Sample thesis topics
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1. Specialization for the tempered fundamental group (algebraic/rigid geometry)

1.1. **Setup and statement.** Let X be a smooth rigid-analytic curve over a local field K, thought of as a **Berkovich space**—the K-analogue of a complex analytic space. Purely for simplicitly, let's assume that K is a finite extension of $\mathbb{Q}_p$. The covering-space theory of X is an interesting combination of **Galois theory** and the **combinatorial topology/metric geometry of trees**.

Example 1.1. Let $\mathbb{B} = \{|x| \leqslant 1\}$ be the unit ball over K. Then, there is the Galois-flavored cover of $\mathbb{B}$, given by $\mathbb{B} \to \mathbb{B}$ sending $x \mapsto x^p - x - 1$ (i.e., Artin–Schreier covers).

Example 1.2. Let $E_q = \mathbb{G}_m^{\mathrm{an}}/q^{\mathbb{Z}}$ be a Tate elliptic curve. Then, the tautological cover $\mathbb{G}^{\mathrm{an}} \to E_q$ is a literal topological covering.

In fact, there can be way wilder covers of rigid-analytic curves–for example, this is what de Jong's papers [dJ95] and [ALY23] are about. To cut down on the complexity, and arrive at a more managable group, Yves Andres in [And03] defined the notion of a **tempered covering space** $Y \to X$. In some sense, this is the minimal category which contains both Galois-theoretic (i.e., finite étale) covers and topological covers. This has corresponding notion of fundamental group, the **tempered fundamental group** written $\pi_1^{\mathrm{temp}}(X)$.

Example 1.3. One has that $\pi_1^{\mathrm{temp}}(E_q) = \widehat{\mathbb{Z}} \times \mathbb{Z}$. The first factor corresponds to the Galois-type coverings given by the multiplication-by-n maps $[n]\colon E_q \to E_q$ for all n. The second factor corresponds to the topological covering $\mathbb{G}_m^{\mathrm{an}} \to E_q$ as mentioned above.

Remark 1.4. Note that in general the tempered fundamental group is not a product of the étale fundamental group and the topological fundamental group as in Example 1.3.

Now, there is a general theme in the covering-space theory of rigid-analytic spaces. Namely, if one thinks of $\mathrm{Sp}(K)$ as being something like a 'punctured disk', where the unpunctured disk is something like $\mathrm{Spf}(\mathcal{O}_K)$ and the originally-missing central point something like $\mathrm{Spec}(k)$, where k is the residue field of K. Then, every time you are able to extend X/K to a so-called **formal model** $\mathfrak{X}/\mathcal{O}_K$ with special fiber $\mathfrak{X}_0/k$, you expect that the 'topology' of $\mathfrak{X}_0$ should inform the 'topology' of X. This is part of a general story called **specialization**. See this survey [You22] of mine and this answer of mine on MathOverflow for an introduction to this circle of ideas.

Such a theory *should* exist for the tempered fundamental group $\pi_1^{\mathrm{temp}}(X)$. In fact, both covers $Y \to X$ n Examples 1.1 and 1.2 'come from the special fiber' in the sense that there are natural models $\mathfrak{X}$ for these respective X and 'covers' in an appropriate sense of $\mathfrak{Y}_0 \to \mathfrak{X}_0$ which 'give rise' to the coverings $Y \to X$ (see the survey/MO answer mentioned above for a precise meaning to this phrase).

The issue historically is it wasn't quite clear what the notion 'cover' $\mathfrak{Y}_0 \to \mathfrak{X}_0$ used here should be. In [ALY22] my collaborators and I solved this question for de Jong's much more general notion of cover. Namely, we showed that if X has a formal model $\mathfrak{X}$, then there is a **specialization map**

$$\mathrm{sp}\colon \pi_1^{\mathrm{dJ}}(X) \to \pi_1^{\mathrm{proet}}(\mathfrak{X}_0),$$

where the target is the **proétale fundamental group** of $\mathfrak{X}_0$ developed by Bhatt–Scholze in [BS15] which classifies so-called **geometric covers** of $\mathfrak{X}_0$.

While this is nice, it has one major drawback: both the de Jong fundamental group and the proétale fundamental group can be quite unwieldy from a computational perspective. In particular, as mentioned above, this unwieldiness of π_1^{dJ} is precisely why Andre originally defined π_1^{temp}. So, for actual applications/computations, it is desirable to have a version of this specialization map for the tempered fundamental group. Note that π_1^{temp} is a *quotient* of π_1^{dJ}, so you can't use π_1^{proet}. So, what should you use? I have strong reason to believe it should be the **SGA3 fundamental group** π_1^{SGA3}. This corresponds to the quotient of π_1^{proet} corresponding to those geometric coveres $\mathfrak{Y}_0 \to \mathfrak{X}_0$ which split completely over an étale cover of $\mathfrak{X}_0$.

Question 1.5. Let X be a rigid-analytic curve over K with formal model $\mathfrak{X}$ and special fiber k. Is there a **specialization map**

$$\mathrm{sp}\colon \pi_1^{\mathrm{temp}}(X) \to \pi_1^{\mathrm{SGA3}}(\mathfrak{X}_0)?$$

If so, what can one say about this? When is this map **surjective**? Can you say anything about **combinatorial loops** in $\mathfrak{X}_0$ and their relation to $\pi_1^{\mathrm{top}}(X)$? What does the induced map on **abelianizations** look like?

There might be multiple ways to approach Question 1.5. It seems possible that one might be able to imitate the method used in [ALY22] in this setup. That said, it would require some interesting geometric input as the method of *op. cit.* doesn't really tell you much about what the relationship is between étale local splitting on the special fiber, and the 'tame covering condition'.

On the other hand, it also seems possible that one may be able to deduce the existence of such a specialization map in the tempered/SGA3 setting from that in the de Jong/proétale setting. Namely, the former objects are naturally quotients of the latter of a very *group-theoretic nature*.

Question 1.6. If one is able to answer Question 1.5, can one provide two proofs: one of a geometric flavor, and one of a group-theoretic flavor?

Finally, a natural question arises in the above story. A positive answer to Question 1.5 would, in particular, give such a specialization map for *any* model $\mathfrak{X}$ of X. That said, there is, in some sense, a *preferred model* of X (assuming its smooth). Namely, up to a finite extension of K, X admits a so-called *semistable model*; see [LÏ6]. There is good reason to believe that for such a semistable model $\mathfrak{X}$ of X, that the specialization map is surjective, and that one may be able to describe the kernel *purely in terms of* X. Namely, it maybe possible to describe the quotient of $\pi_1^{\mathrm{temp}}(X)$ given by specialization in terms of Hübner's notion of *strongly étale morphisms*; see [Hüb21].

Question 1.7. Can one show that if $\mathfrak{X}$ is a semistable model of X, that specialization induces an isomorphism between $\pi_1^{\mathrm{SGA3}}(\mathfrak{X}_0)$ and the quotient of $\pi_1^{\mathrm{temp}}(X)$ corresponding to the (tame infinite) Galois subcategory corresponding to *strongly étale tempered covers*?

1.2. **Prerequisites and a warmup problem.** Let me list out here some prerequisites needed to start working on this problem, roughly in order of how they should be tackled:

(1) You need to understand the algebraic geometry of curves and blowups. For this you could easily get away with Chapters I–IV of Hartshorne (i.e., [Har77]) or an equivalent.
(2) You need to be somewhat familiar with étale fundamental group: the first chapter of [Mil80] and/or the book [Sza09].
(3) You need to have some familiarity with the rigid-analytic geometry of curves—you wouldn't really need much more than the introduction [Tem15]; but you could also benefit from reaching the first four chapters of [Ber93] and/or [LÏ6].

Warmup question 1.8. Try to understand the meaning of *Mumford curves* as explained in [LÏ6, Chapter 2], and try to understand what a specialization map as in Question 1.5 would look like in that situation.

1.3. **Future directions.** After having completed Question 1.5, and any of the later questions, you would be well-prepared to branch out into any of the following research directions:

- **(Higher-dimensional versions)** The tempered fundamental group for higher-dimensional rigid-analytic varieties has been largely left unstudied. This has mostly been because the focus has been on anabelian phenomena (see below), which are better behaved for curves. This project would well-position you to study this higher-dimensional geometry which as been hitherto overlooked, in particular studying the analogue of Question 1.5 in this context.
- **(Anabelian geometry of analytic curves)** There has long been the realization, started in work of Lepage, that for rigid-analytic curve X, one can recover non-trivial information about X from $\pi_1^{\mathrm{temp}}(X)$, i.e., there is a sort of *anabelian geometry* for such X. This has been notably extended in work of Gaulhiac [Gau24], where it is shown that $\pi_1^{\mathrm{temp}}(X)$ determines the so-called *analytic skeleton of* X—this is a topological subspace of X which controls much of the 'topology' of X.

  The methods you use to study Question 1.5 are well-suited to also thinking about these anabelian questions. In fact, it would be interesting to understand anabelian phenomena for π_1^{SGA3} and see if that for the tempered fundamental group can be (at least partially) *pulled back* from the special fiber.
- **(Tropical geometry and rigid-analytic geometry)** There is a strong relationship between tropical geometry and rigid-analytic geometry, as first really exploited by Matt Baker. The point being that the analytic skeleton of the Berkovich analytification of an algebraic curve is the *limit* over all tropicalizations of that curve. This is an active area of research which also intersects the covering space theory of curves quite intensely, e.g., see [Hel23] and the references therein. It would be interesting to see how answers to Question 1.5 interact with this theory.

2. Ekedahl–Oort / zip combinatorics for E_6 (Group theory/combinatorics)

2.1. **Setup and statement.** Shimura varieties are algebraic varieties S over $\mathbb{Q}$ which sit at the intersection of differential geometry, harmonic analysis, algebraic geometry, and number theory. One can think of them as very intricate geometric spaces associated to a (reductive) group G (and some other data) which geometrically encodes the representation theory of G. One of the greatest strengths of these spaces, is that their geometry can be studied quite effectively by group theory and combinatorics. More precisely, in good situations these Shimura varieties admit 'good' models $\mathscr{S}$ over $\mathbb{Z}_p$ and their special fiber, denoted $\overline{\mathscr{S}}$, admits a *period map*

$$\zeta \colon \overline{\mathscr{S}} \longrightarrow G\text{-}\mathbf{Zip}^{\mu}.$$

Here the target is the space of **G-zips of type μ** as first (in full generality) defined in [PWZ11, PWZ15].

The space G-$\mathbf{Zip}^{\mu}$ is of an incredibly explicit, combinatorial nature (more on this shortly). That said, people have generally only given special focus to those G which show up in cases where a connection to Shimura varieties (as mentioned above) was known. In particular, it was not until [MY26] that one could count any groups G of type E_6 amongst those cases. For this reason, the case E_6**-type** G-zip spaces have been completely unstudied. Making progress on the combinatorics of these G-zip spaces would be quite significant as the **first exceptional group case** examined in this regard.

Let me say a little more about the combinatorics that you should study here. To this end, let $k = \overline{\mathbb{F}}_p$, and fix $T \subset B \subset G_k$ (again we will eventually take G to be of type E_6, but for now it can be general). For notational simplicity, we fix the following notation (see a book like [Hum90] or [BB05])

- Δ will be the simple roots associated to (B, T),
- $W = N_{G_k}(T)/T$ the Weyl group,
- S will denote the with simple reflections,
- ℓ will denote the length $\ell(-)$
- $\leqslant$ will denote the Bruhat order.

Given $\mu \in X_*(T)$, set

$$I = I(\mu) := \{\alpha \in \Delta : \langle \alpha, \mu \rangle = 0\}\,.$$

Equivalently, $I(\mu)$ is the set of simple roots of the Levi subgroup $L = Z_{G_k}(\mu)$. We also set

$$W_I = \langle s_\alpha : \alpha \in I(\mu) \rangle\,.$$

and let

$${}^I W := \{w \in W : \ell(w) \leq \ell(uw) \text{ for all } u \in W_I\},$$

i.e., the set of minimal-length representatives for $W_I \backslash W$.

Remark 2.1. Now is maybe a good time to point out that there are essentially *two* cases of groups of type E_6 that are relevant here. The split one E_6^*, and the unramified non-split one usually denoted ${}^2E_6$.

In either case, can make the (geometric) root data more precisely using the Bourbaki labeling of the Dynkin diagram (Planche V). With notation as in there, we have simple roots $\Delta = \{\alpha_1, \dots, \alpha_6\}$ and simple reflections $S = \{s_1, \dots, s_6\}$. For applications to Shimura varieties, one may take μ to be minuscule. In the E_6 situation there are two minuscule fundamental coweights, dual to what are usually written ω_1 and ω_6, interchanged by the outer automorphism of the diagram. In these minuscule cases:

- If $\mu = \omega_1^\vee$, then $I(\mu) = \Delta - \{\alpha_1\}$ and W_I is of type D_5.
- If $\mu = \omega_6^\vee$, then $I(\mu) = \Delta - \{\alpha_6\}$ and W_I is again of type D_5.

The Frobenius σ of G_k. It induces an automorphism of the based root datum, hence a Coxeter system automorphism of (W, S), and thus an automorphism

$$\psi := \sigma|_{W_I} : W_I \longrightarrow W_I.$$

Remark 2.2. In the split case one has $\psi = \mathrm{id}$, while in the unramified nonsplit ${}^2E_6$ case the map ψ is induced by the nontrivial diagram automorphism.In the split case one has $\sigma = \mathrm{id}$ on W, and for ${}^2E_6$ σ is the nontrivial diagram automorphism of the Dynkin diagram of type E_6.

Now, G-zips form an algebraic over $\mathbb{F}_p$:

$$G\text{-}\mathbf{Zip}^\mu \simeq [E_\mu \backslash G_k],$$

where here E_μ is the so-called *zip group* attached to (G, μ) (see [PWZ11]). The important thing for us is that the E_μ-orbits form a finite stratification of G-$\mathbf{Zip}^\mu$ and that(up to harmless normalizations we ignore here), the strata are indexed by ${}^I W$. This is what will allow us to use combinatorics to study geometry.

To this end, let us write $\{Z_w\}_{w \in {}^I W}$ for this stratifcation. The (Zariski) closure relations among these strata are governed by the so-called **zip order** on ${}^I W$ introduced by Pink–Wedhorn–Ziegler in [PWZ11]. More precisely, following [PWZ11, Definition 6.1] let us define a relation $\preceq$ on ${}^I W$ by the following rule:

$$w' \preceq w \quad \Longleftrightarrow \quad \exists\, y \in W_I \text{such that } y\, w'\, \psi(y)^{-1} \leqslant w \;\text{ in the Bruhat order on } W.$$

Then $\preceq$ is a partial order on ${}^I W$, and [PWZ11, Theorem 6.2] implies the closure formula

$$\overline{Z_w} \;=\; \bigcup_{w' \preceq w} Z_{w'}.$$

Thus, for practical computations, deciding whether $Z_{w'} \subset \overline{Z_w}$ reduces to checking whether

$$y\, w'\, \psi(y)^{-1} \leqslant w$$

for some $y \in W_I$.

Question 2.3. Is it possible to give a **computationally simple but conceptual** description of the zip order on ${}^I W$ for (either split or non-split) E_6 in terms of **simple reflections/reduced expressions** or, equivalently, explicit inequalities in the Bruhat order on the Weyl group (or on suitable double cosets)? Can you use this to understand the basic invariants of the poset more conceptually (e.g., maximal elements, covering relation, etc.)?

Remark 2.4. One major payoff of this is that because we have a smooth surjective map $\zeta\colon \overline{\mathscr{S}} \to G\text{-}\mathbf{Zip}^\mu$ (in many cases) as in [MY26] we know that the *EO strata* $\mathscr{S}_w = \zeta^{-1}(Z_w)$ have the same poset structure as that of $G\text{-}\mathbf{Zip}^\mu$. Thus, the above combinatorial description immediatley gives a geometrically useful one for Shimura varieties.

Of course, as E_6 is a single group, and the poset here can have as few as 27 elements, there is a way to start approaching this via *brute force methods*: Given $w, w' \in {}^I W$. Namely, in a CAS/SAS (e.g., Magma, SageMath, or GAP):

(1) Enumerate the elements y of W_I.
(2) For each y, compute $v := yw'\psi(y)^{-1}$ in W.
(3) Test whether $v \leqslant w$ in Bruhat order. If yes, then $w' \preceq w$.

Thus, the goal is not to just have a Hasse diagram for the zip order in some cases, the goal is to have a **conceptual understanding of the order with proofs, not just outputs of algorithms**. This could be quite useful as it will most likely suggest conceptual means of attacks for other exceptional groups which are not so computationally simple (e.g., E_7 and E_8). That said, what is nice about this situation is that you *can use a CAS* to compute things to help you conjecture what is happening at a conceptual level.

Question 2.5. Given a satisfying answer to Question 2.3, can one use it to understand the finer invariants of the zip poset? For example, can one compute:

- whether this zip poset is naturally graded, and compute the rank function,
- the Möbius function,
- characteristic polynomials/interval invariants,
- counts of strata by dimension/length,
- anything that controls inclusion-exclusion for unions of closures.

Remark 2.6 (Computational angle)**.** Even though the final approach should be conceptual, it would be very useful to other people working in this area to have an implementation of your work in a CAS (e.g. in SageMath / GAP / Magma). So, any work you put into that for your own uses will be of use more generally.

2.2. **Prerequisites and a warmup problem.** Let me list out here some prerequisites needed to start working on this problem, roughly in order how they should be tackled:

(1) The only strictly necessary thing is understand reductive groups, root systems, Weyl group combinatorics (length, Bruhat order, etc.). For this you probably don't need anything more than [Hum90] (but [BB05] wouldn't hurt).
(2) Definitions and basic properties of $G\text{-}\mathbf{Zip}^\mu$ and zip strata (and how the indexing works). This you can get by only reading [PWZ11] (which is surprisingly readable).

Warmup question 2.7. Try answering the above questions in the case of most immediate interest to Shimura varieties, but also of least complexity: the case of split E_6 with minuscule cocharacter μ. Even this would be of interest.

2.3. **Future directions.** There are actually quite a few directions you could take this work in, if you so choose:

- **(Applications to Shimura varieties)** Most immediately, once the combinatorics is explicit, one can attempt to use it to study the geometry of mod p reductions of Shimura varieties. Namely, once one has a combinatorial structure of EO strata one can begin to use this to try and decompose the cohomology of Shimura varieties, and use this to study automorphic forms mod p. There are many, many good examples of this, but a particular good blueprint for this (in a case where the combinatorics was already well-understood) is [GK19].
- **(Other exceptional groups)** Having understood what's going on with E_6, one can move on to other exceptional groups, notably E_7. Of course the group theory there is quite different, but you will have a fairly serious command of the general ideas and so will be well-positioned to do this, and potentially the applications to E_7 Shimura varieties. One can even then go beyond this to think about G_2, F_4, or E_8, which are not directly related to Shimura varieties, but are related to their local invariants (moduli spaces of shtukas).

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