Title:

A Novel Approach To Modelling Voluntary Carbon Credit Prices Using A Two Sector Schumpeterian Growth Model And A Jump-Diffusion Process.

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GitHub Repo: https://github.com/Alex-georgalli/VCU-Futures-Pricing-Model

Abstract:

The voluntary carbon market can play a vital role for the transition towards a greener economy and achieving the Paris Agreement goals by increasing funding into investments related to halting climate change. For the market to reach its full potential, however, understanding the underlying dynamics driving the price of reference voluntary carbon credits will be crucial, both for solidifying confidence in the market, and attracting the necessary liquidity providers which will be needed for its operation at a large scale. In the paper I formulate a hypothesis regarding the price evolution of voluntary carbon credits. The contribution is based on linking the expected returns of a reference voluntary carbon credit to the R&D sector and implementing the result using a jump-diffusion model to capture jumps in prices driven from news releases concerning the market. The model simulation suggests improved fit compared to a pure jump-diffusion model when optimized to real voluntary carbon credit price data and provides potential insight on the underlying dynamics of the trends evident in the market.

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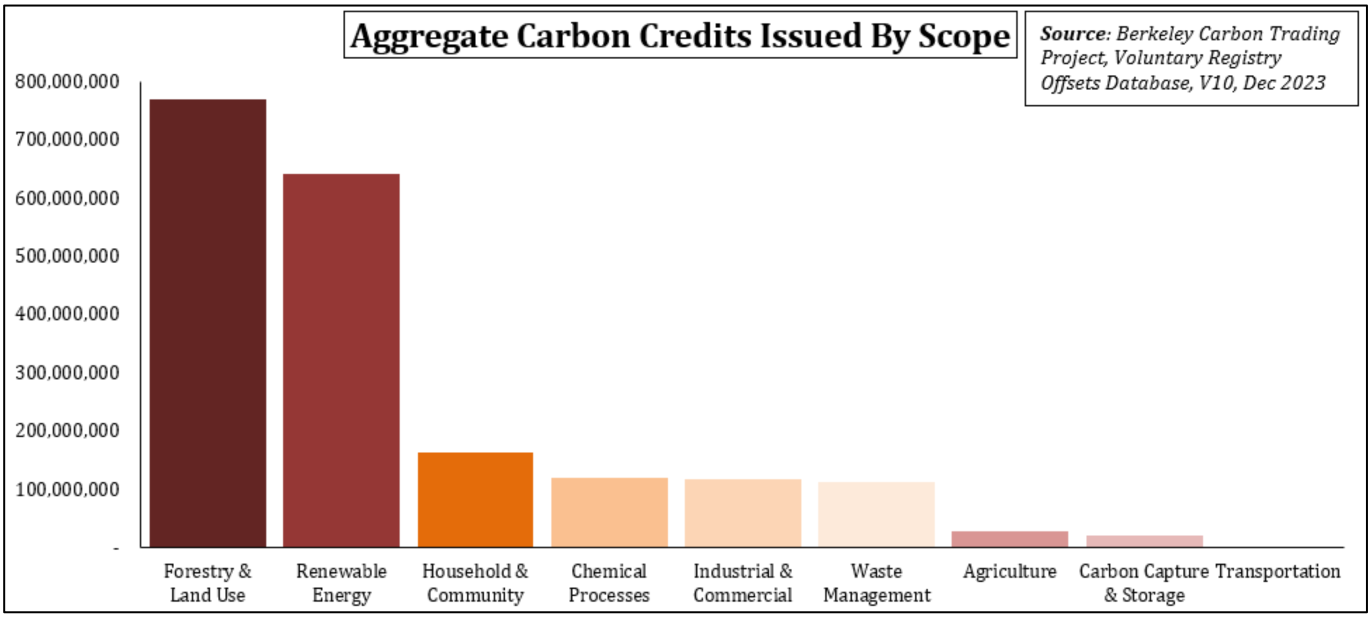
2.1. Voluntary Carbon Market (VCM) Primer:

It is estimated that to achieve the Paris Agreement goals signed in 2015, where it was agreed to limit global warming to 1.5 degrees Celsius, the world must reach net-zero emissions by 2050. As a result of this, an increasing number of firms have made commitments to achieve their own net-zero targets *(TSVCM, 2021)*. By the end of 2022, the cumulative number of companies with validated science-based targets totalled 2,079, with a further 2,151 having commitment to set their targets (SBTi, 2023).

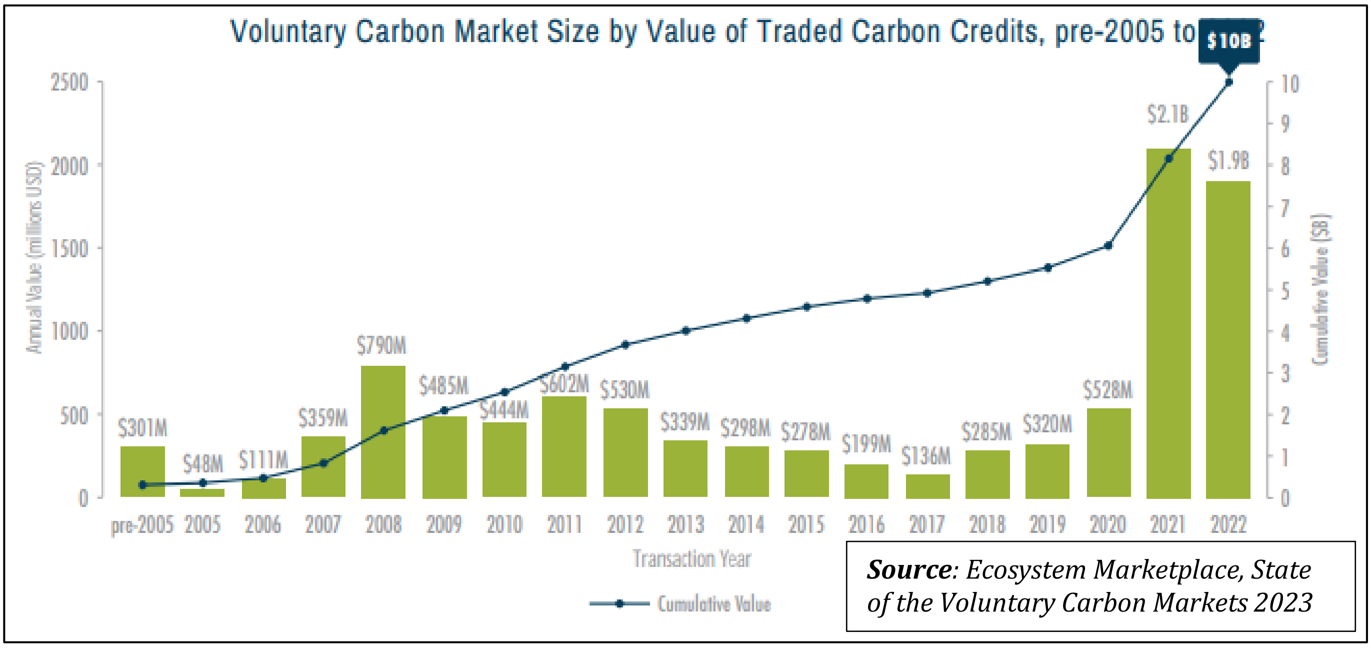
Carbon credits are primarily purchased and retired voluntarily by organizations to compensate (offset) the emissions which they have not yet eliminated from their operations. These voluntary carbon credits are tokens representing the avoidance or removal of greenhouse gas emissions *(CCC, 2022)*. After the Kyoto Protocol in 1997, international carbon credits were made available for purchase under the Clean Development Mechanism and the Joint Implementation mechanism. These mechanisms allowed host countries to sell the credits generated from their emissions reduction projects to other countries, with the credits counting towards the purchasing countries' emissions reduction targets *(CCC, 2022)*. Stemming from the basis of this, independent organisations sprouted in the 1990s. In modern time, the primary independent organizations are Verra (VCS), Gold Standard (GS), Climate Action Reserve (CAR) and American Carbon Registry (ACR) *(CCC, 2022)*. These are all carbon offset registries which develop standardized protocols for carbon projects and act as a third party between project developers and buyers attempting to validate that the offsets sold to buyers achieve their promised environmental impact *(Sullivan, CarbonBetter, 2022)*.

Carbon credits are categorized into two main baskets. Namely, reduction/avoidance and removal credits. The reduction/avoidance credits are generated from projects which either prevent the release of emissions into the atmosphere or reduce the quantity of emissions which are being released. The main credit types included here are nature-based solution (NBS) projects such as Reducing Emissions from Deforestation and forest Degradation (REDD+) and Improved Forest Management (IFM) projects. Removal credit types directly extract emissions such as C02 from the atmosphere and include, but are not limited to, Afforestation, Reforestation, and Revegetation (ARR) projects and Direct Air Carbon Capture and Storage (DACCS).

Most of the carbon credit issuance supply in recent times has come from NBS and Forestry & Land Use projects, a shift from the historic dominance of renewable energy projects. Forestry projects account for ~40% of the aggregate credits issued historically as of December 2023. The following figure displays the total historic issuance of credits broken down by project scope.



Demand in the voluntary carbon market primarily originates from the private sector attempting to fulfil the aforementioned net-zero commitments. The main sectors these participating institutions belong to are financial services, oil and gas, and consumer goods *(CCC, 2022)*. Ecosystem Marketplace (*2023*) estimates the monetary value of credits traded in the VCM to have been $1,873 Million during 2022 - a substantial increase from a decade prior where the trade value was recorded as $530 Million. Despite this sharp rise, the VCM is still relatively small when compared to the size of compliance schemes such as the EU ETS, but also the projected size the market would need to reach if the global economy is to stay on track to achieve the Paris Agreement goals (*CCC, 2022*). The following figure taken from Ecosystem Marketplace *(2023)* illustrates the evolution of the market from pre-2005 up to 2022.



2.2. Motivation:

Despite the voluntary carbon markets (VCM) not being nascent, their growth in market size is recent. The compliance markets have had a plethora of studies and research papers both statistically modelling their prices and fundamentally modelling their expected returns and dynamics. This research will be mentioned in the literature review section. To my knowledge, however, no one has attempted to provide an explanation of the price dynamics evident in the VCM, nor a fundamental model for what is driving their returns. The motivation for attempting to do so in my paper is as follows.

The VCM has the potential to play a sizable positive role in the transition towards a greener economy. This is simply because, even if immense efficiency improvements were made in all polluting operations, there would still be a proportion of emissions which are simply unavoidable. Despite this, the VCM has been plagued with various issues. The most frequently quoted reason for the low level of confidence in the VCM and its slow historic evolution has been centred around quality and transparency issues regarding the credits being issued. To determine the quantity of emission reductions or removals being generated from a carbon project, protocols and methodologies developed from the aforementioned carbon credit registries such as Verra and Gold Standard are deployed. There has been substantial scrutiny around their accuracy and soundness, however, with various articles having been written describing the occurrence of potential over-crediting or just simply poor-quality credits. The impact this has had on the market is substantial, as it has made buyers of these credits fearful of potentially being accused of greenwashing, if credits they had bought and retired were later discovered to be invalid *(Bain & Company, 2023)*.

Besides the quality and transparency concerns regarding these credits however, liquidity in the VCM has also been a major obstacle preventing substantial inflow of institutional capital. Despite the broad variety of carbon credit types, with categorisations including vintages, scope (e.g., Nature-Based, Renewable Energy), removals or avoidance, etc., reference contracts can, and have been, developed. These reference contracts are traded on various exchanges and are usually composed of a basket of carbon credits originating from different projects with some specific characteristics in common. These contracts are bought, sold, and traded both in the spot and forward markets. Both markets offer utility to buyers. More specifically, buyers who want to make yearly purchases to compensate for their current or prior year's emissions, can access the contracts offered on the spot market. Buyers which would like to compensate for their emissions trajectory considering a multi-year forward outlook can access the forward market, helping them manage future price risks *(TSVCM, 2021)*. However, the primary volume of carbon credits is, and will likely continue to be, traded in the over the counter (OTC) market. This is due to the unique and differentiating characteristics of credits originating from specific projects which are hard to encapsulate in standardized contracts and for which buyers have a tendency of specifically seeking out for. A liquid market for the standardized reference contracts with transparent prices, however, will still benefit these OTC transactions. This is because bespoke OTC transactions could use the price of these core carbon contracts as a baseline on which further project specific attributes can be accounted for and priced appropriately *(TSVCM, 2021)*. Therefore, these standardized exchange traded credits can directly aid in improving the pricing efficiency of the credits being settled in the OTC market. Further, an active secondary market for these credits would allow market participants to manage and hedge their risk exposures. In particular, such liquid markets would support longer-term financing for project developers and allow buyers to manage risks that arise from carbon reduction commitments *(TSVCM, 2021)*.

Despite this, these exchange traded contracts are not currently regarded as transparent price signals for the VCM as a whole. In large, this is contributed to the lack of liquidity in these markets driving inefficiencies. Market makers and similar market participants are not present in these markets due to various barriers they face in navigating the VCM *(TSVCM, 2021).* Their presence, however, could allow for a substantial concentration of liquidity, unlocking the benefits mentioned along with it.

Providing a fundamental way of pricing these reference voluntary carbon credit contracts is one of the multiple steps which will be needed to attract liquidity providers into the VCM. Liquidity in the VCM at scale could allow billions of dollars of capital to flow from those making commitments into the hands of those with the ability to reduce or remove carbon (TSVCM, 2021). Wholistically, this is the motivation and intention of the model developed in this paper.

2.3. Paper Introduction:

The model introduced is fundamentally based on one core idea. As the mentioned in the primer, the main source of demand for voluntary carbon credits originates from institutions trying to meet their emission commitments. There is two ways in which institutions can reduce their claimed net carbon emissions which are assumed to be equivalent. Namely, directly reducing these emissions from their operations, with methods such as supply chain management, efficiency improvements in their production process etc, and the purchasing and retirement of carbon offsets. Both methods are costly. Direct reduction of emissions in the production process will require R&D expenditure, with the most prominent example which can be used to display this being petroleum/oil companies investing heavily in research and development of greener methods of operations and products. Carbon offsets on the other hand, have a set price which must be paid for them to be acquired. Since these are two distinct ways of achieving a theoretically identical outcome by the end buyer, the costliness of both methods should be fundamentally interlinked in some way. Further, in a perfectly efficient market, I propose that the cost of these two methods should be identical. This is simply an extrapolation of the efficient market hypothesis. To be more specific, if we assume perfect transparency and quality of carbon credits, an institution attempting to reduce their net carbon footprint would simply choose the most cost-effective way of doing so. If for example carbon offsets are relatively cheaper than R&D, then rational agents should drive up demand for these credits, consequently arbitraging away their cost-efficiency in the process until the prices of these two solutions equalise.

In addition to this, another core idea explored in the paper is that the commitments which these institutions make are not driven by altruism and the "warm glow" effect as explored by Andreoni, 1988. Instead, this is a profit maximizing decision. The choice of R&D expenditure has been deeply explored with Schumpeterian growth model literature. An extension of this model will be used as the starting point following Bezin, 2019, with further modification. From this, I will derive the change in the relative profit between the "clean" and "dirty" sectors for a change in the relative R&D expenditure of these two sectors. This will be assumed to be the expected return of reference carbon credit contracts. The price evolution of these credits will be modelled as a jump diffusion process, introduced by Merton 1976, and following the version deployed by Song Et al., 2019, but having the expected return parameter be replaced with the equation described above. Three additional modifications of the final price evolution equation will also be included for comparison. This will include the original model but modelled as a pure diffusion process (excluding jumps). Additionally, standard pure diffusion and jump diffusion models will be simulated for which the expected carbon return is not based on economic model developed in the paper. These models will all be simulated using Monte Carlo, and, deploying the Powell method, optimized based on real carbon-credit data.

3. Literature Review:

The literature on carbon markets is rich with vast literature having been written regarding carbon trading schemes and traditional compliance schemes in the carbon markets. This literature includes both more fundamentally based approaches for determining the prices or returns for various carbon spot or futures contracts but also empirical studies and forecasting models of various complexities. Despite the compliance carbon markets not being the direct focus of this paper, understanding the work that has been done regarding them will provide useful insight on approaches that could also be deployed for studying the VCM.

First however, I mention the work that has been done for the VCM specifically. For example, empirical studies have been performed using hedonic price functions trying to identify the factors that could explain the large variabilities evident in the prices of different voluntary carbon credits. For example, Conte and Kotchen, 2010, find that providers located in Europe sell offsets at prices approximately 30% higher than their North America or Australasia counterparts, as well as premiums for projects located in developing or least-developed nations, and differences depending on third-party certifications. Other works has been done regarding consumer' willingness for carbon offsetting, such as Tao et al., 2021, and exploring the link between voluntary carbon disclosures and firm value, such as Sun et al., 2022, however, this work is not directly related to this paper. For this reason, I turn to compliance markets for further insight.

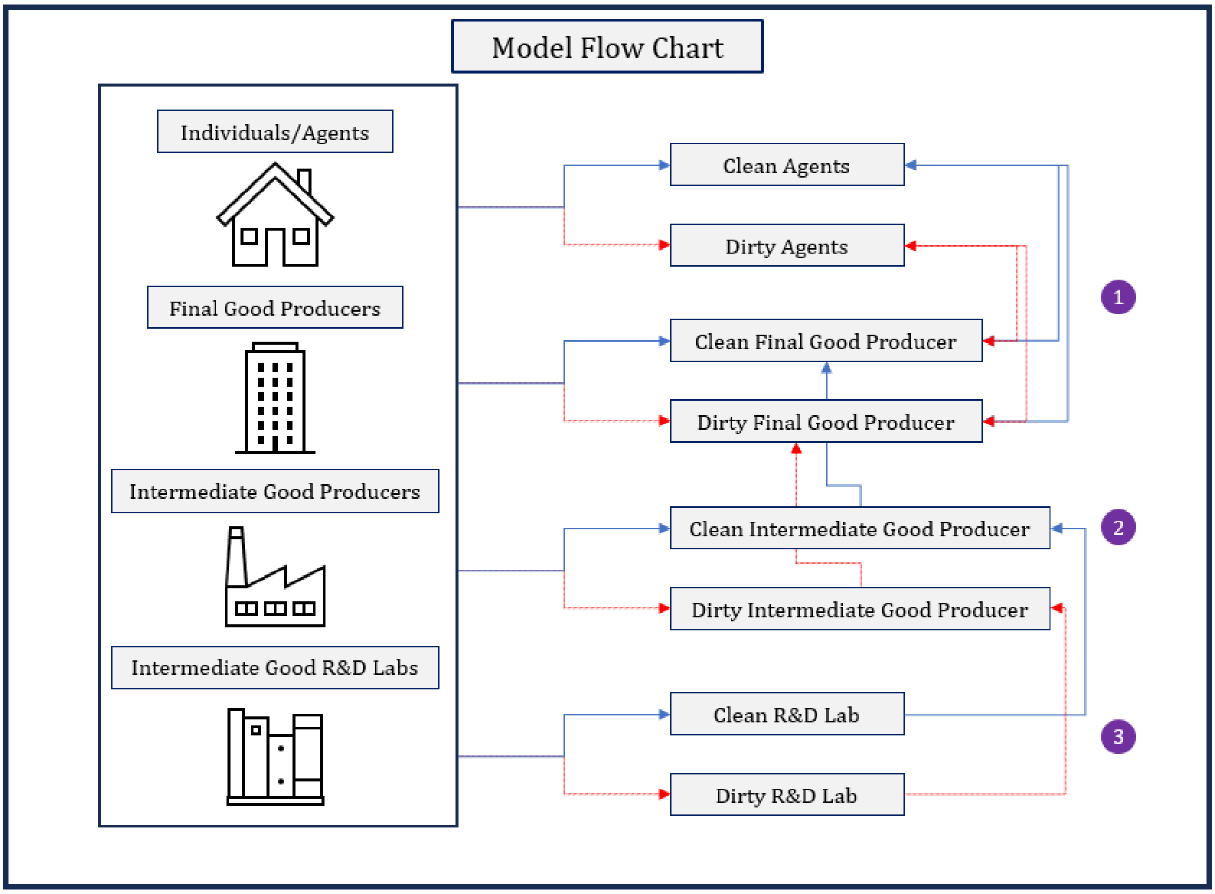
Machine learning and advanced statistical techniques have been used in the forecasting compliance credit prices increasingly in recent times, building from the more traditional approaches used up to now. For example, Zhou et al., 2022, deploy neural network approaches for one-step-ahead based forecasting of the Guangzhou Emissions Trading Scheme (ETS) closing prices based on Complete Ensemble Empirical Mode Decomposition (CEEMAN) and Long Short-Term Memory (LSTM) models. They find that the LSTM model combined with Variational Modal Decomposition (VMD) to be stable and reliable while providing the best average R^2 of 0.982. Along this line of work, Li et al., 2022, note that the carbon price is affected by numerous factors such as policy, macroeconomic environment, energy prices and climate change itself. Based on this, they build from the established literature for solve this multi-factor prediction problem and utilize intrinsic mode function decompositions of the data and deploy a particle swarm optimized extreme learning machine. They predict the prices of carbon credits in various Chinese compliance schemes and present improved prediction accuracy. Ren et al., 2022, propose and test Quantile Group LASSO and Quantile Group SCAD models utilizing a large group of factors for forecasting carbon futures returns, finding superior predictive performance compared to a variety of more traditionally used GARCH, ARMAX, ARMAXS and GARTCHXS models. Through this, they also find that the Brent spot price, the crude oil closing stock in the UK, and the growth of natural gas production in the UK to impact the carbon futures returns in extreme conditions. Chevallier, 2011, takes a non-parametric approach for modelling the price of carbon and specifically BlueNext spot and ECX future contracts, with a focus on the nonlinearities evident in empirical data. Chevallier finds the conditional mean functions of spot prices to be nonlinear, while his use of conditional volatility functions highlights and capture the strong heteroskedastic and asymmetric behaviour in the prices, matching the parametric literature. Adekoya, 2021, takes on a novel approach for predicting the EU carbon allowance prices using a variety of different energy (oil, gas, coal) prices. Adekoya find robust out-of-sample prediction results and concludes that the forecast of future carbon allowance prices should not only encapsulate its past values, but in addition, the energy markets. In summary, empirical literature regarding the compliance carbon markets has managed to produce very accurate forecasting models for the credit prices. However, this empirical work is for the most part focused on analysing statistical relationships, with little being said regarding the fundamental underlying dynamics driving these relationships.

Stochastic jump-diffusion models have also been deployed in compliance carbon credit literature. Borovkov et al. 2011 is an example of this, where they derive and provide a numerical example of a jump-diffusion model with a compensated Poisson process to evaluate the evolution of the underlying carbon credits. Further, Pan et al., 2022, focus on the actual detection of carbon asset price jumps driven from information shocks. They do so by improving on the tradition LM test method utilizing double parameter optimization and their results display that the downside risks in the carbon markets are significant. Song et al., 2019, take a focus on the more nascent Chinese compliance carbon market for which they develop a stochastic model with jumps applied for price prediction. However, the main differentiating factor here is their analyses of different allowance policies and the affects they would have on prices. To do so, they develop a model for the final corporate profit for a firm involved in the market based on trading costs and income. They conclude that their model can provide a robust theoretical strategy implication for firms involved in the market and demand-related policy makers. This is not the only example of more theoretically based models. Mirzaee et al., 2022, use game theory to capture the interactions and motivations between players in cap-and-trade emission schemes based on profit-maximizing strategies. Further, they incorporate stochastics to better describe the reality of choices under uncertainty and the fact that agents must make complicated choices with conflicts of interest among different parties. Yong et al., 2024 follow in a similar vein of tackling the carbon market using game theory and rent-seeking behaviours, with an analysis of carbon emission verification. They analyse the incentives and condition required for falsely reporting carbon emission and further incorporate prospect theory to introduce bounded rationality in the participants. Together, this work provides a closer theory for understanding the price evolution dynamics for carbon credits. Jump-diffusion models explain the evolution as a continuous stochastic prosses with the use of jumps being justified as substantial information shocks in the market. The theoretical piece in the case of Song et al., 2019, but also the game-theoretic approaches help justify the underlying behaviours and motivations of the market participants within these markets painting a clearer picture of the underlying mechanics.

Based on this, I will deploy a jump-diffusion model using a compound Poisson process as is done by Song et al., 2019. My contribution to this will be endogenizing the expected return parameter of this model in a way that is suitable for describing the dynamics of the voluntary carbon market specifically. The endogenization will be done using a modified version of the two sector, Schumpeterian growth model used by Bezin, 2019, in an attempt to explain the evolutionary price path of voluntary carbon credits as a series of utility and profit maximizing exercises of rational market agents.

4. The Economic Model:

I would like to explicitly state that the model shown here is following Bezin, 2019. Bezin, 2019, deploys this model to explore policy implications regarding economic growth and environmental preservation with heavy focus being placed on the theory of green preferences derived from a cultural transmission mechanism. For the scope of this paper, the cultural transmission mechanism is switched off to reduce complexity. Further, as will be shown, I deviate from Bezin, 2019, by implementing R&D expenditure into the R&D sector's profit maximization problem, while also redefining the equation for the evolution of environmental quality. This is done to form a relationship between R&D spending, the profit that this spending generates, and the environmental impact it indirectly causes. I will suggest that the relationship between these



three components to be what governs the expected return on voluntary carbon credits. The following flow Chart is used to give a break-down of the sectors of the economic model which will be deployed.

The economy consists of four distinct elements. Namely, individuals - which are the consumers of the economy's final output - the final good producers, the intermediary good producers - which produce and provide intermediary goods to the final good producers - and the R&D labs. The R&D labs are responsible for innovations in this economy. The exact meaning and structure of this is explained in the next subsections.

4.1 Demand for clean and dirty goods:

Let as suppose that we have two, final output, consumption goods, x_{jt}, where $j = (c, d)$, with c goods corresponding to "clean" goods and d goods denoting "dirty" goods. Further, assume our model economy has two types of agents, G and B which denotes "good" and "bad" respectively. Both these agents consume a combination of clean and dirty final consumption goods. This consumption is set as the sole variables of their utility function, with their utility functions being increasing in with the consumption of both, at a diminishing rate. Their distinguishing characteristic, however, is the marginal utility gained from the consumption of clean goods. G type agents gain a strictly higher utility from the consumption of clean goods than B type agents do, which is represented by θ^G.
We assume the following functional form for the utility of agents type G and B respectively *(Bezin, 2019)*:

$$U^G(x_{ct}, x_{dt}) = \ln\left[\left(\theta^G x_{ct}^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}\right] \quad (1a)$$

$$U^B(x_{ct}, x_{dt}) = \ln\left[\left(x_{ct}^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}\right] \quad (1b)$$

Here, x_{ct} and x_{dt} denote the quantities of clean and dirty goods respectively. We set $\theta^G > 1$, hence imposing the assumption that G type agents have an outsized utility gain from the consumption of x_{ct} compared to B type agents. Here, ϵ is the elasticity of substitution between the clean and dirty good and is defined such that $\epsilon > 1$. Next, we define the budget constraint of these two agents. They both face an identical budget constraint which given by *(Bezin, 2019)*:

$$I_t = p_{ct}x_{ct} + p_{dt}x_{dt} \quad (2)$$

Where I_t denotes the agent's total income and p_{ct} and p_{dt} are the prices of clean and dirty goods respectively, at time t. Rearranging equation (2) and substituting into equations $(1a)$ and $(1b)$ yields an unconstraint optimization problem for the optimal, utility maximizing consumption levels for type G and B agents as follow.

For agent type G we have:

$$U^G(x_{ct}, x_{dt}) = \ln\left[\left(\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}\right]$$

$$x_{dt}^G = \frac{I_t}{p_{dt}}\frac{1}{\left(1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right)} \quad (3a)$$

Substituting $(3a)$ back into the budget constraint (2) yields the optimum level of clean-good consumption for the type G agent:

$$x_{ct}^G = \frac{I_t}{p_{ct}}\left[\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon}{1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon}\right] \quad (4a)$$

The full derivation is provided in the Appendix, part 2.1.

Performing the identical operation for agent type B, we derive:

$$x_{dt}^B = \frac{I_t}{p_{dt}}\left[\frac{1}{1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right] \quad (3b)$$

$$x_{ct}^B = \frac{I_t}{p_{ct}}\left[\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}{1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right] \quad (4b)$$

Let q_t denote the proportion of G agents out of the total agent population, with $(1 - q_t)$ denoting the proportion of B type agents accordingly. Aggregate demand for the clean and dirty goods are denoted X_{ct} and X_{dt} respectively and are simply the population proportionate weighted sum of the optimal consumption levels derived above as follow *(Bezin, 2019)*.

$$X_{ct} = \frac{I_t}{p_{ct}}\left[q_t\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon}{1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon} + (1 - q_t)\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}{1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right] \quad (5a)$$

$$X_{dt} = \frac{I_t}{p_{dt}}\left[q_t\frac{1}{1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon} + (1 - q_t)\frac{1}{1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right] \quad (5b)$$

4.2 Supply of Clean and Dirty Final Goods:

We define the final good production functions as follows:

$$Y_{jt} = \Omega(E_t)L_{jt}^{1-a}\int_0^1 A_{jkt}^{1-a}z_{jkt}^a dk = \left\{\begin{array}{l} Y_{ct} = \Omega(E_t)L_{ct}^{1-a}\int_0^1 A_{ckt}^{1-a}z_{ckt}^a dk \\ Y_{dt} = \Omega(E_t)L_{dt}^{1-a}\int_0^1 A_{dkt}^{1-a}z_{dkt}^a dk \end{array}\right\} \quad (6)$$

First, we assume that there is perfect competition in the final good market. To produce the clean and dirty final output consumption goods, the firms use labour, L_{ct} and L_{dt} respectively, as inputs, as well as intermediate goods, z_{ct} and z_{dt}, respectively which we will interpret as the quantity of machinery used for the production process. We assume that there is a spectrum of sector specific intermediate goods machinery, normalized between 0 and 1, used in the production process, with each specific machine denoted by k having its own respective productivity. A_{ct} and A_{dt} represent aggregate productivities of the clean and dirty intermediate goods respectively and are defined as *(Bezin, 2019)*:

$$A_{jt} = \int_0^1 A_{jkt}\, dk \quad (7)$$

Finally, $\Omega(E_t)$ represents some function of $\Omega(\cdot)$ E_t, which is the stock of environmental quality. It assumed that when $E_t = 0$, $\Omega(E_t) = 0$ implying that output will be 0 in both sectors and is justified with the narrative that no production can happen if environmental quality if completely degraded.
Following, we can set the wage level as the numeraire and derive the first order condition with respect to labour *(Bezin, 2019)*.

$$\frac{\partial Y_{jt}}{\partial L_{jt}} = \begin{Bmatrix} \Omega(E_t)(1-a)L_{ct}^{-a} \int_0^1 A_{ckt}^{1-a}\, z_{ckt}^{a} dk = 1 \\ \Omega(E_t)(1-a)L_{dt}^{-a} \int_0^1 A_{dkt}^{1-a}\, z_{dkt}^{a} dk = 1 \end{Bmatrix} \quad (8)$$

We consider the output produced using machine k defined as:

$$Y_{jkt} = \Omega(E_t) L_{jt}^{1-a} A_{jkt}^{1-a} z_{jkt}^{a} = \begin{Bmatrix} Y_{ct} = \Omega(E_t) L_{ct}^{1-a} A_{ckt}^{1-a} z_{ckt}^{a} \\ Y_{dt} = \Omega(E_t) L_{dt}^{1-a} A_{ckt}^{1-a} z_{ckt}^{a} \end{Bmatrix}$$

The first order condition with respect to z_{jkt} read:

$$\frac{\partial Y_{jkt}}{\partial z_{jkt}} = \begin{Bmatrix} a\Omega(E_t) L_{ct}^{1-a} A_{ckt}^{1-a} z_{ckt}^{a-1} = \dfrac{p_{ckt}}{p_{ct}} \\ a\Omega(E_t) L_{dt}^{1-a} A_{dkt}^{1-a} z_{dkt}^{a-1} = \dfrac{p_{dkt}}{p_{dt}} \end{Bmatrix} = \begin{Bmatrix} a\Omega(E_t) L_{ct}^{1-a} A_{ckt}^{1-a} \dfrac{p_{ct}}{p_{ckt}} = z_{ckt}^{1-a} \\ a\Omega(E_t) L_{dt}^{1-a} A_{dkt}^{1-a} \dfrac{p_{dt}}{p_{dkt}} = z_{dkt}^{1-a} \end{Bmatrix}$$

Where p_{ckt} is the price of machine k. Rearranging, we can see the optimal level of z_{jkt} in equation (9).

$$z_{jkt} = \begin{Bmatrix} \left[\Omega(E_t) \dfrac{a p_{ct}}{p_{ckt}}\right]^{\frac{1}{1-a}} A_{ckt} L_{ct} \\ \left[\Omega(E_t) \dfrac{a p_{dt}}{p_{dkt}}\right]^{\frac{1}{1-a}} A_{dkt} L_{dt} \end{Bmatrix} \quad (9)$$

Further, by equalizing the two prongs of equation (8):

$$\Omega(E_t)(1-a)L_{ct}^{-a} \int_0^1 A_{ckt}^{1-a}\, z_{ckt}^{a} dk = \Omega(E_t)(1-a)L_{dt}^{-a} \int_0^1 A_{dkt}^{1-a}\, z_{dkt}^{a} dk$$

$$L_{ct}^{-a} \int_0^1 A_{ckt}^{1-a}\, z_{ckt}^{a} dk = L_{dt}^{-a} \int_0^1 A_{dkt}^{1-a}\, z_{dkt}^{a} dk$$

Substituting in the optimal z_{jkt} shown in equation (9):

$$L_{ct}^{-a} \int_0^1 \left\{ A_{ckt}^{1-a} \left(\left[\Omega(E_t) \frac{a p_{ct}}{p_{ckt}}\right]^{\frac{1}{1-a}} A_{ckt} L_{ct} \right)^{a} \right\} dk = L_{dt}^{-a} \int_0^1 \left\{ A_{dkt}^{1-a} \left(\left[\Omega(E_t) \frac{a p_{dt}}{p_{dkt}}\right]^{\frac{1}{1-a}} A_{dkt} L_{dt} \right)^{a} \right\} dk$$

$$(p_{ct})^{\frac{a}{1-a}} \int_0^1 \left\{ A_{ckt} \left(\frac{1}{p_{ckt}}\right)^{\frac{a}{1-a}} \right\} dk = (p_{dt})^{\frac{a}{1-a}} \int_0^1 \left\{ A_{dkt} \left(\frac{1}{p_{dkt}}\right)^{\frac{a}{1-a}} \right\} dk$$

And applying equation (7):

$$\frac{A_{ct}}{A_{dt}} \int_0^1 \left\{ \left(\frac{1}{p_{ckt}}\right)^{\frac{a}{1-a}} \right\} dk = \left(\frac{p_{dt}}{p_{ct}}\right)^{\frac{a}{1-a}} \int_0^1 \left\{ \left(\frac{1}{p_{dkt}}\right)^{\frac{a}{1-a}} \right\} dk$$

$$\left(\frac{A_{ct}}{A_{dt}}\right)^{-\frac{1-a}{a}} = \left(\frac{p_{ct}}{p_{dt}}\right) \left[\frac{\int_0^1 \left\{ \left(\frac{1}{p_{dkt}}\right)^{\frac{a}{1-a}} \right\} dk}{\int_0^1 \left\{ \left(\frac{1}{p_{ckt}}\right)^{\frac{a}{1-a}} \right\} dk} \right]^{-\frac{1-a}{a}}$$

$$\left(\frac{A_{ct}}{A_{dt}}\right)^{-\frac{1-a}{a}} = \left(\frac{p_{ct}}{p_{dt}}\right) \left[\frac{\int_0^1 \left\{ p_{dkt}{}^{-\frac{a}{1-a}} \right\} dk}{\int_0^1 \left\{ p_{ckt}{}^{-\frac{a}{1-a}} \right\} dk} \right]^{-\frac{1-a}{a}}$$

$$\left(\frac{A_{ct}}{A_{dt}}\right)^{-\frac{1-a}{a}} = \frac{p_{ct}}{p_{dt}} \left(\frac{p_{ct}}{p_{dt}}\right)^{\frac{1-a}{a}} = \left(\frac{p_{ct}}{p_{dt}}\right)^{\frac{1}{a}}$$

$$\left(\frac{A_{ct}}{A_{dt}}\right)^{-(1-a)} = \frac{p_{ct}}{p_{dt}} \quad (10)$$

We arrive to equation (10), showing that, at the optimum, the prices of the two final goods are inversely proportional to their respective productivities.

4.3 Intermediary good producers:

In the intermediary good sector, each producer of a machine k is the sole producer of that machine in any given period t. The producer of machine k uses ψz_{jkt} units of labour to produce a unit of machinery z_{jkt}, where $0 < \psi < 1$. Given this, and that labour supply is inelastic, labour market clearing requires *(Bezin, 2019)*:

$$L_{ct} + L_{dt} + \psi \int_0^1 z_{ckt}\, dk + \psi \int_0^1 z_{dkt}\, dk \leq 1 \quad (11)$$

Next, we formulate the machine producer's profit as:

$$\pi_{jkt} = p_{jkt} z_{jkt} - \psi z_{jkt}$$

For which we can substitute in the optimal levels of demand from the final good producers derived in equation (9)

$$\pi_{jkt} = \begin{Bmatrix} p_{ckt}\left[\Omega(E_t)\dfrac{ap_{ct}}{p_{ckt}}\right]^{\frac{1}{1-a}} A_{ckt}L_{ct} - \psi\left[\Omega(E_t)\dfrac{ap_{ct}}{p_{ckt}}\right]^{\frac{1}{1-a}} A_{ckt}L_{ct} \\ p_{dkt}\left[\Omega(E_t)\dfrac{ap_{dt}}{p_{dkt}}\right]^{\frac{1}{1-a}} A_{dkt}L_{dt} - \psi\left[\Omega(E_t)\dfrac{ap_{dt}}{p_{dkt}}\right]^{\frac{1}{1-a}} A_{dkt}L_{dt} \end{Bmatrix} \quad (12)$$

The intermediary good producers of machine k will set their prices such that they maximize their profit. Therefore, the first order condition is given from:

$$\frac{\partial \pi_{jkt}}{\partial p_{jkt}} = \left[\Omega(E_t)\frac{ap_{jt}}{p_{jkt}}\right]^{\frac{1}{1-a}} A_{jkt}L_{jt} - p_{jkt}\frac{1}{1-a}\left[\Omega(E_t)\frac{ap_{jt}}{p_{jkt}}\right]^{\frac{a}{1-a}} A_{jkt}L_{jt}\Omega(E_t)\frac{ap_{jt}}{p_{jkt}^2} + \psi\frac{1}{1-a}\left[\Omega(E_t)\frac{ap_{jt}}{p_{jkt}}\right]^{\frac{a}{1-a}} A_{jkt}L_{jt}\Omega(E_t)\frac{ap_{jt}}{p_{jkt}^2} = 0$$

Rearranging and simplifying:

$$\left[\Omega(E_t)\frac{ap_{jt}}{p_{jkt}}\right]^{\frac{1}{1-a}} A_{jkt}L_{jt} = \frac{1}{1-a}\left[\Omega(E_t)\frac{ap_{jt}}{p_{jkt}}\right]^{\frac{a}{1-a}} A_{jkt}L_{jt}\Omega(E_t)\frac{ap_{jt}}{p_{jkt}^2}[p_{jkt}-\psi]$$

$$\left[\Omega(E_t)\frac{ap_{jt}}{p_{jkt}}\right]^{\frac{1}{1-a}} = \frac{1}{1-a}\left[\Omega(E_t)\frac{ap_{jt}}{p_{jkt}}\right]^{\frac{a}{1-a}} \Omega(E_t)\frac{ap_{jt}}{p_{jkt}^2}[p_{jkt}-\psi]$$

$$\Omega(E_t)\frac{ap_{jt}}{p_{jkt}} = \frac{1}{1-a}\Omega(E_t)\frac{ap_{jt}}{p_{jkt}^2}[p_{jkt}-\psi]$$

$$\begin{Bmatrix} p_{ckt} = \dfrac{1}{1-a}[p_{ckt}-\psi] \\ p_{dkt} = \dfrac{1}{1-a}[p_{dkt}-\psi] \end{Bmatrix}$$

$$\begin{Bmatrix} p_{ckt} = \dfrac{\psi}{a} \\ p_{dkt} = \dfrac{\psi}{a} \end{Bmatrix}$$

We can let $\psi = a^2$, such that:

$$\begin{Bmatrix} p_{ckt} = a \\ p_{dkt} = a \end{Bmatrix} \quad (13)$$

Substituting the equilibrium price from equation (13) and $\psi = a^2$ into equation (12) yields the profit level for the intermediate producer at the optimum as follows *(Bezin, 2019)*:

$$\pi_{jkt} = \begin{Bmatrix} a[\Omega(E_t)p_{ct}]^{\frac{1}{1-a}}A_{ckt}L_{ct} - a^2[\Omega(E_t)p_{ct}]^{\frac{1}{1-a}}A_{ckt}L_{ct} \\ a[\Omega(E_t)p_{dt}]^{\frac{1}{1-a}}A_{dkt}L_{dt} - a^2[\Omega(E_t)p_{dt}]^{\frac{1}{1-a}}A_{dkt}L_{dt} \end{Bmatrix}$$

Simplifying to:

$$\pi_{jkt} = \begin{Bmatrix} a(1-a)[\Omega(E_t)p_{ct}]^{\frac{1}{1-a}}A_{ckt}L_{ct} \\ a(1-a)[\Omega(E_t)p_{dt}]^{\frac{1}{1-a}}A_{dkt}L_{dt} \end{Bmatrix}$$

From here, we can generalize for all machine producer in sector j:

$$\pi_{jt} = \begin{Bmatrix} a(1-a)[\Omega(E_t)p_{ct}]^{\frac{1}{1-a}}A_{ct}L_{ct} \\ a(1-a)[\Omega(E_t)p_{dt}]^{\frac{1}{1-a}}A_{dt}L_{dt} \end{Bmatrix} \quad (13)$$

4.4 Innovation (R&D) sector:

In the R&D sector, labs spend R&D expenditure to produce research which may lead to innovation. This innovation, if

successful, will increase the productivity of the intermediary good, machine k, being produced in the economy. When this is the case, that R&D lab will now possess a superior machine than the previous machine k producer, allowing it to become the new monopolist during that period. If the R&D efforts are a failure, then the productivity level of machine k remains unchanged, and the sole monopolist of that machine is assumed to be randomly allocated to some producer.
Let the probability of successful innovation on machine k in sector j be given by μ_{jkt}. Additionally, we assume that a successful innovation increases the productivity parameter by a factor η such that *(Bezin, 2019)*:

$$A_{jkt} = (1+\eta)A_{jkt-1} \quad (14)$$

Given this, we can calculate the expectation of for the level of A_{jkt} with the following:

$$A_{jkt} = \begin{Bmatrix} \mu_{ckt}(1+\eta)A_{ckt-1} + (1-\mu_{ckt})A_{ckt-1} \\ \mu_{dkt}(1+\eta)A_{dkt-1} + (1-\mu_{dkt})A_{dkt-1} \end{Bmatrix} = \begin{Bmatrix} \mu_{ckt}\eta A_{ckt-1} + A_{ckt-1} \\ \mu_{dkt}\eta A_{dkt-1} + A_{dkt-1} \end{Bmatrix}$$

Simplifying to:

$$A_{jkt} = \begin{Bmatrix} (1+\mu_{jkt}\eta)A_{ckt-1} \\ (1+\mu_{jkt}\eta)A_{dkt-1} \end{Bmatrix} \quad (15)$$

From here, we can generalize for all machines by setting $A_{ckt-1} = A_{ct-1}$, $A_{dkt-1} = A_{dt-1}$ and $\mu_{jkt} = \mu_{jt}$ yielding equation (16).

$$A_{jt} = \begin{Bmatrix} (1+\mu_{jt}\eta)A_{ct-1} \\ (1+\mu_{j't}\eta)A_{dt-1} \end{Bmatrix} \quad (16)$$

Therefore, the intertemporal path of relative productivity of the clean sector to the dirty sector is shown as:

$$\frac{A_{ct}}{A_{dt}} = \frac{1+\mu_{ct}\eta}{1+\mu_{dt}\eta}\frac{A_{ct-1}}{A_{dt-1}} \quad (18)$$

We can also now see that the profit of the machine producer at the optimum, derived in equation (13) can be re-written as:

$$\pi_{jt} = \begin{Bmatrix} a(1-a)\left[\Omega(E_t)p_{jt}\right]^{\frac{1}{1-a}}(1+\eta)A_{jt-1}L_{jt} & with\ prob.\ \mu_{jt} \\ a(1-a)\left[\Omega(E_t)p_{jt}\right]^{\frac{1}{1-a}}A_{jt-1}L_{jt} & with\ prob.\ (1-\mu_{jt}) \end{Bmatrix}$$

If the $R\&D$ lab is successful at the innovation, they become the sole monopolist of the machine, and capture π_{jt} with probability of μ_{jt}. Otherwise, the $R\&D$ lab, with probability $(1-\mu_{jt})$ fails and received no positive payoff. We can therefore define the expected profit of the $R\&D$ lab in sector j, Π_{jt}, as given by equation (19):

$$\Pi_{jt} = \mu_{jt}\pi_{jt} - c(R_{jt}) \quad (19)$$

Where $c(R_{jt})$ is the cost function of R&D expenditure dependent on R_{jt}. I assume the following functional forms:

$$c(R_{jt}) = \ln(1+R_{jt}) + \frac{1}{1+R_{jt}} - 1$$

And,

$$\mu_{jt} = \frac{\left(\frac{A_{jt-1}}{A_{j't-1}}\right)^{-1}}{1+\left(\frac{A_{jt-1}}{A_{j't-1}}\right)^{-1}}\frac{R_{jt}}{1+R_{jt}} \quad (20)$$

$c(R_{jt})$ is defined such that $c(R_{jt}) = 0$ when $R_{jt} = 0$ and $c'(R_{jt}) > 0$. $c''(R_{jt})$ is increasing at first with it then changing to decreasing. The intuition behind this is the existence of significant early R&D cost such as initial set up, purchasing of equipment/facilities etc., with it tapering off past this phase. μ_{jt} is composed of two positively defined sigmoid functions. These sigmoid functions are bounded between 0 and 1 (as we assume that $R_{jt-1}, \frac{A_{jt-1}}{A_{j't-1}} > 1$), therefore guaranteeing a suitable value for the probability μ_{jt} without further restrictions on model parameters. We can see that μ_{jt} is increasing in R_{jt-1} and $A_{j't-1}$ and decreasing in A_{dt-1}. The productivity parameters of each sector display spill-over effects. A higher productivity in sector j' increases the likelihood of success in sector j as we assume that this productivity to some extend spills over to this sector. Additionally, a higher productivity in sector j decreases the likelihood of success in sector j, assumes that innovation (increases in the productivity parameter) becomes increasingly difficult. The case for μ_{jt} being increasing with R_{jt-1} is that previous R&D expenditure has persistent probability enhancing effects on the following period of the knowledge, skills and insight gained from the previous period can be reused.

For ease of notation Let $\tilde{A}_{jt-1}(\cdot) = \frac{\left(\frac{A_{jt-1}}{A_{j't-1}}\right)^{-1}}{1+\left(\frac{A_{jt-1}}{A_{j't-1}}\right)^{-1}}$ such that:

$$\mu_{jt} = \tilde{A}_{jt-1}(\cdot)\frac{R_{jt}}{1+R_{jt}}$$

Hence, we take on the following functional form for Π_{jt} which was given in equation (19):

$$\Pi_{jt} = \tilde{A}_{jt-1}(\cdot)\frac{R_{jt}}{1+R_{jt}}\pi_{jt} - \left[\ln(1+R_{jt}) + \frac{1}{1+R_{jt}} - 1\right] \quad (21)$$

R&D labs will choose their expenditure, R_{jt}, optimally such that their expected profit is maximized. Therefore, first order condition with respect to R_{jt} reads:

$$\frac{\partial \mu_{jt}}{\partial R_{jt}} = \tilde{A}_{jt-1}(\cdot)\frac{1}{\left(1+R_{jt}\right)^2}$$

$$c\left(R_{jt}\right) = \ln\left(1+R_{jt}\right) + \frac{1}{1+R_{jt}} - 1$$

$$\frac{\partial c\left(R_{jt}\right)}{\partial R_{jt}} = \frac{R_{jt}}{\left(1+R_{jt}\right)^2}$$

$$\frac{\partial \Pi_{jt}}{\partial R_{jt}} = \frac{\partial \mu_{jt}}{\partial R_{jt}}\pi_{jt} - \frac{\partial c\left(R_{jt}\right)}{\partial R_{jt}} = 0$$

$$\frac{\partial \mu_{jt}}{\partial R_{jt}}\pi_{jt} = \frac{\partial c\left(R_{jt}\right)}{\partial R_{jt}}$$

$$\tilde{A}_{jt-1}(\cdot)\frac{1}{\left(1+R_{jt}\right)^2}\pi_{jt} = \frac{\partial c\left(R_{jt}\right)}{\partial R_{jt}}$$

$$\tilde{A}_{jt-1}(\cdot)\pi_{jt} = \frac{\partial c\left(R_{jt}\right)}{\partial R_{jt}}\left(1+R_{jt}\right)^2$$

$$\tilde{A}_{jt-1}(\cdot)\pi_{jt} = \frac{R_{jt}}{\left(1+R_{jt}\right)^2}\left(1+R_{jt}\right)^2$$

$$R_{jt} = \tilde{A}_{jt-1}(\cdot)\pi_{jt} = \begin{Bmatrix}\tilde{A}_{ct-1}(\cdot)\pi_{ct} \\ \tilde{A}_{dt-1}(\cdot)\pi_{dt}\end{Bmatrix} \quad (22)$$

From equation (22) it is straightforward to show that the relative ratio of R&D expenditure between the clean and dirty sectors will be:

$$\frac{R_{ct}}{R_{dt}} = \frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{dt-1}(\cdot)}\frac{\pi_{ct}}{\pi_{dt}}$$

We can also see that the $\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{jt-1}(\cdot)}$ term simplifies nicely to:

$$\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{dt-1}(\cdot)} = \left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}$$

The derivation for this is provided in the Appendix, part 2.4.1.
Hence, we can re-write the relative ratio of R&D expenditure between the clean and dirty sectors as:

$$\frac{R_{ct}}{R_{dt}} = \left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}\frac{\pi_{ct}}{\pi_{dt}} \quad (23)$$

It is worth pointing out the intuitive result shown by equation (23) by re-arranging it as:

$$\frac{A_{ct-1}}{A_{dt-1}}\frac{R_{ct}}{R_{dt}} = \frac{\pi_{ct}}{\pi_{dt}}$$

Taking the clean sector as an example, at the optimum, profitability of the machine producer is set by the level of pre-existing aggregate productivity in the green sector, which is then amplified by the R&D expenditure that was deployed by the R&D lab in time t to become the new monopolist. The same can be said for the dirty sector, highlighting that relative profitability between the two sector is simply drive through productivity-amplified R&D spending between the two sectors.
Returning to equation (21), we now substitute the optimal levels of R_{jt} derived in equation (22) into the profit function which yields:

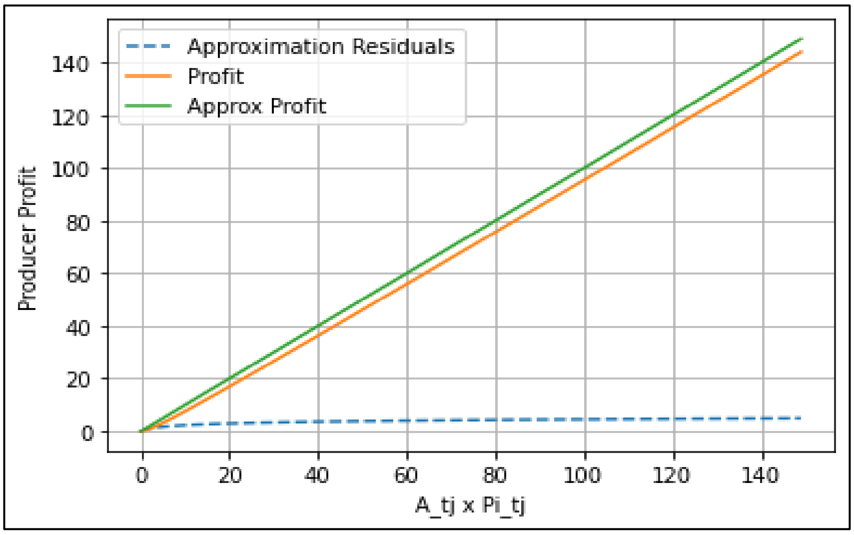
$$\Pi_{jt} = \tilde{A}_{jt-1}(\cdot)\frac{\tilde{A}_{jt-1}(\cdot)\pi_{jt}}{1+\tilde{A}_{jt-1}(\cdot)\pi_{jt}}\pi_{jt} - \left[\ln\left(1+\tilde{A}_{jt-1}(\cdot)\pi_{jt}\right) + \frac{1}{1+\tilde{A}_{jt-1}(\cdot)\pi_{jt}} - 1\right]$$

$$\Pi_{jt} = \frac{\left(\tilde{A}_{jt-1}(\cdot)\pi_{jt}\right)^2}{1+\tilde{A}_{jt-1}(\cdot)\pi_{jt}} - \left[\ln\left(1+\tilde{A}_{jt-1}(\cdot)\pi_{jt}\right) + \frac{1}{1+\tilde{A}_{jt-1}(\cdot)\pi_{jt}} - 1\right]$$

In order to continue from here, we must make the following approximation:

$$\Pi_{jt} \approx \tilde{A}_{jt-1}(\cdot)\pi_{jt} = R_{jt} \quad (24)$$

The effect of this approximation is shown in the following plot:



Despite this approximation seeming crude, and the two lines depicting the actual profit function and the approximated one never converging, the slope of both lines - which will be the important element in the following - is approximated well with this simplification.

Following equation (24), can see that the ratio of R&D lab profits between the clean and dirty sector is approximately given by the ratio of R&D expenditures of the two sectors:

$$\frac{\Pi_{ct}}{\Pi_{dt}} \approx \frac{\tilde{A}_{jt-1}(\cdot)}{\tilde{A}_{j't-1}(\cdot)}\frac{\pi_{jt}}{\pi_{j't}} = \frac{R_{ct}}{R_{dt}} \quad (25)$$

From which it is easy to see that:

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial \frac{R_{ct}}{R_{dt}}} > 0$$

Additionally, we can substitute the functional form we assumed for μ_{jt} into the dynamic equation for the evolution of relative productivity between the sectors derived earlier in equation (18):

$$\frac{A_{ct}}{A_{dt}} = \frac{1+\mu_{ct}\eta}{1+\mu_{dt}\eta}\frac{A_{ct-1}}{A_{dt-1}} \quad (18)$$

$$\frac{A_{ct}}{A_{dt}} = \frac{1+\tilde{A}_{ct-1}(\cdot)\frac{R_{ct}}{1+R_{ct}}\eta}{1+\tilde{A}_{dt-1}(\cdot)\frac{R_{dt}}{1+R_{dt}}\eta}\frac{A_{ct-1}}{A_{dt-1}}$$

Substituting the optimal levels of R_{ct} and R_{dt} from equation (22):

$$\frac{A_{ct}}{A_{dt}} = \frac{1+\tilde{A}_{ct-1}(\cdot)\frac{\tilde{A}_{ct-1}(\cdot)\pi_{ct}}{1+\tilde{A}_{ct-1}(\cdot)\pi_{ct}}\eta}{1+\tilde{A}_{dt-1}(\cdot)\frac{\tilde{A}_{dt-1}(\cdot)\pi_{dt}}{1+\tilde{A}_{dt-1}(\cdot)\pi_{dt}}\eta}\frac{A_{ct-1}}{A_{dt-1}}$$

$$\frac{A_{ct}}{A_{dt}} = \frac{A_{ct-1}}{A_{dt-1}}\left[\left(1+\tilde{A}_{ct-1}(\cdot)\frac{\tilde{A}_{ct-1}(\cdot)\pi_{ct}}{1+\tilde{A}_{ct-1}(\cdot)\pi_{ct}}\eta\right)\left(1+\tilde{A}_{dt-1}(\cdot)\frac{\tilde{A}_{dt-1}(\cdot)\pi_{dt}}{1+\tilde{A}_{dt-1}(\cdot)\pi_{dt}}\eta\right)^{-1}\right]$$

$$\frac{A_{ct}}{A_{dt}} = \frac{A_{ct-1}}{A_{dt-1}}\left[\left(1+\frac{\left(\tilde{A}_{ct-1}(\cdot)\right)^2\eta\pi_{ct}}{1+\tilde{A}_{ct-1}(\cdot)\pi_{ct}}\right)\left(1+\frac{\left(\tilde{A}_{dt-1}(\cdot)\right)^2\eta\pi_{dt}}{1+\tilde{A}_{dt-1}(\cdot)\pi_{dt}}\right)^{-1}\right]$$

$$\frac{A_{ct}}{A_{dt}} = \frac{A_{ct-1}}{A_{dt-1}}\left[\left(\frac{1+\tilde{A}_{ct-1}(\cdot)\pi_{ct}+\left(\tilde{A}_{ct-1}(\cdot)\right)^2\eta\pi_{ct}}{1+\tilde{A}_{ct-1}(\cdot)\pi_{ct}}\right)*\left(\frac{1+\tilde{A}_{dt-1}(\cdot)\pi_{dt}+\left(\tilde{A}_{dt-1}(\cdot)\right)^2\eta\pi_{dt}}{1+\tilde{A}_{dt-1}(\cdot)\pi_{dt}}\right)^{-1}\right]$$

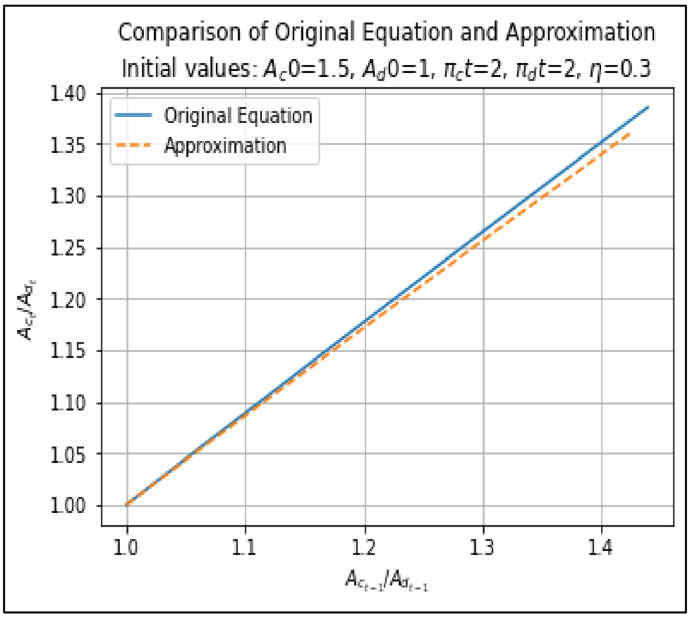
$$\frac{A_{ct}}{A_{dt}} = \frac{A_{ct-1}}{A_{dt-1}} * \left(\left(\frac{1+\tilde{A}_{ct-1}(\cdot)\pi_{ct}\left(1+\eta\tilde{A}_{ct-1}(\cdot)\right)}{1+\tilde{A}_{ct-1}(\cdot)\pi_{ct}}\right)*\left(\frac{1+\tilde{A}_{dt-1}(\cdot)\pi_{dt}\left(1+\eta\tilde{A}_{dt-1}(\cdot)\right)}{1+\tilde{A}_{dt-1}(\cdot)\pi_{dt}}\right)^{-1}\right)$$

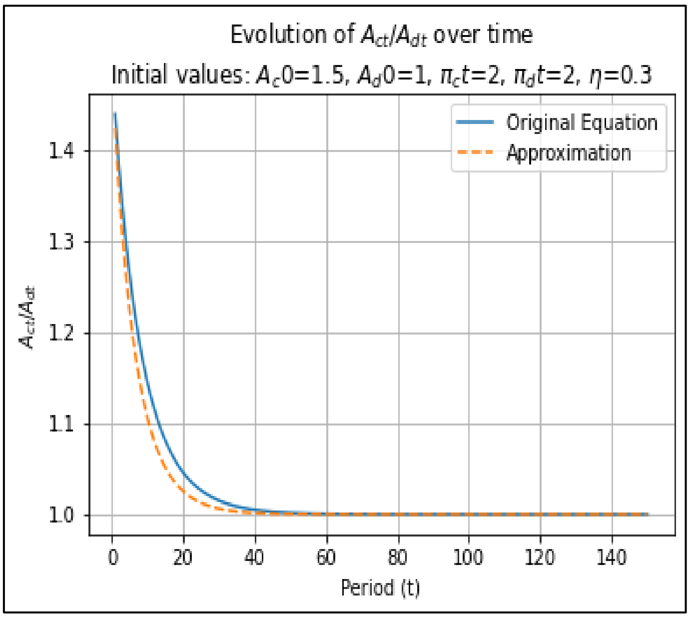
From this step, we can make the following simplifying approximation by removing the 1s appearing in the numerator and denominator of the second and third terms on the RHS:

$$\frac{A_{ct}}{A_{dt}} \approx \frac{A_{ct-1}}{A_{dt-1}} * \left(\left(\frac{\tilde{A}_{ct-1}(\cdot)\pi_{ct}\left(1+\eta\tilde{A}_{ct-1}(\cdot)\right)}{\tilde{A}_{ct-1}(\cdot)\pi_{ct}}\right)\left(\frac{\tilde{A}_{dt-1}(\cdot)\pi_{dt}\left(1+\eta\tilde{A}_{dt-1}(\cdot)\right)}{\tilde{A}_{dt-1}(\cdot)\pi_{dt}}\right)^{-1}\right)$$

$$\frac{A_{ct}}{A_{dt}} \approx \frac{A_{ct-1}}{A_{dt-1}} * \frac{1+\eta\tilde{A}_{ct-1}(\cdot)}{1+\eta\tilde{A}_{dt-1}(\cdot)} \quad (26)$$

The following plots display the impact of this approximation for given π_{ct} and π_{dt} held constant:





For our purposes, this approximation drastically reduces the computation complexity of the problem by avoiding the use of numerical methods that would otherwise be needed. As the plots display, this approximation will not be detrimental to the fundamental functionality of the model.

From here, we let $g(\cdot) = \frac{1+\eta\tilde{A}_{ct-1}(\cdot)}{1+\eta\tilde{A}_{dt-1}(\cdot)}$, such that equation (26) is re-written as:

$$\frac{A_{ct}}{A_{dt}} = g(\cdot)\frac{A_{ct-1}}{A_{dt-1}} \quad (27)$$

Remembering equation (10):

$$\left(\frac{A_{ct}}{A_{dt}}\right)^{-(1-a)} = \frac{p_{ct}}{p_{dt}}$$

Using equation (27) and letting $\gamma = (1-a)(\epsilon-1)$ we can derive equation (28):

$$\left(\frac{A_{ct}}{A_{dt}}\right)^{-(1-a)(1-\epsilon)} = \left(\frac{p_{ct}}{p_{dt}}\right)^{(1-\epsilon)}$$

$$\left(\frac{A_{ct}}{A_{dt}}\right)^{\gamma} = \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} = \left(\frac{p_{ct}}{p_{dt}}\right)^{(1-\epsilon)} \quad (28)$$

The importance of equation (28) is that it allows to compute prices at time t from backwards looking variables.

Let's return to the production function in equation (6) and insert the optimal demand level of machinery derived in equation (9):

$$Y_{jt} = \Omega(E_t)L_{jt}^{1-a}\int_0^1 A_{jkt}^{1-a}z_{jkt}^{a}dk \quad (6)$$

$$z_{jkt} = \left[\Omega(E_t)\frac{ap_{jt}}{p_{jkt}}\right]^{\frac{1}{1-a}} A_{jkt}L_{jt} \quad (9)$$

Remember $p_{jkt} = p_{ckt} = p_{dkt} = a$ according to (13), such that:

$$z_{jkt} = \left[\Omega(E_t)p_{jt}\right]^{\frac{1}{1-a}}A_{jkt}L_{jt}$$

Therefore,

$$Y_{jt} = \Omega(E_t)L_{jt}^{1-a}\int_0^1 \left\{A_{jkt}^{1-a}\left(\left[\Omega(E_t)p_{jt}\right]^{\frac{1}{1-a}}A_{jkt}L_{jt}\right)^{a}\right\}dk$$

$$Y_{jt} = \Omega(E_t)L_{jt}\int_0^1 \left\{A_{jkt}\left[\Omega(E_t)p_{jt}\right]^{\frac{a}{1-a}}\right\}dk$$
$$Y_{jt} = \left[p_{jt}\right]^{\frac{a}{1-a}}\left[\Omega(E_t)\right]^{\frac{1}{1-a}}L_{jt}\int_0^1 \{A_{jkt}\}dk$$
$$Y_{jt} = \left[p_{jt}\right]^{\frac{a}{1-a}}\left[\Omega(E_t)\right]^{\frac{1}{1-a}}L_{jt}\int_0^1 \{A_{jkt}\}dk$$

Remembering (7) $A_{jt} = \int_0^1 \{A_{jkt}\}dk$:

$$Y_{jt} = \left[p_{jt}\right]^{\frac{a}{1-a}}\left[\Omega(E_t)\right]^{\frac{1}{1-a}}A_{jt}L_{jt} \quad (29)$$

And equivalently,

$$L_{jt} = \frac{Y_{jt}}{\left[p_{jt}\right]^{\frac{a}{1-a}}\left[\Omega(E_t)\right]^{\frac{1}{1-a}}A_{jt}} \quad (30)$$

In market clearing equilibrium, we will have that $X_{jt} = Y_{jt}$, such that:

$$L_{jt} = \frac{X_{jt}}{\left[p_{jt}\right]^{\frac{a}{1-a}}\left[\Omega(E_t)\right]^{\frac{1}{1-a}}A_{jt}} \quad (31)$$

We can also show the relative monopolist profit of the machine producers between the clean and dirty good sectors as follows by remembering equation (13):

$$\frac{\pi_{ct}}{\pi_{dt}} = \frac{a(1-a)\left[\Omega(E_t)p_{ct}\right]^{\frac{1}{1-a}}A_{ct}L_{ct}}{a(1-a)\left[\Omega(E_t)p_{dt}\right]^{\frac{1}{1-a}}A_{dt}L_{dt}}$$
$$\frac{\pi_{ct}}{\pi_{dt}} = \frac{\left[\Omega(E_t)p_{ct}\right]^{\frac{1}{1-a}}A_{ct}L_{ct}}{\left[\Omega(E_t)p_{dt}\right]^{\frac{1}{1-a}}A_{dt}L_{dt}}$$

Here, substituting equation (26) yields:

$$\frac{\pi_{ct}}{\pi_{dt}} = \frac{\left[\Omega(E_t)p_{ct}\right]^{\frac{1}{1-a}}A_{ct}\dfrac{X_{ct}}{\left[p_{ct}\right]^{\frac{a}{1-a}}\left[\Omega(E_t)\right]^{\frac{1}{1-a}}A_{ct}}}{\left[\Omega(E_t)p_{dt}\right]^{\frac{1}{1-a}}A_{dt}\dfrac{X_{dt}}{\left[p_{dt}\right]^{\frac{a}{1-a}}\left[\Omega(E_t)\right]^{\frac{1}{1-a}}A_{dt}}}$$
$$\frac{\pi_{ct}}{\pi_{dt}} = \frac{(p_{ct})^{\frac{1}{1-a}}\dfrac{X_{ct}}{\left[p_{ct}\right]^{\frac{a}{1-a}}}}{(p_{ct})^{\frac{1}{1-a}}\dfrac{X_{dt}}{\left[p_{dt}\right]^{\frac{a}{1-a}}}}$$
$$\frac{\pi_{ct}}{\pi_{dt}} = \frac{p_{ct}}{p_{dt}}\frac{X_{ct}}{X_{dt}} \quad (32)$$

Now, substituting the optimal levels of final good demand shown with equations $(5a)$ and $(5b)$ into equation (27):

$$\frac{\pi_{ct}}{\pi_{dt}} = \frac{p_{ct}}{p_{dt}}\frac{\dfrac{I_t}{p_{ct}}\left[q_t\dfrac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^{\epsilon}}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^{\epsilon}} + (1-q_t)\dfrac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right]}{\dfrac{I_t}{p_{dt}}\left[q_t\dfrac{1}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^{\epsilon}} + (1-q_t)\dfrac{1}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right]}$$
$$\frac{\pi_{ct}}{\pi_{dt}} = \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\frac{\dfrac{q_t(\theta^G)^{\epsilon}}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^{\epsilon}} + \dfrac{(1-q_t)}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}}{\dfrac{q_t}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^{\epsilon}} + \dfrac{(1-q_t)}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}} \quad (33)$$

Applying equation (28) to (33):

$$\frac{\pi_{ct}}{\pi_{dt}} = \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\frac{\dfrac{q_t(\theta^G)^{\epsilon}}{1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}} + \dfrac{(1-q_t)}{1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}}}{\dfrac{q_t}{1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}} + \dfrac{(1-q_t)}{1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}}}$$

And simplifying yields:

$$\frac{\pi_{ct}}{\pi_{dt}} = \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} \frac{q_t[(\theta^G)^{\epsilon} - 1] + 1 + (\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}}{-q_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon} - 1] + 1 + (\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}} \quad (34)$$

Derivation is shown in the Appendix, part 2.4.2.
Substituting (34) into (23) yields:

$$\frac{R_{ct}}{R_{dt}} = \left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}\frac{\pi_{ct}}{\pi_{dt}}$$

$$\frac{R_{ct}}{R_{dt}} = \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} \frac{q_t[(\theta^G)^{\epsilon} - 1] + 1 + (\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}}{-q_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon} - 1] + 1 + (\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}} \quad (35)$$

Let:

$$f(\cdot) = \frac{\left(q_t[(\theta^G)^{\epsilon} - 1] + 1 + (\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}{-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon} - 1] + 1 + (\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}}$$

Such that:

$$\frac{R_{ct}}{R_{dt}} = \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(g(\cdot))^{\gamma} f(\cdot)$$

We can show that:

$$\frac{\partial\frac{R_{ct}}{R_{dt}}}{\partial q_t} = \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon} - 1]\left\{\frac{\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon} + 1] + 1 + (\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon} - 1] + 1 + (\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)^2}\right\} \quad (36)$$

$$\frac{\partial\frac{R_{ct}}{R_{dt}}}{\partial q_t} > 0$$

As shown in the Appendix, part 2.4.3.
Since we assumed that:

$$\frac{\Pi_{ct}}{\Pi_{dt}} \approx \frac{\tilde{A}_{jt-1}(\cdot)}{\tilde{A}_{j't-1}(\cdot)}\frac{\pi_{jt}}{\pi_{j't}} = \frac{R_{ct}}{R_{dt}}$$

$$\frac{\partial\frac{R_{ct}}{R_{dt}}}{\partial q_t} \approx \frac{\partial\frac{\Pi_{ct}}{\Pi_{dt}}}{\partial q_t} > 0$$

4.5 Environmental Stock:

Now we turn to the environmental stock of the economy. We will define the equation governing the evolution of environmental stock, E_t as:

$$E_{t+1} = (1+b)E_t - \sigma_c\int_0^1 z_{ckt}\,dk - \sigma_d\int_0^1 z_{dkt}\,dk \quad (37)$$

Where b is the natural regeneration rate of environmental stock, and σ_c and σ_d is the marginal environmental damage caused by all the machines used as inputs from the clean and dirty final good producers respectively.
Additionally, I assume that:

$$\sigma_d = (1+c)\sigma_c$$

Where $c > 0$ such that $\sigma_d > \sigma_c$. In other words, the dirty sector machine consumption has a strictly higher marginal environmental damage affect than the clean sector.
If we now recall equations (9) and (31):

$$z_{jkt} = [\Omega(E_t)p_{jt}]^{\frac{1}{1-a}}A_{jkt}L_{jt} \quad (9)$$

$$L_{jt} = \frac{X_{jt}}{[p_{jt}]^{\frac{a}{1-a}}[\Omega(E_t)]^{\frac{1}{1-a}}A_{jt}} \quad (31)$$

We can derive equation (38), as shown in the Appendix, part 2.5.1:

$$E_{t+1} = (1+b)E_t - \sigma_c I_t \left(q_t \frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon} (\theta^G)^\epsilon}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon} (\theta^G)^\epsilon} + (1-q_t) \frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}} \right) - (1+c)\sigma_c I_t \left(\frac{q_t}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon} (\theta^G)^\epsilon} + \frac{(1-q_t)}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}} \right) \quad (38)$$

From equation (38) we can explore the effect a change in q_t has on environmental stock:

$$\frac{\partial E_{t+1}}{\partial q_t} = -\sigma_c \frac{\partial \int_0^1 z_{ckt} dk}{\partial q_t} - (1+c)\sigma_c \frac{\partial \int_0^1 z_{dkt} dk}{\partial q_t}$$

Which is derived as equation (39) as shown in the Appendix, part 2.5.2.

$$\frac{\partial E_{t+1}}{\partial q_t} = c\sigma_c I_t \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma \left(\frac{(\theta^G)^\epsilon - 1}{\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma (\theta^G)^\epsilon\right]\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right]} \right) > 0 \quad (39)$$

By manipulating the derivatives, we can derive the change in environmental stock caused from a change in relative R&D expenditure between the clean and dirty sectors:

$$\frac{\partial E_{t+1}}{\partial \frac{R_{ct}}{R_{dt}}} = \frac{\partial E_{t+1}}{\partial q_t} \frac{\partial q_t}{\partial \frac{R_{ct}}{R_{dt}}} \quad (40)$$

We have also approximated that:
And since we approximate that:

$$\frac{\Pi_{ct}}{\Pi_{dt}} \approx \frac{R_{ct}}{R_{dt}} \quad (25)$$

We have:

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial \frac{R_{ct}}{R_{dt}}} \approx 1 \quad (41)$$

Which is the change in relative R&D sector due to a change in relative sector R&D expenditure.
I from this I deduce that the change in relative profit for a change in environmental stock must be given by:

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}} / \partial \frac{R_{ct}}{R_{dt}}}{\partial E_{t+1} / \partial \frac{R_{ct}}{R_{dt}}}$$

Which is simply:

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \frac{1}{\partial E_{t+1} / \partial \frac{R_{ct}}{R_{dt}}}$$

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \frac{\partial \frac{R_{ct}}{R_{dt}}}{\partial E_{t+1}}$$

Which we can derive from (40):

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \frac{\partial \frac{R_{ct}}{R_{dt}}}{\partial E_{t+1}} = \frac{\partial q_t}{\partial E_{t+1}} \frac{\partial \frac{R_{ct}}{R_{dt}}}{\partial q_t}$$

Inverting equation (39) yields:

$$\frac{\partial q_t}{\partial E_{t+1}} = \left(\frac{\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma (\theta^G)^\epsilon\right]\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right]}{c\sigma_c I_t \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma [(\theta^G)^\epsilon - 1]} \right)$$

And utilizing equation (36):

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \left(\frac{\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma (\theta^G)^\epsilon\right]\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right]}{c\sigma_c I_t \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma [(\theta^G)^\epsilon - 1]} \right) * \frac{\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma [(\theta^G)^\epsilon - 1]\left(\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma [(\theta^G)^\epsilon + 1] + 1 + (\theta^G)^\epsilon \left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}\right)}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma [(\theta^G)^\epsilon - 1] + 1 + (\theta^G)^\epsilon \left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right)^2}$$

Simplifying to:

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \frac{1}{c\sigma_c I_t} * \left\{ \frac{\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} [(\theta^G)^{\epsilon} + 1] + 1 + (\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{-q_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} [(\theta^G)^{\epsilon} - 1] + 1 + (\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}} \right\}^2 \quad (42)$$

As shown in the Appendix, part 2.5.3.

5. Carbon Credit Price Evolution Model:

The critical idea of this paper is the following. $\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}}$ is the change in relative profit between the two sectors for a change in environmental stock. The change in environmental stock is driven through changes in relative R&D expenditure. The relative R&D expenditure determines which sector produces the most machinery. The machinery of these two sectors pollutes the environment at different rates. This R&D expenditure also directly increases the relative profit of the two sectors. Therefore, the change in relative R&D expenditure causes a direct change in relative profit and a change in the environmental stock. For example, if the clean sector increases its spending relatively more than the dirty sector, it will face an approximately proportional increase in its profits relative to the dirty sector with the side effect of also increasing the environmental stock as more of the economy's output is being produced in the clean sector. Therefore, I interpret $\left(\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} - 1\right)$ as the rate of return to increasing the environmental stock. It is important to note that what maters here is relative profit of the two sectors. If the clean sector increases its R&D expenditure, but the dirty sector does so by more, the environmental stock will still be degraded despite the clean sector's efforts.

Following this, I will assume that since $\left(\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} - 1\right)$ is the rate of return for increasing the environmental stock, and carbon credits are contracts which represent increasing the environmental stock by 1 unit (tC02e), the rate of return on carbon credits, μ_t, in an efficient market, should be identical to $\left(\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} - 1\right)$.

Following Song et al., 2019, Let's assume the Carbon Price follows the following equation:

$$dP_t = \mu_t P_t dt + \sigma P_t dW_t + P_t dJ_t, \qquad 0 < t < T \quad (43)$$

Where μ_t is the expected return of the carbon price, σ is the volatility of the asset price, W_t is the standard geometric Brownian process and J_t encapsulates the jump triggered by publications or news shocks. J_t is a compound Poisson defined as $J_t = \sum_{i=1}^{N_t} Y_i$ and is distributed with intensity λ. Y is a series of identically distributed, random variables which are mutually independent. I assume that these jump amplitudes follow a normal distribution with expected jump amplitude μ_{jump} and variance δ^2. Y_i represents the ith jump amplitude in the series.

For the moment we can ignore jump term for the derivation and state equations (43) as:

$$dP_t = \mu_t P_t dt + \sigma P_t dW_t$$

Stating Itô's Lemma:

$$dX_t = \mu_t dt + \sigma dB_t$$

$$df = \left(\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial x^2}\right) dt + \sigma \frac{\partial f}{\partial x} dB_t$$

And applying it to our simplified equation, we derive:

$$df = \left(\frac{\partial f}{\partial t} + \mu_t P_t \frac{\partial f}{\partial P_t} + P_t^2 \frac{\sigma^2}{2}\frac{\partial^2 f}{\partial {P_t}^2}\right) dt + P_t \sigma \frac{\partial f}{\partial P_t} dW_t$$

If we assume the prices are log-normally distributed, we can let $f = \ln P_t$, such that $\frac{\partial f}{\partial P_t} = \frac{1}{P_t}, \frac{\partial^2 f}{\partial {P_t}^2} = -\frac{1}{P_t^2}, \frac{\partial f}{\partial t} = 0$. Applying these results into the above equation yields the following.

$$d[\ln P_t] = \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \sigma_t dW_t$$

Intergrading from $[0, T]$ and re-arranging:

$$\int_0^T d[\ln P_t] dt = \int_0^T \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \int_0^T \{\sigma dW_t\}$$

$$\ln P_T - \ln P_0 = \int_0^T \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \int_0^T \{\sigma dW_t\}$$

$$\ln P_T = \int_0^T \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \int_0^T \{\sigma dW_t\} + \ln P_0$$

And stating in exponential form:

$$P_T = \exp\left[\int_0^T \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \int_0^T \{\sigma dW_t\} + \ln P_0\right]$$
$$P_T = \exp\left[\int_0^T \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \int_0^T \{\sigma dW_t\}\right] \exp[\ln P_0]$$
$$P_T = P_0 \exp\left[\int_0^T \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \int_0^T \{\sigma dW_t\}\right]$$

Now suppose the jump term appears only once at time t_1. The price at t_1 will be:

$$P_{t_1} = P_0 \exp\left[\int_0^{t_1} \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \int_0^{t_1} \{\sigma dW_t\} + \int_0^{t_1} \{dJ_t\}\right]$$

The change in the credit price from $P_{t_1-\frac{1}{n}}$ to P_{t_1} would be:

$$P_{t_1} - P_{t_1-\frac{1}{n}} = P_0 \exp\left[\int_{t_1-\frac{1}{n}}^{t_1} \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \int_{t_1-\frac{1}{n}}^{t_1} \{\sigma dW_t\} + \int_{t_1-\frac{1}{n}}^{t_1} \{dJ_t\}\right]$$

Where n is the increment of time. When $n \to \infty$, the instantaneous change in price will be driven purely by the jump such that:

$$P_{t_1} - P_{t_1-\frac{1}{n}} = Y_1 P_{t_1-\frac{1}{n}}$$

And we can see that the new price following the jump will simply be:

$$P_{t_1} = P_{t_1-\frac{1}{n}}(1 + Y_1)$$
$$P_{t_1} = P_0(1 + Y_1) \exp\left[\int_0^{t_1} \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \int_0^{t_1} \{\sigma dW_t\}\right]$$

As the jump amplitudes are i.i.d. and mutually exclusive, we can generalize for N_T jumps as:

$$P_T = P_0 \prod_{i=1}^{N_T} (1 + Y_i) \exp\left[\int_0^T \left(\mu_t - \frac{\sigma^2}{2}\right) dt + \int_0^T \{\sigma dW_t\}\right] \quad (44)$$

Where:

$$\mu_t = \frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} - 1$$

I also note that we can easily see from equation (42), the condition:
$\mu_t > 0$ if:

$$\frac{\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} [(\theta^G)^{\epsilon} + 1] + 1 + (\theta^G)^{\epsilon} \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{-q_t \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} [(\theta^G)^{\epsilon} - 1] + 1 + (\theta^G)^{\epsilon} \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}} > (c\sigma_c I_t)^{\frac{1}{2}} \quad (45)$$

The deviation from Song et al., 2019, is that I now implement the economic model developed as the driver of μ_t to shine some light, with economic intuition, on the potential dynamics for the trend in the prices of voluntary carbon credits. I note that equation (44) is modelled in continuous time, but the equation (42) is in discrete time. In practise, this is not necessarily a barrier, as equation (44) is implemented in discrete time following the commonly used Euler discretization scheme as shown in Kelliher, 2022 and Bruti-Liberati & Platen, 2007. Bruti-Liberati & Platen, 2007, show that the simple Euler scheme achieves strong order of convergence and is suitable for the scope of this paper.

6. Data Selection, Model Simulation Methodology and Results:

Taking a numerical approach, I implement equation (44), with μ_t being driven by the economic model using Monte Carlo simulation. Based on the different realizations of the Brownian and Poisson processes, different paths for the evolution of the credit prices are generated. The expected price, $E(P_t)$, is calculated as the average of all the realizations of P_t for every t and mapped out as the mean path of the process. Further, I use real voluntary carbon prices to compare the model's accuracy. Since the model is supposed to provide the price path of a reference carbon contract, assuming no quality, transparency, or liquidity concerns, I choose the Nature-Based solutions Carbon Credit (Vintages 2023-2027) futures expiring in DEC 24 *(YBBZ4 Comdty)* as the contract to simulate. This contract is chosen specifically for various reasons. First, as mentioned in the primer, nature-based and forestry related carbon projects are the most common source of carbon credit supply. Therefore, nature-based projects can be interpreted as a good reference point for which carbon credits prices can be based on. Further, due to the quality concerns surrounding credits issued in the past, vintages 2023-2027 are chosen to try and minimize the influence of these concerns as more recently revised protocols and methodologies have had substantial improvements. Additionally, the Dec 2024 expiration was chosen as these were the most liquid contracts at the time of writing, increasing the likelihood of them being priced efficiently. The data is collected from Bloomberg for 252 trading days of which 75% of are used as training data and 25% of is used for out of time testing. I would like to mention that given the nature and intuition of the economic model, perhaps using a weekly or monthly frequency instead of daily would have been more suitable for this simulation. However, the datasets for most relatively reliable exchange traded voluntary carbon credit contracts do not span far back enough to collect meaningfully large samples using such intervals due to the nascency of these securities.

The way the model is fitted to the price data is by deploying Powell's method, which is a method for finding a local minimum with no need to take the derivatives of the functions. Examples of this method being used for fitting jump-diffusion models can be seen in in Ramezani, 2004, Ramezani, 2006 and Jun, 1999 for reference. More specifically, given initial guess values and appropriate boundaries for all the parameters of the models, I use Powell's method to minimize the Mean Absolute Percentage Error (MAPE) of the expected price path generated from the Monte Carlo simulation compared to the underlying price data. The reason I choose this numerical approach is two-fold. First, this approach does not require taking derivatives of the functions and is less complex than Maximum Likelihood Estimators which are also commonly used in the literature, making it more suitable for the scope of this paper. Second, the choice of fitting the expected path specifically is because the expected path is what is being driven by the economic model, hence, trying to minimize its error will prove more insightful for portraying the effectiveness of the model in this case.

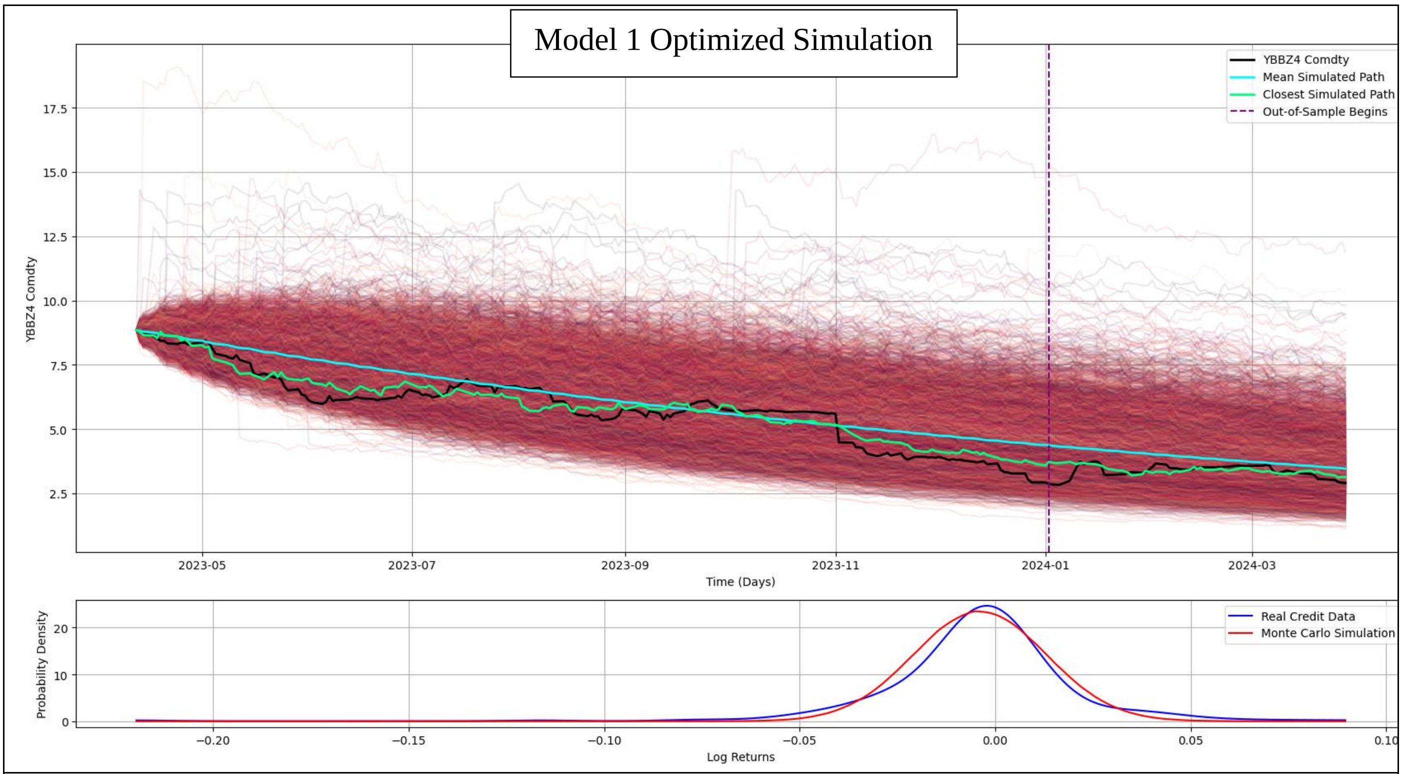
In addition to this, in order to investigate the distribution of the results that are generated from the simulations, using Kernal Density Estimation (KDE), I find the approximated probability density function of the log returns by sampling the Monte Carlo simulation once the parameters are optimized. The same is done for the underlying log return data on the credits, and the effectiveness of the model in replicating the empirical distribution is compared using a Kolmogorov–Smirnov (KS) test statistic.

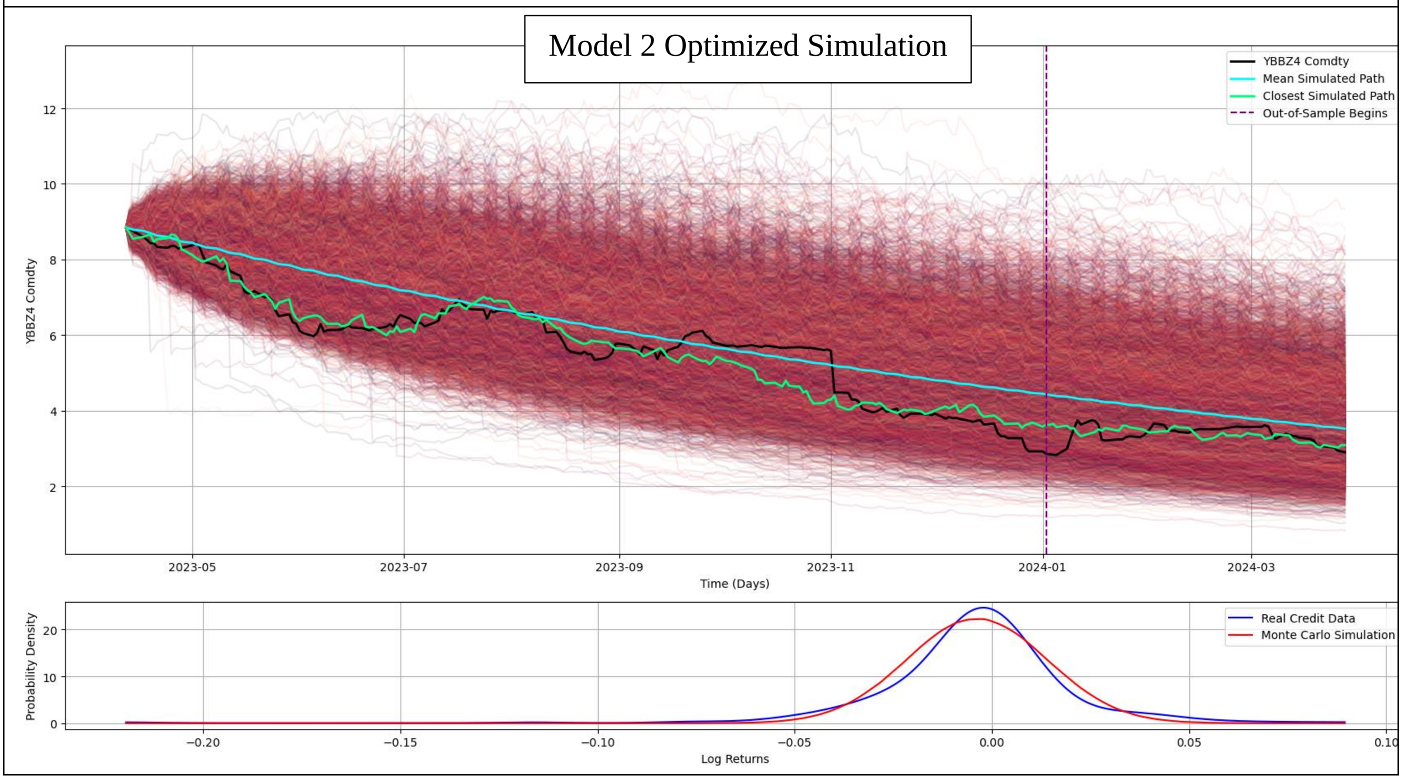
I also include three additional variations of the model. Namely, Model 1 is the model based on equation (44), while Model 2 is identical to Model 1 but instead of μ_t being driven by the economic model, it is set as a constant parameter μ. Models 3 and 4 are identical to Models 1 and 2 respectively but the jump process of the model is switched off, modelling the price evolution as a pure diffusion processes instead. Models 3 and 4 will allow to compare the relevance of the jump process, while comparison between Models 1 and 2 (and between Models 3 and 4) will provide insight on the effectiveness of the economic model in capturing the dynamics of the expected price trajectory.

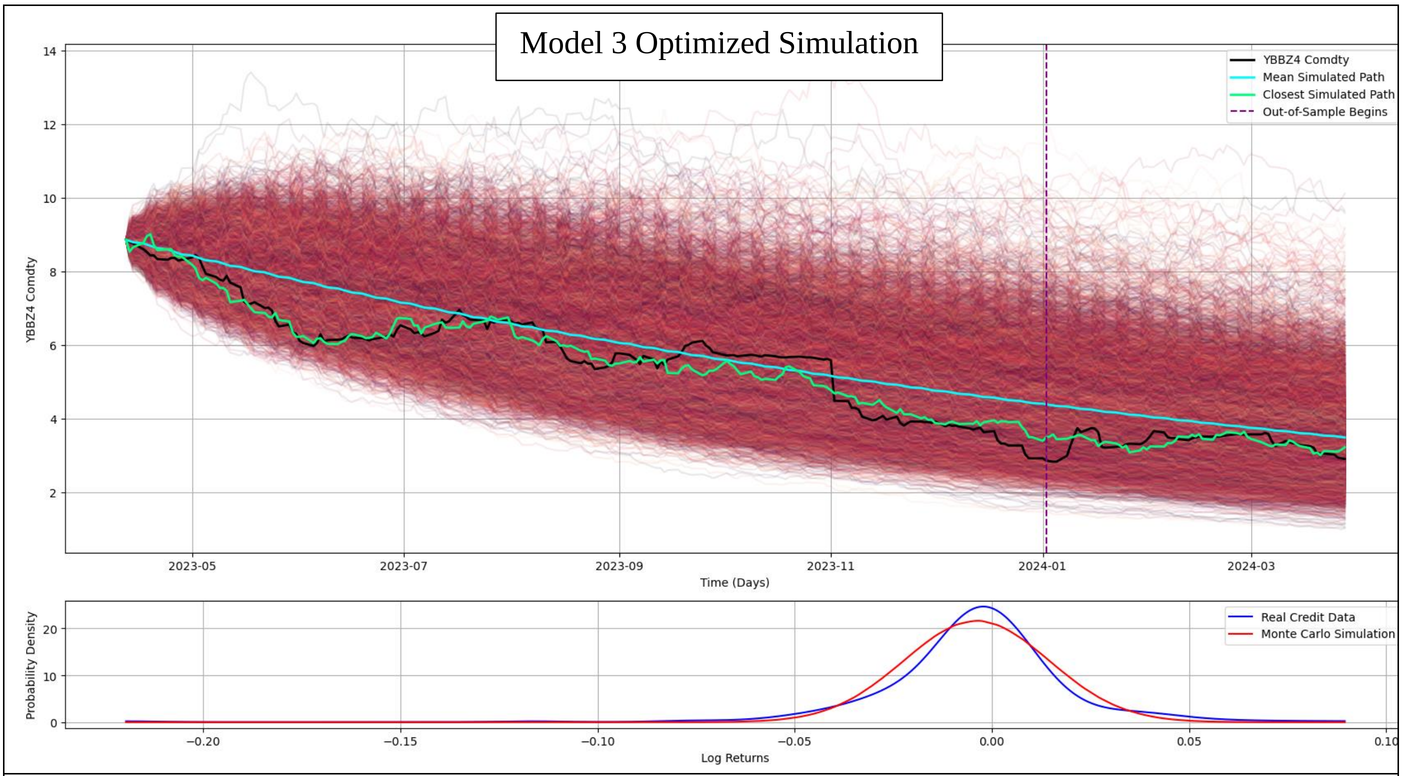
For every iteration calculated using the Powell Method, 10,000 paths are simulated for each model. Once the minimization is complete, the whole process is repeated for a total of 15 runs. The optimized parameters and resulting statistics presented in the following table are the averages of all the results obtained over these 15 runs for all 4 model variations. The code for this is implemented using Python 3 and the total run time for finding the optimized results using this specification for all four models sequentially, on an Intel 12th generation i7 CPU with 16gb of RAM, is approximately 1 hour.

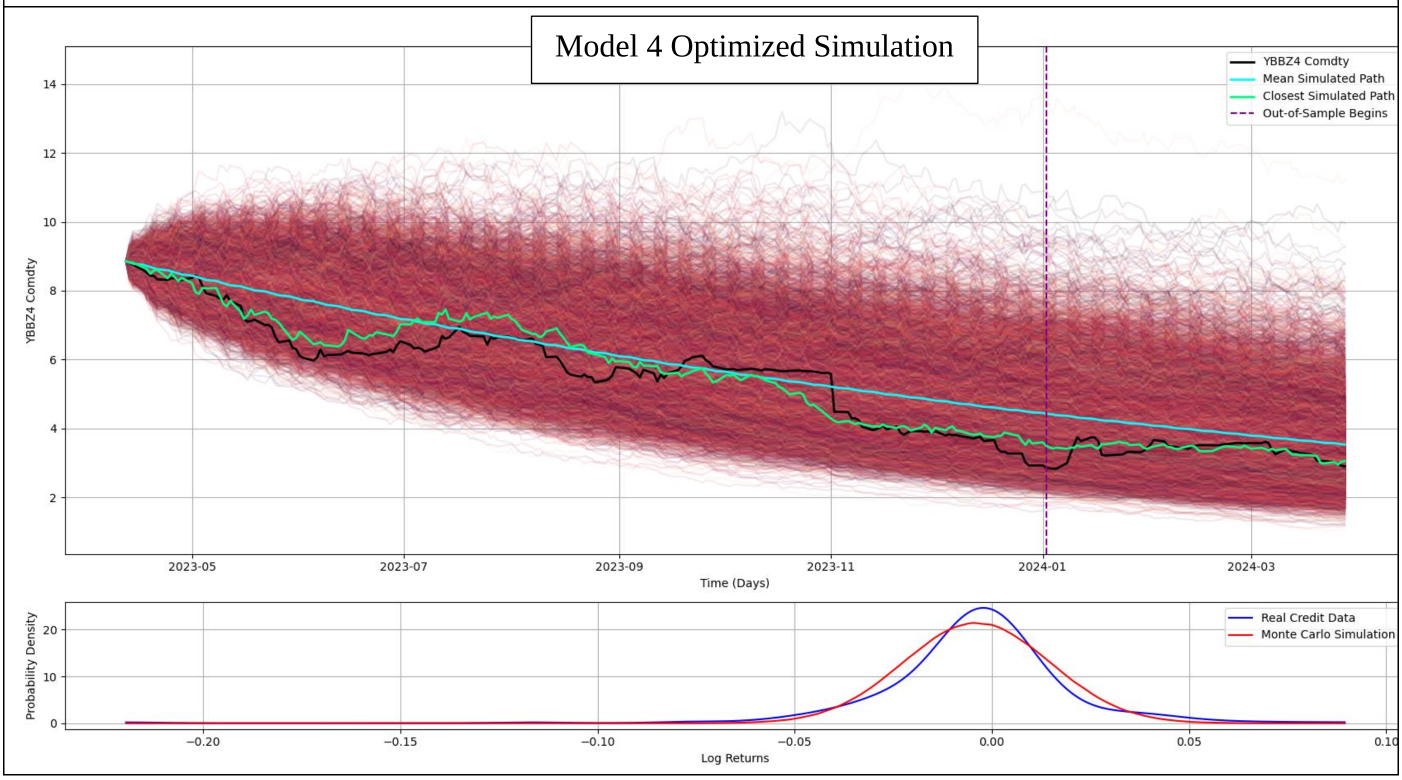
Parameter	Model 1	Model 2	Model 3	Model 4
$\boldsymbol{A_{c0}}$	3.623	-	3.264	-
$\boldsymbol{A_{d0}}$	5.781	-	4.964	-
$\boldsymbol{\eta}$	0.413	-	0.298	-
$\boldsymbol{a}$	0.461	-	0.514	-
$\boldsymbol{\varepsilon}$	1.951	-	1.884	-
$\boldsymbol{\theta^G}$	1.830	-	1.974	-
$\boldsymbol{c}$	1.034	-	0.911	-
$\boldsymbol{\sigma_{carbon}}$	1.613	-	1.344	-
$\boldsymbol{I}$	40.411	-	46.796	-
$\boldsymbol{q}$	0.469	-	0.373	-
$\boldsymbol{\mu}$	-	-0.923	-	-0.928
$\boldsymbol{\sigma}$	0.308	0.280	0.265	0.266
$\boldsymbol{\lambda}$	0.048	0.047	-	-
$\boldsymbol{\mu_{jump}}$	0.135	-0.108	-	-
$\boldsymbol{\delta}$	0.266	0.243	-	-
Expected Path MAPE	**13.58%**	14.19%	13.77%	14.10%
Out-of-Time Expected Path MAPE	**16.87%**	18.23%	17.32%	17.98%
KS Statistic	**0.320 *****	0.359 ***	0.435 ***	0.448 ***

The following plots are the models simulated for 10,000 paths using the above, averaged, optimized parameters. The top plot displays the price evolutions generated by each model and the actual price of the YBBZ4 futures. The bottom subplots are the KDE simulated probability density functions of log returns for the simulated models and real YBBZ4 futures.


Model 1 Optimized Simulation
YBBZ4 Comdty
Mean Simulated Path
Closest Simulated Path
Out-of-Sample Begins
YBBZ4 Comdty
17.5
15.0
12.5
10.0
7.5
5.0
2.5
2023-05
2023-07
2023-09
2023-11
2024-01
2024-03
Time (Days)
Real Credit Data
Monte Carlo Simulation
Probability Density
20
10
0
−0.20
−0.15
−0.10
−0.05
0.00
0.05
0.10
Log Returns


Model 2 Optimized Simulation
YBBZ4 Comdty
Mean Simulated Path
Closest Simulated Path
Out-of-Sample Begins
YBBZ4 Comdty
12
10
8
6
4
2
2023-05
2023-07
2023-09
2023-11
2024-01
2024-03
Time (Days)
Real Credit Data
Monte Carlo Simulation
Probability Density
20
10
0
−0.20
−0.15
−0.10
−0.05
0.00
0.05
0.10
Log Returns

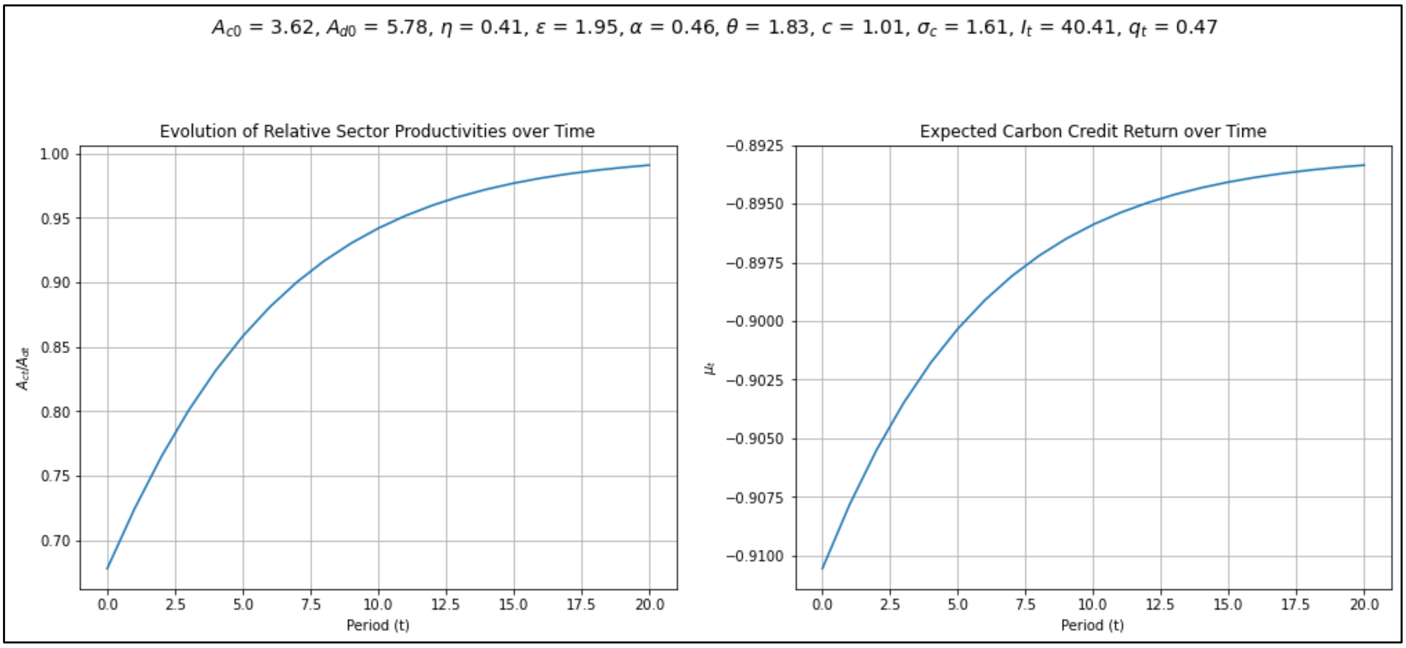




7. Discussion, Model Limitations and Possible Extensions:

From the results presented in section 6 we can first see that Model 1 is the best performing model with respect to both fitting the general price trend of the future contract's price trajectory and at simulating the estimated log returns probability density function. By comparing Model 1 with Model 2 we can see that the economic model being used to drive the expected return parameter in the jump-diffusion model yields improved results in the fit of both the price trajectory and log returns distribution. This is also further validated by comparing the two pure diffusion processes, with Model 3 outperforming Model 4 in both elements. From these results I conclude that the proposed economic engine for modelling the expected price return trajectory of the YBBZ4 futures contract may have some validity. The results are not as clear cut for the relevance of the jump process in this model, however. Model 1 clearly outperforms its diffusion only process counterpart (Model 3) in both fitting the trend and log return distribution, but Model 2 only outperforms Model 4 with respect to fitting the log return distribution. I would like to note however, that I still do believe that the jump process is particularly relevant in the modelling of VCM reference credits, and especially this model. Despite these specific futures contract not displaying any major jumps in the sample period which a simple Brownian process could not replicate, both other VCM credit related securities and a plethora of compliance credits do have tendencies of displaying jump-like price realisations such as the ones shown in Song et al. 2019. Further, since the model make a major assumption regarding these credits having no quality related concerns, allowing for jumps in the model can compensate for this limitation as you could associate quality related news publications which have sharp effects on the credit prices through the jump-process.

I would now also like to use the intuition of the economic model to understand the potential causes of the negative price trajectory evident in YBBZ4 futures and many other exchange traded VCM credit securities alike. Based on the optimized results in section 6, both in Model 1 and 3 the relative productivity at $t = 0, \frac{A_{c0}}{A_{d0}}$, is shown to be approximately 0.6, implying relatively higher initial productivity in the dirty sector. The implication of this on the evolution of relative sector productivities and μ_t are shown in the following two plots.



Given that the relative productivity between the clean and dirty sector starts off as less than 1, the model suggests that the productivities between the sectors in equilibrium should evolve such that relative productivity converges to 1, at a diminishing rate. This is what gives both curves their shape. Fundamentally, this is driven through the knowledge spillover effect which was implemented through the way we defined the probability of successful innovation and the fact that continued innovation becomes less likely the higher the existing productivity level is. Therefore, according to the model, when $A_{ct} < A_{dt}$, the clean sector will face higher probability of successful innovation compared to the dirty sector, as they have outsized benefit from the existing knowledge base of the competing sector, while also having to innovate from a relatively lower level of existing productivity. As seen, however, as relative productivity converges to 1, the expected return of the carbon credits will be increasing but will still converge to a negative value in this case. To understand what is driving this result, I restate condition (45) as:

$\mu_t < 0$ if:

$$\frac{\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}+1]+1+(\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{-q_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}} < (c\sigma_c I_t)^{\frac{1}{2}}$$

Since $\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} = \frac{A_{ct}}{A_{dt}}$, we can re-write as:

$$\frac{\frac{A_{ct}}{A_{dt}}[(\theta^G)^{\epsilon}+1]+1+(\theta^G)^{\epsilon}\left(\frac{A_{ct}}{A_{dt}}\right)^{2}}{-q_t\frac{A_{ct}}{A_{dt}}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\frac{A_{ct}}{A_{dt}}} < (c\sigma_c I_t)^{\frac{1}{2}}$$

As $t \to \infty, \frac{A_{ct}}{A_{dt}} \to 1$:

$$\frac{1+(\theta^G)^{\epsilon}}{-q_t[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}} < \xi^* \quad (46)$$

Where, ξ^* is the threshold value: $\frac{1}{2}(c\sigma_c I_t)^{\frac{1}{2}}$.

We can see that the LHS of the condition is increasing in the excess utility "good" type agents gain from green consumption, θ^G, the elasticity of substitution between the clean and dirty goods, ϵ, and the proportion of green agents, q_t. Therefore, according to this model, if the ICE NBS Vintage 2023-2027 futures are an appropriate proxy for a reference voluntary carbon credit, that is being priced efficiently by the markets such that the return equivalency between the VCM and R&D sector holds, the negative trend in the price is explained through the three stated parameters being insufficiently high. Going further, these parameters all stem from the demand functions of the final consumers, and an increase in any (or all) three parameters would increase the relative demand for clean consumption goods compared to dirty consumption goods. Hence, more generally, the negative price trend in the credit is fundamentally driven by a lack of demand for clean consumption goods compared to dirty consumption goods from the final consumers.

I now turn my attention to some of the limitations of this model. Besides the major underlying assumptions and approximations being made, I would also like to comment on the improved fit shown in the results in section 6. This might simply be a result of the increased number of parameters being incorporated into the pure jump-diffusion model and defining μ_t as a curve instead of a constant, allowing for the Powell method's minimization to fine-tune the fit with more freedom, yielding better results. To this I would like to note that this model is not necessarily an attempt to forecast voluntary carbon credit prices or their return distributions with high accuracy. The papers mention in the literature review achieve vastly superior results utilizing alternative statistical and/or machine learning methods for compliance market credits, which could also be implemented for VCM credits as well. The contribution of this paper is more focused around understanding the theoretical underlying drivers of the price evolution of voluntary carbon credits by hypothesizing that their returns should be fundamentally related to the R&D sector. The two sector Schumpeterian growth which I modify to derive the model is a possible path for deriving this relationship, but other paths could be explored as well. The fitting and parameter estimation of the fully integrated jump-diffusion model is only done to understand what the prices seen in the VCM could be implying using the context of the model and is not a forecasting exercise per se.

Extending the model could be done by endogenizing the three primary parameters θ^G, ϵ and q_t. Bezin, 2019, provides the cultural transmission mechanism for the evolution of q_t and is the most direct way this model could be extended. Beside this, the theoretical findings from the model could be used to conduct an empirical study which could potentially help with validating the effectiveness of the model. More precisely, it is shown that according to the model, $\frac{A_{ct}}{A_{dt}}$ shapes the expected return parameter in the jump-diffusion model. Using proxies for the relative productivities of the two sectors as explanatory variables for carbon credit prices could be done. Proxies can include patent release data which is something being explored in papers such Jee & Srivastav, 2022, with methodologies for categorizing patents into "clean" and "dirty" having been well defined.

8. Conclusion:

The voluntary carbon market can play a vital role for the transition towards a greener, net-neutral economy. The market however, even with its recent boom, has been held back due to various plaguing elements. One of these elements being uncertainty surrounding the price of voluntary carbon credits and the illiquidity evident. In the paper I proposed a novel approach for understanding the price evolution of voluntary carbon credits. The contribution is based on linking the expected returns of a reference voluntary carbon credit to the R&D sector and implementing the result using a jump-diffusion model to capture jumps in prices driven from news releases concerning the market. The model simulation suggests improved price trajectory and return distribution fit compared to a pure jump-diffusion model. The main theoretical conclusion derived from the model is that the negative price trajectory in the VCM is driven by a lack of long-run equilibrium final good demand for the "clean" consumption good. The shape of the expected return curve is also shown to be driven by the relative productivity between the clean and dirty good sectors, forming the basis of possible empirical work that could aid in further understanding the dynamics of the voluntary carbon market.

9. Appendix:

2.1 Demand for clean and dirty goods:

For agent type G:

$$U^G(x_{ct}, x_{dt}) = \ln\left[\left(\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}\right]$$

$$\frac{\partial U^G}{\partial x_{dt}} = \frac{\frac{\partial}{\partial x_{dt}}\left[\left(\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}\right]}{\left(\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}} = 0$$

$$\frac{\frac{\epsilon}{\epsilon-1}\left(\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{1}{\epsilon-1}}\frac{\partial}{\partial x_{dt}}\left[\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}\right]}{\left(\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}\right)^{\frac{\epsilon}{\epsilon-1}}} = 0$$

$$\frac{\frac{\epsilon}{\epsilon-1}\frac{\partial}{\partial x_{dt}}\left[\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}\right]}{\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}} = 0$$

$$\frac{\frac{\epsilon}{\epsilon-1}\frac{\epsilon-1}{\epsilon}\left[\left(-\frac{p_{dt}}{p_{ct}}\right)\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{-\frac{1}{\epsilon}} + x_{dt}^{-\frac{1}{\epsilon}}\right]}{\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}} = 0$$

$$\frac{\left(-\frac{p_{dt}}{p_{ct}}\right)\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{-\frac{1}{\epsilon}} + x_{dt}^{-\frac{1}{\epsilon}}}{\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{\frac{\epsilon-1}{\epsilon}} + x_{dt}^{\frac{\epsilon-1}{\epsilon}}} = 0$$

$$\left(-\frac{p_{dt}}{p_{ct}}\right)\theta^G\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)^{-\frac{1}{\epsilon}} + x_{dt}^{-\frac{1}{\epsilon}} = 0$$

$$x_{dt} = \left(\frac{p_{dt}}{p_{ct}}\right)^{-\epsilon}(\theta^G)^{-\epsilon}\left(\frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}x_{dt}\right)$$

$$x_{dt} = \left(\frac{I_t}{p_{ct}}\right)\frac{1}{\left(\frac{p_{dt}}{p_{ct}}\right)^{\epsilon}(\theta^G)^{\epsilon} + \frac{p_{dt}}{p_{ct}}}$$

$$x_{dt}^G = \frac{I_t}{p_{dt}}\frac{1}{\left(1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^{\epsilon}\right)} \quad (3a)$$

Substituting $(3a)$ back into the budget constraint (2) yields:

$$x_{ct} = \frac{I_t}{p_{ct}} - \frac{p_{dt}}{p_{ct}}\frac{I_t}{p_{dt}}\frac{1}{\left[1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^{\epsilon}\right]}$$

$$x_{ct} = \frac{I_t}{p_{ct}}\left[1 - \frac{1}{\left[1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^{\epsilon}\right]}\right]$$

$$x_{ct}^G = \frac{I_t}{p_{ct}}\left[\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^{\epsilon}}{1 + \left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^{\epsilon}}\right] \quad (4a)$$

2.4.1: Innovation Sector:

$$\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{jt-1}(\cdot)} = \frac{\dfrac{\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}}{1+\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}}}{\dfrac{\left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{-1}}{1+\left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{-1}}}$$

$$\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{jt-1}(\cdot)} = \frac{\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}}{1+\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}} \frac{1+\left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{-1}}{\left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{-1}}$$

$$\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{jt-1}(\cdot)} = \frac{\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}}{\left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{-1}} \frac{1+\left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{-1}}{1+\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}}$$

$$\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{jt-1}(\cdot)} = \left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-2} \frac{1+\left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{-1}}{1+\left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}}$$

$$\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{jt-1}(\cdot)} = \left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{2} * \frac{1+\frac{A_{ct-1}}{A_{dt-1}}}{1+\frac{A_{dt-1}}{A_{ct-1}}}$$

$$\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{jt-1}(\cdot)} = \left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{2} * \frac{\frac{A_{dt-1}+A_{ct-1}}{A_{dt-1}}}{\frac{A_{ct-1}+A_{dt-1}}{A_{ct-1}}}$$

$$\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{jt-1}(\cdot)} = \left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{2} * \frac{A_{dt-1}+A_{ct-1}}{A_{dt-1}} * \frac{A_{ct-1}}{A_{ct-1}+A_{dt-1}}$$

$$\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{jt-1}(\cdot)} = \left(\frac{A_{dt-1}}{A_{ct-1}}\right)^{2} * \frac{A_{ct-1}}{A_{dt-1}}$$

$$\frac{\tilde{A}_{ct-1}(\cdot)}{\tilde{A}_{jt-1}(\cdot)} = \left(\frac{A_{ct-1}}{A_{dt-1}}\right)^{-1}$$

2.4.2: Innovation Sector:

$$\frac{\pi_{ct}}{\pi_{dt}} = \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} \frac{\left[\dfrac{q_t(\theta^G)^{\epsilon}\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)+(1-q_t)\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}\right)}{\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}\right)\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}\right]}{\left[\dfrac{q_t\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)+(1-q_t)\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}\right)}{\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}\right)\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}\right]}$$

$$\frac{\pi_{ct}}{\pi_{dt}} = \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} \frac{q_t(\theta^G)^{\epsilon}\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)+(1-q_t)\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}\right)}{\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}\right)\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}$$
$$* \frac{\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}\right)\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}{q_t\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)+(1-q_t)\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}\right)}$$

$$\frac{\pi_{ct}}{\pi_{dt}} = \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma} \frac{q_t(\theta^G)^{\epsilon}\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)+(1-q_t)\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}\right)}{q_t\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)+(1-q_t)\left(1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}\right)}$$

$$\frac{\pi_{ct}}{\pi_{dt}}=\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\frac{q_t(\theta^G)^{\epsilon}+q_t(\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}-q_t+(1-q_t)\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}+1}{q_t+q_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}-q_t+(1-q_t)\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}+1}$$

$$\frac{\pi_{ct}}{\pi_{dt}}=\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\frac{q_t(\theta^G)^{\epsilon}+q_t(\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}-q_t+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}-q_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}+1}{q_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}-q_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}(\theta^G)^{\epsilon}+1}$$

$$\frac{\pi_{ct}}{\pi_{dt}}=\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\frac{q_t[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}}{-q_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}} \quad (34)$$

2.4.3: Innovation Sector:

$$\frac{\partial\frac{R_{ct}}{R_{dt}}}{\partial q_t}=\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\left[\frac{d}{dq_t}(f(\cdot))\right]$$

$$f(\cdot)=\frac{q_t[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}}{-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}}$$

$$f(\cdot)=\left(q_t[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)*\frac{1}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}$$

$$\frac{\partial f(\cdot)}{\partial q_t}=\frac{[(\theta^G)^{\epsilon}-1]}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}+ \\ *\frac{\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]\left(q_t[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)^2}$$

$$\frac{\partial f(\cdot)}{\partial q_t}=\frac{[(\theta^G)^{\epsilon}-1]\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)^2}+ \\ *\frac{\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]\left(q_t[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)^2}$$

$$\frac{\partial f(\cdot)}{\partial q_t}=[(\theta^G)^{\epsilon}-1]\left\{\frac{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)^2}+ \\ *\frac{\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\left(q_t[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)^2}\right\}$$

$$\frac{\partial f(\cdot)}{\partial q_t}=[(\theta^G)^{\epsilon}-1]\left\{\frac{1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}+\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}[(\theta^G)^{\epsilon}-1]+1+(\theta^G)^{\epsilon}\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{\gamma}\right)^2}\right\}$$

$$\frac{\partial f(\cdot)}{\partial q_t} = [(\theta^G)^\epsilon - 1]\left\{\frac{\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma [(\theta^G)^\epsilon + 1] + 1 + (\theta^G)^\epsilon \left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma [(\theta^G)^\epsilon - 1] + 1 + (\theta^G)^\epsilon \left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right)^2}\right\}$$

Therefore, we can see that:

$$\frac{\partial \frac{R_{ct}}{R_{dt}}}{\partial q_t} = \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma * [(\theta^G)^\epsilon - 1] * \left\{\frac{\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma [(\theta^G)^\epsilon + 1] + 1 + (\theta^G)^\epsilon \left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma [(\theta^G)^\epsilon - 1] + 1 + (\theta^G)^\epsilon \left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right)^2}\right\} > 0 \quad (36)$$

2.5.1 Environmental Stock:

$$z_{jkt} = [\Omega(E_t)p_{jt}]^{\frac{1}{1-a}} A_{jkt} \frac{X_{jt}}{[p_{jt}]^{\frac{a}{1-a}}[\Omega(E_t)]^{\frac{1}{1-a}} A_{jt}}$$

$$z_{jkt} = p_{jt}X_{jt}\frac{A_{jkt}}{A_{jt}}$$

$$\int_0^1 z_{jkt}\,dk = \int_0^1 p_{jt}X_{jt}\frac{A_{jkt}}{A_{jt}}\,dk = p_{jt}X_{jt} = \begin{cases} I_t\left(q_t\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon} + (1-q_t)\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right) \\ I_t\left(q_t\frac{1}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon} + (1-q_t)\frac{1}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right) \end{cases}$$

$$E_{t+1} = (1+b)E_t - \sigma_c I_t\left(q_t\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon} + (1-q_t)\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right)$$
$$- (1+c)\sigma_c I_t\left(q_t\frac{1}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon} + (1-q_t)\frac{1}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right)$$

$$E_{t+1} = (1+b)E_t - \sigma_c I_t\left(q_t\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon} + (1-q_t)\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right)$$
$$- (1+c)\sigma_c I_t\left(\frac{q_t}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon} + \frac{(1-q_t)}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right) \quad (38)$$

2.5.2 Environmental Stock:

$$\frac{\partial \int_0^1 z_{ckt}\,dk}{\partial q_t} = I_t\left(\frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon} - \frac{\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right)$$

$$\frac{\partial \int_0^1 z_{ckt}\,dk}{\partial q_t} = I_t\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\left(\frac{(\theta^G)^\epsilon - 1}{\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right]\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\right]}\right) > 0$$

$$\frac{\partial \int_0^1 z_{dkt}\,dk}{\partial q_t} = I_t\left(\frac{1}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon} - \frac{1}{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}}\right)$$

$$\frac{\partial \int_0^1 z_{dkt}\,dk}{\partial q_t} = I_t\left(\frac{1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon} - \left(1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right)}{\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right]\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\right]}\right)$$

$$\frac{\partial \int_0^1 z_{dkt}dk}{\partial q_t} = I_t\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\left(\frac{1-(\theta^G)^\epsilon}{\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right]\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\right]}\right) < 0$$

$$\frac{\partial E_{t+1}}{\partial q_t} = -\sigma_c I_t\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\left(\frac{(\theta^G)^\epsilon - 1}{\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right]\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\right]}\right) - (1+c)\sigma_c I_t\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\left(\frac{1-(\theta^G)^\epsilon}{\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right]\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\right]}\right)$$

$$\frac{\partial E_{t+1}}{\partial q_t} = I_t\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\left\{-\sigma_c\left(\frac{(\theta^G)^\epsilon - 1}{\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right]\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\right]}\right) - (1+c)\sigma_c\left(\frac{1-(\theta^G)^\epsilon}{\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right]\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\right]}\right)\right\}$$

$$\frac{\partial E_{t+1}}{\partial q_t} = I_t\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\left(\frac{(\theta^G)^\epsilon - 1}{\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right]\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\right]}\right)(-\sigma_c + (1+c)\sigma_c)$$

$$\frac{\partial E_{t+1}}{\partial q_t} = c\sigma_c I_t\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\left(\frac{(\theta^G)^\epsilon - 1}{\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}(\theta^G)^\epsilon\right]\left[1+\left(\frac{p_{ct}}{p_{dt}}\right)^{1-\epsilon}\right]}\right) > 0$$

$$\frac{\partial E_{t+1}}{\partial q_t} = c\sigma_c I_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\left(\frac{(\theta^G)^\epsilon - 1}{\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma(\theta^G)^\epsilon\right]\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right]}\right) > 0 \quad (39)$$

2.5.3 Environmental Stock:

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \left(\frac{\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma(\theta^G)^\epsilon\right]\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right]}{c\sigma_c I_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon - 1]}\right) * \left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon - 1] * \left\{\frac{\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon + 1] + 1 + (\theta^G)^\epsilon\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon - 1] + 1 + (\theta^G)^\epsilon\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right)^2}\right\}$$

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \frac{1}{c\sigma_c I_t}\left(\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma(\theta^G)^\epsilon\right]\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right]\right) * \left\{\frac{\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon + 1] + 1 + (\theta^G)^\epsilon\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon - 1] + 1 + (\theta^G)^\epsilon\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right)^2}\right\}$$

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \frac{1}{c\sigma_c I_t}\left(\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma(\theta^G)^\epsilon\right]\left[1+\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right]\right) * \left\{\frac{\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon + 1] + 1 + (\theta^G)^\epsilon\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon - 1] + 1 + (\theta^G)^\epsilon\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right)^2}\right\}$$

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \frac{1}{c\sigma_c I_t}\left[\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon + 1] + 1 + (\theta^G)^\epsilon\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}\right] * \left\{\frac{\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon + 1] + 1 + (\theta^G)^\epsilon\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{\left(-q_t\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon - 1] + 1 + (\theta^G)^\epsilon\left(g\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma\right)^2}\right\}$$

$$\frac{\partial \frac{\Pi_{ct}}{\Pi_{dt}}}{\partial E_{t+1}} = \frac{1}{c\sigma_c I_t} * \left\{\frac{\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon + 1] + 1 + (\theta^G)^\epsilon\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^{2\gamma}}{-q_t\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma[(\theta^G)^\epsilon - 1] + 1 + (\theta^G)^\epsilon\left(g(\cdot)\frac{A_{ct-1}}{A_{dt-1}}\right)^\gamma}\right\}^2$$

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