Chapter 8: Operators on Complex Vector Spaces

Linear Algebra Done Right (4th Edition), by Sheldon Axler

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Contents

8A: Generalized Eigenvectors and Nilpotent Operators

Lemma 1 (sequence of increasing null spaces)**.** *Suppose $T \in \mathcal{L}(V)$. Then*

$$\{0\} = null\ T^0 \subseteq null\ T^1 \subseteq null\ T^2 \cdots \subseteq null\ T^k \subseteq null\ T^{k+1} \cdots$$

Lemma 2 (equality in the sequence of null spaces)**.** *Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer such that*

$$null\ T^m = null\ T^{m+1}$$

Then

$$null\ T^m = null\ T^{m+1} = null\ T^{m+2} = \cdots$$

Lemma 3 (null space stop growing)**.** *Suppose $T \in \mathcal{L}(V)$. Then*

$$null\ T^{\dim V} = null\ T^{\dim V+1} = null\ T^{\dim V+2} = \cdots$$

Theorem 4 (V is the direct sum of null $T^{\dim V}$ and range $T^{\dim V}$)**.** *Suppose $T \in \mathcal{L}(V)$. Then*

$$V = null\ T^{\dim V} \oplus range\ T^{\dim V}$$

Definition 5 (generalized eigenvector)**.** *Suppose $T \in \mathcal{L}(V)$ and λ is an eigenvalue of T. A vector $v \in T$ is called a **generalized eigenvector** of T corresponding to λ if $v \neq 0$ and*

$$(T - \lambda I)^k v = 0$$

for some positive integer k.

Remark 6. *There is no notion of "generalized eigenvalues" since we do not create new eigenvalues.*

Remark 7. *A nonzero vector $v \in V$ is a generalized eigenvector of T if and only if $(T - \lambda I)^{\dim V} v = 0$*

Theorem 8 (a basis of generalized eigenvectors)**.** *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then there is a basis of V consisting of generalized eigenvectors of T.*

Proposition 9. *Suppose $T \in \mathcal{L}(V)$. Then each generalized eigenvector of T only corresponds to one eigenvalue of T.*

Proposition 10. *Suppose that $T \in \mathcal{L}(V)$. Then every list of generalized eigenvectors of T corresponding to distinct eigenvalues are linearly independent.*

Definition 11 (nilpotent)**.** *An operator is called **nilpotent** if some powers of it equals 0.*

Remark 12. *An operator is nilpotent if every nonzero vector in V is a generalized eigenvector of T corresponding to eigenvalue 0.*

Corollary 13. *Suppose $T \in \mathcal{L}(V)$ is nilpotent. Then $T^{\dim V} = 0$.*

Theorem 14 (eigenvalues of nilpotent operator)**.** *Suppose $T \in \mathcal{L}(V)$.*

(a) If T is nilpotent, then 0 is an eigenvalue of T and T has no other eigenvalues.

(b) If $\mathbb{F} = \mathbb{C}$ and 0 is the only eigenvalue of T, then T is nilpotent.

Theorem 15 (minimal polynomial and upper-triangular matrix of nilpotent operator)**.** *Suppose $T \in \mathcal{L}(V)$. Then the following are equivalent.*

(a) T is nilpotent.

(b) The minimal polynomial of T is z^m for some positive integer m.

(c) There is a basis of V with respect to which the matrix of T has the form

$$\begin{pmatrix} 0 & * & \cdots & * \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & * \\ 0 & 0 & \cdots & 0 \end{pmatrix},$$

where all entries on and below the diagonal equal 0.

Problem 1

Suppose $T \in \mathcal{L}(V)$. Prove that if $\dim \operatorname{null} T^4 = 8$ and $\dim \operatorname{null} T^6 = 9$, then $\dim \operatorname{null} T^m = 9$ for all integers $m \geq 5$.

Proof. Suppose not, then $\dim \operatorname{null} T^5 = 8 = \dim \operatorname{null} T^6$, forming a contradiction. Therefore, the statement holds. $\square$

Problem 2

Suppose $T \in \mathcal{L}(V)$, m is a positive integer, $v \in V$, and $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that $v, Tv, T^2v, \ldots, T^{m-1}v$ is linearly independent.

Proof. Consider

$$a_0 v + a_1 Tv + \cdots + a_{m-1}T^{m-1}v = 0$$

Apply T^{m-1} on both sides yields that

$$a_0 T^{m-1} v = 0$$

and therefore $a_0 = 0$. Note that $v \neq \text{null } T^{m-1}$ and therefore $v \neq \text{null } T^j$ for $j \leq m-1$. Hence, continuing apply the argument above will gets that all $a_i = 0$. $\square$

Problem 3

Suppose $T \in \mathcal{L}(V)$. Prove that

$$V = \text{null } T \oplus \text{range } T \Longleftrightarrow \text{null } T^2 = \text{null } T$$

Proof. $\Rightarrow$ We know that $\text{null } T \subseteq \text{null } T^2$. Take $v \in \text{null } T^2$, then

$$T^2 v = 0 = T(Tv)$$

Therefore $Tv \in \text{null } T$, but $Tv \in \text{range } T$ so $Tv = 0$, which gives that $v \in \text{null } T$.

$\Leftarrow$ Let $v \in (\text{null } T) \cap (\text{range } T)$. Then there exists u s.t. $Tu = v$ and $Tv = 0$. Therefore, we have $T^2u = Tv = 0$ and thus $u \in \text{null } T^2 = \text{null } T$. So $v = T0 = 0$. We've proved the claim. $\square$

Problem 6

Suppose $T \in \mathcal{L}(V)$. Show that

$$V = \text{range } T^0 \supseteq \text{range } T^1 \supseteq \cdots \supseteq \text{range } T^k \supseteq T^{k+1} \supseteq \cdots$$

Proof. Take $v \in \text{range } T^{k+1}$, then we know that there exists $u \in V$ s.t. $v = T^{k+1}u = T^k(Tu)$. Therefore, we have $v \in \text{range } T^k$. $\square$

Problem 9

Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer. Prove that

$$\text{null } T^m = \text{null } T^{m+1} \iff \text{range } T^m = \text{range } T^{m+1}$$

Proof. We know that

$$\dim V = \dim \text{null } T^m + \dim \text{range } T^m = \dim \text{null } T^{m+1} + \dim \text{range } T^{m+1}$$

Therefore, we have

$$\text{null } T^m = \text{null } T^{m+1} \iff \text{range } T^m = \text{range } T^{m+1}$$

□

Problem 12

Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is a generalized eigenvector of T. Prove that there exists $\lambda \in \mathbb{F}$ such that $T - \lambda I$ is nilpotent.

Proof. If T has only one eigenvalue, then it is easy to tell that $T - \lambda I$ is nilpotent for the only eigenvalue λ. Suppose for the contradiction that it has multiple distinct eigenvalues. Then we know that for $v_1 \in G(\lambda_1, T)$ and $v_2 \in G(\lambda_2, T)$ are both invariant under T, but $v = v_1 + v_2 \in G(\lambda, T)$ is also invariant under T. If $\lambda = \lambda_1$ or λ_2, then this contradicts that $\lambda_1 \neq \lambda_2$. If $\lambda \neq \lambda_1$ and $\lambda \neq \lambda_2$, then this contradicts that $G(\lambda_1, T) \cap G(\lambda, T) = \{0\}$. Therefore, there is only one eigenvalue and thus $T - \lambda I$ is nilpotent for the only eigenvaue λ. □

Problem 13

Suppose $S, T \in \mathcal{L}(V)$ and ST is nilpotent. Prove that TS is nilpotent.

Proof. We know $(ST)^k = 0$ for some k. Then

$$(TS)^{k+1} = T(ST)^k S = 0$$

□

Problem 14

Suppose $T \in \mathcal{L}(V)$ is nilpotent and $T \neq 0$. Prove that T is not diagonalizable.

Proof. 0 is the only eigenvalue of T and any nonzero $v \in V$ cannot be represented by an eigenbasis. □

Problem 22

Suppose $T \in \mathcal{L}(\mathbb{C}^5)$ is such that range $T^4 \neq$ range T^5. Prove that T is nilpotent.

Proof. By Problem 9, we have that $\operatorname{null} T^4 \neq \operatorname{null} T^5$ and therefore $\dim \operatorname{null} T^4 < \dim \operatorname{null} T^5 = 5$ where $\dim \mathbb{C}^5 = 5$. Hence, T is nilpotent. $\square$

8B: Generalized Eigenspace Decomposition

Definition 16 (generalized eigenspace, $G(\lambda, T)$)**.** *Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **generalized eigenspace** of T corresponding to λ, denoted by $G(\lambda, T)$, is defined by*

$$G(\lambda, T) = \{v \in V : (T - \lambda I)^k v = 0 \textit{ for some positive integer } k\}.$$

Thus $G(\lambda, T)$ is the set of generalized eigenvectors of T corresponding to λ, along with the 0 vector.

Remark 17. *$E(\lambda, T) \subseteq G(\lambda, T)$ as each eigenvector is a generalized eigenvector.*

Corollary 18 (description of generalized eigenspaces)**.** *Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then $G(\lambda, T) = \textit{null } (T - \lambda I)^{\dim V}$.*

Theorem 19 (generalized eigenspace decomposition)**.** *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ be the distinct eigenvalues of T. Then*

(a) $G(\lambda_k, T)$ is invariant under T for each $k = 1, \ldots, m$;

(b) $(T - \lambda_k I)|_{G(\lambda_k, T)}$ is nilpotent for each $k = 1, \ldots, m$;

(c) $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$.

Definition 20 (multiplicity)**.** *Suppose $T \in \mathcal{L}(V)$. The **multiplicity** of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda, T)$. In other words, the multiplicity of an eigenvalue λ of T equals*

$$\dim \textit{null } (T - \lambda I)^{\dim V}$$

Corollary 21. *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then the sum of the multiplicities of all eigenvalue of T equals $\dim V$.*

Remark 22. *We may know the term **algebraic multiplicity** and **geometric multiplicity** in some books. We have*

$$\textit{algebraic multiplicity of } \lambda = \dim \textit{null } (T - \lambda I)^{\dim V} = \dim G(\lambda, T).$$
$$\textit{geometric multiplicity of } \lambda = \dim \textit{null } (T - \lambda I) = \dim E(\lambda, T).$$

Remark 23. *If V is an inner product space, $T \in \mathcal{L}(V)$ is normal, and λ is an eigenvalue of T, then the algebraic multiplicity of λ equals the geometric multiplicity of λ (i.e. every eigenvector is a generalized eigenvector).*

Definition 24 (characteristic polynomial)**.** *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T, with multiplicities $d_1, \ldots, d_m$. The polynomial*

$$(z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

*is called the **characteristic polynomial** of T.*

Corollary 25. *Suppose* $\mathbb{F} = \mathbb{C}$ *and* $T \in \mathcal{L}(V)$*. Then*

(a) the characteristic polynomial of T *has degree* $\dim V$*;*

(b) the zeros of the characteristic polynomial of T *are the eigenvalues of* T*.*

Theorem 26 (Cayley-Hamilton theorem)**.** *Suppose* $\mathbb{F} = \mathbb{C}, T \in \mathcal{L}(V)$*, and* q *is the characteristic polynomial of* T*. Then* $q(T) = 0$*.*

Corollary 27. *Suppose* $\mathbb{F} = \mathbb{C}$ *and* $T \in \mathcal{L}(V)$*. Then the characteristic polynomial of* T *is a polynomial multiple of the minimal polynomial of* T*.*

Theorem 28. *Suppose* $\mathbb{F} = \mathbb{C}$ *and* $T \in \mathcal{L}(V)$*. Suppose* $v_1, \ldots, v_n$ *is a basis of* V *such that* $\mathcal{M}(T, (v_1, \ldots, v_n))$ *is upper triangular. Then the number of times each eigenvalue* λ *of* T *appears on the diagonal of* $\mathcal{M}(T, (v_1, \ldots, v_n))$ *equals the multiplicity of* λ *as an eigenvalue of* T*.*

Definition 29 (block diagonal matrix)**.** *A* ***block diagonal matrix*** *is a square matrix of the form*

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where $A_1, \ldots, A_m$ *are square matrices lying along the diagonal and all other entries of the matrix equal 0.*

Remark 30. *wrt. an appropriate basis, every operator on a finite-dimensional complex vector space has a matrix of the form.*

Theorem 31 (block diagonal matrix with upper-triangular blocks)**.** *Suppose* $\mathbb{F} = \mathbb{C}$ *and* $T \in \mathcal{L}(V)$*. Let* $\lambda_1, \ldots, \lambda_m$ *be the distinct eigenvalues of* T*, with multiplicities* $d_1, \ldots, d_m$*. Then there is a basis of* V *with respect to which* T *has a block diagonal matrix of the form*

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix},$$

where each A_k *is a* d_k*-by-*d_k *upper-triangular matrix of the form*

$$A_k = \begin{pmatrix} \lambda_k & & * \\ & \ddots & \\ 0 & & \lambda_k \end{pmatrix}$$

Problem 1

Define $T \in \mathcal{L}(\mathbb{C}^2)$ by $T(w,z) = (-z, w)$. Find the generalized eigenspaces corresponding to the distinct eigenvalues of T.

Proof. We have the matrix of T to be

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues for T are $\pm i$. For $\lambda_1 = i$, we have $v_1 = (i, 1)$; for $\lambda_2 = -i$, we have $v_2 = (-i, 1)$. There eigenspace is therefore:

$$E_i = \operatorname{span}\{(i,1)\} \quad E_{-i} = \operatorname{span}\{(-i,1)\}$$

$\square$

Problem 2

Suppose $T \in \mathcal{L}(V)$. Prove that $G(\lambda, T) = G(\frac{1}{\lambda}, T^{-1})$ for every $\lambda \in \mathbb{F}$ with $\lambda \neq 0$.

Proof. WLOG let $v \in G(\lambda, T)$. Then we have that for some k

$$0 = (T - \lambda I)^k$$

We have

$$(\lambda^{-1})^k (T^{-1})^k (T - \lambda I)^k = (\lambda^{-1} T^{-1} (T - \lambda I))^k = (\lambda^{-1} I - T^{-1})^k = 0$$

which shows that $v \in G(\frac{1}{\lambda}, T)$. The other direction follows accordingly. $\square$

Problem 3

Suppose $T \in \mathcal{L}(V)$. Suppose $S \in \mathcal{L}(V)$ is invertible. Prove that T and $S^{-1}TS$ have the same eigenvalues with the same multiplicities.

Proof. Let λ be an eigenvalue of T with multiplicity d. Then we know

$$(T - \lambda I)^d = 0$$

Therefore, we have

$$(S^{-1})^d (T - \lambda I)^d S^d = (S^{-1}(T - \lambda I)S)^d = (S^{-1}TS - \lambda I)^d = 0$$

The converse is proved identically. $\square$

Problem 5

Suppose $T \in \mathcal{L}(V)$ and 3 and 8 are eigenvalues of T. Let $n = \dim V$. Prove that $V = (\operatorname{null} T^{n-2}) \oplus (\operatorname{range} T^{n-2})$.

Proof. This means that the minimal polynomial of T can be written as

$$m_T(x) = (x-3)(x-8)q(x)$$

with max deg $q(x) \leq n-2$. Hence, we have that null T^n = null T^{n-1} = null T^{n-2} and range T^n = range T^{n-1} = range T^{n-2}. Applying P3 from section 8A solves the problem. ☐

Problem 10

Suppose V is a complex inner product space, $e_1, \dots, e_n$ is an orthonormal basis of T, and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of T, each included as many times as its multiplicity. Prove that

$$|\lambda_1|^2 + \cdots + |\lambda_n|^2 \leq \|Te_1\|^2 + \cdots + \|Te_n\|^2$$

Proof.

$$\begin{aligned}
\sum_{i=1}^{n} \|Te_i\|^2 &= \sum_{i=1}^{n} \|U\Sigma V^* e_i\|^2 \\
&= \sum_{i=1}^{n} \|U\Sigma f_i\|^2 \\
&= \sum_{i=1}^{n} \|\Sigma f_i\|^2 \\
&= \sum_{i=1}^{n} \|\lambda_i f_i\|^2 \\
&\geq \sum_{i=1}^{n} |\lambda_i|^2
\end{aligned}$$

by the Bessel's inequality at the last step. ☐

Problem 14

Give an example of an operator on $\mathbb{C}^4$ whose characteristic polynomial equals $z(z-1)^2(z-3)$ and whose minimal polynomial equals $z(z-1)(z-3)$.

Proof. Consider

$$J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

☐

Problem 17

Suppose $\mathbb{F} = \mathbb{C}$ and $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that the characteristic polynomial of P is $z^m(z-1)^n$, where $m = \dim \operatorname{null} P$ and $n = \dim \operatorname{range} P$.

Proof. We know that the projection operator P has eigenvalue 0 and 1 (from definition). By many of our prior exercises, we know that the (generalized eigenspace of) eigenvalue 0 partitions the null space and nonzero ones partitions the range space. You may verify it by yourselves. $\square$

8C: Consequences of Generalized Eigenspace Decomposition

Lemma 32. *Suppose $T \in \mathcal{L}(V)$ is nilpotent. Then $I + T$ has a square root.*

Remark 33. *This lemma holds on both real and complex vector spaces.*

Lemma 34. *Suppose V is a complex vector space and $T \in \mathcal{L}(V)$ is invertible. Then T has a square root.*

Definition 35 (Jordan basis)**.** *Suppose $T \in \mathcal{L}(V)$. A basis of V is called a* Jordan basis *for T if with respect to this basis T has a block diagonal matrix*

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_p \end{pmatrix}$$

in which each A_k is an upper-triangular matrix of the form

$$A_k = \begin{pmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{pmatrix}$$

Theorem 36 (every nilpotent operator has a Jordan basis)**.** *Suppose $T \in \mathcal{L}(V)$ is nilpotent. Then there is a basis of V that is a Jordan basis for T.*

Corollary 37 (Jordan form)**.** *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then there is a basis of V that is a Jordan basis for T.*

Problem 1

Suppose $T \in \mathcal{L}(\mathbb{C}^3)$ is the operator defined by $T(z_1, z_2, z_3) = (z_2, z_3, 0)$. Prove that T does not have a square root.

Proof. Note that for eigenvalue

$$(z_2, z_3, 0) = \lambda(z_1, z_2, z_3)$$

the only solution is $\lambda = 0$ with multiplicity 3. Suppose for contradiction that $S^2 = T$. So any eigenvalue λ of S, λ^2 will be the eigenvalue of T, so S also only has $\lambda = 0$ as its only eigenvalue, indicating that S is nilpotent and $S^3 = 0$. This gives that $T^2 = SS^3 = 0$. However, we in fact have that

$$T^2(z_1, z_2, z_3) = (z_3, 0, 0) \neq 0$$

reaching a contradiction. □

Problem 6

Find a basis of $\mathcal{P}_4(\mathbb{R})$ that is a Jordan basis for the differentiation operator D on $\mathcal{P}_4(\mathbb{R})$ defined by $Dp = p'$.

Proof. Note that the goal here is to find linearly independent $v_1, \ldots, v_5$ s.t. $D(v_1) = 0$ and $D(v_i) = v_{i-1}$. This gives that

$$\left\{1, x, \frac{1}{2}x^2, \frac{1}{6}x^3, \frac{1}{24}x^4\right\}$$

□

Skip the rest of questions.

8D: Trace: A Connection Between Matrices and Operators

Definition 38 (trace of a matrix)**.** *Suppose A is a square matrix with entries in $\mathbb{F}$. The **trace** of A, denoted by trA, is defined to be the diagonal entries of A.*

Proposition 39 (trace of AB equals trace of BA)**.** *Suppose A is an m-by-n matrix and B is an n-by-m matrix. Then*

$$tr(AB) = tr(BA)$$

Lemma 40. *Suppose $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Then*

$$tr\mathcal{M}\left(T, (u_1, \ldots, u_n)\right) = tr\mathcal{M}\left(T, (v_1, \ldots, v_n)\right)$$

Definition 41 (trace of an operator)**.** *Suppose $T \in \mathcal{L}(V)$. The **trace** of T, denoted by trT, is defined by*

$$tr\ T = tr\ \mathcal{M}\left(T, (v_1, \ldots, v_n)\right)$$

where $v_1, \ldots, v_n$ is any basis of V.

Corollary 42. *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then $tr\ T$ equals the sum of the eigenvalues of T, with each eigenvalue included as many times as its multiplicity.*

Corollary 43. *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then $tr\ T$ equals the negative of the coefficient of z^{n-1} in the characteristic polynomial of T.*

Corollary 44. *Suppose V is an inner product space, $T \in \mathcal{L}(V)$, and $e_1, \ldots, e_n$ is an orthonormal basis of V. Then*

$$tr\ T = \langle Te_1, e_1\rangle + \cdots + \langle Te_n, e_n\rangle$$

Theorem 45 (trace is linear)**.** *The function $tr : \mathcal{L}(V) \to \mathbb{F}$ is a linear functional on $\mathcal{L}(V)$ such that*

$$tr\ (ST) = tr\ (TS)$$

for all $S, T \in \mathcal{L}(V)$.

Corollary 46. *There do not exist operators $S, T \in \mathcal{L}(V)$ such that $ST - TS = I$.*

Problem 1

Suppose V is an inner product space and $v, w \in V$. Define an operator $T \in \mathcal{L}(V)$ by $Tu = \langle u, v\rangle w$. Find a formula for tr T.

Proof. Let $e_1, \ldots, e_n$ be the standard orthonormal basis of V. Then we have that

$$\begin{aligned}
\text{tr } T &= \sum_{i=1}^{n} \langle Te_i, e_i\rangle \\
&= \sum_{i=1}^{n} \langle \langle e_i, v\rangle w, e_i\rangle \\
&= \sum_{i=1}^{n} \langle e_i, v\rangle \langle w, e_i\rangle \\
&= \sum_{i=1}^{n} v_i w_i \\
&= v \cdot w
\end{aligned}$$

□

Problem 2

Suppose $P \in \mathcal{L}(V)$ satisfies $P^2 = P$. Prove that

$$\text{tr } P = \dim \text{range } P$$

Proof. Note that $\text{tr}P = \sum_{i=1}^{n} \lambda_i$ where $\lambda_i = 1$ or 0. The multiplicity of $\lambda_i = 1$ determines the $\dim \text{range } P$ and thus gives the desired conclusion. □

Problem 5

Suppose V is an inner product space. Suppose $T \in \mathcal{L}(V)$ is a positive operator and tr $T = 0$. Prove that $T = 0$.

Proof. We know that $\lambda_i \geq 0$ for all i for positive T. Since $\text{tr} = 0$, all eigenvalues are 0 and thus $T = 0$ (as it's self-adjoint by positivity). □

Problem 9

Suppose $T \in \mathcal{L}(V)$ is such that tr $(ST) = 0$ for all $S \in \mathcal{L}(V)$ Prove that $T = 0$.

Proof. Consider orthonormal basis $e_1, \ldots, e_n$ and define S_{ij} to be such that maps e_j to e_i while keep all other zero. Therefore, tr $(S_{ij}T) = T_{ij} = 0$ for all i, j. Hence, we proved $T = 0$. □

Problem 11

Suppose V and W are inner product spaces and $T \in \mathcal{L}(V, W)$. Prove that if $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W, then

$$\text{tr }(T^*T) = \sum_{k=1}^{n} \sum_{j=1}^{m} |\langle Te_k, f_j \rangle|^2$$

Proof. We have that

$$\begin{aligned}
\text{tr }(T^*T) &= \sum_{k=1}^{n} \langle T^*Te_k, e_k \rangle \\
&= \sum_{k=1}^{n} \langle Te_k, Te_k \rangle \\
&= \sum_{k=1}^{n} \langle Te_k, \sum_{j=1}^{m} \langle Te_k, f_j \rangle f_j \rangle \\
&= \sum_{k=1}^{n} \sum_{j=1}^{m} |\langle Te_k, f_j \rangle|^2
\end{aligned}$$

□

Problem 12

Suppose V and W are finite-dimensional inner product spaces.

(a) Prove that $\langle S, T \rangle = \text{tr }(T^*S)$ defines an inner product on $\mathcal{L}(V, W)$.

(b) Suppose $e_1, \ldots, e_n$ is an orthonormal basis of V and $f_1, \ldots, f_m$ is an orthonormal basis of W. Show that the inner product on $\mathcal{L}(V, W)$ from (a) is the same as the standard inner product on $\mathbb{F}^{mn}$, where we identify each element of $\mathcal{L}(V, W)$ with its matrix (with respect to the bases just mentioned) and then with an element of $\mathbb{F}^{mn}$.

Remark 47. *The norm from (a) is called the Frobenius norm or the Hilbert-Schmidt norm.*

Proof. (a) We check each condition manually:

- Positivity: $\langle S, S \rangle = \text{tr}(S^*S) = \sum_{i=1}^{n} (A^*A)_{ii} = \sum_{i=1}^{n} \sum_{j=1}^{m} A^*_{ij} A_{ji} = \sum_{i=1}^{n} \sum_{j=1}^{m} |A_{ij}|^2 \geq 0$ with equality iff $S = 0$.
- Linearity in the first slot: $\langle \lambda S_1 + S_2, T \rangle = \text{tr}(T^*(\lambda S_1 + S_2)) = \lambda \text{tr}(T^*S_1) + \text{tr}(T^*S_2) = \lambda \langle S_1, T \rangle + \langle S_2, T \rangle$

- Conjugate symmetry: $\overline{\langle S, T\rangle} = \overline{\operatorname{tr}(T^*S)} = \operatorname{tr}(\overline{T^*S}) = \operatorname{tr}(S^*T) = \langle T, S\rangle$

(b) The standard inner product on $\mathbb{F}^{mn}$ for the two matrices A, B is

$$\langle A, B\rangle = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ij}\overline{B}_{ij}$$

which is exactly how we define in (a).

$\square$