements $\iota^{U_i}_{a,b}$ form an element of the right hand side by Axiom 1, and hence we obtain an element $\iota_{a,b} \in \pi_1^{\mathrm{alg}}(\mathrm{Spec}(\mathcal{R}_{[a,b]}); \overline{a}, \overline{b})$.

Since neither a nor b is classical and X is smooth, the ring $\mathcal{R}_{[a,b]}$ is a field, i.e. $\mathcal{R}_{[a,b]} = \mathcal{K}_{[a,b]}$. Therefore $\iota_{a,b} \in \pi_1^{\mathrm{alg}}(\mathrm{Spec}(\mathcal{K}_{[a,b]}); \overline{a}, \overline{b})$ corresponds to an identification of the algebraic closures of $\mathcal{K}_{[a,b]}$ in $k(\overline{a})$ and $k(\overline{b})$. We denote by $\overline{\mathcal{K}}_{[a,b]}$ this common value.

If $a \leq b \leq c$ in ℓ, then $\mathcal{K}_{[a,c]} \to \mathcal{K}_{[a,b]} \to k(\overline{a})$ shows that $\overline{\mathcal{K}}_{[a,c]}$ is contained in $\overline{\mathcal{K}}_{[a,b]}$ as subfields of $k(\overline{a})$. Similarly, $\overline{\mathcal{K}}_{[a,c]} \subseteq \overline{\mathcal{K}}_{[b,c]}$ as subfields of $k(\overline{c})$. Moreover, Axiom 2 implies that the following diagram commutes

$$\begin{tikzcd}
k(\overline{a}) & & k(\overline{b}) & & k(\overline{c}) \\
& \overline{\mathcal{K}}_{[a,b]} \arrow[ul] \arrow[ur] & & \overline{\mathcal{K}}_{[b,c]} \arrow[ul] \arrow[ur] & \\
& & \overline{\mathcal{K}}_{[a,c]} \arrow[ul] \arrow[ur] \arrow[uull, bend left] \arrow[uurr, bend right] & &
\end{tikzcd}$$

For $[a,b] \supseteq [a',b']$, we define $\overline{\mathcal{K}}_{[a,b]} \to \overline{\mathcal{K}}_{[a',b']}$ as the composition $\overline{\mathcal{K}}_{[a,b]} \to \overline{\mathcal{K}}_{[a,b']} \to \overline{\mathcal{K}}_{[a',b']}$, which is the same as the composition $\overline{\mathcal{K}}_{[a,b]} \to \overline{\mathcal{K}}_{[a',b]} \to \overline{\mathcal{K}}_{[a',b']}$. This way we obtain an algebraic closure $\overline{\mathcal{K}} = \{\overline{\mathcal{K}}_{[a,b]}\}$ of the ℓ-field $\{\mathcal{K}_{[a,b]}\}$. It is straightforward to check that the geometric interval defined by this algebraic closure is equivalent to $\overline{\ell}$. ☐

Remark 4.4.6. In the above proof, we used the following observation: if ℓ is an interval in X, then only its endpoints can be classical points. To show this, since X is good we may assume that X is affinoid and by Theorem 4.1.3 we may assume that $[X]$ is uniquely arc connected. But, if T is a uniquely arc connected space, $\ell \subseteq T$ is an arc, and $t \in \ell$ is an interior point, then $T \setminus \{t\}$ is disconnected. However, removing a classical point from a connected smooth curve does not make it disconnected (see [Han20, Corollary 2.7]).

Lastly, we combine Lemma 4.3.7 with Proposition 4.4.5 to extend geometric structures from rays to arcs. The corollary below will play an important technical role in our verification that two, ostensibly very different, notions of what we call 'geometric coverings' agree.

Corollary 4.4.7. *Let X be a smooth and separated rigid K-curve and let $i\colon [0,1] \to [X]$ be a parameterized arc. Set $\gamma = i([0,1])$ and $\rho = i([0,1))$. Then, for any structure of a geometric ray $\overline{\rho}$ there exists a structure of a geometric arc $\overline{\gamma}$ on γ whose induced structure on ρ coincides with $\overline{\rho}$ up to equivalence.*

Proof. It is enough to treat the geometric ray induced by $\overline{\rho}$ on $i([\frac{1}{2},1))$. This way, we may assume that no points of ρ are classical points. By Proposition 4.4.5, we may assume that the geometric structure $\overline{\rho}$ comes from an algebraic closure $\overline{K}^{\rho}$ of the $[0,1)$-field $K^{\rho} = \{K_{[a,b]}\}$ on ρ. By Lemma 4.3.7, the algebraic closure $\overline{K}^{\rho}$ extends to an algebraic closure $\overline{K}^{\gamma}$ of K^{γ}. The induced structure of a geometric arc $\overline{\gamma}$ on γ has the required property. □

5. Geometric coverings

In this section, we define geometric coverings, show that they satisfy reasonable geometric properties, and then show that the category of such coverings is a tame infinite Galois category in the sense of [BS15].

In short, a geometric covering is a (étale and partially proper) map of rigid K-spaces that has unique lifting of geometric arcs. This is not quite true, as one needs this property to hold after base change to (nice) curves.

5.1. Lifting of geometric intervals.

Before we define our notion of geometric covering we will need some background results about liftings of geometric intervals. We show that under very mild conditions any finite étale map has unique lifting of geometric arcs. Finally, we show that the property unique lifting of geometric arcs and AVC (Definition 4.1.4) are essentially equivalent for étale and partially proper morphisms of reasonable curves.

Definition 5.1.1. Let $f\colon X' \to X$ be a map of rigid K-spaces and let $\overline{\ell}$ be a geometric interval in X. By a *lifting* of $\overline{\ell}$ to X' we shall mean a geometric interval $\overline{\ell}'$ in X' such that $[f]$ maps ℓ' homeomorphically onto ℓ and $\overline{\ell}$ is equal to the geometric interval $f(\overline{\ell}')$.

It is useful to note that with this definition, if $\overline{\ell}'$ is a lift of the geometric arc $\overline{\ell}$ then for each point x' of ℓ' if we set $x = f(x')$ then the geometric point $\overline{x}'$ (afforded to us by the geometric structure on ℓ') is a lift of the geometric point $\overline{x}$ (with a similar comment). So, in particular, we see $\overline{x}'$ is a point of $Y_{\overline{x}}$ is a canonical way.

We define 'uniqueness' of geometric lifts as follows.

Definition 5.1.2. A map $Y \to X$ of rigid K-spaces satisfies the property of *unique lifting of geometric arcs* if for every geometric arc $\overline{\ell}$ in X with left geometric endpoint $\overline{x}$ and every lifting $\overline{x}'$ of $\overline{x}$ to Y, there exists a unique lifting of $\overline{\ell}$ to Y with left geometric endpoint $\overline{x}'$.

We now observe that unique lifting of geometric arcs implies unique lifting of geometric rays (defined in the analogous way).

Proposition 5.1.3. *Let $Y \to X$ be a map of rigid K-spaces which has unique lifting of geometric arcs. Then, $Y \to X$ has unique lifting of geometric rays.*

Proof. Let $\overline{\rho}$ be a geometric ray in X with left geometric endpoint $\overline{x}$. Let us note then that for all $a < x$ in ρ one has the induced geometric structure $\overline{\rho}_a$ on $\rho_a := [a,x]$, and thus by the uniqueness of lifting of geometric arcs one has a unique lift of $\overline{\rho}_a$ to a geometric arc $\overline{\rho}'_a$ in Y. By uniqueness it is clear that if $a_1 < a_2$ then the induced geometric arc structure from $\overline{\rho}'_{a_1}$ on the sublifting of ρ_{a_2} must equal the lifting $\overline{\rho}'_{a_2}$. Therefore we can uniquely concatenate these to a lift of the geometric ray $\overline{\rho}$, which is clearly unique. □

We would now like to show that one can always lift geometric arcs uniquely along (essentially) any finite étale morphisms.

Proposition 5.1.4. *Let $Y \to X$ be a finite étale map of rigid K-spaces where X is taut. Then $Y \to X$ satisfies unique lifting of geometric arcs.*

Proof. Let $\overline{\gamma}$ be a geometric arc in X, with geometric left endpoint $\overline{x}$, and let $\overline{y} \in Y_{\overline{x}}$ be a lifting of $\overline{x}$ (which one can identify with an element of the point set $Y_{\overline{x}}$ as in §3.2). We denote by x and y the respective anchor points. For every $t \in \gamma$, consider the geometric point

$$\overline{y}(t) = \iota^X_{x,t}(\overline{y}) \in Y_{\overline{t}},$$

and let $y(t) \in [Y]$ be the image of its anchor point. In particular, $\overline{y}(x) = \overline{y}$.

We claim that the map $\gamma \to [Y]$ given by $t \mapsto y(t)$ is continuous. Since this map is clearly a set theoretic lift of the map $\gamma \to [X]$ it is injective and hence its continuity implies its image is an arc γ'_0 in $[Y]$ mapping homeomorphically onto γ.

To verify that this map is continuous let $t_0 \in \gamma$, let $y_0 = y(t_0)$, and let U be an open neighborhood of y_0 in $[Y]$. Let us note that by Proposition 3.7.6 applied to $[f]$ there exists an open neighborhood V of t_0 such that the connected component Y_0 of $[f]^{-1}(V)$ containing y_0 is open and contained in U. Let us set $W = \mathrm{sep}_X^{-1}(V)$ and $C = \mathrm{sep}_Y^{-1}(Y_0)$. Thus C is an overconvergent open subset of Y which is a connected component of Y_W.

Let $[a, b] \subseteq \gamma$ be subarc containing t_0 in its interior and contained in V. By naturality of ι^W with respect to the morphism $C \to Y_W$ in $\mathbf{FÉt}_W$, for every $t \in [a, b]$, the map $\iota^W_{t_0,t} \colon Y_{\overline{t}_0} \to Y_{\overline{t}}$ maps $C_{\overline{t}_0}$ into $C_{\overline{t}}$. By the axioms of a geometric structure, $\overline{y}(t) = \iota^W_{t_0,t}(\overline{y}(t_0)) \in C_{\overline{t}}$, so we have $y(t) \in \mathrm{sep}_Y(C) = Y_0 \subseteq U$ for $t \in [a, b]$, and so $t \mapsto y(t)$ is continuous.

Suppose that $\overline{\gamma}'$ is a lifting of $\overline{\gamma}$ with geometric left endpoint $\overline{y}$. Let $\mathfrak{U}$ be the collection of open subsets $U' \subseteq Y$ such that $U' \to U = f(U')$ is finite étale. Then by Proposition 3.7.5 the set $\mathfrak{U}$ is a γ-basis. Let $[a', b']$ be a subarc of γ' such that $[a', b']$ has some $U' \in \mathfrak{U}$ as an open oc neighborhood and let $[a, b]$ be the corresponding subarc of γ. It follows that $U = f(U')$ is an open oc neighborhood of $[a, b]$ and $f \colon U' \to U$ is finite étale. Let $Z' \to U'$ be a finite étale cover, let $Z = Z' \times_U U'$, and let $\Delta = (\mathrm{id}, f) \colon Z' \to Z$ be the natural map. Then Z is a finite étale cover of U' and Δ is a morphism in the category $\mathbf{FÉt}_{U'}$. We therefore have the commutative diagram

$$\begin{tikzcd}
Z'_{\overline{a}'} \arrow[r, "\iota^{U'}_{a',b'}(Z')"] \arrow[d, "\Delta"'] & Z'_{\overline{b}'} \arrow[d, "\Delta"] \\
Z_{\overline{a}'} \arrow[r, "\iota^{U'}_{a',b'}(Z)"] \arrow[d, equal] & Z_{\overline{b}'} \arrow[d, equal] \\
Z'_{\overline{a}} \arrow[r, "\iota^{U}_{a,b}(Z' \to U)"'] & Z'_{\overline{b}}
\end{tikzcd} \tag{5.1.1}$$

(the top square commutes by naturality of $\iota^{U'}_{a',b'}$, and the bottom one by definition of a lifting of an arc). Since the vertical arrows are injective, this shows that the top map $\iota^{U'}_{a',b'}(Z')$ is uniquely determined by the diagram. By Lemma 4.2.4, this shows uniqueness of the system of étale paths $\iota^{U'}_{a',b'}$ given the underlying arc γ' and lifting of geometric points $\overline{a}' \to \overline{a}$ for $a' \in \gamma'$ mapping to $a \in \gamma$.

Moreover, applying the above to $U' = Z' = Y$, so that $Z = Y \times_X Y$, and the arc $[y, t']$ for some $t' \in \gamma'$ lying over $t \in \gamma$, we obtain a commutative square

$$\begin{tikzcd}
\{\overline{y}\} \arrow[r] \arrow[d] & \{\overline{t}'\} \arrow[d] \\
Y_{\overline{x}} \arrow[r, "\iota^X_{x,t}"'] & Y_{\overline{t}}.
\end{tikzcd}$$

This shows that $\overline{t}' = \iota^X_{x,t}(Y)(\overline{y}) = \overline{y}(t)$ and $t' = y(t)$. Taking images in $[Y]$, we see that the underlying arc γ' equals γ'_0 and that the liftings of geometric points of $\overline{\gamma}$ are also uniquely determined.

It remains to construct the structure of a geometric arc on $\gamma' = \gamma'_0$ with $\overline{t}' = \overline{y}(t)$ as a chosen lift of the geometric point $\overline{t}$. To this end, again by Lemma 4.2.4 it suffices to construct the maps $\iota^{U'}_{a,b}$ for $U' \in \mathfrak{U}$ so that Axioms 1 and 2 of Definition 4.2.1 are satisfied. For $[a', b']$, $U' \in \mathfrak{U}$ and $U = f(U') \subseteq X$ as before we need to check that for every finite étale $Z' \to U'$ the map

$$\iota^U_{a,b}(Z' \to U) \colon Z'_{\overline{a}} \to Z'_{\overline{b}}$$

maps $Z'_{\overline{a}'}$ to $Z'_{\overline{b}'}$. Since $Z'_{\overline{a}'}$ is the preimage of $\overline{a}' \in U_{\overline{a}}$ under $Z'_{\overline{a}} \to U_{\overline{a}}$, and similarly for b, this assertion follows from the functoriality of $\iota^U_{a,b}$ with respect to the morphism $Z' \to U'$ in $\mathbf{F\acute{E}t}_U$. This defines the maps $\iota^{U'}_{a',b'}$. Axioms 1 and 2 of the definition of a geometric arc are easy to verify. □

Using this we can show that uniqueness of geometric arc liftings holds for arbitrary étale and partially proper maps.

Corollary 5.1.5. *Let $f \colon Y \to X$ be an étale and partially proper morphism of rigid K-spaces where X is taut. Let $\overline{\gamma}$ be a geometric arc in X, and let $\overline{x}'$ be a lifting of the left geometric endpoint $\overline{x}$ of $\overline{\gamma}$ to Y. Then, a lifting of $\overline{\gamma}$ to Y with left geometric endpoint $\overline{x}'$ is unique if it exists.*

Proof. Let $\overline{\gamma}'_i$ $(i = 0, 1)$ be two liftings of $\overline{\gamma}$ with left geometric endpoint $\overline{x}'$. Assume first that the underlying arcs γ'_i agree, call this common arc γ'. Let us then note that by Proposition 3.7.5, there exists a finite partition $\gamma' = \bigcup_{j=1}^m \gamma'_j$ and open oc neighborhoods U_i of γ'_j with the property that $f|_{U_j} \colon U_j \to f(U_j)$ is finite étale. Let $\gamma_j = [f](\gamma'_j)$ be the induced subarcs of γ. Note then that by Proposition 5.1.4 and induction on j one has that the geometric structure on each γ'_j induced by $\overline{\gamma}'_i$ must agree. This implies that $\overline{\gamma}'_1 = \overline{\gamma}'_2$ as desired.

Suppose now that γ'_i for $i = 0, 1$ are arbitrary lifts of $\overline{\gamma}$. Let δ' be the connected component $\gamma'_0 \cap \gamma'_1$ containing the anchor point x' of $\overline{x}'$. Since X is taut and f is partially proper we know that Y is also taut by Proposition 3.3.3, and thus $[Y]$ is Hausdorff by Proposition 2.3.2 (a). Thus, δ' is a (closed, possibly degenerate) subarc of both γ'_0 and γ'_1, mapping homeomorphically via $[f]$ onto a subarc δ of γ. Let $\overline{\delta}'_i$ $(i = 0, 1)$ be the induced geometric structure on δ' inherited from $\overline{\gamma}'_i$. Then $\overline{\delta}'_i$ are two geometric lifts of $\overline{\delta}$ with the same underlying arc. Therefore, by the previous paragraph, we have $\overline{\delta}'_0 = \overline{\delta}'_1$.

Denote by $\overline{z}'$ common right geometric endpoint of $\overline{\delta}'_0 = \overline{\delta}'_1$, lying over the right geometric endpoint $\overline{z}$ of δ. If z is the right endpoint of γ, we are done. In any case, by Proposition 3.7.5 there exists an open oc neighborhood U' of z' such that $f \colon U' \to U$ (where $U = f(U')$) is finite étale. If $[x, z] \neq \gamma$, then there exists an a in the interior of $\gamma \cap \mathrm{sep}_X(U)$ with $a \geq z$ such that the preimages of $[z, a]$ in $\mathrm{sep}_Y(U) \cap \gamma'_i$ are subarcs $[z', a'_i]$ $(i = 0, 1)$ with $a'_0 \neq a'_1$. However, this contradicts Proposition 5.1.4 applied to the map $U' \to U$, the geometric subarc of γ supported on $[z, a]$, and the lifting $\overline{z}'$ of $\overline{z}$ to U'. □

While the above pertains to geometric arcs, one can use the above in conjunction with Proposition 4.4.4 and Proposition 5.1.3 to obtain the following useful topological corollary.

Corollary 5.1.6. *Let $f \colon Y \to X$ be a surjective finite étale morphism with X a smooth and separated rigid K-curve. Then, if ℓ is an interval in X, then there exists an interval ℓ' in Y such that $[f]$ maps ℓ' homeomorphically onto ℓ.*

More surprising is the following generalization of this result.

Corollary 5.1.7. *Let $f \colon Y \to X$ be a surjective finite étale morphism where X is a smooth and separated rigid K-curve. Then, for any interval ℓ in X, there exist finitely many intervals ℓ_i*

$(1 \leq i \leq m)$ in Y lifting ℓ such that

$$[f]^{-1}(\ell) = \bigcup_{i=1}^{m} \ell_i.$$

Proof. First, assume that ℓ is an arc. Fix a geometric arc structure $\overline{\ell}$ on ℓ which exists by Proposition 4.4.4. Let us also fix a parameterization $i\colon [0,1] \to [X]$ for ℓ. Let $\overline{x}$ be the left geometric endpoint of $\overline{\ell}$ and consider the geometric fiber $Y_{\overline{x}}$. This contains n geometric points of Y where n is the degree of f. Now, consider the unique geometric arcs $\overline{\ell}_i$ starting from these geometric points and lifting $\overline{\ell}$. Obviously, we have $[f]^{-1}(\ell) \supset \bigcup_j \ell_j$. Conversely, let w be a point of $[f]^{-1}(\ell)$. Pick a geometric point $\overline{w}$ above w that lifts the geometric point of $\overline{\ell}$ that has the maximal point in X above $v = [f](w) \in \ell$ as its anchor point. We can lift $\overline{\ell}|_{[0,v]}$ and $\overline{\ell}|_{[v,1]}$ uniquely to geometric arcs in Y having $\overline{w}$ as a point. The concatenation of the two, however, produces a geometric arc in Y lifting $\overline{\ell}$. This shows $[f]^{-1}(\ell) \subset \bigcup_j \ell_j$.

To prove the geometric ray case, we write the ray as an increasing union of arcs, put geometric arc structures on those arcs in a compatible way and apply the procedure of the first paragraph to those arcs. $\square$

We next show that AVC is strong enough to imply unique lifting of geometric arcs for étale and partially proper morphisms. This provides a vital link between topological and rigid-geometric notions of 'arc lifting' that will be the basis of many of ours later proofs, as well as the utility and ubiquity of morphisms with an arc lifting property.

Proposition 5.1.8. *Let $f\colon Y \to X$ be a map of rigid K-spaces where X is assumed taut, and let $\overline{\gamma}$ be a geometric arc in X with underlying parameterized arc $i\colon [0,1] \to [X]$. Suppose that*

- *f is étale,*
- *f is partially proper,*
- *$[f]$ satisfies AVC with respect to every parameterized subarc of $i\colon [0,1] \to [X]$.*

Then f satisfies unique lifting with respect to $\overline{\gamma}$.

Proof. In this proof, we tacitly use the uniqueness result of Corollary 5.1.5 several times. Let $\overline{x}$ be an endpoint anchored at $i(0)$, and let $\overline{x}'$ be a lift of $\overline{x}$ to Y. For each t in $[0,1]$ let $\overline{\gamma}_t$ be the induced geometric arc structure to $i([0,t])$. Let S be the the set of all $t \in [0,1]$ for which there exists a lifting $\overline{\widetilde{\gamma}}_t$ of $\overline{\gamma}_t$ to a geometric arc in Y with left endpoint $\overline{x}'$. Note that $0 \in S$ and if $t \in S$ then $[0,t] \subseteq S$. Thus, to show that $S = [0,1]$ it suffices to show that

- if $t \in S$ for all $t < t_0$ then $t_0 \in S$,
- if $1 \neq t_0 \in S$ then $t_0 + \varepsilon \in S$ for some $\varepsilon > 0$.

To see the first claim let us then note that the union of the underlying arcs $\widetilde{\gamma}_t$ for $t < t_0$ lifting γ_t is precisely a lift of the parameterized ray $i|_{[0,t_0)}$. By our assumption on AVC relative to parameterized subarcs of i we can extend lift of the parameterized arc $i|_{[0,t_0]}$. Let y' be the point in our lift living over $y = i(t_0)$. Note that by Proposition 3.7.5 there exists an open oc neighborhood U of y' such that $f\colon U \to f(U)$ is finite étale. Since f is étale and partially proper we know that y lies in $\mathrm{int}_X(f(U))$, and thus $i([0,t_0]) \cap \mathrm{sep}_X(V)$ contains $i([t_1,t_0])$ for some $t_1 < t_0$. But, since $f\colon U \to f(U)$ is finite étale, we know by Proposition 5.1.4 that there exists a unique lift of the geometric arc structure on $i([t_1,t_0])$ to a geometric arc of U. This clearly glues to give a geometric lift of $\overline{\gamma}_{t_0}$ as desired.

To show the second claim let $y = i(t_0)$ and let $\overline{\widetilde{\gamma}}_{t_0}$ be our chosen lift of $\overline{\gamma}_{t_0}$. Let y' be the point of $\widetilde{\gamma}_{t_0}$ lying over y. Again by Proposition 3.7.5 there exists open oc neighborhoods U of y' such that $f\colon U \to f(U)$ is finite étale. Since f is étale and partially proper we have that $y \in \mathrm{int}_X(f(U))$ and thus we have that $i([0,1]) \cap \mathrm{sep}_X(f(U))$ contains $i([t_1, t_0 + \varepsilon])$ for some $\varepsilon > 0$. One then proceeds as in the first case. $\square$

Finally, we show that the converse to this result holds in the case of curves.

Proposition 5.1.9. *Let $f\colon Y \to X$ be an étale and partially proper morphism where X is a smooth and separated rigid K-curve. Then, $[f]$ satisfies AVC if and only if f has unique lifting of geometric arcs.*

Proof. If f satisfies AVC then we have already shown in Proposition 5.1.8 that f satisfies unique lifting of geometric arcs. Conversely, let $i\colon [0,1] \to [X]$ be an arc and let $s\colon [0,1) \to [Y]$ be a lifting of the ray. Set $\rho' = s([0,1))$ and $\rho = i([0,1))$. Let us note that since Y is a smooth and separated rigid K-curve we can by Proposition 4.4.4 find a geometric structure $\overline{\rho}'$ on ρ'. Since $[f]$ maps ρ' homeomorphically onto ρ we note then that $f(\overline{\rho}')$ is a geometric structure on ρ. By Corollary 4.4.7 we can extend this to a geometric structure $\overline{\gamma}$ on γ. We then note that by unique lifting of geometric arcs that there exists a lift $\overline{\gamma}'$ of $\overline{\gamma}$.

We claim that γ' is an arc lift of γ extending the ray lift ρ' of ρ. To see this, for each $x < 1$ set

- γ_x to be the arc $i([0,x])$, and $\overline{\gamma}_x$ to be the geometric structure on γ_x induced by $\overline{\gamma}$,
- γ'_x to be the subarc of γ' lifting γ_x, and $\overline{\gamma}'_x$ the geometric structure on γ'_x induced by $\overline{\gamma}'$,
- ρ_x to be the subarc $s([0,x])$ of ρ, and $\overline{\rho}_x$ the geometric structure on ρ_x induced by $\overline{\rho}$.

Note then that $\overline{\rho}_x$ and $\overline{\gamma}'_x$ are both lifts of the geometric arc $\overline{\gamma}_x$. By previous comment we know that f satisfies uniqueness of liftings of geometric arcs, and thus $\overline{\rho}_x = \overline{\gamma}'_x$ and so, in particular, we see that

$$\rho = \bigcup_{x>0} \rho_x = \bigcup_{x>0} \gamma'_x$$

which implies that γ' is, indeed, an arc lift of γ extending ρ as desired. □

5.2. **The definition of geometric coverings and basic properties.** We now seek to define our notion of geometric coverings. In essence, such coverings are étale and partially proper maps which satisfy unique lifting of geometric arcs. But, since the theory of geometric arcs and their lifts only works exceptionally well on smooth and separated rigid K-curves we shall have to use such spaces as the basis of our test objects. Thanks to the contents of Appendix A, such test objects are sufficiently abundant.

But, thanks to Proposition 5.1.9 we may also think of geometric coverings as étale maps which which satisfy both an algebro-geometric valuative criterion (in the form of being partially proper) but also a topological valuative criterion (in the form of satisfying AVC). Again, this requirement will be required to hold precisely over smooth and separated rigid K-curves.

We begin by defining, more rigorously, our test objects of interest.

Definition 5.2.1. Let X be a rigid K-space. A *test curve* in X is the data of a a non-archimedean field extension L of K, a smooth and separated L-curve C, and a map $C \to X$ of adic spaces over K.

We note that if $Y \to X$ is étale and partially proper, and $C \to X$ is a test curve in X, then $Y_C \to Y$ is a test curve in Y.

We now define our notion of geometric coverings. The equivalence of the two conditions follows from Proposition 5.1.9.

Definition 5.2.2. A morphism $Y \to X$ of rigid K-spaces is a *geometric covering* if it is étale, partially proper, and satisfies one of the following two equivalent properties:

(a) for all test curves $C \to X$ the map $Y_C \to C$ has unique lifting of geometric arcs (Definition 5.1.2),
(b) for all test curves $C \to X$ the map $[Y_C] \to [C]$ satisfies AVC (Definition 4.1.4).

We denote the full subcategory of $\mathbf{\acute{E}t}_X$ consisting of geometric coverings of X by $\mathbf{Cov}_X$.

Remark 5.2.3. Note that by Remark 4.1.6 and Corollary 5.1.5 the uniqueness part of (a) and (b) above are automatic.

We will now proceed to show that geometric coverings are closed under many natural operations, and are therefore quite abundant. The first substantive class is singled out by Proposition 5.1.4.

Proposition 5.2.4. *Let $Y \to X$ be a finite étale morphism of rigid K-spaces. Then, $Y \to X$ is a geometric covering.*

Remark 5.2.5. One can also prove that finite étale maps satisfy AVC without the need for geometric arcs, using the corresponding fact about proper continuous maps with discrete fibers (see Remark 4.1.5).

Thus, one sees that the category $\mathbf{F\acute{E}t}_X$ is a full subcategory of $\mathbf{Cov}_X$. By the result below, it also contains the category $\mathbf{UF\acute{E}t}_X$ of disjoint unions of rigid K-spaces finite étale over X.

Proposition 5.2.6. *Let X be a rigid K-space and let $\{Y_i \to X\}$ be a collection of morphisms of rigid K-spaces. Let us set $Y = \coprod_i Y_i$ with its natural map $Y \to X$. Then, $Y \to X$ is a geometric covering if and only if $Y_i \to X$ is a geometric covering.*

In particular, one sees that for a morphism $Y \to X$ of rigid K-spaces to be a geometric covering it is necessary and sufficient to check that for every connected component Y_i of Y that $Y_i \to X$ is a geometric covering.

Proposition 5.2.7. *The property of being a geometric covering is étale local on the target.*

Since $\mathbf{Cov}_X$ contains $\mathbf{UF\acute{E}t}_X$ the étale local nature of geometric coverings implies that $\mathbf{Cov}_X$ in fact contains the étale stackification of $\mathbf{UF\acute{E}t}_X$. This is significant since it implies that $\mathbf{Cov}_X$ contains not only the covering spaces considered in [dJ95b] but also many natural generalizations of those covering spaces (see §6).

To prove Proposition 5.2.7 we will require a few intermediary results which are of independent interest. The first says that the notion of geometric covering is stable under the most basic operations of pullback, composition, and cancellation.

Proposition 5.2.8. *Let X be a rigid K-space. Then, the following statements are true.*

- *(a)* (Composition) *If $f\colon Y \to X$ and $g\colon Z \to Y$ are geometric coverings, then $g \circ f$ is a geometric covering.*
- *(b)* (Pullback) *If $Y \to X$ is a geometric covering and $X' \to X$ is any morphism of rigid K-spaces, then $Y_{X'} \to X'$ is a geometric covering.*
- *(c)* (Change of base field) *If $Y \to X$ is a geometric covering and L is a non-archimedean extension of K, then $Y_L \to X_L$ is a geometric covering.*
- *(d)* (Cancellation) *Let X be a rigid K-space, $Y \to X$ be a geometric covering, and $Y' \to X$ be separated and étale morphism. Then, any X-morphism $Y \to Y'$ is a geometric covering.*

Proof. To see (a) we note that since the properties of étale and partially proper are preserved under composition, it suffices to show that for any test curve $C \to X$ in X that the map $Z_C \to C$ satisfies unique lifting of geometric arcs. But, note that since $Z_C = Z_{Y_C}$, and Y_C is a test curve, one quickly shows that since geometric arcs may be uniquely lifted along $Z_{Y_C} \to Y_C$ and $Y_C \to C$, that they may be uniquely lifted along $Z_C \to C$.

To see (b), we note that since $Y_{X'} \to X'$ is étale and partially proper, it suffices to show that for all test curves $C \to X'$ in X' one has that $(Y_{X'})_C \to C$ satisfies lifting of geometric arcs. But, since $(Y_{X'})_C$ is canonically isomorphic to Y_C as C-spaces, the claim easily follows. The proof of (c) is identical.

For (d), note that by the cancellation principle (see Notation and Conventions) it suffices to check that $\Delta_{Y'/X}\colon Y' \to Y' \times_X Y'$ is a geometric covering. But, since $Y' \to X$ is separated and étale, we know that $\Delta_{Y'/X}$ is a clopen embedding (see [Hub96, Proposition 1.6.8]) and so is a geometric covering by Proposition 5.2.4. □

The other result we shall need to prove Proposition 5.2.7 is the following result which, colloquially, says that the category $\mathbf{Cov}_X$ is closed under taking images.

Proposition 5.2.9. *Let X be a rigid K-space and let $Y \to Y'$ be surjective map of rigid X-spaces. Assume that $Y \to X$ is a geometric covering and that $Y' \to X$ is separated and étale. Then, $Y' \to X$ is a geometric covering.*

Proof. We note that $Y' \to X$ is partially proper, by Lemma 3.7.4. It remains to check that for any test curve $C \to X$ in X that the map $[Y'_C] \to [C]$ satisfies AVC. To ease notation we note that we may assume that X is a test curve and show that AVC holds for $[Y'] \to [X]$. Let $i\colon [0,1] \to [X]$ be an arc and $s'\colon [0,1) \to [Y']$ be a lift of $[0,1)$. By Proposition 5.2.8 (d) we know that $Y \to Y'$ is a geometric covering of smooth and separated rigid K-curves. By combining Proposition 4.4.4 and Proposition 5.1.3 we see that one can, in particular, lift topological rays along $[Y'] \to [Y]$. Thus, one can lift s' to a lift $s\colon [0,1) \to [Y]$. But, since $Y \to X$ is a geometric covering we know that AVC holds for $[Y] \to [X]$, and thus we can lift i to an arc on $[Y]$ extending s. Looking at the composition of this lift with the map $[Y] \to [Y']$ we obtain a lifting of i to $[Y']$ extending s' as desired. □

We now are able to prove Proposition 5.2.7.

Proof of Proposition 5.2.7. By Corollary 3.8.3 we reduce to showing that if $Y \to X$ is a morphism of rigid K-spaces, then whether this map is a geometric covering can be checked on a finite étale Galois cover and can be checked on an affinoid open cover. By Proposition 3.7.1 and Proposition 3.7.3 we know that $Y \to X$ is an étale and partially proper map, and it remains to check that AVC holds for the base change of Y to any test curve $C \to X$.

The claim concerning Galois covers follows easily from previously proven facts. Indeed, denote by $X' \to X$ the Galois cover such that $Y_{X'} \to X'$ is a geometric covering. Since $X' \to X$ is a geometric covering by Proposition 5.2.4 we see from Proposition 5.2.8 (a) that the composition $Y_{X'} \to X$ is a geometric covering. The map $Y_{X'} \to Y$ is surjective, and since $Y \to X$ is étale and partially proper we see by Proposition 5.2.9 that $Y \to X$ is a geometric covering as desired.

To see the claim concerning affinoid open covers, let $\{U_j\}_{j\in J}$ be an affinoid open cover of X such that $Y_{U_j} \to U_j$ is a geometric covering for all j. Taking an affinoid refinement of $\{U_j \times_X C\}$ and replacing X with C, we are then reduced to showing that if X is a smooth and separated curve and $Y \to X$ is an étale and partially proper map, whether this map satisfies AVC can be checked on an affinoid open cover. Since for any arc $\gamma \subseteq [X]$, its preimage $\mathrm{sep}_X^{-1}(\gamma)$ is quasi-compact, we may also assume that X is quasi-compact and the index set J is finite.

Let us note that for every arc γ in X and every affinoid open $U \subseteq X$, the intersection $[U] \cap \gamma$ is closed (because $[U]$ is closed in $[X]$) and has finitely many connected components (by Lemma 5.2.10 below). Let $i\colon [0,1] \to [X]$ be a parameterized arc, and let $s\colon [0,1) \to [Y]$ be a lifting of $i|_{[0,1)}$. Suppose first that $i([0,1]) \subseteq [U_j]$ for some $j \in J$. Since $[Y_{U_j}] \to [U_j]$ satisfies AVC, we can extend s to a map $[0,1] \to [Y_{U_j}]$. Since $[Y_{U_j}] = [Y] \times_{[X]} [U_j]$ by Proposition 2.3.5, we see that $[Y] \to [X]$ satisfies AVC with respect to i.

In any case, $[0,1]$ is covered by the finitely many closed subsets $i^{-1}([U_j])$, and each of them has finitely many connected components, so one of them contains $[1-\varepsilon, 1]$ for some $\varepsilon > 0$. In order to check AVC for a parameterized arc i, we may replace it with its restriction to $[1-\varepsilon, 1]$ for any $\varepsilon > 0$, and then we conclude by our first observation. □

The following lemma, used in the above proof, is the main reason we only use arcs in test curves in our theory of geometric coverings. Indeed, the analogous assertion is false in higher dimensions. See Remark 5.4.11 for a discussion.

Lemma 5.2.10. *Let X be smooth and separated rigid K-curve. Then, for every arc γ in X and every affinoid open $U \subseteq X$, the intersection $[U] \cap \gamma$ is closed and has finitely many connected components.*

Proof. By Proposition 3.5.2, the adic space X is good, and so since γ is quasi-compact and X is separated we may assume that X is affinoid.

By [dJ95b, Lemma 3.1] the Shilov boundary of $[U]$ is finite, and by combining this with [Ber90, Corollary 2.5.13.(ii) and Proposition 3.1.3] it follows that the (topological) boundary $\partial_{[X]}[U]$ of

$[U]$ in $[X]$ is finite. We have tacitly used here that the the Shilov boundary of $[U]$ matches the relative boundary $\partial([U]/\mathcal{M}(k))$ (see [Tem15, Example 3.4.2.5]). As γ and $[U]$ are closed in $[X]$, we have that

$$\partial_\gamma(\gamma \cap [U]) \subset \partial_{[X]}([U]) \cap \gamma.$$

It follows that $\gamma \cap [U]$ can be identified with a closed subset of $[0,1]$ with a finite boundary. But such a subset must be a finite union of closed subintervals, thus the claim follows. $\square$

We end this section by mentioning some miscellaneous geometric properties of geometric coverings which one expects from a good analogue of the topological theory of covering spaces.

Proposition 5.2.11. *Let X be a rigid K-space and let $f\colon Y \to X$ be a geometric covering. Then, $f(Y)$ is a clopen subset of X. In particular, if X is connected and Y is non-empty then f is surjective.*

Proof. By considering the connected components of X, it is enough to show that for X connected and Y non-empty, we have $f(Y) = X$. Suppose now that f is not surjective; we claim that in this case there exists a maximal point $x \notin f(Y)$. Indeed, since $f(Y)$ is an overconvergent open by Proposition 3.7.6, we have $f(Y) = \mathrm{sep}_X^{-1}(\mathrm{sep}(f(Y)))$. By Corollary A.0.5 one can connect any maximal point y in $f(Y)$ to x by a sequence of connected test curves. By possibly changing x, we may thus assume that there exists a connected test curve $C \to X$ which contains points of $f(Y)$ and its complement. We may therefore assume that X is a connected smooth and separated curve.

By Theorem 4.1.3 and Proposition 4.4.4 there exists a geometric arc $\overline{\gamma}$ with left endpoint y and right endpoint x. Let $\overline{y}'$ be a lifting of the geometric point $\overline{y}$ above y. Then, by definition of a geometric covering there exists a lift $\overline{\gamma}'$ of γ. In particular, one of the endpoints of $\overline{\gamma}'$ lifts the point x, which is a contradiction. $\square$

Proposition 5.2.12. *Let X be a rigid K-space and suppose that $Y_i \to X$ for $i = 1, 2$ and $Z \to X$ are geometric coverings. Then, for any X-morphisms $Y_i \to Z$ for $i = 1, 2$ the map $Y_1 \times_Z Y_2 \to X$ is a geometric covering.*

Proof. First, $Y_1 \to Z$ is a geometric covering by Proposition 5.2.8 (d). By Proposition 5.2.8 (b), so is $Y_1 \times_Z Y_2 \to Y_2$. But, since $Y_2 \to X$ is a geometric covering we know that the composition $Y_1 \times_Z Y_2 \to X$ is a geometric covering by Proposition 5.2.8 (a). $\square$

5.3. **A brief recollection of tame infinite Galois categories.** In the next section we shall show that the category of geometric coverings of a rigid space X, with any of its natural fiber functors $F_{\overline{x}}$, is a tame infinite Galois category in the sense of [BS15, §7][12]. Thus, we shall briefly recall the setup of this theory to set notation and prime ourselves for the proof ahead.

In essence, the theory of tame infinite Galois categories seeks to axiomatize the study of the category G-**Set** of discrete sets with a continuous action of a topological group. It is in analogy with the classical theory of Galois categories, where one studies finite sets with a continuous action of a profinite group.

Definition 5.3.1 ([BS15, Definition 7.2.1])**.** Let $\mathcal{C}$ be a category and $F\colon \mathcal{C} \to$ **Set** be a functor (called the *fiber functor*). We then call the pair $(\mathcal{C}, F)$ an *infinite Galois category* if the following properties hold:

- **(IGC1)** The category $\mathcal{C}$ is cocomplete and finitely complete.
- **(IGC2)** Each object X of $\mathcal{C}$ is a coproduct of categorically connected objects of $\mathcal{C}$.
- **(IGC3)** There exists a set S of connected objects of $\mathcal{C}$ which generates $\mathcal{C}$ under colimits.
- **(IGC4)** The functor F is faithful, conservative, cocontinuous, and finitely continuous.

[12] A similar formalism, under different names, appears also in [Lep10a].

The *fundamental group of* $(\mathcal{C}, F)$, denoted $\pi_1(\mathcal{C}, F)$ is the group $\mathrm{Aut}(F)$ endowed with the compact-open topology[13].

In the above we used the terminology that an object Y of a category $\mathcal{C}$ is *categorically connected*. This, by definition, means that if Y' is a non-initial object of $\mathcal{C}$ then every monomorphism $Y' \to Y$ is an isomorphism.

Let $(\mathcal{C}, F)$ be an infinite Galois category. Unlike in the classical theory of (finite) Galois categories, to obtain a workable theory one must bake in the transitivity of the action of $\pi_1(\mathcal{C}, F)$ on $F(X)$ for a categorically connected object X of $\mathcal{C}$ (see e.g. [BS15, Example 7.2.3]).

Definition 5.3.2 ([BS15, Definition 7.2.4])**.** Let $(\mathcal{C}, F)$ be an infinite Galois category. We say that $(\mathcal{C}, F)$ is *tame* if for every categorically connected object X of $\mathcal{C}$ the action of $\pi_1(\mathcal{C}, F)$ on $F(X)$ is transitive.

Example 5.3.3. Let X be a connected scheme (or adic space), and let $\mathbf{UF\acute{E}t}_X$ denote the category of disjoint unions of finite étale coverings of X. For a geometric point $\overline{x}$ of X one obtains a functor $F_{\overline{x}}\colon \mathbf{UF\acute{E}t}_X \to \mathbf{Set}$ given by $F_{\overline{x}}(Y) = \pi_0(Y_{\overline{x}})$. Then $(\mathbf{UF\acute{E}t}_X, F_{\overline{x}})$ is a tame infinite Galois category and $\pi_1(\mathbf{UF\acute{E}t}_X, F_{\overline{x}})$ is the algebraic fundamental group $\pi_1^{\mathrm{alg}}(X, \overline{x})$.

Example 5.3.4. Let X be a sufficiently nice (path connected and semi-locally 1-connected) topological space. Then, for any point x of X one obtains a functor $F_x\colon \mathbf{Cov}_X^{\mathrm{top}} \to \mathbf{Set}$ given by $F_x(Y) = Y_x$. Here $\mathbf{Cov}_X^{\mathrm{top}}$ is the category of usual topological covering spaces. Then, $(\mathbf{Cov}_X^{\mathrm{top}}, F_x)$ is a tame infinite Galois category and $\pi_1(\mathbf{Cov}_X^{\mathrm{top}}, F_x)$ is the usual topological fundamental group $\pi_1^{\mathrm{top}}(X, x)$. This no longer the case for arbitrary spaces.

As in the classical theory of Galois categories, the fundamental group $\pi_1(\mathcal{C}, F)$ of a tame infinite Galois category is a reasonable topological group and the category $\mathcal{C}$ can be identified via F with the category $\pi_1(\mathcal{C}, F)\text{-}\mathbf{Set}$ of discrete sets with a continuous $\pi_1(\mathcal{C}, F)$-action. Making this precise requires the notion of a Noohi group.

Definition 5.3.5 ([BS15, Definition 7.1.1 and Proposition 7.1.5])**.** Let G be a Hausdorff topological group with identity element 1. We say that G is *Noohi* if

- G has a neighborhood basis of 1 given by open (not necessarily normal) subgroups,
- G is Raĭkov complete.[14]

We denote the category of discrete sets with continuous G action by $G\text{-}\mathbf{Set}$.

One can then obtain the upshot of the theory of (tame) infinite Galois categories as follows.

Proposition 5.3.6 ([BS15, Example 7.2.2 and Theorem 7.2.5])**.** *Let $(\mathcal{C}, F)$ be an infinite Galois category and G a Noohi group. Then, the following statements are true.*

(a) The group $\pi_1(\mathcal{C}, F)$ with its compact-open topology is a Noohi group.
(b) The pair $(G\text{-}\mathbf{Set}, F_G)$, where $F_G\colon G\text{-}\mathbf{Set} \to \mathbf{Set}$ is the forgetful functor, is a tame infinite Galois category with a canonical isomorphism $G \simeq \pi_1(G\text{-}\mathbf{Set}, F_G)$.
(c) The natural map $\mathrm{Hom}((\mathcal{C}, F), (G\text{-}\mathbf{Set}, F_G)) \to \mathrm{Hom}_{\mathrm{cnts}}(G, \pi_1(\mathcal{C}, F))$ *is a bijection.*
(d) If $(\mathcal{C}, F)$ is tame then F induces an equivalence $F\colon \mathcal{C} \xrightarrow{\sim} \pi_1(\mathcal{C}, F)\text{-}\mathbf{Set}$.

From Proposition 5.3.6 we see that for a tame infinite Galois category $(\mathcal{C}, F)$ the isomorphism classes of connected objects form a set. It is moreover clear that every object Y isomorphic to a disjoint union of its *categorically connected components* which, by definition, are the connected subobjects (in the sense of [Mac71, §V.7]), excluding the initial object. Using this, it is easy to deduce the following criterion for when a subcategory of a tame infinite Galois category is itself a tame infinite Galois category.

[13]More precisely, for each s in S, where S is as in **(IGC3)**, we endow $\mathrm{Aut}(s)$ with the compact-open topology as in [Stacks, Tag 0BMC]. We then endow $\mathrm{Aut}(F)$ with the subspace topology inherited from the natural map $\mathrm{Aut}(F) \to \prod_{s \in S} \mathrm{Aut}(s)$, which is injective by combining the generating properties of S and the cocontinuity of F from **(IGC4)**.

[14]See [AT08, §3.6] for the definition of Raĭkov complete groups, as well as several equivalent characterizations.

Lemma 5.3.7. *Let $(\mathcal{C}, F)$ be a tame infinite Galois category. Let $\mathcal{C}'$ be a strictly*[15] *full subcategory of $\mathcal{C}$ satisfying the following three properties:*

(a) an object X of $\mathcal{C}'$ is categorically connected as an object of $\mathcal{C}'$ if and only if it is categorically connected as an object of $\mathcal{C}'$,
(b) an object X of $\mathcal{C}$ belongs to $\mathcal{C}'$ if and only if its categorically connected components belong to $\mathcal{C}'$,
(c) the subcategory $\mathcal{C}'$ is closed under (small) colimits and finite limits.

Then, $(\mathcal{C}', F)$ is a tame infinite Galois category.

We give one last consequence of Proposition 5.3.6. Let $(\mathcal{C}, F)$ be a tame infinite Galois category. Let us denote by $\mathcal{C}^{\mathrm{fin}} \subseteq \mathcal{C}$ the full subcategory consisting of objects having finitely many categorically connected components, and by $\mathbf{Ind}(\mathcal{C}^{\mathrm{fin}})$ its ind-completion (see [KS06, Definition 6.1.1]). Since $\mathcal{C}$ is cocomplete, one gets an induced functor $\mathbf{Ind}(\mathcal{C}^{\mathrm{fin}}) \to \mathcal{C}$. Using Proposition 5.3.6 (to reduce to the case where $\mathcal{C} = G\text{-}\mathbf{Set}$) and [KS06, Corollary 6.3.5] the following proposition easily follows.

Proposition 5.3.8. *Let $\mathcal{C}$ be a tame infinite Galois category. Then, the objects of $\mathcal{C}^{\mathrm{fin}}$ are precisely the compact objects of $\mathcal{C}$, and thus the functor $\mathbf{Ind}(\mathcal{C}^{\mathrm{fin}}) \to \mathcal{C}$ is an equivalence.*

5.4. **Geometric coverings form a tame infinite Galois category.** In this section we arrive at the main result of our paper: geometric coverings form a tame infinite Galois category. In particular, we may identify $\mathbf{Cov}_X$ with the category of G-sets for a Noohi group G which we call the *geometric arc fundamental group* of X.

Theorem 5.4.1. *Let X be a connected rigid K-space and let $\overline{x}$ be a geometric point of X. Then, the pair $(\mathbf{Cov}_X, F_{\overline{x}})$ is a tame infinite Galois category.*

From the abstract machinery of the last section, we are able to make the following definition and its subsequent corollary.

Definition 5.4.2. Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then, the Noohi group $\pi_1(\mathbf{Cov}_X, F_{\overline{x}})$ is called the *geometric arc fundamental group* of the pair $(X, \overline{x})$ and is denoted $\pi_1^{\mathrm{ga}}(X, \overline{x})$.

Corollary 5.4.3. *Let X be a connected rigid K-space and let $\overline{x}$ be a geometric point of X. Then, the functor*

$$F_{\overline{x}} \colon \mathbf{Cov}_X \to \pi_1^{\mathrm{ga}}(X, \overline{x})\text{-}\mathbf{Set}$$

is an equivalence of categories.

We now start moving towards the proof of Theorem 5.4.1. The keystone intermediary result is the following statement about the existence of 'paths' between geometric points.

Theorem 5.4.4. *Let X be a connected rigid K-space and let $\overline{x}$ and $\overline{y}$ be two geometric points of X. Then, there exists an isomorphism of functors $F_{\overline{x}} \simeq F_{\overline{y}}$ of functors $\mathbf{Cov}_X \to \mathbf{Set}$.*

Proof. Let us first assume that X is a smooth and separated rigid K-curve. Let $\overline{x}$ and $\overline{y}$ be anchored at x and y respectively. Let us note that if $\overline{x}^{\max}$ is the associated maximal geometric point associated to x, then $F_{\overline{x}} \simeq F_{\overline{x}^{\max}}$ via the definition of these fiber functors and the valuative criterion, and similarly for $\overline{y}$ and $\overline{y}^{\max}$. Thus, we may assume that $\overline{x}$ and $\overline{y}$ are maximal geometric points and so, in particular, that x and y are maximal points of X. Let γ be an arc in X with left endpoint $\mathrm{sep}_X(x)$ and right endpoint $\mathrm{sep}_Y(y)$. Such an arc exists by Theorem 4.1.3. By Proposition 4.4.4 we may upgrade this to a geometric arc $\overline{\gamma}$. Up to replacing $\overline{\gamma}$ by an equivalent arc we may assume that the left geometric endpoint of $\overline{\gamma}$ is $\overline{x}$ and the right geometric endpoint is $\overline{y}$. Let us now define an isomorphism $\eta_{\overline{\gamma}} \colon F_{\overline{x}} \to F_{\overline{y}}$ as follows. For each object Y of $\mathbf{Cov}_X$ we define $\eta_{\overline{\gamma}}(Y) \colon F_{\overline{x}}(Y) \to F_{\overline{y}}(Y)$ to be the map which associates to a geometric point $\overline{x}'$ in $F_{\overline{x}}(Y)$ the right geometric endpoint of the unique geometric arc $\overline{\gamma}'$ lifting $\overline{\gamma}$ with left geometric

[15] I.e. an object of $\mathcal{C}$ isomorphic to an object of $\mathcal{C}'$ is itself an object of $\mathcal{C}'$.

endpoint $\overline{x}'$. It is clear from the definition of uniqueness of geometric arc lifts, applied to both $\overline{\gamma}$ and $\overline{\gamma}^{\mathrm{op}}$, where $\overline{\gamma}^{\mathrm{op}}$ is $\overline{\gamma}$ with the opposite orientation, that this map is bijective. So, it remains to show that this bijection is functorial in Y.

To see this, it suffices to note that if $f\colon Y_1 \to Y_2$ is a morphism in $\mathbf{Cov}_X$ then for a given geometric point $\overline{x}'$ in $F_{\overline{x}}(Y)$ and a lift $\overline{\gamma}'$ of $\overline{\gamma}$ with left geometric endpoint $\overline{x}'$, that $[f]$ maps γ' homeomorphically onto its image and so $f(\overline{\gamma}')$ is a geometric arc lifting $\overline{\gamma}'$ with left geometric endpoint $f(\overline{x}')$. But, we then see that the right geometric endpoint of $f(\overline{\gamma}')$ is on the one hand $f(\eta_{\overline{\gamma}}(\overline{x}'))$, but on the other hand is equal to $\eta_{\overline{\gamma}}(f(\overline{x}'))$. This proves functoriality as desired.

Let us now assume that X is an arbitrary connected rigid K-space. Let $\overline{x}$ and $\overline{y}$ be anchored at x and y respectively. By the same method as above we may assume that $\overline{x}$ and $\overline{y}$ are maximal geometric points, and thus x and y are maximal points. By corollary A.0.5 we can find a sequence of connected test curves in X connecting x and y. By iterating the procedure it suffices to assume that x and y are both contained in the image of a connected test curve $C \to X$. Let $\overline{x}'$ and $\overline{y}'$ be any lifts of $\overline{x}$ and $\overline{y}$ to C. Note then that there is an obvious map

$$\mathrm{Isom}_{\mathbf{Cov}_C}(F_{\overline{x}'}, F_{\overline{y}'}) \to \mathrm{Isom}_{\mathbf{Cov}_X}(F_{\overline{x}}, F_{\overline{y}})$$

obtained via the natural identifications $F_{\overline{x}}(Y) = F_{\overline{x}'}(Y_C)$ and similarly for $\overline{y}$ and $\overline{y}'$. By the previous case we know that $\mathrm{Isom}_{\mathbf{Cov}_C}(F_{\overline{x}'}, F_{\overline{y}'})$ is non-empty, and thus therefore so must $\mathrm{Isom}_{\mathbf{Cov}_X}(F_{\overline{x}}, F_{\overline{y}})$ be as well. The conclusion follows. $\square$

Remark 5.4.5. Note that in contrast to many other instances where tame infinite Galois categories appear in rigid geometry (e.g. [dJ95b]) the proof of the existence of étale paths is purely formal from our setup. This is one marked advantages of the theory of geometric coverings over the categories discussed in §6 where the natural proofs of the existence of étale paths are quite involved (e.g. see the proof of [dJ95b, Theorem 2.9]).

We may now begin in earnest to show that $(\mathbf{Cov}_X, F_{\overline{x}})$ is a tame infinite Galois category. We begin by verifying Axioms **(IGC2)** and **(IGC3)**. Useful in this examination is the following result which shows that categorical connectedness in $\mathbf{Cov}_X$ coincides with naive topological connectedness.

Proposition 5.4.6. *Let X be a connected rigid K-space and let Y be an object of $\mathbf{Cov}_X$. Then, Y is categorically connected if and only if Y is topologically connected. Therefore, Y admits a unique decomposition $Y = \coprod_{i\in I} Y_i$ with each Y_i a categorically connected object in $\mathbf{Cov}_X$.*

Proof. The second statement clearly follows from the first, so it suffices to show this first claim.

Suppose first that Y is categorically connected. Write the connected component decomposition of Y as $Y = \coprod_{i\in I} Y_i$. By Proposition 5.2.6 we know that each $Y_i \to X$ is a geometric covering. One then clearly sees from the assumption that Y is categorically connected that I must be a singleton, and so Y is connected.

Conversely, suppose that Y is connected. Let $Y' \to Y$ be a monomorphism in $\mathbf{Cov}_X$ where Y' is non-empty (and thus non-initial). Since $\mathbf{Cov}_X$ has fibered products by Proposition 5.2.12, $Y' \to Y$ being a monomorphism implies that the canonical map $Y' \to Y' \times_Y Y'$ is an isomorphism (see [Stacks, Tag 01L3]). Again by Proposition 5.2.12 the fibered product in $\mathbf{Cov}_X$ agrees with that in $\mathbf{Rig}_K$, so $Y' \to Y$ is a monomorphism in $\mathbf{Rig}_K$. By Proposition 3.6.1, $Y' \to Y$ is an open embedding. Since $Y' \to Y$ is a geometric covering by Proposition 5.2.8 (d) we deduce from Proposition 5.2.11 that $Y' \to Y$ is surjective, and hence an isomorphism. $\square$

From this, we deduce the following.

Corollary 5.4.7. *Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then, the pair $(\mathcal{C}, F_{\overline{x}})$ satisfies Axiom* **(IGC2)** *and Axiom* **(IGC3)**.

Proof. One sees that Proposition 5.4.6 immediately implies Axiom **(IGC2)**. Axiom **(IGC3)** is elementary and follows from Lemma 3.7.8. We leave the details to the reader. $\square$

We now move on to show that $(\mathbf{Cov}_X, F_{\overline{x}})$ satisfies Axiom **(IGC1)**.

Proposition 5.4.8. *Let X be a connected rigid K-space and let $\overline{x}$ be a geometric point. Then, the category $\mathbf{Cov}_X$ is cocomplete and finitely complete. Consequently, the pair $(\mathbf{Cov}_X, F_{\overline{x}})$ satisfies Axiom* **(IGC1)**.

Proof. We have already verified in Proposition 5.2.6 and Proposition 5.2.12 that $\mathbf{Cov}_X$ has fibered products and arbitrary coproducts. Thus, since $\mathbf{Cov}_X$ has a final object, it suffices to show that $\mathbf{Cov}_X$ has all coequalizers (e.g. see [Mac71, §V.2]).

To do this, let $W \rightrightarrows U$ be a pair of morphisms in $\mathbf{Cov}_X$. Let us consider the induced map $W \to U \times_X U$. By Proposition 5.2.12 the natural morphism $U \times_X U \to X$ is a geometric covering, and since $W \to U \times_X U$ is a morphism over X we deduce from Proposition 5.2.8 (d) that $W \to U \times_X U$ is a geometric covering. In particular, the image R_0 of $W \to U \times_X U$ is clopen by Proposition 5.2.11.

Let R be smallest equivalence relation containing R in $U \times_X U$. It can be obtained in the following way. Define R_1 to be the symmetrization of R_0, i.e. $R_1 = R_0 \cup R_0^{\mathrm{inv}}$, where R_0^{inv} is the image of R_0 via the automorphism of $U \times_X U$ that switches the factors. Next, consider R_1^n defined as the image of the map

$$R_1 \times_{p,U,q} R_1 \times_{p,U,q} R_1 \times_{p,U,q} \ldots \times_{p,U,q} R_1 \to (U \times_X U) \times_{p,U,q} \ldots \times_{p,U,q} (U \times_X U) \xrightarrow{\sim} U \times_X U$$

Here the left hand side is the n-fold fibered product and p (resp. q) is the left (resp. right) projection $U \times_X U \to U$. Similarly to the argument given in the second paragraph, each R_1^n is a clopen subset of $U \times_X U$. Define R to be the union $\bigcup_n R_1^n$. Note that since U is a locally connected topological space, this union will be clopen. Now, the coequalizer of $W \rightrightarrows U$, if it exists, will be isomorphic to the coequalizer of $R \rightrightarrows U \times_X U$, by construction.

Let $\mathcal{F}$ be the sheaf on the étale site of X given by the sheaf quotient $(U \times_X U)/R$. As $R \to U \times_X U$ is a closed immersion, we can apply [War17, Theorem 3.1.5] to get that that $\mathcal{F}$ is representable. Let us write the representing object by Q. By loc. cit. we also have that $Q \to X$ is separated and $U \to Q$ is surjective and étale. We claim that $Q \to X$ is an object of $\mathbf{Cov}_X$. The fact that $Q \to X$ is étale is simple since $U \to X$ is étale and $U \to Q$ is étale and surjective (see Lemma 3.7.2) . So then, one sees that $Q \to X$ is étale and surjective. Since one has a surjection $U \to Q$ over X, we deduce that $Q \to X$ is a geometric covering by Proposition 5.2.9. □

Finally, to show that $(\mathbf{Cov}_X, F_{\overline{x}})$ is an infinite Galois category we need only to show that Axiom **(IGC4)** holds. This is done in the following.

Proposition 5.4.9. *Let X be a connected rigid K-space and let $\overline{x}$ be a geometric point of X. Then, the pair $(\mathbf{Cov}_X, F_{\overline{x}})$ satisfies Axiom* **(IGC4)**.

Proof. Let us first show that $F_{\overline{x}}$ is faithful. Let $Y_1 \to Y_2$ be a morphism in $\mathbf{Cov}_X$. Note then that the graph morphism $Y_1 \to Y_1 \times_X Y_2$ is a geometric covering by combining Proposition 5.2.8 (d) and Proposition 5.2.12. In particular, it is an étale monomorphism, and so an open embedding by Proposition 3.6.1. By Proposition 5.2.11 it has clopen image, and so $Y_1 \to Y_1 \times_X Y_2$ is an isomorphism onto some connected component of $Y_1 \times_X Y_2$. So, suppose that $f, g \colon Y_1 \to Y_2$ are morphisms in $\mathbf{Cov}_X$ such that $F_{\overline{x}}(f) = F_{\overline{x}}(g)$ for some geometric point $\overline{x}$ of X. To show that $f = g$ it suffices to show that the connected components of $Y_1 \times_X Y_2$ corresponding to f and g agree. By Theorem 5.4.4 we have that $F_{\overline{x}}(f) = F_{\overline{x}}(g)$ for all geometric points $\overline{x}$ of X. If $\overline{y}$ is a geometric point of Y_1 lying over the geometric point $\overline{x}$ of X, then the image of this geometric point under the graph map $Y_1 \to Y_1 \times_X Y_2$ for f is $(\overline{y}, F_{\overline{x}}(f)(\overline{y}))$, and similarly for g. In particular, since $F_{\overline{x}}(f) = F_{\overline{x}}(g)$ for all geometric points $\overline{x}$ of X, we see that the maps $Y_1 \to Y_1 \times_X Y_2$ induced by f and g have the same images as desired.

To see that $F_{\overline{x}}$ is conservative, suppose that $f \colon Y_1 \to Y_2$ is a morphism in $\mathbf{Cov}_X$ such that $F_{\overline{x}}(f)$ is a bijection for some geometric point $\overline{x}$. By Proposition 5.2.8 (d), f is étale. Moreover, by Theorem 5.4.4 see that $F_{\overline{x}}(f)$ is a bijection for all geometric points $\overline{x}$ of X. This clearly implies that one has the unique lifting property of geometric points as in Proposition 3.6.1. By Proposition 3.6.1, f is an isomorphism as desired.

To see that $F_{\overline{x}}$ is cocontinuous and finitely continuous, we note that since $\mathbf{Cov}_X$ has a final object it suffices to show that $F_{\overline{x}}$ commutes with arbitrary coproducts, coequalizers, and fibered products (see [Mac71, §V.2]). It clearly commutes with arbitrary coproducts and fibered products, and thus it suffices to show it commutes with coequalizers. Let $W \rightrightarrows U$ be a pair of arrows in $\mathbf{Cov}_X$. Recall that in the proof of Proposition 5.4.8 one identified

$$\mathrm{Coeq}\,(W \rightrightarrows U) \simeq (U \times_X U)/R$$

with notation as in that proposition. Since quotients of étale equivalence relations commute with pullback (cf. [Stacks, Tag 03I4]), for each morphism $V \to X$ of adic spaces, one has that

$$\mathrm{Coeq}\,(W \rightrightarrows U)_V \simeq ((U \times_X U)/R)_V \simeq (U_V \times_V U_V)/R_V \simeq \mathrm{Coeq}\,(W_V \rightrightarrows U_V)$$

Applying this to the case when $V = \overline{x}$ shows that $F_{\overline{x}}$ commutes with coequalizers. □

Thus, the only part of Theorem 5.4.1 left to be proven is the claim that the infinite Galois category $(\mathbf{Cov}_X, F_{\overline{x}})$ is tame. This is relatively simple and shown as follows.

Proposition 5.4.10. *Let X be a connected rigid K-space, $\overline{x}$ a geometric point of X, and Y a categorically connected object of $\mathbf{Cov}_X$. Then, the group $\mathrm{Aut}(F_{\overline{x}})$ acts transitively on $F_{\overline{x}}(Y)$. In particular, the infinite Galois category $(\mathbf{Cov}_X, F_{\overline{x}})$ is tame.*

Proof. As in the beginning of the proof of Proposition 5.4.4, we may assume that $\overline{x}$ is a maximal geometric point. Let $\overline{y}_1$ and $\overline{y}_2$ be elements of $F_{\overline{x}}(Y)$. Since Y is connected we obtain from Theorem 5.4.4 an étale path $\eta\colon F_{\overline{y}_1} \xrightarrow{\sim} F_{\overline{y}_2}$ of functors on $\mathbf{Cov}_Y$. Consider the base change functor $b\colon \mathbf{Cov}_X \to \mathbf{Cov}_Y$. One easily sees that $F_{\overline{y}_i} \circ b \simeq F_{\overline{x}}$. Thus, from the isomorphism η we obtain an isomorphism η' given as the composition

$$F_{\overline{x}} \simeq F_{\overline{y}_1} \circ b \xrightarrow{\eta} F_{\overline{y}_2} \circ b \simeq F_{\overline{x}}$$

Note though that by functoriality of η and the definition of η' one has a commutative diagram

$$\begin{tikzcd}
\{\overline{y}_1\} = F_{\overline{y}_1}(Y) \arrow[r, "\eta(Y)"] \arrow[d, "\Delta_{Y/X}"'] & F_{\overline{y}_2}(Y) = \{\overline{y}_2\} \arrow[d, "\Delta_{Y/X}"] \\
F_{\overline{y}_1}(Y \times_X Y) \arrow[r, "\eta(Y \times_X Y)"] \arrow[d, equal] & F_{\overline{y}_2}(Y \times_X Y) \arrow[d, equal] \\
F_{\overline{x}}(Y) \arrow[r, "\eta'(Y)"] & F_{\overline{x}}(Y)
\end{tikzcd}$$

where $\Delta_{Y/X}\colon Y \to Y \times_X Y$ is the diagonal map. This clearly shows that $\eta(Y)$ takes $\overline{y}_1$ to $\overline{y}_2$. Since $\overline{y}_1$ and $\overline{y}_2$ were arbitrary, the conclusion follows. □

Remark 5.4.11. Why do we need to use restriction to curves to define geometric coverings? And why do we use arcs and not paths? Roughly, the reason is that arcs in higher-dimensional Berkovich spaces can be very wiggly, which may cause trouble at various steps of the proofs. For example, Lemma 5.2.10 is false in higher dimension.

Thus, arcs in curves enjoy some good tameness properties. We expect there to be a more general notion of a tame path in a Berkovich analytic space, as well as the corresponding notion of a tame geometric path in a rigid K-space X. We list below such properties that one should need.

1. If $u\colon [0,1] \to [0,1]$ is piecewise monotone and $p\colon [0,1] \to [X]$ is tame, then $p \circ u$ is tame. In particular, being tame does not depend on the parameterization, and the restriction of a tame path $p\colon [0,1] \to X$ to a closed sub-interval of $[0,1]$ is tame.
2. If $[0,1] = \bigcup_{i=1}^m [x_{i-1}, x_i]$ is a finite partition, then $p\colon [0,1] \to [X]$ is tame if and only if $p|_{[x_{i-1},x_i]}$ is tame for all i. In particular, the concatenation of tame paths is tame.
3. If $p\colon [0,1] \to [X]$ is tame, then for any map $f\colon X \to X'$ of rigid K-spaces, the composition $[f] \circ p\colon [0,1] \to [X']$ is tame. If L is a non-archimedean extension of K and $p\colon [0,1] \to [X_L]$ is tame, then the composition $[0,1] \to [X_L] \to [X]$ is tame.

4. If $U \subseteq X$ is an affinoid open and if $p\colon [0,1] \to [X]$ is a tame path, then $p^{-1}([U])$ has finitely many connected components,
5. If X is connected, then every two points $x, y \in [X]$ are the endpoints of a tame path.

It is likely that if there were a notion satisfying those, then one could replace the equivalent conditions (a) and (b) in Definition 5.2.2 with

(a) $Y \to X$ satisfies unique lifting for tame geometric paths,
(b) $[Y] \to [X]$ satisfies AVC with respect to all tame paths.

and still get a good (if not equivalent) notion of a covering space.

A notion of definable path is hinted at in [HL16], but it is unclear to us whether their theory could yield results for arbitrary Berkovich analytic spaces, as opposed to analytifications of algebraic varieties.

6. Variants of the de Jong fundamental group

Let X be a rigid K-space. In this section, we study the full subcategories $\mathbf{Cov}_X^\tau \subseteq \mathbf{Cov}_X$ consisting of coverings which are τ-locally isomorphic to the disjoint union of finite étale morphisms, where $\tau \in \{\mathrm{oc}, \mathrm{adm}, \text{ét}\}$. The smallest one of these, $\mathbf{Cov}_X^{\mathrm{oc}}$, is the category studied by de Jong in [dJ95b]. In §6.2 below, we show that for X connected they form tame infinite Galois categories.

A natural question, asked on [dJ95b, pg. 106], is whether

$$\mathbf{Cov}_X^{\mathrm{oc}} = \mathbf{Cov}_X^{\mathrm{adm}}.$$

In §6.3, we answer this question negatively by constructing an example of a an object of $\mathbf{Cov}_X^{\mathrm{adm}}$ which is not in $\mathbf{Cov}_X^{\mathrm{oc}}$, where X is an annulus over a field K of positive characteristic. This, in particular, implies that the map of fundamental groups

$$\pi_1^{\mathrm{dJ,adm}}(X, \overline{x}) \to \pi_1^{\mathrm{dJ,oc}}(X, \overline{x})$$

is not an isomorphism. Similar examples should exist in mixed characteristic, though there the analysis is likely more complicated. By the results of our companion paper [ALY21] no such examples can exist in the case of a discretely valued field of equal characteristic zero (at least in the case of smooth base).

6.1. The categories $\mathbf{Cov}_X^\tau$.

Recall that for a rigid K-space X, we denote by $\mathbf{F\acute{E}t}_X$ the category of finite étale spaces over X. Let us denote by $\mathbf{UF\acute{E}t}_X$ the category of disjoint unions of objects of $\mathbf{F\acute{E}t}_X$.

We will consider below the following three Grothendieck topologies $\tau = \mathrm{oc}, \mathrm{adm}, \text{ét}$ on $\mathbf{\acute{E}t}_X$, whose covers are:

- ($\tau = \mathrm{oc}$, for 'overconvergent') open covers by overconvergent opens (see §2.4),
- ($\tau = \mathrm{adm}$, for 'admissible') open covers,
- ($\tau = \text{ét}$) jointly surjective étale morphisms.

which are in increasing order of fineness.

Definition 6.1.1. Let X be a rigid K-space and let $\tau \in \{\mathrm{oc}, \mathrm{adm}, \text{ét}\}$. Then, we define the category $\mathbf{Cov}_X^\tau$ to be the full subcategory of $\mathbf{\acute{E}t}_X$ consisting of those morphisms $Y \to X$ for which there exists a cover $U \to X$ in the τ-topology so that $Y_U \to U$ is an object of $\mathbf{UF\acute{E}t}_U$.

The significance of these categories is as follows:

- $\mathbf{Cov}_X^{\mathrm{oc}}$ is the category of 'étale covering spaces' considered in [dJ95b],
- $\mathbf{Cov}_X^{\mathrm{adm}}$ will be used to answer a question of de Jong posed in [dJ95b] (see the Introduction for a discussion),
- in Theorem 7.5.1 below we show that $\mathbf{Cov}_X^{\text{ét}}$ is equivalent to the category of local systems for the pro-étale topology as in [Sch13].

Remark 6.1.2. The categories $\mathbf{Cov}_X^\tau$ find a place in a hierarchy of categories of 'covering spaces.' Let us denote by **Sep** the category of separated morphisms in $\mathbf{Rig}_K$ (its objects are separated morphisms $Y \to X$, and morphisms are commutative squares), treated as a fibered category over $\mathbf{Rig}_K$. Let us note that **Sep** is closed under arbitrary coproducts (disjoint unions) with fixed base, and is a τ-stack for any Grothendieck topology τ on $\mathbf{Rig}_K$ weaker than the big étale topology (for this latter statement cf. [War17, Corollary 3.1.9]), in particular for $\tau \in \{\mathrm{oc}, \mathrm{adm}, \text{ét}\}$.

For any fibered subcategory $\mathcal{C}$ of **Sep** and any topology τ weaker than the big étale topology on $\mathbf{Rig}_K$ we define the following operations:

- we denote by $L_\tau \mathcal{C}$ the τ-stackification of $\mathcal{C}$ (i.e. the class of separated maps $Y \to X$ for which there exists a τ-cover $U \to X$ such that $(Y_U \to U) \in \mathcal{C}$),
- we denote by $\mathbf{U}\mathcal{C}$ (resp. $\mathbf{U}_{\mathrm{fin}}\mathcal{C}$) the subcategory of **Sep** consisting of arbitrary (resp. finite) coproducts of objects of $\mathcal{C}$ with fixed target.

Note that by the aforementioned properties of **Sep** one has that $L_\tau\mathcal{C}$, $\mathbf{U}\mathcal{C}$, and $\mathbf{U}_{\mathrm{fin}}\mathcal{C}$ are naturally fibered subcategories of **Sep**. Let us also denote by $\mathbf{Isom} \subseteq \mathbf{Sep}$ the category of all isomorphisms.

The above notation allows us to express certain categories of coverings as follows:

- The fibered category $L_{\mathrm{adm}}\mathbf{U}\,\mathbf{Isom}$ is the category of covering spaces, i.e. morphisms of adic spaces which are local isomorphisms and covering spaces in the topological sense.
- The fibered category $L_{\text{ét}}\mathbf{U}_{\mathrm{fin}}\,\mathbf{Isom}$ is the category of finite étale maps.
- The category $\mathbf{Cov}_X^\tau$ is $(L_\tau\,\mathbf{UF\acute{E}t})_{/X} = (L_\tau \mathbf{U} L_{\text{ét}}\mathbf{U}_{\mathrm{fin}}\,\mathbf{Isom})_{/X}$.

As shown in §5.4, the category of geometric coverings **Cov** is closed under disjoint unions and forms an étale stack, i.e. $\mathbf{Cov} = \mathbf{U}\,\mathbf{Cov} = L_{\text{ét}}\,\mathbf{Cov}$. In particular, it contains the above categories (and any combinations of L_τ and $\mathbf{U}$ applied to them). Therefore **Cov** provides a natural framework for talking about all these notions of 'covering spaces' uniformly.

Remark 6.1.3. A similar hierarchy can be built to treat 'coverings' of spaces which are not locally simply connected. We start with the class $\mathcal{C}_0$ of isomorphisms in **Top**, and inductively define for $n \geq 0$:

(1) $\mathcal{C}_{n+\frac{1}{2}}$ to be the class of maps $Y \to X$ for which we can write $Y = \coprod_{i\in I} Y_i$ with each $Y_i \to X$ in $\mathcal{C}_n$,

(2) $\mathcal{C}_{n+1}$ to be the class of maps $Y \to X$ for which there exists an open cover $X = \bigcup_{i\in I} U_i$ such that $Y \times_X U_i \to U_i$ belongs to $\mathcal{C}_{n+\frac{1}{2}}$ for all i.

Thus $\mathcal{C}_1$ is the category of coverings, while $\mathcal{C}_{1\frac{1}{2}}$ is the category of disjoint unions of coverings, and so on. One can extend the definition of the categories $\mathcal{C}_n$ to any ordinal n, and hope that for a fixed target X, the categories $(\mathcal{C}_n)_{/X}$ stabilize for n large enough, giving a good category of generalized coverings of X.

6.2. **The categories $\mathbf{UCov}_X^\tau$ are tame infinite Galois categories.** We would now like to carry through with our plan of using the fact that $(\mathbf{Cov}_X, F_{\overline{x}})$ is a tame infinite Galois category to make similar statements for the categories $\mathbf{Cov}_X^\tau$ for $\tau \in \{\mathrm{oc}, \mathrm{adm}, \text{ét}\}$. Unfortunately, these categories cannot be tame infinite Galois categories since they fail to be closed under disjoint unions. Instead, we consider the categories $\mathbf{UCov}_X^\tau$ of disjoint unions of objects of $\mathbf{Cov}_X^\tau$. This minimal fix works and we show the following result.

Theorem 6.2.1. *Let X be a connected rigid K-space and $\overline{x}$ a geometric point in X. Then, for each $\tau \in \{\mathrm{oc}, \mathrm{adm}, \text{ét}\}$ one has that $(\mathbf{UCov}_X^\tau, F_{\overline{x}})$ is a tame infinite Galois category.*

Proof. By Lemma 5.3.7 it suffices to verify that conditions (a), (b), and (c) hold when $\mathcal{C} = \mathbf{Cov}_X$ and $\mathcal{C}' = \mathbf{UCov}_X^\tau$.

To see that (a) holds it suffices to show, given Proposition 5.4.6, that an object of $\mathbf{UCov}_X^\tau$ is categorically connected if and only if it is connected. But, the method of proof in Proposition 5.4.6 gives that it suffices to prove that for $Y_i \to X$ for $i = 1, 2$ and $Z \to X$ objects of $\mathbf{UCov}_X^\tau$ that the fiber product $Y_1 \times_Z Y_2$ in $\mathbf{Rig}_K$ is an object of $\mathbf{UCov}_X^\tau$.

To show this let $U \to X$ be a τ-cover such that $(Y_i)_U$ for $i = 1, 2$ and Z_U are objects of $\mathbf{UF\acute{E}t}_U$. Let us write

$$(Y_1)_U = \coprod_{i \in I} Y_{1,i}, \quad Z_U = \coprod_{j \in J} Z_j, \quad (Y_2)_U = \coprod_{k \in K} Y_{2,k}$$

where each piece is connected, and thus finite étale, over U. Notice that we have functions $f\colon I \to J$ and $g\colon K \to J$ given by the fact that $(Y_1)_U \to Z_U$ and $(Y_2)_U \to Z_U$ must send connected components into connected components. So then,

$$(Y_1 \times_Z Y_2)_U \simeq (Y_1)_U \times_{Z_U} (Y_2)_U = \coprod_{\substack{(i,j,k) \in I \times J \times K \\ f(i) = k = g(j)}} Y_{1,i} \times_{Z_j} Y_{2,k} \tag{6.2.1}$$

from where the claim follows.

Claim (b) is clear. Indeed, $\mathbf{UCov}_X^\tau$ is closed under disjoint unions, and therefore for an object Y of $\mathbf{Cov}_X$ its connected components belonging to $\mathbf{UCov}_X$ implies it itself belongs to $\mathbf{UCov}_X^\tau$. Conversely, if Y belongs to $\mathbf{UCov}_X^\tau$ and C is any connected component of Y then C_U is a connected component of Y_U for any morphism of rigid K-spaces $U \to X$. In particular, if Y_U is in $\mathbf{UF\acute{E}t}_U$ then C_U is in $\mathbf{UF\acute{E}t}_U$. Taking $U \to X$ to be a τ-cover such that Y_U is in $\mathbf{UF\acute{E}t}_U$ thus shows that C is an object of $\mathbf{UCov}_X^\tau$ as desired.

Finally, to verify condition (c) we note that if suffices to prove this in the case of a colimit or finite limit indexed by a diagram $D\colon I \to \mathbf{UCov}_X^\tau$ which is either a coproduct, coequalizer, or fibered product diagram. The claim in the case of fibered products and disjoint unions has already been addressed earlier in this proof. Thus, it suffices to prove this result in the case when $D\colon I \to \mathbf{UCov}_X^\tau$ is coequalizer diagram. But, as already observed before (cf. [Stacks, Tag 03I4]) the formation of coequalizers commutes with base change. In particular, by considering the base change to an appropriate τ-cover we reduce ourselves to showing the coequalizer of a diagram in $\mathbf{UF\acute{E}t}_X$ taken in $\mathbf{Cov}_X$ is in $\mathbf{UF\acute{E}t}_X$. But, this is clear (cf. [Stacks, Tag 0BN9]). □

From this, we can then define associated fundamental groups for these categories in good conscience.

Definition 6.2.2. Let X be a rigid K-space and $\overline{x}$ a geometric point of X. We define the τ-*adapted de Jong fundamental group* of the pair $(X, \overline{x})$, denoted $\pi_1^{\mathrm{dJ},\tau}(X, \overline{x})$, to be the fundamental group of the tame infinite Galois category $(\mathbf{UCov}_X^\tau, F_{\overline{x}})$.

Remark 6.2.3. As the name indicates, if X is taut and $\overline{x}$ a maximal geometric point then there is a natural isomorphism between the Noohi group $\pi_1^{\mathrm{dJ},\mathrm{oc}}(X, \overline{x})$ and the topological group constructed in [dJ95b] for the Berkovich space $(u(X), u(\overline{x}))$ (in the notation of [Hub96, §8.3]).

Again from the abstract machinery discussed in §5.3 one obtains the following corollary.

Corollary 6.2.4. *Let X be a connected rigid K-space and $\overline{x}$ a geometric point of X. Then for each $\tau \in \{\mathrm{oc}, \mathrm{adm}, \acute{\mathrm{e}}\mathrm{t}\}$ the functor*

$$F_{\overline{x}}\colon \mathbf{UCov}_X^\tau \to \pi_1^{\mathrm{dJ},\tau}(X, \overline{x})\text{-}\mathbf{Set}$$

is an equivalence of categories.

Finally, let us observe that from the natural series of inclusions of tame infinite Galois categories

$$(\mathbf{UCov}_X^{\mathrm{oc}}, F_{\overline{x}}) \subseteq (\mathbf{UCov}_X^{\mathrm{adm}}, F_{\overline{x}}) \subseteq (\mathbf{UCov}_X^{\acute{\mathrm{e}}\mathrm{t}}, F_{\overline{x}}) \subseteq (\mathbf{Cov}_X, F_{\overline{x}})$$

one obtains from Proposition 5.3.6 a series of maps

$$\pi_1^{\mathrm{dJ},\mathrm{oc}}(X, \overline{x}) \longleftarrow \pi_1^{\mathrm{dJ},\mathrm{adm}}(X, \overline{x}) \longleftarrow \pi_1^{\mathrm{dJ},\acute{\mathrm{e}}\mathrm{t}}(X, \overline{x}) \longleftarrow \pi_1^{\mathrm{ga}}(X, \overline{x})$$

Moreover, from [Lar19, Proposition 2.37 (2)] one sees that each of these maps has dense image.

Remark 6.2.5. While not necessary for the proof of Theorem 6.2.1, one can verify that the categories $\mathbf{UCov}_X^\tau$ satisfy many of the properties in §5.2 with two notable exceptions. First, while it is of course true that membership in $\mathbf{Cov}_X^\tau$ can be checked locally for the τ-topology, this property for $\mathbf{UCov}_X^\tau$ is unknown to the authors and almost certainly false. Second, composition fails: if $(Y \to X) \in \mathbf{UCov}_X^\tau$ and $(Z \to Y) \in \mathbf{UCov}_X^\tau$ then it may not be the case that $(Z \to Y) \in \mathbf{UCov}_X^\tau$ (and similarly with $\mathbf{Cov}^\tau$).

Remark 6.2.6. Combining Proposition 5.3.8 and Theorem 6.2.1 we see that $\mathbf{UCov}_X^\tau$ can be described as $\mathbf{Ind}(\mathbf{Cov}_X^{\tau,\mathrm{fin}})$ which is a more categorically intrinsic description of this category.

6.3. **The example.** Let K be a non-archimedean field of characteristic $p > 0$ and let $\varpi \in K$ be a pseudouniformizer. In this subsection, we construct an object $Y \to X$ of $\mathbf{Cov}_X^{\mathrm{adm}}$ which is not contained in $\mathbf{Cov}_X^{\mathrm{oc}}$, where X is the rigid-analytic annulus

$$X = \{|\varpi| \le |x| \le |\varpi|^{-1}\}$$

The covering $Y \to X$ is obtained by gluing two families $Y_n^\pm$ of Artin–Schreier coverings of two annuli

$$U^- = \{|\varpi| \le |x| \le 1\}, \qquad U^+ = \{1 \le |x| \le |\varpi|^{-1}\}$$

which are split over shrinking overconvergent neighborhoods of the intersection

$$C = U^- \cap U^+ = \{|x| = 1\}.$$

We begin with an analysis of Artin–Schreier coverings of $\mathbf{A}_K^{1,\mathrm{an}}$. For a rational number α, we define the affinoid opens

$$D(\alpha) = \{|x| \le |\varpi|^{-\alpha}\} \quad \supseteq \quad S(\alpha) = \{|x| = |\varpi|^{-\alpha}\}.$$

For integers a, b with $b > 0$, we denote by $Y_{a,b}$ the following Artin–Schreier covering of $\mathbf{A}_K^{1,\mathrm{an}}$:

$$Y_{a,b} = \left(\operatorname{Spec} K[x,y]/(y^p - y - \varpi^a x^b)\right)^{\mathrm{an}}.$$

It is a $\mathbf{Z}/p\mathbf{Z}$-torsor over $\mathbf{A}_K^{1,\mathrm{an}}$, and since $\mathbf{Z}/p\mathbf{Z}$ does not have any non-trivial subgroups, the restriction of $Y_{a,b}$ to an affinoid subdomain $U \subseteq \mathbf{A}_K^{1,\mathrm{an}}$ is disconnected if and only if it splits completely over U and if and only if the equation $y^p - y = \varpi^a x^b$ has a solution in $\mathcal{O}(U)$.

Lemma 6.3.1. *Let a, b be integers with $b > 0$ and let α be a rational number. The following are equivalent:*

(a) The covering $Y_{a,b}$ splits over $D(\alpha)$.
(b) The covering $Y_{a,b}$ splits over $S(\alpha)$.
(c) We have $\alpha < a/b$.

Proof. Write $g = \varpi^a x^b$. The unique solution to $y^p - y = -g$ in $K[\![x]\!]$ satisfying $y(0) = 0$ is the power series

$$f = g + g^p + g^{p^2} + \cdots = \sum_{s \ge 1} \varpi^{p^s a} x^{p^s b}.$$

It converges on $D(\alpha)$ (meaning that it is in the image of $\mathcal{O}(D(\alpha)) \to \widehat{\mathcal{O}}_{D(\alpha),0} = K[\![x]\!]$) if and only if $\alpha < a/b$, because $\mathcal{O}(D(\alpha))$ consists of power series $\sum a_n x^n$ with $|a_n| \cdot |\varpi|^{-n\alpha} \to 0$. This shows (a) is equivalent to (c).

Since clearly (a) implies (b), it remains to show (b) implies (a). The ring $\mathcal{O}(S(\alpha))$ consists of Laurent series $f = \sum_{n \in \mathbf{Z}} a_n x^n$ with $|a_n| \cdot |\varpi|^{-n\alpha} \to 0$ as $|n| \to \infty$. It suffices to show that if $f^p - f \in \mathcal{O}(D(\alpha))$, then $f \in \mathcal{O}(D(\alpha))$ (i.e. $a_n = 0$ for $n < 0$). We have $f^p - f$ is equal to $\sum_{n \in \mathbf{Z}} (a_{n/p}^p - a_n) x^n$ where we set $a_{n/p} = 0$ if p does not divide n. Since $f^p - f \in \mathcal{O}(D(\alpha))$, we have $a_{n/p}^p - a_n = 0$ for $n < 0$. It then follows by simple induction that $a_n = 0$ for all $n < 0$. □

Remark 6.3.2. The equivalence of (a) and (b) in Lemma 6.3.1 holds more generally for every finite étale cover of $D(\alpha)$, by the argument in [dJ95b, proof of Proposition 7.5].

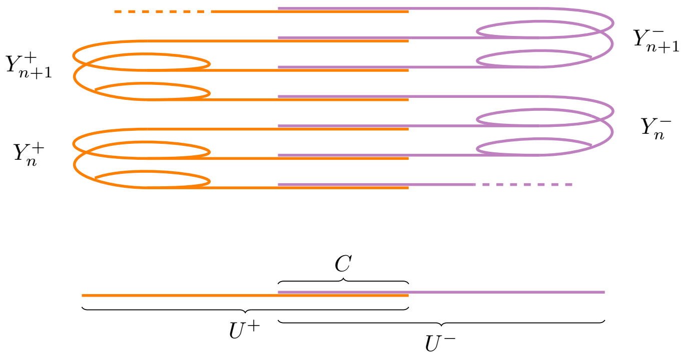

FIGURE 2. Construction of the covering $Y \to X$ (for $p = 3$).

Construction 6.3.3. We use notation as above. We fix two sequences of positive integers $(a_n)_{n\in\mathbf{Z}}$ and $(b_n)_{n\in\mathbf{Z}}$ such that $a_n/b_n > 1$ for all $n \in \mathbf{Z}$ and $\lim_{|n|\to\infty} a_n/b_n = 1$. For $n \in \mathbf{Z}$, we set Y_n to be the restriction of the Artin–Schreier covering Y_{a_n,b_n} to U^+. By Lemma 6.3.1, every Y_n splits completely over $C = S(1)$, while for every $m > 0$ its restriction to $D(1+\frac{1}{m}) \cap U^+$ and $S(1+\frac{1}{m})$ is connected for $|n| \gg 0$. We set $Y^+ = \coprod_{n\in\mathbf{Z}} Y_n^+$, which is an object of $\mathbf{UF\acute{E}t}_{U^+}$.

The automorphism $x \mapsto x^{-1}$ of X induces an isomorphism $i\colon U^- \xrightarrow{\sim} U^+$. We let $Y^- = \coprod_{n\in\mathbf{Z}} Y_n^-$ be the pullback of $Y^+ \to U^+$ under the map i. The restriction of Y_n^- to $S(1-\frac{1}{m})$ is thus connected for $|n| \gg 0$, while Y_n^- splits completely over C for all n.

Label the irreducible components of $Y_n^\pm \times_{U^\pm} C$ by $Z^\pm_{np}, Z^\pm_{np+1}, \ldots, Z^\pm_{np+p-1}$ (in any order); every $Z^\pm_m$ maps isomorphically onto C. Identify $Y_n^+ \times_{U^+} C = \coprod_{m\in\mathbf{Z}} Z_m^+$ with $Y_n^- \times_{U^-} C = \coprod_{m\in\mathbf{Z}} Z_m^-$ by identifying Z_m^+ with Z_{m-1}^- for all $m \in \mathbf{Z}$ (see Figure 2). This defines an étale morphism $Y \to X$ whose restriction to U^+ (resp. U^-) is Y^+ (resp. Y^-), in particular $Y \to X$ it is an object of $\mathbf{Cov}_X^{\mathrm{adm}}$.

Proposition 6.3.4. *The map $Y \to X$ given in Construction 6.3.3 is not a de Jong covering space (i.e. not an object of $\mathbf{Cov}_X^{\mathrm{oc}}$).*

Proof. Let V be a connected overconvergent open subset of X, which by shrinking we may assume satisfies that $V \cap U^\pm$ is connected containing the Gauss point η of C (in other words, the Gauss point of the unit disc $\{|x| \le 1\}$, which happens to belong to C). It suffices to show that $Y_V \to V$ contains a connected open subset with infinite fiber size.

By Lemma 6.3.5 below, V contains $S(1+\frac{1}{m}) \cup S(1-\frac{1}{m})$ for some $m > 0$. Since for $n \gg 0$, say $n > n_0$, the restriction of $Y_n^\pm$ to $S(1 \pm \frac{1}{m})$ is connected, the restriction $Y_n^\pm \cap Y_V$ of $Y_n^\pm$ to $V \cap U^\pm$ is connected as well.

Let Y' be the image in Y of the union of Y_n^+ and Y_n^- for all $n > n_0$, which is an open subset of Y. We claim that $Y'_V = Y' \cap Y_V$ is connected. This will give the required assertion since $Y'_V \to V$ has infinite fiber size. To see the claim, it suffices to note that in the infinite sequence

$$Y^+_{n_0+1} \cap Y_V, \quad Y^-_{n_0+1} \cap Y_V, \quad Y^+_{n_0+2} \cap Y_V, \quad Y^-_{n_0+2} \cap Y_V, \quad Y^+_{n_0+3} \cap Y_V, \quad \ldots$$

each set is a connected open subset of $Y' \cap Y_V$ with a non-empty intersection with the subsequent one (which can be seen by considering the fiber over a point of C). Consulting Figure 2 again might help the reader with the last step. $\square$

Lemma 6.3.5. *Every overconvergent neighborhood of the Gauss point η of C inside X contains $S(1+\frac{1}{m}) \cup S(1-\frac{1}{m})$ for some $m > 0$.*

Proof. We can replace X with the ambient disc $D = D(|\varpi|^{-1})$. As in the proof of Theorem 4.1.3, it suffices to prove the analogous claim for the Berkovich disc D^{Berk}. By [BR10, Proposition 1.6], a basis of the topology for D^{Berk} is given by sets of the form $D(a,r)^- \setminus \bigcup_{i=1}^n D(a_i, r_i)$ where $D(a,r)$ (resp. $D(a,r)^-$) denotes the open (resp. closed) disc with center a and radius r, and where we allow $r > 0$ or $n = 0$. We note that for every a, r we have either $D(a, r) = D(0, r)$ (if $|a| \leq |r|$) or $D(a, r) \subseteq S(0, |a|)$ where $S(0, |a|) = \{|x| = |a|\}$ (if $|a| > |r|$), and similarly for $D(a, r)^{-}$.

Let V be an open subset of D^{Berk} containing (the image of) η which is of the above form. By the above observation, we may assume that $a = 0$, $r > \rho$, $a_1 = 0$, $r_1 < \rho$, and $D(a_i, r_i) \subseteq S(0, |a_i|)$ for $i \geq 2$. Let $\rho' \in (\rho, r)$ be such that the interval (ρ, ρ') does not contain any of the r_i, then the annulus $\{\rho < |x| < \rho'\}$ is contained in V. Taking preimages in D we obtain the desired claim. $\square$

Remark 6.3.6. As we observed above, the cover $Y \to X$ from Construction 6.3.3 has the property that Y_C its splits into finite étale components Y_n. Moreover, while each Y_n individually extends to an overconvergent open neighborhood V_n of C, these overconvergent opens V_n intersect (essentially) at C itself. In other words, the failure of finding an overconvergent open V contaning C such that Y_V is in $\mathbf{UF\acute{E}t}_V$ is related to shrinking behavior of the largest overconvergent open neighborhood of C over which each Y_n extends. This was not mistake.

In fact, one can show the following result:

> *Let X be a taut rigid K-space and let $f\colon Y \to X$ be étale and partially proper. Let $W \subseteq Y_U$ be a quasi-compact open subset such that $f|_W\colon W \to f(W) = U$ is finite étale. Then, there exists an overconvergent open neighborhood U' of U and an open subset $W' \subseteq Y_{U'}$ finite étale over U' such that $W' \cap Y_U = W$.*

One can see this result as saying that while $\mathbf{Cov}_X^{\mathrm{oc}}$ is, in general, smaller than $\mathbf{Cov}_X^{\mathrm{adm}}$, they cannot be distinguished by looking at the extension behavior of their finite étale split components.

7. The category $\mathbf{Cov}_X^{\text{ét}}$ and locally constant sheaves in the pro-étale topology

In this section we show that if X is an adic space then there is a natural equivalence between the category $\mathbf{Cov}_X^{\text{ét}}$, whose definition is the same as in Definition 6.1.1, and the category $\mathbf{Loc}(X_{\text{proét}})$ of locally constant sheaves on the pro-étale topology as in [Sch13]. The proof will proceed by first studying locally constant sheaves on the category of pro-finite G-sets for a pro-finite group G, using this to deduce a result about locally constant sheaves on the pro-finite étale category $X_{\text{profét}}$, and then finally using this describe locally constant sheaves on $X_{\text{proét}}$.

7.1. Locally constant sheaves.

Since this section is concerned with locally constant sheaves on various sites, we quickly recall the definitions and notations we use for such objects.

Let $\mathcal{C}$ be a site. Following [SGA4, Tome I, Exposé IV, §4.3], we call an object of $\mathbf{Sh}(\mathcal{C})$ *constant* if it is in the essential image of the pullback functor p^* for the unique morphism of topoi $p\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Set}$. Equivalently, an object of $\mathbf{Sh}(\mathcal{C})$ is constant, and isomorphic to $p^*(S)$, if and only if it is isomorphic to the sheafification $\underline{S}$ (or $\underline{S}_{\mathcal{C},S}$ if we want to emphasize $\mathcal{C}$) of the constant presheaf associated to S. For a morphism of topoi $f\colon \mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$, the equality $f^*(\underline{S}_{\mathcal{D}}) = \underline{S}_{\mathcal{C}}$ holds.

Let us denote by $\mathbf{1}$ the final object of $\mathbf{Sh}(\mathcal{C})$. Following [SGA4, Tome III, Exposé IX, §2.0], we say that an object $\mathcal{F}$ is *locally constant* if there exists a cover $\{U_i \to \mathbf{1}\}$ in the canonical topology on $\mathbf{Sh}(\mathcal{C})$ (see [SGA4, Tome I, Exposé II, §2.5]) such that for all i the restriction $\mathcal{F}|_{U_i}$ is a constant as an object of $\mathbf{Sh}(\mathcal{C})/U_i$. Suppose that $\mathcal{C}$ has a final object X. In this case an object $\mathcal{F}$ of $\mathbf{Sh}(\mathcal{C})$ is locally constant if and only if there exists a cover $\{U_i \to X\}$ such that $\mathcal{F}|_{U_i}$ is a constant sheaf in $\mathbf{Sh}(\mathcal{C}/U_i)$ (e.g. apply [SGA4, Tome I, Exposé III, Proposition 5.4] and [SGA4, Tome I, Exposé II, Corollaire 4.1.1 and Proposition 4.3]). Let us denote for a site $\mathcal{C}$ the full subcategory of locally constant objects of $\mathbf{Sh}(\mathcal{C})$ by $\mathbf{Loc}(\mathcal{C})$. For a morphism of topoi $\mathbf{Sh}(\mathcal{C}) \to \mathbf{Sh}(\mathcal{D})$, the pullback of a locally constant sheaf is locally constant.

7.2. The pro-étale site of a rigid space and classical sheaves.

We begin by recalling the pro-étale site $X_{\text{proét}}$ of an adic space X. Moreover, we single out a class of sheaves in $\mathbf{Sh}(X_{\text{proét}})$, the analogue of classical sheaves from [BS15], which play an important role in the proof of our main result.

The pro-étale site. Before we define the pro-étale site, we give some notational conventions concerning the pro-completion of a category. Namely, if $\mathcal{C}$ is a category we denote by $\mathbf{Pro}(\mathcal{C})$ the pro-completion of $\mathcal{C}$ as in [KS06, §6]. We abbreviate the notation for a general object $F : \mathcal{I} \to \mathcal{C}$ of $\mathbf{Pro}(\mathcal{C})$ to $\{U_i\}_{i\in\mathcal{I}}$ (or just $\{U_i\}$) where $F(i) = U_i$. We freely identify $\mathcal{C}$ as a full subcategory of $\mathbf{Pro}(\mathcal{C})$ by sending U to the constant diagram $\{U\}$ which we shorten to U. To distinguish such objects of $\mathcal{C}$ from a general object of $\mathbf{Pro}(\mathcal{C})$ we write the latter with the sans serif font (e.g. U).

Let us denote by $\mathbf{Pro\acute{E}t}_X$ the full subcategory of $\mathbf{Pro}(\mathbf{\acute{E}t}_X)$ consisting of those objects U such that $\mathsf{U} \to X$ is pro-étale in the sense of [Sch13, Definition 3.9]. Let us note that every object of $\mathbf{Pro\acute{E}t}_X$ has a presentation of the form $\{U_i\}_{i\in\mathcal{I}}$ where

- $\mathcal{I}$ has a final object 0 such that $U_0 \to X$ is étale,
- the maps $U_j \to U_i$ for $i \geqslant j \geqslant 0$ are finite étale and surjective.

When speaking of a presentation of an object of $\mathbf{Pro\acute{E}t}_X$ we shall assume it is of this form. By [Sch13, Lemma 3.10 vii)] the category $\mathbf{Pro\acute{E}t}_X$ admits all finite limits, a fact we use without further comment.

As in [Sch13, §3], for an object $\mathsf{U} = \{U_i\}$ we denote by $|\mathsf{U}|$ the topological space $\varprojlim |U_i|$ and call it the *underlying topological space* of U. The formation of the underlying topological space is functorial in U, and thus independent of presentation. We call an object U of $\mathbf{Pro\acute{E}t}_X$ *quasi-compact and quasi-separated* if its underlying space $|\mathsf{U}|$ is. We denote the subcategory of quasi-compact and quasi-separated objects of $\mathbf{Pro\acute{E}t}_X$ by $\mathbf{Pro\acute{E}t}_X^{\mathrm{qcqs}}$. Utilizing [Sch13, Proposition 3.12 i), ii), and v)] one can show that $\mathbf{Pro\acute{E}t}_X^{\mathrm{qcqs}}$ is closed under fiber products and every object of $\mathbf{Pro\acute{E}t}_X$ has a cover in $X_{\mathrm{pro\acute{e}t}}$ by objects of $\mathbf{Pro\acute{E}t}_X^{\mathrm{qcqs}}$ which moreover can be assumed to be open embeddings on the underlying topological spaces. We denote by $X_{\mathrm{pro\acute{e}t}}^{\mathrm{qcqs}}$ the induced site structure on $\mathbf{Pro\acute{E}t}_X^{\mathrm{qcqs}}$ from $X_{\mathrm{pro\acute{e}t}}$. By [SGA4, Tome I, Exposé III, Thèoremé 4.1] the natural morphism of topoi $\mathbf{Sh}(X_{\mathrm{pro\acute{e}t}}) \to \mathbf{Sh}(X_{\mathrm{pro\acute{e}t}}^{\mathrm{qcqs}})$ is an equivalence.

We define the *pro-étale site* of X, denoted $X_{\mathrm{pro\acute{e}t}}$, to be the site whose underlying category is $\mathbf{Pro\acute{E}t}_X$ and whose topology is as in [BMS19, §5.1]. The inclusion functor $\mathbf{\acute{E}t}_X \to \mathbf{Pro\acute{E}t}_X$ preserves finite limits (e.g. dualize [KS06, Corollary 6.1.6]), and is a continuous morphism of categories with Grothendieck topologies. So, by [Stacks, Tag 00X6] we obtain an induced morphism of sites $X_{\mathrm{pro\acute{e}t}} \to X_{\mathrm{\acute{e}t}}$. We denote by ν_X, or ν when X is clear from context, the induced morphism of topoi $\mathbf{Sh}(X_{\mathrm{pro\acute{e}t}}) \to \mathbf{Sh}(X_{\mathrm{\acute{e}t}})$.

Classical sheaves. Using the morphism of topoi ν_X, we can define the appropriate notion of 'classical sheaf' as in [BS15, Definition 5.1.3].

Definition 7.2.1. A sheaf $\mathcal{G}$ in $\mathbf{Sh}(X_{\mathrm{pro\acute{e}t}})$ is called *classical* if it is in the essential image of the pullback functor $\nu^*\colon \mathbf{Sh}(X_{\mathrm{\acute{e}t}}) \to \mathbf{Sh}(X_{\mathrm{pro\acute{e}t}})$.

As the pullback functor ν_X^* is quite inexplicit we would like to give a more usable description of the values of these pullbacks on quasi-compact and quasi-separated objects. To make this precise, for a sheaf $\mathcal{F}$ in $\mathbf{Sh}(X_{\mathrm{\acute{e}t}})$ let us denote by $\mathcal{F}^{(\to)}$ the association

$$\mathcal{F}^{(\to)}(\{U_i\}) = \varinjlim \mathcal{F}(U_i)$$

which is easily seen to be an element of $\mathbf{PSh}(X_{\mathrm{pro\acute{e}t}})$. In fact, one can check that $\mathcal{F}^{(\to)}$ is the presheaf pullback of $\mathcal{F}$, and thus there is a natural map $\mathcal{F}^{(\to)} \to \nu_X^*(\mathcal{F})$ of objects of $\mathbf{PSh}(X_{\mathrm{pro\acute{e}t}})$.

Proposition 7.2.2 ([Sch13, Lemma 3.16])**.** *For any sheaf $\mathcal{F}$ in $\mathbf{Sh}(X_{\mathrm{\acute{e}t}})$ the map $\mathcal{F}^{(\to)} \to \nu_X^*(\mathcal{F})$ is a bijection when evaluated on any object of $X_{\mathrm{pro\acute{e}t}}^{\mathrm{qcqs}}$.*

Remark 7.2.3. Note that while the cited result assumes that $\mathcal{F}$ is abelian, the proof remains valid for computing the global sections for general sheaves of sets. The use of injective sheaves is only to address the higher cohomology groups. Compare also [BS15, Lemma 5.1.2].

Using this, we obtain a structure theorem for classical sheaves (cf. [BS15, Lemma 5.1.2]).

Proposition 7.2.4. *For any sheaf $\mathcal{F}$ in $\mathbf{Sh}(X_{\text{ét}})$, the unit map $\mathcal{F} \to \nu_*\nu^*\mathcal{F}$ for the adjunction $\nu^* \dashv \nu_*$ is an isomorphism. Therefore, the functor $\nu^*\colon \mathbf{Sh}(X_{\text{ét}}) \to \mathbf{Sh}(X_{\text{proét}})$ is fully faithful with essential image those $\mathcal{F}$ such that the counit map $\nu^*\nu_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.*

Proof. The latter statements follow from general category theory (see [Mac71, §IV.1, Theorem 1] and [Stacks, Tag 07RB]). To see the first statement we note that by [SGA4, Tome I, Exposé III, Thèoremé 4.1] it suffices to show that the map $\mathcal{F} \to \nu_*\nu^*\mathcal{F}$ is an isomorphism when evaluated on any quasi-compact and quasi-separated object of $X_{\text{ét}}$. But, this then follows immediately from Proposition 7.2.2. □

Arguing as in [BS15, Lemma 5.1.4], we obtain the following corollary.

Lemma 7.2.5. *Let $\mathcal{F}$ be an object of $\mathbf{Sh}(X_{\text{proét}})$. If there exists a covering $\{\mathsf{U}_i \to X\}$, and for each i a classical sheaf $\mathcal{F}_i$ such that $\mathcal{F}|_{\mathsf{U}_i} \simeq \mathcal{F}_i|_{\mathsf{U}_i}$, then $\mathcal{F}$ is classical. In particular, locally constant sheaves are classical.*

Using this and 2.1.4 we can give a concrete description of constant sheaves on $X_{\text{proét}}$.

Proposition 7.2.6. *For any set S the natural map of sheaves $\underline{S}_{X_{\text{proét}}} \to \mathrm{Hom}_{\text{cnts}}(\pi_0(|-|), S)$ is a bijection when applied to any object of $X^{\text{qcqs}}_{\text{proét}}$.*

We now use the above discussion to shed further light on representable presheaves on $X_{\text{proét}}$. For an object Y of $\mathbf{\acute{E}t}_X$ we denote by $h_{Y,\text{ét}}$ its corresponding representable sheaf (see [Hub96, (2.2.2)]). Similarly, for an object Y of $\mathbf{Pro\acute{E}t}_X$ we denote by $h_{\mathsf{Y},\text{proét}}$ its corresponding representable presheaf, and by $h^\sharp_{\mathsf{Y},\text{proét}}$ its sheafification. The sheafification is necessary, as the following example shows.

Example 7.2.7. Let X be the disjoint union of copies X_n of $\mathrm{Spa}(K)$ indexed by $n \geq 0$, and let L be a non-trivial finite separable extension of K. We set Y to be the disjoint union of $Y_n = \mathrm{Spa}(L)$, with the natural étale map $Y \to X$. Let $\mathsf{U} = \{U_i\}$ be the following object of $X_{\text{proét}}$ indexed by $i \geq 0$: the space U_i is the disjoint union of $U_{i,n}$ indexed by $n \geq 0$, where $U_{i,n} = X_n$ for $n > i$ and $U_{i,n} = Y_n$ for $n \leq i$. Note that for every i, we have $\mathrm{Hom}_X(U_i, Y) = \emptyset$, and hence $h_{Y,\text{proét}}(\mathsf{U})$ is empty. Then $\{Y_n \to \mathsf{U}\}_{n\geq 0}$ forms a pro-étale cover which violates the sheaf condition for $h_{Y,\text{proét}}$.

That said, while $X_{\text{proét}}$ is not subcanonical, we can use Proposition 7.2.2 to show that the sheafification of a representable presheaf on $X_{\text{proét}}$ is a sheaf when restricted to $X^{\text{qcqs}}_{\text{proét}}$.

Proposition 7.2.8. *Let Y be an object of $\mathbf{Pro\acute{E}t}_X$. Then, the natural map $h_{\mathsf{Y},\text{proét}} \to h^\sharp_{\mathsf{Y},\text{proét}}$ a bijection when evaluated on any element of $\mathbf{Pro\acute{E}t}^{\text{qcqs}}_X$. In particular, the site $X^{\text{qcqs}}_{\text{proét}}$ is subcanonical.*

Proof. By Lemma 7.2.9 below, it suffices to show that $h_{\mathsf{Y},\text{proét}}$ is a sheaf when restricted to $X^{\text{qcqs}}_{\text{proét}}$. But, if $\mathsf{Y} = \{Y_i\}$ then $h_{\mathsf{Y},\text{proét}} = \varprojlim h_{Y_i,\text{proét}}$ where this inverse limit is taken in $\mathbf{PSh}(X_{\text{proét}})$. Since the inverse limit of sheaves, taken in the category of presheaves, is a sheaf, we're restricted to showing that for all i the restriction of $h_{Y_i,\text{proét}}$ to $X^{\text{qcqs}}_{\text{proét}}$ is a sheaf. But, since $h_{Y_i,\text{proét}} = h^{(\to)}_{Y_i,\text{ét}}$ this follows from Proposition 7.2.2. □

Lemma 7.2.9. *Let $\mathcal{C}$ be a site and let $\mathcal{B} \subseteq \mathcal{C}$ be a full subcategory closed under fiber products and such that every object V of $\mathcal{C}$ admits a covering family $\{U_\alpha \to V\}_{\alpha\in I}$ with each U_α an object of $\mathcal{B}$. Let $\mathcal{F}$ be a presheaf on $\mathcal{C}$ whose restriction to $\mathcal{B}$ is a sheaf, i.e. which satisfies the sheaf condition for coverings of $\{U_\alpha \to U\}_{\alpha\in I}$ with U and all U_α objects of $\mathcal{B}$, and let $\mathcal{F}^\#$ be its sheafification. Then, for every object U of $\mathcal{B}$, the natural map $\mathcal{F}(U) \to \mathcal{F}^\#(U)$ is an isomorphism.*

Proof. Let us recall the construction of $\mathcal{F}^\#$ from [Stacks, Tag 00W1]. For a covering family $\mathcal{V} = \{V_\alpha \to V\}_{\alpha\in I}$ in $\mathcal{C}$, we denote by $H^0(\mathcal{V}, \mathcal{F})$ the corresponding equalizer. For an object V of $\mathcal{C}$, we denote by $\mathcal{J}_V$ the category of coverings of V: its objects are covering families

$\{V_\alpha \to V\}_{\alpha \in I}$ and morphisms are refinements. We note that by [Stacks, Tag 00W7] and [Stacks, Tag 00W6] that the diagram $H^0(-,\mathcal{F})\colon \mathcal{J}_V \to \mathbf{Set}$ is filtered (see [Stacks, Tag 002V]). We define $\mathcal{F}^+(V) = \varinjlim_{\mathcal{J}_V} H^0(\mathcal{V},\mathcal{F})$. This is a presheaf endowed with a natural transformation $\mathcal{F} \to \mathcal{F}^+$. The sheafification of $\mathcal{F}$ is identified with the composition $\mathcal{F} \to \mathcal{F}^+ \to (\mathcal{F}^+)^+$.

For V an object of $\mathcal{C}$, consider the full subcategory $\mathcal{J}'_V$ of $\mathcal{J}_V$ consisting of coverings whose elements are in $\mathcal{B}$. One can check that the diagram $H^0(-,\mathcal{F})\colon \mathcal{J}'_V \to \mathbf{Set}$ is cofinal in the diagram $H^0(-,\mathcal{F})\colon \mathcal{J}_V \to \mathbf{Set}$. Therefore $\mathcal{F}^+(V) = \varinjlim_{\mathcal{J}'_V} H^0(\mathcal{V},\mathcal{F})$. This colimit depends only on the values of $\mathcal{F}$ on V, and if V is itself an object of $\mathcal{B}$, then $\mathcal{F}^+(V) = \mathcal{F}(V)$. Therefore $\mathcal{F}^\#(V) = (\mathcal{F}^+)^+(V) = \mathcal{F}^+(V) = \mathcal{F}(V)$. $\square$

7.3. **Sheaves on G-pFSet and $X_{\text{profét}}$.** In this subsection we establish a version of our main result for the category G-**pFSet** of profinite spaces with continuous G-action where G is a profinite group. More precisely, we describe the locally constant sheaves on G-**pFSet** in terms of 'representing objects'. We then use this to obtain a similar description of the locally constant sheaves on the pro-finite étale site $X_{\text{profét}}$ of an adic space X.

The site G-pFSet. Let G be a profinite topological group. Let us denote by G-**FSet** the category of finite (discrete) spaces with a continuous action of G. One has a natural identification **Pro**(G-**FSet**) with G-**pFSet** where the latter category, by definition, is the category of profinite topological spaces equipped with a continuous action of G.

We endow G-**pFSet** with a Grothendieck topology where the coverings $\{S_i \to S\}$ are jointly surjective maps such that each map $S_i \to S$ satisfies a condition similar to that in the definition of the pro-étale site as in [BMS19, §5.1] (with '$\mathsf{U}_0 \to \mathsf{U}$ étale' replaced by '$\mathsf{U}_0 \to \mathsf{U}$ is the pullback of a morphism in G-**FSet**' and '$\mathsf{U}_\mu \to \mathsf{U}_{<\mu}$ is finite étale surjective' is replaced by '$\mathsf{U}_\mu \to \mathsf{U}_{<\mu}$ is the pullback of a finite surjective map in G-**FSet**').

The following claim is not obvious, and is pivotal in the proof of Proposition 7.3.3 below.

Proposition 7.3.1 ([Sch16, (1)])**.** *Let G be a profinite group. Then, $\{G \to *\}$ is a cover.*

The following result, while being of independent interest, is most useful for its implication concerning the site $X_{\text{profét}}$ (see Proposition 7.3.4).

Proposition 7.3.2. *Let G be a profinite group. Then, the site G-**pFSet** has all finite colimits and is subcanonical.*

Proof. Since one has an equivalence G-**pFSet** $\simeq$ **Pro**(G-**FSet**) and G-**FSet** has all finite colimits, the existence of all finite colimits in G-**pFSet** follows from [KS06, Corollary 6.1.17 i)].

We now show G-**pFSet** is subcanonical. Let S be an element of G-**pFSet**, we wish to show that h_S is a sheaf. Note though that we can write $S = \varprojlim S_i$ where S_i are finite objects of G-**pFSet**. Since $h_S = \varprojlim_i h_{S_i}$ where the right hand side is a limit in the category of presheaves, and the limit in the category of presheaves where the constituents are sheaves is a sheaf, we are reduced to showing that h_S is a sheaf when S is finite. This case is handled at the beginning of the proof of Proposition 7.3.3 below. $\square$

Locally constant sheaves on G-pFSet. We now describe the category **Loc**(G-**pFSet**) in terms of the category G-**Set** of discrete sets with a continuous action of G.

Proposition 7.3.3. *Let G be a profinite topological group. Then, the functor*

$$G\text{-}\mathbf{Set} \to \mathbf{Loc}\,(G\text{-}\mathbf{pFSet})\,, \qquad T \mapsto (\mathcal{F}_T : S \mapsto \mathrm{Hom}_{\mathrm{cnts},G}(S,T))$$

is an equivalence of categories.

Proof. We first show that $\mathcal{F}_T$ is a sheaf. As covers in G-**pFSet** are jointly surjective, it follows that the sheaf condition is satisfied for the presheaf $S \mapsto \mathrm{Hom}_G(S,T)$ on G-**pFSet**. One needs to check that continuity condition is local, which can be checked as follows. Any cover $\{U_i \to U\}$ in G-**pFSet** has a finite subcover. Passing to such subcover and considering $\coprod_i U_i \to U$, we can assume the map to be a surjective map of compact spaces. It is then automatically a quotient map of topological spaces, thus the continuity of functions can be tested after pullback to $\coprod_i U_i$.

For S (resp. T) an object of G-**pFSet** (resp. G-**Set**), we will denote the underlying set of S (resp. T) with trivial G-action by S_{ta} (resp. T_{ta}) which is an object of G-**pFSet** (resp. G-**Set**). We claim that $\mathcal{F}_T$ is trivialized after restricting to G seen as an element of G-**pFSet**. Indeed, the slice category G-**pFSet**$/G$ is equivalent to the category of profinite sets $*$-**pFSet** via $(S \xrightarrow{f} G) \mapsto (f^{-1}(1_G))$. Under this equivalence, the pushforward of $\mathcal{F}_T$ is the sheaf on $*$-**pFSet** given by $U \mapsto \mathrm{Maps}_{\mathrm{cnts}}(U, T)$, which is precisely a constant sheaf in $*$-**pFSet**.

Suppose now that $\mathcal{F}$ is a locally constant sheaf on G-**pFSet**. We wish to show there exist T in G-**Set** such that $\mathcal{F} = \mathcal{F}_T$. To show this it is first useful to see that such a $\mathcal{F}$ satisfies a 'classicality' condition. More precisely, that the obvious map $\operatorname{colim} \mathcal{F}(S) \to \mathcal{F}(S_i)$, where $S = \varprojlim S_i$ where S_i are finite G-sets, is a bijection. To show such an equality it suffices to check it after a cover (cf. [BS15, Lemma 5.1.4]). This reduces us to the constant case, which is trivial.

We define our candidate for T as follows. As a discrete set, we set $T = \mathcal{F}(G)$. The G-action on $T = \mathcal{F}(G)$ is defined using the map $G \to \mathrm{Aut}_{G\text{-}\mathbf{pFSet}}(G)^{\mathrm{op}}$ given by $h \mapsto (g \mapsto g \cdot h)$ (where we view G as an element of G-**pFSet** by acting on the left). To see that this action is continuous we must show the equality $\mathcal{F}(G) = \operatorname{colim} \mathcal{F}(G/U)$, as U travels over the open normal subgroups of G, but this follows from the classicality condition.

We now verify that $\mathcal{F}_T$ is isomorphic to $\mathcal{F}$. By Lemma 7.3.1 the map $G \to *$ is a cover, and thus $G \times S \to S$ is a cover for any S an object of G-**pFSet**. Observe that $G \times S_{\mathrm{ta}} \to S$ defined by $(g, s) \mapsto gs$ is isomorphic to $G \times S \xrightarrow{\mathrm{pr}_S} S$ in G-**pFSet**$/S$ via the map $(g, s) \mapsto (g, gs)$. Similarly, $G \times G_{\mathrm{ta}} \times S_{\mathrm{ta}}$ is isomorphic to $G \times G \times S$ via the map $(g, h, s) \mapsto (g, gh, gs)$. We thus have the following isomorphism of diagrams.

$$\begin{tikzcd}
S \arrow[d, equal] & G \times S_{\mathrm{ta}} \arrow[l] \arrow[d, "\wr"] & G \times G_{\mathrm{ta}} \times S_{\mathrm{ta}} \arrow[l, shift left] \arrow[l, shift right] \arrow[d, "\wr"] \\
S & G \times S \arrow[l] & G \times G \times S. \arrow[l, shift left] \arrow[l, shift right]
\end{tikzcd} \tag{7.3.1}$$

By the classicality of $\mathcal{F}$ we have a canonical identification

$$\mathcal{F}(G \times S_{\mathrm{ta}}) = \operatorname*{colim}_i \mathcal{F}(G \times S_i) = \mathrm{Maps}(S_i, T) = \mathrm{Hom}_{\mathrm{cnts}}(S_{\mathrm{ta}}, T)$$

after presenting $S_{\mathrm{ta}} = \lim_i S_i$ for finite (discrete) S_i, and the middle equality follows from the canonical identifications $\mathcal{F}(G \times S_i) = \mathcal{F}(\coprod_{s \in S_i} G) = \prod_{s \in S_i} \mathcal{F}(G) = \mathrm{Maps}(S_i, T)$. Consider the exact sequence of sets

$$\mathcal{F}(S) \to \mathcal{F}(G \times S_{\mathrm{ta}}) \rightrightarrows \mathcal{F}(G \times G_{\mathrm{ta}} \times S_{\mathrm{ta}})$$

obtained by using the identification given in Equation (7.3.1), the observation that as an object of G-**Set** we have that $(G \times S) \times_S (G \times S)$ is isomorphic to $G \times G \times S$, and the sheaf sequence for the cover $G \times S \to S$. We then make the identifications

$$\mathcal{F}(G \times S_{\mathrm{ta}}) = \mathrm{Hom}_{\mathrm{cnts}}(S_{\mathrm{ta}}, T), \qquad \mathcal{F}(G \times G_{\mathrm{ta}} \times S_{\mathrm{ta}}) = \mathrm{Hom}_{\mathrm{cnts}}(G_{\mathrm{ta}} \times S_{\mathrm{ta}}, T) \tag{7.3.2}$$

as above. As the maps $G \times G_{\mathrm{ta}} \times S_{\mathrm{ta}} \rightrightarrows G \times S_{\mathrm{ta}}$ are explicitly given by $(g, h, s) \mapsto (g, s)$ and $(g, h, s) \mapsto (gh, h^{-1}s)$, and by the definition of the action of G on T, we see that the corresponding maps $\mathrm{Hom}_{\mathrm{cnts}}(S_{\mathrm{ta}}, T) \rightrightarrows \mathrm{Hom}_{\mathrm{cnts}}(G_{\mathrm{ta}} \times S_{\mathrm{ta}}, T)$ are given by $f \mapsto f \circ \mathrm{pr}_{S_{\mathrm{ta}}}$ and $f \mapsto ((h, s) \mapsto h \cdot f(h^{-1} \cdot s))$. Using this, and the sequence given in Equation (7.3.2) we get the canonical identification of $\mathcal{F}(S)$ and $\mathrm{Hom}_{\mathrm{cnts},G}(S, T) = \mathcal{F}_T(S)$, as desired.

We now proceed to verify that our functor, which we denote by ψ for convenience, is fully faithful. Let $T_1, T_2 \in G$-**Set** and let $f, g : T_1 \to T_2$ be two maps in G-**Set**. For any t in T_1, let ϵ_t be the unique G-equivariant map between G and T_1 that maps 1_G to t. It is automatically continuous, and thus gives an element of $\mathcal{F}_{T_1}(G)$. If $f \neq g$ and t is such that $f(t) \neq g(t)$ then one computes that $\psi(f)(\epsilon_t) \neq \psi(g)(\epsilon_t)$. This proves faithfullness. Let $\alpha \colon \mathcal{F}_{T_1} \to \mathcal{F}_{T_2}$ be a morphism in $\mathbf{Loc}(X_{\mathrm{prof\acute{e}t}})$. In particular, this gives

$$\alpha_G \colon \mathrm{Hom}_{\mathrm{cnts},G}(G, T_1) = \mathcal{F}_{T_1}(G) \to \mathcal{F}_{T_2}(G) = \mathrm{Hom}_{\mathrm{cnts},G}(G, T_2).$$

Let ϵ_t be as before. Define $f(t) = \alpha_G(\epsilon_t)(1_G)$. Then one checks that f is an element of $\mathrm{Hom}_{G\text{-}\mathbf{Set}}(T_1, T_2)$ and that $\alpha = \psi(f)$. Thus, the functor ψ is full. $\square$

The site $X_{\text{profét}}$. We now consider the pro-finite étale site of an adic space X. Consider $\mathbf{Pro}(\mathbf{F\acute{E}t}_X)$ as a full subcategory of $\mathbf{Pro\acute{E}t}_X$ and endow it with a Grothendieck topology as in [BMS19, §5.1] (with '$\mathsf{U}_0 \to \mathsf{U}$ étale' replaced by '$\mathsf{U}_0 \to \mathsf{U}$ finite étale').

Our main technique for studying $X_{\text{profét}}$ is the relationship with the site $\pi_1^{\text{alg}}(X, \overline{x})\text{-}\mathbf{pFSet}$.

Proposition 7.3.4 ([Sch13, Proposition 3.5],[Sch16])**.** *Let X be a connected adic space and let $\overline{x}$ be a geometric point of X. The functor*

$$X_{\text{profét}} \to \pi_1^{\text{alg}}(X, \overline{x})\text{-}\mathbf{pFSet}, \qquad \{U_i\} \mapsto \varprojlim F_{\overline{x}}(U_i)$$

is an equivalence of sites.

From this and Proposition 7.3.2 we immediately deduce the following.

Corollary 7.3.5. *The site $X_{\text{profét}}$ has all finite colimits and is subcanonical.*

Localy constant sheaves on $X_{\text{profét}}$. We end this section by describing all the objects of $\mathbf{Loc}(X_{\text{profét}})$ using Proposition 7.3.3. To do so, let us define for any object $\mathsf{Y} = \{Y_j\}$ of $\mathbf{Pro}(\mathbf{\acute{E}t}_X)$ the presheaf $h_{\mathsf{Y},\text{profét}}$ on $X_{\text{profét}}$ obtained by restricting $h_{\mathsf{Y},\text{proét}}$ to $X_{\text{profét}}$. More explicitly, we have the following formula.

$$h_{\mathsf{Y},\text{profét}}(\{U_i\}) = \varprojlim_j \varinjlim_i \mathrm{Hom}_{\mathbf{\acute{E}t}_X}(U_i, Y)$$

We define $h^\sharp_{\mathsf{Y},\text{profét}}$ to be the sheafification of this presheaf. The following result then follows from Proposition 7.3.4, Proposition 7.3.3, and the fact that $\pi_1^{\text{alg}}(X, \overline{x})\text{-}\mathbf{Set} \simeq \mathbf{UF\acute{E}t}_X$.

Proposition 7.3.6. *For all Y in $\mathbf{UF\acute{E}t}_X$, the presheaf $h_{Y,\text{profét}}$ is a sheaf and the functor*

$$\mathbf{UF\acute{E}t}_X \to \mathbf{Loc}(X_{\text{profét}}), \qquad Y \mapsto h_{Y,\text{profét}}$$

is an equivalence of categories.

Proof. It suffices to verify the first claim. We may assume that X is connected. The presheaf $h_{Y,\text{profét}}$ corresponds via the equivalence in Proposition 7.3.4 to the sheaf $\mathcal{F}_T$ from Proposition 7.3.3 with $T = Y_{\overline{x}}$, and thus is a sheaf. $\square$

7.4. Sheaves on $X_{\text{proét}}$ which are pro-finite étale locally constant.

We now to upgrade Proposition 7.3.6 to a result about sheaves on $X_{\text{proét}}$ which are pro-finite étale locally constant.

Comparing $\mathbf{Sh}(X_{\text{proét}})$ and $\mathbf{Sh}(X_{\text{profét}})$. To carry out this special case of our main theorem we must first be able to compare locally constant sheaves on $X_{\text{proét}}$ and $X_{\text{profét}}$. To this end, note that the inclusion $\mathbf{Pro}(\mathbf{F\acute{E}t}_X) \to \mathbf{Pro\acute{E}t}_X$ is a continuous functor of categories with Grothendieck topologies. Since this morphism preserves fiber products we deduce from [Stacks, Tag 00X6] that we get an induced map of sites $X_{\text{proét}} \to X_{\text{profét}}$. We denote the induced morphism of topoi $\mathbf{Sh}(X_{\text{proét}}) \to \mathbf{Sh}(X_{\text{profét}})$ by θ_X or just θ when X is clear from context. We then have the following comparison result (cf. [AG16, Proposition VI.9.18]).

Proposition 7.4.1. *Let X be a quasi-compact and quasi-separated adic space. Then, the unit map $\mathbf{1} \to \theta_*\theta^*$ for the adjunction $\theta^* \dashv \theta_*$ is an isomorphism. Therefore, the functor $\theta^* \colon \mathbf{Sh}(X_{\text{profét}}) \to \mathbf{Sh}(X_{\text{proét}})$ is fully faithful with essential image those sheaves $\mathcal{F}$ for which the counit map $\theta^*\theta_*\mathcal{F} \to \mathcal{F}$ is an isomorphism.*

Proof. Again, the latter statements follow from general category theory. To show the first statement begin by noting that θ^*, being a left adjoint, commutes with colimits. We claim that θ_* also commutes with filtered colimits. By [SGA4, Tome II, Exposé VI, Thèoremé 5.1] it suffices to prove that θ is a coherent morphism between coherent topoi. The fact that $\mathbf{Sh}(X_{\text{proét}})$ is coherent is verified in [Sch13, Proposition 3.12 (vii)], and the fact that $\mathbf{Sh}(X_{\text{profét}})$ is coherent follows easily from this. To show that θ is coherent let $\mathsf{V} = \{V_j\}$ be an object of $X_{\text{profét}}$. This system consists of spectral spaces and the transition maps are quasi-compact. We deduce from Proposition 2.1.4 that $|\mathsf{V}|$ is quasi-compact and quasi-separated from where we are finished by [Sch13, Proposition 3.12 (iv)].

Fix a sheaf $\mathcal{F}$ in $\mathbf{Sh}(X_{\text{proét}})$. We now show that the unit map $\mathcal{F} \to \theta^*\theta_*\mathcal{F}$ is an isomorphism. By Corollary 7.3.5 the category $X_{\text{profét}}/\mathcal{F}$ is filtered, where this category means the subcategory of the slice category $\mathbf{PSh}(X_{\text{profét}})/\mathcal{F}$ consisting of representable presheaves (see [SGA4, Tome I, Exposé I, 3.4.0]). So then, we have by [SGA4, Tome I, Exposé II, Corollaire 4.1.1] that the natural map

$$\varinjlim_{\mathsf{Y} \in \mathbf{Pro}(\mathbf{F\acute{E}t}_X)/\mathcal{F}} h_{\mathsf{Y},\text{profét}} \to \mathcal{F}$$

is an isomorphism where we do not need to sheafify $h_{\mathsf{Y},\text{profét}}$ thanks to Corollary 7.3.5. Since θ^* and θ_* both commute with filtered colimits we've thus reduced to showing that the unit map is an isomorphism when evaluated on representable sheaves $h_{\mathsf{Y},\text{profét}}$. A quick computation using the adjunction property and [Stacks, Tag 04D3] shows this is equivalent to the natural map $h_{\mathsf{Y},\text{profét}}(\mathsf{V}) \to h^\sharp_{\mathsf{Y},\text{proét}}(\mathsf{V})$ being a bijection. But, this follows from Proposition 7.2.8. $\square$

The following will also be used in the sequel.

Proposition 7.4.2. *Let X be a quasi-compact and quasi-separated adic space and let Y be an object of $\mathbf{UF\acute{E}t}_X$. Then, $\theta^* h_{Y,\text{profét}} \simeq h^\sharp_{Y,\text{proét}}$.*

Proof. Write $Y = \varinjlim_j Y_j$ with Y_j objects of $\mathbf{F\acute{E}t}_X$ and the transition maps are open embeddings. Since θ^* commutes with colimits, we're reduced to showing that $h_{Y,\text{profét}} \simeq \varinjlim_j h_{Y_j,\text{profét}}$ and $h^\sharp_{Y,\text{proét}} \simeq \varinjlim_j h^\sharp_{Y_j,\text{proét}}$. The former is clear since every object of $X_{\text{profét}}$ is quasi-compact and quasi-separated, and the latter is clear by combining Proposition 7.2.8 and [SGA4, Tome I, Exposé III, Thèoremé 4.1]. $\square$

Locally constant sheaves on $X_{\text{proét}}$ trivialized on a pro-finite étale cover. We now establish the aforementioned special case of our main theorem, classifying locally constant sheaves on $X_{\text{proét}}$ which become constant on a pro-finite étale cover of X.

Proposition 7.4.3. *Let X be a quasi-compact and quasi-separated adic space. Then, the functor*

$$\mathbf{UF\acute{E}t}_X \to \mathbf{Loc}(X_{\text{proét}}), \qquad Y \mapsto h^\sharp_{Y,\text{proét}}$$

is fully faithful with essential image those objects $\mathcal{F}$ of $\mathbf{Loc}(X_{\text{proét}})$ which become constant on a pro-finite étale cover of X.

Proof. We may assume that X is connected. To see this functor has image contained in the correct subcategory of $\mathbf{Loc}(X_{\text{proét}})$ let Y be an object of $\mathbf{UF\acute{E}t}_X$. As observed in the proof of Proposition 7.3.6 the sheaf $h_{Y,\text{profét}}$ corresponds, under Proposition 7.3.4 to $\mathcal{F}_T$ where $T = Y_{\overline{x}}$. From loc. cit. we see that $h_{Y,\text{profét}}$ is an object of $\mathbf{Loc}(X_{\text{profét}})$. In particular, $\theta^* h_{Y,\text{profét}}$ is an object of $\mathbf{Loc}(X_{\text{proét}})$ trivialized on a pro-finite étale cover of X and so we're done by Proposition 7.4.2. Moreover, our functor is fully faithful since it can be described, by Proposition 7.2.8, as ν^* restricted to $\mathbf{UF\acute{E}t}_X$ from where we deduce fully faithfulness by Proposition 7.2.4.

To show our functor is essentially surjective let $\mathcal{F}$ an object of $\mathbf{Sh}(X_{\text{proét}})$ trivialized on a pro-finite étale cover of X. In particular, it is trivialized on a fixed universal pro-finite Galois cover $\tilde{X}$ of X which, under the equivalence in Proposition 7.3.4, corresponds to $\pi_1^{\text{alg}}(X, \overline{x})$ acting on itself by left multiplication. By combining Proposition 7.3.4 and Proposition 7.3.3, there exists some Y in $\mathbf{UF\acute{E}t}_X$ and an isomorphism $\psi : h_{Y,\text{profét}} \xrightarrow{\sim} \theta_*\mathcal{F}$ of objects of $\mathbf{Sh}(X_{\text{profét}})$. One then obtains a morphism $h^\sharp_{Y,\text{proét}} \xrightarrow{\sim} \theta^* h_{Y,\text{profét}} \xrightarrow{\sim} \theta^*\theta_*\mathcal{F} \to \mathcal{F}$ where the first isomorphism comes from Proposition 7.4.2, the second map is $\theta^*\psi$, and the last map is the counit map. We claim that this is an isomorphism. It remains to check that the counit map $\phi : \theta^*\theta_*\mathcal{F} \to \mathcal{F}$ is an isomorphism. To see this, first observe that since this is a morphism of sheaves and $\tilde{X} \to X$

is a cover in $X_{\text{proét}}$, it is enough to check that $\phi_{|\tilde{X}}$ is an isomorphism. As both sheaves are locally constant and trivialized when restricted to the connected cover $\tilde{X}$, it is enough to check that $\phi(\tilde{X})\colon \theta^*\theta_*\mathcal{F}(\tilde{X}) \to \mathcal{F}(\tilde{X})$ is a bijection. But this can be checked after applying θ_*. The map obtained from unit and counit $\theta_* \to \theta_*\theta^*\theta_* \to \theta_*$ is the identity morphism (see [Mac71, §IV.1, Theorem 1]). Moreover, by Proposition 7.4.1, we know that the first morphism in this composition is an isomorphism. It follows that the second morphism is an isomorphism too. Thus, $\theta_*\theta^*\theta_*\mathcal{F}(\tilde{X}) = \theta_*\mathcal{F}(\tilde{X})$, as desired. □

7.5. **Main result.** We now arrive at the main result of this section. Informally, this result says the locally constant sheaves on the pro-étale site of X are represented by objects of $\mathbf{Cov}_X^{\text{ét}}$.

Theorem 7.5.1. *Let X be an adic space. Then, the functor*

$$\mathbf{Cov}_X^{\text{ét}} \to \mathbf{Loc}(X_{\text{proét}}), \quad Y \mapsto h^\sharp_{Y,\text{proét}}$$

is an equivalence of categories.

Proof. Suppose first that $Y \to X$ is an object of $\mathbf{Cov}_X^{\text{ét}}$. We claim that the object $h^\sharp_{Y,\text{proét}}$ of $\mathbf{Sh}(X_{\text{proét}})$ lies in the subcategory $\mathbf{Loc}(X_{\text{proét}})$. But, there exists an étale cover $\{U_i \to X\}$ such that U_i is affinoid for all i, and such that Y_{U_i} is an object of $\mathbf{UF\acute{E}t}_{U_i}$ for all i. Since $h^\sharp_{Y,\text{proét}}$ restricted to U_i is $h^\sharp_{Y_{U_i},\text{proét}}$ in $\mathbf{Sh}(X_{\text{proét}}/U_i) = \mathbf{Sh}((U_i)_{\text{proét}})$ we know from Proposition 7.4.3 that $h^\sharp_{Y,\text{proét}}$ restricted to each U_i is locally constant. Thus, $h^\sharp_{Y,\text{proét}}$ itself is locally constant.

Note then that by Proposition 7.2.8 our functor is nothing but ν_X^*, it is thus fully faithful by Proposition 7.2.4. Given this, and the fact both the source $\mathbf{Cov}_X^{\text{ét}}$ and the target $\mathbf{Loc}(X_{\text{proét}})$ naturally form stacks on X for the étale topology (the former by [War17, Corollary 3.1.9]), it is enough to show essential surjectivity étale locally on X. In particular, it is enough to assume X is affinoid and show that every sheaf of sets $\mathcal{F}$ of $X_{\text{proét}}$ which becomes constant on a pro-finite étale cover of X comes from an object of $\mathbf{UF\acute{E}t}_X$. This follows directly from Proposition 7.4.3. □

Remark 7.5.2. If X is a connected rigid K-space and $\overline{x}$ a geometric point of X, one sees from Theorem 7.5.1 and Theorem 6.2.1 that $(\mathbf{ULoc}(X_{\text{proét}}), F_{\overline{x}})$, where $\mathbf{ULoc}(X_{\text{proét}})$ denotes the category of disjoint unions of locally constant sheaves on $X_{\text{proét}}$ and where $F_{\overline{x}}$ is the fiber functor, is a tame infinite Galois category with fundamental group $\pi_1^{\text{dJ,ét}}(X, \overline{x})$. The version of the pro-étale topology we are working with in this section is quite coarse, which allows for such a result (compare with [BS15, Example 7.3.12]).

Appendix A. Curve-connectedness of rigid varieties

Definition A.0.1. A rigid K-space is *curve-connected* if for every two classical points x_0, x_1 of X there exists a sequence of morphisms $C_i \to X$ $(i = 0, \ldots, n)$ over K where each C_i is a connected affinoid rigid K-curve and we have $x_0 \in \operatorname{im}(C_0 \to X)$, $x_1 \in \operatorname{im}(C_n \to X)$, and $\operatorname{im}(C_i \to X) \cap \operatorname{im}(C_{i-1} \to X) \neq \emptyset$ for $i = 1, \ldots, n$.

We note that using [Con99, Theorem 3.3.6], one can assume that the curves C_i are smooth over K.

Proposition A.0.2. *Let X be a connected rigid K-space. Then, X is curve-connected.*

The analogous result was proven by de Jong in [dJ95a, Theorem 6.1.1] for quasi-compact rigid K-spaces in the case when K is discretely valued. Our proof follows his, but requires some non-trivial alterations due to the fact that $\mathcal{O}_K$ is non-Noetherian. It is also worth pointing out that Berkovich obtained similar results for Berkovich spaces (see [Ber07, Theorem 4.1.1]) which, in the language of adic spaces, are partially proper over $\operatorname{Spa}(K)$.

Proof. Idea of the proof: If X is a connected affine variety, one can use the Bertini theorem to find a connected hypersurface passing through two given points. In the situation at hand, if $X = \mathrm{Spa}(A)$ is affinoid, we can construct a suitable connected hyperplane section of its reduction $\widetilde{X} = \mathrm{Spec}(\widetilde{A})$, and lift it to the formal model $\mathrm{Spf}(A^\circ)$. The main difficulty, resolved carefully below, is to ensure that the resulting hypersurface remains connected on the generic fiber.

Step 1. [*We may assume that X is affinoid*] Let U be the union of all connected quasi-compact opens containing x_0. Then U is (open and) closed: if $y \notin U$ and V is a connected affinoid neighborhood of y, then $V \cap U = \emptyset$, otherwise $V \cup U$ is quasi-compact and connected. Since X is connected, $x_1 \in U$, i.e. x_0 and x_1 admit a connected quasi-compact neighborhood W. Write $W = \bigcup_{i=1}^n X_i$ with X_i affinoid, which we may assume are connected and $X_i \cap X_{i-1} \neq \emptyset$ (in particular, the intersection has classical points), and it is enough to show the result for the X_i.

Step 2. [*We may assume that X is geometrically connected and geometrically normal*] In the proof, we can freely pass to a finite extension L of K, because every connected component of X_L surjects onto X. By [Con99, §3.2], we may assume that X is geometrically connected. By [Con99], affinoid K-algebras are excellent and we have the normalization map $X^{\mathrm{n}} \to X$, a finite surjective morphism. By [Con99, Theorem 3.3.6], after passing to a finite extension of K we may therefore assume that X is geometrically normal.

Notation. We write $X = \mathrm{Spa}(A)$, denote by $A^\circ \subseteq A$ the subring of powerbounded elements.

Step 3. [*Reduced Fiber Theorem*] By the Reduced Fiber Theorem [GR66, BLR95], passing to a finite extension of K we may assume that A° is a topologically finitely presented $\mathcal{O}_K$-algebra such that $A^\circ \otimes k$ is geometrically reduced and the formation of A° is compatible with further finite extensions of K.

Notation. We write $\widetilde{A} = A^\circ \otimes k$ and $\widetilde{X} = \mathrm{Spec}(\widetilde{A})$, and set $d = \dim X$. We denote the irreducible components of $\widetilde{X}$ by $Z_1, \ldots, Z_r$.

Note that we have $d = \dim Z_i$ for all i. To see this, let $\widetilde{U}_i \subseteq \widetilde{X}$ be an affine open subset contained in Z_i, so that $\dim \widetilde{U}_i = \dim Z_i$, and let $U_i \subseteq \mathrm{Spf}(A^\circ)$ be the corresponding open formal subscheme. Then the generic fiber $U_{i,\eta}$ is an affinoid subdomain of X and hence has dimension d. We conclude by [Con06, Theorem A.2.1].

Step 4. [*Noether normalization*] Since $\widetilde{X}$ is geometrically reduced, Lemma A.0.3 below yields a finite and generically étale morphism $\tilde{f} \colon \widetilde{X} \to \mathbf{A}^d_k$ (if k is finite, we might need to pass to a finite extension of K). Choosing lifts of $\tilde{f}^*(x_i)$ to A°, we lift $\tilde{f}$ to a map

$$f \colon \mathrm{Spec}(A^\circ) \to \mathrm{Spec}(\mathcal{O}_K\langle x_1, \ldots, x_d\rangle),$$

which we claim is finite, and étale at the points of $\widetilde{X}$ where $\tilde{f}$ is étale. Then the morphism $f_\eta \colon X \to \mathbf{D}^d_K$ induced by $K\langle x_1, \ldots, x_d\rangle \to A$ is finite as well.

We begin by showing finiteness. By [FK18, Chapter I, Proposition 4.2.3] it suffices to show that $f_0^* \colon (\mathcal{O}_K/\varpi)[x_1, \ldots, x_d] \to A^\circ/\varpi$ is finite. Let us note then that f_0 is a thickening of $\tilde{f}$ in the sense of [Stacks, Tag 04EX] and that A°/ϖ is of finite presentation over $\mathcal{O}_K/\varpi$. Thus by [Stacks, Tag 0BPG], the finiteness of $\tilde{f}$ implies the finiteness of f_0. Let now $x \in \widetilde{X}$ be a point where $\tilde{f}$ is étale; we will show that f is étale at x. Let $U = \mathrm{Spec}(B)$ be an affine open neighborhood of $f(x)$ such that the induced map $\tilde{f} \colon \widetilde{V} \to \widetilde{U}$ is étale, where $V = f^{-1}(U) = \mathrm{Spec}(C)$. It suffices to show that $B \to C$ is étale. Let $\widehat{B}$, $\widehat{C}$ be the ϖ-adic completions of B and C; since $B \to \widehat{B}$ is faithfully flat by Gabber's lemma [Bos14, 8.2/2] and $\widehat{C} = C \otimes_B \widehat{B}$, it suffices to show that $\widehat{B} \to \widehat{C}$ is étale. Note that since this map is finite, and thus also of finite presentation by [Bos14, Theorem 7.3/4], it suffices to check that it is weakly étale (in the sense of [BS15, Definition 4.1.1]) by [BS15, Theorem 1.3]). But, by [FK18, Proposition I, 5.3.11] this is the same thing as checking that $\widehat{B}/\varpi^n = B/\varpi^n \to \widehat{C}/\varpi^n = C/\varpi^n$ is weakly étale, or equivalently étale, for all $n \geq 1$. Since B/ϖ^n and C/ϖ^n are flat and of finite presentation over $\mathcal{O}_K/\varpi^n$, by [Stacks, Tag 0CF4] applied to the thickening $\mathrm{Spec}(k) \to \mathrm{Spec}(\mathcal{O}_K/\varpi^n)$ this is true because $\widetilde{B} \to \widetilde{C}$ is étale.

Lemma A.0.3 (Generically étale Noether normalization lemma)**.** *Let Y be an affine scheme of finite type over a field k which is geometrically reduced and whose irreducible components have the same dimension d. Then (possibly after passing to a finite extension of k if k is finite) there exists a finite and generically étale map $Y \to \mathbf{A}^d_k$.*

Proof. This follows from the proof of [Stacks, Tag 0CBK] applied to the generic points of Y (the fact that we have several generic points does not change the argument). $\square$

Step 5. We claim that it is enough to show:

If $d \geq 2$, there is a hyperplane $H \subseteq \mathbf{D}^d_K$ such that $f_\eta^{-1}(H) \subseteq X$ is connected.

Here, by a hyperplane we mean the zero set $H = V(h)$ of a linear form $h = \sum_{i=1}^d a_i x_i - a$ with $a_i, a \in \mathcal{O}_K$ with a_i not all zero in k.

The argument for this is exactly as in [dJ95a]: we prove the entire theorem by induction on $d = \dim X \geq 0$. Since the assertion is clear for $d \leq 1$, we assume $d \geq 2$. For the induction step, we find H as in the claim, and since $f_\eta^{-1}(H)$ is curve-connected by the induction assumption, it suffices to connect a given classical point $x \in X$ to a point $y \in f_\eta^{-1}(H)$ by a curve in X. Let $p\colon \mathbf{D}^d_K \to H$ be a linear projection (so $p|_H = \mathrm{id}_H$); then $C = p^{-1}(p(f(x)))$ is a connected curve (in fact, isomorphic to $\mathbf{D}^1_L$ for a finite extension L/K) connecting $f(x)$ with H. Let C' be the connected component of $f^{-1}(C)$ containing x. We claim that $\dim C' = 1$; otherwise, x is an isolated point of $f^{-1}(C)$, which is impossible because the map $\mathrm{Spec}(A) \to \mathrm{Spec}(K\langle x_1, \ldots, x_d\rangle)$ is open by [Stacks, Tag 0F32][16]. The map $C' \to C$ is finite, and hence surjective, and therefore C' connects x with a point in $f^{-1}(H)$.

In the following steps, we shall construct a hyperplane $\widetilde{H} \subseteq \mathbf{A}^d_k$ and show that every hyperplane $H \subseteq \mathbf{D}^d_K$ lifting $\widetilde{H}$ has connected preimage in X.

Step 6. [*Construction of H*] For a decomposition $I \cup J = \{1, \ldots, r\}$ with I, J non-empty and disjoint, we set

$$Z_{I,J} = \left(\bigcup_{i \in I} Z_i\right) \cap \left(\bigcup_{j \in J} Z_j\right),$$

treated as a reduced subscheme of $\widetilde{X}$. We claim as in [dJ95a, §6.4] that $\dim Z_{I,J} = d-1$ for every such I, J. To this end, note that $\widetilde{X} \setminus Z_{I,J}$ is disconnected, and hence so is $\mathfrak{U}_\eta \subseteq X$ where $\mathfrak{U} \subseteq \mathrm{Spf}(A^\circ)$ is the open formal subscheme supported on $\widetilde{X} \setminus Z_{I,J}$. But, if $\dim Z_{I,J} < d-1$, then by [L76, Satz 2] we have $\mathcal{O}(\mathfrak{U}_\eta) = A$, which does not have non-trivial idempotents.

We denote by $\widetilde{T} \subseteq \widetilde{X}$ the closed subscheme where $\tilde{f}$ is not étale, and by $\widetilde{S} \subseteq \widetilde{X}$ the closed subscheme where $\tilde{f}$ is not flat. By construction, $\dim \widetilde{T} < d$. By Miracle Flatness [Stacks, Tag 00R4], $\widetilde{S}$ is the non-Cohen–Macaulay locus of $\widetilde{X}$, see [Stacks, Tag 00RE]. Since $\widetilde{X}$ is reduced, it is S_1 [Stacks, Tag 031R], and therefore we have $\dim \widetilde{S} < d-1$.

By the Bertini theorem [Jou83, Théorème 6.3], a generic hyperplane (which exists after replacing k by a finite extension if k is finite) $\widetilde{H} = V(\tilde{h}) \subseteq \mathbf{A}^d_k$, $\tilde{h} = \sum_{i=1}^d \tilde{a}_i x_i - \tilde{a}$ ($\tilde{a}_i, \tilde{a} \in k$) satisfies the following properties:

1. The intersections $\tilde{f}^{-1}(\widetilde{H}) \cap Z_i$ ($i = 1, \ldots, r$) are irreducible of dimension $d-1$ and generically étale over $\widetilde{H}$.
2. For every decomposition $I \cup J = \{1, \ldots, r\}$ with I and J non-empty and disjoint, $\tilde{f}^{-1}(\widetilde{H}) \cap Z_{I,J}$ has a component of dimension $d-2$, and all such components are generically flat over $\widetilde{H}$.

[16] Indeed, if $I \subseteq B = K\langle x_1, \ldots, x_d\rangle$ is the ideal of C, then x is an isolated point of $f^{-1}(C) = \mathrm{Spa}(A/IA)$ if and only if it is an isolated point of $\mathrm{Spec}(A/IA)$, as both mean that the local ring $(A/IA)_x$ is Artinian. If $U \subseteq \mathrm{Spec}(A)$ is an open subset with $U \cap \mathrm{Spec}(A/IA) = \{x\}$, then its image in $\mathrm{Spec}(B)$ is an open neighborhood of $f(x)$, and hence it contains the generic point of $\mathrm{Spec}(B/I)$, contradiction.

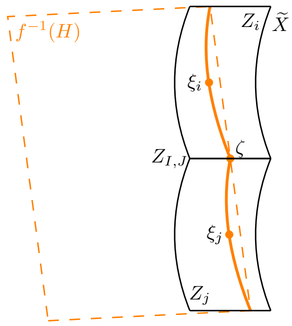

FIGURE 3. The intersection $f^{-1}(H) \cap \widetilde{X} = \tilde{f}^{-1}(\widetilde{H})$.

Indeed, [Jou83, Théorème 6.3(1b,4)] shows that an open set of hyperplanes will have the property that $\tilde{f}^{-1}(\widetilde{H}) \cap Z_i$ are irreducible of dimension $d-1$. To ensure étaleness in (1) it suffices to choose $\widetilde{H}$ not contained in $\tilde{f}(\widetilde{T})$. If $V \subseteq Z_{I,J}$ is an irreducible component of dimension $d-1$, then by [Jou83, Théorème 6.3(1b)] for a generic $\widetilde{H}$, the preimage $\tilde{f}^{-1}(\widetilde{H}) \cap V$ is of dimension $d-2$. To ensure flatness in (2), we choose a hyperplane whose intersection with $\tilde{f}(V)$ is not contained in $\tilde{f}(\tilde{S})$.

We let $h = \sum_{i=1}^{d} a_i x_i - a$ $(a_i, a \in \mathcal{O}_K)$ be any lifting of h, and set $H = V(h) \subseteq \mathbf{D}_K^d$. In the remaining two steps, we shall prove that $f_\eta^{-1}(H) \subseteq X$ is connected. By the claim in Step 5, this will finish the proof. To this end, if $f_\eta^{-1}(H)$ is disconnected, then we have $\mathrm{Spec}(A^\circ/hA^\circ) = T_1 \cup T_2$ for two non-empty closed subsets T_1, T_2 such that $T_1 \cap T_2$ is non-empty and set-theoretically contained in $\widetilde{X}$. In the final two steps, we shall derive a contradiction. See Figure 3 for a reference.

Step 7. [*$T_1 \cap T_2$ does not contain any $\tilde{f}^{-1}(\widetilde{H}) \cap Z_i$*] Let ξ_i be the generic point of $\tilde{f}^{-1}(\widetilde{H}) \cap Z_i$; we will show that $\xi_i \notin T_1 \cap T_2$. It is enough to show that $A^\circ_{\xi_i}/hA^\circ_{\xi_i}$ is a domain. Indeed, since $\dim(T_1) = \dim(T_2) = d$ if $\xi_i \in T_1 \cap T_2$ then $\mathrm{Spec}(A^\circ_{\xi_i}/hA^\circ_{\xi_i})$ would contain multiple components of $\mathrm{Spec}(A^\circ/hA^\circ)$, but this would contradict that $A^\circ_{\xi_i}/hA^\circ_{\xi_i}$ is a domain. To see that $A^\circ_{\xi_i}/hA^\circ_{\xi_i}$ is a domain, consider the localization R of $\mathcal{O}_K\langle x_1, \ldots, x_d\rangle$ at the generic point of $\widetilde{H}$ and the map $R \to A^\circ_{\xi_i}$ induced by f. Since f is étale in a neighborhood of ξ_i, the induced map $R/hR \to A^\circ_{\xi_i}/hA^\circ_{\xi_i}$ is the composition of an étale morphism and a localization; in particular, it is weakly étale. Since $\mathcal{O}_K\langle x_1, \ldots, x_d\rangle/h \simeq \mathcal{O}_K\langle y_1, \ldots, y_{d-1}\rangle$ is normal, so is R/hR. Thus, by [Stacks, Tag 0950], we conclude that $A^\circ_{\xi_i}/hA^\circ_{\xi_i}$ is normal as well, in particular it is a domain.

Step 8. [*The contradiction*] It follows from the previous step that if we set

$$I = \{i \,:\, \xi_i \in T_1\}, \qquad J = \{j \,:\, \xi_j \in T_2\}$$

then these form a non-trivial partition of $\{1, \ldots, r\}$. Therefore $T_1 \cap T_2 \cap \widetilde{X}$ is contained in $Z_{I,J}$. Note though that since T_1 and T_2 both have codimension one in $\mathrm{Spec}(A^\circ)$ the intersection $T_1 \cap T_2 \cap \widetilde{X}$ necessarily contains a component V of $\tilde{f}^{-1}(\widetilde{H}) \cap Z_{I,J}$. This component can be taken to have codimension two in $\widetilde{X}$, and by our choice of hyperplane, we know that this component is generically Cohen–Macaulay and so not contained in $\widetilde{S}$.

Let ζ be the generic point of V. By the previous paragraph we know that ζ is not contained in $\widetilde{S}$ and therefore $\widetilde{A}_\zeta$ is a two-dimensional, Cohen–Macaulay, local ring. Moreover, $\widetilde{h}$ is a nonzerodivisor of $\widetilde{A}_\zeta$ contained in its maximal ideal. Note that since $\widetilde{A}_\zeta/\widetilde{h}\widetilde{A}_\zeta$ is not zero-dimensional its maximal ideal contains some nonzerodivisor. Let $\widetilde{g}$ in $\widetilde{A}_\zeta$ be the lift of such an element. Then, $(\widetilde{g}, \widetilde{h})$ is a regular sequence in $\widetilde{A}_\zeta$. Let ϖ be the pseudouniformizer from above and choose a lift g of $\widetilde{g}$ in A°_ζ. Note then that (ϖ, g, h) is a regular sequence in A°_ζ. Indeed, this is clear since $A^\circ_\zeta/\varpi A^\circ_\zeta$ is a local ring and (g, h) have images in this ring that, in the further quotient ring, $\widetilde{A}_\zeta$ have images forming a regular sequence. Therefore (ϖ, g) is a regular sequence in the two-dimensional $A_\zeta^\circ/hA_\zeta^\circ$. The fact that we can permute a regular sequence follows from the argument given in [Stacks, Tag 00LJ]. Indeed, while $A_\zeta^\circ/hA_\zeta^\circ$ is not Noetherian it is coherent (e.g. by [FK18, Chapter 0, Corollary 9.2.8]) and so the annihilator of any element of $A_\zeta^\circ/hA_\zeta^\circ$ is finitely generated, which is all the referenced argument requires.

By Lemma A.0.4 below we know that $W = \mathrm{Spec}(A_\zeta^\circ/hA_\zeta^\circ)\setminus V(\varpi, g)$ is connected. Note though that since $T_1 \cup T_2 = \mathrm{Spec}(A^\circ/hA^\circ)$ by construction, that $T_1' \cup T_2' = W$ where $T_i' = T_i \cap W$. Each T_i' is closed in W, and we have that $T_1' \cap T_2' \subseteq V(\varpi) \cap W$. However, we have $V(\varpi) \cap W) = \varnothing$: note that $V(\varpi) \cap W = \mathrm{Spec}(\widetilde{A}_\zeta/\widetilde{h}\widetilde{A}_\zeta) \setminus V(\widetilde{g})$ and this set is contained in the union of ξ_i's that are in the intersection $T_1 \cap T_2$. As shown above, the set of those ξ_i's is empty. Thus, T_1' and T_2' form a disconnection of W. Contradiction. □

Lemma A.0.4. *Let R be a local ring, let $x, y \in R$ be such that (x, y) and (y, x) are regular sequences, and set $W = \mathrm{Spec}(R) \setminus V(x, y)$. Then $\Gamma(W, \mathcal{O}_W) = R$; in particular, W is connected.*

Proof. Since $W = D(x) \cup D(y)$, we have $\Gamma(W, \mathcal{O}_W) = \ker(R_x \times R_y \to R_{xy})$. As x is a nonzerodivisor, $R \to R_x$ is injective, and hence so is $R \to \Gamma(W, \mathcal{O}_W)$. To show it is surjective, let us take an element $(a/x^n, b/y^m) \in R_x \times R_y$ in the kernel, i.e. $(xy)^N(ay^m - bx^n) = 0$. Since xy is a nonzerodivisor, we have $ay^m = bx^n$. We may assume that $n = 0$ or that $a \notin xR$. If $n > 0$, then we have $ay^m = 0$ in R/xR, and hence $a \in xR$ since y is a nonzerodivisor in R/xR. Therefore $n = 0$, analogously $m = 0$, and hence $a = b$. □

We would like to bootstrap this up to connect points on connected rigid K-spaces which are not necessarily classical, and we are able to do this at the expense of base field extension.

Corollary A.0.5. *Let X be a connected rigid K-space. Fix maximal points x and y in X. Then, there exists a complete extension L of K and smooth connected affinoid L-curves C_i with maps $C_i \to X$ such that for all i we have that $\mathrm{im}(C_i \to X) \cap \mathrm{im}(C_{i+1} \to X)$ is non-empty, $x \in \mathrm{im}(C_1 \to X)$, and $y \in \mathrm{im}(C_m \to X)$.*

Proof. As in the second step of the proof of Proposition A.0.2 we may assume that X is geometrically connected. Let us then note that by Gruson's theorem ([Gru66, Théorème 1]) one has that the completed tensor product $k(x)\widehat{\otimes}_K k(y)$ is non-zero. Thus, by taking a point of $\mathcal{M}(k(x)\widehat{\otimes}_K k(y))$ (which exists by [Ber90, Theorem 1.2.1]) and looking at its residue field we get a valued extension L of K containing both $k(x)$ and $k(y)$. This gives two maps $\mathcal{M}(L) \rightrightarrows X$. They give rise to two L-points x' (resp. y') of X_L mapping to x (resp. y). We then note that by Proposition A.0.2 there exists smooth, connected, affinoid curves $C_1, \ldots, C_n$ over L and morphisms $C_i \to X_L$ satisfying $x' \in \mathrm{im}(C_1 \to X_L)$, $y' \in \mathrm{im}(C_n \to X_L)$, and $\mathrm{im}(C_i \to X_L) \cap \mathrm{im}(C_{i+1} \to X_L)$ is non-empty. Clearly then the compositions $C_i \to X_L \to X$ satisfy the desired properties. □

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Institute of Mathematics of the Polish Academy of Sciences
ul. Śniadeckich 8, 00-656 Warsaw, Poland
Email address: pachinger@impan.pl, marcin.lara@impan.pl, ayoucis@impan.pl