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§0 Themes for this week

- Understand structural thinking,
- Understand abstract notions of order

§1 Posets

e.g.) $\mathbb{R}$ w/ relation $\leq$

e.g.) $\mathbb{Z}$ w/ relation $|$

e.g.) $P(X)$ w/ relation $\subseteq$

3) **Def:** Let X be a set. A *partial order* on X is a relation $\leq$ on X which is

- reflexive ($x \leq x \ \forall x \in X$)
- transitive ($x \leq y,\ y \leq z \Rightarrow x \leq z$)
- *anti-symmetric* i.e., if $x \leq y$ and $y \leq x \Rightarrow x = y$.

We write $x < y$ if $x \leq y$ and $x \neq y$.

"$\leq$" = "$\subseteq$"

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NB: $1 \mid -1 = 1 \cdot -1$, $-1 \mid 1 = -1 \cdot -1$

So $(\mathbb{Z}, \mid)$ is not a partial order

but $(\mathbb{N}, \mid)$ is.

e.g.] $\mathbb{R}^{\mathbb{R}}$ ($Y^X = \{f : X \to Y\}$) has a partial order

where $f \leq g \iff (f(x) \leq g(x) \ \forall x \in \mathbb{R})$.

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e.g. Take $X = \mathbb{N}^2$ w/ the lexicographic ordering

$$(a,b) \leq_{lex} (c,d) \iff \begin{cases} a < c & \text{if } a \neq c \\ b \leq d & \text{if } a = c \end{cases}$$

$(3,7) \overset{?}{\leq}_{lex} (4,7)$ ✓

$(4,13) \overset{?}{\leq}_{lex} (4,798)$ ✓

$(3,7) \overset{?}{\leq}_{lex} (3,8)$ ✓

$(3,8) \leq_{lex} (3,7)$ ✗

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Def'n: A partially ordered set (a poset) is a set equipped w/ a partial ordering, i.e., $(X, \leq)$.

Visually (Hasse diagram):

$\exists$ an arrow $x \to y$ iff $x < y$ and $\nexists\, z$ s.t. $x < z < y$.

e.g. $(\mathbb{R}, \leq)$ Not useful

Hasse diagram — NO ARROWS!

7] Draw the Hasse diagram for $\{x \in \mathbb{N} : x \mid 20\}$ w/ $\mid$.

20
10
2
5
1

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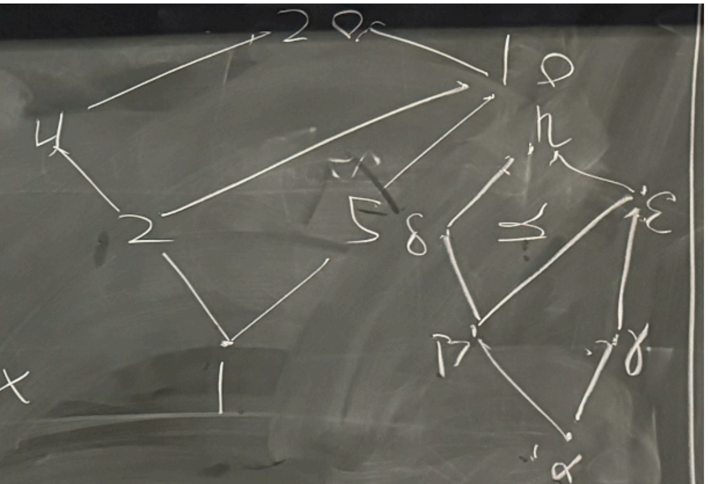
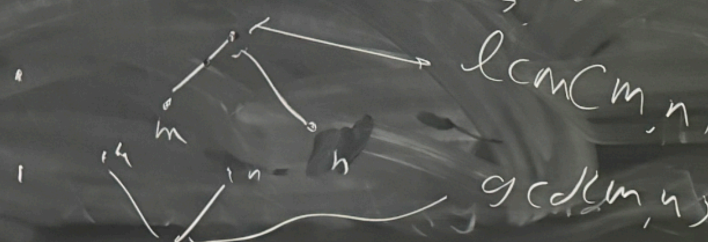
$a \mid 20 \iff \exists x \quad 20 = a \cdot x$



Remark: • In these type of diagrams the things that sit immediately above 1 : prime divisors



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§2 Monotone maps

<u>Defn</u>: Let $(X, \leq)$ and $(Y, \preceq)$ be posets.
A map $f: X \to Y$ is

- <u>monotone</u> if $x_1 \leq x_2 \Rightarrow f(x_1) \preceq f(x_2)$
- <u>order presesving</u> if $f(x_1) \preceq f(x_2) \Rightarrow x_1 \leq x_2$
- <u>embedding</u> if its Monotone and order presevin
- an <u>isomorphism</u> if it's a surj. emb

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- **Example:** monotone but not order preserving

- **Example:** order preserving but not monotone.

- **Claim:** An order preserving map f is inj.

Pf: Assume that $f(x_1) = f(x_2)$. Then $f(x_1) \leq f(x_2)$
$\Longrightarrow x_1 \leq x_2$. Also, $f(x_2) \leq f(x_1) \Rightarrow x_2 \leq x_1$. So $x_1 = x_2$ ▨