Spacecraft Mission

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This report will cover all parts of assignment one in the Aerospace 720 course

1 Orbital Propagation

1.1 Solving Kepler's Equation

I) We need to solve Kepler's Equation using numerical methods. Using the Newton-Rasphon Method we can take the eccentricity and mean anamoly as inputs and numerically solve for the eccentric anamoly.

```
def Kepler(e, M, tol = 1e-12, max_i = 1000):
    E = M                           # Guess solution
    for i in range(max_i):
        f_E = E - e * np.sin(E) - M  # Define the
            function in terms of f(E) = 0
         f_prime = 1 - e * np.cos(E) # Derive the
             function in terms of E
        del_E = f_E / f_prime
        E_new = E - del_E           # Calculate the
            new eccentric anamoly
        if np.abs(del_E) < tol:     # If the value is
            within the set tolerance
            theta = 2*np.arctan(np.tan(E_new/2) *
                ((1+e)/(1-e))**(0.5))
            return theta            # Return true
                anamoly
        E = E_new
```

II) If we set the tolerance to $1e-12$, we can compute the true anamoly of the asteroid at t_0 and $t_0 + 100$ days. A_ae0 is the OBJ data of the asteroid, it is an array.

```
trueAnamoly_asteroidt_0 = Kepler(A_ae0[2], A_ae0[6])
meanAnamolyt_100 = get_mean_anamoly(100*(3600*24),
    A_ae0[6], A_ae0[1])
trueAnamoly_asteroidt_100 = Kepler(A_ae0[2],
    meanAnamolyt_100)
```

Printing these values gives that the true anamoly $\theta_{t_0} = 1.4246$ and $\theta_{t_0+100} = 2.1369$. Where these answers are in radians.

III) Now I have created a function that takes in a state of orbital elements and returns the position and velocity vectors at that point. It uses a rotation matrix to convert from the perifocal frame to the ECI frame. This is defined via i, ω, and Ω terms and is calculated using the defind matricies in the appendix.

```
def COE2RV(arr, mu):
    a, e, i, Omega, omega, theta_var = arr[0:6]
    h = np.sqrt(mu * a * (1 - e**2))
    r = a*(1-(e**2))/(1 + e*np.cos(theta_var))

    arr_r = np.array([r*np.cos(theta_var),
        r*np.sin(theta_var), 0])
    arr_v = (mu/h)* np.array([-np.sin(theta_var), e +
        np.cos(theta_var), 0])

    # Rotate position and velocity from perifocal to
        inertial frame using the
    # transfomration matrix
    R_matrix = rotation_matrix(i, Omega, omega)

    r_ijk = R_matrix @ arr_r
    v_ijk = R_matrix @ arr_v
    return r_ijk, v_ijk
```

Using this code we can output the state vector at some time t. The first three values are the x, y, z positions in **km**. The last three are the velocity values in the x, y, z direction in **km/s**.

$$\textbf{At } t_0: \quad \bar{\mathbf{X}} = \begin{bmatrix} x = -1.1694365\text{e+}08 \\ y = 1.53462780\text{e+}08 \\ z = -6.7446087\text{e+}06 \\ v_x = -3.1710203\text{e+}01 \\ v_y = -3.6285380\text{e+}00 \\ v_z = -1.8931546\text{e+}00 \end{bmatrix}$$

$$\textbf{At } t_0 + 100: \quad \bar{\mathbf{X}} = \begin{bmatrix} x = -3.2057997\text{e+}08 \\ y = 6.72659396\text{e+}07 \\ z = -1.8991445\text{e+}07 \\ v_x = -1.6964807\text{e+}01 \\ v_y = -1.2943780\text{e+}01 \\ v_z = -1.0284663\text{e+}00 \end{bmatrix}$$

IV) Next, I have written a function called "Ephemeris". It returns the position and velocity at some time t.

```python
def Ephemeris(t, OBJdata, mu):
    time, a, e, i, Omega, omega, mean_anamoly =
        OBJdata[0:7]
    nu_t = (mu / (a**3))**0.5
    t = t - t_0_days*days_convert
    mean_anamoly_t = mean_anamoly + nu_t * (t)

    h = np.sqrt(mu * a * (1 - e**2))
    theta_var = Kepler(e, mean_anamoly_t)
    r = a*(1-(e**2))/(1 + e*np.cos(theta_var))
    arr_r = np.array([r*np.cos(theta_var),
        r*np.sin(theta_var), 0])
    arr_v = (mu/h)* np.array([-np.sin(theta_var), e +
        np.cos(theta_var), 0])
    R_matrix = rotation_matrix(i, Omega, omega)
    r_ijk = R_matrix @ arr_r
    v_ijk = R_matrix @ arr_v

    return r_ijk, v_ijk
```

Using this code we can calculate the position and velocity vectors in the Sun's frame of reference.

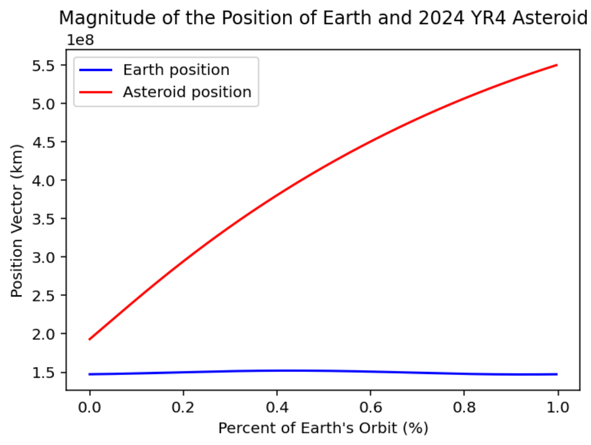


Figure 1: Position vectors of Earth and the Asteroid for a full Earth orbital period

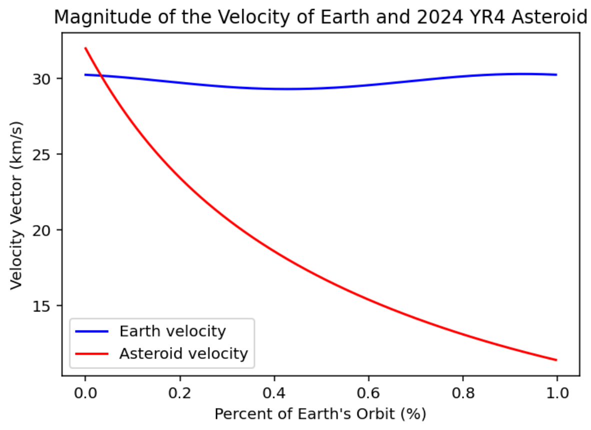


Figure 2: Velocity vectors of Earth and the Asteroid for a full Earth orbital period

Next we can plot the seperation of the two bodies over ten years. Doing this we get the following graph

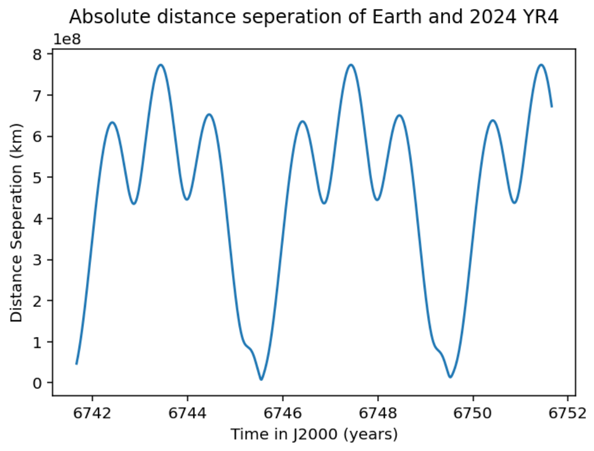


Figure 3: Distance between Earth and 2024 YR4 asteroid in kilometers

We can see how the distance is sinusoidal in nature and repeats in an oscillatory fashion. This graph shows three distinct peaks, where the middle one is the greatest. We can examine the sydonic period of the two bides by examing the equation of the two periods $\frac{1}{T_{syd}} = |\frac{1}{T_{Earth}} - \frac{1}{T_{Asteroid}}|$. And using the orbital periods, we can see that roughly every 1.33 years, the two bodies are at their closest approach. This lines up well with the graph produced, showing the magnitude in their seperation.

1.2 Numerical Integration

To derive the necessary state function we have:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{1}$$

From here we know that $\frac{dv}{dt} = \dot{r}$ giving:

$$\frac{d\mathbf{v}}{dt} = -\frac{\mu}{r^3}\mathbf{r} \tag{2}$$

Expanding each vector as three-dimensional components in x, y, z:

$$\frac{dx}{dt} = v_x, \frac{dy}{dt} = v_y, \frac{dz}{dt} = v_z \tag{3}$$

$$\frac{dv_x}{dt} = -\frac{\mu}{r^3}x, \frac{dv_y}{dt} = -\frac{\mu}{r^3}y, \frac{dv_z}{dt} = -\frac{\mu}{r^3}z \tag{4}$$

Where $r = \sqrt{x^2 + y^2 + z^2}$
We can now define a state vector $\bar{\mathbf{X}}$

$$\bar{\mathbf{X}} = \begin{bmatrix} x \\ y \\ z \\ v_x \\ v_y \\ v_z \end{bmatrix} \tag{5}$$

Finally, deriving this state vector gives the following:

$$\dot{\bar{\mathbf{X}}} = \begin{bmatrix} v_x \\ v_y \\ v_z \\ -\frac{\mu}{r^3}x \\ -\frac{\mu}{r^3}y \\ -\frac{\mu}{r^3}z \end{bmatrix} \tag{6}$$

Now we can use a function to define the right-hand side of this equation

```
def TBP_ECI(t, state_X, mu):
    x, y, z, vx, vy, vz = state_X # Unpack state
        vector
    r = np.sqrt(x**2 + y**2 + z**2) # Compute radius
    ax, ay, az = -mu * x / r**3, -mu * y / r**3, -mu
        * z / r**3 # Acceleration components
    return [vx, vy, vz, ax, ay, az] # Return
        derivatives
```

With this function we can use *SciPy's* integration feature with solve_ivp. We can pass through a set of initial conditions: a position and velocity vector. It passes through the gravitational parameter, μ as an argument when solving the differential system. Furthermore, it uses the Runge-Kutta 45 method to integrate.

```
r0 = np.linalg.norm(X0[:3]) # Initial distance from
     Earth's center (km)
v0 = np.linalg.norm(X0[3:]) # Initial speed (km/s)
a = 1/(2/r0 - v0**2/mu_earth) # Semi-major axis (km)
T = 2 * np.pi * np.sqrt(a**3/mu_earth) # Orbital
     period (s)

# Set integration time span for two orbital periods
t_start = 0
t_end = 2 * T # Two orbital periods
time_step = 10 # Output every 10 seconds
t_eval = np.arange(t_start, t_end, time_step)

# Solve the system using solve_ivp with strict
     tolerances
solution = solve_ivp(
    TBP_ECI, (t_start, t_end), X0, t_eval=t_eval,
        method='RK45',
    args=(mu_earth,), rtol=1e-12, atol=1e-12
)

# Extract components
x, y, z = solution.y[0], solution.y[1], solution.y[2]
vx, vy, vz = solution.y[3], solution.y[4],
     solution.y[5]
```

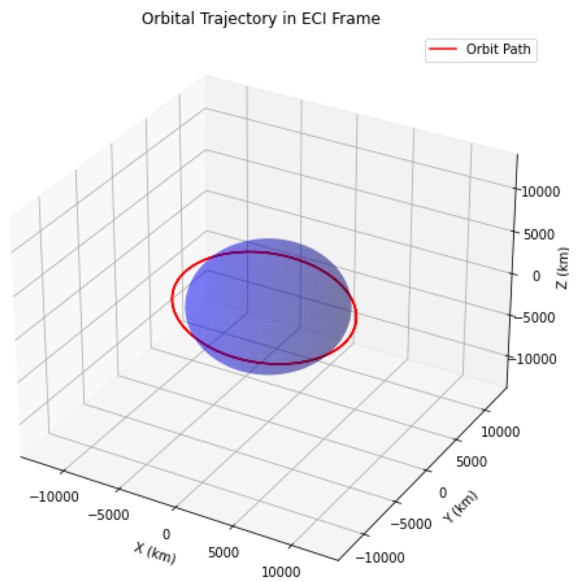


Figure 4: Orbital trajectory in ECI frame with initial conditions given from X_0

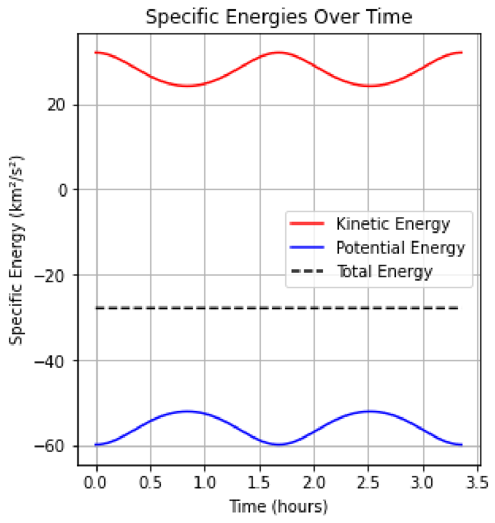


Figure 5: How Kinetic, Potential, and Total energies change throughout the orbit

This graph shows the sinusoidal nature of the different specific energies. Importantly, we can see how the kinetic and potential energies always sum to the same value. This ensures that the total energy remains constant, which we would expect as this is a closed system with no external forces.

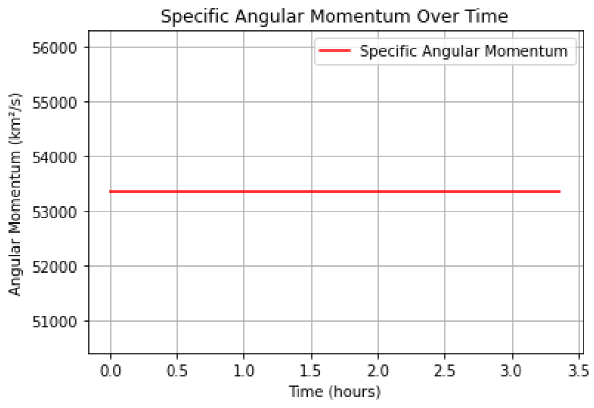


Figure 6: Graph shows the change in the angular momentum over the orbit is zero

Here we see this graph depicting how the angular momentum changes over time. It shows how the angular momentum stays at a constant value and remains constant. This makes sense because the angular momentum is defined as the cross product between the position and velocity vector.

$$\frac{dh}{dt} = \frac{d}{dt}(\bar{r} \times \dot{\bar{r}}) \implies \frac{dh}{dt} = \bar{r} \times \frac{d\dot{\bar{r}}}{dt} \qquad (1)$$

$$\frac{dh}{dt} = \bar{r} \times -\frac{\mu}{r^3}\bar{r} \implies \frac{dh}{dt} = 0 \qquad (2)$$

So yes, these diagrams make physical sense, based upon the conservation laws of both Energy and Angular Momentum

2 Orbital maneuvers and mission design

2.1 Reachability Analysis

```
def RV2COE(state_x, mu):
    r_vec = state_x[0:3]
    v_vec = state_x[3:6]
    r_mag = norm(r_vec)
    v_mag = norm(v_vec)

    a = r_mag / (2 - (r_mag*v_mag**2/mu))
    h = np.cross(r_vec, v_vec)
    h_mag = norm(h)
    e_vec = np.cross(v_vec, h) / mu - r_vec / r_mag
    e_mag = norm(e_vec)
    i = np.arccos(h[2]/h_mag)

    n_vec = np.cross(k_hat, h)
    n_mag = norm(n_vec)
    n_hat = n_vec / n_mag
    Omega_raan = np.arccos(n_hat[0])
    if n_hat[1] < 0:
        Omega_raan = 2*np.pi - Omega_raan

    omega = np.arccos(np.dot(n_vec, e_vec)/(n_mag *
         e_mag))
    if e_vec[2] < 0:
        omega = 2*np.pi - omega

    cos_theta = np.dot(r_vec, e_vec) / (r_mag * e_mag)
    cos_theta = np.clip(cos_theta, -1.0, 1.0)
    theta = np.arccos(cos_theta)
    if np.dot(r_vec, v_vec) < 0:
        theta = 2*np.pi - theta

    return np.array([a, e_mag, i, Omega_raan, omega,
         theta])
```

Using this function, we can report the COE state for the initial condition. Inputting the vector $\bar{X}$, returns the following elements.

$$\textbf{At} \quad X_0\text{: COE} = \begin{bmatrix} a = 7.17813700\text{e}{+03} \\ e = 7.00000000\text{e}{-02} \\ i = 1.67551608\text{e}{+00} \\ \Omega = 3.49065850\text{e}{-01} \\ \omega = 7.85398163\text{e}{-01} \\ \theta = 6.28318529\text{e}{+00} \end{bmatrix}$$

```
def rotate_matrix(state_x):
    r_vec = state_x[0:3]
    v_vec = state_x[3:6]

    r_hat = r_vec/(norm(r_vec))


    h_vec = np.cross(r_vec, v_vec)
    h_hat = h_vec / np.linalg.norm(h_vec)
    t_hat = np.cross(h_hat, r_hat)

    rotation = np.column_stack((r_hat, t_hat, h_hat))

    return rotation
```

With this rotation matrix now defined at every point along the orbit, I can define a function named "impulse". This function takes a rotation matrix, direciton and an initial state as input parameters. It calculates the impulse in the direction of either, radius, transverse or normal (which are calculated via 3-D basis vectors). Then the rotation matrix is applied to the impulse to convert it from the RTN frame to the ECI frame. It adds this vector to the velocity components of the state and then calculates the orbital elements of this final state and the inputted initial state to find the difference between components.

```
def impulse(r_matrix, direct, initial_state):
    dv_ = np.dot(delta_v, direct)

    impulse_eci = r_matrix @ dv_

    state_final = initial_state.copy()
    state_final[3:] += impulse_eci
    oElements_initial = RV2COE(initial_state,
         mu_earth)
    oElements_final = RV2COE(state_final, mu_earth)

    coe_diff = oElements_final - oElements_initial

    return coe_diff, oElements_final
```

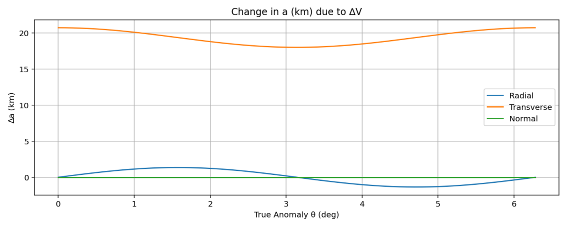
Using these combinations of functions we can apply an impulse in the three different directions and calculate the changes.

Impulse Type	Δa	Δe	Δi
Radial	0.012927	1.28e−5	0
Transverse	20.7374	0.00267900	0
Normal	0.012927	1.67e−6	8.85e−4

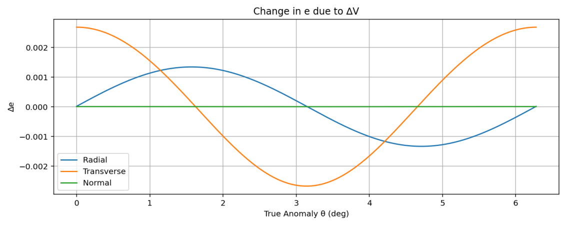
Table 1: Changes in the semi-major axis, eccentricity and inclination

$\Delta\Omega$	$\Delta\omega$	ΔM
0	-0.0191214	-6.26406
0	0	0.0000000149012
0.000889606	0.0000933805	0.0000000149012

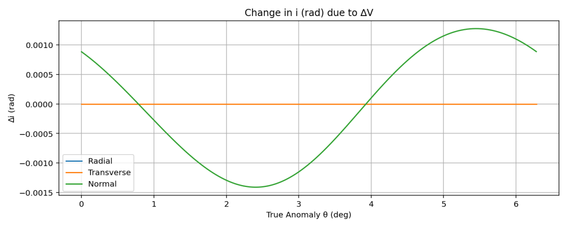
Table 2: Changes in Ω, ω and mean anamoly



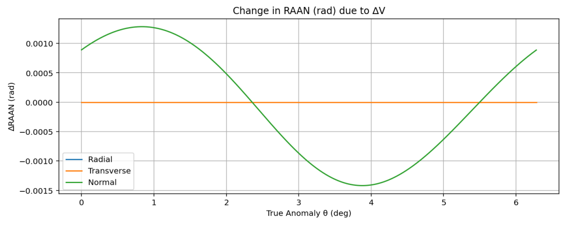
(a) Graph 1



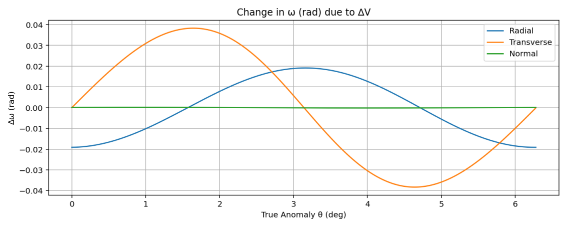
(b) Graph 2



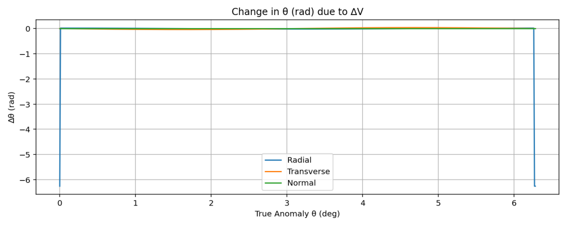
(c) Graph 3



(d) Graph 4



(e) Graph 5



(f) Graph 6

Figure 7: Six graphs showing how the classical orbital elements change with impulses

Using these graphs we can examine which θ values maximise Δi. Using the Signal library apart of Python and its corrosponding find_peaks_cwt function, the peaks of Δi are -0.0014 and 0.0013 radians. These corresponding θ values are 2.36 and 5.46 radians. Understanding that the argument of latitude, u, is defined as $\theta + \omega$, and observing that at these theta values, $\omega = 0.79$ and 0.79 radians. Hence, $u = 3.145$ and 6.245 radians. These numbers represent the apogee and perigee respectively (which make sense because the perigee and apogee occur at π and 2π). Therefore, for maximum change to the inclination, one should provide a normal impulse at the perigee and apogee.

2.2 Lunar Mission Design

Now we can analyse what happens when the apogee of the transfer orbit is increased passed the orbital radius of the moon. By increasing this distance we are completeing a non-tangential transfer, where the mission will have a lower transfer time, but will arrive with some non-zero radial velocity.

Using numpy's matrix-oriented math in Python, calculations can be swift looking at the range of apogee values.

```
parking_altitude = 220
parking_radius = parking_altitude + radius_earth

radius_apogee = r_mean*np.arange(1.1, 1.4, 0.01)
semi_major_axis = 0.5*(parking_radius + radius_apogee)
transfer_e =
    (radius_apogee-parking_radius)/(radius_apogee+parking_radius)
cos_theta_A = (semi_major_axis * (1 - transfer_e**2)
    / r_mean - 1) / transfer_e
theta_2 = np.arccos(cos_theta_A)
semi_latus_rectum = semi_major_axis * (1 -
    transfer_e**2)

v_radial = np.sqrt(mu_earth / semi_latus_rectum) *
    transfer_e * np.sin(theta_2)
v_transverse = np.sqrt(mu_earth / semi_latus_rectum)
    * (1 + transfer_e * np.cos(theta_2))
normed_apogee = radius_apogee/r_mean
```

Now we have the true anamoly, radial, and transverse velocities inside a matrix array for every apogee value normalized to the moon's orbital radius. Printing these components, we can see how these change as we increase the distance.

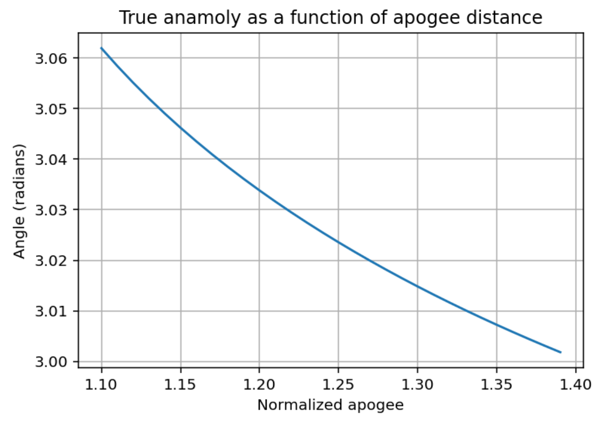


Figure 8: Graph depicting how the true anamoly decreases as the apogee distance increases

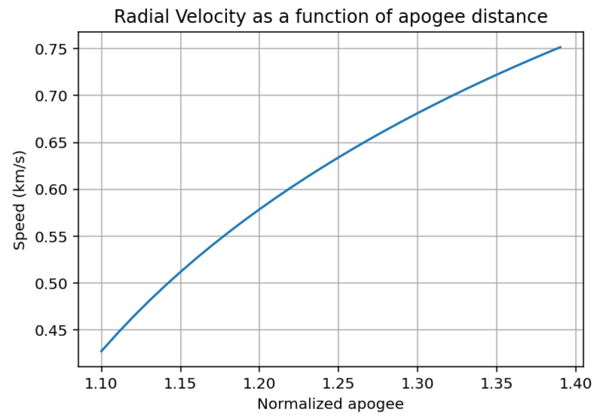


Figure 9: Graph depicting how the radial velocity increases as the apogee distance increases

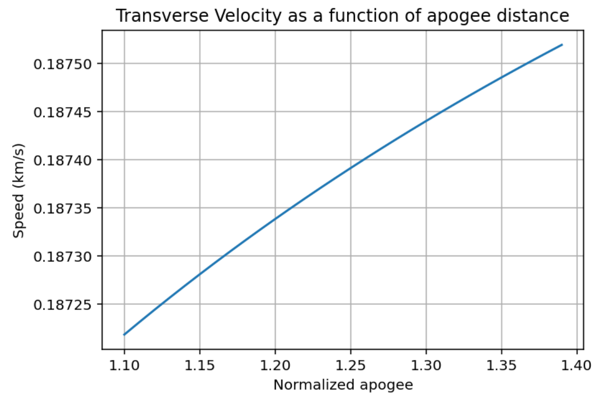


Figure 10: Graph depicting how the transverse velocity increases as the apogee distance increases

We can see from these figures how the radial component of the velocity changes much more significantly that the transverse component. This is because the radial component is related to the orbital energy equation. When the semi-major axis is significantly increased, to reach beyond the Moon, most of that additional energy manifests as radial velocity because it directly contributes to increasing the orbital energy and apogee height. By comparison, the transverse velocity is primarily responsible for maintaining orbital angular momentum, so does not change as violently.

Next we can observe how the apogee and transfer time are related. Writing some more code we can examine this, again using numpy's matrix operations.

```
theta_eccentric = 2*np.arctan(np.tan(theta_2/2) *
     ((1-transfer_e)/(1+transfer_e))**(0.5))
theta_mean = theta_eccentric -
     transfer_e*np.sin(theta_eccentric)
time_total = time_orbit(semi_major_axis, mu_earth)
delta_t = (time_total / (2 * np.pi)) * theta_mean /
     days_convert
```

This gives the transfer time for every apogee value, which we can plot as the following.

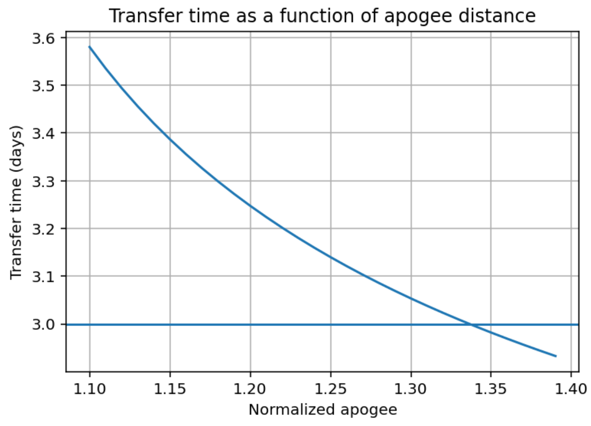


Figure 11: Graph showing how the transfer time decreases as the apogee value increases

This plot clearly shows how the transfer time decreases as the apogee value increases. This makes sense because increasing the apogee point, means the spacecraft is intercepting the moon at an earlier stage of its orbit. There is a horizontal line corrosponding to $t = 3$ days. We can look at the intercept of this line with the plot to see that for a transfer time of three days, an apogee value of $1.34 \cdot r_m$ is required

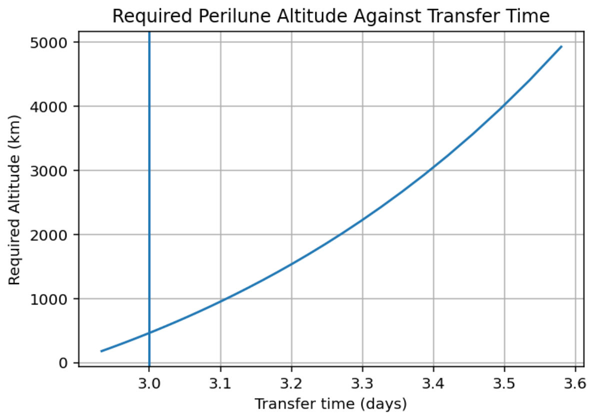


Figure 12: Distance between Earth and 2024 YR4 asteroid in kilometers

3 Appendix