GLUING TRIPLES

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Abstract. For a complete discrete valuation field K, we show that one may always glue a separated formal algebraic space $\mathfrak{X}$ over $\mathcal{O}_K$ to a separated algebraic space X_K over K along an open immersion of rigid spaces $j\colon \mathfrak{X}^{\mathrm{rig}} \to (X_K)^{\mathrm{an}}$ producing a separated algebraic space X over $\mathcal{O}_K$. This process gives rise to an equivalence between such 'gluing triples' $(X^\circ, \mathfrak{X}, j)$ and separated algebraic spaces X over $\mathcal{O}_K$, giving a non-affine version of the Beauville–Laszlo theorem. Moreover, such results holds over any excellent base. The proof is a combination of Nagata compactification theorem for algebraic spaces and of Artin's contraction theorem. We give many examples and applications of this idea.

Contents

1. Introduction

In this note we deal with the following basic problem (as well as its variants).

> *Let K be a non-archimedean field with valuation ring $\mathcal{O}$, and let X be a variety over K. What is the minimal data necessary to describe all $\mathcal{O}$-models $\mathcal{X}$ of X?*

Our answer uses formal and rigid geometry, and is hinted at in multiple earlier works (e.g., [2], [4], and [8]). To illustrate the idea, we consider a basic example from Bruhat–Tits theory.

Example 1.1 (Models of GL_n)**.** Let us assume K is discretely valued. Let $G = \mathrm{GL}(V)$ for a finite dimensional vector space V over K. An $\mathcal{O}$-lattice $\Lambda \subseteq V$ gives rise to the model $\mathcal{G} = \mathrm{Aut}(\Lambda)$ of G over $\mathcal{O}$. Choosing an isomorphism $\Lambda \simeq \mathcal{O}^n$, the image of $\mathcal{G}(\mathcal{O})$ in $G(K)$ is the set

$$\mathbf{G}(K) = \{(a_{ij}) \in \mathrm{GL}_n(K) \,:\, |a_{ij}| \leqslant 1\}$$

of K-points of a rigid analytic affinoid subgroup $\mathbf{G}$ of G^{an}. This comes with a reduction map $\mathrm{red}\colon \mathbf{G}(K) \to \mathcal{G}(k)$, where k is the residue field of K. One may think of $\mathcal{G}$ as the result of

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'gluing' G to $\mathcal{G}_k$ along $\mathbf{G}$, with the 'glue' being provided by this reduction map. For example, this 'gluing' has the correct rational points $G(K) \sqcup \mathcal{G}(k)$. Another smooth affine model (the *Iwahori model*) $\mathcal{I}$ of G is obtained by gluing the affinoid subgroup

$$\mathbf{I}(K) = \{(a_{ij}) \in \mathrm{GL}_n(K) \,:\, |a_{ij}| \leqslant 1 \text{ and } |a_{ij}| \leqslant |\pi| \text{ when } i < j\}$$

(where π is a uniformizer) to the upper triangular subgroup $\mathcal{I}_k \subseteq \mathcal{G}_k$ along the natural reduction map. In fact, all smooth affine models of G arise from such a gluing procedure.

The procedure of Example 1.1 applies in general when appropriately formulated. To a locally of finite type $\mathcal{O}$-scheme $\mathcal{X}$, one attaches a triple $t(\mathcal{X}) = (\mathcal{X}_K, \widehat{\mathcal{X}}, j_{\mathcal{X}})$ consisting of

- its generic fiber $\mathcal{X}_K$,
- its π-adic formal completion $\widehat{\mathcal{X}}$ (where π is a pseudouniformizer of K),
- a natural morphism rigid analytic spaces over K
$$j_{\mathcal{X}} \colon \widehat{\mathcal{X}}^{\mathrm{rig}} \to \mathcal{X}_K^{\mathrm{an}},$$
where $\widehat{\mathcal{X}}^{\mathrm{rig}}$ is the rigid generic fiber of $\widehat{\mathcal{X}}$ and $\mathcal{X}_K^{\mathrm{an}}$ is the analytification of $\mathcal{X}_K$.

We treat $t(\mathcal{X})$ as an object of the category of triples $(X, \mathfrak{X}, j)$ consisting of a K-scheme X, a formal $\mathcal{O}$-scheme $\mathfrak{X}$, and a morphism of rigid spaces $j \colon \mathfrak{X}^{\mathrm{rig}} \to X^{\mathrm{an}}$. Intuitively, $\mathcal{X}$ should be described as the effect of gluing $\mathcal{X}_K$ to $\widehat{\mathcal{X}}$ along $\widehat{\mathcal{X}}^{\mathrm{rig}}$. While such a pushout does not literally make sense, we can show the following result.

Proposition 1.2 (see Corollary 4.4)**.** The functor t is fully faithful.

It is a natural question if, or to what extent, the functor t is an equivalence. For simplicity, let us only consider separated schemes, and let the target category be that of triples $(X, \mathfrak{X}, j)$ where the map j is an open embedding. In [21, Example 5.3] one finds a smooth proper *algebraic space* $\mathcal{X}$ over $\mathbb{Z}_p$ such that both $\mathcal{X}_{\mathbb{Q}_p}$ and $\mathcal{X}_{\mathbb{F}_p}$ are projective schemes (K3 surfaces). Therefore the corresponding triple $(\mathcal{X}_{\mathbb{Q}_p}, \widehat{\mathcal{X}}, j_{\mathcal{X}})$ is not in the essential image of t. This example shows that the question is better formulated in the realm of algebraic spaces. Somewhat surprisingly, it is always possible to *glue* such triples into algebraic spaces.

To state our result properly, we require some setup. Let S be an excellent algebraic space and $S_0 \subseteq S$ a closed subspace (e.g., $(S, S_0) = (\mathrm{Spec}(\mathcal{O}), V(\pi))$. Set $\widehat{S}$ to be the formal completion of S along S_0, S° to be $S \setminus S_0$, and use similar notation for algebraic S-spaces $\mathcal{X}$. Denote by $\mathbf{AlgSp}_S^{\mathrm{sep}}$ the category of separated algebraic spaces locally of finite type over S. Let $\mathbf{Trip}_{(S,S_0)}^{\mathrm{sep}}$ denote the category of *separated gluing triples* $(X, \mathfrak{X}, j)$ where:

- X is a locally of finite type separated algebraic space over S°,
- $\mathfrak{X}$ is a locally of finite type separated formal algebraic space over $\widehat{S}$,
- $j \colon \mathfrak{X}^{\mathrm{rig}} \to X^{\mathrm{an}}$ is an open embedding of rigid algebraic spaces,[1]

(see §2.1 for a recollection of these concepts in this generality).

Theorem 1.3 (The Gluing Theorem, see Theorem 2.18)**.** The functor

$$t \colon \mathbf{AlgSp}_S^{\mathrm{sep}} \longrightarrow \mathbf{Trip}_{(S,S_0)}^{\mathrm{sep}}, \qquad t(\mathcal{X}) = (\mathcal{X}^\circ, \widehat{\mathcal{X}}, j_{\mathcal{X}})$$

is an equivalence of categories, with an explicit quasi-inverse *gluing functor* g.

When S is affine and one restricts to those affine-like gluing triples (i.e., with X and $\mathfrak{X}$ affine), one can view this result as a manifestation of the Beauville–Laszlo gluing theorem (e.g., in the form presented in [28, Tag 0F9Q]). When one allows S to be arbitrary, but still restricts to affine-like gluing triples the Gluing Theorem is a generalization of the results in [4]. Thus, in general, we view it as a generalization of the Beauville–Laszlo theorem to accommodate non-affine objects, which *only makes sense in a rigid-analytic setting.*

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[1]In fact, by a result of Conrad–Temkin in [11] and Warner in [29] (also announced in [13]), under the separatedness hypotheses both $\mathfrak{X}^{\mathrm{rig}}$ and X^{an} are representable by rigid spaces.

The Gluing Theorem is quite clarifying with respect to well-known phenomena in arithmetic geometry: see Remarks 2.26, 2.29, 2.31, and 5.15. In addition, the ideas surrounding the Gluing Theorem play a central role in [17], and appear implicitly in Bruhat–Tits theory (see [19, §2.10]).

Idea of proof and conditions on S. Many of our results, including the fully faithfulness portion of Theorem 1.3 requires only that (S, S_0) is of Type (N)/(V) (i.e., S is Noetherian or the spectrum of a rank 1 valuation ring). But, our proof of essential surjectivity uses

(a) the Artin contraction theorem (see Theorem 4.8) which (with some work) handles the case when the gluing triple is 'proper-like', and
(b) Nagata compactification for algebraic spaces (as in [9]) to reduce to the 'proper-like' setting.

Roughly, (a) explains our restriction to S excellent, and (b) explains our separation hypotheses.

We believe that both the conditions of excellence and separatedness can be significantly weakened. To this first point, we are able to slightly weaken the excellence condition (e.g., allowing $(S, S_0) = (\mathrm{Spec}(\overline{\mathbb{Z}}_p), (p)))$ by an argument using approximation of formal models, and are hopeful that these methods can be souped up to handle the majority of cases of type (N)/(V). It may also be possible to directly apply Artin's criterion in the case of Type (V) as such rings satisfy Artin approximation (e.g., see [26]).

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Outline of the article. In §2.1–2.2 we review the necessary background in rigid geometry necessary to define the category of gluing triples in the generality we require. In §2.3 we discuss some illustrative examples of triples and give some applications of Theorem 1.3 to clarify them. In §2.4 we explain how to single out those gluing triples whose gluing is a scheme (opposed to an algebraic space), generalizing results from [8]. In §3 we prove coherent gluing: that for an algebraic space $\mathcal{X}$, coherent sheaves on $\mathcal{X}$ and $t(\mathcal{X})$ are the same. In §4 we prove Theorem 1.3. Finally, in §5 we explain that gluing of G-torsors (i.e., that G-torsors on $\mathcal{X}$ and $t(\mathcal{X})$ are the same) is a consequence of the Gluing Theorem, and similar results hold for finite étale covers.

Notation and conventions.

- All *(formal) algebraic spaces* in this article are assumed **quasi-separated**.
- A non-archimedean field is a non-discrete complete topological field K whose topology is induced by a rank 1-valuation $|\cdot|$. We often use $\mathcal{O}_K$ to denote the valuation ring of K.
- For a Huber ring A, we shorten the notation $\mathrm{Spa}(A, A^\circ)$ to $\mathrm{Spa}(A)$.
- A locally spectral space is called *coherent* if it is quasi-compact and quasi-separated,

2. Gluing triples

In this section, we formalize the notion of gluing triples over a general base S, define the associated 'triples functor' from algebraic spaces to gluing triples, and establish some basic properties of such objects. Additionally, we formulate the main result of this article: the Gluing Theorem (see Theorem 2.18).

2.1. Formal algebraic spaces and their rigid locus.

Formal algebraic spaces and their rigid locus. Before defining gluing triples, we need to summarize the theory of formal algebraic spaces, their rigid locus, and the their relationship to rigid algebraic spaces.

2.1.1. *Setup and notation.* In this subsection we fix various notations, terminology, and definitions that will be used throughout the rest of the paper.

Base data. We begin by fixing the base for which the objects we study will live over.

Notation 2.1. Fix S to be a coherent scheme and S_0 a finitely presented closed subscheme with ideal sheaf of definition $\mathcal{I}$. We set

- $S^\circ = S \setminus S_0$,
- $S_n = V(\mathcal{I}^{n+1})$ for $n \geqslant 0$, a finitely presented closed subscheme of S,
- $\widehat{S} = \varinjlim_n S_n$ a formal scheme.

We will primarily be interested in pairs (S, S_0) of the following form.

Definition 2.2. With notation as in Notation 2.1 we say the pair (S, S_0) is of type

- *(N)* if S is Noetherian,
- *(V)* if $(S, S_0) = (\mathrm{Spec}(\mathcal{O}), V(\varpi))$ for a rank one valuation ring $\mathcal{O}$ and pseudouniformizer ϖ,
- *(N)/(V)* if it is either type (N) or type (V),

We say that (S, S_0) is further *Jacobson* if S_0 is a Jacobson scheme.

Formal algebraic spaces. We denote a formal scheme (or algebraic space) using fraktur letters like $\mathfrak{X}$, or the symbol $\widehat{X}$ even if the symbol X is not yet and/or is nowhere defined. We also recall our convention that all formal schemes (or algebraic spaces) are **quasi-separated**.

Notation 2.3. If $\mathcal{I}$ is an ideal sheaf of definition of $\mathfrak{X}$, then for any $n \geqslant 0$ we denote by $\mathfrak{X}_n$ the locally (topologically) ringed space $(|\mathfrak{X}|, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})$ (which is a scheme if $\mathfrak{X}$ is quasi-compact).

In the sequel we shall use the following notation for various categories associated to $\mathfrak{X}$.

Notation 2.4. Fix a formal scheme $\mathfrak{X}$ with ideal sheaf of definition $\mathcal{I}$.

(i) We denote by $\mathbf{FSch}_{\mathfrak{X}}$ (or just $\mathbf{Sch}_{\mathfrak{X}}$ if $\mathfrak{X}$ is a scheme) the category of formal $\mathfrak{X}$-schemes.

(ii) For a property $\bullet$ of formal $\mathfrak{X}$-schemes, $\mathbf{FSch}^{\bullet}_{\mathfrak{X}} \subseteq \mathbf{FSch}_{\mathfrak{X}}$ denotes the full subcategory of such objects. We apply this mostly when $\bullet$ is in the following list of abbreviations:

- lft: locally of finite type,
- adm: admissible (i.e., locally of finite type and $\mathcal{I}$-torsionfree),
- ft: finite type,
- sep: separated
- coh: coherent objects,

Similarly, for properties $\bullet$ and $\blacksquare$ we use $\mathbf{FSch}^{\bullet,\blacksquare}_{\mathfrak{X}}$ to denote the full subcategory of objects satisfying both $\bullet$ and $\blacksquare$.

(iii) We denote by $\mathbf{FAlgSp}^{\mathrm{lft}}_{\mathfrak{X}}$ the category of locally of finite type formal algebraic $\mathfrak{X}$-spaces, and we use $\mathbf{FAlgSp}^{\bullet}_{\mathfrak{X}}$ and $\mathbf{FAlgSp}^{\bullet,\blacksquare}_{\mathfrak{X}}$ as in (ii).

Rigid algebraic spaces. We denote an analytic adic space (or rigid algebraic space) using caligraphic letters like $\mathcal{X}$, or often by the symbol $\widehat{X}^{\mathrm{rig}}$ (whose meaning will be soon clear) even if the symbol X is not yet and/or is nowhere defined.

In the sequel we shall use the following notation for various categories associated to $\mathcal{X}$.

Notation 2.5. Let $\mathcal{X}$ be an analytic adic space.

(i) We denote by $\mathbf{Rig}_{\mathcal{X}}$ the category of rigid $\mathcal{X}$-spaces (i.e., locally of finite type adic $\mathcal{X}$-spaces).

(ii) Our conventions for $\mathbf{Rig}^{\bullet}_{\mathcal{X}} \subseteq \mathbf{Rig}_{\mathcal{X}}$ are the same as in (ii) of Notation 2.4 with $\bullet \in \{\mathrm{ft}, \mathrm{sep}, \mathrm{coh}\}$.

(iii) $\mathbf{RigAlgSp}_{\mathcal{X}}$ denotes the category of rigid algebraic $\mathcal{X}$-spaces,[2] and we use $\mathbf{RigAlgSp}^{\bullet}_{\mathcal{X}}$ and $\mathbf{RigAlgSp}^{\bullet,\blacksquare}_{\mathcal{X}}$ as in (ii).

[2]A *rigid algebraic $\mathcal{X}$space* is a (quasi-separated) sheaf $\mathcal{F}$ on $\mathbf{Rig}_{\mathcal{X}}$ equipped with the étale topology such that there exists a (representable) étale surjection $\mathcal{U} \to \mathcal{F}$ for some rigid $\mathcal{X}$-space $\mathcal{U}$.

2.1.2. *Rigid locus of formal algebraic spaces.* We now very briefly call the notion of the rigid locus of a formal scheme. In the following we use the property *locally universally rigid-Noetherian* as in of Definition [13, Chapter I, Definition 2.1.7]) (e.g., $\widehat{S}$ where (S, S_0) is of type (N)/(V)).

There is a unique functor

$$(-)^{\mathrm{ad}}\colon \left\{\begin{matrix}\text{Locally universally rigid-Noetherian}\\ \text{formal schemes}\end{matrix}\right\} \to \left\{\begin{matrix}\text{Adic}\\ \text{spaces}\end{matrix}\right\},$$

uniquely characterized by preserving open embeddings/coverings and such that there is a functorial identification $\mathrm{Spf}(A)^{\mathrm{ad}} = \mathrm{Spa}(A)$ (e.g., combine [15, Proposition 4.1] with [30]). On the other hand, there is the *analytic locus* functor

$$(-)_{\mathrm{an}}\colon \left\{\begin{matrix}\text{Adic}\\ \text{spaces}\end{matrix}\right\}^{*} \to \left\{\begin{matrix}\text{Analytic}\\ \text{adic spaces}\end{matrix}\right\},$$

where the asterisk on the source denotes that we are only considering adic morphisms (in the sense of [15, §3]). Namely, for an adic space X the analytic locus $X_{\mathrm{an}} \subseteq X$ is the open subset consisting of analytic points x (i.e., those x such that $k(x)$ is not discrete). We may compose the functors $(-)^{\mathrm{ad}}$ and $(-)_{\mathrm{an}}$ to obtain the *rigid locus* functor

$$(-)_{\mathrm{an}} \circ (-)^{\mathrm{ad}} =: (-)^{\mathrm{rig}}\colon \left\{\begin{matrix}\text{Locally universally rigid-Noetherian}\\ \text{formal schemes}\end{matrix}\right\}^{*} \to \left\{\begin{matrix}\text{Adic}\\ \text{spaces}\end{matrix}\right\},$$

where again the asterisk means restricting only to adic morphisms. If $\mathfrak{X} = \mathrm{Spf}(A)$ where A has (π) as an ideal of definition, then $\mathfrak{X}^{\mathrm{rig}} = \mathrm{Spa}(A[1/\pi])$ (see [13, Chapter II, §A.4.(b)–§A.4.(d)]).

The rigid locus functor extends to, and has quite pleasant properties on, the category of lft formal algebraic $\mathfrak{X}$-spaces. To make this precise, we recall the following basic definitions.

Definition 2.6. Let $\mathfrak{f}\colon \mathfrak{Y}' \to \mathfrak{Y}$ be a morphism in $\mathbf{FSch}^{\mathrm{lft}}_{\mathfrak{X}}$. We say that $\mathfrak{f}$ is an

- *admissible blowup* if there is an isomorphism of $\mathfrak{Y}$-spaces $\mathfrak{Y}' \simeq V(\mathcal{J}\text{-torsion}) \subseteq \mathrm{Bl}_{\mathcal{J}}(\mathfrak{Y})$ where $\mathrm{Bl}_{\mathcal{J}}(\mathfrak{Y})$ is the formal blowup at the open ideal sheaf $\mathcal{J} \subseteq \mathcal{O}_{\mathfrak{Y}}$,
- an *admissible modification* if $\pi_1 \circ \mathfrak{f} = \pi_2$ for a diagram of admissible blowups

$$\mathfrak{Y}' \xleftarrow{\pi_1} \mathfrak{Z} \xrightarrow{\pi_2} \mathfrak{Y}.$$

If $\mathfrak{f}\colon \mathfrak{Y}' \to \mathfrak{Y}$ is a morphism in $\mathbf{FAlgSp}^{\mathrm{lft}}_{\mathfrak{X}}$ we say that $\mathfrak{f}$ is an

- *admissible modification* if the morphism of formal schemes $\mathfrak{Y}' \times_{\mathfrak{Y}} \mathfrak{Z} \to \mathfrak{Z}$ is an admissible modification for any morphism $\mathfrak{Z} \to \mathfrak{Y}$ from an lft formal $\mathfrak{X}$-scheme $\mathfrak{Z}$.

Let W denote the class of admissible modifications of formal algebraic spaces. This is left multiplicative in the sense of [28, Tag 04VB]. In particular, for any full subcategory $\mathcal{C} \subseteq \mathbf{FAlgSp}^{\mathrm{lft}}_{\mathfrak{X}}$ the localization $\mathcal{C}[W^{-1}]$ (which would be denoted by $W^{-1}\mathcal{C}$ in loc. cit.) is well-defined.

Proposition 2.7. *If $\mathfrak{X}$ is locally universally rigid-Noetherian, there exists a unique functor*

$$(-)^{\mathrm{rig}}\colon \mathbf{FAlgSp}^{\mathrm{lft}}_{\mathfrak{X}} \to \mathbf{RigAlgSp}_{\mathfrak{X}^{\mathrm{rig}}}$$

extending the rigid locus functor on representable objects, and such that the natural map

$$\mathfrak{U}^{\mathrm{rig}}/\mathfrak{R}^{\mathrm{rig}} \to (\mathfrak{U}/\mathfrak{R})^{\mathrm{rig}}$$

is an isomorphism for any étale equivalence relation $\mathfrak{U} \rightrightarrows \mathfrak{R}$. This functor sends the class W of admissible modifications $\mathfrak{Y}' \to \mathfrak{Y}$ to isomorphisms and induces an equivalence of categories

$$\mathbf{FAlgSp}^{\mathrm{ft,adm}}_{\mathfrak{X}}[W^{-1}] \xrightarrow[\text{incl.}]{\sim} \mathbf{FAlgSp}^{\mathrm{ft}}_{\mathfrak{X}}[W^{-1}] \xrightarrow[(-)^{\mathrm{rig}}]{\sim} \mathbf{RigAlgSp}^{\mathrm{coh}}_{K}.$$

Remark 2.8. Working with objects finite type over some fixed base is crucial for the fully faithfulness in Proposition 2.7. Let $K \subseteq L$ be a finite extension of non-archimedean fields such that $\mathcal{O}_L$ is not a finite $\mathcal{O}_K$-algebra (e.g., see [5, §6.4.1, Proposition 2] and the succeeding discussion). Write $L = K[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ where f_i belong to $\mathcal{O}_K[x_1, \ldots, x_n]$ and set $A = \mathcal{O}_K\langle x_1, \ldots, x_n\rangle/(f_1, \ldots, f_m)$. Then

$$\mathrm{Spf}(\mathcal{O}_L)^{\mathrm{rig}} \simeq \mathrm{Spa}(L) \simeq \mathrm{Spf}(A)^{\mathrm{rig}},$$

but, it is simple to check that there is no common admissible blowup of $\mathrm{Spf}(\mathcal{O}_L)$ and $\mathrm{Spf}(A)$, and so they are not isomorphic in $\mathbf{FSch}_{\mathcal{O}_K}[W^{-1}]$. Note that A, but not $\mathcal{O}_L$, is topologically of finite type over $\mathcal{O}_K$.

We will often use the following terminology in the sequel.

Terminology 2.9. Let $\mathcal{Y}$ (resp. $f\colon \mathcal{Y}' \to \mathcal{Y}$) be an object (resp. morphism) of $\mathbf{RigAlgSp}_{\mathfrak{X}^{\mathrm{rig}}}$.

- A *model* of $\mathcal{Y}$ is an lft formal algebraic $\mathfrak{X}$-space $\mathfrak{Y}$ and an identification $\mathfrak{Y}^{\mathrm{rig}} \simeq \mathcal{Y}$.
- A *model* of f is a morphism $\mathfrak{f}\colon \mathfrak{Y}' \to \mathfrak{Y}$ in $\mathbf{FAlgSp}^{\mathrm{lft}}_{\mathfrak{X}}$ together with an identification $f \simeq \mathfrak{f}^{\mathrm{rig}}$.
- We say a model $\mathfrak{Y}$ (resp. $\mathfrak{f}$) of $\mathcal{Y}$ (resp. f) is *admissible* (resp. *coherent*) if $\mathfrak{Y}$ (resp. the source and target are both) admissible (resp. coherent).

We finally observe a well-known omnibus result saying that essentially all reasonable properties of $\mathfrak{f}$ are inherited by $\mathfrak{f}^{\mathrm{rig}}$. And, in many cases, f satisfying a property is equivalent to having some model $\mathfrak{f}$ with this property.

Proposition 2.10. *Suppose that $\mathfrak{X}$ is locally of type (N)/(V).*[3] *Let P be one of the following properties of morphisms*

i) quasi-compact,	*iv) finite,*	*vii) flat,*[4]
ii) open embedding,	*v) separated,*	*viii) faithfully flat,*[5]
iii) closed embedding,	*vi) proper,*	*ix) étale.*

Then, if a morphism $\mathfrak{f}$ in $\mathbf{FSch}^{\mathrm{ft}}_{\mathfrak{X}}$ is P, then the induced map $\mathfrak{f}^{\mathrm{rig}}$ is P (for cases vii) and viii) we must assume that $\mathfrak{X}$ is Jacobson). Moreover, except in case ix), if a morphism f in $\mathbf{Rig}^{\mathrm{ft}}_{\mathfrak{X}^{\mathrm{rig}}}$ is P, then it has a coherent admissible formal model $\mathfrak{f}$ which is P.

2.1.3. *Specialization map.* Let us fix $\mathfrak{X}$ to be a locally universally rigid-Noetherian formal scheme.

Notation 2.11. Let $\mathfrak{Y}$ be an lft formal algebraic $\mathfrak{X}$-space (resp. rigid algebraic space $\mathcal{Y}$). We define the *underlying space* to have set

$$|\mathfrak{Y}| := \{x\colon\, \mathrm{Spec}(k_x) \to \mathfrak{Y} : k_x \text{ is a field}\} / \sim$$

$$\left(\text{resp. } |\mathcal{Y}| := \left\{x\colon \mathrm{Spa}(k_x, k_x^+) \to \mathcal{Y} : (k_x, k_x^+) \text{ is an an affinoid field}\right\}\right)$$

where $x \sim y$ if they can be dominated by a common $z\colon\, \mathrm{Spec}(k_z) \to \mathfrak{Y}$ (resp. $z\colon \mathrm{Spa}(k_z, k_z^+) \to \mathcal{Y}$). As in [28, Tag 03BX], we may uniquely, functorially topologize this set so that it agrees with the usual underlying space in the representable case.

W now formulate the existence of a specialization map with reasonable properties, whose proof follows easily by bootstrapping from [13, Chapter II, Theorem 3.1.2] and [13, Chapter II, Theorem 3.1.5]. Below we let $(-)_{\mathrm{Ét}}$ denote the big étale topos on the relevant object with its natural structure sheaves.

Proposition/Definition 2.12. *Suppose that $\mathfrak{X}$ is locally universally rigid-Noetherian.*

(1) There exists a unique natural transformation of functors $\mathbf{FAlgSp}^{\mathrm{lft}}_{\mathfrak{X}} \to \mathbf{Top}$

$$\mathrm{sp}_\bullet\colon |\bullet^{\mathrm{rig}}| \to |\bullet|,$$

such that for $\mathfrak{Y} = \mathrm{Spf}(A)$ with ideal of definition $(\pi) \subseteq A$, one has that

$$\mathrm{sp}_{\mathfrak{Y}}\colon \mathrm{Spa}(A[1/\pi]) \to \mathrm{Spf}(A), \quad \nu \mapsto \{a \in A : \nu(a) < 1\}, \tag{2.1.1}$$

called the specialization map. *For any object $\mathfrak{Y}$ of $\mathbf{FAlgSp}^{\mathrm{lft}}_{\mathfrak{X}}$ the map $\mathrm{sp}_{\mathfrak{Y}}$ is quasi-compact and closed, and is surjective if $\mathfrak{Y}$ is admissible.*

[3] i.e., locally Noetherian or locally topologically of finite type over a complete rank 1 valuation ring.

[4] Flatness is only defined in [16] in theorem environments, but is the expected thing, that the morphism of (rational) structure sheaves is flat (e.g., see [31, Definition B.4.1]).

[5] i.e., flat and surjective.

(2) For representable $\mathfrak{Y}$, the specialization map uniquely upgrades to a functorial morphism

$$\mathrm{sp}_{\mathfrak{Y}}\colon (\mathfrak{Y}^{\mathrm{rig}}, \mathcal{O}^{+}_{\mathfrak{Y}^{\mathrm{rig}}}) \to (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}})$$

of locally topologically ringed spaces so that on global sections (2.1.1) is the map $A \to A[1/\pi]^{\circ}$.

(3) For general $\mathfrak{Y}$, specialization gives rise to a morphism of ringed topoi

$$\mathrm{sp}_{\mathfrak{Y}}\colon (\mathfrak{Y}^{\mathrm{rig}}_{\text{Ét}}, \mathcal{O}^{+}_{\mathfrak{Y}^{\mathrm{rig}}}) \to (\mathfrak{Y}_{\text{Ét}}, \mathcal{O}_{\mathfrak{Y}}),$$

whose underlying morphism of sites associates sends $\mathfrak{Z} \to \mathfrak{Y}$ to $\mathfrak{Z}^{\mathrm{rig}} \to \mathfrak{Y}^{\mathrm{rig}}$.

2.2. Gluing triples.

We now define the category of gluing triples in the generality of Theorem 2.18, generalizing the notion which appeared in the introduction. Throughout this subsection we use notation as in Notation 2.1, but we assume that (S, S_0) type (N)/(V).

2.2.1. *Analytification and rigid locus of completion.* Let $X \to S$ be an lft algebraic S-space. We write X° for the algebraic S°-space $X \times_S S^{\circ}$.

Terminology 2.13. Associated to X are the following two objects:

- the *completion* of $X \to S$, denoted $\widehat{X} \to \widehat{S}$, as in [13, Chapter II, §6.3.(f)],
- the *analytification* of $X^{\circ} \to S^{\circ}$, denoted $(X^{\circ})^{\mathrm{an}} \to \widehat{S}^{\mathrm{rig}}$, defined to be the quotient $(U^{\circ})^{\mathrm{an}}/(R^{\circ})^{\mathrm{an}}$, for any presentation $X^{\circ} = U^{\circ}/R^{\circ}$.

Let us further clarify the meaning of the analytification (e.g., considered in [10]). For an lft S°-scheme Y° we denote by $(Y^{\circ})^{\mathrm{an}}$ the rigid $\widehat{S}^{\mathrm{rig}}$-space

$$(Y^{\circ})^{\mathrm{an}} := Y^{\circ} \times_{S^{\circ}} \widehat{S}^{\mathrm{rig}},$$

as in [15, Proposition 3.8]. Here the morphism of ringed spaces

$$(\widehat{S}^{\mathrm{rig}}, \mathcal{O}_{\widehat{S}^{\mathrm{rig}}}) \to (S^{\circ}, \mathcal{O}_{S^{\circ}}) \tag{2.2.1}$$

is uniquely induced from the composition of the natural maps

$$(\widehat{S}^{\mathrm{rig}}, \mathcal{O}_{\widehat{S}^{\mathrm{rig}}}) \to (\widehat{S}^{\mathrm{ad}}, \mathcal{O}_{\widehat{S}^{\mathrm{ad}}}) \to (\widehat{S}, \mathcal{O}_{\widehat{S}}) \to (S, \mathcal{O}_S),$$

as $\mathcal{I}\mathcal{O}_{\widehat{S}^{\mathrm{rig}}}$ is the unit ideal. The fact that $(X^{\circ})^{\mathrm{an}}$ this does not depend on the choice of presentation $X^{\circ} = U^{\circ}/R^{\circ}$ is simple (c.f. [10, Lemma 2.2.1]).

Proposition 2.14 (cf. [16, Proposition 1.9.6])**.** *There exists a unique natural transformation of functors* $\mathbf{FAlgSp}^{\mathrm{lft}}_S \to \mathbf{RigAlgSp}_{\widehat{S}^{\mathrm{rig}}}$

$$j_X\colon \widehat{X}^{\mathrm{rig}} \to (X^{\circ})^{\mathrm{an}}, \quad X \in \mathbf{FAlgSp}^{\mathrm{lft}}_S,$$

such that when X is a scheme the following square of locally ringed spaces commutes

$$\begin{tikzcd} \widehat{X}^{\mathrm{rig}} \arrow[r, "j_X"] \arrow[d] & (X^{\circ})^{\mathrm{an}} \arrow[d] \\ \widehat{X} \arrow[r] & X. \end{tikzcd}$$

Moreover, the map j_X is étale. If $X \to S$ is a separated, proper, or representable morphism then j_X open embedding, isomorphism, or locally open embedding, respectively.

2.2.2. *Gluing triples and the Gluing Theorem.* We now come to the category of gluing triples.

Definition 2.15. The category $\mathbf{Trip}_{(S,S_0)}$ of *gluing triples* over (S, S_0) consists of triples $(X^{\circ}, \widehat{X}, j_X)$ where

- X° is an lft algebraic S°-space,
- $\widehat{X}$ is an lft formal algebraic $\widehat{S}$-space,
- $j_X\colon (\widehat{X})_{\mathrm{rig}} \to (X^{\circ})^{\mathrm{an}}$ is an étale morphism,

and has morphisms $(Y^{\circ}, \widehat{Y}, j_Y) \to (X^{\circ}, \widehat{X}, j_X)$ given by a pair of morphisms $Y^{\circ} \to X^{\circ}$ and $\widehat{Y} \to \widehat{X}$ in the appropriate categories and commuting with the j-maps.

We will often use the following terminology concerning gluing triples.

Terminology 2.16. Let $(X^\circ, \widehat{X}, j_X)$ be a gluing triple over (S, S_0).

- An *open cover* of $(X^\circ, \widehat{X}, j_X)$ is a collection of morphisms $\{(U_i^\circ, \widehat{U}_i, j_{U_i}) \to (X^\circ, \widehat{X}, j_X)\}$ in $\mathbf{Trip}_{(S,S_0)}$ such that $U_i^\circ \to X^\circ$ and $\widehat{U} \to \widehat{X}$ are open embeddings for all i.
- We say that $(X^\circ, \widehat{X}, j_X)$ is *separated* if X° and $\widehat{X}$ are separated and j_X is an open embedding. We write $\mathbf{Trip}^{\mathrm{sep}}_{(S,S_0)}$ for the full subcategory of separated gluing triples.

The category $\mathbf{Trip}_{(S,S_0)}$ has all fiber products, computed in the obvious way. Moreover, if $(T, T_0) \to (S, S_0)$ is a morphism of type (N)/(V) pairs as in Notation 2.1, with $|T_0| = |T \times S_0| \subseteq |T|$, then there is a natural base change functor $\mathbf{Trip}_{(S,S_0)} \to \mathbf{Trip}_{(T,T_0)}$.

Proposition 2.14 allows us to realize algebraic spaces over S as gluing triples.

Definition 2.17. The functor

$$t\colon \mathbf{AlgSp}^{\mathrm{lft}}_S \to \mathbf{Trip}_{(S,S_0)}, \qquad X \mapsto (X^\circ, \widehat{X}, j_X),$$

is called the *triples functor over* S, and restrict to a functor

$$t\colon \mathbf{AlgSp}^{\mathrm{lft,sep}}_S \to \mathbf{Trip}^{\mathrm{sep}}_{(S,S_0)}.$$

It is simple to see that the triples functor t preserves all fiber products and t is compatible with base change along a morphism of pairs $(T, T_0) \to (S, S_0)$.

We come now to the Gluing Theorem, whose proof will occupy the entirety of §4. If $(X^\circ, \widehat{X}, j_X)$ is a gluing triple with $t(X) \simeq (X^\circ, \widehat{X}, j_X)$ we say that X is the *gluing* X° and $\widehat{X}$ via j_X. The Gluing Theorem says such a gluing always exists when S is a G-scheme (see [28, Tag 07GG]).

Theorem 2.18 (The Gluing Theorem)**.** *If (S, S_0) is of Type (N)/(V), then the triples functor*

$$t\colon \mathbf{AlgSp}^{\mathrm{lft,sep}}_S \to \mathbf{Trip}^{\mathrm{sep}}_{(S,S_0)}, \qquad X \mapsto (X^\circ, \widehat{X}, j_X).$$

is fully faithful. Furthermore, if S is a G-scheme (e.g., S is excellent) it is an equivalence of categories with quasi-inverse given by the gluing functor

$$g\colon \mathbf{Trip}^{\mathrm{sep}}_{(S,S_0)} \to \mathbf{AlgSp}^{\mathrm{lft,sep}}_S, \quad (X^\circ, \widehat{X}, j_X) \mapsto \left(T \mapsto \mathrm{Hom}_{\mathbf{Trip}_{(S,S_0)}}(t(T), (X^\circ, \widehat{X}, j_X)\right).$$

2.2.3. *Properties of triples and morphisms of triples.* We now give an omnibus result that shows that the triples functor t reflects and preserves most reasonable properties of morphisms.

Definition 2.19. Let P be a property of morphisms of (formal) algebraic spaces. A morphism $(Y^\circ, \widehat{Y}, j_Y) \to (X^\circ, \widehat{X}, j_X)$ is said to be P if both $Y^\circ \to X^\circ$ and $\widehat{Y} \to \widehat{X}$ are.

Remark 2.20. With an eye towards the flow of the article we have included Proposition 2.21 here, but the proof relies on later material (namely §3 and §4). That said, there is no circularity.

Proposition 2.21. *Suppose that $f\colon Y \to X$ is a morphism of lft algebraic S-spaces. Then, for the following properties P:*

i) quasi-compact,	*v) (locally) quasi-finite,*	*ix) flat,*
ii) surjective,	*vi) isomorphism,*	*x) étale,*
iii) open embedding,	*vii) finite,*	*xi) smooth,*
iv) closed embedding,	*viii) separated,*	

f is P if and only if $t(f)$ is P.

Proof. As all these properties are stable/local for the étale topology, by considering $U \times_X Y \to U$ for an étale cover $U \to X$ where U is a lft S-scheme, we may assume X and Y are representable.

For all of these properties P, it's evident if f satisfies P then $f^\circ\colon Y^\circ \to X^\circ$ satisfies P, and that $f_n\colon Y_n \to X_n$ satisfies P for all n, where $X_n = X_{S_n}$, and similarly for Y_n. Thus, to verify that $t(f)$ satisfies P it suffices to show that $\widehat{f}\colon \widehat{Y} \to \widehat{X}$ is P if and only if f_n is P for all n. But, in all cases this is true as is seen from

i) obvious,
ii) obvious,
iii) Prop. 4.4.2 + xi),
iv) Prop. 4.3.6,
v) obvious,
vi) obvious,
vii) Prop. 4.2.3,
viii) Prop. 4.6.9,
ix) Prop. 4.8.1,
x) Prop. 5.3.11,
xi) Prop. 5.3.18,

where each reference is to a result in [13, Chapter I].

So, suppose now that $t(f)$ is P. The fact that f is P is obvious for i), ii), v). Let us then observe that every other claim follows easily once we establish vii) and ix). Namely, xi) and xii) easily follow once ix) is established from [28, Tag 01V9]. Moreover, once xi) is known, the case of iii) follows easily: if $t(f)$ is an open embedding, then f is étale, and so by [28, Tag 02LC] it suffices to show that f is universally injective. But, by [28, Tag 01S4] it suffices to show that Δ_f is surjective, which can clearly be checked on the level of $t(f)$. Then, from iii) and ii) we see that vi) follows as an isomorphism is a surjective open embedding. Now suppose that vii) is established. Then, to show iv) it suffices by [28, Tag 03BB] that f is a monomorphism. But, this means showing that Δ_f is an isomorphism, which follows easily from vi). Finally, to show viii) we must show that Δ_f is a closed embedding, but this follows from iv).

To address case ix), we assume $Y = \operatorname{Spec}(B)$ and $X = \operatorname{Spec}(A)$ and show that $A \to B$ is flat. By assumption, $\operatorname{Spec}(B) - V(IB) \to \operatorname{Spec}(A) - V(IA)$ and $\widehat{A} \to \widehat{B}$ are flat (see [13, Chapter I, Proposition 4.8.1]). Suppose that $\mathfrak{n}$ is a point of $V(IB)$ and let $\mathfrak{m}$ be its image in $V(IA)$. As $A_\mathfrak{m}/I^n A_\mathfrak{m} \to B_\mathfrak{n}/I^n B_\mathfrak{n}$ is flat, it suffices to verify that the conditions of [13, Chapter 0, Corollary 8.3.9] are satisfied. Evidently the pair $(B_\mathfrak{n}, I)$ is Zariskian, and so it suffices to verify that the pairs $(A_\mathfrak{m}, IA_\mathfrak{m})$ and $(B_\mathfrak{n}, IB_\mathfrak{n})$ are **(APf)** as in [13, Chapter 0, §7.(c)]. For type (N) this follows from [13, Chapter 0, Proposition 7.4.14]. For Type (V), it suffices by [13, Chapter 0, Corollary 8.2.17] to show that $(B_\mathfrak{n}, IB_\mathfrak{n})$ is pseudo-adhesive, but by [13, Chapter 0, Proposition 8.5.7] it further suffices to show that (B, IB) is pseudo-adhesive. But, $(\mathcal{O}, (\varpi))$ is universally pseudo-adhesive (see [30, Theorem 2.16] and [13, Chapter 0, Theorem 9.2.1]) from where the claim follows.

Finally, to show case vii), i.e., finite, let us observe that we may write $Y^\circ = \underline{\operatorname{Spec}}(\mathcal{A}^\circ)$ for a coherent $\mathcal{O}_{X^\circ}$-algebra $\mathcal{A}^\circ$, and similarly we may write $\widehat{Y} = \underline{\operatorname{Spf}}(\widehat{\mathcal{A}})$ for a coherent $\mathcal{O}_{\widehat{X}}$-algebra $\widehat{\mathcal{A}}$ (see [13, Chapter I, Proposition 4.2.6]). By setup the pullback of $\mathcal{A}^\circ$ and $\widehat{\mathcal{A}}$ to $\widehat{X}^{\mathrm{rig}}$ is isomorphic, and so by Proposition 3.8 we may find a coherent $\mathcal{O}_X$-algebra $\mathcal{A}$ inducing $\mathcal{A}^\circ$ and $\widehat{\mathcal{A}}$. Then, $\underline{\operatorname{Spec}}(\mathcal{A}) \to X$ is a finite morphism, and by construction $t(\underline{\operatorname{Spec}}(\mathcal{A}))$ is isomorphic to Y. By Corollary 4.4, $\underline{\operatorname{Spec}}(\mathcal{A})$ is isomorphic to Y as an X-scheme, from where the claim follows. □

2.3. **Examples and applications.** We illustrate the Gluing Theorem (see Theorem 2.18) by giving several novel examples of gluing triples and the algebraic space they glue to. When applicable, we also remark on natural applications of the gluing theorem that the example more generally suggests.

We begin by showing how simple gluing triples can give rise to quite exotic-looking schemes.

Example 2.22. Consider the gluing triple $(X^\circ, \widehat{X}, j_X)$ where $X^\circ = \mathbb{A}^1_{\mathbb{Q}_p}$, $\mathfrak{X} = \widehat{\mathbb{A}}^1_{\mathbb{Z}_p} \sqcup \widehat{\mathbb{A}}^1_{\mathbb{Z}_p}$ and j_X is the natural embedding

$$\mathfrak{X}^{\mathrm{rig}} \xrightarrow{\sim} \left\{x \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_p} : |x| \leqslant 1\right\} \sqcup \left\{x \in \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_p} : |x - 1/p| \leqslant 1\right\} \subseteq \mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_p}.$$

In this case one may show that $g(X^\circ, \mathfrak{X}, j_X) \simeq \operatorname{Spec}(A)$ where

$$A = \{f(x) \in \mathbb{Z}_p[x] : f(x + 1/p) \in \mathbb{Z}_p[x]\},$$

which is not even evidently a finitely generated $\mathbb{Z}_p$-algebra (although see Proposition 2.37).

Example 2.22 produces a strange affine scheme by gluing two simple affine objects together. But it is not in general even true that the result of gluing an affine formal scheme to an affine scheme results in an affine object (see Proposition 2.33 for the missing conditions).

Example 2.23. Let $\mathbb{A}^1_{\mathbb{Z}_p} = \operatorname{Spec}(\mathbb{Z}_p[x])$. Then, $X = \mathbb{A}^1_{\mathbb{Z}_p} - V(p, x)$ satisfies $t(X) = (\mathbb{A}^1_{\mathbb{Z}_p}, \widehat{\mathbb{G}}_{m,\mathbb{Z}_p}, j_X)$ where j_X is the natural inclusion of the unit circle into $\mathbb{A}^{1,\mathrm{an}}_{\mathbb{Q}_p}$.

In fact, it is not even true that the result of gluing a formal scheme to a scheme along a rigid analytic open is *even a scheme*, i.e., need not be a representable algebraic space.

Example 2.24. In [21, Example 5.3] there is constructed a (separated) K3 surface X *in algebraic* $\mathbb{Z}_p$*-spaces* with the property that $X_{\mathbb{Q}_p}$ and $\widehat{X}$ is a scheme and a formal scheme, respectively. In particular, one has $X = g(X_{\mathbb{Q}_p}, \widehat{X}, j_X)$ is *not representable.*

Such examples seem obscure but naturally come up in studying good reduction phenomena as in the case of loc. cit. In fact, the following shows that scheme-like gluing triples with non-representable gluing even show up when addressing the following simple question: if X is a smooth proper $\mathbb{Q}$-scheme such that $X_{\mathbb{Q}_p}$ has good reduction (i.e., admits a smooth proper model over $\mathbb{Z}_p$), does X have good reduction at p (i.e., admit a smooth proper model over $\mathbb{Z}_{(p)}$)?

Example 2.25. In [23, Warning 3.5.79] there is constructed a smooth quartic surface $X_{\mathbb{Q}} \subseteq \mathbb{P}^3_{\mathbb{Q}}$ such that $X_{\mathbb{Q}}$ *does not* admit a smooth proper model over $\mathbb{Z}_{(p)}$, but *does* admit a smooth proper model $X_{\mathbb{Z}_p}$ over $\mathbb{Z}_p$. So $(X_{\mathbb{Q}}, \widehat{X}_{\mathbb{Z}_p}, \mathrm{can.})$ gives another separated gluing triple over $\mathbb{Z}_{(p)}$ with both components representable with $g(X_{\mathbb{Q}}, \widehat{X}_{\mathbb{Z}_p}, \mathrm{can.})$ a non-representable algebraic space.

Remark 2.26. As observed in Example 2.25 (see [23, Remark 3.5.80]), the Gluing Theorem (or more to the point, the results of [2]) shows that if K is a discrete valuation field with excellent valuation ring $\mathcal{O}$, and X_K a smooth proper K variety such that $X_{\widehat{K}}$ has good reduction over $\widehat{\mathcal{O}}$, then X_K has good reduction *in algebraic spaces* over $\mathcal{O}$, but not necessarily in schemes. More generally, the Gluing Theorem shows the following.

Proposition 2.27. *Let K be a discrete valuation field with resaonable valuation ring $\mathcal{O}$. Then, if $X_{\widehat{\mathcal{O}}}$ is a $\widehat{\mathcal{O}}$-model of $X_{\widehat{K}}$ then there exists an* algebraic space *$\mathcal{O}$-model X of X_K with $\widehat{X} \simeq \widehat{X}_{\widehat{\mathcal{O}}}$.*

Next we give simple example illustrating that a theory like that of gluing triples is quite useful in conceptualizing Néron models, as already observed in (for example) [8].

Example 2.28. Let K be a discrete valuation field with valuation ring $\mathcal{O}$ and uniformizer π, and set $\mathfrak{X} = \bigsqcup_{n\in\mathbb{Z}} \widehat{\mathbb{G}}_{m,\widehat{\mathcal{O}}}$, a separated lft formal group scheme over $\mathrm{Spf}(\widehat{\mathcal{O}})$ (with multiplication given by naturally identifying $\mathfrak{X} \simeq \widehat{\mathbb{G}}_{m,\widehat{\mathcal{O}}} \times \underline{\mathbb{Z}}$). There is a natural embedding group rigid $\widehat{K}$-spaces $j\colon \mathfrak{X}_\eta \hookrightarrow \mathbb{G}^{\mathrm{an}}_{m,\widehat{K}}$ embedding the n^{th} component via

$$\widehat{\mathbb{G}}^{\mathrm{rig}}_{m,\widehat{\mathcal{O}}} \xrightarrow{\sim} \{x \in \mathbb{G}^{\mathrm{an}}_{m,\widehat{K}} : |x| = |\pi^n|\} \subseteq \mathbb{G}^{\mathrm{an}}_{m,\widehat{K}}.$$

This gives rise to a gluing triple $(\mathfrak{X}, \mathbb{G}_{m,K}, j)$, and it's easy to see that $g(\mathfrak{X}, \mathbb{G}_{m,K}, j)$ is the Néron model of $\mathbb{G}_{m,K}$ over $\mathcal{O}$. This example makes very clear why Néron models only commute with unramified base change, as ramified base change would alter the map j.

Remark 2.29. In the case of abelian varieties, observations concerning the construction of Néron models by triples-like considerations have already been made by Bosch–Lütkebohmert in [8]. It would be interesting to see the analogous construction for Néron models of tori carried out.

We also remark that the Gluing Theorem conceptually simplifies the construction in [8]. In our terminology and with notation as in Example 2.28, in [8] they construct for an abelian variety A over K a gluing triple $(A, \mathfrak{A}, j)$ so that if $(A, \mathfrak{A}, j) = t(\mathcal{A})$ for some group $\mathcal{O}$-scheme $\mathcal{A}$, then $\mathcal{A}$ is the Néron model of A. Without the Gluing Theorem, the construction of $\mathcal{A}$ is quite involved. But, using the Gluing Theorem, $g(A, \mathfrak{A}, j)$ is a group algebraic $\mathcal{O}$-space which is automatically a group $\mathcal{O}$-scheme by [24, Theorem 3.3.1].

Finally, we point out that the Gluing Theorem gives one a recognition principle for when a formal scheme is algebraizable, which applies in some otherwise mysterious situations.

Example 2.30. Let $\mathcal{E}$ be the minimal model of an elliptic curve E over K with split multiplicative reduction. The special fiber $\mathcal{E} \simeq \mathbf{P}^1_k/(0 \sim \infty)$, where k is the residue field of K, admits a $\mathbf{Z}$-covering space $\mathcal{X}_k \to \mathcal{E}_k$ by an infinite chain of $\mathbf{P}^1_k$'s. Let $\mathfrak{X} \to \mathfrak{E}$ be the unique lift of this map to an étale map of formal schemes. The generic fiber of $\mathfrak{X}$ is $\mathfrak{X}^{\mathrm{rig}} = \mathbf{G}^{\mathrm{an}}_{m,K}$, and hence by the Gluing

Theorem there exists an algebraic space locally of finite type X over $\mathcal{O}_K$ with generic fiber $\mathbf{G}_{m,K}$ and formal completion $\widehat{X}$. Such a space (in fact, a scheme) has been explicitly described in [12, §IV.1], and is called the *Tate curve.*

Remark 2.31. Example 2.30 showcases an interesting (but immediate) corollary of the Gluing Theorem. To state this precisely, let S be an excellent algebraic space and S_0 a closed subspace. Fix a separated lft formal algebraic $\widehat{S}$-space $\mathfrak{X}$, and let $\mathcal{A}(\mathfrak{X})$ denote the category of *algebraizations* of $\mathfrak{X}$ (i.e., the category of pairs (X, ι_X) where X is a separated lft algebraic S-space and $\iota_X\colon \mathfrak{X} \xrightarrow{\sim} \widehat{X}$ an isomorphism). On the other hand, let $\mathcal{E}(\mathfrak{X}_\eta)$ denote the category of pairs (X°, j_X) where X° is a separated lft algebraic S°-space and $j_X\colon \mathfrak{X}_\eta \to (X^\circ)^{\mathrm{an}}$ an open embedding.

Proposition 2.32. *The functor*

$$\mathcal{A}(\mathfrak{X}) \to \mathcal{E}(\mathfrak{X}_\eta), \quad (X, \iota_X) \mapsto (\widehat{X}, j_X \circ \iota_X^{\mathrm{rig}})$$

is an equivalence with quasi-inverse given by sending (X°, j_X) to $g(X^\circ, \mathfrak{X}, j_X)$. In particular, $\mathfrak{X}$ is algebraizable if and only if $\mathfrak{X}_\eta$ can be open embedded into (the analytification) of a separated lft algebraic S°-space.

2.4. **Characterization of schemes.** In this section we discuss a characterization of (affine) schemes in terms of gluing triples, i.e., a characterization of *(affine) schematic gluing triples.* The fact that this discussion is not trivial is illustrated by Examples 2.23, 2.24, and 2.25. For simplicity, we restrict ourselves to the setting $(S, S_0) = (\mathrm{Spec}(R), V(\pi))$ of Type (N).

The first observation is a characterization of the gluing triples associated to affine schemes. Namely, it shows that the explanation for the phenomenon witnessed in Example 2.23 is that the map $\mathbb{Q}_p[x] \to \mathbb{Q}_p\langle x^{\pm 1}\rangle$ does not have dense image.

Proposition/Definition 2.33. *The triples functor t induces an equivalence between the category of ft affine R-schemes and that of* affine gluing triples over (S, S_0)*: $(X^\circ, \widehat{X}, j_X)$ such that*

- *(a) $X^\circ = \mathrm{Spec}(A)$ is affine,*
- *(b) $\widehat{X} = \mathrm{Spf}(\widehat{B})$ is affine,*
- *(c) $j\colon \mathrm{Spa}(\widehat{B}[1/\pi]) \to \mathrm{Spec}(A)^{\mathrm{an}}$ is an open immersion,*
- *(d) and $j^*\colon A \to B$ has (π-adically) dense image.*

The following is a trivial consequence of Proposition/Definition 2.33.

Corollary/Definition 2.34. *The triples functor t an equivalence between the category of separated lft R-schemes and that of* schematic gluing over (S, S_0)*: triples $(X^\circ, \widehat{X}, j)$ such that $(X^\circ, \widehat{X}, j_X)$ admits an open cover by affine gluing triples $(X_i^\circ, \widehat{X}_i, j_i)$.*

Remark 2.35. A characterizations similar to that in Proposition/Definition 2.33 already occurs in [8, Remark 1.6] with two caveats. First, they restrict to cases of Type (V) but, more seriously, they assume their triples have reduced special fiber. This considerably simplifies the proof of the key Proposition 2.37 below.

Before we continue on to the proof of Proposition/Definition 2.33 we observe that combining the Gluing Theorem (see Theorem 2.18), [28, Tag 03XX], and Proposition 2.21 gives a more practical criterion for proving that a separated gluing triple is schematic.

Proposition 2.36. *If S is a G-scheme, then a separated gluing triple $(X^\circ, \widehat{X}, j_X)$ over (S, S_0) is schematic if it admits a separated and locally quasi-finite map to a schematic gluing triple.*

Now, the fully faithfullness of the functor t in Proposition/Definition 2.33 will follow from later considerations (see Corollary 4.4), and so we focus now on essential surjectivity. This is a simple consequence of the following.

Proposition 2.37. *Let A be a finitely generated $R[1/\pi]$-algebra, let B be a topologically finitely generated $\widehat{R}$-algebra, and set $C = B[1/\pi]$. Let $j^*\colon A \to C$ be a $R[1/\pi]$-algebra homomorphism with* *dense image for which the induced map* $\mathrm{Spa}(C) \to \mathrm{Spec}(A)^{\mathrm{an}}$ *is an open immersion. Define the ring* D *as the pull-back*

$$\begin{tikzcd} D \arrow[r] \arrow[d] & B \arrow[d] \\ A \arrow[r, "j^*"'] & C \end{tikzcd}$$

Then, D *is a finitely generated* R*-algebra with* $A = D[1/\pi]$ *and* $\widehat{D} = B$*.*

Proof. We break the proof into three steps.

Step 1. We shall find elements $x_1, \ldots, x_n$ in D satisfying the following properties:

(1) their images in A generate it as a $R[1/\pi]$-algebra,
(2) their images in B topologically generate it as an $\widehat{R}$-algebra,
(3) the open subset $\mathrm{Spa}(C)$ of $\mathrm{Spec}(A)^{\mathrm{an}}$ is cut out by the inequalities $|x_i| \leqslant 1$ $(i = 1, \ldots, n)$,
(4) the π-torsion of B (which is the kernel of $B \to C$) is generated as an ideal by a subset $\{x_1, \ldots, x_r\}$, and $x_1, \ldots, x_r$ map to zero in A.

Let $\{y_i'\}$ be any finite set of generators for A over $R[1/\pi]$. Note that $\{\pi^N y_i'\}$ is also a set of generators and, as B has open image in C, have image in C lying in the image of B for $N \gg 0$. Choose such an N and $\{y_i\}$ be a subset of D mapping to $\{\pi^N y_i'\}$ in A. Similarly, let $\{\hat{z}_j\}$ be a finite set of topological generators for B over $\widehat{R}$. Then, for any other elements $\widehat{z}_j'$ the set $\{\widehat{z}_j + \pi\widehat{z}_j'\}$ topologically generates B, as j^* has dense image, must also have image in C lying in the image of A. Choose such $\{\widehat{z}_j'\}$ and let $\{z_j\}$ be a subset of D mapping to $\{\widehat{z}_j + \pi\widehat{z}_j'\}$. Write $\{u_1, \ldots, u_m\} := \{y_i\} \cup \{z_j\}$.

Observe that by construction the map $R[1/\pi][U_1, \ldots, U_m] \to A$ with $U_k \mapsto u_k$ is surjective. Let us write I for its kernel. Consider the map

$$C' := \widehat{R}[1/\pi]\langle U_k\rangle / I\widehat{R}[1/\pi]\langle U_k\rangle = A \otimes_{R[1/\pi][U_k]} \widehat{R}[1/\pi]\langle U_k\rangle \to C, \tag{2.4.1}$$

which is surjective by construction. The space $\mathrm{Spa}(C')$ is a Weierstrass affinoid subdomain of $\mathrm{Spec}(A)^{\mathrm{an}}$ defined by $|u_k|_\pi \leqslant 1$. Indeed, this follows as $\mathrm{Spec}(C')$ is the pullback of the Weierstrass domain $\mathrm{Spa}(\widehat{R}[1/\pi]\langle U_k\rangle) \to \mathbb{A}^{n,\mathrm{an}}_{\widehat{R}[1/\pi]}$ along the Zariski closed embedding $\mathrm{Spec}(A)^{\mathrm{an}} \hookrightarrow \mathbb{A}^{n,\mathrm{an}}_{\widehat{R}[1/\pi]}$.

As $\mathrm{Spa}(C) \to \mathrm{Spec}(A)^{\mathrm{an}}$ is also an open embedding, we deduce that the map $\mathrm{Spa}(C) \to \mathrm{Spa}(C')$ induced by (2.4.1) is an open immersion. But since this morphism is also a closed immersion, we must have a decomposition $C' = C \times C''$ where $C' \to C$ is the first projection. Consider the element $t' = (0, 1/\pi) \in C \times C'' = C'$. Since $A \to C'$ has dense image by construction, we may find an element t_0 in A whose image is close enough to t' so that $|t_0|_\pi \leqslant 1$ on $\mathrm{Spa}(C)$ and $|t_0|_\pi > 1$ everywhere on $\mathrm{Spa}(C'')$. Observe that, in particular, $j^*(t_0)$ lies in the image of $B \to C$, and so we may choose some t in D mapping to it.

Let $\{w_1, \ldots, w_k\}$ be generators of the π-torsion $B[\pi^\infty]$ which is finitely generated as R is of Type (N). Set $\{x_1, \ldots, x_n\} = \{u_1, \ldots, u_m, t, w_1, \ldots, w_k\}$.

Step 2. We shall prove that $\{x_1, \ldots, x_n\}$ in D as in Step 1 generate D as an R-algebra, assuming that B is π-torsion free. To this end, consider the commutative diagram

$$\begin{tikzcd} R[x] \arrow[rr, "\delta"] \arrow[dr] \arrow[dd] & & D \arrow[dr] \arrow[dd] & \\ & R\langle x\rangle \arrow[rr, two heads, "\beta"] \arrow[dd] & & B \arrow[dd] \\ R[1/\pi][x] \arrow[rr, two heads, "\alpha"] \arrow[dr] & & A \arrow[dr] & \\ & R[1/\pi]\langle x\rangle \arrow[rr, two heads, "\gamma"] & & C \end{tikzcd}$$

Our goal is to show that the top arrow δ is surjective. Fix d in D and denote by a, b, c its images in A, B, C, and let U, V, W be their preimages in $R[1/\pi][x]$, $R\langle x\rangle$, $R[1/\pi]\langle x\rangle$. Since the left square is cartesian as well, we need to show that the images of U and V in W intersect. To this end, it is enough to show two assertions:

(i) The image of U in W is dense.
(ii) The image of V in W is open (and nonempty).

Let us denote by I, J, K the kernels of α, β, γ. Thus

$$U = \tilde{\alpha} + I, \qquad V = \tilde{\beta} + J, \qquad W = \tilde{\gamma} + K$$

for arbitrary lifts $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ of a, b, c. In particular, showing (i) and (ii) is equivalent to showing the density of the image of I in K and the openness of the image of J in K.

For the first assertion, we note first that condition (3) means that

$$C \simeq A \otimes_{R[1/\pi][x]} R[1/\pi]\langle x\rangle.$$

This is equivalent to saying that $K = I \cdot R[1/\pi]\langle x\rangle$. Thus the image of I is dense in K. For the second assertion, the assumed injectivity of $B \to C$ implies that $J = K \cap R\langle x\rangle$. Since $R\langle x\rangle$ is open in $R[1/\pi]\langle x\rangle$, J is open in K.

Step 3. We now show that D is finitely in general. Since B is Noetherian, there exists an $N \geqslant 0$ such that $B[\pi^\infty] = B[\pi^N]$. In this case, we have $B \simeq B' \times_{B'''} B''$, where

$$B' = B/B[\pi^\infty] = \mathrm{im}(B \to C), \qquad B'' = B/\pi^N B, \qquad B''' = B' \otimes_B B'' = B'/\pi^N B'.$$

Therefore, setting $D' = A \times_C B'$, we have

$$D \xrightarrow{\sim} D' \times_{B'''} B''.$$

Now D', B', and B''' are finitely generated R-algebras. Moreover, the maps $D \to D'$ and $D \to B''$ are surjective. Therefore by [28, Tag 00IT], the ring D is finitely generated over R. □

3. Coherent sheaves and gluing triples

In this section we show an equivalence of categories between coherent sheaves on an algebraic space X and on its associated triple $t(X)$, which is a crucial input in our proof of the Gluing Theorem. Throughout we fix a pair (S, S_0) as in Notation 2.1 which we assume to be of type (N)/(V). Since the rings in question are coherent (see Lemma 3.1(1)), coherent $\mathcal{O}_X$-modules coincide with finitely presented $\mathcal{O}_X$-modules.

3.1. Gluing for modules over an affine triple.

In this subsection we recall the gluing of results of Artin (see [2, Theorem 2.6]) in the Noetherian case and Beauville–Laszlo (see [3] and the generalization in [28, Tag 0FGM]) in the principal case.

Throughout we fix an affine ft S-scheme $X = \mathrm{Spec}(A)$ and write $\mathcal{I}(\mathrm{Spec}(A)) = I$, an ideal of definition $I \subseteq A$. Then $\widehat{X} = \mathrm{Spf}(\widehat{A})$ and we set $s(\widehat{X}) := \mathrm{Spec}(\widehat{A}) \setminus V(I\widehat{A})$, an X°-scheme. The following lemma summarizes the ring-theoretic properties of A.

Lemma 3.1. *Suppose that* $(\mathrm{Spec}(A), V(I))$ *is of Type (N)/(V), and write* $X = \mathrm{Spec}(A)$ *and* $X^\circ = \mathrm{Spec}(A) \setminus V(I)$. *then the following properties hold.*

(1) The pair (A, I) *is universally pseudo-adhesive, and the ring* A *is coherent,*[6]
(2) the map $X^\circ \sqcup \mathrm{Spec}(\widehat{A}) \to X$ *is faithfully flat,*
(3) for any finite A*-module* M *the natural map* $M \otimes_A \widehat{A} \to \widehat{M}$ *is an isomorphism,*
(4) for any finite A*-module* M, *the natural map* $M[I^\infty] \to \widehat{M}[I^\infty]$ *is a bijection,*

[6]Recall (e.g., see [13, Chapter 0, Definitions 8.5.1 and 8.5.4]), that a ring/ideal pair (A, I) is called *universally pseudo-adhesive* if for any finite type A-algebra R the scheme $\mathrm{Spec}(R) \setminus V(IR)$ is Noetherian, and for any finitely generated R-module M the module $M[I^\infty]$ is bounded, i.e., there exists some $n \geqslant 0$ such that $I^n M[I^\infty] = 0$. Moreover, A is called *coherent* if every finitely generated ideal is finitely presented.

Proof. To prove (1), we first assume that (S, S_0) is of type (N). In this case A is Noetherian, and so evidently $\mathrm{Spec}(A) - V(I)$ is Noetherian. Moreover, The fact that $M[I^\infty]$ is bounded for a finitely generated A-module M is trivial as M is Noetherian, and so $M[I^\infty]$ is finitely generated. In the case when (S, S_0) is of type (V) the claim follows as the pair $(\mathcal{O}, (\varpi))$ is universally pseudo-adhesive by [30, Theorem 2.16] and/or [13, Chapter 0, Theorem 9.2.1]. The coherence of A is clear in Type (N), and follows from [13, Chapter 0, Corollary 9.2.9] in case of Type (V).

Given (1), we see that (3) follows from [13, Chapter 0, Proposition 8.2.17], and in fact this reference also proves (2). Indeed, evidently $X^\circ \sqcup \mathrm{Spec}(\widehat{A}) \to X$ is surjective and and $X^\circ \to X$ is flat, so it suffices to explain why $A \to \widehat{A}$ is flat, but this follows from loc. cit.

Finally, we prove (4). Observe that we have a short exact sequence

$$0 \to M[I^\infty] \to M \to Q \to 0,$$

where Q is a finitely generated and I-torsionfree A-module. By work of Raynaud–Gruson (see [6, §7.4, Theorem 4]) Q is then finitely presented, and so $M[I^\infty]$ is actually finitely generated (see [28, Tag 0519]). As $A \to \widehat{A}$ is flat and $M \otimes_A \widehat{A} \simeq \widehat{M}$, we obtain an exact sequence

$$0 \to M[I^\infty] \otimes_A \widehat{A} \to \widehat{M} \to Q \otimes_A \widehat{A} \to 0.$$

As $M[I^\infty]$ and Q are finitely generated A-modules, we have isomorphisms

$$M[I^\infty] \otimes_A \widehat{A} \simeq A[I^\infty]^\wedge, \quad Q \otimes_A \widehat{A} \simeq \widehat{Q}, \tag{3.1.1}$$

where all completions are I-adic. But $M[I^\infty]$ is I-torsion and so $M[I^\infty]^\wedge = M[I^\infty]$. So, all in all, we have the exact sequence

$$0 \to M[I^\infty] \to \widehat{M} \to \widehat{Q} \to 0.$$

So, it suffices to observe that $\widehat{Q}$ is I-torsionfree. By (3.1.1) this follows as $A \to \widehat{A}$ is flat. $\square$

In the affine setting, the category $\mathbf{Coh}(\mathrm{Spec}(A))$ of coherent sheaves on $\mathrm{Spec}(A)$ is replaced by the category $\mathbf{Mod}^{\mathrm{fp}}(A)$ of finitely-presented A-modules. The analogue for the gluing triple $t(A)$ is given by the following.

Definition 3.2. Denote by $\mathbf{Mod}^{\mathrm{fp}}(t(A))$ the category of triples $(\mathcal{M}^\circ, \widehat{M}, \iota_M)$ where

(1) $\mathcal{M}^\circ$ is a coherent sheaf on X°,
(2) $\widehat{M}$ is a finitely presented $\widehat{A}$-module,
(3) $\iota_M \colon \mathcal{M}^\circ_{s(\widehat{X})} \xrightarrow{\sim} \widetilde{\widehat{M}}|_{s(\widehat{X})}$ is an isomorphism of coherent sheaves on $s(\widehat{X})$,

with a morphism $(\mathcal{M}^\circ, \widehat{M}, \iota_M) \to (\mathcal{N}^\circ, \widehat{N}, \iota_N)$ a pair $f \colon \mathcal{M}^\circ \to \mathcal{N}^\circ$ and $g \colon \widehat{M} \to \widehat{N}$ of morphisms (in the appropriate categories) that commute with the ι-isomorphisms in the obvious sense.

We will often abuse notation concerning a triple $(\mathcal{M}^\circ, \widehat{M}, \iota_M)$ and let $\widehat{\mathcal{M}}^\circ$ denote the common (via the identification ι_M) coherent sheaf on $s(\widehat{X})$.

Proposition/Definition 3.3. *The functor*

$$t \colon \mathbf{Mod}^{\mathrm{fp}}(A) \to \mathbf{Mod}^{\mathrm{fp}}(t(A))$$

is an equivalence of A-linear abelian $\otimes$-categories, with quasi-inverse given by the gluing functor

$$g \colon \mathbf{Mod}^{\mathrm{fp}}(t(A)) \to \mathbf{Mod}^{\mathrm{fp}}(t(A)), \quad (\mathcal{M}^\circ, \widehat{M}, \iota_M) \mapsto \mathcal{M}^\circ(X^\circ) \times_{\widehat{\mathcal{M}}^\circ(s(\widehat{X}))} \widehat{M}.$$

Proof. In type (N), this has been established by Artin in [2, Theorem 2.6]. In order to cover type (V) as well, we aim to employ the enhanced version of the Beauville–Laszlo theorem in [28, Tag 0BP2]. Type (N) already covered, we may assume that the ideal of definition I is principal, generated by an element π. Assertions (3) and (4) of Lemma 3.1 imply that every finitely generated A-module is glueable (as defined in [28, Tag 0BNI]). It remains to show that an A-module M is finitely presented if both $\widehat{M}$ and $M[1/\pi]$ are. Since the map $A \to \widehat{A} \times A[1/\pi]$ is faithfully flat by Lemma 3.1, we conclude by [28, Tag 05B0]. $\square$

An immediate corollary of this result (and [13, Chapter II, Proposition 6.6.5]) is the following, where when I is locally principal we denote by $A[1/I]$ the global sections of the affine (see [28, Tag 07ZT]) scheme $\mathrm{Spec}(A) \setminus V(I)$.

Corollary 3.4. *When I is locally principal the natural map*

$$A \to A[1/I] \times_{\widehat{A}[1/I]} \widehat{A}$$

is an isomorphism of rings. More generally, the natural map

$$\mathcal{O}(X) \to \mathcal{O}(X^\circ) \times_{\mathcal{O}(\widehat{X}^{\mathrm{rig}})} \mathcal{O}(\widehat{X}), \tag{3.1.2}$$

is an isomorphism of rings.

3.2. **Coherent sheaves and triples.** We now move on to patching together the affine-local results of §3.1. In essence this is giving a globalization of the previously mentioned results of Artin and Beauville–Laszlo, which can only happen involving analytic geometry, as well as generalization of similar ideas in [4].

As a means of setting notation/terminology, we first recall the definition of coherent sheaves on algebraic spaces of various varieties. In the following, $(-)_{\text{Ét}}$ denotes the big étale site.

Definition 3.5. Let $\mathscr{X}$ be an lft algebraic S-space, lft formal algebraic $\widehat{S}$-space, or a rigid algebraic $\widehat{S}^{\mathrm{rig}}$-space. We define the category of *coherent sheaves on* $\mathscr{X}$, denoted $\mathbf{Coh}(\mathscr{X})$, to be the full subcategory of $\mathbf{Mod}(\mathscr{X}_{\text{Ét}}, \mathcal{O}_{\mathfrak{X}})$ which are finitely presented in the sense of [28, Tag 01BN].

As per usual, we are denoting by $\mathcal{O}_{\mathscr{X}}$ the sheaf on $\mathscr{X}_{\text{Ét}}$ defined by

$$\mathcal{O}_{\mathscr{X}}(\mathscr{U}) := \varprojlim_{\mathscr{U} \to \mathscr{X}} \mathcal{O}_{\mathscr{U}}(\mathscr{U}) \tag{3.2.1}$$

with $\mathscr{U} \to \mathscr{X}$ ranges over morphisms to $\mathscr{X}$ from representable objects. If $\mathscr{X}$ is representable, then coherent sheaves satisfy flat descent (see [28, Tag 023T], [13, Chapter I, Proposition 6.1.11], and [7, Theorem 2.1]). So, for any étale surjection $\mathscr{U} \to \mathscr{X}$ we have a natural equivalence

$$\mathbf{Coh}(\mathscr{X}) \to \text{2-Eq}\Big(\mathbf{Coh}(\mathscr{U}) \rightrightarrows \mathbf{Coh}(\mathscr{U} \times_{\mathscr{X}} \mathscr{U}) \mathrel{\substack{\rightarrow\\[-0.6ex]\rightarrow\\[-0.6ex]\rightarrow}} \mathbf{Coh}(\mathscr{U} \times_{\mathscr{X}} \mathscr{U} \times_{\mathscr{X}} \mathscr{U})\Big), \tag{3.2.2}$$

where $\mathbf{Coh}(\mathscr{U})$ can be seen as sheaves on the underlying topological space for representable $\mathscr{U}$.

Reducing to the affine situation allows one to make the following observation.

Lemma 3.6. *For $\mathscr{X}$ as in Definition 3.5, $\mathbf{Coh}(\mathscr{X})$ is an $\mathcal{O}_{\mathscr{X}}$-linear abelian $\otimes$-category.*

Observe that we have the following two natural functors.

(1) For an lft formal algebraic $\widehat{S}$-space $\widehat{X}$, we have by Proposition/Definition 2.12 a morphism

$$\mathrm{sp}_{\widehat{X}} \colon (\widehat{X}^{\mathrm{rig}}_{\text{Ét}}, \mathcal{O}^+_{\widehat{X}^{\mathrm{rig}}}) \to (\widehat{X}_{\text{Ét}}, \mathcal{O}_{\widehat{X}}),$$

of ringed topoi, functorial in $\widehat{X}$, which induces a natural functor

$$(-)^{\mathrm{rig}} \colon \mathbf{Coh}(\widehat{X}) \to \mathbf{Coh}(\widehat{X}^{\mathrm{rig}}), \quad \mathcal{E} \mapsto \mathcal{E}^{\mathrm{rig}} = \mathrm{sp}^*_{\widehat{X}}(\mathcal{E}) \otimes_{\mathcal{O}^+_{\widehat{X}^{\mathrm{rig}}}} \mathcal{O}_{\widehat{X}^{\mathrm{rig}}}.$$

(2) Similarly, if X° is an lft algebraic S°-space, then there is a natural morphism of ringed topoi

$$(-)^{\mathrm{an}} \colon ((X^\circ)^{\mathrm{an}}_{\text{Ét}}, \mathcal{O}_{(X^\circ)^{\mathrm{an}}}) \to (X^\circ_{\text{Ét}}, \mathcal{O}_{X^\circ}),$$

which induces a natural functor

$$(-)^{\mathrm{an}} \colon \mathbf{Coh}(X^\circ) \to \mathbf{Coh}.((X^\circ)^{\mathrm{an}}).$$

We are now prepared to define coherent sheaves on gluing triples over (S, S_0).

Definition 3.7. Let $(X^\circ, \widehat{X}, j_X)$ be a gluing triple over (S, S_0). Then, we define the category $\mathbf{Coh}(X^\circ, \widehat{X}, j_X)$ of *coherent sheaves* to sit in the following 2-Cartesian diagram of categories:

$$\begin{tikzcd} \mathbf{Coh}(X^\circ, \widehat{X}, j_X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{Coh}(X^\circ) \arrow[d, "j_X^* \circ (-)^{\mathrm{an}}"] \\ \mathbf{Coh}(\widehat{X}) \arrow[r, "(-)^{\mathrm{rig}}"'] & \mathbf{Coh}(\widehat{X}^{\mathrm{rig}}), \end{tikzcd}$$

i.e., the category of triples $(\mathcal{E}^\circ, \widehat{\mathcal{E}}, \iota_{\mathcal{E}})$ where $\mathcal{E}^\circ$ is a coherent $\mathcal{O}_{X^\circ}$-module, $\widehat{\mathcal{E}}$ is a coherent $\mathcal{O}_{\widehat{X}}$-module, and $\iota_{\mathcal{E}} \colon j_X^*(\mathcal{E}^\circ)^{\mathrm{an}} \xrightarrow{\sim} \widehat{\mathcal{E}}^{\mathrm{rig}}$ is an isomorphism of coherent $\mathcal{O}_{\widehat{X}^{\mathrm{rig}}}$-modules.

Let us observe that if X is a lft algebraic S-space, then there is a *triples functor*

$$t \colon \mathbf{Coh}(X) \to \mathbf{Coh}(t(X)), \quad \mathcal{E} \mapsto (\mathcal{E}^\circ, \widehat{\mathcal{E}}, \mathrm{can.}).$$

Here we denote by

(i) $(-)^\circ$ the pullback functor along the morphism of schemes $X^\circ \to X$,

(ii) $\widehat{(-)}$ the pullback functor along the morphism of ringed topoi $\widehat{(-)} \colon (\widehat{X}_{\text{Ét}}, \mathcal{O}_{\widehat{X}}) \to (X_{\text{Ét}}, \mathcal{O}_X)$,

(iii) and can.: $(\mathcal{E}^\circ)^{\mathrm{rig}} \xrightarrow{\sim} j_X^\circ \widehat{\mathcal{E}}^{\mathrm{rig}}$ is obtained as both are naturally the pullback of $\mathcal{E}$ along the morphism of ringed topoi $(\widehat{X}^{\mathrm{rig}}_{\text{Ét}}, \mathcal{O}_{\widehat{X}^{\mathrm{rig}}}) \to (X_{\text{Ét}}, \mathcal{O}_X)$.

The functor t is clearly 2-functorial in X.

Proposition 3.8. *Let X be an lft separated algebraic S-space. Then, the triples functor*

$$t \colon \mathbf{Coh}(X) \to \mathbf{Coh}(t(X)),$$

is an $\mathcal{O}_X$-linear $\otimes$-equivalence of abelian categories.

Proof. Observe that coherent sheaves on gluing triples satisfy étale descent in the following sense, which is a trivial consequence of Proposition 2.10 and étale descent for coherent sheaves on algebraic spaces, formal algebraic spaces, and rigid algebraic spaces.

Lemma 3.9. *Suppose that $(U^\circ, \widehat{U}, j_U) \to (X^\circ, \widehat{X}, j_X)$ is an étale surjection. Let us write $(U_1^\circ, \widehat{U}_1, j_{U_1})$ and $(U_2^\circ, \widehat{U}_2, j_{U_2})$ for the self fiber product and self triple fiber product of $(U^\circ, \widehat{U}, j_U)$ over $(X^\circ, \widehat{X}, j_X)$, respectively. Then, the natural functor*

$$\mathbf{Coh}(X^\circ, \widehat{X}, j_X) \to \text{2-Eq}\left(\mathbf{Coh}(U^\circ, \widehat{U}, j_U) \rightrightarrows \mathbf{Coh}(U_1^\circ, \widehat{U}_1, j_{U_1}) \substack{\rightarrow\\[-1ex]\rightarrow\\[-1ex]\rightarrow} \mathbf{Coh}(U_2^\circ, \widehat{U}_2, j_{U_2})\right),$$

is an equivalence of categories.

Using this and étale descent for algebraic spaces as in (3.2.2), we quickly reduce to the case when $X = \mathrm{Spec}(A)$. Let us write $I = \mathcal{I}(\mathrm{Spec}(A))$, an ideal of definition of A. Recall then that $s(\widehat{X})$ is defined to be $\mathrm{Spec}(\widehat{A}) \setminus V(I\widehat{A})$. There is a natural morphism of ringed spaces $(\widehat{X}^{\mathrm{rig}}, \mathcal{O}_{\widehat{X}^{\mathrm{rig}}}) \to (s(\widehat{X}), \mathcal{O}_{s(\widehat{X})})$ which induces a pullback $\mathcal{O}_X$-linear $\otimes$-equivalence of abelian categories of coherent sheaves (see [13, Chapter II, Proposition 6.6.5]). From this we see that the case for $X = \mathrm{Spec}(A)$ reduced precisely to Proposition/Definition 3.3. $\square$

Corollary 3.10. *Let X be an lft separated algebraic S-space. Then, the functor*

$$t \colon \mathbf{Vect}(X) \to \mathbf{Vect}(t(X)),$$

is a Quillen exact $\mathcal{O}_X$-linear $\otimes$-equivalence of categories with Quillen exact quasi-inverse.

Proof. Given Proposition 3.8, it suffices to show that a coherent $\mathcal{O}_X$-module $\mathcal{E}$ is a vector bundle if and only if $t(\mathcal{E})$ is. But, we may quickly reduce to the case when $X = \mathrm{Spec}(A)$ is affine. Let us write $M = \mathcal{E}(X)$. Note that $\widehat{\mathcal{E}}$ is a vector bundle if and only if $\widehat{E}(\mathrm{Spf}(\widehat{A})) = \widehat{M}$ is a finite projective $\widehat{A}$-module (e.g., see the proof of [18, Proposition A.13]). As $\widehat{M} = M \otimes_A \widehat{A}$ by Lemma 3.1, we are thus reduced to the following claim: M is a finite projective A-module if and only if

$\widehat{M}|_{X^\circ}$ and $M \otimes_A \widehat{A}$ is a vector bundle on X° and a finite projective $\widehat{A}$-module, respectively. But, given Lemma 3.1 this follows from [28, Tag 05A9]. $\square$

4. Proof of the Gluing Theorem

We now turn to the proof of the Gluing Theorem (see Theorem 2.18). For ease of reading, we have broken this section down into subsections isolating distinct portions of the proof. Throughout this section we fix (S, S_0) as in Notation 2.1, which we always assume to be of type (N)/(V).

4.1. **Explicit reconstruction in the schematic case.**

We first observe a more down-to-earth case of the Gluing Theorem. Namely, we wish to show that if X is an lft S-scheme then one may explicitly reconstruct the ringed space $(X, \mathcal{O}_X)$ from the triple $t(X)$.

Definition 4.1. Let $(X^\circ, \widehat{X}, j_X)$ be gluing triple over (S, S_0). The *underlying topological space*, denoted $|(X^\circ, \widehat{X}, j_X)|$, has underlying set $|X^\circ| \sqcup |\widehat{X}|$ and topology with closed subsets those of the form $C \cup \mathrm{sp}(C^{\mathrm{an}} \cap \widehat{X}^{\mathrm{rig}}) \cup D$, where C is a closed subset of X° and D a closed subset of $|\widehat{X}|$.

It is clear that the underlying topological space determines a functor

$$|\cdot|\colon \mathbf{Trip}_{(S,S_0)} \to \mathbf{Top},$$

and that there is a natural map of topological spaces $\alpha_X\colon |t(X)| \to |X|$.

Proposition 4.2. *Let X be an lft algebraic S-space. Then, the natural map*

$$\alpha_X\colon |t(X)| \to |X|$$

is a homeomorphism.

Proof. As the map α_X is continuous and bijective, it suffices to show this map is closed.

First assume that X is a scheme. It suffices to prove that if C is a closed subset of X°, then $C \sqcup \mathrm{sp}(C^{\mathrm{an}} \cap \widehat{X}^{\mathrm{rig}})$ is a closed subset of X. As admissible blowups are closed maps and the specialization map is functorial, this quickly reduces to the case when $X = \mathrm{Spec}(A)$ where $\mathcal{I}(X) = I$ is principal. Suppose that $C = V(J)$ for $J \subseteq \mathcal{O}(X^\circ)$ an ideal. We claim that $C \sqcup \mathrm{sp}(C^{\mathrm{an}} \cap \widehat{X}^{\mathrm{rig}}) = V(J^c)$ where J^c is the pullback of J along the map $A \to \mathcal{O}(X^\circ)$. This reduces to showing that $\mathrm{sp}(V(J^c)^{\mathrm{an}} \cap \widehat{X}^{\mathrm{rig}}) = V(I, J^c)$. Set $B = A/J^c$, then this is equivalent to the claim that $\mathrm{sp}(\mathrm{Spf}(\widehat{B})^{\mathrm{rig}}) = \mathrm{Spec}(B/IB)$. By [13, Chapter II, Proposition 3.1.5] it suffices to show that $\widehat{B}$ is I-torsionfree. By Lemma 3.1, $\mathrm{Spec}(\widehat{B}) \to \mathrm{Spec}(B)$ is flat , and so it suffices to show that B is I-torsion free. But, $B = A/J^c$ injects into $\mathcal{O}(X^\circ)/J$ which is I-torsion free.

To show closedness in general, let $U \to X$ be an étale surjection where U is a scheme so that $|U| \to |X|$ is quotient map. By functoriality of the specialization map, the pullback of $C \sqcup \mathrm{sp}(C^{\mathrm{an}} \cap \widehat{X}^{\mathrm{rig}})$ under $|U| \to |X|$ is $C' \sqcup \mathrm{sp}((C')^{\mathrm{an}} \cap \widehat{U}^{\mathrm{rig}})$, where C' is the preimage of C under $U^\circ \to X^\circ$. This set is closed by the previous paragraph, and thus so is $C \sqcup \mathrm{sp}(C^{\mathrm{an}} \cap \widehat{X}^{\mathrm{rig}})$. $\square$

Define the full-subcategory $\mathscr{C} \subseteq \mathbf{Trip}_{(S,S_0)}$ to consist of those gluing triples $(X^\circ, \widehat{X}, j_X)$ where X° is a S°-scheme, $\widehat{X}$ is a formal $\widehat{S}$-scheme, and j_X is an open embedding. Define a functor

$$g\colon \mathscr{C} \to \mathbf{RS}_S, \quad g(X^\circ, \widehat{X}, j_X) = (t(X), \mathcal{O}_{t(X)}).$$

Here $\mathbf{RS}_S$ is the category of ringed spaces over S, and we define the sheaf $\mathcal{O}_{t(X)}$ as follows:

$$\mathcal{O}_{t(X)}(t(U)) := \mathcal{O}(U^\circ) \times_{\mathcal{O}(\widehat{U}^{\mathrm{rig}})} \mathcal{O}(\widehat{U}),$$

It is clear that this defines a sheaf on X, and that g really is a functor.

Proposition 4.3. *For an lft S-scheme X, there is a natural isomorphism of ringed spaces over S*

$$\alpha_X\colon (t(X), \mathcal{O}_{t(X)}) \xrightarrow{\sim} (X, \mathcal{O}_X).$$

Proof. By Proposition 4.2, α_X is a homeomorphism, so it suffices to show that the natural map $\mathcal{O}_X \to (\alpha_X)_* \mathcal{O}_{t(X)}$ is an isomorphism on affine open subsets. But, this is Corollary 3.4. $\square$

Corollary 4.4. *The functor*

$$t\colon \mathbf{Sch}_S^{\mathrm{lft}} \to \mathbf{Trip}_{(S,S_0)}$$

is fully faithful.

4.2. **Proof of fully faithfulness.** We now aim to show that for any separated lft algebraic spaces X and Y that the natural map

$$\mathrm{Hom}(X,Y) \to \mathrm{Hom}(t(X),t(Y)), \tag{4.2.1}$$

is a bijection. We do this in two steps.

Step 1: when X is a scheme. For injectivity, let $U \to Y$ be an étale surjection with U a lft separated S-scheme. If $f_i\colon X \to Y$ $(i=1,2)$ are morphisms with $t(f_1)=t(f_2)$ then $t(f_1')=t(f_2')$, where $f_i'\colon U\times_Y X \to U$ are the induced morphisms. But, as $U\times_Y X$ is a scheme, this implies by Corollary 4.4 that $f_1'=f_2'$ and, as $U\times_X Y\to X$ is a surjection, that $f_1=f_2$ as desired.

For surjectivity, let $f\colon t(X)\to t(Y)$ be a morphism. As X and Y are separated the graph morphisms $\Gamma_{f^\circ}\colon X^\circ \to X^\circ\times_{S^\circ} Y^\circ$ and $\Gamma_{\widehat{f}}\colon \widehat{X}\to\widehat{X}\times_{\widehat{S}}\widehat{Y}$ are closed embeddings, and so define ideal sheaves $\mathcal{I}_{f^\circ}\subseteq \mathcal{O}_{X^\circ\times_{S^\circ}Y^\circ}$ and $\mathcal{I}_{\widehat{f}}\subseteq\mathcal{O}_{\widehat{X}\times_{\widehat{S}}\widehat{Y}}$. As $f^\circ|_{\widehat{X}^{\mathrm{rig}}}=\widehat{f}|_{\widehat{X}^{\mathrm{rig}}}$ it's clear that $\mathcal{I}_{f^\circ}$ and $\mathcal{I}_{\widehat{f}}$ induce the same ideal sheaf of $\widehat{X}^{\mathrm{rig}}\times_{\widehat{S}^{\mathrm{rig}}}\widehat{Y}^{\mathrm{rig}}$, so define an object $\mathcal{I}_f$ in $\mathbf{Coh}(t(X\times_S Y))$ in the sense of Definition 3.7. By Proposition 3.8 this corresponds to a unique coherent sheaf $\mathcal{J}$ (which is an ideal sheaf as an equivalence preserves subobjects) on $X\times_S Y$ with $t(\mathcal{J})=\mathcal{I}_f$.

Consider $V(\mathcal{J})\subseteq X\times_S Y$. The projection map furnishes a morphism $V(\mathcal{J})\to X$ which is separated (as each $V(\mathcal{J})$ and X is) and quasi-finite (as this may be checked over X° and $\widehat{X}$, where it is an isomorphism). Thus, $V(\mathcal{J})$ is a scheme by [28, Tag 03XX]. Observe then that there is a natural isomorphism $\iota\colon t(X)\to t(V(\mathcal{J}))$ and thus by Corollary 4.4 there exists a unique isomorphism $i\colon X\to V(\mathcal{J})$ such that $t(i)=\iota$. Composing i with the projection $X\times_S Y\to Y$ produces a morphism $g\colon X\to Y$ which, by construction, satisfies $t(g)=f$, as desired.

Step 2: the general case. By [27, Exposé I, Corollaire 4.1.1] we may write $X=\varinjlim X_i$ in the category of presheaves on S-schemes. In particular, we have an identification

$$\mathrm{Hom}(X,Y)\simeq \varprojlim_{X_i\to X}\mathrm{Hom}(X_i,Y).$$

But, using **Step 1** one may directly check that one also has an identification

$$\mathrm{Hom}(t(X),t(Y))\simeq \varprojlim_{X_i\to X}\mathrm{Hom}(t(X_i),t(Y)).$$

The morphism from (4.2.1) is then obtained by passing to the limit over the morphsims

$$\mathrm{Hom}(X_i,Y)\to\mathrm{Hom}(t(X_i),t(Y)),$$

which are each bijections by **Step 1**, and thus we are done.

4.3. **Proof of essential surjectivity in the G-scheme case.** Our proof of essential surjectivity will employ results of Artin, ultimately relying on Artin approximation which requires excellence-like (i.e., G-scheme) hypotheses. To this end, we now restrict to the case when S is a G-scheme.

4.3.1. *Artin's contraction theorem.* We begin by recalling the main theorem of [2], and the improvement in [28, Tag 0GIB]. Our exposition will follow that of [28, Tag 0GH7].

Definition 4.5. A *formal contraction problem* over S is a quadruple of data $(X',T',\widehat{X},\widehat{g})$ where

- X' is a lft algebraic S-space,
- T' is a closed subset of $|X'|$,
- $\widehat{X}$ is an lft formal algebraic $\widehat{S}$-space,
- $\widehat{g}\colon \widehat{X'}_{T'}\to\widehat{X}$ is proper, rig-étale (see [28, Tag 0AQM]), and both $\widehat{g}$ and its diagonal are rig-surjective (see [28, Tag 0AQQ]).

We will only need the following very special case of a formal contraction problem.

Example 4.6. Let X' be a lft algebraic S-space and let $\widehat{g}\colon \widehat{X}' \to \widehat{X}$ be a formal modification of lft formal algebraic $\widehat{S}$-spaces. Then, $(X', |\widehat{X}'|, \widehat{X}, \widehat{g})$ is a formal contraction problem. Indeed, it suffices to show that $\widehat{g}$ is rig-étale and that $\widehat{g}$ and its diagonal are rig-surjective. But, by asssumption, $\widehat{g}^{\mathrm{rig}}$ is an isomorphism, and thus so is its diagonal. It is easy to see this implies that $\widehat{g}$ and its diagonal are rig-surjective (cf. [13, Chapter II, Lemma 3.3.7]). To prove that $\widehat{g}$ is rig-étale, we are reduced to the case of admissible blowups by combining [13, Chapter II, Corollary 2.1.5] and part (4) of [28, Tag 0GCZ], but this case is simple.

Definition 4.7. Let $(X', T', \widehat{X}, \widehat{g})$ be a formal contraction problem over S. Then, a *potential solution* to this formal contraction problem is a quadruple (X, T, g, ι) where

- X is a lft algebraic S-space,
- $g\colon X' \to X$ a proper morphism of formal algebraic $\widehat{S}$-spaces,
- $T \subseteq |X|$ a closed subset,
- $\iota\colon \widehat{X}_T \xrightarrow{\sim} \widehat{X}$ is an isomorphism of formal $\widehat{S}$-schemes.

We say a potential solution (X, T, g, ι) is a *solution* if

- $T' = g^{-1}(T)$,
- $g\colon X' \setminus T' \to X \setminus T$ is an isomorphism,
- the completion of g recovers, via ι, the map $\widehat{g}$.

In [2] it is shown that every formal contraction problem has a solution when S is excellent. This result was strengthened in [28, Tag 0GH7] to the setting of G-schemes (see [28, Tag 07GH]).

Theorem 4.8 (Artin's contraction theorem, see [2, Theorem 3.1] and [28, Tag 0GIB])**.** *Suppose that S is a G-scheme. Then, every formal contraction problem has a solution.*

4.3.2. *Essential surjectivity in the G-scheme case.* Assume now that S is a G-scheme and that $(X^\circ, \widehat{X}, j_X)$ is a separated gluing triple over (S, S_0).

By [9, Theorem 1.2], we may find a proper morphism of algebraic spaces $\overline{X} \to S$ and a dense open embedding $X^\circ \to \overline{X}^\circ$. As $\overline{X} \to S$ is proper, we know from Proposition 2.14 that $j_{\overline{X}}$ is an isomorphism. In particular, we have an open embedding

$$\widehat{X}^{\mathrm{rig}} \xrightarrow{j_X} (X^\circ)^{\mathrm{an}} \to (\overline{X}^\circ)^{\mathrm{an}} \xrightarrow{j_{\overline{X}}^{-1}} \widehat{\overline{X}}^{\mathrm{rig}}. \tag{4.3.1}$$

Thus, there exists admissible modifications $g\colon \widehat{Y} \to \widehat{X}$ and $\widehat{Y}' \to \widehat{\overline{X}}$ and an open embedding $\widehat{Y} \to \widehat{Y}'$ which recovers (4.3.1) on applying $(-)^{\mathrm{rig}}$.

By [28, Tag 0GDU] there exists a unique X°-modification $Y' \to \overline{X}$ whose completion recovers $\widehat{Y}' \to \widehat{\overline{X}}$. Let $|Y| \subseteq |Y'|$ be the union of

$$|X^\circ| \subseteq |\overline{X}^\circ| = |(Y')^\circ|, \quad \text{and} \quad |\widehat{Y}| \subseteq |\widehat{Y}'| \subseteq |Y'|.$$

Then using Proposition 4.2 one sees that $|Y|$ is an open subset of $|Y'|$ and so corresponds to a unique open algebraic subspace Y of Y' (see [28, Tag 03BZ]).

The quadruple $(Y, |\widehat{Y}|, \widehat{X}, g)$ is then a formal contraction problem in the sense of Definition 4.5. As S is a G-scheme we see from Theorem 4.8 that this formal contraction problem has a solution in the sense of Definition 4.8. Write this solution as (X, T, g, ι). As T is the image of $|\widehat{Y}|$ it is clear that $T = |\widehat{X}|$ and so $t(X) = (X^\circ, \widehat{X}, j_X)$ as desired.

5. Gluing of torsors and finite étale covers

In this final section we use the Gluing Theorem (see Theorem 2.18) to show that one can glue both torsors (called *torsor gluing*) and finite étale covers along gluing triples. We use the latter case to deduce an integral analogue of the Riemann existence theorem.

5.1. **Gluing torsors.** It is useful to know that not only does one have gluing for coherent sheaves (see Proposition 3.8), but also that one gluing for G-torsors for groups over X. For $G = \mathrm{GL}_{n,X}$ this essentially reduces to Corollary 3.10. Conversely, if G comes by base change from an affine flat group scheme over a Dedekind base one can use a Tannakian approach to deduce gluing for G-torsors from Corollary 3.10 (c.f. [18, Proposition A.17 and Theorem A.18]). That said, this torsor gluing can be proven for more general G, and independent of coherent gluing (in fact recovering Corollary 3.10 as a special case).

Throughout this section let us fix G to be a separated, flat, and lft group algebraic S-space.

5.1.1. *General definitions.* As we will be thinking about G-torsors in several geometric contexts, we first recall the general definition of a G-torsor on an S-ringed topos.

Definition 5.1. Let $\mathscr{X}$ be a topos and $\mathcal{O}$ an S-algebra object of $\mathscr{X}$. Then we have the sheaf

$$G_{\mathcal{O}} \colon \mathscr{X} \to \mathbf{Grp}, \quad T \mapsto G(\mathcal{O}(T)).$$

The category $\mathbf{Tors}_G(\mathscr{X}, \mathcal{O})$ (or just $\mathbf{Tors}_G(\mathscr{X})$ when $\mathcal{O}$ is clear from context) consists of objects Q of $\mathscr{X}$ equipped with a right action $G \times Q \to Q$ such that

(1) Q is a G-pseudotorsor (i.e., $Q \times G \to Q \times Q$ given by $(q, g) \mapsto (q, qg)$ is an isomorphism),
(2) Q is locally non-empty (i.e., each $Q(T_i)$ is non-empty for a cover $\{T_i \to *\}$ of the final object),

and where morphisms are G-equivariant morphisms in $\mathscr{X}$.

If $f \colon (\mathscr{X}, \mathcal{O}) \to (\mathscr{X}', \mathcal{O}')$ is a morphism of S-ringed topoi, then we have a functor

$$f^* \colon \mathbf{Tors}_G(\mathscr{X}') \to \mathbf{Tors}_G(\mathscr{X}), \quad Q \mapsto f^{-1}(Q) \times^{f^{-1}(G_{\mathcal{O}'})} G_{\mathcal{O}},$$

where $f^{-1}(G_{\mathcal{O}'}) \to G_{\mathcal{O}}$ is the natural morphism of groups, and $(-)\times^{(-)}(-)$ denotes the contracted product, as in [14, Chapitre III, Proposition 1.4.6]. Similarly, if $\varphi \colon G \to H$ is a morphism of separated, flat, and lft group S-schemes, then we have the morphism

$$\varphi_* \colon \mathbf{Tors}_G(\mathscr{X}) \to \mathbf{Tors}_H(\mathscr{X}), \quad Q \mapsto Q \times^{G_{\mathcal{O}}} H_{\mathcal{O}},$$

where $G_{\mathcal{O}} \to H_{\mathcal{O}}$ is the natural morphism induced by φ.

5.1.2. *Torsor gluing.* To formulate the torsor gluing we first specialize Definition 5.1 to the cases of our particular interest.

Notation 5.2. Let X, $\widehat{X}$, and $\widehat{X}^{\mathrm{rig}}$ be an lft algebraic S-space, formal algebraic S-space, and rigid algebraic $\widehat{S}^{\mathrm{rig}}$-space, respectively. Then, we abbreviate:

(1) $\mathbf{Tors}_G(X_{\mathrm{fppf}}, \mathcal{O}_X)$ to $\mathbf{Tors}_G(X)$.
(2) $\mathbf{Tors}_G(\widehat{X}^{\mathrm{adic}}_{\mathrm{fl}}, \mathcal{O}_{\widehat{X}})$ to $\mathbf{Tors}_G(\widehat{X})$ (where $\widehat{X}^{\mathrm{adic}}_{\mathrm{fl}}$ is the adic flat site, c.f., [18, §A.4]),
(3) $\mathbf{Tors}_G(\widehat{X}^{\mathrm{rig}}_{\mathrm{fl}}, \mathcal{O}_{\widehat{X}^{\mathrm{rig}}})$ to $\mathbf{Tors}_G(\widehat{X}^{\mathrm{rig}})$ (where $\widehat{X}^{\mathrm{rig}}_{\mathrm{fl}}$ is the flat site of $\widehat{X}^{\mathrm{rig}}$),[7]
(4) $\mathbf{Tors}_G(\widehat{X}^{\mathrm{rig}}_{\mathrm{fl}}, \mathcal{O}^+_{\widehat{X}^{\mathrm{rig}}})$ to $\mathbf{Tors}_{G^+}(\widehat{X}^{\mathrm{rig}})$.

If $\widehat{X}$ is an lft formal algebraic $\widehat{S}$-space, then we have the following two operations on torsors.

(i) From Proposition 2.10 we have a natural morphism of S-ringed topoi

$$\mathrm{sp}_{\widehat{X}} \colon (\widehat{X}^{\mathrm{rig}}_{\mathrm{fl}}, \mathcal{O}_{\widehat{X}^{\mathrm{rig}}}) \to (\widehat{X}_{\mathrm{fl}}, \mathcal{O}_{\widehat{X}}),$$

and thus we obtain a functor

$$\mathrm{sp}^*_{\widehat{X}} \colon \mathbf{Tors}_G(\widehat{X}) \to \mathbf{Tors}_{G^+}(\widehat{X}^{\mathrm{rig}}).$$

(ii) If X° is a lft algebraic S°-space, then the tautological map of S-ringed topoi

$$((X^\circ)^{\mathrm{an}}_{\mathrm{fl}}, \mathcal{O}_{(X^\circ)^{\mathrm{an}}}) \to (X^\circ_{\mathrm{fl}}, \mathcal{O}_{X^\circ}),$$

induces a functor

$$(-)^{\mathrm{an}} \colon \mathbf{Tors}_G(X^\circ) \to \mathbf{Tors}_G((X^\circ)^{\mathrm{an}}).$$

[7] i.e., the category of (representable) rigid $\widehat{X}^{\mathrm{rig}}$-spaces with covers given by jointly surjective flat maps of rigid $\widehat{X}^{\mathrm{rig}}$-spaces.

In our situation, we can more concretely describe the group sheaves resulting from the S-ringed topoi appearing in Notation 5.2.

Lemma 5.3. *The groups $\widehat{G}$, $(G^\circ)^{\mathrm{an}}$, and $\widehat{G}^{\mathrm{rig}}$ represent $G_{\mathcal{O}_{\widehat{X}}}$, $G_{\mathcal{O}_{(X^\circ)^{\mathrm{an}}}}$, and $G_{\mathcal{O}^+_{\widehat{X}^{\mathrm{Rig}}}}$, respectively.*

With this set up, we can define the category of G-torsors on a gluing triple over S.

Definition 5.4. Let $(X^\circ, \widehat{X}, j_X)$ be a gluing triple over S. We then define the category $\mathbf{Tors}_G(X^\circ, \widehat{X}, j_X)$ of *G-torsors over* $(X^\circ, \widehat{X}, j_X)$ to consist of triples $(Q^\circ, \widehat{Q}, \iota_Q)$ where

- Q° is an object of $\mathbf{Tors}_G(X^\circ)$,
- $\widehat{Q}$ is an object of $\mathbf{Tors}_G(\widehat{X})$,
- ι_Q is a morphism of sheaves $\mathrm{sp}^*_{\widehat{X}}(\widehat{Q}) \to j^*_X((Q^\circ)^{\mathrm{an}})$ equivariant for the map of group sheaves $G^{\mathrm{rig}} \to (G^\circ)^{\mathrm{an}}$ or, equivalently, an isomorphism
$$\mathrm{sp}^*_{\widehat{X}}(\widehat{Q}) \times^{\widehat{G}^{\mathrm{rig}}} (G^\circ)^{\mathrm{an}} \xrightarrow{\sim} j^*_X((Q^\circ)^{\mathrm{an}}).$$

Suppose now that X is a separated lft algebraic S-space. Then, there is a *triples functor*
$$\tau\colon \mathbf{Tors}_G(X) \to \mathbf{Tors}_G(t(X)), \quad Q \mapsto (Q^\circ, \widehat{Q}, \mathrm{can.}).$$
Here $\widehat{Q}$ is the pullback of Q along the morphism of S-ringed topoi $(\widehat{X}_{\mathrm{fl}}, \mathcal{O}_{\widehat{X}}) \to (X_{\mathrm{fl}}, \mathcal{O}_X)$, Q° is the pullback of of Q along $S^\circ \to S$, and can. is the natural identification
$$\mathrm{sp}^*_{\widehat{X}}(\widehat{Q}) \times^{\widehat{G}^{\mathrm{rig}}} (Q^\circ)^{\mathrm{an}} \xrightarrow{\sim} j^*_X((Q^\circ)^{\mathrm{an}}),$$
using the fact that both source and target can be identified with the pullback of Q along the morphism of ringed spaces $(\widehat{X}^{\mathrm{rig}}_{\mathrm{fl}}, \mathcal{O}_{\widehat{X}^{\mathrm{rig}}}) \to (X_{\mathrm{fl}}, \mathcal{O}_X)$.

Proposition 5.5. *Let X be a separated lft algebraic S-space. Then, the functor*
$$\tau\colon \mathbf{Tors}_G(X) \to \mathbf{Tors}_G(t(X)), \quad Q \mapsto (Q^\circ, \widehat{Q}, \mathrm{can.}).$$
is an equivalence of categories.

Proof. By [28, Tag 04SK], any G-torsor on X, X°, or $\widehat{X}$ is representable by a (formal) algebraic space.[8] Moreover, as being separated and lft is a flat local property, one sees that these algebraic spaces must be separated and lft.

Fix a G-torsor $(Q^\circ, \widehat{Q}, \iota_Q)$ over the separated gluing triple $(X^\circ, \widehat{X}, j_X)$, where we may view Q° and $\widehat{Q}$ as both a sheaf and (formal) algebraic space. From this perspective we may identify
$$\mathrm{sp}^*_{\widehat{X}}(\widehat{Q}) \simeq \widehat{Q}^{\mathrm{rig}}, \quad j^*_X((Q^\circ)^{\mathrm{an}}) \simeq (Q^\circ)^{\mathrm{an}}|_{\widehat{X}^{\mathrm{rig}}},$$
where the latter object is an open rigid algebraic $\widehat{S}^{\mathrm{rig}}$-subspace of $(Q^\circ)^{\mathrm{an}}$. The map ι_Q can then be interpreted as a map $\widehat{Q}^{\mathrm{rig}} \to (Q^\circ)^{\mathrm{an}}|_{\widehat{X}^{\mathrm{rig}}}$. This is an open embedding, since we may check flat locally which reduces us to the claim that $j_G\colon \widehat{G}^{\mathrm{rig}} \to (G^\circ)^{\mathrm{an}}|_{\widehat{X}^{\mathrm{rig}}}$ is an open embedding. In particular, we see that $(Q^\circ, \widehat{Q}^\circ, \iota)$ naturally gives rise to a gluing triple in $\mathbf{Trip}^{\mathrm{sep}}_{(X,X_0)}$.

Consider $Q = g(Q^\circ, \widehat{Q}, \iota)$, with g as in Theorem 2.18, an lft algebraic S-space. We claim that this naturally comes with a right action of G. Indeed, by loc. cit. it suffices to show produce an action map
$$(Q^\circ, \widehat{Q}, \iota) \times (G^\circ, \widehat{G}, j) \to (Q^\circ, \widehat{Q}, \iota)$$
in $\mathbf{Trip}_{(X,X_0)}$. There are natural action maps
$$Q^\circ \times G^\circ \to Q^\circ, \qquad \widehat{Q} \times \widehat{G} \to \widehat{Q},$$
and the fact that these maps agree over $\widehat{X}^{\mathrm{rig}}$ follows from the equivariance of the map ι. Moreover, as the natural maps
$$Q^\circ \times G^\circ \to Q^\circ \times Q^\circ, \qquad \widehat{Q} \times \widehat{G} \to \widehat{Q} \times \widehat{Q},$$

[8]In the case of the formal scheme $\widehat{X}$, one applies this result to each reduction $\widehat{X}_n$, and passes to the colimit.

are isomorphisms, we also deduce that the action $Q \times G \to Q$ endows Q with the structure of a G-pseudotorsor. Thus, to finish it suffices to show that the map $Q \to X$ is faithfully flat. But, thanks to Proposition 2.21 this reduces to the claims that $Q^\circ \to X^\circ$ and $\widehat{Q} \to \widehat{X}$ are faithfully flat, which is true by assumption. Thus, one sees that Q is a torsor with $\tau(Q) \simeq (Q^\circ, \widehat{Q}, \iota)$. Thus, τ is essentially surjective. Fully faithfulness follows from similar considerations. □

5.2. **Gluing finite étale covers and an integral Riemann existence theorem.** Finally, we use gluing triples to give a finer understanding of the finite étale covers of an algebraic S-space. For simplicitly, we assume throughout this subsection that S is a G-scheme.

Definition 5.6. Let $(X^\circ, \widehat{X}, j_X)$ be a separated gluing triple over (S, S_0). Then, we define the category $\mathbf{F\acute{E}t}(X^\circ, \widehat{X}, j_X)$ of *finite étale covers of* to be the full subcategory of objects $(Y^\circ, \widehat{Y}, j_Y)$ of $\mathbf{Trip}_{(X,X_0)}$ whose (X, X_0, j_X)-structure morphism is finite étale.

The following is an immediate corollary of Theorem 2.18 together with Proposition 2.21.

Proposition 5.7. *Let X be a separated lft algebraic S-space. Then, the functor*

$$t\colon \mathbf{F\acute{E}t}(X) \to \mathbf{F\acute{E}t}(t(X)),$$

is an equivalence of categories.

Proposition 5.7 implies a mixed characteristic analogue of the Riemann existence theorem in rigid geometry (see [20]). Throughout the rest of this section, we fix the following data:

- K a DVF field of char. $p \geqslant 0$,
- k is the residue field K,
- ϖ a uniformizer of K,
- $X \to \mathrm{Spec}(\mathcal{O}_K)$, separated and lft.

We associate to X an adic space over $\mathrm{Spa}(\mathcal{O}_K)$ built from the associated triple $t(X)$ in a natural way, and which *differs* from the normal 'analytification' of X:

$$X^{\mathrm{ad}} := X \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spa}(\mathcal{O}_K),$$

as in [16, Proposition 3.8], in the special fiber.

Definition 5.8. We define the adic space $X^{\backslash\mathrm{ad}}$ over $\mathrm{Spa}(\mathcal{O}_K)$ by the following construction:

$$X^{\backslash\mathrm{ad}} := (X^\circ)^{\mathrm{an}} \sqcup_{j_X, \widehat{X}^{\mathrm{rig}}, \mathrm{incl.}} \widehat{X}^{\mathrm{ad}},$$

which is well-defined as j_X and the natural inclusion $\widehat{X}^{\mathrm{rig}} \to \widehat{X}^{\mathrm{ad}}$ are open embeddings.

Remark 5.9. Such spaces are closely related to recent work in the theory of Shimura varieties, e.g., see [22] as well as [18] and [17]. Indeed, if one considers the v-sheaf $(X^{\backslash\mathrm{ad}})^\lozenge$ associated to $X^{\backslash\mathrm{ad}}$ as in [25, §18.1] this agrees with the v-sheaf $X^{\backslash\lozenge}$ from [22, Definition 2.1.9].

There is a natural morphism $\iota_X\colon X^{\backslash\mathrm{ad}} \to X^{\mathrm{ad}}$ corresponding to the two natural maps

$$(X^\circ)^{\mathrm{an}} \simeq (X^{\mathrm{ad}})_\eta \to X^{\mathrm{ad}}, \qquad (g, f^{\mathrm{ad}}) = \widehat{X}^{\mathrm{ad}} \to X^{\mathrm{ad}} = X \times_{\mathrm{Spec}(\mathcal{O}_K)} \mathrm{Spa}(\mathcal{O}_K),$$

where $f\colon \widehat{X} \to \mathrm{Spf}(\mathcal{O}_K)$ is the structure morphism, and g is the morphism of locally ringed spaces given by the composition $\widehat{X}^{\mathrm{ad}} \to \widehat{X} \to X$.

Example 5.10. If $X = \mathbf{A}^1_{\mathcal{O}_K}$, then the map $\iota_X\colon X^{\backslash\mathrm{ad}} \to X$ is not an isomorphism. On the residue field k, ι_X is the natural morphism $\mathrm{Spa}(k[x], k[x]) \to \mathrm{Spa}(k[x], \mathbf{Z})$.

Let C be an algebraically closed non-archimedean extension of K, and let $x\colon \mathrm{Spec}(\mathcal{O}_C) \to X$ be morphism over $\mathcal{O}_K$. as $\mathcal{O}_C$ has no finiteétale covers, we have natural fiber functors

$$F_x\colon \mathbf{F\acute{E}t}(X) \to \mathbf{Set}, \qquad F_{x^{\backslash\mathrm{ad}}}\colon \mathbf{F\acute{E}t}(X^{\backslash\mathrm{ad}}) \to \mathbf{Set}.$$

Denote by $\mathbf{F\acute{E}t}(X)^{\prime p}$ the full subcategory of $\mathbf{F\acute{E}t}(X)$ consisting of finite étale covers of prime-to-p order, and similarly for $\mathbf{F\acute{E}t}(X^{\backslash\mathrm{ad}}))^{\prime p}$.[9] Then, the pairs

$$(\mathbf{F\acute{E}t}(X), F_x),\ (\mathbf{F\acute{E}t}(X)^{\prime p}, F_x), \quad (\mathbf{F\acute{E}t}(X^{\backslash\mathrm{ad}}), F_{x^{\backslash\mathrm{ad}}}),\ (\mathbf{F\acute{E}t}(X^{\backslash\mathrm{ad}}))^{\prime p}, F_{x^{\backslash\mathrm{ad}}})$$

[9] If $p = 0$ then we interpret this condition as vacuous.

are all Galois categories.[10] We write

$$\pi_1^{\text{ét}}(X,x),\ \pi_1^{\text{ét}}(X,x)^{\prime p},\quad \pi_1^{\text{ét}}(X^{\backslash\mathrm{ad}},x^{\backslash\mathrm{ad}}),\ \pi_1(X^{\backslash\mathrm{ad}},x^{\backslash\mathrm{ad}})$$

for their corresponding Galois groups.

Lemma 5.11 (cf. [16, Example 1.6.6])**.** *The functor* $(-)^{\backslash\mathrm{ad}}$ *induces functors*

$$(-)^{\backslash\mathrm{ad}}\colon \mathbf{F\acute{E}t}(X)\to\mathbf{F\acute{E}t}(X^{\backslash\mathrm{ad}}),\quad (-)^{\backslash\mathrm{ad}}\colon \mathbf{F\acute{E}t}(X)^{\prime p}\to\mathbf{F\acute{E}t}(X^{\backslash\mathrm{ad}})^{\prime p}$$

which have the property that $F_{x^{\backslash\mathrm{ad}}}\circ(-)^{\backslash\mathrm{ad}}\simeq F_x$.

In particular, we obtain morphisms of pro-finite groups

$$\pi_1^{\text{ét}}(X,x)\to\pi_1^{\text{ét}}(X^{\backslash\mathrm{ad}},x^{\backslash\mathrm{ad}}),\quad \pi_1(X,x)^{\prime p}\to\pi_1^{\text{ét}}(X^{\backslash\mathrm{ad}},x^{\backslash\mathrm{ad}})^{\prime p},$$

which are called *comparison morphisms*.

Proposition 5.12. *The morphism* $\mathbf{F\acute{E}t}(X)^{\prime p}\to\mathbf{F\acute{E}t}(X^{\backslash\mathrm{ad}})^{\prime p}$ *is an equivalence. Thus, the prime-to-p comparison morphism*

$$\pi_1(X,x)^{\prime p}\to\pi_1^{\text{ét}}(X^{\backslash\mathrm{ad}},x^{\backslash\mathrm{ad}})^{\prime p}$$

is an isomorphism.

Lemma 5.13. *The functor* $(-)^{\mathrm{ad}}$ *induces an equivalences of categories*

$$(-)^{\mathrm{ad}}\colon \mathbf{F\acute{E}t}(\widehat{X})\to\mathbf{F\acute{E}t}(\widehat{X}^{\mathrm{ad}}),\quad (-)^{\mathrm{ad}}\colon \mathbf{F\acute{E}t}(\widehat{X})^{\prime p}\to\mathbf{F\acute{E}t}(\widehat{X}^{\mathrm{ad}})^{\prime p}.$$

Proof. The functor $(-)^{\mathrm{ad}}\colon \mathbf{F\acute{E}t}(\widehat{X})\to\mathbf{F\acute{E}t}(\widehat{X}^{\mathrm{ad}})$ is fully faithful for any X and is a stack for the Zariski topology on X. Thus, we may reduce to the case when $X=\mathrm{Spf}(A)$. But, then

$$\mathbf{F\acute{E}t}(\mathrm{Spf}(A))\simeq\mathbf{F\acute{E}t}(\mathrm{Spec}(A))\simeq\mathbf{F\acute{E}t}(\mathrm{Spa}(A)),$$

explicitly sending $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ to $\mathrm{Spf}(B)\to\mathrm{Spf}(A)$ and $\mathrm{Spa}(B)\to\mathrm{Spa}(A)$ (B receiving the ϖ-adic topology), respectively. The latter being an equivalence follows as in [16, Example 1.6.6 ii)], and the former by combining [13, Chapter I, Proposition 4.2.1] and [1, Lemma A.13]. $\square$

Proof of Proposition 5.12. We claim that there is a commutative diagram

$$\begin{tikzcd}
\mathbf{F\acute{E}t}(X)^{\prime p} \arrow[r] \arrow[drr, bend right=15, "\sim"'] & \mathbf{F\acute{E}t}(X^{\backslash\mathrm{ad}})^{\prime p} \arrow[r, "\sim"] & \mathbf{F\acute{E}t}(X_\eta^{\mathrm{an}},\widehat{X}^{\mathrm{ad}},j_X)^{\prime p} \\
 & & \mathbf{F\acute{E}t}(X_\eta,\widehat{X},j_X)^{\prime p}, \arrow[u, "\wr"']
\end{tikzcd}$$

with the marked arrows being equivalences of categories. This implies the desired claim.

Here $\mathbf{F\acute{E}t}(X_\eta^{\mathrm{an}},\widehat{X}^{\mathrm{ad}},j_X)^{\prime p}$ means the category of triples $(\mathcal{Y},\mathfrak{Y},j)$ consisting of a prime-to-p finite étale cover $\mathcal{Y}\to X_\eta^{\mathrm{an}}$, a prime-to-$p$ finite étale cover $\mathfrak{Y}\to\widehat{X}^{\mathrm{ad}}$, and j an isomorphism of their restrictions to $\widehat{X}_\eta^{\mathrm{ad}}=\widehat{X}^{\mathrm{rig}}$. The horizontal equivalence is then the obvious one, taking a finite étale cover $\mathscr{Y}\to X^{\backslash\mathrm{ad}}$ and associating to it $(\mathscr{Y}_{X_\eta^{\mathrm{an}}},\mathscr{Y}_{\widehat{X}^{\mathrm{ad}}},\mathrm{can.})$. The vertical map associates to $(Y_\eta,\widehat{Y},j)$ the triple $(Y_\eta^{\mathrm{an}},\widehat{Y}^{\mathrm{ad}},j)$, and the curved arrow is just the functor t from Proposition 5.7 and so, in particular, an equivalence. As this diagram obviously commutes, it suffices to verify that this vertical arrow is an equivalence as claimed. But, this follows easily by combining Lemma 5.13 and [20, Theorem 4.1]. $\square$

Corollary 5.14. *For a prime-to-p finite étale cover* $\widehat{f}\colon\widehat{Y}\to\widehat{X}$*, we have an equivalence*

$$\left\{\begin{matrix}\text{Finite étale algebraizations}\\ Y\to X\text{ of }\widehat{f}\end{matrix}\right\}\to\left\{\begin{matrix}\text{Finite étale covers}\\ Y_\eta\to X_\eta\text{ with }Y_\eta^{\mathrm{an}}\times_{X_\eta^{\mathrm{an}}}\widehat{X}_\eta\simeq\widehat{f}_\eta\end{matrix}\right\},$$

$$(Y\to X)\mapsto(Y_\eta\to X_\eta).$$

[10] Here we are abusing terminology. Namely, these are not Galois categories unless X and $\widehat{X}^{\mathrm{ad}}$ are connected, but we really mean here the corresponding Galois categories of finite étale covers on the connected components determined by x and $x^{\backslash\mathrm{ad}}$.

So, $\widehat{f}$ algebraizes if and only if $\widehat{Y}_\eta \to \widehat{X}_\eta$ extends to a finite étale cover of $(X^\circ)^{\mathrm{an}}$.

Remark 5.15. Grothendieck's specialization theorem says that if $X \to \mathrm{Spec}(\mathcal{O}_K)$ is proper then every finite étale cover $Y_k \to X_k$ deforms uniquely to a finite étale cover $Y \to X$. It is instructive to think about this result in two steps:

Step 1: a finite étale cover $Y_k \to X_k$ deforms uniquely to a finite étale cover $\widehat{Y} \to \widehat{X}$,

Step 2: a finite étale cover $\widehat{Y} \to \widehat{X}$ uniquely algebraizes to a finite étale cover $Y \to X$.

It is **Step 2** that requires $X \to \mathrm{Spec}(\mathcal{O}_K)$ to be proper, as the content of **Step 1** holds for any $X \to \mathrm{Spec}(\mathcal{O}_K)$ (e.g., see [1, Proposition 3.6]). As $\widehat{X}^{\mathrm{rig}} = (X^\circ)^{\mathrm{an}}$ when $X \to \mathrm{Spec}(\mathcal{O}_K)$ is proper (see Proposition 2.14), one may view Corollary 5.14 as precisely qualifying the failure of Grothendieck's specialization in the non-proper case, where generally $\widehat{X}^{\mathrm{rig}} \subsetneq (X^\circ)^{\mathrm{an}}$.

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