Some examples of Cantor's algorithm

Cantor's Theorem

Theorem. *Let* $(X, \leqslant)$ *be a countable dense linear endless poset. Then* $(X, \leqslant)$ *is order-isomorphic to* $(\mathbb{Q}, \leqslant)$.

Cantor's proof uses a *back-and-forth* construction which I have decided to call this *Cantor's algorithm*.

Let me roughly recall how this works (please see your course notes for the precise construction).Fix enumerations

$$X = \{x_0, x_1, x_2, \dots\}, \qquad \mathbb{Q} = \{q_0, q_1, q_2, \dots\}.$$

We build finite order-preserving bijections (partial maps $X \to \mathbb{Q}$, partial meaning only defined on parts of the domain and codomain)

$$f_n : A_n \longrightarrow B_n, \qquad A_0 = B_0 = \varnothing, \qquad f_0 = \varnothing,$$

such that

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots, \qquad B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots,$$

and each f_{n+1} extends f_n. The limit map

$$f = \bigcup_{n=0}^{\infty} f_n : X \longrightarrow \mathbb{Q}$$

will then be an order-isomorphism.

Rule at stage n:

- If $n = 2k + 1$ (odd, with $k \geqslant 0$), ensure that $x_k \in A_n$.
- If $n = 2k$ (even, with $k \geqslant 1$), ensure that $q_k \in B_n$.

If the required element is already included, we do nothing. Otherwise we extend f_n in the unique order-preserving way, using the density, linearity, and endlessness of X and $\mathbb{Q}$ (see Lemma 1 and Lemma 2 from the course notes).

Below we illustrate this idea in the case when $(X, \leqslant)$ is *also* the usual ordered set $(\mathbb{Q}, \leqslant)$. We will see then how the function produced by Cantor's algorithm really depends on the enumeration we pick, can do nothing at some stages, and certainly does not send x_k to q_k.

Example Run 1

We take the enumerations

$$x_0 = 0,\ x_1 = 1,\ x_2 = -1,\ x_3 = 2,\ x_4 = -2, \ldots$$

$$q_0 = 0,\ q_1 = \frac{1}{2},\ q_2 = -\frac{1}{2},\ q_3 = 1,\ q_4 = -1, \ldots$$

Stage 0

$A_0 = B_0 = \varnothing,\ \ f_0 = \varnothing.$

Stage 1 (odd)

Ensure $x_0 = 0 \in A_1$. Choose an arbitrary unused rational for its image (not necessarily q_0). Let

$$A_1 = \{0\}, \qquad B_1 = \{1\}, \qquad f_1(0) = 1 = q_3.$$

Stage 2 (even)

Ensure $q_1 = \frac{1}{2} \in B_2$. Since $\frac{1}{2} \leqslant 1 = f_1(0)$, pick some $x < 0$, e.g. $x_2 = -1$, and define

$$A_2 = \{-1, 0\}, \qquad B_2 = \left\{\frac{1}{2}, 1\right\}, \qquad f_2(-1) = \frac{1}{2}.$$

Stage 3 (odd)

Ensure $x_1 = 1 \in A_3$. Since $1 > 0$, choose a rational > 1, say 2:

$$A_3 = \{-1, 0, 1\}, \qquad B_3 = \left\{\frac{1}{2}, 1, 2\right\}, \qquad f_3(1) = 2.$$

Stage 4 (even)

Ensure $q_2 = -\frac{1}{2} \in B_4$. Since $-\frac{1}{2} \leqslant \frac{1}{2} = f_2(-1)$, choose some $x < -1$, e.g. $x_4 = -2$:

$$A_4 = \{-2, -1, 0, 1\}, \qquad B_4 = \left\{-\frac{1}{2}, \frac{1}{2}, 1, 2\right\}, \qquad f_4(-2) = -\frac{1}{2}.$$

Stage 5 (odd)

Ensure $x_2 = -1 \in A_5$. It is already in A_4, so do nothing: $A_5 = A_4$, $B_5 = B_4$, $f_5 = f_4$.

Stage 6 (even)

Ensure $q_3 = 1 \in B_6$. It is already in B_5, so do nothing: $A_6 = A_5$, $B_6 = B_5$, $f_6 = f_5$.

At this point the partial map is

$$-2 \mapsto -\frac{1}{2}, \qquad -1 \mapsto \frac{1}{2}, \qquad 0 \mapsto 1, \qquad 1 \mapsto 2.$$

Example Run 2

We now use different enumerations:

$$x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = -\frac{1}{2},\ x_3 = 1,\ x_4 = -1, \ldots$$

$$q_0 = 0,\ q_1 = 1,\ q_2 = -1,\ q_3 = \frac{1}{2},\ q_4 = -\frac{1}{2}, \ldots$$

Stage 1 (odd)

Ensure $x_0 = 0 \in A_1$ and send it to q_0:

$$A_1 = \{0\}, \qquad B_1 = \{0\}, \qquad f_1(0) = 0.$$

Stage 2 (even)

Ensure $q_1 = 1 \in B_2$. Since $1 > 0$, choose some $x > 0$, e.g. $x_3 = 1$:

$$A_2 = \{0, 1\}, \qquad B_2 = \{0, 1\}, \qquad f_2(1) = 1.$$

Stage 3 (odd)

Ensure $x_1 = \frac{1}{2} \in A_3$. It lies strictly between 0 and 1, so we choose a rational between 0 and 1, namely $q_3 = \frac{1}{2}$:

$$A_3 = \{0, \tfrac{1}{2}, 1\}, \qquad B_3 = \left\{0, \tfrac{1}{2}, 1\right\}, \qquad f_3\left(\tfrac{1}{2}\right) = \tfrac{1}{2}.$$

Stage 4 (even)

Ensure $q_2 = -1 \in B_4$. Since $-1 < 0$, pick $x < 0$, e.g. $x_4 = -1$:

$$A_4 = \{-1, 0, \tfrac{1}{2}, 1\}, \qquad B_4 = \{-1, 0, \tfrac{1}{2}, 1\}, \qquad f_4(-1) = -1.$$

Stage 5 (odd)

Ensure $x_2 = -\frac{1}{2} \in A_5$. It lies between -1 and 0, so pick a rational between -1 and 0, namely $q_4 = -\frac{1}{2}$:

$$A_5 = \{-1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1\}, \qquad B_5 = \{-1, -\tfrac{1}{2}, 0, \tfrac{1}{2}, 1\}, \qquad f_5(-\tfrac{1}{2}) = -\tfrac{1}{2}.$$

Stage 6 (even)

Ensure $q_3 = \frac{1}{2} \in B_6$. It is already present, so do nothing.

The partial map is now

$$-1 \mapsto -1, \qquad -\frac{1}{2} \mapsto -\frac{1}{2}, \qquad 0 \mapsto 0, \qquad \frac{1}{2} \mapsto \frac{1}{2}, \qquad 1 \mapsto 1.$$

Summary

These two runs demonstrate that:

- the back–and–forth construction does *not* force x_k to map to q_k,
- at many stages we genuinely do nothing because the required element is already in A_n or B_n,
- different enumerations (and choices of intermediate points) give very different partial maps, all of which extend to (different) order-isomorphisms $(X, \leqslant) \simeq (\mathbb{Q}, \leqslant)$.