Robótica Móvil
un enfoque probabilístico

Filtro de Bayes – Filtro de Kalman

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Basado en material de Wolfram Bugard, Dieter Fox y Cyrill Stachniss

Repaso de filtro de Bayes

$$Bel(x_t) = \eta\, p(z_t \mid x_t) \int p(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

- Predicción

$$\overline{Bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) Bel(x_{t-1}) dx_{t-1}$$

- Corrección

$$Bel(x_t) = \eta\, p(z_t \mid x_t) \overline{Bel}(x_t)$$

Repaso de filtro de Bayes

$$Bel(x_t) = \eta\ P(z_t \mid x_t) \int P(x_t \mid u_t, x_{t-1})\ Bel(x_{t-1})\ dx_{t-1}$$

1. Algoritmo **Bayes_filter**(*Bel(x),d*):
2. $\eta = 0$
3. If *d* is a perceptual data item *z* then
4. For all *x* do
5. $Bel'(x) = P(z \mid x) Bel(x)$
6. $\eta = \eta + Bel'(x)$
7. For all *x* do
8. $Bel'(x) = \eta^{-1} Bel'(x)$
9. Else if *d* is an action data item *u* then
10. For all *x* do
11. $Bel'(x) = \int P(x \mid u, x')\ Bel(x')\ dx'$
12. Return *Bel'(x)*

Filtro de Kalman

- Filtro de Bayes con **Gaussianas**
- Desarrollado a fines de los 1950's
- Filtro de Bayes más utilizado en la práctica
- Usado en economía, pronóstico climático, navegación satelital, robótica, etc.

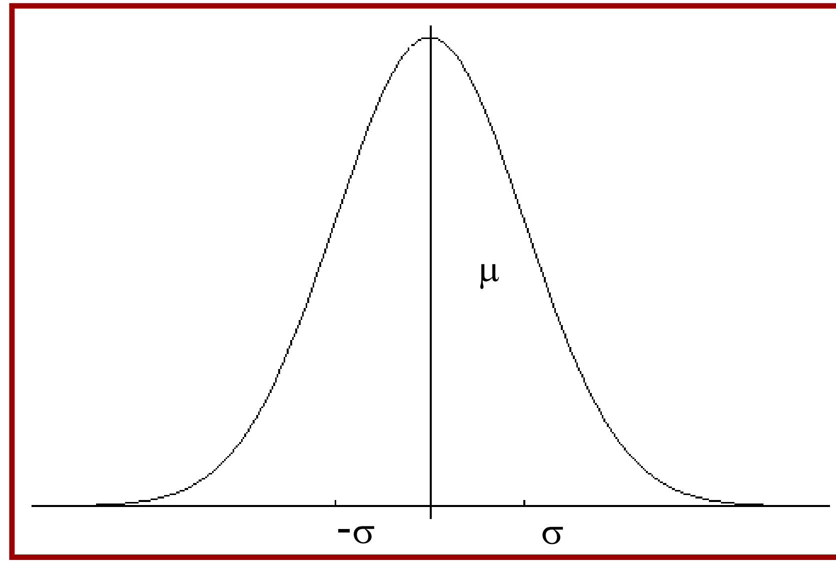
- El algoritmo del filtro de Kalman es sólo algunas **multiplicaciones matriciales**!

Gaussianas

$$p(x) \sim N(\mu, \sigma^2):$$

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

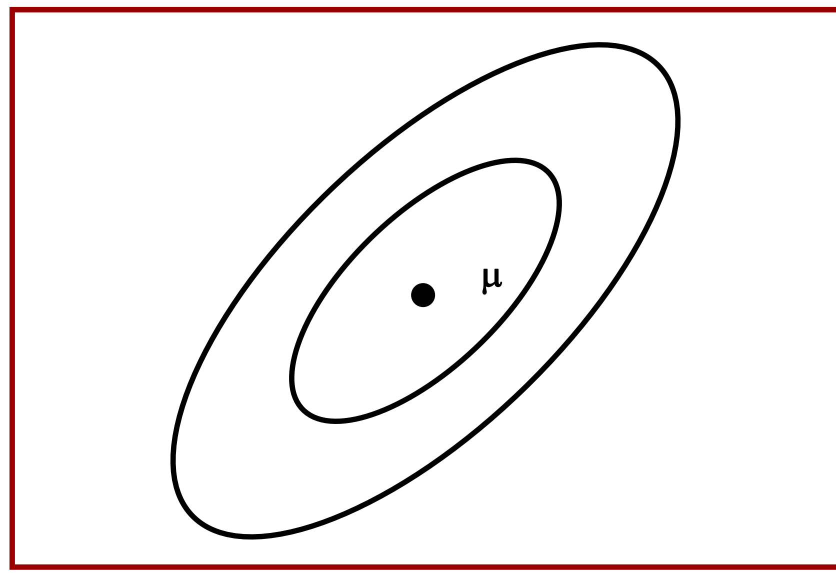
Una variable



$$p(\mathbf{x}) \sim N(\mu, \Sigma):$$

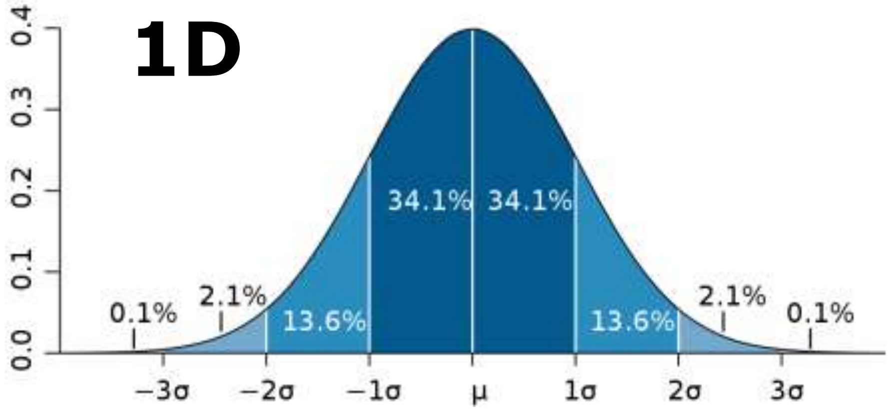
$$p(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\mathbf{x}-\mu)^t \Sigma^{-1}(\mathbf{x}-\mu)}$$

Multivariable



Gaussianas

1D



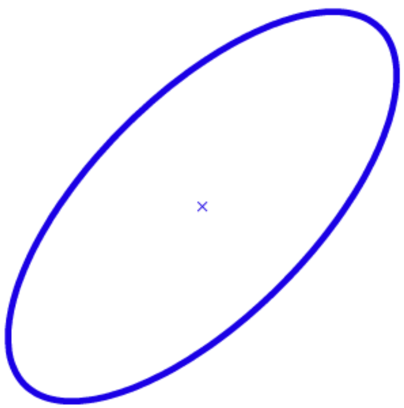
2D

$$C = \begin{bmatrix} 0.020 & 0.013 \\ 0.013 & 0.020 \end{bmatrix}$$

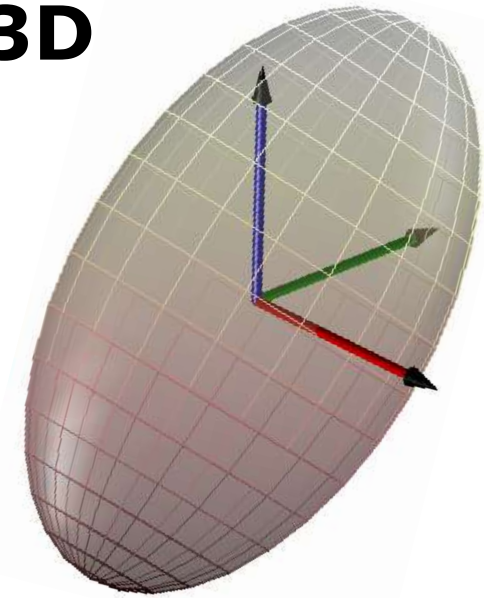
$$\lambda_1 = 0.007$$

$$\lambda_2 = 0.033$$

$$\rho = \sigma_{XY} / \sigma_X \sigma_Y = 0.673$$



3D



Propiedades de Gaussianas

- Caso univariable

$$\left.\begin{array}{l} X \sim N(\mu,\sigma^2) \\ Y = aX + b \end{array}\right\} \Rightarrow \quad Y \sim N(a\mu + b, a^2\sigma^2)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1,\sigma_1{}^2) \\ X_2 \sim N(\mu_2,\sigma_2{}^2) \end{array}\right\} \Rightarrow p(X_1)\cdot p(X_2) \sim N\left(\frac{\sigma_2{}^2}{\sigma_1{}^2+\sigma_2{}^2}\mu_1 + \frac{\sigma_1{}^2}{\sigma_1{}^2+\sigma_2{}^2}\mu_2, \quad \frac{1}{\sigma_1{}^{-2}+\sigma_2{}^{-2}}\right)$$

Propiedades de Gaussianas

- Caso multivariable

$$\left.\begin{array}{r} X \sim N(\mu, \Sigma) \\ Y = AX + B \end{array}\right\} \Rightarrow \quad Y \sim N(A\mu + B, A\Sigma A^T)$$

$$\left.\begin{array}{l} X_1 \sim N(\mu_1, \Sigma_1) \\ X_2 \sim N(\mu_2, \Sigma_2) \end{array}\right\} \Rightarrow p(X_1) \cdot p(X_2) \sim N\left(\frac{\Sigma_2}{\Sigma_1 + \Sigma_2} \mu_1 + \frac{\Sigma_1}{\Sigma_1 + \Sigma_2} \mu_2, \quad \frac{1}{\Sigma_1^{-1} + \Sigma_2^{-1}} \right)$$

(donde la división "–" denota inversión de la matriz)

- Seguirá siendo **Gaussiana** si empezamos con Gaussianas y hacemos sólo **transformaciones afines y productos**

Filtro de Kalman Discreto

Estima el estado x de un proceso de tiempo discreto que es gobernado por una ecuación lineal estocástica en diferencias:

$$\boxed{x_t = A_t x_{t-1} + B_t u_t + e_t}$$

con una medición:

$$\boxed{z_t = C_t x_t + d_t}$$

Componentes del filtro de Kalman

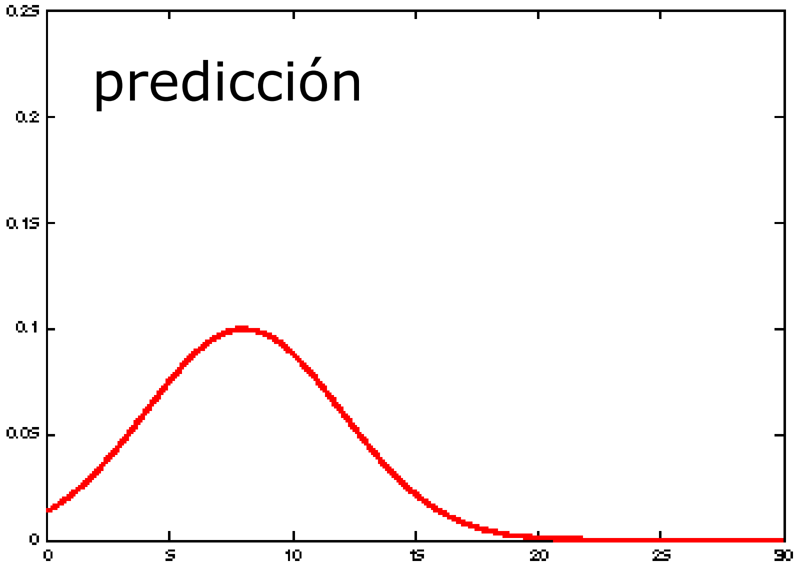
$\boxed{A_t}$ Matriz $(n \times n)$ que describe cómo el estado evoluciona de t-1 a t sin control ni ruido.

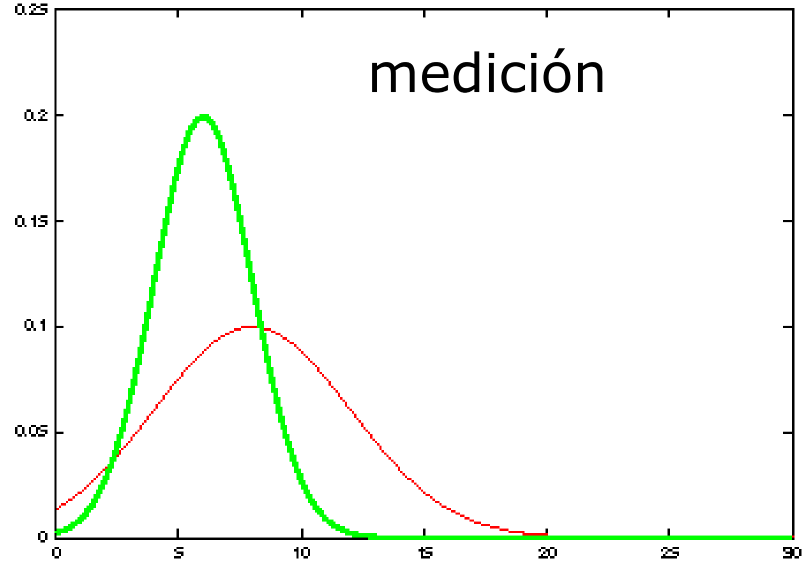
$\boxed{B_t}$ Matriz $(n \times l)$ que describe como el control u_t cambia el estado de t-1 a t.

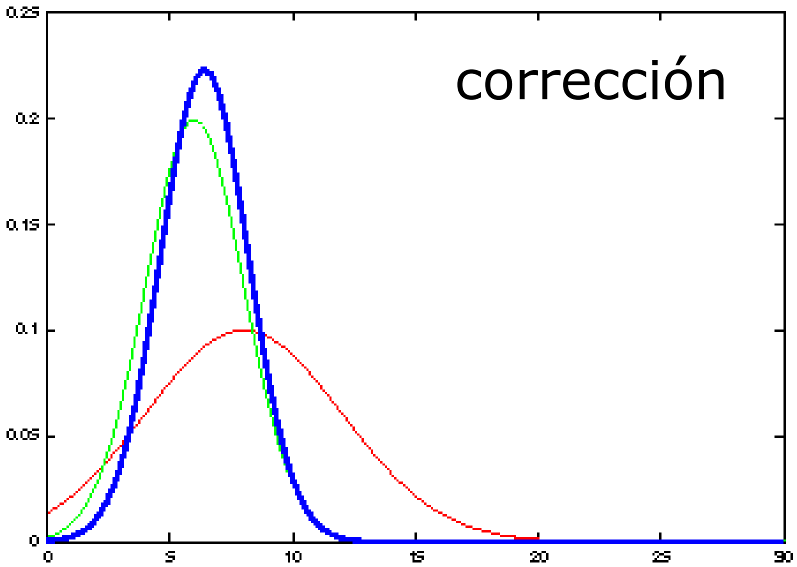
$\boxed{C_t}$ Matriz $(k \times n)$ que describe como mapear el estado x_t a una observación z_t.

$\boxed{e_t}$ $\boxed{d_t}$ Variables aleatorias que representan el ruido de proceso y de medición que se asumen independientes y normalmente distribuidas con covarianzas Q_t y R_t respectivamente.

Filtro de Kalman en 1D

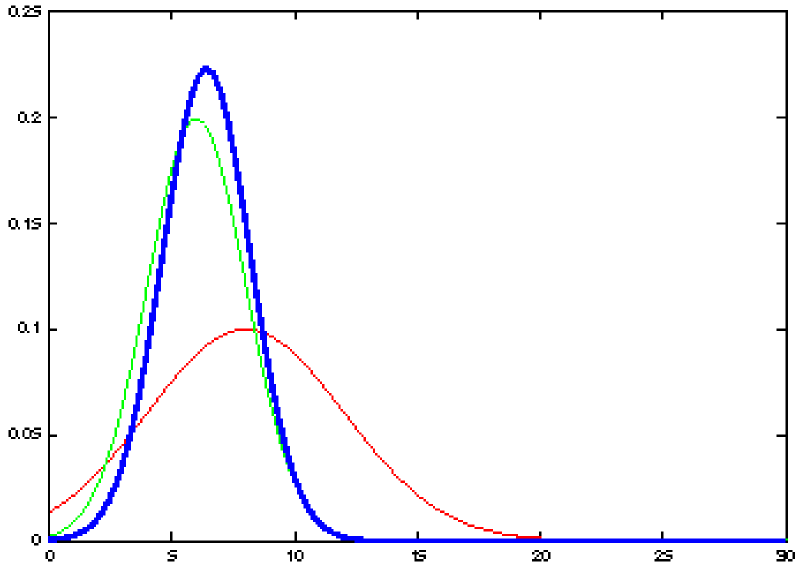






Es un promedio ponderado

Filtro de Kalman en 1D



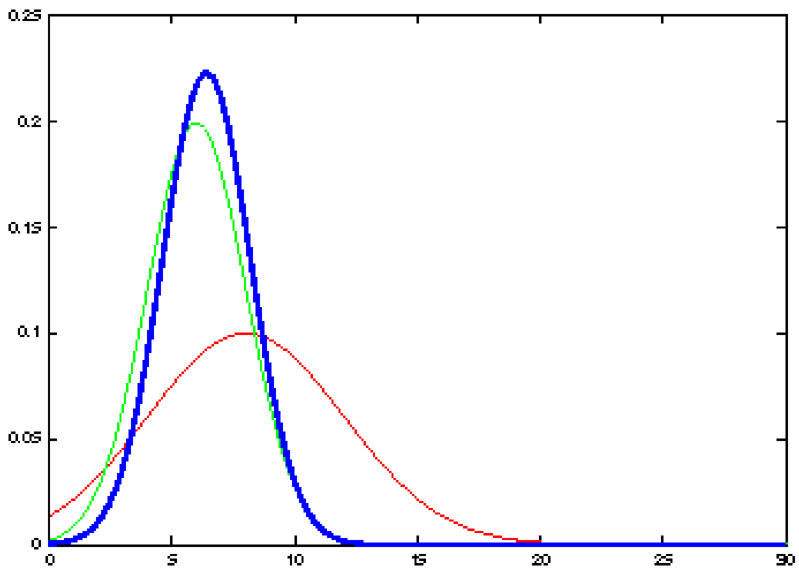
Cómo obtener la curva azul?

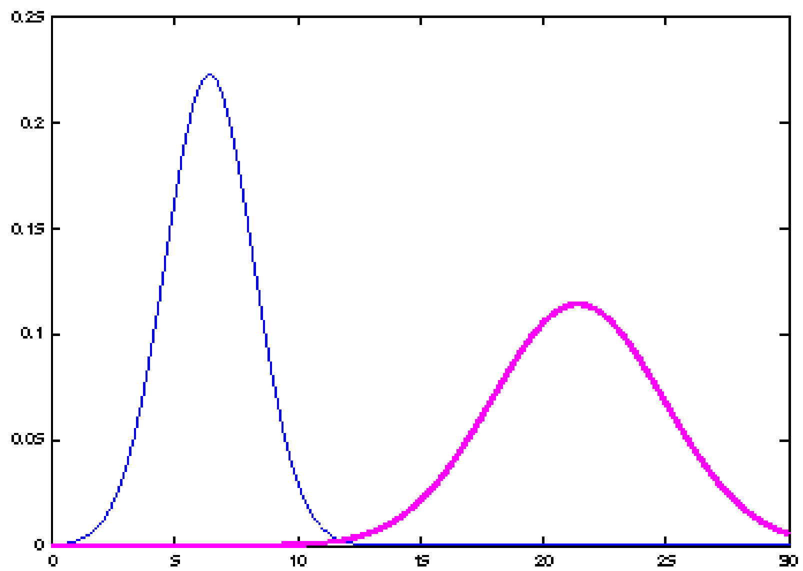
Paso de corrección del filtro de Kalman

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \sigma_t^2 = (1 - K_t C_t)\bar{\sigma}_t^2 \end{cases} \quad \text{con} \quad K_t = \frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2 + \bar{\sigma}_{obs,t}^2}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \quad \text{con} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$

Filtro de Kalman en 1D





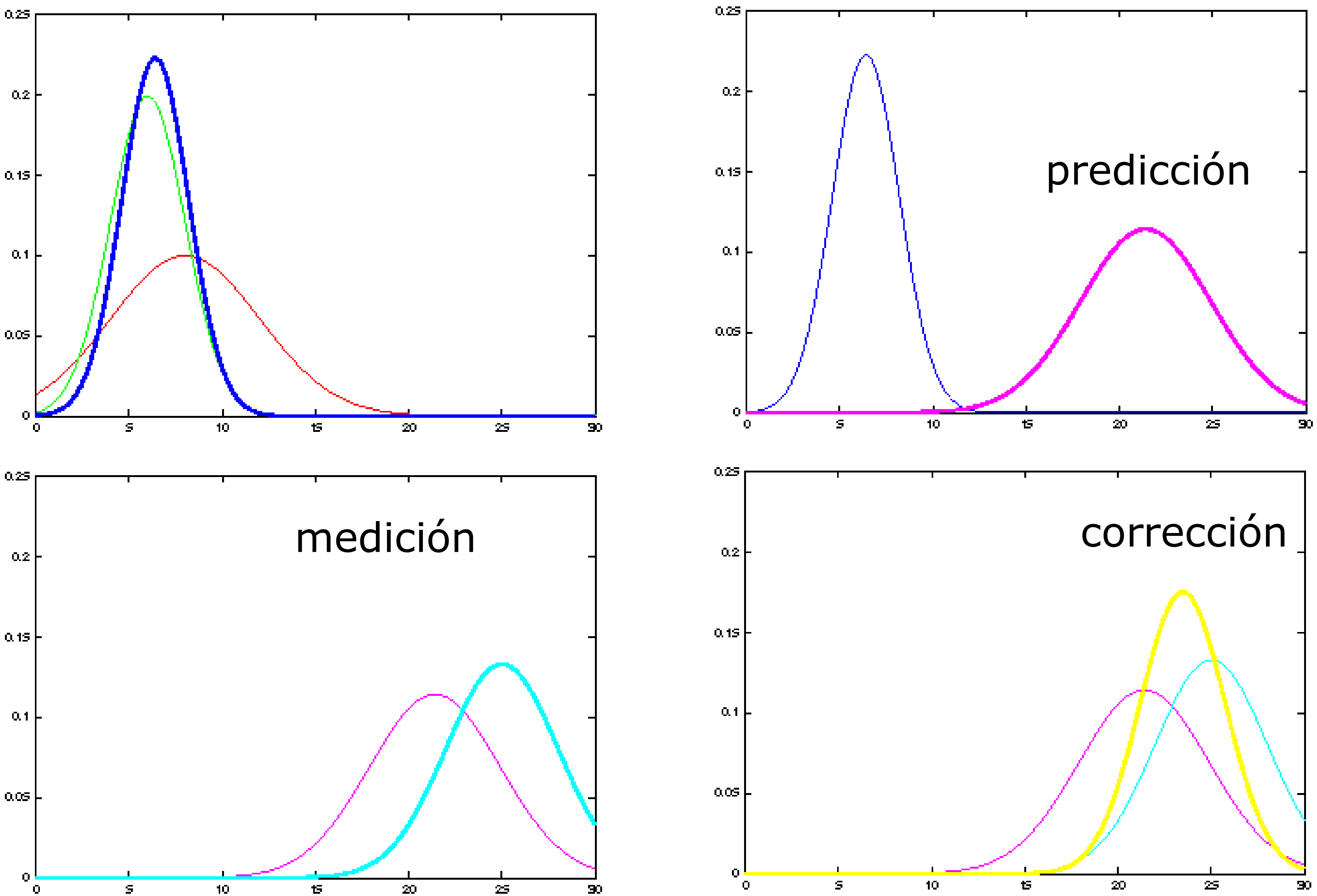
$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t \end{cases}$$

Cómo obtener la curva magenta?

Paso de predicción de estado

Filtro de Kalman en 1D



Sistemas Gaussianos Lineales: Inicialización

El Belief inicial está normalmente distribuido:

$$\boxed{bel(x_0) = N\left(x_0; \mu_0, \Sigma_0\right)}$$

Sistemas Gaussianos Lineales: Dinámica

La dinámica son funciones lineales del estado y el control más ruido aditivo:

$$\boxed{x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t}$$

$$\boxed{p(x_t \mid u_t, x_{t-1}) = N\left(x_t; A_t x_{t-1} + B_t u_t, Q_t\right)}$$

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1})\, bel(x_{t-1})\, dx_{t-1}$$

$$\Downarrow \qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, Q_t\right) \quad \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

Sistemas Gaussianos Lineales: Dinámica

$$\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) \qquad bel(x_{t-1}) dx_{t-1}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N\left(x_t; A_t x_{t-1} + B_t u_t, Q_t\right) \sim N\left(x_{t-1}; \mu_{t-1}, \Sigma_{t-1}\right)$$

$$\Downarrow$$

$$\overline{bel}(x_t) = \eta \int \exp\left\{-\frac{1}{2}(x_t - A_t x_{t-1} - B_t u_t)^T Q_t^{-1} (x_t - A_t x_{t-1} - B_t u_t)\right\}$$

$$\exp\left\{-\frac{1}{2}(x_{t-1} - \mu_{t-1})^T \Sigma_{t-1}^{-1} (x_{t-1} - \mu_{t-1})\right\} dx_{t-1}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t \end{cases}$$

Sistemas Gaussianos Lineales: Observaciones

Las observaciones son funciones lineales del estado, más ruido aditivo:

$$\boxed{z_t = C_t x_t + d_t}$$

$$\boxed{p(z_t \mid x_t) = N(z_t; C_t x_t, R_t)}$$

$$\boxed{\begin{array}{llll} bel(x_t) = & \eta & p(z_t \mid x_t) & \overline{bel}(x_t) \\ & & \Downarrow & \Downarrow \\ & & \sim N(z_t; C_t x_t, R_t) & \sim N(x_t; \overline{\mu}_t, \overline{\Sigma}_t) \end{array}}$$

Sistemas Gaussianos Lineales: Observaciones

$$bel(x_t) = \quad \eta \quad p(z_t \mid x_t) \qquad \overline{bel}(x_t)$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\sim N(z_t; C_t x_t, R_t) \qquad \sim N(x_t; \bar{\mu}_t, \bar{\Sigma}_t)$$

$$\Downarrow$$

$$bel(x_t) = \eta \exp\left\{-\frac{1}{2}(z_t - C_t x_t)^T R_t^{-1}(z_t - C_t x_t)\right\} \exp\left\{-\frac{1}{2}(x_t - \bar{\mu}_t)^T \bar{\Sigma}_t^{-1}(x_t - \bar{\mu}_t)\right\}$$

$$bel(x_t) = \begin{cases} \mu_t = \bar{\mu}_t + K_t(z_t - C_t\bar{\mu}_t) \\ \Sigma_t = (I - K_t C_t)\bar{\Sigma}_t \end{cases} \qquad \text{with} \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$$

Algoritmo de filtro de Kalman

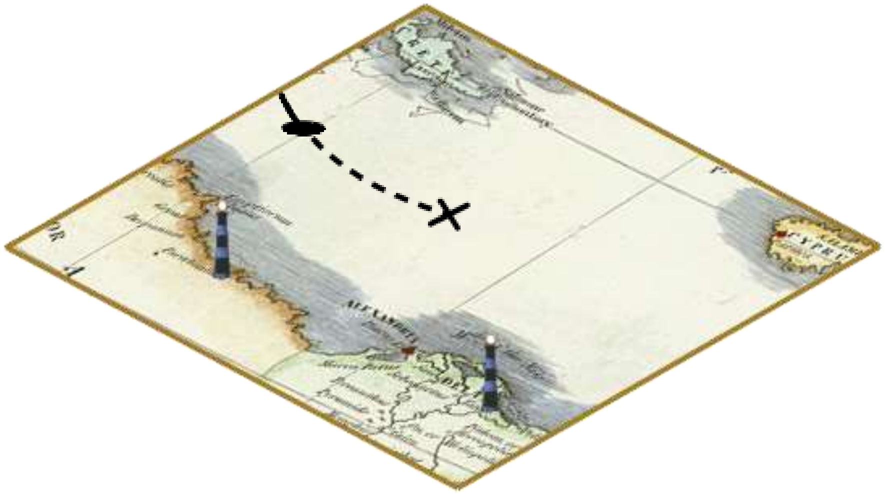
1. Algoritmo **Kalman_filter**($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

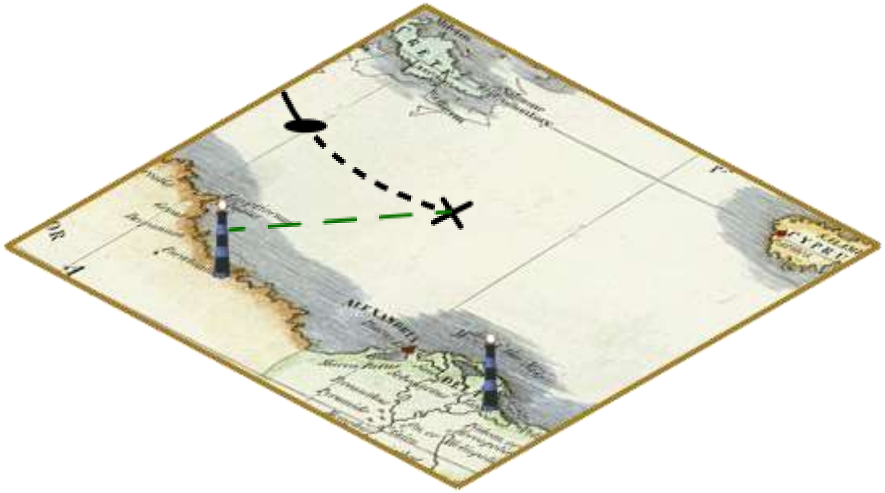
2. Predicción:
3. $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$
4. $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t$

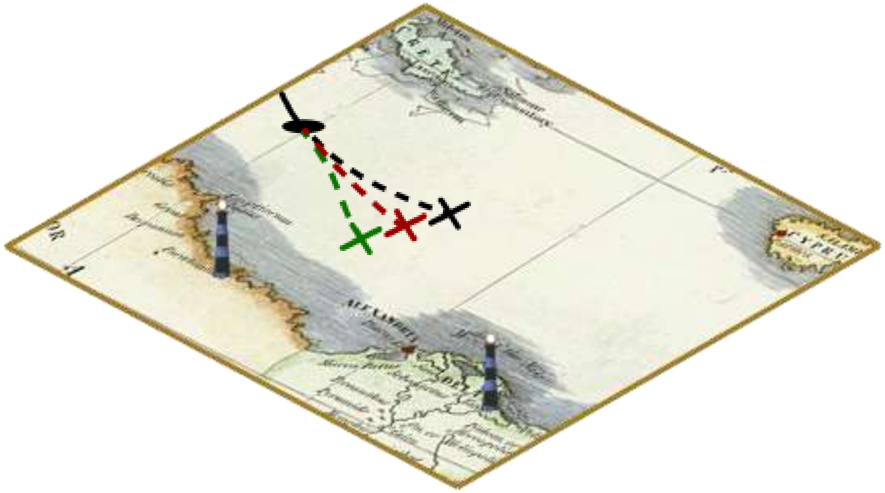
5. Corrección:
6. $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + R_t)^{-1}$
7. $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$
8. $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$

9. Return μ_t, Σ_t

Algoritmo de filtro de Kalman





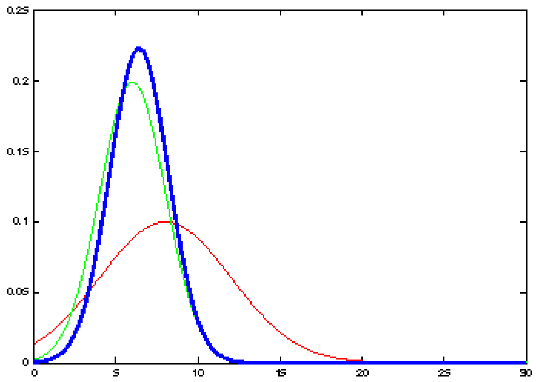


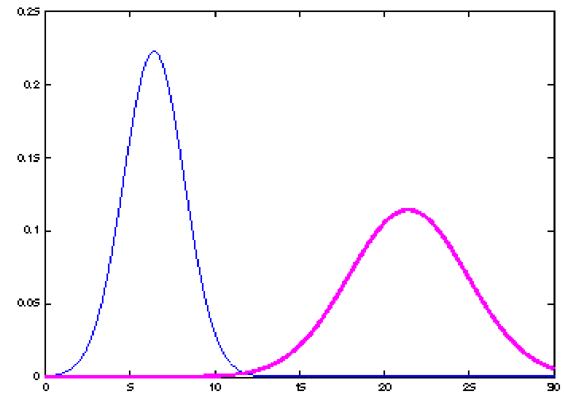
Ciclo predicción-Corrección

Predicción

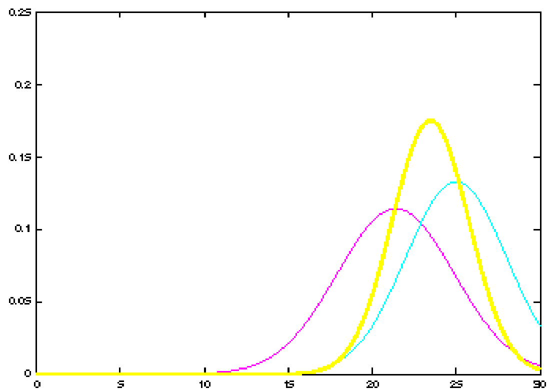
$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = a_t \mu_{t-1} + b_t u_t \\ \bar{\sigma}_t^2 = a_t^2 \sigma_t^2 + \sigma_{act,t}^2 \end{cases}$$

$$\overline{bel}(x_t) = \begin{cases} \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t \\ \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + Q_t \end{cases}$$





Ciclo predicción-Corrección



$$bel(x_t)=\begin{cases}\mu_t=\bar{\mu}_t+K_t(z_t-\bar{\mu}_t)\\ \sigma_t^2=(1-K_t)\bar{\sigma}_t^2\end{cases}, K_t=\frac{\bar{\sigma}_t^2}{\bar{\sigma}_t^2+\bar{\sigma}_{obs,t}^2}$$

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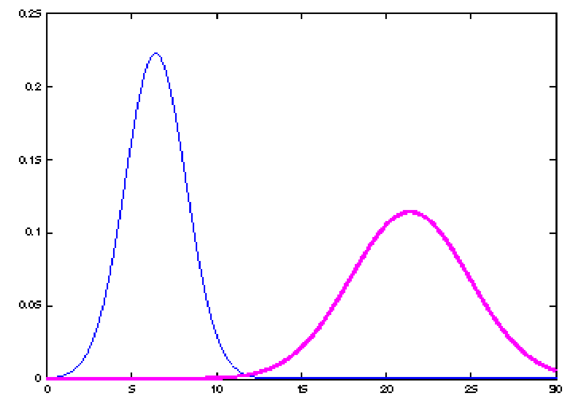

Corrección

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Resumen del filtro de Kalman

- Dos parámetros describen el belief del estado del sistema
- **Muy eficiente:** Polinómico en la dimensión de la medición k y la dimensión del estado n:

$$O(k^{2.376} + n^2)$$

- **Óptimo para sistemas Gaussianos lineales**!
- Pero: La mayoría de los sistemas robóticos son **no-lineales**!
- Sólo puede modelar beliefs unimodales