

Chapter 2: Finite-Dimensional Vector Spaces

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2A: Span and Linear Independence

Definition 1 (Linear Combination). A linear combination of a list v_1, \dots, v_m of vectors in V is a vector of the form

$$a_1v_1 + \dots + a_mv_m$$

where $a_1, \dots, a_m \in \mathbb{F}$.

Definition 2 (Span). The set of all linear combinations of list of vectors v_1, \dots, v_m in V is called the span of v_1, \dots, v_m , denoted by $\text{span}(v_1, \dots, v_m)$. In other words,

$$\text{span}(v_1, \dots, v_m) = \{a_1v_1 + \dots + a_mv_m : a_1, \dots, a_m \in \mathbb{F}\}$$

The span of the empty list $()$ is defined to be $\{0\}$.

Theorem 3. The span of a list of vectors in V is the smallest subspace of V containing all vectors in the list.

Definition 4 (Spans). If $\text{span}(v_1, \dots, v_m)$ equals V , we say the list v_1, \dots, v_m spans V .

Definition 5 (Finite-dimensional vector space). A vector space is called finite-dimensional if some list of vectors in it spans the space.

Definition 6 (polynomial). A function $p: \mathbb{F} \rightarrow \mathbb{F}$ is called a polynomial with coefficients in \mathbb{F} if there exist $a_0, \dots, a_m \in \mathbb{F}$ s.t.

$$(z) = a_0 + a_1z + a_2z^2 + \dots + a_mz^m$$

for all $z \in \mathbb{F}$.

$\mathcal{P}(\mathbb{F})$ is the set of all polynomials with coefficients in \mathbb{F} .

Definition 7 (Linear independence). A list of vectors v_1, \dots, v_m in V is called linearly independent if the only choice of $a_1, \dots, a_m \in \mathbb{F}$ that makes

$$a_1v_1 + \dots + a_mv_m = 0$$

is $a_1 = \dots = a_m = 0$.

The empty list $()$ is also declared to be linearly independent.

Lemma 8. Suppose v_1, \dots, v_m is a linearly dependent list in V . Then there exists $k \in \{1, 2, \dots, m\}$ s.t.

$$v_k \in \text{span}(v_1, \dots, v_{k-1})$$

Furthermore, if k satisfies the condition above and the k^{th} term is removed from v_1, \dots, v_m , then the span of the remaining list equals $\text{span}(v_1, \dots, v_m)$.

Lemma 9 (length of linearly independent list \leq length of spanning list). In a finite-dimensional vector space, the length of every linearly independent list of vectors is less than or equal to the length of every spanning list of vectors.

Problem 1

Find a list of four distinct vectors in \mathbb{F}^3 whose span equals

$$\{(x, y, z) \in \mathbb{F}^3 : x + y + z = 0\}$$

Proof. Example: $(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, -1, -1)$. \square

Problem 2

Prove or give a counterexample: If v_1, v_2, v_3, v_4 spans V , then the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

also spans V .

Proof. Take any $v \in V$, then we have $v = a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + (a_1 + a_2 + a_3)(v_3 - v_4) + (a_1 + a_2 + a_3 + a_4)v_4$. Conversely, any linear combination of these vectors still belong to V . \square

Problem 3

Suppose v_1, \dots, v_m is a list of vectors in V . For $k \in \{1, \dots, m\}$, let

$$w_k = v_1 + \dots + v_k$$

Show that $\text{span}(v_1, \dots, v_m) = \text{span}(w_1, \dots, w_m)$

Proof. Take $v = \sum_{i=1}^m a_i v_i$ from l.h.s, then we can write

$$\begin{aligned} v &= a_1v_1 + \dots + a_mv_m \\ &= (a_1 - a_m)v_1 + \dots + (a_{m-1} - a_m)v_{m-1} + a_mw_m \\ &= (a_1 - a_m - a_{m-1})v_1 + \dots + (a_{m-2} - a_{m-1} - a_m)v_{m-2} + (a_{m-1} - a_m)w_{m-1} + a_mw_m \\ &= \sum_{i=1}^m \left(a_i - \sum_{j=i+1}^m a_j \right) w_i \in \text{r.h.s} \end{aligned}$$

Conversely, take $w = \sum_{i=1}^m b_i w_i = \sum_{i=1}^m (b_i \sum_{j=1}^i c_j) v_j \in \text{l.h.s.}$ \square

Problem 4

- (a) Show that a list of length one in a vector space is linearly independent if and only if the vector in the list is not 0.
- (b) Show that a list of length two in a vector space is linearly independent if and only if neither of the two vectors in the list is a scalar multiple of the other.

Proof. (a) In order for the only way to write $av = 0$ is to ensure $v \neq 0$.

(b) $av_1 + bv_2 = 0$. \Rightarrow the only solution is $a = b = 0$ so v_1 cannot be a multiple of v_2 ; vice versa. \Leftarrow same reason. \square

Problem 8

Suppose v_1, v_2, v_3, v_4 is linearly independent in V . Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

Proof.

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + a_3(v_3 - v_4) + a_4v_4 = a_1v_1 + (-a_1 + a_2)v_2 + (-a_2 + a_3)v_3 + (-a_3 + a_4)v_4$$

The only solution is $a_1 = 0, -a_1 + a_2 = 0, -a_2 + a_3 = 0, -a_3 + a_4 = 0$. \square

Problem 9

Prove or give a counter example: If v_1, \dots, v_m is a linearly independent list of vectors in V , then

$$5v_1 - 4v_2, v_2, v_3, \dots, v_m$$

is also linearly independent.

Proof.

$$\begin{aligned} a_1(5v_1 - 4v_2) + a_2v_2 + \dots + a_mv_m \\ = 5a_1v_1 + (-4a_1 + a_2)v_2 + \dots + a_mv_m \end{aligned}$$

The only solution is $5a_1 = -4a_1 + a_2 = \dots = a_m = 0$. \square

Problem 10

Prove or give a counterexample: If v_1, \dots, v_m is a linearly independent list of vectors in V and $\lambda \in \mathbb{F}$ with $\lambda \neq 0$, then $\lambda v_1, \dots, \lambda v_m$ is linearly independent.

Proof.

$$\sum_i^m \lambda a_i v_i = 0$$

The only solution is $a_i = 0$ for all i . \square

Problem 12

Suppose v_1, \dots, v_m is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \dots, v_m + w$ is linearly dependent, then $w \in \text{span}(v_1, \dots, v_m)$.

Proof. We know the only solution to $\sum_i^m a_i v_i$ is all $a_i = 0$. Now we have that non-zero a_i for solving $\sum_i^m a_i v_i + \sum_i^m a_i w = 0$. Thus $w = \frac{\sum_i^m a_i v_i}{\sum_i^m a_i}$ which completes the proof. \square

Problem 13

Suppose v_1, \dots, v_m is linearly independent in V and $w \in V$. Show that

$$v_1, \dots, v_m, w \text{ is linearly independent} \iff w \notin \text{span}(v_1, \dots, v_m)$$

Proof. \Rightarrow By contradiction, if w in the span then it can be written as linear combination for some nonzero a_i and thus they are linearly dependent.

\Leftarrow Similarly. \square

Problem 17

Prove that V is infinite-dimensional if and only if there is a sequence v_1, v_2, \dots of vectors in V such that v_1, \dots, v_m is linearly independent for every positive integer m .

Proof. \Rightarrow There doesn't exist any finite list of vectors that span the space. For the sake of contradicting assumes for every sequence v_1, \dots of vectors in $V \exists m$ such that v_1, \dots, v_m is linearly dependent. Then this means we can construct the basis for the vector space as follows: select $v_i \in V$ s.t. v_i and v_1, \dots, v_{i-1} are linearly independent. This means that there exists m s.t. v_1, \dots, v_m that spans V , forming a contradiction.

\Leftarrow Suppose for contradiction. Then there exists a span, contradicting the linear independence claim for every m . \square

Problem 18

Prove \mathbb{F}^∞ is infinite-dimensional.

Proof. We apply Problem 17. Construct v_i to be the vector that has 1 on the i -th coordinate and 0 elsewhere. clearly v_1, v_2, \dots are s.t. v_1, \dots, v_m is linearly independent for every positive integer m . \square

2B: Bases

Definition 10 (basis). *A basis of V is a list of vectors in V that is linearly independent and spans V .*

Theorem 11 (criterion for basis). *A list v_1, \dots, v_m of vectors in V is a basis of V if and only if every $v \in V$ can be written uniquely in the form*

$$v = \sum_i^m a_i v_i$$

where $a_i \in \mathbb{F}$.

Lemma 12 (every spanning list contains a basis). *Every spanning list in a vector space can be reduced to a basis of the vector space.*

Lemma 13. *Every finite-dimensional vector space has a basis.*

Lemma 14. *Every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space.*

Lemma 15. *Suppose V is finite-dimensional and \mathcal{U} is a subspace of V . Then there is a subspace W of V such that $V = \mathcal{U} \oplus W$.*

Problem 1

Find all vector spaces that have exactly one basis.

Proof. The only answer is $\{0\}$. Otherwise, for any basis v one can get av for $a \neq 0, a \neq 1$. \square

Problem 4

1. Let \mathcal{U} be the subspace of \mathbb{C}^5 defined by $\mathcal{U} = \{(z_1, z_2, z_3, z_4, z_5) \in \mathbb{C}^5 : 6z_1 = z_2, z_3 + 2z_4 + 3z_5 = 0\}$ Find a basis of \mathcal{U} .
2. Extend the basis to a basis in \mathbb{C}^5 .
3. Find a subspace \mathcal{W} of \mathbb{C}^5 s.t. $\mathbb{C}^5 = \mathcal{U} \oplus \mathcal{W}$.

Proof. 1. $(z_1, 6z_1, -2z_4 - 3z_5, z_4, z_5) \Rightarrow \{(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1)\}$
 2. $\{(1, 6, 0, 0, 0), (0, 0, -2, 1, 0), (0, 0, -3, 0, 1), (0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\}$
 3. $\mathcal{W} = \text{span}(\{(0, 1, 0, 0, 0), (0, 0, 1, 0, 0)\})$

\square

Problem 5

Suppose V is finite-dimensional and \mathcal{U}, \mathcal{W} are subspaces of V such that $V = \mathcal{U} + \mathcal{W}$. Prove that there exists a basis of V consisting of vectors in $\mathcal{U} \cup \mathcal{W}$.

Proof. Let $\{v_i\}_{i=1}^m$ denote the basis for the vector space V . By definition we have $v_i = u_i + w_i$ for some u_i, w_i . Then we have the spanning set of the vector space V $\sum_i^m a_i(u_i + w_i)$, which can be reduced to a basis by the lemma. \square

Problem 7

Suppose v_1, v_2, v_3, v_4 is a basis of V . Prove that

$$v_1 + v_2, v_2 + v_3, v_3 + v_4, v_4$$

is also a basis of V .

Proof. We know v_1, v_2, v_3, v_4 is linearly independent and spans V .

$$a_1(v_1 + v_2) + a_2(v_2 + v_3) + a_3(v_3 + v_4) + a_4 v_4 = a_1 v_1 + (a_1 + a_2)v_2 + (a_2 + a_3)v_3 + (a_3 + a_4)v_4$$

which shows the linear independence. For proving spanning, let $v \in V$ then

$$v = \sum_{i=1}^4 a_i v_i = a_1(v_1 + v_2) + (a_2 - a_1)(v_2 + v_3) + (a_3 - a_2)(v_3 + v_4) + (a_4 - a_3)v_4$$

\square

Problem 8

Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and \mathcal{U} is a subspace of V such that $v_1, v_2 \in \mathcal{U}$ and $v_3 \notin \mathcal{U}$ and $v_4 \notin \mathcal{U}$, then v_1, v_2 is a basis of \mathcal{U} .

Proof. Take $V = \mathbb{R}^4$ and the standard basis. Consider $\mathcal{U} = \{(x_1, x_2, x_3, kx_3)\}$, then we disprove the claim. \square

Problem 10

Suppose \mathcal{U} and \mathcal{W} are subspaces of V s.t. $V = \mathcal{U} \oplus \mathcal{W}$. Suppose also that u_1, \dots, u_m is a basis of \mathcal{U} and w_1, \dots, w_n is a basis of \mathcal{W} . Prove that

$$u_1, \dots, u_m, w_1, \dots, w_n$$

is a basis of V .

Proof. We know that this set is linearly independent (otherwise violating the direct sum assumption) so it suffices to prove the spanning. Let $v \in V$, then $v = u + w = \sum_{i=1}^m a_i u_i + \sum_{j=1}^n b_j w_j$. \square

2C: Dimension

Lemma 16 (basis length does not depend on basis). *Any two bases of a finite-dimensional vector space have the same length.*

Definition 17 (dimension). • *The dimension of a finite-dimensional vector space is the length of any basis of the vector space.*

• *The dimension of a finite-dimensional vector space V is denoted by $\dim V$.*

Corollary 18. *If V is finite-dimensional and \mathcal{U} is a subspace of V , then $\dim \mathcal{U} \leq \dim V$.*

Corollary 19. *Suppose V is finite-dimensional. Then every linearly independent list of vectors in V of length $\dim V$ is a basis of V .*

Corollary 20. *Suppose that V is finite-dimensional and \mathcal{U} is a subspace of V such that $\dim \mathcal{U} = \dim V$. Then $\mathcal{U} = V$.*

Corollary 21. *Suppose that V is finite-dimensional. Then every spanning list of vectors in V of length $\dim V$ is a basis of V .*

Theorem 22 (dimension of a sum). *If V_1 and V_2 are subspaces of a finite-dimensional vector space, then*

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

Problem 1

Show that the subspaces of \mathbb{R}^2 are precisely $\{0\}$, all lines in \mathbb{R}^2 containing the origin and \mathbb{R}^2 .

Proof. We know $\dim(\mathbb{R}^2) = 2$ so the subspace dimension is either 0 ($\{0\}$) or 1 (then this means it has to be lines crossing the origin). \square

Problem 5

- (a) Let $\mathcal{U} = \{p \in \mathcal{P}_4(\mathbb{F}) : p(2) = p(5)\}$. Find a basis of \mathcal{U} .
- (b) Extend the basis in (a) to a basis of $\mathcal{P}_4(\mathbb{F})$.
- (c) Find a subspace \mathcal{W} of $\mathcal{P}_4(\mathbb{F})$ s.t. $\mathcal{P}_4(\mathbb{F}) = \mathcal{U} \oplus \mathcal{W}$.

Proof. (a) This means that

$$a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 = a_0 + 5a_1 + 25a_2 + 125a_3 + 625a_4$$

Solving this gives that $a_1 = -7a_2 - 39a_3 - 203a_4$. So we can write that

$$\begin{aligned} p(x) &= a_0 + (-7a_2 - 39a_3 - 203a_4)x + a_2x^2 + a_3x^3 + a_4x^4 \\ &= a_0 + a_2(x^2 - 7x) + a_3(x^3 - 39x) + a_4(x^4 - 203x) \end{aligned}$$

The basis now becomes $\{1, x^2 - 7x, x^3 - 39x, x^4 - 203x\}$

(b) $\{1, x, x^2 - 7x, x^3 - 39x, x^4 - 203x\}$

(c) $\mathcal{W} = \{x\}$ \square

Problem 8

Suppose v_1, \dots, v_m is linearly independent in V and $w \in V$. Prove that

$$\dim \text{span}(v_1 + w, \dots, v_m + w) \geq m - 1$$

Proof. We claim that

$$v_{i+1} - v_i \in \text{span}(v_1 + w, \dots, v_m + w) \text{ for all } m \geq i \geq 2$$

as $v_{i+1} - v_i = (v_{i+1} + w) - (v_i + w)$. Thus $v_2 - v_1, \dots, v_m - v_{m-1}$ is in the $\text{span}(v_1 + w, \dots, v_m + w)$. We've proved that $v_2 - v_1, \dots, v_m - v_{m-1}$ is linearly independent and this list has $m - 1$ vectors and thus we've proved the claim. \square

Problem 9

Suppose m is a positive integer and $p_0, p_1, \dots, p_m \in \mathcal{P}(\mathbb{F})$ are such that each p_k has degree k . Prove that p_0, p_1, \dots, p_m is a basis of $\mathcal{P}_m(\mathbb{F})$.

Proof. It's easy to see that this list is linearly independent. Take any element in $\mathcal{P}_m(\mathbb{F})$, we can decompose that by degrees and get each component to be some multiple of p'_i s. \square

Problem 10

Suppose m is a positive integer. For $0 \leq k \leq m$, let

$$p_k(x) = x^k(1-x)^{m-k}$$

Show that p_0, \dots, p_m is a basis of $\mathcal{P}_m(\mathbb{F})$.

Proof. It suffices to prove that $p_k(x)$ are linearly independent. We know from binomial theorem that

$$(x+y)^m = \sum_{i=0}^m \binom{m}{i} x^i y^{m-i}$$

Applying this identity here we get that

$$p_k(x) = x^k \sum_{i=0}^{m-k} \binom{m-k}{i} 1^{m-i} (-1)^i x^i = \sum_{i=0}^{m-k} a_i x^{k+i}$$

where $a_i = \binom{m-k}{i} (-1)^i$. For the polynomial to be identically 0 ($\sum_{k=0}^m c_k p_k(x) = 0$), for each power x^i we need the coefficient to be 0:

$$\sum_{k=0}^{\min(i,m)} c_k \binom{m-k}{i-k} (-1)^{i-k} = 0$$

We can prove the only solution for this is all $c_k = 0$ by induction on m .

The base case is trivial. Assume the statement holds for $k = m-1$, then we try to prove for $p_m(x)$, where we have $P(x) - c_m p_m(x) = \sum_{k=0}^{m-1} c_k p_k(x) = 0$. This means that $c_m = 0$ and we've proved the linear independence. \square

Problem 11

Suppose \mathcal{U} and \mathcal{W} are both four-dimensional subspaces of \mathbb{C}^6 . Prove that there exist two vectors in $\mathcal{U} \cap \mathcal{W}$ such that neither of these vectors is a scalar multiple of the other.

Proof. $\dim(\mathcal{U} \cap \mathcal{W}) = \dim(\mathcal{U}) + \dim(\mathcal{W}) - \dim(\mathcal{U} + \mathcal{W}) \geq 8 - 6 = 2$

So there must exist two linearly independent vectors in the intersection. \square

Problem 12

Suppose that \mathcal{U} and \mathcal{W} are subspaces of \mathbb{R}^8 such that $\dim \mathcal{U} = 3$, $\dim \mathcal{W} = 5$ and $\mathcal{U} + \mathcal{W} = \mathbb{R}^8$. Prove that $\mathbb{R}^8 = \mathcal{U} \oplus \mathcal{W}$.

Proof. Similar to problem 11, we can get that $\dim(\mathcal{U} \cap \mathcal{W}) = 0$. \square

Problem 14

Suppose V is a ten-dimensional vector space and V_1, V_2, V_3 are subspaces of V with $\dim V_1 = \dim V_2 = \dim V_3 = 7$. Prove that $V_1 \cap V_2 \cap V_3 \neq \{0\}$.

Proof.

$$\dim((V_1 \cap V_2) + V_3) = \dim(V_1 \cap V_2) + \dim V_3 - \dim(V_1 \cap V_2 \cap V_3)$$

and also that

$$\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$$

This gives that

$$\dim(V_1 \cap V_2 \cap V_3) = \dim V_1 + \dim V_2 + \dim V_3 - \dim(V_1 + V_2) - \dim((V_1 \cap V_2) + V_3)$$

Note that from question we know $\dim V_1 + \dim V_2 + \dim V_3 = 21 > 20 = 2 \dim V$

Hence we know

$$\dim(V_1 \cap V_2 \cap V_3) > (\dim(V) - \dim(V_1 + V_2)) + (\dim(V) - \dim((V_1 \cap V_2) + V_3)) > 0$$

We've thus proved the claim. \square

Problem 16

Suppose V is finite-dimensional and \mathcal{U} is a subspace of V with $\mathcal{U} \neq V$. Let $n = \dim V$ and $m = \dim \mathcal{U}$. Prove that there exist $n - m$ subspaces of V , each of dimension $n - 1$, whose intersection equals \mathcal{U} .

Proof. To show existence, we can start with the basis for $\mathcal{U} : u_1, \dots, u_m$. We extend the basis to $V : \{u_1, \dots, u_m, v_1, \dots, v_{n-m}\} := K$. Construct the subspace $V_i = \text{span}\{K \setminus \{v_i\}\}$. Then we know that $\bigcap_i V_i = \text{span}\{u_1, \dots, u_m\}$. \square

Problem 17

Suppose that V_1, \dots, V_m are finite-dimensional subspaces of V . Prove that $V_1 + \dots + V_m$ is finite-dimensional and

$$\dim(V_1 + \dots + V_m) \leq \dim V_1 + \dots + \dim V_m$$

Proof. We prove by induction on m . The base case is trivial. Assume the statement hold for k , then for $k + 1$, we have that (Denote $V_1 + \dots + V_k = M_k$)

$$\dim(M_k + V_{k+1}) \leq \dim(V_1) + \dots + \dim(V_{k+1})$$

which is finite. \square

Problem 18

Suppose V is finite-dimensional with $\dim V = n \geq 1$. Prove that there exist one-dimensional subspaces V_1, \dots, V_n of V such that

$$V = V_1 \oplus \cdots V_n$$

Proof. We know there are basis $\{v_1, \dots, v_n\}$ for V . Hence we can construct

$$V_i = \text{span}\{v_i\}$$

□

Problem 19

Prove or give a counter example:

$$\begin{aligned} \dim(V_1 + V_2 + V_3) &= \dim V_1 + \dim V_2 + \dim V_3 \\ &\quad - \dim(V_1 \cap V_2) - \dim(V_1 \cap V_3) - \dim(V_2 \cap V_3) \\ &\quad + \dim(V_1 \cap V_2 \cap V_3) \end{aligned}$$

Proof. We know that

$$\begin{aligned} \dim((V_1 + V_2) + V_3) &= \dim(V_1 + V_2) + \dim(V_3) - \dim((V_1 + V_2) \cap V_3) \\ &= \dim(V_1) + \dim(V_2) + \dim(V_3) - \dim(V_1 \cap V_2) - \dim((V_1 + V_2) \cap V_3) \end{aligned}$$

Here we can get that

$$\begin{aligned} \dim((V_1 + V_2) \cap V_3) &= \dim((V_1 \cap V_3) + (V_2 \cap V_3)) \\ &= \dim(V_1 \cap V_3) + \dim(V_2 \cap V_3) - \dim(V_1 \cap V_2 \cap V_3) \end{aligned}$$

By substituting this back, we can get the desired solution. □