APPLIED PROBABILITY FOR MATHEMATICAL FINANCE REPORT

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Abstract. This report explores the evaluation of American put options through a binomial tree model, emphasizing the variability of profit and loss (P&L) and stopping times across different volatility and risk parameters. By simulating 10,000 paths, the study analyzes how early exercise strategies, driven by market conditions and model dynamics, impact hedging and risk management decisions. The findings illustrate the critical role of volatility and the risk-free rate in determining the optimal timing of option exercises and the associated financial outcomes.

1. Introduction

American put options provide the holder the right, but not the obligation, to sell a specific asset at a predetermined price before or at the option's expiration. This flexibility makes their valuation particularly complex but also highly strategic for managing financial risks. The valuation of these options in a risk-neutral environment assumes that the underlying asset prices are driven by a geometric Brownian motion, modified to include discrete time steps and binary outcomes at each step. This study employs a modified binomial tree model, which approximates the continuous movements of asset prices with a discrete-time, discrete-state approach. Key to this approach is the adherence to the martingale property under the risk-neutral measure, ensuring no arbitrage opportunities exist. This model serves as the foundation for simulating asset price paths, calculating exercise boundaries, and evaluating the option's response to various market conditions through the determination of risk-neutral probabilities and backward induction techniques.

2. Methodolgy

The methodology section detailed the foundational model specifications for the binomial tree approach used to evaluate the American put option. Key parameters such as the strike price, initial stock price, maturity, volatility, risk-free rate, and expected return were established. The construction of the binomial tree involved calculating the up and down factors and the risk-neutral probabilities, essential for simulating the asset price paths and determining the early exercise boundaries. Subsequently, a backward induction process was utilized for option valuation, considering both the exercise and continuation values at each node. The comprehensive analysis included simulations of asset price paths under different scenarios of early and no early exercise, accompanied by the hedging strategies required in each case.

2.1. **Problem 1.**

Under the risk-neutral measure $\mathbb{Q}$, the discounted asset price must be a martingale. The bank account process is given by:

$$B_t = e^{rt}.$$

Thus, the discounted asset price $\frac{S_t}{B_t}$ satisfies:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{S_{t_k}}{B_{t_k}}\middle|\mathcal{F}_{t_{k-1}}\right] = \frac{S_{t_{k-1}}}{B_{t_{k-1}}}.$$

Given the asset price dynamics:

$$S_{t_k} = S_{t_{k-1}} e^{r\Delta t + \sigma\sqrt{\Delta t}\epsilon_k},$$

we can express the discounted asset price as:

$$\mathbb{E}^{\mathbb{Q}}[S_{t_k}|\mathcal{F}_{t_{k-1}}] = S_{t_{k-1}} e^{r\Delta t}.$$

Therefore, the martingale condition becomes:

$$\mathbb{E}^{\mathbb{Q}}[S_{t_k}|\mathcal{F}_{t_{k-1}}] = S_{t_{k-1}} e^{r\Delta t}\mathbb{E}^{\mathbb{Q}}[e^{\sigma\sqrt{\Delta t}\epsilon_k}].$$

which implies:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma\sqrt{\Delta t}\epsilon_k}\right] = 1.$$

Since $\epsilon_k \in \{+1, -1\}$, we have:

$$\mathbb{E}^{\mathbb{Q}}\left[e^{\sigma\Delta t\epsilon_k}\right] = Q(\epsilon_k = +1)e^{\sigma\sqrt{\Delta t}} + Q(\epsilon_k = -1)e^{-\sigma\sqrt{\Delta t}} = 1.$$

Let $p = Q(\epsilon_k = +1)$. Then:

$$pe^{\sigma\sqrt{\Delta t}} + (1-p)e^{-\sigma\sqrt{\Delta t}} = 1.$$

Solving for p:

$$p\left(e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}\right) = 1 - e^{-\sigma\sqrt{\Delta t}},$$

$$p = \frac{1 - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

Therefore, the risk-neutral probabilities for Problem 2 are

$$\mathbb{Q}(\epsilon_k = +1) = \frac{1 - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

and

$$\mathbb{Q}(\epsilon_k = -1) = \frac{e^{\sigma\sqrt{\Delta t}} - 1}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}$$

2.2. **Problem 2.**

2.2.1. *Model Specification.*

The binomial tree model setup for the American put option involves the following specifications and assumptions:

- Initial stock price $S_0 = 10$
- Strike price $K = 10$
- Time to maturity $T = 1$ year
- Risk-free rate $r = 0.02$
- Volatility $\sigma = 0.20$
- Expected return $\mu = 0.05$
- Number of steps $N = 5000$

2.2.2. *Binomial Tree Construction.*

1. Up and down factors: The up (u) and down (d) factors are crucial for the binomial tree construction, derived as:

$$u = e^{r\Delta t + \sigma\sqrt{\Delta t}},$$
$$d = e^{r\Delta t - \sigma\sqrt{\Delta t}}.$$

where $\Delta t = \frac{T}{N}$ represents the time increment per step.

2. Probability calculations:

- Risk-neutral probabilities are calculated to ensure that the expected stock price under the risk-neutral measure equals the stock price compounded at the risk-free rate:

$$p = \frac{1 - e^{-\sigma\sqrt{\Delta t}}}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}},$$
$$q = 1 - p = \frac{e^{\sigma\sqrt{\Delta t}} - 1}{e^{\sigma\sqrt{\Delta t}} - e^{-\sigma\sqrt{\Delta t}}}.$$

- Real-world probabilities reflect the expected real-world movements of the underlying asset based on its expected return:

$$p_{\text{real}} = \frac{1}{2}\left(1 + \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right),$$
$$q_{\text{real}} = \frac{1}{2}\left(1 - \frac{(\mu - r) - \frac{1}{2}\sigma^2}{\sigma}\sqrt{\Delta t}\right)$$

3. Stock price tree:

Using the up and down factors, the model builds a binomial tree of stock prices. For each node (i, j), the stock price is computed as:

$$S_{t_{i,j}} = S_0 \times u^{i-j} \times d^j$$

where i represents the time step, and j represents the number of down moves.

2.2.3. *Backward Induction for Option Valuation.*

The model uses backward induction to compute the value of the American put option at each node of the binomial tree. The value at maturity is the intrinsic value of the option:

$$\text{Option value at maturity} = \max(K - S_T, 0)$$

Moving backward through the tree, at each node, the model calculates:

- Hold Value: The expected value of holding the option, which is the discounted expected future payoff:

$$\text{Hold value} = e^{-r\Delta t} \times (p \times V_{\text{up}} + q \times V_{\text{down}}),$$

where V_{up} and V_{down} are the option values at the up and down nodes in the next time step.

- Exercise Value: The value of exercising the option immediately at a particular node:

$$\text{Exercise value} = \max(K - S_{t_{i,j}}, 0)$$

The model compares the hold value and the exercise value and takes the maximum, as the holder of an American option can choose to exercise early.

2.2.4. *Delta Calculation.*

The delta of the option, which measures the sensitivity of the option's value to changes in the underlying asset price, is calculated at each node using:

$$\Delta = \frac{V_{\text{up}} - V_{\text{down}}}{S_{\text{up}} - S_{\text{down}}}$$

This value is used to determine the appropriate hedging strategy.

2.2.5. *Exercise Boundary Determination.*

The model identifies the early exercise boundary by tracking the stock prices at each time step where it is optimal to exercise the option. These prices represent the boundary at which the intrinsic value exceeds the hold value, leading to early exercise.

2.2.6. *Paths Simulation and P&L Analysis.*

- Simulating Realized Paths: The model simulates real-world stock price paths using the practical probabilities p_{real} and q_{real}. Each path represents a sequence of up and down moves over the life of the option.
- P&L and Stopping Time Calculation: For each simulated path, the model calculates the profit and loss (P&L) and the stopping time, which is the time at which the option is exercised. The P&L is computed as the difference between the strike price and the stock price at the exercise time, and the stopping time reflects when the option is exercised early (or at maturity).

3. Results

3.1. **Problem 2(a).**

Figure 1 shows that the exercise boundary increases as the option nears its expiration. This is a common characteristic for American put options, which hold the flexibility of being exercised at any time up to and including the day of expiry. The trend of initial stability and gradual increase suggests that the option's intrinsic value and the urgency of exercising it grow as the certainty of the underlying asset's declining value becomes more pronounced. The exercise boundary provides crucial guidance for option holders regarding when to exercise based on the evolving market price of the underlying asset. It effectively maps out the stock price below which the holder should consider exercising to maximize the option's value, particularly as the expiration nears and uncertainty about future price movements increases.

Figure 2 illustrates a scenario where the stock price prompts an early exercise of the option at t=0.50 years. The lower boundary path is deliberately structured to hit the earliest exercise price at this halfway point, leading to a sharp drop in stock price which triggers an early exercise, marked by a red dot. Conversely, the random path shows substantial fluctuations before reaching the early exercise decision at the same time point. This demonstrates varying stock behaviors under similar conditions, highlighting how different curves can achieve the same exercise outcome but with differing risk levels and volatility. This visualization underscores the influence of stock price dynamics on option exercise strategies and the financial implications of early option exercise.

Figure 3 shows the dynamic hedging strategies for American put options at t=0.50 years. The lower boundary path maintains a delta near -1.0, indicating a stable, almost fully short hedging

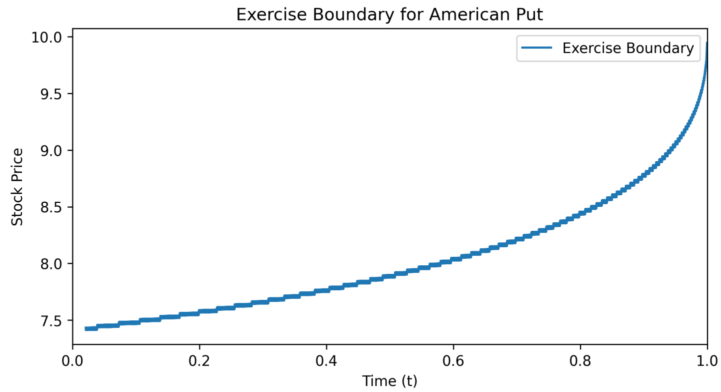


Figure 1. The exercise boundary as a function of t for both approaches.

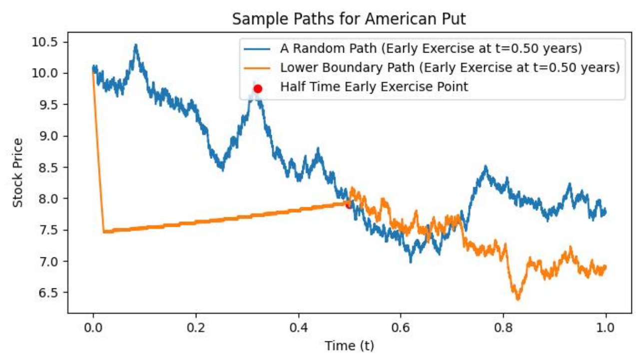


Figure 2. The option is exercised early.

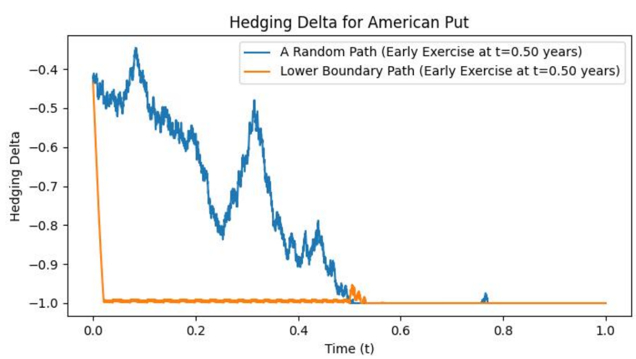


Figure 3. The hedging strategy of the option exercising early.

strategy with minimal rebalancing needed due to low volatility up to the exercise point. In contrast, the random path exhibits significant delta fluctuations, demonstrating a more volatile hedging demand that requires active management to mitigate risks associated with unstable stock price movements. These varying delta paths highlight the complexities of options trading and the need for adaptive hedging strategies in response to rapid price changes, illustrating different risk management challenges in stable versus volatile market conditions.

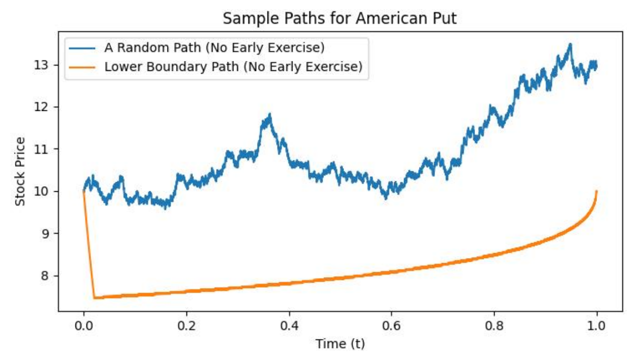


FIGURE 4. The option is not exercised.

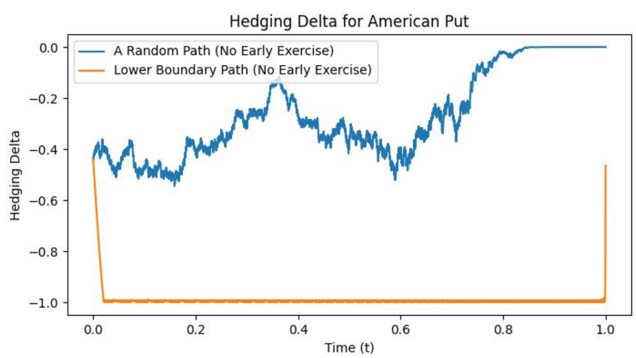


FIGURE 5. The hedging strategy of the option is not exercising.

Figure 4 displays simulated stock price paths for American put options held until maturity without early exercise. The lower boundary path shows a gradual stock price increase, adhering to the least favorable conditions that prevent early exercise, as designed by an algorithm to keep the price above a critical early exercise threshold. In contrast, the random path shows significant volatility with wide fluctuations but also remains above the early exercise boundary throughout its term. This path reflects common market conditions where despite volatility, the criteria for early exercise aren't met due to inadequate intrinsic value or other market influences. Both paths demonstrate different risk management strategies, with the lower boundary path indicating a conservative approach, while the random path illustrates the challenges of managing options amidst typical market fluctuations.

Figure 5 shows the delta hedging strategies for American put options held until maturity without early exercise. The lower boundary path maintains a nearly constant delta close to -1, indicating minimal price changes and a stable hedging requirement, reflecting a scenario with the underlying price consistently low, keeping the option deep in-the-money. In contrast, the random path exhibits significant delta volatility due to substantial price movements, necessitating frequent hedging adjustments. This path's delta stabilizes as it nears maturity, aligning with the put option's increasing intrinsic value. The stable delta suggests a cost-effective hedging approach with minimal adjustments, while the volatile delta indicates a more dynamic and costly strategy, underscoring the need for ongoing risk management and adjustments in response to market fluctuations.

Figure 6, 7, and 8 showcase the collection of plots of American put options across a range of volatilities and risk-free rates.

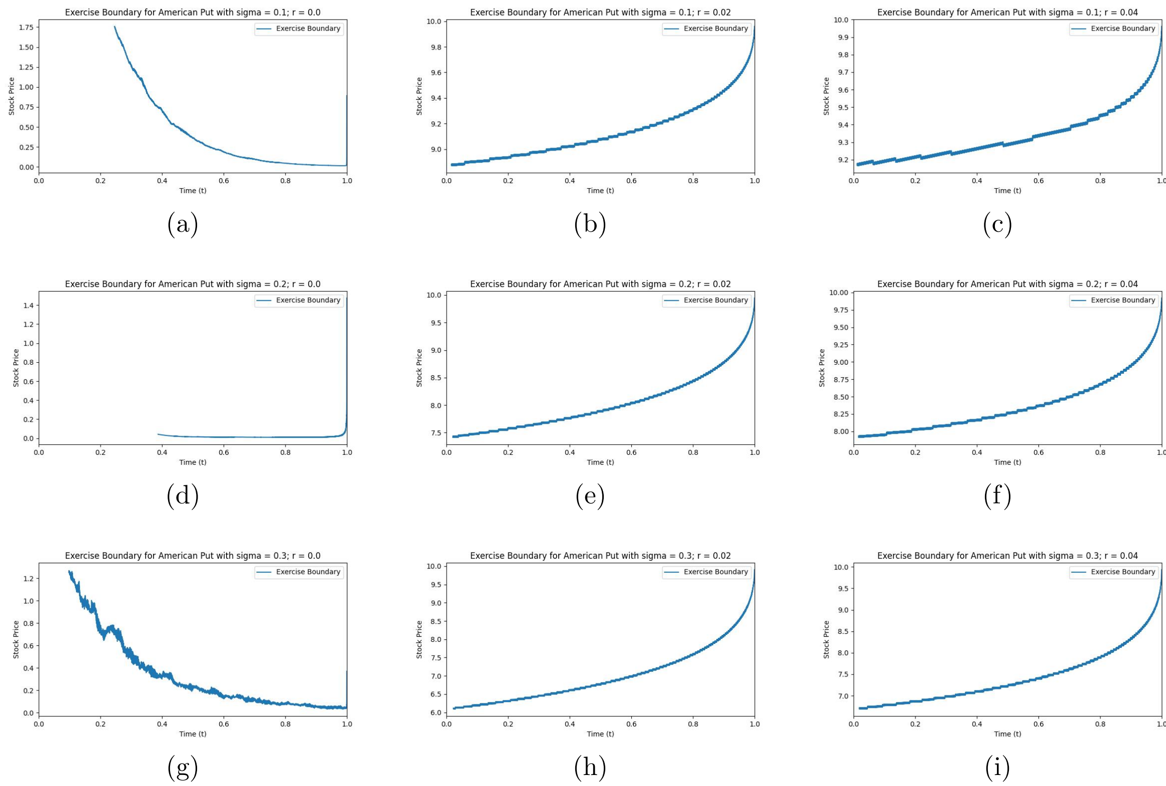


FIGURE 6. Exercise boundary plots (a) σ=0.1; r=0.0 (b) σ=0.1; r=0.02 (c) σ=0.1; r=0.04 (d) σ=0.2; r=0.0 (e) σ=0.2; r=0.02 (f) σ=0.2; r=0.04 (g) σ=0.3; r=0.0 (h) σ=0.3; r=0.02 (i) σ=0.3; r=0.04

Figure 6 display how the optimal exercise price evolves from t=0 to t=1 year of the option. The behavior of the exercise boundary in these graphs is crucial for understanding strategic option exercises. For lower volatilities, the option is less likely to be exercised early unless the stock price falls significantly. Conversely, for higher volatilities, the option's protective value is recognized earlier, making it optimal to hold the option unless a significant decrease in stock price justifies early exercise. The increase in risk-free interest rates raises the opportunity cost of holding the option, which can justify earlier exercise as the option nears maturity.

Figure 7 illustrate the behavior of stock prices for an American put option under various simulated paths, considering different combinations of volatility and risk-free rate. Higher volatility increases the option's value due to the greater potential for the stock price to move below the strike price, increasing the probability of profitable early exercise. This results in a wider range of scenarios where early exercise becomes optimal as the expiration approaches, evident from the paths that deviate more as volatility increases. Higher risk-free rate decreases the present value of the option's strike price, potentially making early exercise less favorable unless the stock price falls sufficiently below the strike price.

Figure 8 showcase the hedging delta for an American put option under different volatility and risk-free rate scenarios, illustrating how these factors affect the dynamic hedging requirements across the lifespan of the option. Higher volatility leads to more significant fluctuations in delta, indicating that more active management is required to maintain an effective hedge. The graphs show more obvious spikes and troughs in the delta as σ increases, highlighting the greater risk and

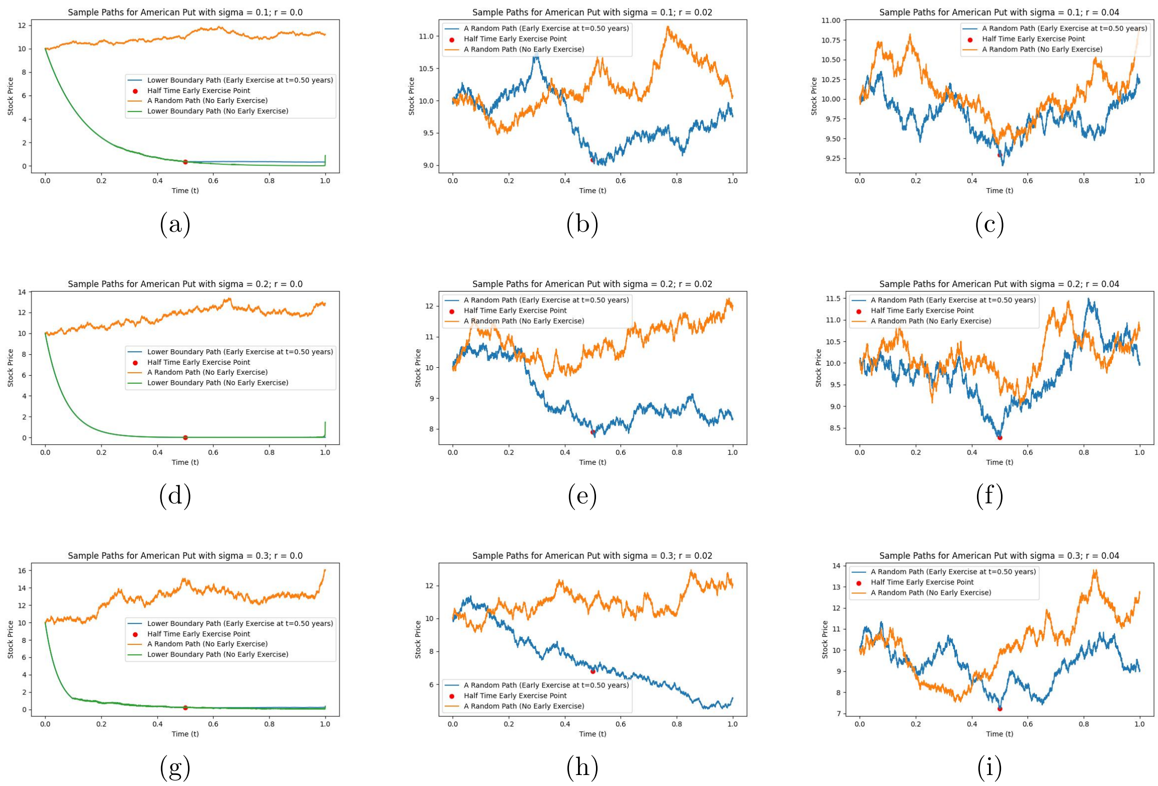


FIGURE 7. Sample prices path plots (a) σ=0.1; r=0.0 (b) σ=0.1; r=0.02 (c) σ=0.1; r=0.04 (d) σ=0.2; r=0.0 (e) σ=0.2; r=0.02 (f) σ=0.2; r=0.04 (g) σ=0.3; r=0.0 (h) σ=0.3; r=0.02 (i) σ=0.3; r=0.04

protective action required in more volatile markets. Changes in the risk-free rate influence the option's valuation and thus the delta. Higher rates tend to reduce the present value of the exercise price, which can affect the decision to exercise early and the corresponding delta adjustments.

When the risk-free rate is zero, several factors may contribute to the unusual behavior observed in the graphs: The absence of a discount factor keeps future cash flows unchanged, making option valuations highly sensitive to immediate stock price fluctuations. Without the economic incentive of earning interest, option holding strategies pivot solely on the stock's intrinsic value. This lack of interest influence prompts more volatile, less predictable early exercise decisions and erratic hedging strategies. Additionally, the delta becomes extremely reactive to price changes, lacking the stabilizing effect typically provided by positive risk-free returns.

3.2. **Problem 2(b).**

Figure 9 shows the P&L distribution for an American put option based on 10,000 simulated paths using a binomial tree model. Given the volatility of 0.20, the range of stock price movements is considerable, leading to diverse early exercise outcomes as reflected in the spread of the P&L distribution. The positive risk-free rate and expected return encourage holding the option unless significant profit is apparent, hence why a notable portion of the distribution is centered around lower positive values, suggesting cautious early exercises.

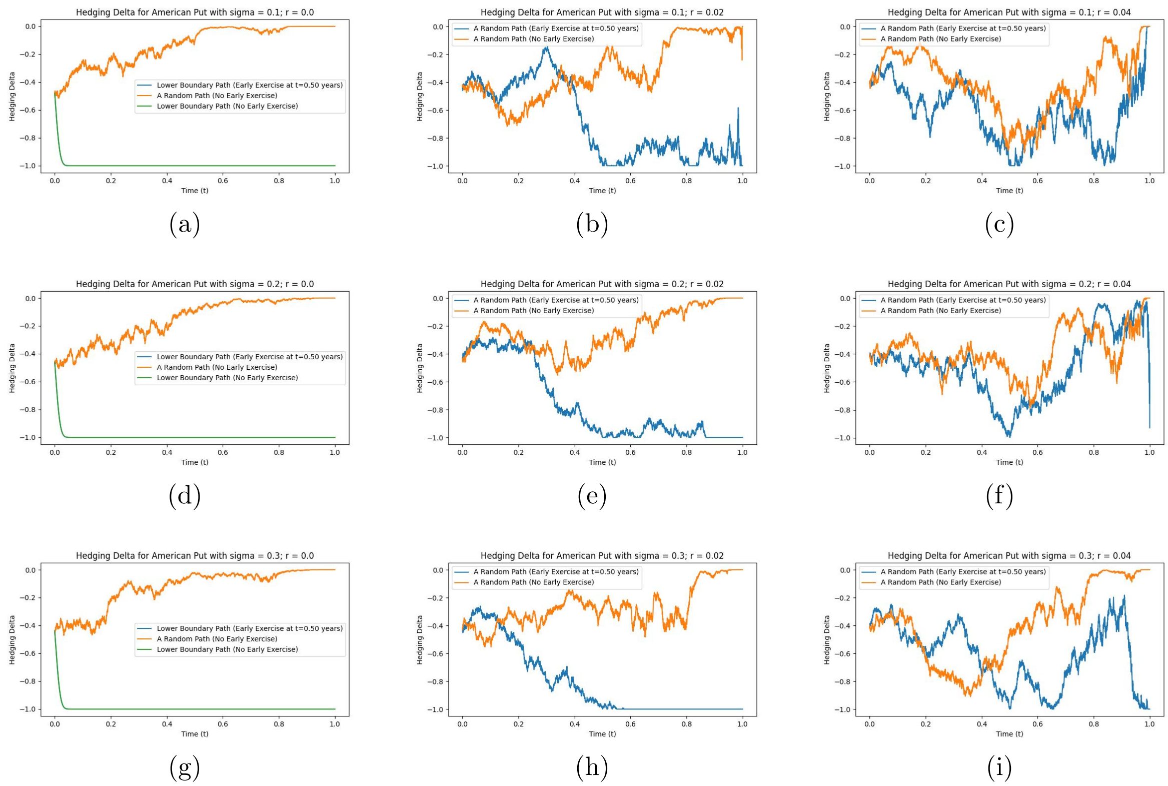


Figure 8. Delta path plots (a) σ=0.1; r=0.0 (b) σ=0.1; r=0.02 (c) σ=0.1; r=0.04 (d) σ=0.2; r=0.0 (e) σ=0.2; r=0.02 (f) σ=0.2; r=0.04 (g) σ=0.3; r=0.0 (h) σ=0.3; r=0.02 (i) σ=0.3; r=0.04

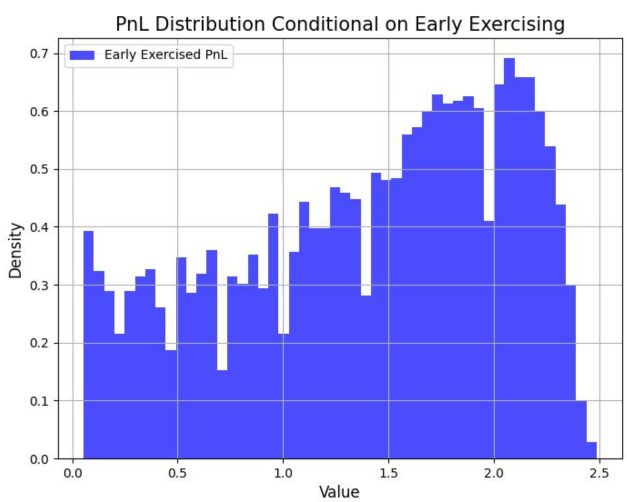


Figure 9. P&L Distribution with Point Mass at 0

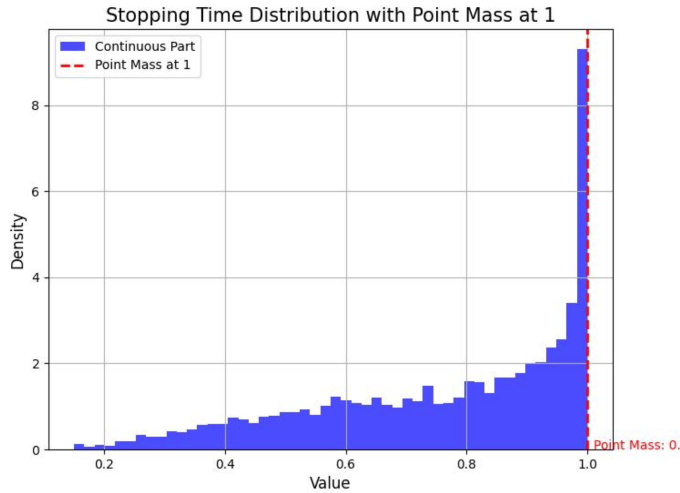


FIGURE 10. Stopping Time Distribution with Point Mass at 1

Figure 10 shows the stopping time distribution for an American put option, distinguishing between continuous exercise times and a discrete point mass at 1, indicating the option was held until maturity. Most stopping times are continuous, showing varied optimal exercise times before maturity. However, a significant point mass at 1, denoted as 'Point Mass at 1', occurs in numerous simulations where conditions favored holding the option until expiration. This suggests that often, the best strategy was not early exercise, influenced by the model's dynamics like stock price movements and volatility. The distribution also shows other stopping times where early exercise was justified, demonstrating the dependency of exercise decisions on stock price dynamics, volatility, strike price, and maturity. The probability of exercise to account for the point mass at 1 is 51%.

Figure 11 and 12 illustrate the profit and loss (P&L) and stopping time distributions for an American put option under different levels of volatility and risk-free rates.

Figure 11 shows the spread and frequency of profitable exercises, which vary depending on both σ and r. The P&L distributions tend to be narrower with most of the outcomes concentrated around lower values. This suggests that when market volatility is low, the benefits derived from exercising the option early are limited, usually resulting in smaller profits. As volatility increases, the distributions broaden and shift towards higher values, indicating larger variances in outcomes. This is consistent with higher market uncertainty allowing for potentially higher profits if the option is exercised optimally in favorable conditions. Across the different levels of volatility, increasing the risk-free rate generally results in a shift towards higher values in the P&L distribution. This is because a higher risk-free rate increases the opportunity cost of holding the option, thus potentially increasing the value gained from exercising the option under favorable conditions. The computed probability of early exercise reflects the likelihood that exercising the option before maturity is more beneficial than holding it to expiration. For higher volatility and lower risk-free rates, the probability of early exercise tends to increase, correlating with the observed shifts in the P&L distributions towards more favorable outcomes.

Figure 12 represents the fraction of paths where the option is held until expiration. At lower volatilities, the stopping times are densely packed towards maturity. This indicates that the option is more often exercised at or near the end of its term. As volatility increases, the distribution

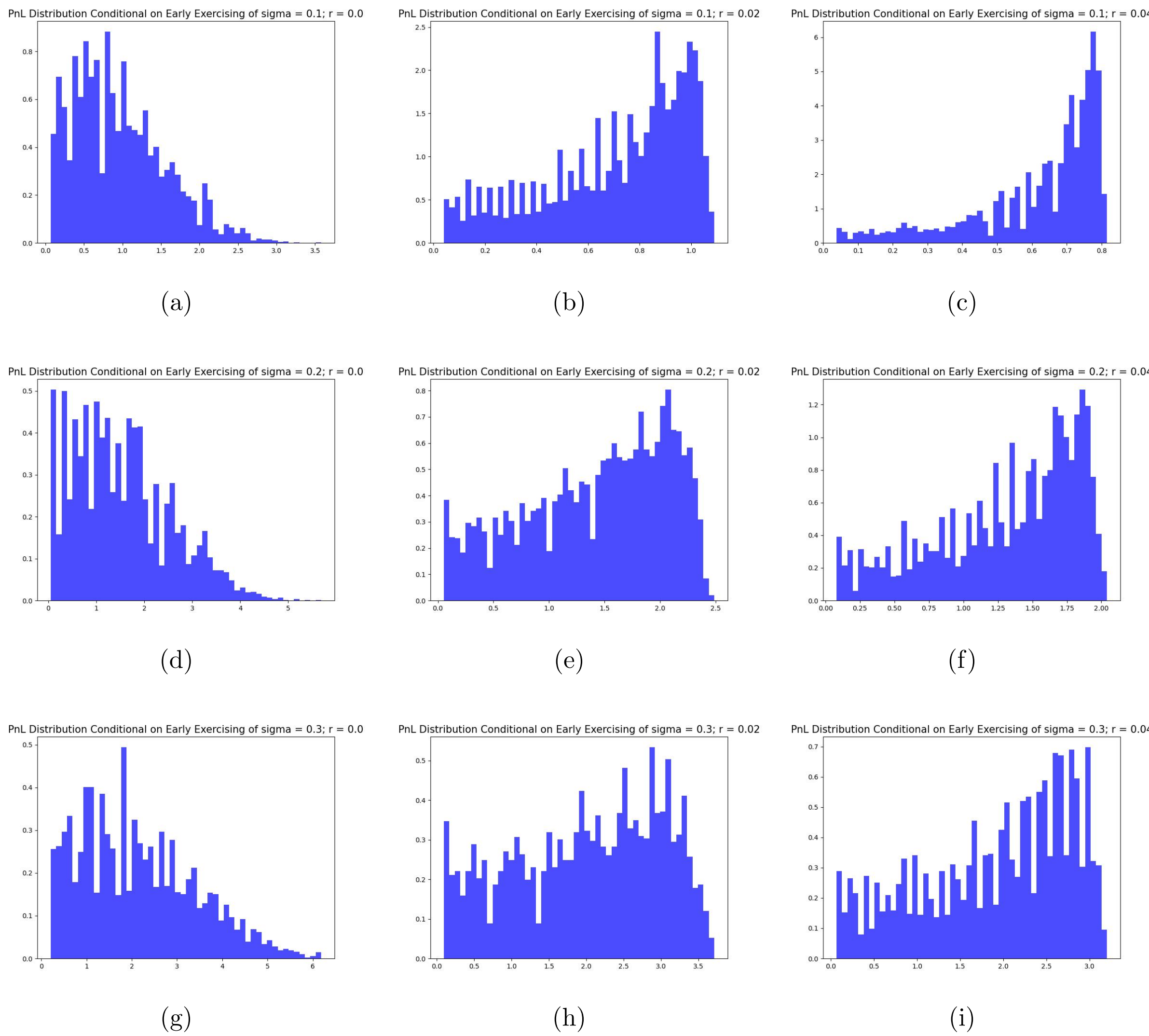
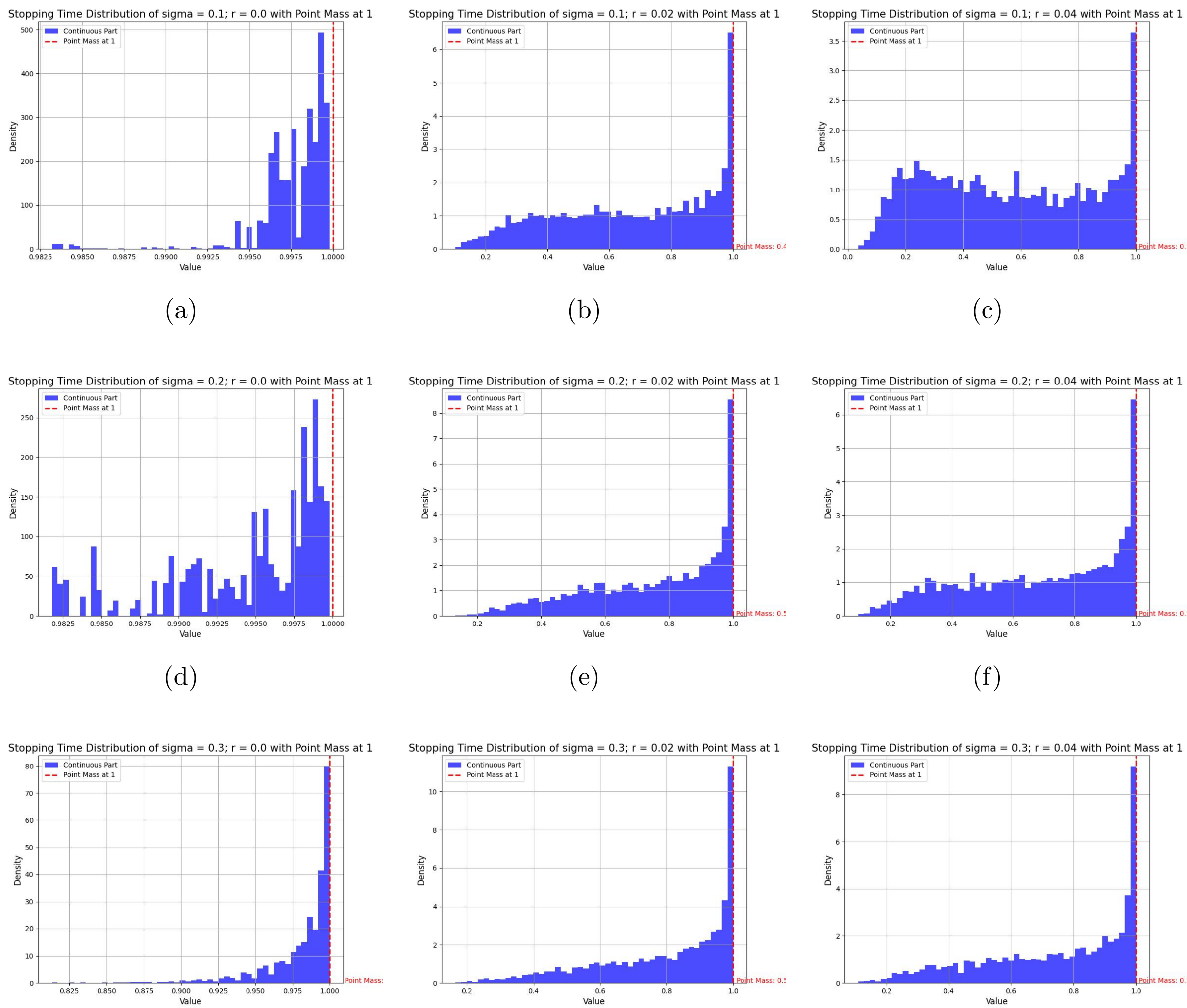


FIGURE 11. P&L distribution plots (a) σ=0.1; r=0.0 (b) v=0.1; r=0.02 (c) σ=0.1; r=0.04 (d) σ=0.2; r=0.0 (e) σ=0.2; r=0.02 (f) σ=0.2; r=0.04 (g) σ=0.3; r=0.0 (h) σ=0.3; r=0.02 (i) σ=0.3; r=0.04

spreads out, showing a broader range of stopping times. Increasing the risk-free rate generally pushes the stopping time distribution to earlier exercises. This is visible as the spread of the continuous part of the histogram moves from right to left.

Figure 13 indicate that as volatility increases (from $\sigma = 0.1$ to 0.3), the continuous part of the P&L distribution broadens. When the realized volatility is lower than the volatility used to establish the exercise boundary σ=0.10 and σ=0.15), the resulting P&L distributions tend to have more density towards the lower value end. This reflects smaller price movements in the underlying asset, leading to less profitable opportunities for early exercise. The distribution when the realized volatility matches the modeling volatility (σ=0.20) shows a more balanced spread across the range of P&L values. This indicates a good alignment between the model's expectations and the actual market behavior, yielding a diverse set of profitable outcomes. At higher volatilities (σ=0.25 and σ=0.30), the distributions widen significantly with heavier tails and higher peaks. This suggests



(a) (b) (c)

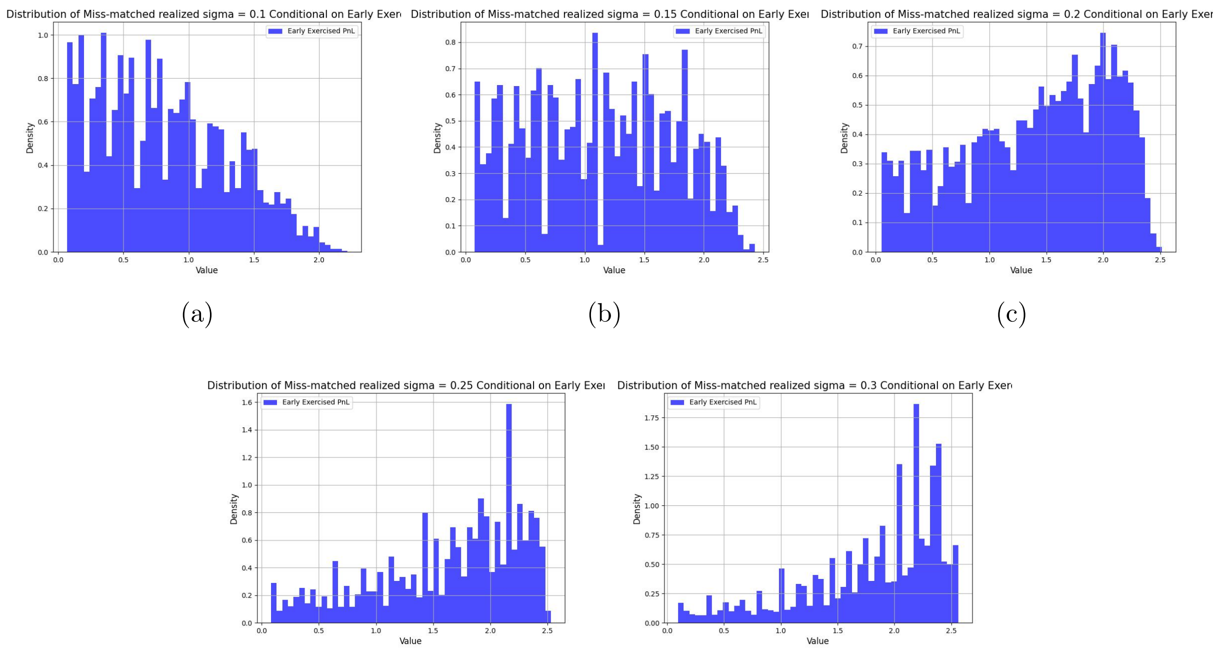
(d) (e) (f)

(g) (h) (i)

FIGURE 12. Stopping time distribution plots (a) σ=0.1; r=0.0 (b) σ=0.1; r=0.02 (c) σ=0.1; r=0.04 (d) σ=0.2; r=0.0 (e) σ=0.2; r=0.02 (f) σ=0.2; r=0.04 (g) σ=0.3; r=0.0 (h) σ=0.3; r=0.02 (i) σ=0.3; r=0.04

that larger movements in the underlying asset's price provide more opportunities for substantial profits through early exercise. However, this also introduces more variability in the outcomes, reflecting the increased risk associated with higher volatility.

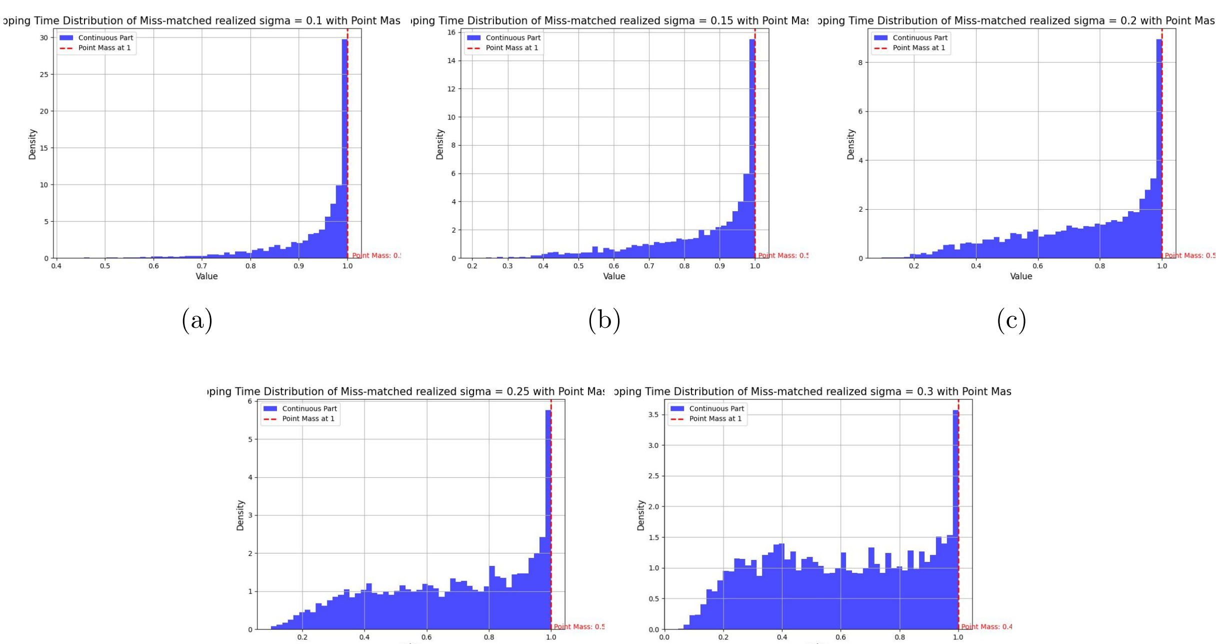
Figure 14 demonstrates that as volatility increases, the continuous part of the distribution becomes more spread out and less peaked near the maturity. This suggests that higher volatility increases the uncertainty about the optimal exercise time, leading to a wider range of exercise times being observed. Higher interest rates tend to shift the distribution towards earlier exercise times. This can be attributed to the time value of money; as the interest rate increases, the opportunity cost of holding the option increases, making earlier exercise more attractive.



(a) (b) (c)

(d) (e)

FIGURE 13. Miss-match vol P&L distribution plots (a) σ=0.1 (b) σ=0.15 (c) σ=0.2 (d) σ=0.25 (e) σ=0.3



(a) (b) (c)

(d) (e)

FIGURE 14. Miss-match vol Stopping time distribution plots (a) σ=0.1 (b) σ=0.15 (c) σ=0.2 (d) σ=0.25 (e) σ=0.3

4. Conclusions

The binomial tree model's exploration of American put options reveals key insights into the impact of volatility and risk-free rates on option pricing, exercise timing, and hedging effectiveness. Higher volatility increases the potential profit from early exercises due to significant price fluctuates but also demands sophisticated hedging strategies. Similarly, higher risk-free rates prompt earlier exercise decisions by increasing the opportunity cost of holding options. These findings are instrumental for traders to refine strategies based on market conditions, though it's crucial to note the model's limitations such as its reliance on constant parameter assumptions. This study offers valuable guidelines for trading decisions and risk management, emphasizing the practical application of financial models in evolving market scenarios.

Contribution: Gloria wrote the Problem 1 and the written report of this project. Alex implemented the coding parts of Problem 2.

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