Math 138 – Midterm

October 22, 2025

Instructions:

- You have 2 hours to complete this exam.
- No external resources are allowed.
- Do not hesitate to ask for clarification on exam questions.

Question 1. (15 pts)

Spot the error in the following proof, and explain why it is wrong.

Proposition. *Every horse is the same color.*

Proof. We prove by induction that every collection S of n horses has the property that all horses in S have the same color, which clearly implies the proposition. When $n = 1$ this is trivially true as S only contains one horse. Suppose the claim is true for S of size n, and let T be a set of horses of size $n + 1$. Choose any horse h in T, and consider the set $T - \{h\}$. This is a set of n horses, so by the induction hypothesis all horses in $T - \{h\}$ have the same color. Let h' be a different horse in T and consider $T - \{h'\}$. Again by induction all horses in $T - \{h'\}$ have the same color. Let h'' be any horse different than h and h'. Then, as both h' and h'' belong to $T - \{h\}$ they have the same color, and as both h and h'' both belong to $T - \{h'\}$ they both have the same color. So, h, h', and h'' all have the same color. As h, h', and h'' were arbitrary we deduce that all horses in T have the same color as desired. □

Question 2. (15 pts)

Prove that for all $n \geqslant 0$, the equality

$$F_0 + F_1 + F_2 + \cdots + F_n = F_{n+2} + 1$$

Question 3. (20 pts)

Suppose that $p \neq q$ are primes. Show that $\sqrt[3]{pq^2}$ is not rational.

Question 4. 20 pts

Let A and B be subsets of a set S. Prove that

$$(A \cup B) \triangle (A \cap B) = A \triangle B.$$

Question 5. (30 pts)

1. **(20 pts)**

   Prove by induction that the number of subsets of $\{1, \ldots, n\}$ is 2^n. (*Hint: prove that if S is the collection of subsets of $\{1, \ldots, n+1\}$ containing $n+1$, and T is the collection not containing $n+1$, then S and T are in 1-to-1 correspondence.*)

2. **(10 pts)**

   Use 1. to show that

$$2^n = \sum_{k=0}^{n} \binom{n}{k}.$$

   (*Note: you can use the claim from 1. to solve 2. even if you didn't solve 1.*)