Chapter 5: Eigenvalues and Eigenvectors

Linear Algebra Done Right (4th Edition), by Sheldon Axler

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Contents

5A: Invariant Subspaces

Definition 1 (operator)**.** *A linear map from a vector space to itself is called an **operator**.*

Definition 2 (invariant subspace)**.** *Suppose $T \in \mathcal{L}((V))$. A subspace U of V is called **invariant** under T if $Tu \in U$ for every $u \in U$.*

Remark 3. *Four common invariant subspaces of $T \in \mathcal{L}(V)$ are:*
$\{0\}, V, \textit{null}\, T, \textit{range}\, T$. *To discover invariant subspaces other than $\{0\}, V$ (the simplest possible nontrivial one is the invariant subspace of dimension one) motivates the concept of eigenspaces.*

Definition 4 (eigenvalue)**.** *Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an **eigenvalue** of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.*

Remark 5. *Note that here we can think of $U = \{\lambda v \colon \lambda \in \mathbb{F}\} = \textit{span}(v)$ as the one-dimensional invariant subspace spanned by the eigenvector v.*

Lemma 6 (equivalent conditions to be an eigenvalue)**.** *Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then the following are equivalent:*

(a) λ is an eigenvalue of T.

(b) $T - \lambda I$ is not injective.

(c) $T - \lambda I$ is not surjective.

(d) $T - \lambda I$ is not invertible.

Definition 7 (eigenvector)**.** *Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an **eigenvector** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.*

Remark 8. *A vector $v \in V$ with $v \neq 0$ is an eigenvector of T corresponding to λ if and only if $v \in \textit{null}\,(T - \lambda I)$.*

$$Tv = \lambda v \iff (T - \lambda I)v = 0$$

Lemma 9 (linearly independent eigenvectors)**.** *Suppose $T \in \mathcal{L}(V)$. Then every list of eigenvectors of T corresponding to distinct eigenvalues of T is linearly independent.*

Theorem 10 (operator cannot have more eigenvalues than dimension of vector space)**.** *Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.*

Definition 11 (T^m)**.** *Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.*

- *$T^m \in \mathcal{L}(V)$ is defined by $T^m = \underbrace{T \cdots T}_{m \text{ times}}$.*

- *T^0 is defined to be the identity operator I on V.*
- *If T is invertible with inverse T^{-1}, then $T^{-m} \in \mathcal{L}(V)$ is defined by*

$$T^{-m} = (T^{-1})^m.$$

Definition 12 ($p(T)$)**.** *Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by*

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$. Then $p(T)$ is the operator on V defined by

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m.$$

Remark 13. *If we fix an operator $T \in \mathcal{L}(V)$, then the function from $\mathcal{P}(\mathbb{F})$ to $\mathcal{L}(V)$ given by $p \mapsto p(T)$ is linear.*

Definition 14 (product of polynomials)**.** *If $p, q \in \mathcal{P}(\mathbb{F})$, then $pq \in \mathcal{P}(\mathbb{F})$ is the polynomial defined by*

$$(pq)(z) = p(z)q(z)$$

for all $z \in \mathbb{F}$.

Theorem 15. *Suppose $p, q \in \mathcal{P}(\mathbb{F})$ and $T \in \mathcal{L}(V)$. Then*

(a) $(pq)(T) = p(T)q(T)$*;*

(b) $p(T)q(T) = q(T)p(T)$*.*

Definition 16 (null space and range of $p(T)$ are invariant under T)**.** *Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Then null $p(T)$ and range $p(T)$ are invariant under T.*

Problem 1

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V.

(a) Prove that if $U \subseteq \operatorname{null} T$, then U is invariant under T.

(b) Prove that if $\operatorname{range} T \subseteq U$, then U is invariant under T.

Proof. (a) Suppose for contradiction that U is not invariant under T. Then this means that there exists $u \in U$ such that $Tu \notin U$. We know that $Tu = 0 \in \operatorname{null} T$. As $U \in \operatorname{null} T$, this forms a contradiction.

(b) Take $u \in U$, then $Tu \in \operatorname{range} T \subseteq U$, and thus $Tu \in U$.

□

Problem 3

Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of every collection of subspaces of V invariant under T is invariant under T.

Proof. Suppose $\{V_i\}_{i=1}^m$ for positive integer m is a collection of invariant subspaces. We hope to prove that $\bigcap_{i=1}^m V_i$ is invariant.

Take any $v \in \bigcap_{i=1}^m V_i$, then we know that $Tv \in V_i$ for each i and thus $Tv \in \bigcap_{i=1}^m V_i$. □

Problem 5

Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is defined by $T(x, y) = (-3y, x)$. Find the eigenvalues of T.

Proof. We have that

$$T(x, y) = (-3y, x) = \lambda(x, y)$$

Then this means that $-3y = \lambda x, x = \lambda y$. Solving this gives that $\lambda^2 = -3$ and thus $\lambda = \pm\sqrt{3}i$. □

Problem 7

Define $T \in \mathcal{L}(\mathbb{F}^3)$ by $T(z_1, z_2, z_3) = (2z_2, 0, 5z_3)$. Find all eigenvalues and eigenvectors of T.

Proof.

$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3) = \lambda(z_1, z_2, z_3)$$

Then $\lambda z_1 = 2z_2, \lambda z_2 = 0, \lambda z_3 = 5z_3$. This gives that $\lambda = 5$ or 0. For $\lambda = 0$, the eigenvector would be $\{(z_1, 0, z_3) \colon z_1, z_3 \in \mathbb{F}\}$. For $\lambda = 5$, we have that $\{(0, 0, z_3) \colon z_3 \in \mathbb{F}\}$. □

Problem 9

Define $T\colon \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $Tp = p'$. Find all eigenvalues and eigenvectors of T.

Proof.

$$Tp = p' = \lambda p$$

This means that $\deg p' = \deg p$ so $\deg p = 0$. In this case, the only satisfying solution is $\lambda = 0$ and correspondingly $v = \{(c)\colon c \in \mathbb{F}\}$. $\square$

Problem 11

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\alpha \in \mathbb{F}$. Prove that there exists $\delta > 0$ such that $T - \lambda I$ is invertible for all $\lambda \in \mathbb{F}$ such that $0 < |\alpha - \lambda| < \delta$.

Proof. Let λ^* be the eigenvalue of T which is closest to α, i.e. $\lambda^* = \min_\lambda\{|\lambda - \alpha|\}$. We can take $\delta = |\alpha - \lambda^*|/2$, then we know that under the condition $T - \lambda I$ is injective and thus reaching the desired result. $\square$

Problem 13

Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible.

(a) Prove that T and $S^{-1}TS$ have the same eigenvalues.

(b) What is the relationship between the eigenvectors of T and the eigenvectors of $S^{-1}TS$?

Proof. (a) Notice that $S^{-1}v$ is an eigenvector of $S^{-1}TS$ for $v \in V$. For the forward direction, take λ to be the eigenvalue of T. Then we know that $Tv = \lambda v$ for all $v \in V$. Let's consider $S^{-1}TS(S^{-1}u) = S^{-1}Tu = S^{-1}\lambda u = \lambda(S^{-1}u)$. Therefore, λ is also an eigenvalue for $S^{-1}TS$. Conversely, take λ to be the eigenvalue of $S^{-1}TS$. Then we have that $S^{-1}TS(S^{-1}v) = S^{-1}Tv = \lambda S^{-1}v$. Multiplying S on both sides yield that $Tv = \lambda v$. Thus λ is also an eigenvalue of T.

(b) v is an eigenvector of $S^{-1}TS$ iff $u = Sv$ is an eigenvector of T.

$\square$

Problem 15

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{F}$. Show that λ is an eigenvalue of T if and only if λ is an eigenvalue of the dual operator $T' \in \mathcal{L}(V')$.

Proof. Forward direction, take λ to be an eigenvalue of T, then we have that $Tv = \lambda v$. Then take any $\varphi \in V'$, we have that $T'(\varphi)(v) = \varphi \circ Tv = \lambda\varphi(v)$.

Backward direction, take λ to be an eigenvalue of T', then we know there exists nonzero $\varphi \in V'$ such that $T'\varphi(v) = \varphi T(v) = \lambda\varphi(v)$ This implies that

$$\varphi(Tv - \lambda v) = 0$$

Suppose for contradiction that λ is not an eigenvalue of T, then this means that $T - \lambda I$ is invertible and therefore its image equals the whole space V. Then this means that φ is a zero functional, forming a contradiction. Hence, $Tv = \lambda v$ and thus λ is an eigenvalue of T as well.

□

Problem 16

Suppose $v_1, \ldots, v_n$ is a basis of V and $T \in \mathcal{L}(V)$. Prove that if λ is an eigenvalue of T, then

$$|\lambda| \leq n \max\{|\mathcal{M}(T)_{j,k}| \colon 1 \leq j, k \leq n\},$$

where $\mathcal{M}(T)_{j,k}$ denotes the entry in row j, column k of the matrix of T with respect to the basis $v_1, \ldots, v_n$.

Proof. We know that

$$Tv_k = \sum_{j=1}^{n} \mathcal{M}(T)_{j,k} v_j = \lambda v_k$$

and that by Triangular inequality we have

$$|Tv_k| \leq \sum_{j=1}^{n} |\mathcal{M}(T)_{j,k} v_j| \leq M \sum_{j=1}^{n} |v_j|$$

where we take $M = \max\{|\mathcal{M}(T)_{j,k}| \colon 1 \leq j, k \leq n\}$. Here we take k to be the index such that maximizes the norm, i.e. $k = \max_k |v_k|$. Then we have that

$$|Tv_k| = |\lambda||v_k| \leq nM|v_k|$$

which gives us that

$$|\lambda| \leq nM$$

□

Problem 18

Suppose $\mathbb{F} = \mathbb{R}, T \in \mathcal{L}(V)$, and $\lambda \in \mathbb{C}$. Prove that λ is an eigenvalue of the complexification T_C if and only if $\overline{\lambda}$ is an eigenvalue of T_C, where $T_C \colon \mathcal{L}(V_C)$ is defined by

$$T_C(u + iv) = Tu + iTv$$

for all $u, v \in V$.

Proof. Let $v_1, \ldots, v_n$ be basis of V_C. Suppose λ is an eigenvalue of T_C and $v = \sum_{i=1}^n a_i v_i$ is the corresponding eigenvector. Then we have that

$$\lambda v = Tv = \sum_{i=1}^n a_i T v_i$$

Taking conjugates yields that

$$\overline{\lambda}\overline{v} = \overline{\sum_{i=1}^n a_i T v_i} = \sum_{i=1}^n \overline{a}_i \overline{T v_i}$$

Let M be the matrix of T with real entries, and then we have

$$\overline{T v_i} = \overline{\sum_{j=1}^n M_{j,i} v_j} = \sum_{j=1}^n M_{j,i} \overline{v}_j = T\overline{v}_i$$

Substitute this back to the previous equation gives that

$$\overline{\lambda}\overline{v} = T\left(\overline{\sum_{i=1}^n a_i v_i}\right) = T\overline{v}$$

Hence $\overline{\lambda}$ is also an eigenvector. $\square$

Problem 19

Show that the forward shift operator $T \in \mathcal{L}(\mathbb{F}^\infty)$ defined by

$$T(z_1, z_2, \cdots) = (0, z_1, z_2, \cdots)$$

has no eigenvalues.

Proof. We have that

$$(0, z_1, z_2, \cdots) = \lambda(z_1, z_2, \cdots)$$

So we either have $\lambda = 0$ or $z_1 = 0 = z_2 = \cdots$. In the first case, we have that $z_1 = 0 = z_2 = \cdots$ and thus there is no nonzero eigenvector. In the second case, there is also no nonzero eigenvectors. $\square$

Problem 21

Suppose $T \in \mathcal{L}(V)$ is invertible.

(a) Suppose $\lambda \in \mathbb{F}$ with $\lambda \neq 0$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

(b) Prove that T and T^{-1} have the same eigenvectors.

Proof.

$$Tv = \lambda v \quad \Longleftrightarrow \quad T^{-1}Tv = T^{-1}\lambda v \quad \Longleftrightarrow \quad \frac{1}{\lambda}v = T^{-1}v$$

□

Problem 23

Suppose V is finite-dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

Proof. $\Rightarrow$ Let λ, v be the eigenvalue and eigenvector of ST. Then take $u = Tv$, we have that

$$TS(u) = TS(Tv) = T(ST)(v) = T(\lambda v) = \lambda(Tv) = \lambda u$$

Thus λ is also the eigenvalue of TS.

Conversely, let λ, v be eigenvalue and eigenvector of TS. Then take $u = Sv$, we have that

$$ST(u) = ST(Sv) = S(TS)(v) = S(\lambda v) = \lambda(SV) = \lambda u$$

Thus λ is also the eigenvalue of ST. □

Problem 24

Suppose A is an n-by-n matrix with entries in $\mathbb{F}$. Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $Tx = Ax$, where elements of $\mathbb{F}^n$ are thought of as n-by-1 column vectors.

(a) Suppose the sum of the entries in each row of A equals 1. Prove that 1 is an eigenvalue of T.

(b) Suppose the sum of entries in each column of A equals 1. Prove that 1 is an eigenvalue of T.

Proof. Let $x = (1, \ldots, 1)^\top$ be the all-one vector. Then we have that

$$(Tx)_{i,:} = (Ax)_{i,:} = \sum_{j=1}^{n} A_{i,j}x_j = \sum_{j=1}^{n} A_{i,j} = 1$$

Therefore, $Tx = x$ and thus 1 is an eigenvalue of T.

For the column case, we can simply take $A = A^\top$ and we know the eigenvalues of A equal $A^\top$. □

Problem 25

Suppose $T \in \mathcal{L}(V)$ and u, w are eigenvectors of T such that $u + w$ is also an eigenvector of T. Prove that u and w are eigenvectors of T corresponding to the same eigenvalue.

Proof. Suppose not. Then there exists λ_u and λ_w such that

$$Tu = \lambda_u u \quad Tw = \lambda_w w$$

We also know that

$$T(u + w) = Tu + Tw = \lambda_u u + \lambda_w w = \lambda(u + w)$$

for some λ. Solving this gives that $\lambda = \lambda_u = \lambda_w$. $\square$

Problem 26

Suppose $T \in \mathcal{L}(V)$ is such that every nonzero vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

Proof. We claim that $Tv = \lambda v$ for all nonzero $v \in V$, i.e. $T = \lambda I$. Let's get $v_1, v_2 \in V$. First we consider the case that they are linearly dependent, meaning that $v_1 = cv_2$ for some c. We have that

$$\lambda_1 v_1 = Tv_1 = cTv_2 = c\lambda_2 v_2 = \lambda_2 v_1$$

Therefore $\lambda_1 = \lambda_2$. Next we consider the case that they are linearly independent, then we have that

$$\lambda_{1+2}(v_1 + v_2) = T(v_1 + v_2) = Tv_1 + Tv_2 = \lambda_1 v_1 + \lambda_2 v_2$$

Therefore $\lambda_{1+2} = \lambda_1 = \lambda_2$, completing the proof. $\square$

Problem 28

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has at most $1 + \dim \operatorname{range} T$ distinct eigenvalues.

Proof. Suppose T has m distinct eigenvalues. For nonzero λ_k, we have that

$$T\left(\frac{1}{\lambda_k} v_k\right) = v_k$$

Thus at most $m - 1$ distinct eigenvectors are in range T, so $m - 1 \leq \dim \operatorname{range} T$ and $m = 1 + \dim \operatorname{range} T$ at most. $\square$

Problem 33

Suppose $T \in \mathcal{L}(V)$ and m is a positive integer.

(a) Prove that T is injective if and only if T^m is injective.

(b) Prove that T is surjective if and only if T^m is surjective.

Proof. (a) $\Rightarrow$ Given T is injective, take $v \in \operatorname{null} T^m$. Then we know that $T^m(v) = T(T^{m-1}v) = 0$, which implies that $T^{m-1}v = 0$. This can be recursively deduced to that $Tv = 0$ and thus $v = 0$. Therefore T^m is injective.

$\Leftarrow$ Conversely, take $v \in \operatorname{null} T$, then we have $T^m(v) = T^{m-1}(Tv) = T^{m-1}(0) = 0$. Since T^m is injective, $v = 0$.

(b) $\Rightarrow$ Given T is surjective, then $\dim rangeT = \dim V$. We also have $\dim \operatorname{range} T^2 = \dim V$ as T is surjective. Therefore, T^m is surjective.

$\Leftarrow$ Let $w \in V$, then there exists $v \in V$ such that $T^m(v) = w$. Then we have that let $u = T^{m-1}(v)$, then $Tu = w$. Thus T is also surjective. □

Problem 34

Suppose V is finite-dimensional and $v_1, \dots, v_m \in V$. Prove that the list $v_1, \dots, v_m$ is linearly independent if and only if there exists $T \in \mathcal{L}(V)$ such that $v_1, \dots, v_m$ are eigenvectors of T corresponding to distinct eigenvalues.

Proof. $\Rightarrow$ We can define $Tv_k = kv_k$ and extend T to V.

$\Leftarrow$ This is proved by Theorem in the book. □

Problem 35

Suppose that $\lambda_1, \dots, \lambda_n$ is a list of distinct real numbers. Prove that the list $e^{\lambda_1 x}, \dots, e^{\lambda_n x}$ is linearly independent in the vector space of real-valued functions on $\mathbb{R}$.

Proof. Following the hint, let $V = \operatorname{span}(e^{\lambda_1 x}, \dots e^{\lambda_n x})$. Define an operator $D \in \mathcal{L}(V)$ by $Df = f'$. Then we have that

$$De^{\lambda_k x} = \lambda_k e^{\lambda_k x}$$

Here naturally λ_k is a distinct eigenvalue with corresponding eigenvector $e^{\lambda_k x}$. By P34, we complete the proof. □

Problem 37

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(S) = TS$$

for each $S \in \mathcal{L}(V)$. Prove that the set of eigenvalues of T equals the set of eigenvalues of $\mathcal{A}$.

Proof. $\Rightarrow$ Let λ be an eigenvalue of T and v be the corresponding eigenvector of T. Suppose $v_1, \ldots, v_n$ is the basis of V, we can define $S \in \mathcal{L}(V)$ such that

$$Sv_i = v$$

for all i. Here we have that

$$\begin{aligned}
\mathcal{A}(S)(v) &= TS(v) \\
&= TS\sum_{i=1}^{n} a_i v_i \\
&= T\sum_{i=1}^{n} a_i S v_i \\
&= \sum_{i=1}^{n} a_i T v \\
&= \lambda \sum_{i=1}^{n} a_i v \\
&= \lambda \sum_{i=1}^{n} S(a_i v_i) \\
&= (\lambda S)(v)
\end{aligned}$$

Hence λ is also an eigenvalue of $\mathcal{A}(S)$.

$\Leftarrow$ Consider

$$\mathcal{A}(S) = TS = \lambda S$$

Applying v on both sides yields that

$$T(Sv) = \lambda(Sv)$$

Therefore, λ is also an eigenvalue of T. $\square$

Problem 38

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V invariant under T. The *quotient operator* $T/U \in \mathcal{L}(V/U)$ is defined by

$$(T/U)(v + U) = Tv + U$$

for each $v \in V$.

(a) Show that the definition of T/U makes sense and show that T/U is an operator on V/U.

(b) Show that each eigenvalue of T/U is an eigenvalue of T.

Proof. (a) Suppose $v-w \in U$. As U is invariant under T, $T(v-w) = Tv - Tw \in U$. So we have that $Tv + U = Tw + U$ and thus it is well-defined. To show that T/U is an operator, we can just check its linearity. $(T/U)((\lambda v_1 + U) + (v_2 + U)) = (T/U)((\lambda v_1 + v_2) + U) = \lambda_1 T v_1 + U + T v_2 + U = \lambda_1 T/U(v_1 + U) + T/U(v_2 + U)$.

(b) Let λ be the eigenvalue of T/U, then we have

$$(T/U)(v+U) = Tv + U = \lambda v + U$$

This means that

$$Tv = \lambda v + u$$

for some $u \in U$. Now suppose we also have $u' \in U$, then

$$T(v + u') = \lambda v + u + Tu'$$

We hope to find u' s.t. $u + Tu' = \lambda u'$, equivalently $(\lambda I - T)u' = u$.

Since U is invariant under T, it is also invariant under $T - \lambda I$. Here if $T - \lambda I$ is not invertible on U, then we are done (λ would be an eigenvalue). If it is, then we can define $u' = -(T - \lambda I)^{-1} u$, completing the proof. □

Problem 39

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an eigenvalue if and only if there exists a subspace of V of dimension $\dim V - 1$ that is invariant under T.

Proof. $\Rightarrow$ By the hypothesis, there exists $\lambda \in \mathbb{F}$, $v \in V$ with $v \neq 0$ s.t. $Tv = \lambda v$. We claim that

$$v^\perp = \{u \in V : v^\top u = 0\}$$

is invariant under T of dimension $n-1$. Take any $u \in v^\perp$, then $v^\top T(u) = (v^\top T)(u) = \lambda v^\top u = 0$. Hence $T(u) \in v^\perp$. This set has dimension $\dim V - 1$ as it is the orthogonal complement of a dimensional 1 subspace.

$\Leftarrow$ Let U be the the invariant subspace of $\dim V - 1$ under T. Let $\{w_1, \ldots, w_{n-1}\}$ be its basis. Then we can extend the basis to $\{w_1, \ldots, w_{n-1}, v\}$ for some $v \in V \setminus U$. Hence, we have that

$$T(v) = \sum_{i=1}^{n-1} a_i w_i + cv$$

for some scalar $c \in \mathbb{F}$. We claim that c is the eigenvalue of T. Suppose not, then we can get that

$$(T - cI)(v) = \sum_{i=1}^{n-1} a_i w_i$$

Then this means that $v \in U$, reaching a contradiction. □

Problem 40

Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Suppose $p \in \mathcal{P}(\mathbb{F})$ is a polynomial. Prove that
$$p(STS^{-1}) = Sp(T)S^{-1}$$

Proof. We note that

$$(STS^{-1})^m = (STS^{-1})(STS^{-1})\cdots(STS^{-1}) = ST^mS^{-1}$$

Therefore,

$$\begin{aligned} p(STS^{-1}) &= a_0I + a_1(STS^{-1}) + \cdots + a_m(STS^{-1})^m \\ &= a_0I + a_1(STS^1) + \cdots + a_m(ST^mS^{-1}) \\ &= S(a_0I + a_1T + \cdots + a_mT^m)S^{-1} \\ &= Sp(T)S^{-1} \end{aligned}$$

□

Problem 41

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T. Prove that U is invariant under $p(T)$ for every polynomial $p \in \mathcal{L}(\mathbb{F})$.

Proof. Take $u \in U$, then

$$p(T)(u) = a_0I(u) + a_1T(u) + \cdots + a_mT^m(u)$$

Now it suffices to prove that $T^m(u) \in U$ for all positive integer. To see this, we can make an inductive argument on m and gets the desired conclusion. □

5B: The Minimal Polynomial

Theorem 17 (existence of eigenvalues)**.** *Every operator on a finite-dimensional nonzero complex vector space has an eigenvalue.*

Definition 18 (monic polynomial)**.** *A* ***monic polynomial*** *is a polynomial whose highest-degree coefficient equals* 1.

Theorem 19 (existence, uniqueness, and degree of minimal polynomial)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then there is a unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0$. Furthermore,* $\deg p \leq \dim V$.

Definition 20 (minimal polynomial)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then the* ***minimal polynomial*** *of T is the unique monic polynomial $p \in \mathcal{P}(\mathbb{F})$ of smallest degree such that $p(T) = 0$.*

Remark 21. *This means to find the smallest positive integer m such that*

$$c_0 I + c_1 T + \cdots + c_{m-1} T^{m-1} = -T^m$$

has a solution $c_0, c_1, \ldots, c_{m-1} \in \mathbb{F}$. A way to solve this is to pick $v \in V$ with $v \neq 0$ and consider that

$$c_0 v + c_1 T v + \cdots + c_{\dim V - 1} T^{\dim V - 1} v = -T^{\dim V} v.$$

Theorem 22 (eigenvalues are the zeros of the minimal polynomial)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$.*

(a) The zeros of the minimal polynomial of T are the eigenvalues of T.

(b) If V is a complex vector space, then the minimal polynomial of T has the form

$$(z - \lambda_1) \cdots (z - \lambda_m).$$

where $\lambda_1, \ldots, \lambda_m$ is a list of all eigenvalues of T, possibly with repetitions.

Theorem 23 ($q(T) = 0 \iff q$ is a polynomial multiple of the minimal polynomial)**.** *Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $q \in \mathcal{P}(\mathbb{F})$. Then $q(T) = 0$ if and only if q is a polynomial multiple of the minimal polynomial of T.*

Theorem 24 (minimal polynomial of a restriction operator)**.** *Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and U is a subspace of V that is invariant under T. Then the minimal polynomial of T is a polynomial multiple of the minimal polynomial of $T|_U$.*

Corollary 25. *Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is not invertible if and only if the constant term of the minimal polynomial of T is 0.*

Theorem 26 (even-dimensional null space)**.** *Suppose $\mathbb{F} = \mathbb{R}$ and V is finite-dimensional. Suppose also that $T \in \mathcal{L}(V)$ and $b, c \in \mathbb{R}$ with $b^2 < 4c$. Then* $\dim$ *null* $(T^2 + bT + cI)$ *is an even number.*

Theorem 27. *Every operator on an odd-dimensional vector space has an eigenvalue.*

Problem 1

Suppose $T \in \mathcal{L}(V)$. Prove that 9 is an eigenvalue of T^2 if and only if 3 or -3 is an eigenvalue of T.

Proof. $\Rightarrow$ Given 9 is an eigenvalue of T^2, then this means that $(T^2 - 9I) = (T - 3I)(T + 3I) = 0$. So either 3 or -3 is an eigenvalue of T.

$\Leftarrow$ We know there exists nonzero $v \in V$ s.t. $Tv = \pm 3v$, so $T(Tv) = T(\pm 3v) = \pm 3T(v) = (\pm 3)^2 v = 9v$. $\square$

Problem 2

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of V invariant under T is either $\{0\}$ or infinite-dimensional.

Proof. Suppose for contradiction that there exists finite-dimensional nontrivial subspace U of V such that is invariant under T. So we know that by theorem 5.31 and 5.22 $T|_U$ has eigenvalue and so does T. This forms a contradiction to the hypothesis. $\square$

Problem 3

Suppose n is a positive integer and $T \in \mathcal{L}(\mathbb{F}^n)$ is defined by

$$T(x_1, \ldots, x_n) = (x_1 + \cdots + x_n, \ldots, x_1 + \cdots + x_n)$$

(a) Find all eigenvalues and eigenvectors of T.

(b) Find the minimal polynomial of T.

Proof. (a) Let v be such that

$$(x_1 + \cdots + x_n, \cdots, x_1 + \cdots + x_n) = (\lambda x_1, \ldots, \lambda x_n)$$

This means $\sum_{i=1}^n x_i = \lambda x_j = \lambda x_k$ for all j, k. If $\lambda = 0$, then $\sum_{i=1}^n x_i = 0$ and the eigenvector would be $\{(x_1, \ldots, x_n) \colon \sum_{i=1}^n x_i = 0\}$. If $\lambda \neq 0$, then we have that $x_i = x_j$ for all i, j and thus $\lambda = n$, the corresponding eigenvector is that $\{(a, \ldots, a) \colon a \in \mathbb{F}\}$.

(b) We know the eigenvalues are $0; n$, so it must be of the form

$$p(\lambda) = \lambda(\lambda - n)$$

$\square$

Problem 4

Suppose $\mathbb{F} = \mathbb{C}, T \in \mathcal{L}(V), p \in \mathcal{P}(\mathbb{C})$, and $\alpha \in \mathbb{C}$. Prove that α is an eigenvalue of $p(T)$ if and only if $\alpha = p(\lambda)$ for some eigenvalue λ of T.

Proof. $\Rightarrow$ Given α is an eigenvalue of $p(T)$, then we know that

$$p(z) - \alpha = c(z - \lambda_1) \cdots (z - \lambda_m)$$

for some λ_i's. Note that since p is nonconstant $c \neq 0$. Then this means that one of $z - \lambda_i = 0 \iff T - \lambda_i I = 0$. Then this means $p(\lambda_i) - \alpha = 0$ so $p(\lambda_i) = \alpha$.

$\Leftarrow$ Conversely, we have that for some nonzero v to be the eigenvector of T

$$\begin{aligned} p(T)(v) &= (a_0 I + a_1 T + \cdots + a_m T^m)(v) \\ &= (a_0 I + a_1 \lambda + \cdots + a_m \lambda^m)(v) \\ &= \alpha(v) \end{aligned}$$

□

Problem 6

Suppose $T \in \mathcal{L}(\mathbb{F}^2)$ is defined by $T(w, z) = (-z, w)$. Find the minimal polynomial of T.

Proof. Pick $v = e_1$ then $Tv = (0, 1)$ and $T^2(v) = (-1, 0)$. We have that

$$(1 + T^2)(v) = 0$$

□

Problem 8

Suppose $T \in \mathcal{L}(\mathbb{R}^2)$ is the operator of counterclockwise rotation by 1^o. Find the minimal polynomial of T.

Proof. We know that the matrix represented by the operator is that

$$T = \begin{bmatrix} \cos(1^\circ) & -\sin(1^\circ) \\ \sin(1^\circ) & \cos(1^\circ) \end{bmatrix}$$

Let $v = e_1, Tv = (\cos 1^\circ, \sin 1^\circ)$ and $T^2 v = (\cos^2 1^\circ - \sin^2 1^\circ, 2\cos 1^\circ \sin 1^\circ)$. Hence, we have that

$$\begin{aligned} (-1)1 + (2\cos 1^\circ)\cos 1^\circ &= \cos^2 1^\circ - \sin^2 1^\circ \\ (-1)0 + (2\cos 1^\circ)\sin 1^\circ &= 2\cos 1^\circ \sin 1^\circ \end{aligned}$$

Then the minimal polynomial is

$$p(T) = T^2 - 2\cos 1^\circ T - I$$

□

Problem 9

Suppose $T \in \mathcal{L}(V)$ is such that with respect to some basis of V, all entries of the matrix of T are rational numbers. Explain why all coefficients of the minimal polynomial of T are rational numbers.

Proof. Let M be the matrix of T and d be the degree of its minimal polynomial. Consider

$$\mathbf{A} = \Big(\text{vec}(I), \text{vec}(M), \dots, \text{vec}(M^d)\Big)$$

Then we know that $\mathbf{A}$ is rational and we can partition $\mathbf{A}$ into first d column and last 1 column where the last column is a linear combination of the first d linearly independent ones. This further implies that we can find a solution using Gaussian elimination, where its entries are therefore all rational as well, and it is indeed the coefficients of the minimal polynomial. $\square$

Problem 10

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$. Prove that

$$\text{span}(v, Tv, \dots, T^m v) = \text{span}(v, Tv, \dots, T^{\dim V - 1} v)$$

for all integers $m \geq \dim V - 1$.

Proof. Denote $A = \text{span}(v, Tv, \dots, T^m v)$ and $B = \text{span}(v, Tv, \dots, T^{\dim V-1} v)$.

$\Rightarrow$ Since the list A has length greater than $\dim V$, the elements after $T^{\dim V-1}v$ can be written as linear combination of the previous terms (aka terms in B). Therefore any element in A can be written as linear combination of elements in B so $A \subseteq B$.

$\Leftarrow$ This direction is trivial. $\square$

Problem 12

Define $T \in \mathcal{L}(\mathbb{F}^n)$ by $T(x_1, x_2, x_3, \dots, x_n) = (x_1, 2x_2, 3x_3, \dots, nx_n)$. Find the minimal polynomial of T.

Proof. Let $v = (1, 0, 0, \dots, 0)$. Then $T^k v = e_1$ for all $1 \leq k \leq n$. Thus we have that

$$p(T) = T^n - I$$

$\square$

Problem 13

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Prove that there exists a unique $r \in \mathcal{P}(\mathbb{F})$ such that $p(T) = r(T)$ and $\deg r$ is less than the degree of the minimal polynomial of T.

Proof. Let $m_T(x)$ be the minimal polynomial of T. By the division algorithm, there exists polynomial $q(x)$ and $r(x)$ with $\deg r < \deg m_T$ such that

$$p(x) = q(x)m_T(x) + r(x)$$

Evaluating at T then yields that

$$p(T) = r(T)$$

Here we prove the uniqueness of r. Suppose for contradiction that there exists another $r'(x)$ such that

$$p(T) = r(T) = r'(T)$$

If we now define $r''(x) = r(x) - r'(x) = 0 = r''(T)$, then it must be a multiple of $m_T(x)$ by the definition of the minimal polynomial. However, as its degree is less than m_T, this forms a contradiction, completing the proof. $\square$

Problem 15

Suppose V is a finite-dimensional complex vector space with $\dim V > 0$ and $T \in \mathcal{L}(V)$. Define $f \colon \mathbb{C} \to \mathbb{R}$ by

$$f(\lambda) = \dim \operatorname{range}\,(T - \lambda I).$$

Prove that f is not a continuous function.

Proof. Suppose λ is an eigenvalue of T and consider a sequence $\{\lambda_k\}$ converging to λ, where λ_k's are not eigenvalues of T (you may verify such a sequence exists). For λ_k near λ, we have that $f(\lambda_k) = \dim V - \dim \operatorname{null}\,(T - \lambda_k I) = \dim V$, where $f(\lambda) < \dim V$. Hence, f is discontinuous at λ and therefore not a continuous function. $\square$

Problem 17

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and p is the minimal polynomial of T. Suppose $\lambda \in \mathbb{F}$. Show that the minimal polynomial of $T - \lambda I$ is the polynomial q defined by $q(z) = p(z + \lambda)$.

Proof. First we can see that $q(T - \lambda I) = p(T) = 0$. Now it suffices to prove the minimality condition. Suppose there exists another monic polynomial $r(z)$ of lesser degree than $q(z)$ such that $r(T - \lambda I) = 0$. We can now define $s(x) = r(x - \lambda)$ and then $s(T) = r(T - \lambda I) = 0$ with $\deg s < \deg p$, contradicting the minimal polynomial assumption of p. Therefore, we complete the proof. $\square$

Problem 19

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\mathcal{E}$ be the subspace of $\mathcal{L}(V)$ defined by

$$\mathcal{E} = \{q(T) \colon q \in \mathcal{P}(\mathbb{F})\}.$$

Prove that $\dim \mathcal{E}$ equals the degree of the minimal polynomial of T.

Proof. Note that $\{a_0 I, a_1 T, a_2 T^2, a_3 T^3, \ldots, a_{n-1} T^{n-1}\} \coloneqq A$ is a basis of $\mathcal{E}$ with dimension n. Suppose $\deg m_t = m$ for notational purposes. We have that

$$\begin{aligned} A \text{ is linearly independent } &\iff \text{Express } T^n \text{ as linear combination of terms in } A \\ &\iff \text{Minimal degree of } m_T \text{ is } n \end{aligned}$$

□

Problem 21

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial has degree at most $1 + \dim \operatorname{range} T$.

Proof. Let $k = \dim \operatorname{range} T, n = \dim V$. We note that $T(\operatorname{range} T) \subseteq \operatorname{range} T$, so T naturally induces a well-defined linear operator $S \in \mathcal{L}(\operatorname{range} T)$ by restriction ($Sv = Tv \forall v \in \operatorname{range} T$). Consider the minimal polynomial $m_S(x)$ of S. We know that $\deg m_S \leq \dim \operatorname{range} T = k$. Under the construction, we also have $T^2 = TS$, which implies that T behaves like S on range T. Note for any $v \in \operatorname{null} T, T(v) = 0$.

Now we can construct a polynomial that annihilates T on V. Define $q(x) = x \cdot m_S(x)$. We claim that $q(T) = 0$. This is true since for $v \in \operatorname{null} T, q(T)(v) = T \cdot m_S(T)(v) = T(0) = 0$. For $v \in \operatorname{range} T$, we have $m_S(T)(v) = m_S(S)(v) = 0$ and therefore $q(T)(v) = 0$. Note that under this construction, we have $\deg q \leq k + 1$. Since the minimal polynomial divides p, its degree is also upper bounded by $1 + \dim \operatorname{range} T$.

□

Problem 22

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if $I \in \operatorname{span}(T, T^2, \ldots, T^{\dim V})$.

Proof. $\Rightarrow$ This means that

$$I = a_0 T + a_1 T^2 + \cdots + a_n T^n$$

with $n = \dim V$. Rewriting the equation gives that (for the largest nonzero degree), we have

$$-I + \frac{a_0}{a_m} T + \frac{a_1}{a_m} T^2 + \cdots + T^m = 0$$

Therefore the constant term of minimal polynomial of T is not 0 and thus T is invertible.

$\Leftarrow$ We know that

$$I = a_0T + a_1T^2 + \ldots + a_nT^n = T(a_0I + a_1T + \ldots + a_nT^{n-1})$$

Define $S = (a_0I + a_1T + \ldots + a_nT^{n-1})$ so $TS = I$. At the same time,

$$ST = (a_0I + a_1T + \ldots + a_nT^{n-1})T = a_0T + a_1T^2 + \ldots + a_nT^n = I$$

So T is invertible. $\square$

Problem 23

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $n = \dim V$. Prove that if $v \in V$, then $\operatorname{span}(v, Tv, \ldots, T^{n-1}v)$ is invariant under T.

Proof. We know that $Tv, \ldots, T^{n-1}v \in \operatorname{span}(v, Tv, \ldots, T^{n-1}v)$, so it suffices to prove that $T^nv \in \operatorname{span}(v, Tv, \ldots, T^{n-1}v)$. To see this, notice that the list

$$v, Tv, \ldots, T^{n-1}v, T^nv$$

has length more than n and can be reduced to at least n terms. Therefore, we complete the proof. $\square$

Problem 28

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the minimal polynomial of $T' \in \mathcal{L}(V')$ equals the minimal polynomial of T.

Let p' and p be the minimal polynomial of T' and T respectively. We know that $T'(\varphi) = \varphi \circ T$. We have that

$$p'(\varphi) = (c_0I + c_1T' + \cdots c_k(T')^k)(\varphi) = \varphi \circ (c_0I + c_1T + \cdots + c_kT^k) = \varphi \circ p = 0$$

which essentially shows that the minimality condition is iff for p' and p, otherwise it would reach a contradiction that one is the minimal polynomial.

Problem 29

Show that every operator on a finite-dimensional vector space of dimension at least two has an invariant subspace of dimension two.

Proof. We prove this claim through the induction on $n = \dim V$. For base case, we have $\dim V = 2$, then as V is invariant, the claim is satisfied.

For inductive case, suppose the statement holds for V with dimension $n - 1$. Then consider V with dimension n. Take any nonzero $v \in V$ and consider $V \setminus \operatorname{span}(v)$, which is a space of dimension $n - 1$, so it has an invariant subspace of dimension two, which is still a subspace of V, completing the proof. $\square$

5C: Upper-Triangular Matrices

Definition 28 (matrix of an operator, $\mathcal{M}(T)$)**.** *Suppose $T \in \mathcal{L}(V)$. The* ***matrix of*** *T with respect to a basis $v_1, \ldots, v_n$ of V is the n-by-n matrix*

$$\mathcal{M}(T) = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix}$$

whose entries $A_{j,k}$ are defined by

$$Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n.$$

The notation $\mathcal{M}(T, (v_1, \ldots, v_n))$ is used if the basis is not clear from the context.

Remark 29. *Operators have square matrices.*

Definition 30 (diagonal of a matrix)**.** *The* ***diagonal*** *of a square matrix consists of the entries on the line from the upper left corner to the bottom right corner.*

Definition 31 (upper-triangular matrix)**.** *A square matrix is called* ***upper triangular*** *if all entries below the diagonal are 0.*

Theorem 32 (conditions for upper-triangular matrix)**.** *Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then the following are equivalent.*

(a) The matrix of T with respect to $v_1, \ldots, v_n$ is upper triangular.

(b) span$(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$.

(c) $Tv_k \in$ span$(v_1, \ldots, v_k)$ for each $k = 1, \ldots, n$.

Lemma 33. *Suppose $T \in \mathcal{L}(V)$ and V has a basis with respect to which T has an upper-triangular matrix with diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then*

$$(T - \lambda_1 I) \cdots (T - \lambda_n I) = 0$$

Theorem 34 (determination of eigenvalues from upper-triangular matrix)**.** *Suppose $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.*

Remark 35. *Main proof technique: $(T - \lambda_k)Iv_k \in$ span$(v_1, \ldots, v_{k-1})$ for the matrix of T to be upper-triangular.*

Lemma 36 (necessary and sufficient condition to have an upper-triangular matrix)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V if and only if the minimal polynomial equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$.*

Theorem 37 (if $\mathbb{F} = \mathbb{C}$, then every operator on V has an upper-triangular matrix)**.** *Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.*

Problem 1

Prove or give a counter example: If $T \in \mathcal{L}(V)$ and T^2 has an upper-triangular matrix with respect to some basis of V, then T has an upper-triangular matrix with respect to some basis of V.

Proof. Since T^2 has an upper triangular matrix wrt. some basis of V, its minimal polynomial equals $p_{T^2}(x) = (z - \lambda_1) \cdots (z - \lambda_m)$ for some $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$. This means that $p_{T^2}(T^2) = (T^2 - \lambda_1 I) \cdots (T^2 - \lambda_m I) = (T - \sqrt{\lambda_1} I)(T + \sqrt{\lambda_1} I) \cdots (T - \sqrt{\lambda_m} I)(T + \sqrt{\lambda_m} I) = 0$. This means that the minimal polynomial of T can also be in the form of an upper-triangular matrix.

$\square$

Problem 3

Suppose $T \in \mathcal{L}(V)$ is invertible and $v_1, \ldots, v_n$ is a basis of V with respect to which the matrix of T is upper triangular, with $\lambda_1, \ldots, \lambda_n$ on the diagonal. Show that the matrix of T^{-1} is also upper triangular with respect to the basis $v_1, \ldots, v_n$, with

$$\frac{1}{\lambda_1}, \ldots, \frac{1}{\lambda_n}$$

on the diagonal.

Proof. We first show that T^{-1} is also upper triangular wrt. the same basis. To see this, we know that $Tv_k \in \operatorname{span}(v_1, \ldots, v_k)$ for each k, or equivalently,

$$T(v_k) = \lambda_k v_k + \sum_{j=1}^{k-1} A_{j,k} v_j$$

Let $w_k = T^{-1} v_k$, then we know that $Tw_k = v_k \in \operatorname{span}(v_1, \ldots, v_k)$. So this implies that $w_k \in \operatorname{span}(v_1, \ldots, v_k)$ by property of T. Thus this completes the first part of the proof. For the second part, notice that $TT^{-1} = I$ and since the two matrices are both upper-triangular, by rule of matrix multiplication, $T_{ii} T_{ii}^{-1} = 1$ and thus the entries on the diagonal of T^{-1} is the inverse of T's, completing the proof. $\square$

Problem 6

Suppose $\mathbb{F} = \mathbb{C}$, V is finite-dimensional, and $T \in \mathcal{L}(V)$. Prove that if $k \in \{1, \ldots, \dim V\}$, then V has a k-dimensional subspace invariant under T.

Proof. We have proved in the book that T has an upper-triangular matrix with respect to some basis of V. Then by equivalent conditions of upper-triangular matrix, we know that for some basis $v_1, \ldots, v_n$, the $\operatorname{span}(v_1, \ldots, v_k)$ is invariant under T for each $k = 1, \ldots, n$, completing the proof. $\square$

Problem 7

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$, and $v \in V$.

(a) Prove that there exists a unique monic polynomial p_v of smallest degree such that $p_v(T)v = 0$.

(b) Prove that the minimal polynomial of T is a polynomial multiple of p_v.

Proof. (a) First we prove the existence. Consider the list

$$v, Tv, \ldots, T^n v$$

which is a linearly dependent list in $\dim V = n$ space. Therefore we can always reduce this list to a smallest degree such that

$$T^m v = \sum_{k=0}^{m-1} \frac{a_k}{a_m} T^m v$$

Next we prove uniqueness. Suppose there exists another monic polynomial of minimal degree s such that $s(T)(v) = 0$. Then we can consider

$$0 = p(T)(v) - s(T)(v) = (p - s)(T)(v)$$

So this polynomial $(p - s)$ also satisfies the required condition and it has smaller degree than p or s, reaching a contradiction.

(b) Let m be the minimal polynomial of T. By the division algorithm, we have

$$m(z) = q(z)p(z) + r(z)$$

for some polynomial q and r with $\deg r < \deg p$. We claim that $r = 0$. This is true as

$$0 = m(T)(v) = qp(T)(v) + r(T)(v) = r(T)(v)$$

Thus $r(T)(v) = 0$. If r is non-zero map, then this means there exists a monic polynomial of even lesser degree than p that also maps v to 0, contradicting that p is the unique smallest monic polynomial. Therefore we complete the proof. □

Problem 9

Suppose B is a square matrix with complex entries. Prove that there exists an invertible square matrix A with complex entries such that $A^{-1}BA$ is an upper-triangular matrix.

Proof. Consider the linear map T represented by the matrix B. Since the field is taken over $\mathbb{C}$, we know that T has an upper-triangular matrix with respect to some basis of V. Then let A be such change-of-basis matrix we can indeed find the desired matrix. □

Problem 10

Suppose $T \in \mathcal{L}(V)$ and $v_1, \dots, v_n$ is a basis of V. Show that the following are equivalent.

(a) The matrix of T with respect to $v_1, \dots, v_n$ is lower triangular.

(b) $\operatorname{span}(v_k, \dots, v_n)$ is invariant under T for each $k = 1, \dots, n$.

(c) $Tv_k \in \operatorname{span}(v_k, \dots, v_n)$ for each $k = 1, \dots, n$.

Proof. $(a) \Rightarrow (b)$ This means that $Tv_j \in \operatorname{span}(v_j, \dots, v_n)$. If $j \geq k$, then we have that

$$Tv_j \in \operatorname{span}(v_k, \dots, v_n)$$

$(b) \Rightarrow (c)$ This holds by definition.

$(c) \Rightarrow (a)$ This means when writing each Tv_k as a linear combination of the basis vectors $v_1, \dots, v_n$, we need to use only the vectors $v_k, \dots, v_n$. Hence, all entries above the diagonal of $\mathcal{M}(T)$ are 0 and thus it is an lower-triangular matrix. □

Problem 12

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ has an upper-triangular matrix with respect to some basis of V, and U is a subspace of V that is invariant under T.

(a) Prove that $T|_U$ has an upper-triangular matrix with respect to some basis of U.

(b) Prove that the quotient operator T/U has an upper-triangular matrix with respect to some basis of V/U.

Proof. We know that there exists a basis $v_1, \dots, v_n$ such that $Tv_k \in \operatorname{span}(v_1, \dots, v_k)$ for each k.

(a) We can always find a subset of the basis such that $v_1, \dots, v_m$ is the basis of U (WLOG we make the numbering easy). Then for the list we always have that $Tv_k \in \operatorname{span}(v_1, \dots, v_k)$ for each k and thus $T|_U$ has an upper-triangular matrix.

(b) Consider $\{v_{m+1} + U, \dots, v_n + U\}$, which is the basis of V/U. Note that we still have $T(v_k + U) \in \operatorname{span}(v_{m+1} + U, \dots, v_k + U)$, hence T/U also has an upper-triangular matrix. □

Problem 14

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has an upper-triangular matrix with respect to some basis of V if and only if the dual operator T' has an upper-triangular matrix with respect to some basis of the dual space.

Proof. From 5B problem 28 we know that the minimal polynomial of T' equals the minimal polynomial of T. Then this means that they have the same roots and therefore the same form of $(z-\lambda_1)\cdots(z-\lambda_m)$ if they having upper-triangular matrix. The iff is directly deduced from the iff of the relation between each one's minimal polynomial. $\square$

5D: Diagonalizable Operators

Definition 38 (diagonal matrix)**.** *A* ***diagonal matrix*** *is a square matrix that is 0 everywhere except possibly on the diagonal.*

Remark 39. *The entries on the diagonal are precisely the eigenvalue of the operator.*

Definition 40 (diagonalizable)**.** *An operator on V is called* ***diagonalizable*** *if the operator has a diagonal matrix with respect to some basis of V.*

Remark 41. *Diagonalization may require a different basis.*

Definition 42 (eigenspace, $E(\lambda, T)$)**.** *Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The* ***eigenspace*** *of T corresponding to λ is the subspace $E(\lambda, T)$ of V defined by*

$$E(\lambda, T) = \textit{null}\ (T - \lambda I) = \{v \in V : Tv = \lambda v\}.$$

Hence $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

Remark 43. *λ is an eigenvalue of T if and only if $E(\lambda, T) \neq \{0\}$.*

Theorem 44. *Suppose $T \in \mathcal{L}(V)$ and $\lambda_1, \ldots, \lambda_m$ are distinct eigenvalues of T. Then*

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum. Furthermore, if V is finite-dimensional, then

$$\dim E(\lambda_1, T) + \cdots + \dim(\lambda_m, T) \leq \dim V$$

Theorem 45 (conditions equivalent to diagonalizability)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Then the following are equivalent.*

(a) T is diagonalizable.

(b) V has a basis consisting of eigenvectors of T.

(c) $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$.

(d) $\dim V = \dim E(\lambda_1, T) + \cdots + \dim E(\lambda_m, T)$.

Corollary 46 (enough eigenvalues implies diagonalizability)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues. Then T is diagonalizable.*

Theorem 47 (necessary and sufficient condition for diagonalizability)**.** *Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Then T is diagonalizable if and only if the minimal polynomial of T equals $(z - \lambda_1) \cdots (z - \lambda_m)$ for some list of* ***distinct*** *numbers $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$.*

Corollary 48. *Suppose $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T. Then $T|_U$ is a diagonalizable operator on U.*

Definition 49 (Gershgorin disks)**.** *Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Let A denote the matrix of T with respect to this basis. A **Gershgorin disk** of T with respect to the basis $v_1, \ldots, v_n$ is a set of the form*

$$\{z \in \mathbb{F} \colon |z - A_{j,j}| \leq \sum_{k=1 k \neq j}^{n} |A_{j,k}|\},$$

where $j \in \{1, \ldots, n\}$.

Remark 50. *Intuition: if the nondiagonal entries of A are small, then each eigenvalue of T is near a diagonal entry of A.*

Theorem 51 (Gershgorin disk theorem)**.** *Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then each eigenvalue of T is contained in some Gershgorin disk of T with respect to the basis $v_1, \ldots, v_n$.*

Problem 1

Suppose V is a finite-dimensional complex vetor space and $T \in \mathcal{L}(V)$.

(a) Prove that if $T^4 = I$, then T is diagonalizable.

(b) Prove that if $T^4 = T$, then T is diagonalizable.

(c) Give an example of an operator $T \in \mathcal{L}(\mathbb{C}^2)$ such that $T^4 = T^2$ and T is not diagonalizable.

Proof. (a) The polynomial $p(x) = x^4 - 1$ annihilates T ($p(T) = 0$) has four distinct roots. The minimal polynomial of T divides p and therefore T is diagonalizable.

(b) The polynomial $p(x) = x^4 - x = x(x-1)(x^2+1)$ annihilates T has four distinct roots. The minimal polynomial of T divides p and therefore T is diagonalizable.

(c) Consider

$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Then we have that

$$T^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = T^4$$

□

Problem 3

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that if the operator T is diagonalizable, then $V = \operatorname{null} T \oplus \operatorname{range} T$.

Proof. By equivalent characterizations, there exists $v_1, \ldots, v_n$ to be the eigenbasis of V. Let $\lambda_1, \ldots, \lambda_n$ be the corresponding eigenvalues. We can partition these vectors into corresponding basis for null T and range T. Naturally, we can partition these eigenbasis into two parts: let $\lambda_1, \ldots, \lambda_m$ denotes the eigenbasis with 0 eigenvalues and $\lambda_{m+1}, \ldots, \lambda_n$ denotes the eigenbasis with nonzero eigenvalues. We relabel the eigenvectors accordingly. We claim that null $T = \operatorname{span}(v_1, \ldots, v_m)$ and range $T = \operatorname{span}(v_{m+1}, \ldots, v_n)$. The proof is complete once we complete proving our claim.

For proving the null space, $\Rightarrow$ take $v \in \operatorname{null} T$, then we have that

$$\begin{aligned} Tv &= T\sum_{i=1}^{m} a_i v_i + T\sum_{i=m+1}^{n} a_i v_i \\ &= \sum_{i=m+1}^{n} (a_i\lambda_i) v_i = 0 \end{aligned}$$

By linear independence of eigenbasis (as $\lambda_i \neq 0$ for $m+1 \leq i \leq n$), we have that $a_i = 0$ for all $m+1 \leq i \leq n$. Thus $v \in \operatorname{span}(v_1, \ldots, v_m)$. $\Leftarrow$ for the other direction, it follows naturally by construction.

For proving the range space, $\Rightarrow$ take $v \in \operatorname{range} T$, then there exists $u \in V$ s.t.

$$v = Tu = \sum_{i=m+1}^{n} (a_i \lambda_i) v_i$$

Therefore, $v \in \operatorname{span}(v_{m+1}, \ldots, v_n)$. $\Leftarrow$ for the other direction, it follows by construction. $\square$

Problem 4

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.

(a) $V = \text{null } T \oplus \text{range } T$.

(b) $V = \text{null } T + \text{range } T$.

(c) $\text{null } T \cap \text{range } T = \{0\}$.

Proof. $(a) \Rightarrow (b)$ Trivial.

$(b) \Rightarrow (c)$ We know that

$$\dim V = \dim(\text{null } T + \text{range } T) = \dim \text{null } T + \dim \text{range } T - \dim \text{null } T \cap \text{range } T$$

At the same time, $\dim V = \dim \text{null } T + \dim \text{range } T$, hence

$$\dim \text{null } T \cap \text{range } T = 0$$

$(c) \Rightarrow (a)$ This can be proved similarly as above. $\square$

Problem 5

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if

$$V = \text{null } (T - \lambda I) \oplus \text{range } (T - \lambda I)$$

for every $\lambda \in \mathbb{C}$.

Proof. $\Rightarrow$ Given T is diagonalizable, then we know that $T - \lambda I$ is also diagonalizable (the matrix of that is diagonal). Hence, by P3, we get the desired result.

$\Leftarrow$ The minimal polynomial of T can be written as $(z - \lambda_1)^{n_1} \ldots (z - \lambda_m)^{n_m}$ for some distinct distinct $\lambda_1, \ldots, \lambda_m \in \mathbb{C}$. If $n_1 = \cdots = n_m = 1$, then we are done. WLOG suppose $n_1 > 1$. Then take arbitrary $v \in V$, we have

$$\prod_{k=1}^{m} (T - \lambda_k I)^{n_k}(v) = (T - \lambda_1 I)(T - \lambda_1 I)^{n_1 - 1} \prod_{k=2}^{m} (T - \lambda_k I)^{n_k}(v) = 0$$

This implies that $(T-\lambda_1 I)^{n_1-1}\prod_{k=2}^{m}(T-\lambda_k I)^{n_k} \in \text{null }(T-\lambda_1 I)\cap\text{range }(T-\lambda_1 I) = \{0\}$. Therefore, this reach a contradiction to the minimal polynomial we previously get. $\square$

Problem 6

Suppose $T \in \mathcal{L}(\mathbb{F}^5)$ and $\dim E(8,T) = 4$. Prove that $T-2I$ or $T-6I$ is invertible.

Proof. We know that

$$\dim E(8,T) + \dim E(2,T) + \dim E(6,T) \leq 5$$

Since $\dim(8,T) = 4$, then we have that

$$\dim E(2,T) = 0 \quad \text{or} \quad \dim E(6,T) = 0$$

In other words, $T-2I$ or $T-6I$ is invertible. $\square$

Problem 7

Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that

$$E(\lambda,T) = E(\frac{1}{\lambda},T^{-1})$$

for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

Proof. We know that

$$Tv = \lambda v \quad \iff \quad v = T^{-1}\lambda v \quad \iff \quad \frac{1}{\lambda}v = T^{-1}v$$

$\square$

Problem 8

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Let $\lambda_1,\ldots,\lambda_m$ denote the distinct nonzero eigenvalues of T. Prove that

$$\dim E(\lambda_1,T) + \cdots + \dim E(\lambda_m,T) \leq \dim \text{range } T.$$

Proof. It suffices to show that

$$E' = E(\lambda_1,T) + \cdots + E(\lambda_m,T) = E(\lambda_1,T) \oplus \cdots \oplus E(\lambda_m,T) \subseteq \text{range }(T)$$

To see this, let $v \in E(\lambda_k,T)$, then we know that $Tv = \lambda_k v$ and that $v = T\frac{1}{\lambda_k}v \in$ range T. Hence, we complete the proof. $\square$

Problem 9

Suppose $R, T \in \mathcal{L}(\mathbb{F}^3)$ each have 2,6,7 as eigenvalues. Prove that there exists an invertible operator $S \in \mathcal{L}(\mathbb{F}^3)$ such that $R = S^{-1}TS$.

Proof. This means that R, T are both diagonalizable and thus invertible. Thus we can define a change-of-basis operator S that is invertible (one may provide more details here). □

Problem 12

Suppose $T \in \mathcal{L}(\mathbb{C}^3)$ is such that 6 and 7 are eigenvalues of T. Furthermore, suppose T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$. Prove that there exists $(z_1, z_2, z_3) \in \mathbb{C}^3$ such that

$$T(z_1, z_2, z_3) = (6 + 8z_1, 7 + 8z_2, 13 + 8z_3).$$

Proof. This means that T only has 6, 7 as the eigenvalues and so 8 is not an eigenvalue of T. Equivalently, $T - 8I$ is invertible and thus there exists $(z_1, z_2, z_3) \in \mathbb{C}^3$ such that $(T - 8I)(z_1, z_2, z_3) = (6, 7, 13)$. Rewriting the equation gives the desired result. □

Problem 13

Suppose A is a diagonal matrix with distinct entries on the diagonal and B is a matrix of the same size as A. Show that $AB = BA$ if and only if B is a diagonal matrix.

Proof. $\Rightarrow$ We can simply examine each entry of AB and BA, note that

$$(AB)_{ij} = \sum_{k=1}^{n} A_{i,k}B_{k,j} = A_{i,i}B_{i,j}$$

and

$$(BA)_{ij} = \sum_{k=1}^{n} B_{i,k}A_{k,j} = A_{j,j}B_{i,j}$$

Since $A_{i,i} \neq A_{j,j}$, $B_{i,j} = 0$ for $i \neq j$. Hence we have B is diagonal.

$\Leftarrow$ This naturally holds by matrix multiplication. □

Problem 14

(a) Give an example of a finite-dimensional complex vector space and an operator T on that vector space such that T^2 is diagonalizable but T is not diagonalizable.

(b) Suppose $\mathbb{F} = \mathbb{C}$, k is a positive integer, and $T \in \mathcal{L}(V)$ is invertible. Prove that T is diagonalizable if and only if T^k is diagonalizable.

Proof. (a) Consider the matrix of T to be

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and let $\mathbb{F} = \mathbb{R}$. Then $T^2 = 0$ which is diagonalizable.

(b) $\Rightarrow$ By P13, we know that since T is diagonal, T^2 is diagonal, and $T(T^2)$ is diagonal. Recursively applying the argument yields that T^m is diagonal and therefore diagonalizable.

$\Leftarrow$ We know that $p(x) = (x^n - \lambda_1) \cdots (x^n - \lambda_k)$ annihilates T where $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of T^m. Since all λ_i are distinct, $x^n - \lambda_i$ and $x^n - \lambda_j$ do not share any common root for $i \neq j$. Then since the minimal polynomial of T divides $p(x)$, it also does not have any repeated roots and therefore T is also diagonalizable. $\square$

Problem 16

Suppose $T \in \mathcal{L}(V)$ is diagonalizable. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T. Prove that a subspace U of V is invariant under T if and only if there exist subspaces $U_1, \ldots, U_m$ of V such that $U_k \subseteq E(\lambda_k, T)$ for each k and $U = U_1 \oplus \cdots \oplus U_m$.

Proof. $\Leftarrow$ Take any $u = u_1 + \cdots + u_m \in U$, then $Tu = Tu_1 + \cdots + Tu_m = \lambda_1 u_1 + \cdots + \lambda_m u_m \in U$.

$\Rightarrow$ Define $U_k = E(\lambda_k, T) \cap U$. Clearly $U_k \subseteq E(\lambda_k, T)$. Since U is invariant under T, U_k is also invariant. By 5.65 we know that $T|_{U_k}$ is also diagonalizable. Since we know that $E(\lambda_i, T) \cap E(\lambda_j, T) = \{0\}$ for $i \neq j$, $U_j \cap U_i = \{0\}$ for $i \neq j$ as well. Now it suffices to show that $U \subseteq U_1 \oplus \cdots \oplus U_m$. Take $u \in U \subset V$, since we know that $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$, $u \in E_k$ for some k, therefore completing the proof. $\square$

Problem 18

Suppose that $T \in \mathcal{L}(V)$ is diagonalizable and U is a subspace of V that is invariant under T. Prove that the quotient operator T/U is a diagonalizable operator on V/U.

Proof. First we know that there exists eigenbasis $v_1, \ldots, v_n$. We can first partition these basis into $v_1, \ldots, v_m$ for U and then $v_{m+1} + U, \ldots, v_n + U$ would be the basis for V/U. Notice that here we have $(T/U)(v_j + U) = T(v_j) + U + \lambda_j v_j + U$, with the same eigenbasis so preserving the diagonalizability. $\square$

Problem 20

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is diagonalizable if and only if the dual operator T' is diagonalizable.

Proof. WLOG we go from the forward direction $\Rightarrow$. Let $v_1, \ldots, v_n$ be the eigenbasis and $\varphi_1, \ldots, \varphi_n$ be the corresponding dual basis. Then we have that

$$(T'\varphi_i)(v_j) = \varphi_i \circ Tv_j = \lambda_j \varphi_i(v_j) = \lambda_i \delta_{ij}$$

Hence, we have that $T'\varphi_i = \lambda_i \varphi_i$ for each i and thus T' is diagonalizable. $\square$

Problem 22

Suppose $T \in \mathcal{L}(V)$ and A is an n-by-n matrix that is the matrix of T with respect to some basis of V. Prove that if

$$|A_{j,j}| > \sum_{k=1,k\neq j}^{n} |A_{j,k}|$$

for each $j \in \{1, \ldots, n\}$, then T is invertible.
In other words, the implication is that if the diagonal entries of the matrix of T are large enough compared to non-diagonal ones, then T is invertible.

Proof. Equivalently, we aim to prove that 0 is not an eigenvalue of A. From the Gershgorin disk theorem, we know that every eigenvalue of A lies within at least one of the Gershgorin disk

$$\left\{ z \in \mathbb{F} \colon |z - A_{j,j}| \leq \sum_{k=1 k\neq j}^{n} |A_{j,k}| \right\}$$

for $j \in \{1, \ldots, n\}$. From the question, we can see that none of the eigenvalues are 0 and therefore A is strictly diagonally dominant and hence invertible. $\square$

5E: Commuting Operators

Definition 52 (commute).

- *Two operators S and T on the same vector space **commute** if $ST = TS$.*
- *Two square matrices A and B of the same size **commute** if $AB = BA$.*

Lemma 53 (commuting operators correspond to commuting matrices)**.** *Suppose $S, T \in \mathcal{L}(V)$ and $v_1, \ldots, v_n$ is a basis of V. Then S and T commute if and only if $\mathcal{M}(S, (v_1, \ldots, v_n))$ and $\mathcal{M}(T, (v_1, \ldots, v_n))$ commute.*

Lemma 54 (eigenspace is invariant under commuting operator)**.** *Suppose $S, T \in \mathcal{L}(V)$ commute and $\lambda \in \mathbb{F}$. Then $E(\lambda, S)$ is invariant under T.*

Theorem 55 (simultaneous diagonalizablity $\Longleftrightarrow$ commutativity)**.** *Two diagonalizable operators on the same vector space have diagonal matrices with respect to the same basis if and only if the two operators commute.*

Lemma 56 (common eigenvector for commuting operators)**.** *Every pairs of commuting operators on a finite-dimensional nonzero complex vector space has a common eigenvector.*

Lemma 57 (commuting operators are simultaneously upper triangular)**.** *Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then there is a basis of V with respect to which both S and T have upper-triangular matrices.*

Theorem 58 (eigenvalues of sum and product of commuting operators)**.** *Suppose V is a finite-dimensional complex vector space and S, T are commuting operators on V. Then*

- *Every eigenvalue of $S + T$ is an eigenvalue of S plus an eigenvalue of T,*
- *Every eigenvalue of ST is an eigenvalue of S times an eigenvalue of T.*

Problem 2

Suppose $\mathcal{E}$ is a subset of $\mathcal{L}(V)$ and every element of $\mathcal{E}$ is diagonalizable. Prove that there exists a basis of V with respect to which every element of $\mathcal{E}$ has a diagonal matrix if and only if every pair of element of $\mathcal{E}$ commutes.

Proof. $\Rightarrow$ This follows by products of diagonal matrices.

$\Leftarrow$ We are given that every pair of element of $\mathcal{E}$ commutes. Pick any $A \in \mathcal{E}$ and let $E(\lambda) = \{v \in V : Av = \lambda v\}$ be the λ-eigenspace of A. We know that as A is diagonalizable,

$$V = \bigoplus_{\lambda \in \mathbb{F}} E(\lambda)$$

Take another element $B \in \mathcal{E}$, then we know that $E(\lambda)$ is invariant under B and thus $B|_{E(\lambda)}$ is diagonalizable by 5.75. This means that we can further get that $E(\lambda, \mu) = \{v \in E(\lambda) : Bv = \mu v\} = \{v \in V : Av = \lambda v, Bv = \mu v\}$, and we have that

$$V = \bigoplus_{\lambda, \mu \in \mathbb{F}} E(\lambda, \mu)$$

Applying the argument recursively yields that

$$V = \bigoplus_{\{\lambda_i\}_{i=1}^{\infty} :\, \lambda_i \in \mathbb{F}} E(\{\lambda_i\}_{i=1}^{\infty})$$

where $E(\{\lambda_i\}_{i=1}^{\infty})$ consist of all vectors v such that $T_i v = \lambda_i v$ for commuting operators $T_i \in \mathcal{E}$. Note that even notionally we make ∞ this is still a finite set since the collection of finite(ly nonzero eigenvalues) is still finite. Hence, we derive a common eigenbasis and finish the proof. $\square$

Problem 3

Suppose $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Suppose $p \in \mathcal{P}(\mathbb{F})$.

(a) Prove that null $p(S)$ is invariant under T.

(b) Prove that range $p(S)$ is invariant under T.

Proof. (a) Let $p(S) = a_0 I + a_1 S + a_2 S^2 + \cdots + a_m S^m$. Take $v \in \operatorname{null} p(S)$, then this means that $p(S)(v) = 0$. We have that

$$p(S)(Tv) = T(p(S)(v)) = T(0) = 0$$

Hence, $Tv \in \operatorname{null} p(S)$.

(b) Take $v \in \operatorname{range} p(S)$, then this means there exists $u \in V$ such that $p(S)(u) = v$. Consider that

$$p(S)(Tu) = T(p(S)u) = Tv$$

Therefore, $Tv \in \operatorname{range} p(S)$. $\square$

Problem 5

Prove that a pair of operators on a finite-dimensional vector space commute if and only if their dual operators commute.

Proof. Take $\phi \in V'$ and $v \in V$. Then $\Rightarrow$ assume $TS = ST$, we have that

$$(T^*S^*\phi)(v) = S^*\phi(Tv) = \phi(STv)$$

At the same time

$$(S^*T^*\phi)(v) = T^*\phi(SV) = \phi(TSv)$$

Hence, we have that

$$T^*S^* = S^*T^*$$

.

$\Leftarrow$ Assume $T^*S^* = S^*T^*$, then

$$(T^*S^*\phi)(v) = \phi(STv) = \phi(TSv) = (S^*T^*\phi)(v)$$

Since this holds for all $\phi \in V', v \in V$, $ST = TS$. □

Problem 6

Suppose V is a complex vector space, $S, T \in \mathcal{L}(V)$ commute. Prove that there exist $\alpha, \lambda \in \mathbb{C}$ such that

$$\text{range } (S - \alpha I) + \text{range } (T - \lambda I) \neq V.$$

Proof. Let $v_1, \dots, v_n$ be the same basis such that $\mathcal{M}(S), \mathcal{M}(T)$ are diagonal. Then we let $\alpha = \mathcal{M}(S)_{1,1}$ and $\lambda = \mathcal{M}(T)_{1,1}$. Then this means that

$$\text{range } (S - \lambda I) + \text{range } (T - \lambda I) = V \setminus \text{span}(v_1) \neq V.$$

□