MMF 1928 PROJECT 2 - DYNAMIC HEDGING

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Abstract. This project covers the incorporation of Delta and Delta-Gamma hedging techniques in the framework of Black-Scholes options and their performance comparison. The research focuses on the profit and loss distributions among 5,000 paths, using different monthly options expiration, and considering the commission costs related to the stocks and the options. Examine factors including varying drift rates μ and discrepancies between assumed and actual volatility levels σ to understand their impact on hedging efficiency. The analysis demonstrates the benefits of utilizing gamma-sensitive hedging strategies, provided that uncertainty and misjudged volatility are present. These findings provide practical implications for risk management and derivative pricing in financial markets.

1. Introduction

Dynamic hedging strategies are indispensable for risk management, particularly in case one uses the Black-Scholes model. Our project compares and assesses delta hedging and delta-gamma hedging methods to find out their potential effects on reducing any possible profit or loss variations under different cases.
Incorporating daily hedging activities with both the stock and the at-the-money call option of the same underlying, let's consider the stock priced at \$10, and the sale of 10,000 units of a $\frac{1}{4}$-year call option. Transaction costs are included, which account for market realities like changing volatility and fixed parameters (μ = 10%, σ = 25%, r = 5%).
The study addresses 5,000 price paths and evaluates the hedging performance. Aiming at enhancing awareness on the relationship between the trading of deltas and gamma hedging strategies, the paper explores this topic and suggests their use in volatile markets that are non-linear.

2. Model Setup

Suppose that an asset price process $S = (S_t)_{t\geq 0}$ follows the Black-Scholes model. The asset's current price is \$10, we have sold 10000 units of an at-the-money $\frac{1}{4}$ year call on this asset, and we wish to hedge it with delta-hedging strategy. Call this option g.

We also trade in an at-the-money call with 0.3 year (call this option h), the stock (S_t) and the bank account (B_t) to construct a portfolio and conduct delta-gamma hedging.

As well, we will account for transaction costs by assuming that equity transaction is charged \$0.005 per share and option transaction is charged \$0.005 per option.

Only integer value of stocks and options are allowed to be traded.

The remaining model parameters are, $\mu = 10\%, \sigma = 25\%, r = 5\%$.

3. Methodology

3.1. Path Simulation.

By assuming that price of the underlying asset follows Geometric Brownian Motion (GBM), we could simulate 5000 paths of stock price by using formula,

$$S_{t+\Delta t} = S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\cdot Z_t} \tag{1}$$

Where:

- S_t is the current asset price.
- μ is the drift (expected return of the asset).
- σ is the volatility.
- Δt is the time increment (1/252 for daily steps, assuming 252 trading days in a year).
- $Z_t \sim \mathcal{N}(0,1)$ is a standard normal random variable.

3.2. Delta Hedging.

Following the path simulation, we consider using delta hedging to manage the risk of our portfolio. To be more specific, delta hedging is a risk management strategy that involves adjusting the position in the underlying asset to offset changes in the value of an options portfolio, ensuring it remains neutral to small price movements in the underlying asset.
Consider an underlying asset S_t,

$$dS_t = \mu_t^S S_t dt + \sigma_t^S S_t dW_t$$

Where: μ_t^S and σ_t^S can depend on time t and asset S_t.

And the bank account,

$$dB_t = rB_t dt$$

Where: r is the risk-free rate, $B_0 = 1$.
Where:

- r is the risk-free rate.
- $B_0 = 1$
- $B_{t+\Delta t} = B_t e^{r\Delta t}$

We construct our portfolio with α_t units of underlying asset S_t with a current price of \$10, β_t units of B and -10000 units of call option g_t which is an at-the-money $\frac{1}{4}$-year call option on the underlying asset.

in the Black-Scholes Model framework where

$$C(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$
$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

We could derive that delta of call option (g_t) is,

$$\Delta^g = \frac{\partial g(t, S_t)}{\partial S_t} = N(d_1) \tag{2}$$

Such that we could derive delta of the whole portfolio as,

$$\Delta^V = \alpha_t \Delta^S + \beta_t \Delta^B - 10000\Delta^g$$

Where:

- $\Delta^S = 1$
- $\Delta^B = 0$

The objective of delta hedging strategy is to construct a delta-neutral portfolio for the purpose of hedging the risk of holding an option by taking positions in the underlying asset and bank account. Therefore, we let,

$$\Delta^V = \alpha_t - 10000\Delta^g = 0$$

Therefore, we need to long $10000\Delta^g$ units of stock to achieve a delta-neutral portfolio.

In order to compute the profit and loss (P&L) of the hedging strategy, we need to compute initial portfolio value V_0 and terminal portfolio value V_T.
Precisely,

$$V_0 = \alpha_0 S_0 + B_0 - 10000 g_0 \tag{3}$$

Where: $B_0 = 10000g_0 - 0.005 \times 10000 - \alpha_0 S_0$

For the purpose of achieving a delta-neutral portfolio, we need to trade stock to re-balance our portfolio. Specifically, number of shares of stocks traded daily is,

$$\Delta\alpha = \alpha_t - \alpha_{t-1}$$

Known that transaction cost for equity is \$0.005 per share, we could derive the bank account would be,

$$B_t = B_{t-1}e^{r\Delta t} - |\Delta\alpha| \cdot 0.005 - \Delta\alpha \cdot S_t$$

Then we could derive portfolio value at maturity as,

$$V_T = \alpha_T S_T + B_T - 10000 g_T \tag{4}$$

Where:

- $g_T = max(S_T - K_g, 0)$
- $B_T = B_{T-1}e^{r\Delta t} - |\Delta\alpha| \cdot 0.005 - \Delta\alpha \cdot S_T$

The profit and loss (P&L) of the portfolio at maturity date could be expressed as,

$$P\&L = V_T - V_0$$

3.3. **Delta-Gamma Hedging.**
In Black-Scholes Model framework, we could express gamma of a call option as,

$$\Gamma^h = \frac{\partial^2 h(t, S_t)}{\partial S_t^2} = \frac{N'(d_1)}{S_t \sigma \sqrt{T}} \tag{5}$$

The objective of delta-gamma hedging strategy is to construct a portfolio that hedges against both first-order (Δ) and second-order (Γ) changes in option prices.

In an effort to achieve a gamma-neutral portfolio, we introduce a second option, $h(t, S_t)$, an at-the-money call option with maturity 0.3 year, into the portfolio. We want to have a both delta-neutral and gamma-neutral portfolio that,

$$\Delta^V = \alpha_t \Delta^S + \beta_t \Delta^B - 10000\Delta^g + \eta_t \Delta^h = 0$$
$$\Gamma^V = \alpha_t \Gamma^S + \beta_t \Gamma^B - 10000\Gamma^g + \eta_t \Gamma^h = 0$$

In order to compute P&L of the hedging strategy, we also need to figure out portfolio value at initial time (t = 0) and at maturity time (t = T).

To start with,

$$V_0 = \alpha_0 S_0 + B_0 - 10000 g_0 + \eta_0 h_0 \tag{6}$$

Where: $B_0 = 10000 g_0 - 0.005 \times 10000 - 0.005 \cdot |\eta_0| - \eta_0 h_0 - \alpha_0 S_0$

For the sake of achieving a delta-neutral and gamma-neutral portfolio, we need to adjust our position in both stock and option h from day to day.

$$\Delta\alpha = \alpha_t - \alpha_{t-1}$$
$$\Delta\eta = \eta_t - \eta_{t-1}$$

Given that we are charged \$0.005 per share on equity transactions and \$0.005 per option on option transactions, we can then derive portfolio value at maturity as,

$$V_T = \alpha_T S_T + B_T - 10000 g_T + \eta_T h_T \tag{7}$$

Where:

- $g_T = max(S_T - K_g, 0)$
- $h_T = max(S_T - K_h, 0)$
- $B_T = B_{T-1} e^{r\Delta t} - |\Delta\alpha| \cdot 0.005 - \Delta\alpha \cdot S_T - |\Delta\eta| \cdot 0.005 - \Delta\eta \cdot h_T$.

Then we could derive the expression to compute P&L of delta-gamma hedging strategy.

$$P\&L = V_T - V_0$$

4. Results

4.1. Impacts of Variation in μ.

4.1.1. *Plotting P&L Distributions with Varied μ.*

μ represents drift of the underlying asset in the real-world measure. By letting μ equal to various values including {-0.2, -0.1, 0, 0.1, 0.2, 0.4, 0.6}, we could simulate different stock paths for testing the hedging strategies. Comparisons of P&L distribution between delta and delta-gamma hedging are demonstrated in Figure 1.

[0] Raw code to solve these problems can be viewed by clicking on the link: https://github.com/Alex-Zhang-074/PricingTheory/tree/main/Project%202

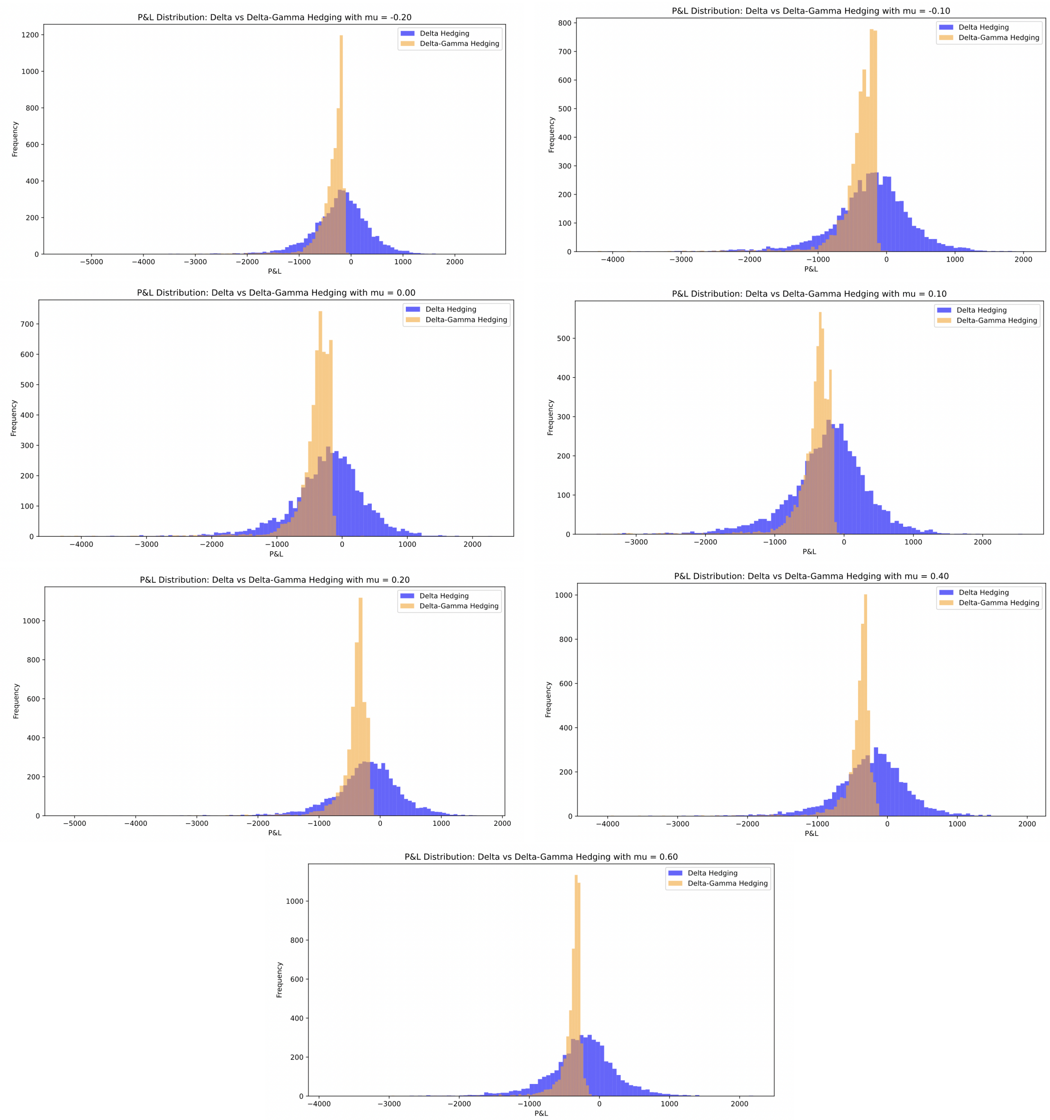

FIGURE 1. Delta vs Delta-Gamma with Varied μ

4.1.2. *Observations on P&L Distribution Plots.*
By the above plots generated with varied drift values of the underlying asset, we could observe that,

- **Tail Behavior:** As μ gets greater, P&L distributions of delta hedging strategy tend to have heavier tails, meaning that it is more likely to have large gains or losses during large price movements; P&L distributions of delta-gamma hedging also tend to have broader tails, while slighter than that of gamma hedging, meaning the outcomes of delta-gamma hedging is more stable in large price movements.

- **Mean of P&L Distribution:** As the value of μ changes in the plots above, the mean of P&L distribution of delta hedging strategy also changes significantly in the same direction as μ, meaning that the mean of the P&L distribution of delta hedging aligns closely with

μ; the mean of P&L distribution of delta-gamma hedging shifts moderately when μ varies, also in the same direction as the shift in μ.

4.1.3. *Financial Intuition.*

Considering the computational difference between delta and delta-gamma hedging, we could explain the above observations with financial intuition.

- **Delta hedging** aims to achieve a delta-neutral portfolio, meaning that it only neutralizes linear exposure to changes in price of the underlying asset. In this sense, mean of its P&L distribution will be significantly impacted by price changes, the fatter tail of the distribution also exhibits its high sensitivity to price movements caused by variation in μ.

- **Delta-gamma hedging** takes both delta and gamma into account, guaranteeing a higher hedging accuracy and reducing its exposure to large price movements. The lighter tail of its P&L distribution could demonstrate that it is less sensitive to price changes. Moreover, the second-order adjustment makes delta-gamma hedging strategy more robust to non-linear price changes, and thus its mean does not shift significantly with the variation of μ.

Overall, we conclude that compared with delta hedging strategy, delta-gamma hedging strategy is more robust to price movements.

4.2. **Hedging Position Paths Plots for ITM and OTM Cases.**

4.2.1. *Plotting Hedge Position Paths.*

Following the analysis of influences of μ variations, we consider sample stock paths that ends in-the-money and out-of-the-money at maturity date.

To be more specific, a call option ends in-the-money means that at maturity time, stock price is greater than strike price of the call option, resulting in the option having intrinsic value.

$$Payoff_{ITM} = S_T - K > 0$$

On the other hand, ending out-of-the-money means that at maturity time, stock price is smaller than strike price of the call option, making the option have no intrinsic value.

$$Payoff_{OTM} = S_T - K \leq 0$$

By assuming stock paths follow Geometric Brownian Motion, we simulate stock paths with the expression,

$$S_{t+\Delta t} = S_t \cdot e^{\left(\mu - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}\cdot Z_t}, Z \sim N(0,1)$$

Then we generate two stock paths corresponding to conditions of end in-the-money and out-of-the-money respectively. Then we compute our position changes in stock and options to achieve delta and delta-gamma hedging and compare the position trajectories in both conditions. Plots of position trajectories are shown in Figure 2.

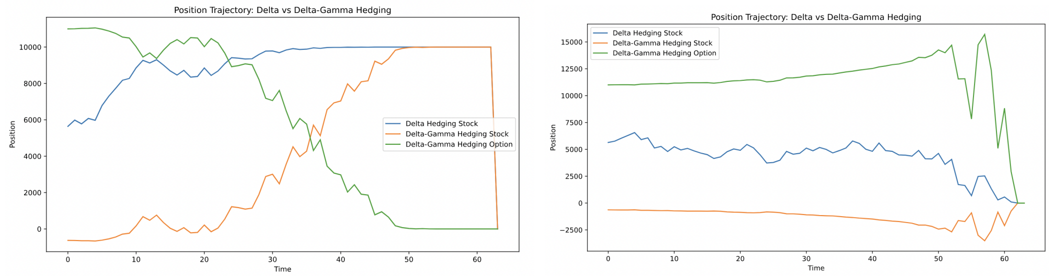

FIGURE 2. Position Trajectory - Ending ITM (left) and OTM (right)

4.2.2. *Observations on Plots.*
By the above position trajectory plots, we could observe that,

- **Delta hedging (ITM):** Stock position gradually increase to 10000, reflecting the rising delta of the option approaches to 1. Then it suddenly drop to 0 at maturity time.
- **Delta hedging (OTM):** Stock position gradually decrease as the delta of the option approaches to 0 due to OTM.
- **Delta-Gamma hedging (ITM):** Stock position aggressively increases when the option goes deeper in the money and delta of the option approaches to 1. While the option position decreases, showing an inverse relationship to the stock position, for the purpose of achieving delta and gamma neutral portfolio.
- **Delta-Gamma heding (OTM):** Stock position gradually decreases as the option goes out of the money with time, then the stock position drops aggressively when the option loses time value and eventually goes to 0 at maturity. Same as the aforementioned analysis, option position moves in the inverse direction as the stock position, and eventually converge to 0 at maturity.

4.3. **Impact of Variation in Real-World Volatility on P&L.**

4.3.1. *Plotting P&L Distribution with Varied Real-World Volatility.*
Then we take real-world volatility into account and compare the P&L distribution of delta and delta-gamma hedging strategies. We suppose that the real-world $\mathbb{P} \in \{20\%, 22\%, ..., 30\%\}$, we sell option g using $\sigma_g = 25\%$, and hedging by assuming $\sigma_h = 25\%$.

To start with, we incorporate real-world volatility (σ_{real}) to simulate stock paths with Geometric Brownian Motion (GBM) illustrated in equation (1). Then we could derive Δ^g by subbing $\sigma_g = 25\%$ in equation (2), and we could further derive portfolio delta. Then we could compute portfolio value at initial time and maturity time by using equation (3) and (4). P&L of delta hedging strategy with real-world σ could then be derived with,

$$P\&L_{real} = V_T - V_0$$

We plug in $\sigma_h = 25\%$ to equation (2) and (5) then figure out Δ^h and Γ^h. Then we could compute the initial and maturity value of the portfolio with equation (6) and (7). P&L of delta-gamma hedging strategy with real-world σ could then be derived with,

$$P\&L_{real} = V_T - V_0$$

Plots of P&L distributions with different values of σ_{real} is shown in Figure 3.

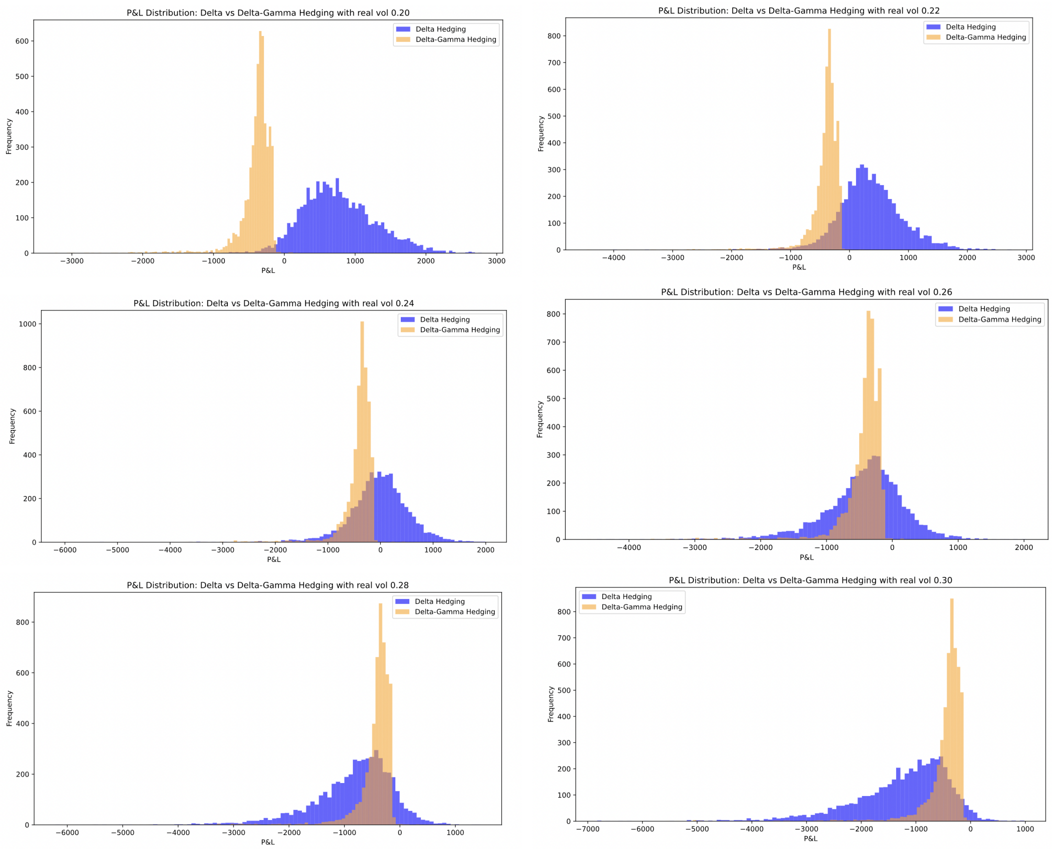

FIGURE 3. Delta vs Delta-Gamma with Varied Real Volatility

4.3.2. *Observations on P&L Distribution Plots.*
By the above plots generated with varied real-world volatility, we could observe that,

- **Impacts of σ_{real} on Distribution Location** As σ_{real} gets greater, P&L distribution of delta hedging strategy shifts more significantly than that of delta-gamma hedging strategy.

- **Impacts of σ_{real} on Distribution skewness:** P&L distribution of delta hedging strategy changes from right-skewed to left-skewed as real-world volatility goes from low to high, while that of delta-gamma hedging strategy keeps being left-skewed.

4.3.3. *Financial Intuition.*
Considering the conditions when real-world volatility is lower than or grater than the assumed volatility, we could explain the above observations with financial intuition.

- **Low real-world volatility ($\sigma_{real} < 25\%$):** When real-world volatility is lower than the assumed volatility, our hedging strategies have overestimated price variability, which would lead to systematic losses. The shift in both distributions to the negative side has demonstrated this.
- **High real-world volatility ($\sigma_{real} > 25\%$):** When real-world volatility is higher than the assumed volatility, our hedging strategies have underestimated price variability, which would lead to tail risk. The expansion of the tail of P&L distribution of both strategies

supports this intuition, in the sense that under-hedging occurs and the outcomes of hedging strategies become unstable.

- **Comparison between Delta and Delta-Gamma Hedging:** When volatility mismatch happens, delta-gamma hedging exhibits less pronounced shift in P&L distributions and narrower tails. That is because delta-gamma hedging is adjusted for both delta and gamma, making it more robust to volatility mismatches.

5. Conclusions

In this project we explored the construction of delta and delta-gamma hedging strategies and their effectiveness under several market conditions using the Black-Scholes framework. We evaluate the performance delta and delta-hedging strategies by comparing their profit and loss (P&L) distributions under varying drift rates (μ) and different levels of real-world volatility (σ_{real}). We have also probed into cases when the option ends in-the-money and out-of-the-money, and plotted the position trajectory of both strategies to compare their position adjustments.

Close scrutiny reveals that while delta hedging is simpler to implement, it is highly sensitive to market fluctuations including price movements and volatility mismatches since it only hedges against first-order risks (Δ). Delta-gamma hedging has a higher hedging accuracy and efficiency, especially in the volatile and unpredictable market, due to its involvement of an additional layer of adjustment (Γ).

The results emphasize the significance of applying proper hedging strategies based on market conditions and characteristics of underlying assets.

6. Contribution Sheet

Surui (Alex) Zhang completed the tasks of writing code to solve the problems, interpreting the code results and revising the report.

Jingwen (Vivien) Shao completed the tasks of mathematical derivations for the problems, interpreting the code results and composing the report.

Contributor Signatures

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JS.

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