Chapter 3: Linear Maps

Linear Algebra Done Right (4th Edition), by Sheldon Axler

Last updated: September 4, 2024

Contents

3A: Vector Space of Linear Maps

Definition 1 (Linear Map). *A* linear map *from V to W is a function $T\colon V \to W$ with the following properties:*

- ***Additivity**: $T(u+v) = Tu + Tv$ for all $u, v \in V$.*
- ***Homogeneity**: $T(\lambda v) = \lambda T(v)$ for all $\lambda \in \mathbb{F}$ and $v \in V$.*

Notation: $\mathcal{L}(V,W), \mathcal{L}(V)$

- The set of linear maps from V to W is denoted by $\mathcal{L}(V,W)$.
- The set of linear maps from V to V is denoted by $\mathcal{L}(V)$.

Lemma 2 (linear map basis lemma). *Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_n \in W$. Then there exists a unique linear map $T\colon V \to W$ such that*

$$Tv_k = w_k$$

for each $k = 1, \ldots, n$.

Definition 3 (additional and scalar multiplication on $\mathcal{L}(V,W)$). *Suppose $S, T \in \mathcal{L}(V,W)$ and $\lambda \in \mathbb{F}$. The sum $S+T$ and the product λT are the linear maps from V to W defined by*

$$(S+T)(v) = Sv + Tv \text{ and } (\lambda T)(v) = \lambda(Tv)$$

for all $v \in V$.

Remark 4. *$\mathcal{L}(V,W)$ is a vector space.*

Definition 5 (product of linear maps). *If $T \in \mathcal{L}(U,V)$ and $S \in \mathcal{L}(V,W)$, then the* product *$ST \in \mathcal{L}(U,W)$ is defined by*

$$(ST)(u) = S(Tu)$$

for all $u \in U$.

Remark 6 (algebraic properties of product of linear maps). *We have associativity, identity, and distributive properties whenever such properties are defined.*

Theorem 7 (linear maps take 0 to 0). *Suppose T is a linear map from V to W. Then $T(0) = 0$.*

Problem 1

Suppose $b, c \in \mathbb{R}$. Define $T\colon \mathbb{R}^3 \to \mathbb{R}^2$ by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$$

Show that T is linear if and only if $b = c = 0$.

Proof. T is linear $\iff T(x, y, z = 0) \iff b = 0$

In addition, let's only consider the second coordinate, then we have

$$T(x_1 + x_2, y_1 + y_2, z_1 + z_2)_2 = 6(x_1 + x_2) + c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2)$$

which only equals $T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$ if $c = 0$. $\square$

Problem 3

Suppose that $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$. Show that there exist scalars $A_{j,k} \in \mathbb{F}$ for $j = 1, \ldots, m$ and $k = 1, \ldots, n$ such that

$$T(x_1, \ldots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \ldots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

for every $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Proof. Let $\{u_1, \ldots, u_n\}$ denote the standard basis of $\mathbb{F}^n$. We have that

$$Tu_i = (A_{1,i}, \ldots, A_{m,i})$$

Take arbitrary $(x_1, \ldots, x_n) \in \mathbb{F}^n$, we have that

$$Tx_iu_i = x_i(A_{1,i}, \ldots, A_{m,i})$$

thus we have that

$$T(x_1, \cdots, x_n) = (A_{1,1}x_1 + \cdots + A_{1,n}x_n, \cdots, A_{m,1}x_1 + \cdots + A_{m,n}x_n)$$

$\square$

Problem 4

Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_m$ is a list of vectors in V such that $Tv_1, \ldots, Tv_m$ is a linearly independent list in W. Prove that $v_1, \ldots, v_m$ is linearly independent.

Proof. This means that the only solution to $\sum_{i=1}^m a_i T(v_i) = \sum_{i=1}^m T(a_i v_i) = 0$ is all $a_i = 0$. However, as we know $T(0) = 0$ so $a_i v_i = 0$ and thus we've proved the claim. $\square$

Problem 7

Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V)$, then there exists $\lambda \in \mathbb{F}$ such that $Tv = \lambda v$ for all $v \in V$.

Proof. Since $\dim V = 1$, every $v \in V$ can be expressed as $\lambda v'$ for some other $v' \in V$. As $Tv \in V$, we have $Tv = \beta v$. Then $Tv = T\lambda v' = \beta\lambda v' = \beta v$ □

Problem 8

Give an example of a function $\phi\colon \mathbb{R}^2 \to \mathbb{R}$ such that

$$\phi(av) = a\phi(v)$$

for all $a \in \mathbb{R}$ and all $v \in \mathbb{R}^2$ but ϕ is not linear.

Proof. Consider $f(x_1, x_2) = x_1^2/x_2$ if $x_2 \neq 0$ o.w. 0, then $f(a(x_1, x_2)) = ax_1^2 = af(x_1, x_2)$. However, $f((x_1 + y_1, x_2 + y_2)) = (x_1 + y_1)^2/(x_2 + y_2) \neq f(x_1, x_2) + f(y_1, y_2)$. □

Problem 11

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for all $S \in \mathcal{L}(V)$.

Proof. $\Rightarrow$ We can express $T = \lambda I$. Then $S(\lambda I) = \lambda S = \lambda IS = TS$.

$\Leftarrow$ Let $v_1, \ldots, v_m$ be a basis of V. Pick S_i such that $S_i(\sum_{i=1}^m a_i v_i) = a_i v_i$, which is clearly a linear operator. Then we have that

$$S_i T(v) = TS_i(v)$$
$$S_i \sum_{j=1}^m b_j v_j = T(a_i v_i)$$
$$b_i v_i = a_i T(v_i)$$

This shows that for all v_i, there exists λ_i such that $T(v_i) = \lambda_i v_i$. Next we show that such λ_i does not depend on i. Construct S_{ij} subtly such that $S_{ij} \sum_{k=1}^n a_k v_k = a_j v_i + a_i v_j$.

Then we have that

$$S_{ij} T v = TS_{ij} v$$
$$S_{ij} \left(\sum_{k=1}^n \lambda_k a_k v_k \right) = T(a_j v_i + a_i v_j)$$
$$\lambda_j a_j v_i + \lambda_i a_i v_j = \lambda_i a_j v_i + \lambda_j a_i v_j$$

This shows that $\lambda_i = \lambda_j$ for all i, j and thus we've shown that $T = \lambda I$ for some λ. □

Problem 12

Suppose U is a subspace of V with $U \neq V$. Suppose $S \in \mathcal{L}(U, W)$ and $S \neq 0$. Define $T \colon V \to W$ by

$$Tv = \begin{cases} Sv & \text{if } v \in U \\ 0 & \text{if } v \in V \text{ and } v \notin U \end{cases}$$

Prove that T is not a linear map on V.

Proof. Take $u \in U$ such that $u \notin \mathrm{Null}(S)$. Take $v \in V \setminus U$, then $u + v \in V \setminus U$. This means that

$$T(u+v) = S(u+v) = 0 \neq T(u) + T(v) = Su$$

□

Problem 13

Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

Proof. Note that there exists subspace P s.t. $V = P \oplus U$. For all $v \in V$, $v = p + u$ for some p, u. Then define $T(v) = Su + p$. Clearly we have that $T(u) = Su$ for all $u \in U$. It now only suffices to prove T is a linear map. Homogeneity is trivial to show. For additivity, $T(v_1 + v_2) = T(u_1 + u_2 + p_1 + p_2) = S(u_1 + u_2) + p_1 + p_2 = (Su_1 + p_1) + (Su_2 + p_2) = Tv_1 + Tv_2$. □

Problem 14

Suppose V is finite-dimensional with $\dim V > 0$, and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

Proof. Recall the definition of infinite-dimension from ch2 P17:

V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \dots$ of vectors in V such that $v_1, \dots, v_m$ is linearly independent for every positive integer m.

W being infinite-dimensional implies this. Denote the sequence by $w_1, w_2, \dots$ Let $v_1, \dots, v_m$ be the basis for V. Define a sequence of linear operators as follows: $T_k \in \mathcal{L}(V, W)$ such that $T_k(v) = v_k$. Then we have that for every positive integer m, the solution for the following equation is all $a_i = 0$.

$$a_1 T_1(v) + \cdots + a_m T_m(v) = a_1 v_1 + \cdots + a_m v_m = 0$$

This shows that $\mathcal{L}(V, W)$ is infinite-dimensional. □

Problem 15

Suppose $v_1, \ldots, v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \ldots, w_m \in W$ such that no $T \in \mathcal{L}(V, W)$ satisfies $Tv_k = w_k$ for each $k = 1, \ldots, m$.

Proof. Take $w_1, \ldots, w_m$ to be linearly independent list of vectors in W. Then we have that

$Ta_1v_1 + \cdots Ta_mv_m = a_1w_1 + \cdots a_mw_m = 0$

The only solution is that $a_i = 0$ for all i, but this contradicts that $v_1, \ldots, v_m$ is linearly dependent. □

Problem 16

Suppose V is finite-dimensional with $\dim V > 1$. Prove that there exist $S, T \in \mathcal{L}(V)$ such that $ST \neq TS$.

Proof. Let $v_1, \ldots, v_m$ be the basis of V. Define $S(v) = S(\sum_{i=1}^m a_iv_i) = \sum_{i=1}^m a_{m-i}v_i$ and $T(v) = a_1v_1$. Then we have that

$$STv = Sa_1v_1 = a_1v_1$$

but

$$TSv = T\sum_{i=1}^m a_{m-i}v_i = a_mv_1$$

□

Problem 17

Suppose V is finite-dimensional. Show that the only two-sided ideas of $\mathcal{L}(V)$ are $\{0\}$ and $\mathcal{L}(V)$, where we define that a subspace $\mathcal{E}$ of $\mathcal{L}(V)$ is called two-sided ideal if $TE \in \mathcal{E}$ and $ET \in \mathcal{E}$ for all $E \in \mathcal{E}$ and all $T \in \mathcal{L}(V)$.

Proof. It's easy to verify that $\{0\}$ and $\mathcal{L}(V)$ are two-sided ideal of $\mathcal{L}(V)$. Suppose for the sake of contradiction that such $\mathcal{E}$ exists. Let $e_1, \ldots, e_m$ be its basis and let $e_1, \ldots e_m, e_{m+1}, \ldots, e_n$ be the basis for V. Define $T(v) = T(\sum_{i=1}^n a_ie_i) = \sum_{i=1}^n a_ie_{n-i}$ to be a linear map in V (ez to verify), and $E_j(u) = E_j(\sum_{k=1}^m b_ke_k) = b_je_j$. We have reached the contradicting example such that $TE_j(v) = a_je_{n-j} \in V \setminus \mathcal{E}$. □

3B: Null Spaces and Ranges

Definition 8 (null space)**.** *For* $T \in \mathcal{L}(V, W)$*, the* null space *of* T*, denoted by* *null* T*, is the subset of* V *consisting of those vectors that* T *maps to 0:*

$$\textit{null}\ T = \{v \in V : Tv = 0\}$$

Corollary 9. *Suppose* $T \in \mathcal{L}(V, W)$*, then null* T *is a subspace of* V*.*

Definition 10 (injective)**.** *A function* $T: V \to W$ *is called* ***injective*** *if* $Tu = Tv$ *implies* $u = v$*.*

Theorem 11. *Let* $T \in \mathcal{L}(V, W)$*. Then* T *is injecive if and only if null* $T = \{0\}$*.*

Definition 12 (range)**.** *For* $T \in \mathcal{L}(V, W)$*, the* ***range*** *of* T *is the subset of* W *consisting of those vectors that are equal to* Tv *for some* $v \in V$*:*

$$\textit{range}\ T = \{Tv : v \in V\}$$

Corollary 13. *Suppose* $T \in \mathcal{L}(V, W)$*, then range* T *is a subspace of* W*.*

Definition 14 (surjective)**.** *A function* $T: V \to W$ *is called* ***surjective*** *if its range equals* W*.*

Theorem 15 (fundamental theorem of linear map)**.** *Suppose* V *is finite-dimensional and* $T \in \mathcal{L}(V, W)$*. Then range* T *is finite-dimensional and*

$$\dim V = \dim \textit{null}\ T + \dim \textit{range}\ T$$

Proof. Let $u_1, \ldots, u_m$ be a basis of null T and let $u_1, \ldots, u_m, v_1, \ldots, v_n$ be a basis of V. It now suffices to prove dim range $T = n$. Let $v \in T$, then

$$v = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$$

We apply T on both sides to get that

$$Tv = \sum_{j=1}^{n} b_j T v_j$$

which shows that $Tv_1, \ldots, Tv_n$ spans range T and thus it's finite-dimensional. To show they are linearly independent, we have

$$\sum_{j=1}^{n} b_j T v_j = T \sum_{j=1}^{n} b_j v_j = 0$$

The only solution is that all $b_j = 0$. □

Corollary 16 (linear map to a lower-dimen space is not injective)**.** *Suppose* V *and* W *are finite-dimensional vector spaces such that* $\dim V > \dim W$*. Then no linear map from* V *to* W *is injective.*

Corollary 17 (linear map to a higher-dimen space is not surjective)**.** *Suppose* V *and* W *are finite-dimensional vector spaces such that* $\dim V < \dim W$*. Then no linear map from* V *to* W *is surjective.*

Application Consider the system of linear equation defined by the map $T\colon \mathbb{F}^n \to \mathbb{F}^m$:

$$T(x_1, \ldots, x_n) = \left(\sum_{k=1}^{n} A_{1,k} x_k, \ldots, \sum_{k=1}^{n} A_{m,k} x_k \right)$$

Corollary 18 (homogeneous systems of linear equations)**.** *A homogeneous system of linear equations with more variables than equations has nonzero solutions.*

Corollary 19 (inhomogeneous system fo linear equations)**.** *An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.*

Problem 1

Give an example of a linear map T with dim null $T = 3$ and dim range $T = 2$.

Proof. Consider $T(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, x_4, x_5)$.

It's easy to verify that dim null $T = 3$ and applying the theorem of linear map solves the problem. □

Problem 2

Suppose $S, T \in \mathcal{L}(V)$ are such that range $S \subseteq$ null T. Prove that $(ST)^2 = 0$.

Proof. This means that take $x \in V$, $T(S(x)) = 0$. We have $(ST)^2 = STST = 0$. □

Problem 3

Suppose $v_1, \ldots, v_m$ is a list of vectors in V. Define $T \in \mathcal{L}(\mathbb{F}^m, V)$ by

$$T(z_1, \ldots, z_m) = z_1 v_1 + \cdots + z_m v_m$$

(a) what property of T corresponds to $v_1, \ldots, v_m$ spanning V?
(b) what property of T corresponds to $v_1, \ldots, v_m$ being linearly independent?

Proof. (a) surjective (range $= V$)

(b) injective ($z_1, \ldots, z_m$ is identically zero if and only if $z_1 v_1 + \cdots + z_m v_m = 0$) □

Problem 4

Show that $\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) \colon \dim \operatorname{null} T > 2\}$ is not a subspace of $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$.

Proof. Try to come up with a counterexample. Consider $f(e_1, e_2, e_3, e_4, e_5) = (e_1, 0, 0, e_4)$ and $g(e_1, e_2, e_3, e_4, e_5) = (0, e_2, e_3, 0)$. Then we have that null $f = \{0, e_2, e_3, 0, e_5\}$ and null $g = \{e_1, 0, 0, e_4, e_5\}$ so both in $\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)\colon \dim \text{null } T > 2\}$. However, $f + g(e_1, e_2, e_3, e_4, e_5) = \{(e_1, e_2, e_3, e_4)\}$ and this means their dim null = 5 - 4 = 1 ¡ 3. □

Problem 7

Suppose V and W are finite-dimensional and $2 \leq \dim V \leq \dim W$. Show that $\{T \in \mathcal{L}(V, W)\colon T \text{ is not injective}\}$ is not a subspace of $\mathcal{L}(V, W)$.

Proof. T is not injective means that $\dim V > \dim W$, which contradicts the assumption in the question. □

Problem 9

Suppose $T \in \mathcal{L}(V, W)$ is injective and $v_1, \ldots, v_n$ is linearly independent in V. Prove that $Tv_1, \ldots, Tv_n$ is linearly independent in W.

Proof.

$$\sum_{i=1}^{n} z_i v_i = 0 \Longleftrightarrow \sum_{i=1}^{n} z_i T(v_i) = 0$$

□

Problem 10

Suppose $v_1, \ldots, v_n$ spans V and $T \in \mathcal{L}(V, W)$. Show that $Tv_1, \ldots, Tv_n$ spans range T.

Proof. Take v in V, then we know $v = \sum_{i=1}^{n} a_i v_i$. Take win range T. Then we know there exists $v \in V$ such that $w = T(v)$ and thus $w = T(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i T(v_i)$. □

Problem 11

Suppose that V is finite-dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace $\mathcal{U}$ of V such that

$$\mathcal{U} \cap \text{null } T = \{0\} \text{ and range } T = \{Tu\colon u \in \mathcal{U}\}$$

Proof. Since we know null T is a subspace of V, then there exists $\mathcal{U}$ such that $\mathcal{U} \oplus \text{null } T = V$. We can define $\mathcal{U}$ to be such case. To finish the proof, let $v = u + t, u \in \mathcal{U}, t \in nullT$, then $rangeT = \{T(u + t) = Tu\colon u \in \mathcal{U}\}$. □

Problem 16

Suppose V and W are both finite-dimensional. Prove that there exists an injective linear map from V to W if and only if $\dim V \leq \dim W$.

Proof. $\Rightarrow$ $T \in \mathcal{L}(V, W)$ is injective. Then this means that $\dim V = \dim \operatorname{range} T \leq \dim W$.

$\Leftarrow$ By assumption, we can define a linear map T such that $Tv_i = w_i$ where v_i, w_i are the respective basis of V and W. Then we have that

$$\sum_{i=1}^{n} a_i T v_i = \sum_{i=1}^{n} a_i w_i = 0 \Longleftrightarrow a_i \text{ are identically zero}$$

This means that null T is $\{0\}$ and thus it is injective. $\square$

Problem 17

Suppose V and W are both finite-dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim V \geq \dim W$.

Proof. $\Rightarrow$ $\dim V \geq \operatorname{range} T = \dim W$.

$\Leftarrow$ By assumption we can define a linear map T such that $Tv_i = w_i$ for $1 \leq i \leq \dim W$ and $Tv_i = 0$ for $\dim W \leq i \dim V$. This mean that take $w \in W$, then $w = \sum_{i=1}^{\dim W} a_i w_i$. At the same time, we know that for all w_i, there exists v_i s.t. $Tv_i = w_i$, therefore we have $w = \sum_{i=1}^{\dim W} a_i T v_i$ and thus T is surjective. $\square$

Problem 18

Suppose V and W are finite-dimensional and that $\mathcal{U}$ is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that null $T = \mathcal{U}$ if and only if $\dim \mathcal{U} \geq \dim V - \dim W$.

Proof. We know $\dim V = \operatorname{null} T + \operatorname{range} T$ and $\operatorname{range} T \leq \dim W$. Therefore, $\dim \mathcal{U} = \dim \operatorname{null} T \geq \dim V - \dim W$. $\square$

Problem 19

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity operator on V.

Proof. T is injective $\Longleftrightarrow$ $T(v)$ is unique for $v \in V$. $\Longleftrightarrow$ We can define $S\colon STv = v$. $\square$

Problem 21

Suppose V is finite-dimensional, $T \in \mathcal{L}(V, W)$, and $\mathcal{U}$ is a subspace of W. Prove that $\mathcal{T} = \{v \in V : Tv \in \mathcal{U}\}$ is a subspace of V and

$$\dim \mathcal{T} = \dim \operatorname{null} T + \dim(\mathcal{U} \cap \operatorname{range} T)$$

Proof. To prove subspace, we can simply follow by definition. We can define $S \in \mathcal{L}(\mathcal{T}, \mathcal{U})$ such that $Sv = Tv$ for all $v \in \mathcal{T}$. Then we have that range(S) $\in \mathcal{U} \cap$ range(T) and that null T = null S. Hence we have proved the claim. $\square$

Problem 22

Suppose $\mathcal{U}$ and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(\mathcal{U}, V)$. Prove that

$$\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T$$

Proof. We can write that

$$\text{null } ST = \{u \in \mathcal{U} \colon T(u) \in \text{null } (S)\}$$

By the previous question, we know that

$$\dim \text{null } ST = \dim \text{null } T + \dim \text{null } S \cap \text{range } T \leq \dim \text{null } T + \dim \text{null } S$$

□

Problem 23

Suppose $\mathcal{U}$ and V are finite-dimensional vector spaces and $S \in \mathcal{L}(V, W)$ and $T \in \mathcal{L}(\mathcal{U}, V)$. Prove that

$$\dim \text{range } ST \leq \min\{\dim \text{range } S, \dim \text{range } T\}$$

Proof. First note that $\dim \text{range } ST \leq \dim \text{range } S$. This because range $ST \subseteq$ range S.

To prove $\dim \text{range } ST \leq \dim \text{range } T$, we have that $\dim \mathcal{U} = \dim \text{null } T + \dim \text{range } T = \dim \text{null } ST + \dim \text{range } ST$. Since we know $\dim \text{null } T \geq \dim \text{null } ST$, $\dim \text{range } ST \leq \dim \text{range } T$. □

Problem 24

(a) Suppose $\dim V = 5$ and $S, T \in \mathcal{L}(V)$ are such that $ST = 0$. Prove that $\dim \text{range } TS \leq 2$.
(b) Give an example of $S, T \in \mathcal{L}(\mathbb{F}^5)$ with $ST = 0$ and $\dim \text{range } TS = 2$.

Proof. (a) $\dim \text{null } ST = 5 \leq \dim \text{null } T + \dim \text{null } S$

We also know that $5 = \dim \text{null } T + \dim \text{range } T = \dim \text{range } S + \dim \text{null } S$ and thus we have that $\dim \text{range } T + \dim \text{range } S \leq 5$. This implies that $\min\{\dim \text{range } T, \dim \text{range } S\} \leq 2$. Hence, by applying P23, we finish the proof.

(b) Consider $T(x_1, x_2, x_3, x_4, x_5) = (x_3, x_4, 0, 0, 0), S(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, x_4, x_5)$. Then we have that $ST(x_1, x_2, x_3, x_4, x_5) = S(x_3, x_4, 0, 0, 0) = (0, 0, 0, 0, 0)$ while $TS(x_1, x_2, x_3, x_4, x_5) = T(0, 0, x_3, x_4, x_5) = (x_3, x_4, 0, 0, 0)$. □

Problem 25

Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that null $S \subseteq$ null T if and only if there exists $E \in \mathcal{L}(W)$ such that $T = ES$.

Proof. null $S \subseteq$ null $T \Rightarrow \dim \text{range } T \leq \dim \text{range } S \Rightarrow$ let $s_1, \ldots, s_n$ be basis of range S and $t_1, \ldots, t_m$ be basis of range T where $m \leq n$. We can always define $E \in \mathcal{L}(W)$ such that $E(s_i) = t_i$ for all $1 \leq i \leq m$ and 0 otherwise.

The other direction is trivial. □

Problem 26

Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that range $S \subseteq$ range T if and only if there exists $E \in \mathcal{L}(V)$ such that $S = TE$.

Proof. range $S \subseteq$ range $T \Rightarrow$ Take $v_1, \ldots, v_m$ to be the basis of V, define the linear map $Sv_i = s_i$ for $s_i \in$ range $S \subseteq$ range T. Then there exists $u_1, \ldots, u_m$ such that $Tu_i = s_i$ for all i. Then we can define the linear map $Ev_i = u_i$ for all i such that $S = TE$.

The other direction is trivial. □

Problem 27

Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V =$ null $P \oplus$ range P.

Proof. Take $v \in$ null $(P) \cap$ range (P). Then this means that $P(v) = 0$ and there exists u s.t. $v = P(u)$. At the same time, $P(v) = P(P(u)) = P(u) = 0 = v$. So we have that null $P \cap$ range $P = \{0\}$. Since P is defined on $\mathcal{L}(P)$, we have that $V =$ null $P \oplus$ range P. □

3C: Matrices

Definition 20 (matrix, $\mathbf{A}_{j,k}$)**.** *Suppose m and n are nonnegative integers. An m-by-n matrix* matrix $\mathbf{A}$ *is a rectangular array of elements of $\mathbb{F}$ with m rows and n columns:*

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{1,1} & \cdots & \mathbf{A}_{1,n} \\ \vdots & & \vdots \\ \mathbf{A}_{m,1} & \cdots & \mathbf{A}_{m,n} \end{pmatrix}$$

where the notation $\mathbf{A}_{j,k}$ denotes the entry in row j and column k.

Definition 21 (matrix of a linear map, $\mathcal{M}(T)$)**.** *Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. The* matrix of T *with respect to these bases is the m-by-n matrix $\mathcal{M}(T)$ whose entries $\mathbf{A}_{j,k}$ are defined by*

$$Tv_k = \mathbf{A}_{1,k}w_1 + \cdots + \mathbf{A}_{m,k}w_m$$

.

If the bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ are not clear from the context, then the notation $\mathcal{M}(T, (v_1, \ldots, v_n), (w_1, \ldots, w_m))$ is used.

Remark 22. *The k-th column of $\mathcal{M}(T)$ consists of the scalars needed to write Tv_k as a linear combination of $w_1, \ldots, w_m$:*

$$Tv_k = \sum_{j=1}^{m} \mathbf{A}_{j,k}w_j$$

Remark 23. *If T is a linear map from n-dimensional vector space to an m-dimensional vector space, then $\mathcal{M}(T)$ is an m-by-n matrix.*

Corollary 24 (Matrix addition and scalar multiplication)**.** *Suppose $S, T \in \mathcal{L}(V, W)$ and $\lambda \in \mathbb{F}$. Then $\mathcal{M}(\lambda S + T) = \lambda\mathcal{M}(S) + \mathcal{M}(T)$.*

For m and n positive integers, the set of all m-by-n matrices with entries in F is denoted by $\mathbb{F}^{m,n}$.

Theorem 25. *Suppose m and n are positive integers. $\mathbb{F}^{m,n}$ is a vector space of dimension mn.*

Definition 26 (matrix multiplication)**.** *Suppose* $\mathbf{A}$ *is an m-by-n matrix and* $\mathbf{B}$ *is an n-by-p matrix. Then AB is defined to be the m-by-p matrix whose entry in row j, column k, is given by the equation*

$$(\mathbf{AB})_{j,k} = \sum_{r=1}^{n} \mathbf{A}_{j,r}\mathbf{B}_{r,k}$$

In words, the entry in row j, column k of $\mathbf{AB}$ *is computed by taking row j of* $\mathbf{A}$ *and column k of* $\mathbf{B}$.

Motivations. Let $v_1, \ldots, v_n$ to be the basis of V, $w_1, \ldots, w_m$ to be the basis of W, and $u_1, \ldots, u_p$ to be the basis of U. Consider linear maps $T: U \to V$ and $S: V \to W$. Suppose $\mathcal{M}(T) = \mathbf{A}$ and $\mathcal{M}(S) = \mathbf{B}$. For $1 \leq k \leq p$, we have

$$
\begin{aligned}
(ST)u_k &= S\left(\sum_{r=1}^{n} \mathbf{B}_{r,k} v_r\right) \\
&= \sum_{r=1}^{n} \mathbf{B}_{r,k}(Sv_r) \\
&= \sum_{r=1}^{n} \mathbf{B}_{r,k} \sum_{j=1}^{m} \mathbf{A}_{j,r} w_j \\
&= \sum_{j=1}^{m} \left(\sum_{r=1}^{n} \mathbf{A}_{j,r}\mathbf{B}_{r,k}\right) w_j
\end{aligned}
$$

Theorem 27 (matrix of product of linear maps)**.** *If $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$, then $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$.*

Definition 28 ($\mathbf{A}_{j,\cdot}, \mathbf{A}_{\cdot,k}$)**.** *Suppose $\mathbf{A}$ is an m-by-n matrix*

- *If $1 \leq j \leq m$, then $\mathbf{A}_{j,\cdot}$ denotes the 1-by-n matrix consisting of row j of $\mathbf{A}$.*
- *If $1 \leq k \leq n$, then $\mathbf{A}_{\cdot,k}$ denotes the m-by-1 matrix consisting of column k of $\mathbf{A}$.*

Corollary 29 (entry of matrix product equals row times column)**.** *Suppose $\mathbf{A}$ is an m-by-n matrix and $\mathbf{B}$ is an n-by-p matrix. Then*

$$(\mathbf{AB})_{j,k} = \mathbf{A}_{j,\cdot}\mathbf{B}_{\cdot,k}$$

if $1 \leq j \leq m$ and $1 \leq k \leq p$. In other words, the entry in row j, column k, of $\mathbf{AB}$ equals (row j of $\mathbf{A}$) times (column k of $\mathbf{B}$).

Corollary 30 (column of matrix product equals matrix times column)**.** *Suppose $\mathbf{A}$ is an m-by-n matrix and $\mathbf{B}$ is an n-by-p matrix. Then*

$$(\mathbf{AB})_{\cdot,k} = \mathbf{A}\mathbf{B}_{\cdot,k}$$

if $1 \leq k \leq p$. In other words, column k of $\mathbf{AB}$ equals $\mathbf{A}$ times column k of $\mathbf{B}$.

Corollary 31 (linear combination of columns)**.** *Suppose $\mathbf{A}$ is an m-by-n matrix and $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ is an n-by-1 matrix. Then*

$$\mathbf{Ab} = \mathbf{b}_1\mathbf{A}_{\cdot,1} + \cdots + \mathbf{b}_n\mathbf{A}_{\cdot,n}.$$

In other words, $\mathbf{Ab}$ *is a linear combination of the columns of* $\mathbf{A}$*, with the scalars that multiply the columns coming from* $\mathbf{b}$.

Theorem 32 (matrix multiplication as linear combination of columns)**.** *Suppose* C *is an* m*-by-*c *matrix and* R *is a* c*-by-*n *matrix.*

- *If* $k \in \{1, \ldots, n\}$*, then column* k *of* CR *is a linear combination of the columns of* C*, with the coefficients of this linear combination coming from columns* k *of* R*.*
- *If* $j \in \{1, \ldots, m\}$*, then row* j *of* CR *is a linear combination of the rows of* R*, with the coefficients of this linear combination coming from row* j *of* C*.*

Definition 33 (column rank, row rank)**.** *Suppose* $\mathbf{A}$ *is an* m*-by-*n *matrix with entries in* $\mathbb{F}$*.*

- *The column rank of* $\mathbf{A}$ *is the dimension of the span of the columns of* $\mathbf{A}$ *in* $\mathbb{F}^{m,1}$*.*
- *The row rank of* $\mathbf{A}$ *is the dimension of the span of the rows of* $\mathbf{A}$ *in* $\mathbb{F}^{1,n}$*.*

Definition 34 (transpose, $\mathbf{A}^\top$)**.** *The transpose of a matrix* $\mathbf{A}$*, denoted by* $\mathbf{A}^\top$*, is the matrix obtained from* $\mathbf{A}$ *by interchanging rows and columns. Specifically, if* $\mathbf{A}$ *is an* m*-by-*n *matrix, then* $\mathbf{A}^\top$ *is the* n*-by-*m *matrix whose entries are given by the equation*

$$(\mathbf{A}^\top)_{k,j} = \mathbf{A}_{j,k}$$

Lemma 35 (column-row factorization)**.** *Suppose* $\mathbf{A}$ *is an* m*-by-*n *matrix with entries in* $\mathbb{F}$ *and column rank* $c \geq 1$*. Then there exists an* m*-by-*c *matrix* $\mathbf{C}$ *and a* c*-by-*n *matrix* $\mathbf{R}$*, both with entries in* $\mathbb{F}$*, such that* $\mathbf{A} = \mathbf{CR}$*.*

Proof. Each column of $\mathbf{A}$ is an m-by-1 matrix. The list $\mathbf{A}_{\cdot,1}, \ldots, \mathbf{A}_{\cdot,n}$ of columns of $\mathbf{A}$ can be reduced to a basis of the span of the columns of $\mathbf{A}$. This basis has length c. The c columns in this basis can be put together to form an m-by-c matrix $\mathbf{C}$.

If $k \in \{1, \ldots, n\}$, then column k of $\mathbf{A}$ is a linear combination of the columns of $\mathbf{C}$. Make the coefficients of this linear combination into column k of a c-by-n matrix that we call $\mathbf{R}$. Then $\mathbf{A} = \mathbf{CR}$. □

Theorem 36 (column rank equals row rank)**.** *Suppose* $\mathbf{A} \in \mathbb{F}^{m,n}$*. Then the column rank of* $\mathbf{A}$ *equals the row rank of* $\mathbf{A}$*.*

Proof. Let c denote the column rank of $\mathbf{A}$. Let $\mathbf{A} = \mathbf{CR}$ be the column-row factorization of $\mathbf{A}$ given by the proof before, where $\mathbf{C}$ is m-by-c and $\mathbf{R}$ is c-by-n. Then the column-row factorization lemma tells us that every row of $\mathbf{A}$ is a linear combination of the rows of $\mathbf{R}$. Because $\mathbf{R}$ has c rows, this implies that the row of $\mathbf{A}$ is less than or equal to the column rank c of $\mathbf{A}$.

To prove the other direction, we can do the same thing to $\mathbf{A}^\top$ and then we can get that column rank of $\mathbf{A}$ = row rank of $\mathbf{A}^\top \leq$ column rank of $\mathbf{A}^\top$ = row rank of $\mathbf{A}$ which we proved above. □

Problem 1

Suppose $T \in \mathcal{L}(V,W)$. Show that with respect to each choice of bases of V and W, the matrix of T has at least $\dim \operatorname{range} T$ nonzero entries.

Proof. Let $v_1, \ldots, v_n$ be the basis of V and $w_1, \ldots, w_m$ be the basis of W. Suppose there exists a matrix $\mathbf{A} = \mathcal{M}(T)$ of T has less than $\dim \operatorname{range} T$ nonzero entries. This means that $\mathbf{A}$ has at most $\dim \operatorname{range} T - 1$ nonzero columns so this implies that $\dim \operatorname{range} T < \dim \operatorname{range}\ -1$ which forms a contradiction. $\square$

Problem 2

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V,W)$. Prove that $\dim \operatorname{range} T = 1$ if and only if there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ equal 1.

Proof. $\Rightarrow$ We solve this problem through careful construction of the basis. That is, we will construct basis $v_1, \ldots, v_n$ of V and $w_1, \ldots, w_m$ of W such that $Tv_i = w_1 + \cdots + w_m$ for all i. We can achieve this again mainly because $\dim \operatorname{range} T = 1$. let $u_1, \ldots, u_m$ be a set of arbitrary basis of W. Take $w \in \operatorname{range} T$ so we have $w = \sum_{i=1}^m a_i u_i$. We consider all the coefficients a_i as follows: take an index set I such that for all $i \in I, a_i \neq 0$ and we have $|I| = c$. Take arbitrary $j \in I$ and we construct the basis as follows: let $w_j = (a_j - (m-r))u_j, w_k = a_k u_k$ for all $k \in I, k \neq j$, and $w_k = u_k + u_j$ for all $k \notin I$. Then we have that $w = \sum_{l=1}^m w_l$. Since we know $w \in \operatorname{range} T$, then there exists $v_1 \in V$ s.t. $T(v_1) = w$. Let $v_2, \ldots, v_n$ be the basis of null T. Since we know $\dim V = \operatorname{null} T + 1$ so $v_1, v_2, \ldots, v_n$ constitutes a basis of V. In this way, we successfully constructs the basis such that all entries of $\mathcal{M}(T)$ equal 1.

$\Leftarrow$ There exists basis $v_1, \ldots, v_n$ of V and $w_1, \ldots, w_m$ of W such that all entries of $\mathcal{M}(T)$ equal 1. This means that $Tv_1 = \cdots = Tv_n = w_1 + \cdots + w_m$. So we first have that $\dim \operatorname{range} T \neq 0$. To prove $\dim \operatorname{range} T = 1$, take arbitrary $u_1, u_2 \in \operatorname{range} T$. Then we know that $u_1 = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i(w_1 + \cdots + w_n)$ and similarly $u_2 = \sum_{i=1}^n b_i(w_1 + \cdots + w_n)$. Here, we have that

$$u_1 = \left(\sum_i^n a_i\right) / \left(\sum_i^n b_i\right) u_2$$

and thus $\dim \operatorname{range} T$ has to be 1-dimensional since every two arbitrary vector is simply a scalar multiple of each other. $\square$

Problem 3

Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Show that if $S, T \in \mathcal{L}(V,W)$ and $\lambda \in \mathbb{F}$, then $\mathcal{M}(\lambda S + T) = \lambda \mathcal{M}(S) + \mathcal{M}(T)$.

Proof.

$$\begin{aligned}(\lambda S + T)v_k &= \lambda S(v_k) + T(v_k) \\ &= \lambda \sum_{i=1}^{m} \mathbf{A}_{i,k} w_i + \sum_{i=1}^{m} \mathbf{B}_{i,k} w_i\end{aligned}$$

□

Problem 5

Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that there exists a basis of V and a basis of W such that with respect to these bases, all entries of $\mathcal{M}(T)$ are 0 except that the entries in row k, column k equal 1 if $1 \leq k \leq \dim \operatorname{range} T$.

Proof. Let $\dim V = n, \dim W = m, \dim \operatorname{range} T = k$. The main idea of the proof is to construct basis $v_1, \ldots, v_n$ and $w_1, \ldots, w_m$ such that $Tv_i = w_i$ for all $1 \leq i \leq k$ and zero otherwise. (So the constructed T satisfies the requirement).

We proceed with obtaining the basis $w_1, \ldots, w_k$ of range T and extend this $w_1, \ldots, w_k, \ldots, w_m$ to the basis of W. We know for all $1 \leq i \leq k$, there exists $v_i \in V$ s.t. $Tv_i = w_i$. We claim that $v_1, \ldots, v_k$ are linearly independent. To prove this, see $\sum_{j=1}^{k} a_j v_j = 0 \Rightarrow \sum_{j=1}^{k} a_j T(v_j) = \sum_{j=1}^{k} a_j w_j = 0$. Similar to the proof in the fundamental theorem of linear map, we extend the basis to V through considering the null space. We further claim that (let $K = \operatorname{span}(v_1, \ldots, v_k)$) $V = K \oplus \operatorname{null} T$ (which is easy to prove). Hence, extending the basis from K with the basis from null T completes the proof. □

Problem 6

Suppose $v_1, \ldots, v_m$ is a basis of V and W is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $w_1, \ldots, w_n$ of W such that all entries in the first column of $\mathcal{M}(T)$ [with respect to these bases] are 0 except for possibly a 1 in the first row, first column.

Proof. Let $w_1 = Tv_1$ and extend w_1 to basis of W. Then we automatically obtains T that gets the desired property. □

Problem 7

Suppose $w_1, \ldots, w_n$ is a basis of W and V is finite-dimensional. Suppose $T \in \mathcal{L}(V, W)$. Prove that there exists a basis $v_1, \ldots, v_m$ of V such that all entries in the first of $\mathcal{M}(T)$ [wrt. these bases] are 0 except for possibly a 1 in the first row, first column.

Proof. Take arbitrary $u_1 \in V$ and we have that $Tu_1 = \sum_{i=1}^{n} a_{i,1} w_i$. Take $v_1 = u_1 / a_{1,1}$, then $Tv_1 = w_1 + \sum_{j=2}^{n} b_{j,1} w_j$. We can extend $v_1, u_2, \ldots, u_m$ to be the basis of V. Then consider

$$Tu_j = \sum_{k=1}^{n} b_{k,j} w_k$$

Consider to let $b_{1,j} = 0$ for all $2 \leq j \leq n$. To do this, let $v_j = u_j - b_{1,j}v_1$, then we have that $T(v_j) = T(u_j - b_{1,j}v_1) = \sum_{k=2}^{n} c_{k,j}w_k$. It now left to verify that $v_1, v_2, \ldots, v_m$ is the basis.

$$\begin{aligned} q_1v_1 + \cdots + q_mv_m &= 0 \\ q_1v_1 + q_2(u_2 - b_{1,2}v_1) + \cdots + q_m(u_m - b_{1,m}v_1) &= 0 \\ (q_1 - (q_2b_{1,2} + \cdots + q_mb_{1,m}))v_1 + q_2u_2 + \cdots + q_mu_m &= 0 \end{aligned}$$

The only solution is that $q_1 - (q_2b_{1,2} + \cdots + q_mb_{1,m}) = q_2 = \cdots = q_m = 0$ and thus we have completed the proof. □

Problem 8

Suppose $\mathbf{A}$ is an m-by-n matrix and $\mathbf{B}$ is an n-by-p matrix. Prove that

$$(\mathbf{AB})_{j,\cdot} = \mathbf{A}_{j,\cdot}\mathbf{B}$$

for each $1 \leq j \leq m$. In other words, show that row j of $\mathbf{AB}$ equals (row j of $\mathbf{A}$) times $\mathbf{B}$.

Proof. We know that

$$(\mathbf{AB})_{j,k} = \sum_{i=1}^{n} \mathbf{A}_{j,i}\mathbf{B}_{i,k}$$

For instance $(\mathbf{AB})_{j,1} = \sum_{i=1}^{n} \mathbf{A}_{j,i}\mathbf{B}_{i,1} = \mathbf{A}_{j,\cdot}\mathbf{B}_{\cdot,1}$. Similarly, $(\mathbf{AB})_{j,k} = \mathbf{A}_{j,\cdot}\mathbf{B}_{\cdot,k}$ and thus by treating this we have

$$(\mathbf{AB})_{j,\cdot} = \mathbf{A}_{j,\cdot}\mathbf{B}$$

□

Problem 9

Suppose $\mathbf{a} = (a_1, \cdots, a_n)$ is a 1-by-n matrix and $\mathbf{B}$ is an n-by-p matrix. Prove that

$$\mathbf{aB} = a_1\mathbf{B}_{1,\cdot} + \cdots + a_n\mathbf{B}_{n,\cdot}$$

In other words, show that $\mathbf{aB}$ is a linear combination of the rows of $\mathbf{B}$, with the scalars that multiply the rows coming from $\mathbf{a}$.

Proof. By definition of matrix multiplication, the entry in the column k of the l.h.s. equals

$$(\mathbf{aB})_{1,k} = a_1\mathbf{B}_{1,k} + \cdots + a_n\mathbf{B}_{n,k}$$

which equals the right-hand side. □

Problem 11

Prove that the distributive property holds for matrix addition and multiplication. In other words, suppose $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}$ are matrices whose sizes are such that $\mathbf{A}(\mathbf{B}+\mathbf{C})$ and $(\mathbf{D}+\mathbf{E})\mathbf{F}$ make sense. Explain this and prove that

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB} + \mathbf{AC} \qquad (\mathbf{D}+\mathbf{E})\mathbf{F} = \mathbf{DF} + \mathbf{EF}$$

Proof. We let $\mathbf{A}, \mathbf{D}, \mathbf{E}$ to be m-by-n matrix and $\mathbf{B}, \mathbf{C}, \mathbf{F}$ to be n-by-p matrix. Then all the matrix multiplication make sense. We proceed to prove the equality:

$$\begin{aligned}
(\mathbf{A}(\mathbf{B}+\mathbf{C}))_{j,k} &= \sum_{i=1}^{n} \mathbf{A}_{j,i}(\mathbf{B}+\mathbf{C})_{i,k} \\
&= \sum_{i=1}^{n} \mathbf{A}_{j,i}(\mathbf{B}_{i,k}+\mathbf{C}_{i,k}) \\
&= \sum_{i=1}^{n} (\mathbf{A}_{j,i}\mathbf{B}_{i,k}) + (\mathbf{A}_{j,i}\mathbf{C}_{i,k}) \\
&= \mathbf{AB} + \mathbf{AC}
\end{aligned}$$

For the other one, we have that

$$\begin{aligned}
((\mathbf{D}+\mathbf{E})\mathbf{F})_{j,k} &= \sum_{i=1}^{n} (\mathbf{D}+\mathbf{E})_{j,i}\mathbf{F}_{i,k} \\
&= \sum_{i}^{n} (\mathbf{D}_{j,i}\mathbf{F}_{i,k}) + (\mathbf{E}_{j,i}\mathbf{F}_{i,k}) \\
&= \mathbf{DF} + \mathbf{EF}
\end{aligned}$$

□

Problem 13

Suppose $\mathbf{A}$ is an n-by-n matrix and $1 \leq j, k \leq n$. Show that the entry in row j, column k of $\mathbf{A}^3$ (which is defined to mean $\mathbf{AAA}$) is

$$\sum_{p=1}^{n} \sum_{r=1}^{n} \mathbf{A}_{j,p}\mathbf{A}_{p,r}\mathbf{A}_{r,k}$$

Proof.

$$\begin{aligned}
(\mathbf{A}(\mathbf{A}\mathbf{A}))_{j,k} &= \sum_{p}^{n} \mathbf{A}_{j,p}(\mathbf{A}\mathbf{A})_{p,k} \\
&= \sum_{p}^{n} \mathbf{A}_{j,p} \sum_{r=1}^{n} \mathbf{A}_{p,r}\mathbf{A}_{r,k} \\
&= \sum_{p}^{n} \sum_{r}^{n} \mathbf{A}_{j,p}\mathbf{A}_{p,r}\mathbf{A}r, k
\end{aligned}$$

□

Problem 14

Suppose m and n are positive integers. Prove that the function $\mathbf{A} \mapsto \mathbf{A}^\top$ is a linear map from $\mathbf{F}^{m,n}$ to $\mathbf{F}^{n,m}$.

Proof. Let T denote the linear map such that $T(\mathbf{A}) = \mathbf{A}^\top$. Then we have that

$$T(\lambda\mathbf{A} + \mathbf{B}) = (\lambda\mathbf{A} + \mathbf{B})^\top = \lambda\mathbf{A}^\top + \mathbf{B}^\top = \lambda T(\mathbf{A}) + T(\mathbf{B})$$

□

Problem 15

Prove that if $\mathbf{A}$ is an m-by-n matrix and $\mathbf{C}$ is an n-by-p matrix, then

$$(\mathbf{A}\mathbf{C})^\top = \mathbf{C}^\top\mathbf{A}^\top$$

Proof.

$$\begin{aligned}
((\mathbf{A}\mathbf{C})^\top)_{j,k} &= (\mathbf{A}\mathbf{C})_{k,j} \\
&= \sum_{i=1}^{n} \mathbf{A}_{k,i}\mathbf{C}_{i,j}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\mathbf{C}^\top\mathbf{A}^\top)_{j,k} &= \sum_{i=1}^{n} \mathbf{C}^\top_{j,i}\mathbf{A}^\top_{i,k} \\
&= \sum_{i=1}^{n} \mathbf{A}_{k,i}\mathbf{C}_{i,j}
\end{aligned}$$

□

Problem 16

Suppose $\mathbf{A}$ is an m-by-n matrix with $\mathbf{A} \neq 0$. Prove that the rank of $\mathbf{A}$ is 1 if and only if there exist $(c_1, \ldots, c_m) \in \mathbb{F}^m$ and $(d_1, \ldots, d_n) \in \mathbb{F}^n$ such that $\mathbf{A}_{j,k} = c_j d_k$ for every $j = 1, \ldots, m$ and every $k = 1, \ldots, n$.

Proof. $\Rightarrow$ If rank of $\mathbf{A}$ is 1, then this means that the span of the columns of $\mathbf{A}$ is 1-dimensional, take $\mathbf{A}_{\cdot,1} \in \mathbb{F}^m$ from the span. (Note that here we assume $\mathbf{A}_{\cdot,1} \neq 0$ ow. take the first nonzero one) Then we can let $(c_1, \ldots, c_m) = \operatorname{vec}(\mathbf{A}_{\cdot,1})$. and let For every other columns, they are the scalar multiple of the the first one, so there exists $d_2, \ldots, d_n$ s.t. $\mathbf{A}_{j,k} = c_j d_k$ (we take $d_1 = 1$).

$\Leftarrow$ If we denote $\mathbf{c} = (c_1, \ldots, c_m)$, then

$$\mathbf{A} = \begin{pmatrix} d_1\mathbf{c} & \cdots & d_n\mathbf{c} \end{pmatrix}$$

so each column is a scalar multiple of the other and thus $\mathbf{A}$ has rank 1. $\square$

Problem 17

Suppose $T \in \mathcal{L}(V)$, and $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Prove that the following are equivalent:
(a) T is injective.
(b) The column of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{n,1}$.
(c) The columns of $\mathcal{M}(T)$ span $\mathbb{F}^{n,1}$.
(d) The rows of $\mathcal{M}(T)$ span $\mathbb{F}^{1,n}$.
(e) The rows of $\mathcal{M}(T)$ are linearly independent in $\mathbb{F}^{1,n}$.
Here $\mathcal{M}(T)$ means $\mathcal{M}(T, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$.

Proof. Let $\mathbf{A}$ represents the matrix of $\mathcal{M}(T)$.

$(a) \Rightarrow (b)$: $a_1\mathbf{A}_{\cdot,1} + \cdots + a_n\mathbf{A}_{\cdot,n} = 0$. We also know that $Tv_i = \sum_{j=1}^n \mathbf{A}_{j,i} u_j$. Then we have that

$$T(a_1v_1 + \cdots + a_nv_n) = \sum_{i=1}^n a_i T(v_i) = \sum_{i=1}^n a_i \sum_{j=1}^n \mathbf{A}_{j,i} u_j = \sum_{j=1}^n \left(\sum_{i=1}^n a_i \mathbf{A}_{j,i} \right) u_j = 0$$

To prove all a_i is zero, we note that $a_1v_1 + \cdots + a_nv_n \in \operatorname{null} T$ and since T is injective, the only possible case is that all $a_i = 0$ and thus we prove the claim.

$(b) \Rightarrow (c)$ The columns of $\mathcal{M}(T)$ forms a basis and thus span the space.

$(c) \Rightarrow (d)$ Column rank of $\mathcal{M}(T)$ is n and thus row rank is also n and thus the rows span $\mathbb{F}^{1,n}$.

$(d) \Rightarrow (e)$ Same as above.

$(e) \Rightarrow (a)$ The only solution to $a_1\mathbf{A}_{1,\cdot} + \cdots + a_n\mathbf{A}_{n,\cdot} = 0$ is all $a_i = 0$. Suppose for the sake of contradiction that T is not injective so there exists nonzero $v \in \operatorname{null} T$. We know that $v = \sum_{i=1}^n a_i v_i$ and that $Tv_i = \sum_{j=1}^n \mathbf{A}_{j,i} u_j$. Then

$$Tv = \sum_{i=1}^n a_i T(v_i) = \sum_{i=1}^n a_i \sum_{j=1}^n \mathbf{A}_{j,i} u_j = \sum_{i=1}^n \left(\sum_{j=1}^n a_i \mathbf{A}_{j,i} \right) u_j = 0$$

and thus $\sum_{j=1}^{n} a_i \mathbf{A}_{j,i} = 0$ for all j. We know not all $a_i = 0$ since v is non-trivial and thus the solution contradicts that the rows are all linearly independent. Proof completed. $\square$

3D: Invertibility and Isomorphisms

Definition 37 (invertible, inverse).

- *A linear map $T \in \mathcal{L}(V, W)$ is called* invertible *if there exists a linear map $S \in \mathcal{L}(W, V)$ such that ST equals the identity operator on V and TS equals the identity operator on W.*
- *A linear map $S \in \mathcal{L}(W, V)$ satisfying $ST = I$ and $TS = I$ is called an* inverse *of T.*

Theorem 38. *An invertible linear map has a unique inverse.*

If T is invertible, then its inverse is denoted by T^{-1}.

Theorem 39. *A linear map is invertible if and only if it is injective and surjective.*

Proof. $\Rightarrow$ We have an invertible linear map T that $Tu = Tv$. Then we have

$$u = T^{-1}Tu = T^{-1}Tv = v$$

To show T is surjective, we have that for any $w \in W, w = T(T^{-1}w)$.

$\Leftarrow$ Define S such that for each $w \in W$, $S(w)$ is the unique element s.t. $T(S(w)) = w$. (we can do this due to injectivity and surjectivity). Then we have that $T(ST)v = (TS)Tv = Tv$ and thus $STv = v$ and thus $ST = I$. So we suffices the identity operator condition. It's easy to show that S is a linear map. ☐

Theorem 40. *Suppose that V and W are finite-dimensional vector spaces, $\dim V = \dim W$, and $T \in \mathcal{L}(V, W)$. Then*

$$T \text{ is invertible } \iff T \text{ is injective } \iff T \text{ is surjective.}$$

Corollary 41. *Suppose V and W are finite-dimensional vector spaces of the same dimension, $S \in \mathcal{L}(W, V)$, and $T \in \mathcal{L}(V, W)$. Then $ST = I$ if and only if $TS = I$.*

Definition 42 (Isomorphism). *An **Isomorphism** is an invertible linear map. Two vector spaces are called **isomorphic** if there is an isomorphism from one vector space onto the other one.*

Theorem 43 (dimension shows whether vector spaces are isomorphic). *Two finite-dimensional vector spaces over $\mathbb{F}$ are isomorphic if and only if they have the same dimension.*

Theorem 44. *Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then $\mathcal{M}$ is an isomorphism between $\mathcal{L}(V, W)$ and $\mathbb{F}^{m,n}$.*

Corollary 45. *Suppose V and W are finite-dimensional. Then $\mathcal{L}(V, W)$ is finite-dimensional and*

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$

Definition 46 (matrix of a vector, $\mathcal{M}(v)$)**.** *Suppose $v \in V$ and $v_1, \ldots, v_n$ is a basis of V. Then **matrix of** v with respect to the basis is the n-by-1 matrix*

$$\mathcal{M}(v) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

where $b_1, \ldots, b_n$ are the scalars such that

$$v = b_1 v_1 + \cdots + b_n v_n$$

Remark 47. *The matrix $\mathcal{M}(v)$ of a vector $v \in V$ depends on the basis $v_1, \ldots, v_n$ and v. We can think of elements of V as relabeled to be n-by-1 matrices, i.e. $V \mapsto \mathbb{F}^{n,1}$.*

Corollary 48. *Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Let $1 \leq k \leq n$. Then kth column of $\mathcal{M}(T)$, which is denoted by $\mathcal{M}(T)_{\cdot,k}$, equals $\mathcal{M}(Tv_k)$*

Theorem 49 (linear maps act like matrix multiplication)**.** *Suppose $T \in \mathcal{L}(V, W)$ and $v \in V$. Suppose $v_1, \ldots, v_n$ is a basis of V and $w_1, \ldots, w_m$ is a basis of W. Then*

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$$

Proof.

$$\begin{aligned} \mathcal{M}(Tv) &= b_1 \mathcal{M}(Tv_1) + \cdots + b_n \mathcal{M}(Tv_n) \\ &= b_1 \mathcal{M}(T)_{\cdot,1} + \cdots + b_n \mathcal{M}(T)_{\cdot,n} \\ &= \mathcal{M}(T)\mathcal{M}(v) \end{aligned}$$

□

Remark 50. *Each m-by-n matrix $\mathbf{A}$ induces a linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$, namely the matrix multiplication function that takes $x \in \mathbb{F}^{n,1}$ to $\mathbf{A}x \in \mathbb{F}^{m,1}$. We can think of every linear map (from a finite vector space to another) as a matrix multiplication map after suitable relabeling via the isomorphisms given by $\mathcal{M}$. Specifically, if $T \in \mathcal{L}(V, W)$ and we identify $v \in V$ with $\mathcal{M}(v) \in \mathbb{F}^{n,1}$, then the result above says that we can identify Tv with $\mathcal{M}(T)\mathcal{M}(v)$.*

Theorem 51 (dimension of range T equals column rank of $\mathcal{M}(T)$)**.** *Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then $\dim$ range T equals the column rank of $\mathcal{M}(T)$.*

Theorem 52 (change-of-basis-formula)**.** *Suppose $T \in \mathcal{L}(V)$. Suppose $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are bases of V. Let*

$$\mathbf{A} = \mathcal{M}(T, (u_1, \ldots, u_n)) \text{ and } \mathbf{B} = \mathcal{M}(T, (v_1, \ldots, v_n))$$

and $\mathbf{C} = \mathcal{M}(I, (u_1, \ldots, u_n), (v_1, \ldots, v_n))$. Then

$$\mathbf{A} = \mathbf{C}^{-1}\mathbf{B}\mathbf{C}.$$

Proof. ☐

Problem 1

Suppose $T \in \mathcal{L}(V, W)$ is invertible. Show that T^{-1} is invertible and

$$(T^{-1})^{-1} = T.$$

Proof. T^{-1} is invertible because there exists T such that $TT^{-1} = T^{-1}T = I$.

$$T^{-1}T = TT^{-1} = I$$

so $(T^{-1})^{-1} = T$. ☐

Problem 2

Suppose $T \in \mathcal{L}(U, V)$ and $S \in \mathcal{L}(V, W)$ are both invertible linear maps. Prove that $ST \in \mathcal{L}(U, W)$ is invertible and that $(ST)^{-1} = T^{-1}S^{-1}$.

Proof. $(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = I = T^{-1}S^{-1}ST$. ☐

Problem 3

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that the following are equivalent.
(a) T is invertible.
(b) $Tv_1, \ldots, Tv_n$ is a basis of V for every basis $v_1, \ldots, v_n$ of V.
(c) $Tv_1, \ldots, Tv_n$ is a basis of V for some basis $v_1, \ldots, v_n$ of V.

Proof. $(a) \Rightarrow (b)$ It only suffice to prove linear independence. We can show this

$$a_1Tv_1 + \cdots + a_nTv_n = 0 \Longleftrightarrow a_1v_1 + \cdots + a_nv_n = 0$$

since T is injective and thus the only solution is all a_i are identically zero.

$(b) \Rightarrow (c)$ Trivial.

$(c) \Rightarrow (a)$ By the linear map lemma, there exists $S \in \mathcal{L}(V)$ such that $S(Tv_i) = v_i$ for all i. Such S is the inverse of T (one can verify) and thus T is invertible. ☐

Problem 5

Suppose V is finite-dimensional, U is a subspace of V, and $S \in \mathcal{L}(U, V)$. Prove that there exists an invertible linear map T from V to itself such that $Tu = Su$ for every $u \in U$ if and only if S is injective.

Proof. $\Rightarrow$ Since T is invertible, we can define T^{-1} restricted to U, where $S^{-1}u = T^{-1}u$ for all $u \in U$. Then S has an inverse and thus is injective.

$\Leftarrow$ Let $v_1, \dots v_m$ be the basis of U and $v_1, \dots, v_m, x_1 \dots, v_n$ be the extended basis of V. Define $Tv_i = Sv_i$ and $Tx_i = x_i$. Then T is injective and thus invertible. $\square$

Problem 6

Suppose that W is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that null S = null T if and only if there exists an invertible $E \in \mathcal{L}(W)$ such that $S = ET$.

Proof. $\Rightarrow$ Suppose V is finite-dimensional for simplicity (one can do more work for relaxing this assumption). Let $V = \text{null } S \oplus C = \text{null } T \oplus C$ for some C. Then we know that $T|_C \colon C \to \text{range }(T(C))$ and $S|_C \colon C \to \text{range }(S(C))$ is invertible. Then there exists an invertible map $\hat{E} \colon \text{range }(T(C)) \to \text{range }(S(C))$ as $\dim \text{range } T(C) = \text{range }(S(C)) = \dim V$. Extending this map to W solves the problem (take bases from range $T(C)$ and extend to W with "identity" map).

$\Leftarrow$ Take $v \in \text{null } T$, then $S(v) = E(Tv) = E(0) = 0$ and thus $v \in \text{null } S$. Conversely, take $v \in \text{null } S$, then $T(v) = E^{-1}S(v) = 0$ so $v \in \text{null } T$. Thus null S = null T. $\square$

Problem 7

Suppose that V is finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that range S = range T if and only if there exists an invertible $E \in \mathcal{L}(V)$ such that $S = TE$.

Proof. $\Rightarrow$ By 3B P26, we know that there exists $\tilde{E} \in \mathcal{L}(V)$ s.t. $S = T\tilde{E}$ as range $S \subseteq$ range T. range S = range T implies that $\dim \text{null } S = \dim \text{range } T$. Then we can define an invertible map $\bar{E}$ from null S to null T. We can define the map E as follows: Denote $V = \text{null } S \oplus P$, then $E(v) = \tilde{E}(v_1) + \bar{E}(v_2)$ for $v = v_1 + v_2$, where $v_1 \in P, v_2 \in \text{null } S$ and now $\tilde{E} = \tilde{E}|_P$. Then we have that $TE(v) = T(\tilde{E}(v_1) + \bar{E}(v_2)) = S(v_1) + S(v_2) = S(v)$. Now, it left to verify that our proposed E is invertible.

To show this, we mainly aim at proving range $\tilde{E} \cap$ range $\bar{E} = \{0\}$, as if we have this, we know that $E(v) = \tilde{E}(v_1) + \bar{E}(v_2) = 0 \Rightarrow \tilde{E}(v_1) = \bar{E}(v_2) = 0$. We know $\bar{E}$ is invertible so $v_2 = 0$. To see $\tilde{E}$ is injective, we can see that for any $v_1 \in \text{null } \tilde{E}$, we have $Sv_1 = T\tilde{E}v_1 = T0 = 0$ and thus $v_1 \in \text{null } S \cap P = 0$, so $v_1 = 0$ and thus $nul(E) = \{0\}$, E is injective and therefore invertible.

To complete the proof, let $u \in \text{range } \tilde{E} \cap \text{range } \bar{E}$, then we know there exists $v_1 \in P, v_2 \in \text{null } S$ s.t. $u = \tilde{E}(v_1) = \bar{E}(v_2)$. We also know that $u \in \text{null } T$ as range $\bar{E}$ = null T. Now, we have that $S(v_1) = T\tilde{E}(v_1) = T\bar{E}(v_2) = 0$ and thus $v_1 \in \text{null } S$. So $v_1 \in P \cap \text{null } S = \{0\}$ and thus we have that $u = \{0\}$, finishing the proof.

$\Leftarrow$ Take any $s \in \text{range } S$, then there exists $E(s)$ s.t. $T(E(s)) = s$. Conversely, take any $t \in \text{range } T$, then there exists $v \in V$ s.t. $t = Tu = TEE^{-1}v = S(E^{-1}v)$.

Thus range $S =$ range T. $\square$

Problem 8

Suppose V and W are finite-dimensional and $S, T \in \mathcal{L}(V, W)$. Prove that there exist invertible $E_1 \in \mathcal{L}(V)$ and $E_2 \in \mathcal{L}(W)$ such that $S = E_2 T E_1$ if and only if $\dim \operatorname{null} S = \dim \operatorname{null} T$.

Proof. $\Rightarrow$ We know from 3B exercises that $\dim \operatorname{range} ST \leq \min(\dim \operatorname{range} S, \dim \operatorname{range} T)$. Applying here gets that $\dim \operatorname{range} S \leq \dim \operatorname{range} T$. Consider that $T = E_2^{-1} S E_1^{-1}$, then we can also derive $\dim \operatorname{range} T \leq \dim \operatorname{range} S$ and thus $\dim \operatorname{range} S = \dim \operatorname{range} T$. By the fundamental theorem of linear map, we can get that $\dim \operatorname{null} S = \dim \operatorname{null} T$.

$\Leftarrow$ We know $\dim \operatorname{null} S = \dim \operatorname{null} T$, so there exists an isomorphism $\tilde{E}_1 \colon \operatorname{null} S \mapsto \operatorname{null} T$. Extending $\tilde{E}_1 \colon \operatorname{null} S \mapsto V$ still preserves its injectivity. Therefore, by P5, we have an invertible $E_1 \colon V \mapsto V$ such that $E_1 u = \tilde{E}_1 u$ for all $u \in \operatorname{null} S$. Now we intend to show that $\operatorname{null} S = \operatorname{null} TE_1$, as proving so would imply there exists an invertible $E_2 \in \mathcal{L}(W)$ such that $S = E_2 T E_1$ by P6. To see this, take $s \in \operatorname{null} S$, then $TE_1(s) = T(\tilde{E}_1 s) = 0$ and thus $s \in \operatorname{null} TE_1$. To see the other direction, take $t \in \operatorname{null} TE_1$, then we know $E_1 t \in \operatorname{null} T = \operatorname{range} \tilde{E}_1$. Thus there exists $v \in \operatorname{null} S$ s.t. $E_1 t = \tilde{E}_1 v = E_1 v$. As E_1 is invertible, $t = v \in \operatorname{null} S$ and thus we have shown $\operatorname{null} S = \operatorname{null} TE_1$, completing the proof. $\square$

Problem 9

Suppose V is finite-dimensional and $T \colon V \to W$ is a surjective linear map of V onto W. Prove that there is a subspace U of V such that $T|_U$ is an isomorphism of U onto W.

Proof. T being surjective means that $\dim V \geq \dim W$. Take any subspace of V that has $\dim W$, then there is an isomorphism between this subspace and W. $\square$

Problem 10

Suppose V and W are finite-dimensional and U is a subspace of V. Let

$$\mathcal{E} = \{T \in \mathcal{L}(V, W) \colon U \subseteq \operatorname{null} T\}.$$

(a) Show that $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.
(b) Find a formula for $\dim \mathcal{E}$ in terms of $\dim V, \dim W$, and $\dim U$.

Proof. (a) Take $T_1, T_2 \in \mathcal{E}$ and $\lambda \in \mathbb{F}$, then consider null $(\lambda T_1 + T_2)$. Let $u \in U$, then $\lambda T_1 + T_2(u) = \lambda T_1(u) + T_2(u) = 0 + 0 = 0$ and thus we have $\lambda T_1 + T_2 \in \mathcal{E}$ and thus $\mathcal{E}$ is a subspace of $\mathcal{L}(V, W)$.

(b) Following the hint, define $\Phi \colon \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ by $\Phi(T) = T|_U$. Then $\Phi \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(U, W))$. We try to show the range and null space of Φ.

We first claim null $\Phi = \mathcal{E}$. To see this, take $T \in$ null Φ, then we know $\Phi(T) = T|_U = 0 \in \mathcal{L}(U, W)$, meaning that U is the subset of null T, so null $\Phi \subseteq \mathcal{E}$. Conversely, take $T \in \mathcal{E}$, then we have that for all $u \in U$, $Tu = 0$, then we have that $T|_U = 0$. This shows that $\mathcal{E} \subseteq$ null Φ.

Next we claim range $\Phi = \mathcal{L}(U, W)$. Take any $S \in \mathcal{L}(U, W)$, then we can naturally always extend S to $T \in \mathcal{L}(V, W)$ by setting $Tu = Su$ for all $u \in U$. The converse direction is trivial.

Thus we have that $\dim \mathcal{E} = \dim \operatorname{null} \Phi$ and $\dim \operatorname{range} \Phi = \dim \mathcal{L}(U, W) = \dim U \dim W$ and $\dim \mathcal{L}(V, W) = \dim V \dim W = \dim \operatorname{null} \Phi + \dim \operatorname{range} \Phi$, thus we derive that

$$\dim V \dim W = \dim \mathcal{E} + \dim U \dim W.$$

□

Problem 14

Prove or give a counterexample: If V is finite-dimensional and $R, S, T \in \mathcal{L}(V)$ are such that RST is surjective, then S is injective.

Proof. We know that

$$\dim \operatorname{range} RST = \dim \operatorname{range} V \leq \min\{\dim \operatorname{range} R, \dim \operatorname{range} S, \dim \operatorname{range} T\}$$

This means that $\dim RST \leq \dim \operatorname{range} S$. If $\dim \operatorname{null} S > 0$, then $\dim \operatorname{range} S < \dim V$ and thus contradicts the assumption that RST is surjective. □

Problem 15

Suppose $T \in \mathcal{L}(V)$ and $v_1, \ldots, v_m$ is a list in V such that $Tv_1, \ldots, Tv_m$ spans V. Prove that $v_1, \ldots, v_m$ spans V.

Proof. We can reduce the list to basis and use previous conclusions. Since $Tv_1, \ldots, Tv_m$ spans V, we can reduce the list to $Tv_1, \ldots, Tv_n$ such that the list is the basis of V. By previous results, $v_1, \ldots, v_n$ is also linearly independent and its dimension equals the $\dim V$ and thus is the basis of V. It thus spans V. Extending the list to $v_1, \ldots, v_m$ also spans V. □

Problem 16

Prove that every linear map from $\mathbb{F}^{n,1}$ to $\mathbb{F}^{m,1}$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$, then there exists an m-by-n matrix $\mathbf{A}$ such that $Tx = \mathbf{A}x$ for every $x \in \mathbb{F}^{n,1}$.

Proof. We can simply consider the standard basis of $\mathbb{F}^{n,1}$ and $\mathbb{F}^{m,1}$. Then Let $\mathbf{A}$ be the matrix of T wrt these bases. Then we have that by definition,

$$Tx = \mathcal{M}(Tx) = \mathcal{M}(T)\mathcal{M}(x) = \mathbf{A}x$$

□

Problem 17

Suppose V is finite-dimensional and $S \in \mathcal{L}(V)$. Define $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$ by

$$\mathcal{A}(T) = ST$$

for $T \in \mathcal{L}(V)$.
(a) Show that $\dim \operatorname{null} \mathcal{A} = \dim V \dim \operatorname{null} S$.
(b) Show that $\dim \operatorname{range} \mathcal{A} = \dim V \dim \operatorname{range} S$.

Proof. (a) We prove this by showing $\operatorname{null} \mathcal{A} = \mathcal{L}(V, \operatorname{null} S)$. Take $A \in \operatorname{null} \mathcal{A}$, then we know that $\mathcal{A}(A) = SA = 0$ and thus $Av \in \operatorname{null} S$ for all $v \in V$. For the other direction, take $A \in \mathcal{L}(V, \operatorname{null} S)$, then $SA(v) = \mathcal{A}(v) = 0$ and thus $A \in \operatorname{null} \mathcal{A}$.

(b) We know $\dim(\mathcal{L}(V)) = \dim \operatorname{null} \mathcal{A} + \dim \operatorname{range} \mathcal{A}$. Hence

$$\dim \operatorname{range} \mathcal{A} = \dim V(\dim V - \dim \operatorname{null} S) = \dim V \dim \operatorname{range} S.$$

□

Problem 18

Show that V and $\mathcal{L}(\mathbb{F}, V)$ are isomorphic vector spaces.

Proof. For simplicity, assume V is finite-dimensional, then $\dim V = \dim \mathbb{F} \dim V = \dim \mathcal{L}(\mathbb{F}, V)$ and thus they are isomorphic. One might do a more careful construction for infinite-dimensional V. □

Problem 19

Suppose V is finite-dimensional and $T \in \mathcal{L}(V)$. Prove that T has the same matrix with respect to every basis of V if and only if T is a scalar multiple of the identity operator.

Proof. $\Rightarrow$ We know $Tv_k = \sum_{j=1}^{n} A_{j,k} v_k = \sum_{j=1}^{n} 2A_{j,k} v_k = T(2v_k)$. This means that $A_{j,k} = 0$ for $j \neq k$. When $j = k$, we have that $Tv_k = A_{k,k} v_i$ for some arbitrary ordering of the same basis. Then this means that $A_{i,i} = A_{j,j}$ for all i, j. Thus, T is a scalar multiple of the identity operator.

$\Leftarrow$ This follows by the definition of the identity operator. □

3E: Products and Quotients of Vector Spaces

Definition 53 (product of vector spaces)**.** *Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$.*

- *The **product** $V_1 \times \cdots \times V_m$ is defined by*

$$V_1 \times \cdots \times V_m = \{(v_1, \dots, v_m) \colon v_1 \in V_1, \dots, v_m \in V_m\}.$$

- *Addition on $V_1 \times \cdots \times V_m$ is defined by*

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m).$$

- *Scalar multiplication on $V_1 \times \cdots \times V_m$ is defined by*

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m).$$

Theorem 54 (product of vector spaces is a vector space)**.** *Suppose $V_1, \dots, V_m$ are vector spaces over $\mathbb{F}$. Then $V_1 \times \cdots \times V_m$ is a vector space over $\mathbb{F}$.*

Theorem 55 (dimension of a product is the sum of dimensions)**.** *Suppose $V_1, \dots, V_m$ are finite-dimensional vector spaces. Then $V_1 \times \cdots \times V_m$ is finite-dimensional and*

$$\dim(V_1 \times \cdots \times V_m) = \dim V_1 + \cdots + \dim V_m.$$

Lemma 56 (products and direct sums)**.** *Suppose that $V_1, \dots, V_m$ are subspaces of V. Define a linear map $\Gamma \colon V_1 \times \cdots \times V_m \to V_1 + \cdots + V_m$ by*

$$\Gamma(v_1, \dots, v_m) = v_1 + \cdots + v_m.$$

Then $V_1 + \cdots + V_m$ is a direct sum if and only if Γ is injective.

Theorem 57 (a sum is a direct sum if and only if dimensions add up)**.** *Suppose V is finite-dimensional and $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if*

$$\dim(V_1 + \cdots + V_m) = \dim V_1 + \cdots + \dim V_m.$$

Definition 58 ($v + U$, translation)**.** *Suppose $v \in V$ and $U \subset V$. Then $v + U$ is the subset of V defined by*

$$v + U = \{v + u \colon u \in U\}.$$

*The set $v + U$ is said to be a **translate** of U.*

Definition 59 (**quotient space**, V/U)**.** *Suppose U is a subspace of V. Then the **quotient space** V/U is the set of all translates of U. Thus*

$$V/U = \{v + U \colon v \in V\}.$$

Remark 60. *If* $U = \{(x, 2x) \in \mathbb{R}^2 \colon x \in R\}$*, then* $\mathbb{R}^2/U$ *is the set of all lines in* $\mathbb{R}^2$ *that have slope 2.*

Lemma 61 (two translates of a subspace are equal or disjoint)**.** *Suppose* U *is a subspace of* V *and* $v, w \in V$*. Then*

$$v - w \in U \Longleftrightarrow v + U = w + U \Longleftrightarrow (v + U) \cap (w + U) = \emptyset$$

Definition 62 (addition and scalar multiplication on V/U)**.** *Suppose* U *is a subspace of* V*. Then* ***addition*** *and* ***scalar multiplication*** *are defined on* V/U *by*

$$\begin{aligned}(v + U) + (w + U) &= (v + w) + U\\ \lambda(v + U) &= (\lambda v) + U\end{aligned}$$

for all $v, w \in V$ *and all* $\lambda \in \mathbb{F}$*.*

Theorem 63 (quotient space is a vector space)**.** *Suppose* U *is a subspace of* V*. Then* V/U*, with the operations of addition and scalar multiplication as defined above, is a vector space.*

Definition 64 (quotient map)**.** *Suppose* U *is a subspace of* V*. The* ***quotient map*** $\pi\colon V \to V/U$ *is the linear map defined by*

$$\pi(v) = v + U$$

for each $v \in V$*.*

Theorem 65 (dimension of quotient space)**.** *Suppose* V *is finite-dimensional and* U *is a subspace of* V*. Then*

$$\dim V/U = \dim V - \dim U$$

Definition 66. *Suppose* $T \in \mathcal{L}(V, W)$*. Define* $\tilde{T}\colon V/(\mathit{null}\ T) \to W$ *by*

$$\tilde{T}(v + \mathit{null}\ T) = Tv.$$

Another interpretation of quotient space is on **congruence of subspaces**. The definition goes as follows:

If Y is a subspace of X, then two vectors $x_1, x_2 \in X$ are **congruent modulo** Y, denoted $x_1 \cong x_2(\mod Y)$ if $x_1 - x_2 \in Y$.

The quotient space X/Y denotes the set of equivalence classes in X, modulo Y by defining that

$$\{x\} + \{z\} \coloneqq \{x + z\} \quad a\{x\} \coloneqq \{ax\}$$

So here one can think of the zero vector of $v + U$ defined as the equivalence class that contains that zero vector $0 \in V$:

$$0 + U = \{0 + u \colon u \in U\} = U.$$

This means that the zero element in the quotient space V/U is simply the subspace U, and this equivalence class contains all vectors differ from 0 by an element of U, meaning it is precisely U itself, so $\pi(v) = 0$ if and only if $v \in U$.

Remark 67. *The output of $\pi(v)$ is the equivalence class $v+U$, which is the set of all vectors in V that are equivalent to v under the subspace U.*

For $\tilde{T}$, the input to it is an equivalent class $v + \text{null }(T)$ in the quotient space $V/\text{null } T$. It represents all vectors in V that differ from v by a vector in null T. Its output is Tv, which is an element in W.

The null space is

$$\text{null }(\tilde{T}) = \{v + \text{null }(T) \colon \tilde{T}(v + \text{null }(T)) = 0\}.$$

Since we know that $\tilde{T}(v + \text{null } T) = Tv$, we have that

$$\tilde{T}(v + \text{null } T) = 0 \iff Tv = 0.$$

so $v \in \text{null } T$ and thus null $\tilde{T} = \{\text{null } T\}$. We have that range $\tilde{T} =$ range T.

Theorem 68. *Suppose $T \in \mathcal{L}(V, W)$. Then*
(a) $\tilde{T} \circ \pi = T$, where π is the quotient map of V onto $V/null\ T$;
(b) $\tilde{T}$ is injective;
(c) range $\tilde{T}$ = range T;
(d) $V/null\ T$ and range T are isomorphic vector spaces.

Remark 69. *One might think of the domain of $\tilde{T}$ as the non-trivial domain of T and the range as the normal image of T.*

Problem 1

Suppose T is function from V to W. The *graph* of T is the subset of $V \times W$ defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}.$$

Prove that T is a linear map if and only if the graph of T is a subspace of $V \times W$.

Proof. The two properties closely relate to each other:

$$(v_1, Tv_1) + (v_2, Tv_2) \in V \times W : v \in V \iff Tv_1 + Tv_2 = T(v_1 + v_2)$$

and

$$(\lambda v, T\lambda v) \in V \times W : v \in V \iff T\lambda v = \lambda Tv$$

□

Problem 2

Suppose that $V_1, \ldots, V_m$ are vector spaces such that $V_1 \times \cdots \times V_m$ is finite-dimensional. Prove that V_k is finite-dimensional for each $k = 1, \ldots, m$.

Proof. Recall the definition of infinite-dimension from ch2 P17:

V is infinite-dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $v_1, \ldots, v_m$ is linearly independent for every positive integer m.

Suppose for the sake of contradiction that there exists V_k to be infinite-dimensional. Then this means that there is a sequence $v_{k,1}, v_{k,2}, \ldots$ of vectors in V_k such that $v_{k,1}, \ldots, v_{k,m}$ is linearly independent for every positive integer m. We can build such sequence for the product space as well. More specifically, consider the sequence $0, \ldots, v_{k,j}, \ldots, 0$ where we fill 0 for each of other V_j's and only leave the sequence from V_k. In this case, we can see that the product space is infinite-dimensional, contradicting the claim. □

Problem 3

Suppose $V_1, \ldots, V_m$ are vector spaces. Prove that $\mathcal{L}(V_1 \times \cdots \times V_m, W)$ and $\mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W)$ are isomorphic vector spaces.

Proof. We define a permutation invariant set function $S : \mathcal{L}(V_1, W) \times \cdots \times \mathcal{L}(V_m, W) \to \mathcal{L}(V_1 \times \cdots \times V_m, W)$ such that

$$S(T_1, \ldots, T_m) = T_1 + \cdots + T_m$$

We will show that S is an isomorphism (it's linear). For injectivity, null S means that for any arbitrary input $T_1 + \cdots + T_m = 0$, which implies that $T_i = 0$ for all i. For surjectivity, take $\phi \in \mathcal{L}(V_1 \times \cdots \times V_m, W)$, we can define

$T_i(v_i) = \phi((0, \ldots, v_i, \ldots, 0))$ where i means for the space V_i. Then we can naturally get that $\phi((v_1, \ldots, v_m)) = \phi(\sum_{i=1}^m e_i^\top v_i) = \sum_{i=1}^m T_i(v_i)$ and thus we finish the proof. $\square$

Problem 4

Suppose $W_1, \ldots, W_m$ are vector spaces. Prove that $\mathcal{L}(V, W_1 \times \cdots \times W_m)$ and $\mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m)$ are isomorphic vector spaces.

We can simply define a concatenate operator $S\colon \mathcal{L}(V, W_1) \times \cdots \times \mathcal{L}(V, W_m) \to \mathcal{L}(V, W_1 \times \cdots \times W_m)$ such that

$$S(T_1, \ldots, T_m) = (T_1, \ldots, T_m)$$

We prove this linear map is an isomorphism. The injective property can be proved using the same logic as above. For surjectivity, take $\phi \in Lc(V, W_1 \times \cdots \times W_m)$, then we can naturally obtain $T_k = \pi_k \phi$ where π_k means projecting the k-th component of the input. Note that $T_k \in \mathcal{L}(V, W_k)$ and therefore we finish the proof.

Problem 5

For m a positive integer, define V^m by

$$V^m = \underbrace{V \times \cdots \times V}_{m \text{ times}}.$$

Prove that V^m and $\mathcal{L}(\mathbb{F}^m, V)$ are isomorphic vector spaces.

Proof.

$$\dim(V^m) = m \dim V = \dim(\mathcal{L}(\mathbb{F}^m, V)).$$

$\square$

Problem 6

Suppose that v, x are vectors in V and that U, W are subspaces of V such that $v + U = x + W$. Prove that $U = W$.

Proof. Note that $v = x + w$ for some $w \in W$ if we take $u = 0 \in U$ and similarly $x = v + u$ for some $u \in U$. Hence $v - x \in W$ and $x - v \in U$. Then we have that take $u \in U, u = (x + w) - v = (x - v) + w \in W$. Similarly, take $w \in W, w = (v + u) - x = (v - x) + u \in U$. Hence $W = U$. $\square$

Problem 9

Prove that a nonempty subset A of V is a translate of some subspace of V if and only if $\lambda v + (1 - \lambda) w \in A$ for all $v, w \in A$ and all $\lambda \in \mathbb{F}$.

Proof. $\Rightarrow$ Suppose $A = x + U$ for some subspace $U \subseteq V$. Then take $a_1 = x + u_1 \in A$ and $a_2 = x + u_2 \in A$. We have that

$$\lambda a_1 + (1-\lambda)a_2 = \lambda x + \lambda u_1 + (1-\lambda)x + (1-\lambda)u_2 = x + (\lambda u_1 + (1-\lambda)u_2) \in A$$

$\Leftarrow$ take any $x \in A$ and define the subspace such that

$$U \coloneqq (-x) + A$$

first $0 = -x + x \in U$. Next, take $v_1 - x$ and $v_2 - x \in U$. We have that $v_1 - x + v_2 - x = (-x) + (2v_1 - x)/2 + (2v_2 - x)/2 \in A$ as $2v_i - x \in A$ by taking $\lambda = 2$. Finally, $\lambda(-x + v) = (-x) + (\lambda v + (1-\lambda)x) \in A$. ☐

Problem 10

Suppose $A_1 = v + U_1$ and $A_2 = w + U_2$ for some $v, w \in V$ and some subspaces U_1, U_2 of V. Prove that the intersection $A_1 \cap A_2$ is either a translate of some subspace of V or is the empty set.

Proof. If $A_1 \cap A_2$ is not empty, then $S = A_1 \cap A_2$ is a subspace such that $v + U_1|_S = w + U_2|_S$, so we know that $U_1|_S = U_2|_S$ by Problem 6. Hence it is a translate of some subspace of V. Otherwise, it is the empty set. ☐

Problem 11

Suppose $U = \{(x_1, x_2, \ldots) \in \mathbb{F}^\infty : x_k \neq 0$ for only finitely many $k\}$.
(a) Show that U is a subspace of $\mathbb{F}^\infty$.
(b) Prove that $\mathbb{F}^\infty / U$ is infinite-dimensional.

Proof. (a) all zero, where $x_k \neq 0$ for zero k's (finitely many) is an element of U. Next, take $(x_1, x_2, \ldots)$ and $(y_1, y_2, \ldots) \in U$. Then $(x_1 + y_1, x_2 + y_2, \ldots)$ will have finitely many nonzero since each of them has only finitely many nonzero entries. Same holds for $\lambda(x_1, x_2, \ldots)$.

(b) Let's consider a "standard basis" $\{v_i\}_{i=1}^\infty$ that only has 0 on i-th spot and all 1 otherwise. We can see that for all m, $v_1, \ldots, v_m$ is linearly independent. And so does $v_1 + U, \ldots, v_m + U$ since each $v_i \notin U$, ☐

Problem 12

Suppose $v_1, \ldots, v_m \in V$. Let

$$A = \{\lambda_1 v_1 + \cdots + \lambda_m v_m : \lambda_1, \ldots, \lambda_m \in F \text{ and } \lambda_1 + \cdots + \lambda_m = 1\}.$$

(a) Prove that A is a translate of some subspace of V.
(b) Prove that if B is a translate of some subspace of V and $\{v_1, \ldots, v_m\} \subseteq B$, then $A \subseteq B$.
(c) Prove that A is a translate of some subspace of V of dimension less than m.

Proof. (a) We try to show this through using the conclusion from Problem 9. Let $a_1 = \sum_{i=1}^m \alpha_i v_i$ and $a_2 = \sum_{i=1}^m \gamma_i v_i$ s.t. $\sum_{i=1}^m \alpha_i = \sum_{i=1}^m \gamma_i = 1$. We have that

$$\lambda a_1 + (1-\lambda)a_2 = \sum_{i=1}^m (\lambda\alpha_i + (1-\lambda)\gamma_i)v_i$$

We can show that

$$\sum_{i=1}^m (\lambda\alpha_i + (1-\lambda)\gamma_i) = \lambda\sum_{i=1}^m \alpha_i + (1-\lambda)\sum_{i=1}^m \gamma_i = 1$$

Thus $\lambda a_1 + (1-\lambda)a_2 \in A$ and thus A is a translate of some subspace of V.

(b) Let $B = w + Y$ for some subspace $Y \subseteq V$. Then we have that $v_k = w + y_k$. Take $x \in A$, then we know that

$$x = \sum_{i=1}^m \lambda_i v_i = \sum_{i=1}^m \lambda_i(w + y_i) = w + \sum_{i=1}^m \lambda_i y_i \in B$$

(c) Write $A = w + U$. Denote $B = \operatorname{span}\{v_1, \ldots, v_m\}$. Then by (b) we know that $A \subseteq B$. We will show that $\dim U < m$. The statement is trivial if $v_1, \ldots, v_m$ is not linearly independent. Let's consider the case that it's indeed linearly independent. Here we claim that $w \notin U$ and thus $\dim U < m$. To see this, the only way to write $\sum_{i=1}^m \lambda_i v_i = 0$ is to let all $\lambda_i = 0$, but in this case the vectors will not be in A, so $0 \notin A$. Suppose for the sake of contradiction that $w \in U$. This implies that $0 = w + (-w) \in w + U = A$, forming a contradiction. Hence we finish the proof. □

Problem 13

Suppose U is a subspace of V such that V/U is finite-dimensional. Prove that V is isomorphic to $U \times (V/U)$.

Proof.

$$\dim V = \dim U + (\dim V - \dim U) = \dim U + \dim(V/U).$$

□

Problem 14

Suppose U and W are two subspaces of V and $V = U \oplus W$. Suppose $w_1, \ldots, w_m$ is a basis of W. Prove that $w_1 + U, \ldots, w_m + U$ is a basis of V/U.

Proof. First we show that the list is linearly independent. We consider the following system:

$$0 + U = \lambda_1(w_1 + U) + \cdots + \lambda_m(w_m + U) = (\lambda_1 w_1 + \cdots + \lambda_m w_m) + U$$

This means that $\lambda_1 w_1 + \cdots + \lambda_m w_m \in U$ and since we know that $w_1, \ldots, w_m \notin U$, we have that the only solution is $\lambda_i = 0$ for all i.

Next we show that the list spans V/U. Take arbitrary $v + U \in V/U$, then we have that $v = u + w$ for some $w = \sum_{i=1}^m \lambda_i w_i$. Then we have that

$$v + U = \sum_{i=1}^m \lambda_i w_i + U = \sum_{i=1}^m \lambda_i (w_i + U).$$

Thus the list spans the quotient space. □

Problem 15

Suppose U is a subspace of V and $v_1 + U, \ldots, v_m + U$ is a basis of V/U and $u_1, \ldots, u_n$ is a basis of U. Prove that $v_1, \ldots, v_m, u_1, \ldots, u_n$ is a basis of V.

Proof. We first know that

$$\dim V = \dim V/U + \dim U.$$

So it only suffices to prove the list either spans V or is linearly independent. Take any $v \in V$. Then we have that

$$v + U = \lambda_1 v_1 + \cdots + \lambda_m v_m + U.$$

Thus $v - \sum_{i=1}^m \lambda_i v_i \in U$. So

$$v = \sum_{i=1}^m \lambda_i v_i + \sum_{j=1}^n \alpha_j u_i.$$

Hence, we finish the proof. □

Problem 16

Suppose $\varphi \in \mathcal{L}(V, \mathbb{F})$ and $\varphi \neq 0$. Prove that $\dim V/(\operatorname{null} \varphi) = 1$.

Proof. We know that $\dim \operatorname{range} \varphi = 1$ and thus

$$\dim V/\operatorname{null} \varphi = \dim V - \dim \operatorname{null} \varphi = \dim \operatorname{range} \varphi = 1.$$

□

Problem 17

Suppose that U is a subspace of V such that $\dim V/U = 1$. Prove that there exists $\varphi \in \mathcal{L}(V, \mathbb{F})$ such that $\operatorname{null} \varphi = U$.

Proof. We make construction as follows: (1) there exists a natural isomorphism $T\colon V/U \to \mathbb{F}$. We further define $S\colon V \to V/U$ by $S(v) = v + U$. Then we show the map

$$\varphi = TS$$

satisfies the requirement.

First take $u \in U$. Then we have that

$$\varphi(u) = TS(u) = T(u + U) = T(0) = 0$$

and thus $u \in \text{null } \varphi$.

Conversely take $u \in \text{null } \varphi$, then

$$0 = \varphi(u) = T(S(u)) = T^{-1}T(S(u)) = S(u).$$

Thus $u \in \text{null } S = U$. We've finished the proof. □

Problem 18

Suppose U is a subspace of V such that V/U is finite-dimensional.
(a) Show that if W is a finite-dimensional subspace of V and $V = U + W$, then $\dim W \geq \dim V/U$.
(b) Prove that there exists a finite-dimensional subspace W of V such that $\dim W = \dim V/U$ and $V = U \oplus W$.

Proof. (a) We know that

$$\dim W = \dim V + \dim(U \cap V) - \dim U \geq \dim V - \dim U = \dim V/U.$$

(b) We will construct the space W through some manipulation with the basis. Since we are given V/U, we start with the basis $w_1 + U, \ldots, w_m + U$ of that. Note that $\{w_1, \ldots, w_m\}$ is linearly independent as the only solution is all-zero. We can now define

$$W = \text{span}\{w_1, \ldots, w_m\}.$$

Then we have that $\dim W = \dim V/U$ and now it suffices to verify $V = U \oplus W$. To see this, one only needs to show $W \cap U = \{0\}$ as we've proved that $V = W + U$ in previous questions (of similar construction). Take arbitrary $v \in W \cap U$. Then we have that

$$v = \sum_{i=1}^{m} \lambda_i w_i$$

and that

$$v + U = \sum_{i=1}^{m} \lambda_i w_i + U = 0 + U$$

The only solution is all $\lambda_i = 0$. □

Problem 19

Suppose $T \in \mathcal{L}(V, W)$ and U is a subspace of V. Let π denote the quotient map from V onto V/U. Prove that there exists $S \in \mathcal{L}(V/U, W)$ such that $T = S \circ \pi$ if and only if $U \subseteq \operatorname{null} T$.

Proof. $\Rightarrow$ Take $u \in U$, then we have that

$$T(u) = S\pi(u) = S(u + U) = S(0) = 0$$

and thus $u \in \operatorname{null} T$.

$\Leftarrow$ We define $S \colon V/U \to W$ by

$$S(v + U) = T(v)$$

It now only suffices to show that this map is valid for the equivalence mapping. Take $v_1 + U = v_2 + U$. Then we have that $v_1 - v_2 \in U \subseteq \operatorname{null} T$ and thus $T(v_1 - v_2) = 0 = Tv_1 - Tv_2$ and thus $S(v_1 + U) = Tv_1 = Tv_2 = S(v_2 + U)$. This map is clearly linear and thus we have $T = S \circ \pi$. $\square$

3F: Duality

Definition 70 (linear functional)**.** *A **linear functional** on V is a linear map from V to $\mathbb{F}$. In other words, a linear functional is an element of $\mathcal{L}(V, \mathbb{F})$.*

Definition 71 (dual space, V')**.** *The **dual space** of V, denoted by V', is the vector space of all linear functionals on V. In other words, $V' = \mathcal{L}(V, \mathbb{F})$.*

Lemma 72. *Suppose V is finite-dimensional. Then V' is also finite-dimensional and*

$$\dim V' = \dim V.$$

Definition 73 (dual basis)**.** *If $v_1, \ldots, v_n$ is a basis of V, then the **dual basis** of $v_1, \ldots, v_n$ is the list $\varphi_1, \ldots, \varphi_n$ of elements of V', where each φ_j is the linear functional on V such that*

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

Theorem 74 (dual basis gives coefficients for linear combination)**.** *Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the dual basis. Then*

$$v = \varphi_1(v)v_1 + \cdots + \varphi_n(v)v_n$$

for each $v \in V$.

Theorem 75 (dual basis is a basis of the dual space)**.** *Suppose V is finite-dimensional. Then the dual basis of a basis of V is a basis of V'.*

Definition 76 (dual map, T')**.** *Suppose $T \in \mathcal{L}(V, W)$. The **dual map** of T is the linear map $T' \in \mathcal{L}(W', V')$ defined for each $\varphi \in W'$ by*

$$T'(\varphi) = \varphi \circ T.$$

Corollary 77 (algebraic properties of dual map)**.** *Suppose $T \in \mathcal{L}(V, W)$. Then*
(a) $(S + T)' = S' + T'$ for all $S \in \mathcal{L}(V, W)$;
(b) $(\lambda T)' = \lambda T'$ for all $\lambda \in \mathbb{F}$;
(c) $(ST)' = T'S'$ for all $S \in \mathcal{L}(W, U)$.

The goal of this section is to describe null T' and range T' in terms of range T and null T.

Definition 78 (annihilator, U^0)**.** *For $U \subseteq V$, the **annihilator** of U, denoted by U^0, is defined by*

$$U^0 = \{\varphi \in V' \colon \varphi(u) = 0 \text{ for all } u \in U\}.$$

Remark 79. *U^0 is a subspace of V'.*

Theorem 80 (dimension of the annihilator)**.** *Suppose V is finite-dimensional and U is a subspace of V. Then*

$$\dim U^0 = \dim V - \dim U.$$

Lemma 81. *Suppose V is finite-dimensional and U is a subspace of V. Then*
(a) $U^0 = 0 \Longleftrightarrow U = V$;
(b) $U^0 = V' \Longleftrightarrow U = \{0\}$.

Lemma 82 (the null space of T')**.** *Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then*
(a) $\text{null } T' = (\text{range } T)^0$;
(b) $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$.

Theorem 83 (T surjective is equivalent to T' injective)**.** *Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then*

$$T \text{ is surjective } \Longleftrightarrow T' \text{ is injective.}$$

Lemma 84 (the range of T')**.** *Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then*
(a) $\dim \text{range } T' = \dim \text{range } T$;
(b) $\text{range } T' = (\text{null } T)^0$.

Theorem 85 (T surjective is equivalent to T' injective)**.** *Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then*

$$T \text{ is surjective } \Longleftrightarrow T' \text{ is injective.}$$

Theorem 86 (matrix of T' is transpose of matrix of T)**.** *Suppose V and W are finite-dimensional and $T \in \mathcal{L}(V, W)$. Then*

$$\mathcal{M}(T') = (\mathcal{M}(T))^\top$$

Proof. Let $v_1, \ldots, v_n$ be basis of V and $w_1, \ldots, w_m$ be basis of W. Let $\varphi_1, \ldots, \varphi_n$ be the dual basis of V' and $\psi_1, \ldots, \psi_m$ be the dual basis of W'.

Let $A = \mathcal{M}(T)$ and $C = \mathcal{M}(T')$. Suppose $1 \le j \le m$ and $1 \le k \le n$. From the definition of $\mathcal{M}(T')$ we have

$$T'(\psi_j) = \sum_{r=1}^{n} C_{r,j} \varphi_r$$

Then applying both sides of the equation above to v_k gives that

$$\begin{aligned}(\psi_j \circ T)(v_k) &= \sum_{r=1}^{n} C_{r,j} \varphi_r(v_k) \\ &= C_{k,j}\end{aligned}$$

At the same time, we have that

$$\begin{aligned}(\psi_j \circ T)(v_k) &= \psi_j(Tv_k) \\ &= \psi_j\left(\sum_{r=1}^{m} A_{r,k} w_r\right) \\ &= \sum_{r=1}^{m} A_{r,k}\psi_j(w_r) \\ &= A_{j,k}\end{aligned}$$

Here we have that $C_{k,j} = A_{j,k}$ and thus $C = A^{\top}$. Hence, $\mathcal{M}(T') = (\mathcal{M}(T))^{\top}$, as desired. $\square$

We can use duality to provide an alternative proof for the rank of matrix:

Theorem 87 (column rank equals row rank)**.** *Suppose* $\mathbf{A} \in \mathbb{F}^{m,n}$. *Then the column rank of* $\mathbf{A}$ *equals the row rank of* $\mathbf{A}$.

Problem 3

Suppose V is finite-dimensional and $v \in V$ with $v \neq 0$. Prove that there exists $\varphi \in V'$ such that $\varphi(v) = 1$.

Proof. The main requirement is to ensure that the constructed φ is valid. Note that there exists subspace $W \subseteq V$ such that $V = W \oplus \operatorname{span}\{v\}$. Thus we can define $\varphi(u) = i$ where $u = w + iv$ for $u \in V, w \in W, i \in \mathbb{F}$. Here φ is a valid linear map and $\varphi(v) = 1$. □

Problem 6

Suppose $\varphi, \beta \in V'$. Prove that null $\varphi \subseteq$ null β if and only if there exists $c \in \mathbb{F}$ such that $\beta = c\varphi$.

Proof. $\Rightarrow$ If $\varphi = 0$, then it trivially holds. If $\varphi \neq 0$, then we can first take $v_0 \in V$ s.t. $\psi(v_0) \neq 0$, then define $c = \beta(v_0)/\varphi(v_0)$, then we have that $\beta(v) = \beta(v_0 + u) = \beta(v_0) = c\varphi(v_0)$ for some $u \in$ null β.

$\Leftarrow$ Take $v \in$ null φ, then $\beta(v) = c\varphi(v) = 0$. Thus null $\varphi \subseteq$ null β. □

Problem 8

Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the dual basis of V'. Define $\Gamma\colon V \to \mathbb{F}^n$ and $\Lambda\colon \mathbb{F}^n \to V$ by

$$\Gamma(v) = (\varphi_1(v), \ldots, \varphi_n(v)) \text{ and } \Lambda(a_1, \ldots, a_n) = a_1 v_1 + \cdots + a_n v_n.$$

Explain why Γ and Λ are inverses of each other.

Proof. Take $a_1, \ldots, a_n \in \mathbb{F}^n$, then

$$\begin{aligned}\Gamma(\Lambda(a_1, \ldots, a_n)) &= \Gamma(a_1 v_1 + \cdots + a_n v_n) \\ &= (\varphi_1(a_1 v_1), \ldots, \varphi_n(a_n v_n)) \\ &= (a_1, \ldots, a_n)\end{aligned}$$

Similarly, take $v = a_1 v_1 + \cdots + a_n v_n$, we can get that

$$\Lambda(\Gamma(v)) = v$$

□

Problem 9

Suppose m is a positive integer. Show that the dual basis of the basis $1, x, \ldots, x^m$ of $\mathcal{P}_m(\mathbb{R})$ is $\varphi_0, \varphi_1, \ldots, \varphi_m$, where

$$\varphi_k(p) = \frac{p^{(k)}(0)}{k!}.$$

Proof. We consider different cases. If $j = k$, then

$$\varphi_k(x^j) = \frac{(x^j)^{(k)}}{k!} = \frac{k!}{k!} = 1$$

If $j < k$, then

$$\varphi_k(x^j) = \frac{(x^j)^{(k)}}{k!} = \frac{0^{j-k} j!/(j-k)!}{k!} = 0$$

If $j > k$, then

$$\varphi_k(x^j) = \frac{(x^j)^{(k)}}{k!} = \frac{0}{k!} = 0$$

Hence, we have that

$$\varphi_k(x_j) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{o.w.} \end{cases}$$

□

Problem 11

Suppose $v_1, \ldots, v_n$ is a basis of V and $\varphi_1, \ldots, \varphi_n$ is the corresponding dual basis of V'. Prove that

$$\psi = \psi(v_1)\varphi_1 + \cdots + \psi(v_n)\varphi_n.$$

Proof. By property of dual basis, we have that

$$\psi = a_1\varphi_1 + \cdots + a_n\varphi_n$$

Apply v_k on both sides give that

$$\begin{aligned} \psi(v_k) &= a_1\varphi_1(v_k) + \cdots + a_n\varphi_n(v_k) \\ &= a_k \end{aligned}$$

Substitute this back gives the desired equality. □

Problem 13

Show that the dual map of the identity operator on V is the identity operator on V'.

Proof. Take arbitrary $f \in V'$, then we have that

$$I'(f)(v) = f \circ I(v) = f(v)$$

for all $v \in V$. □

Problem 16

Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that

$$T' = 0 \iff T = 0.$$

Proof. This is obvious from $\dim \operatorname{range} T' = \dim \operatorname{range} T$. □

Problem 19

Suppose $U \subseteq V$. Explain why

$$U^0 = \{\varphi \in V' : U \subseteq \text{null } \varphi\}$$

Proof. This follows from definition. □

Problem 20

Suppose V is finite-dimensional and U is a subspace of V. Show that

$$U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

Proof. Denote $K = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}$.

Take $u \in U$ and $\varphi \in U^0$, then by definition $\varphi(u) = 0$, and thus $u \in K$.

Conversely, take $\varphi \in U^0$. Suppose for the sake of contradiction that there exist $v \notin U$ but $v \in K$, then $\varphi(v) \neq 0$, contradicting that $v \in K$, thus completing the proof. □

Problem 21

Suppose V is finite-dimensional and U and W are subspaces of V.
(a) Prove that $W^0 \subseteq U^0$ if and only if $U \subseteq W$.
(b) Prove that $W^0 = U^0$ if and only if $U = W$.

Proof. (a) $\Rightarrow$ We know that there exists $\psi \in V'$ such that null $\psi = W$. Then $\psi \in W^0$ and thus $\psi \in U^0$. This means that $U \subseteq W = \text{null } \psi$.

$\Leftarrow$ Take $\psi \in W^0$. So we know $\psi(w) = 0$ for all $w \in W$. Suppose for the sake of contradiction that $\psi \notin U^0$, then there exists $u \in U$ s.t. $\psi(u) \neq 0$. However, as $u \in W$, we have reached a contradiction.

(b) $W^0 = U^0 \iff W_0 \subseteq U_0$ and $U_0 \subseteq W_0 \iff U \subseteq W$ and $W \subseteq W \iff U = W$. □

Problem 22

Suppose V is finite-dimensional and U and W are subspaces of V.
(a) Show that $(U + W)^0 = U^0 \cap W^0$.
(b) Show that $(U \cap W)^0 = U^0 + W^0$.

Proof. Note that we have

$$(U+W)^0 = \{\varphi \in V' \colon \varphi(v) = 0 \text{ for every } v \in U+W\}$$
$$(U \cap W)^0 = \{\varphi \in V' \colon \varphi(v) = 0 \text{ for every } v \in U \cap W\}$$

(a) Take $\varphi \in (U+W)^0$, then we know that $\varphi(u) = 0$ and $\varphi(w) = 0$ for all $u \in U, w \in W$. Thus $(U+W)^0 \subseteq U^0 \cap W^0$. Conversely, take $\varphi \in U^0 \cap W^0$, then we know that for every $v = u + w, \varphi(v) = \varphi(u) + \varphi(w) = 0$, therefore $U^0 \cap W^0 \subseteq (U+W)^0$.

(b) Take $\varphi_1 \in U^0, \varphi_2 \in W^0$, then we have that $\varphi_1 + \varphi_2(v) = \varphi_1(v) + \varphi_2(v) = 0$ for $v \in U \cap W$, therefore $U^0 + W^0 \subseteq (U \cap W)^0$. It suffices now to show that the dimension equal each other.

$$\dim((U \cap W)^0) = \dim(V) - \dim(U \cap W)$$

Note that

$$\begin{aligned}
\dim(U^0 + W^0) &= \dim(U^0) + \dim(W^0) - \dim(U^0 \cap W^0) \\
&= (\dim(V) - \dim(U)) + (\dim(V) - \dim(W)) - \dim((U+W)^0) \\
&= 2\dim(V) - (\dim(U) + \dim(W)) - (\dim(V) - \dim(U+W)) \\
&= \dim(V) - (\dim(U) + \dim(W) - \dim(U+W)) \\
&= \dim(V) - \dim(U \cap W)
\end{aligned}$$

Thus we have completed the proof. □

Problem 23

Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_m \in V'$. Prove that the following three sets are equal to each other.
(a) $\operatorname{span}(\varphi_1, \ldots, \varphi_m)$.
(b) $((\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m))^0 := A$.
(c) $\{\varphi \in V' \colon (\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m) \subseteq \operatorname{null} \varphi\} := B$.

Proof. $(a) \Rightarrow (b)$ take $\varphi = \sum_{i=1}^m a_i\varphi_i \in \operatorname{span}(\varphi_1, \ldots, \varphi_m)$. Then we know that for every $v \in (\operatorname{null} \varphi_1) \cap \cdots \cap (\operatorname{null} \varphi_m)$, $\varphi(v) = \sum_{i=1}^m a_i\varphi_i(v) = 0$.

$(b) \Rightarrow (c)$ Take $\varphi \in A$, then by P19 this holds.

$(c) \Rightarrow (a)$ Take $\varphi \in B$, then we know by P3 there exists $c \in \mathbb{F}$ such that $\varphi = c\varphi_i$ for all $1 \leq i \leq m$. This means that $\varphi \in \operatorname{span}(\varphi_1, \ldots, \varphi_m)$. □

Problem 24

Suppose V is finite-dimensional and $v_1, \ldots, v_m \in V$. Define a linear map $\Gamma \colon V' \to \mathbb{F}^m$ by $\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m))$.
(a) Prove that $v_1, \ldots, v_m$ spans V if and only if Γ is injective.
(b) Prove that $v_1, \ldots, v_m$ is linearly independent if and only if Γ is surjective.

Proof. (a) $\Rightarrow$ Take $v = \sum_{i=1}^{m} a_i v_i$ and $\varphi \in \text{null } \Gamma$. We aim to prove that $\varphi = 0$. We have that $\Gamma(\varphi) = (\varphi(v_1), \ldots, \varphi(v_m)) = 0$ so $\varphi(v_1) = \cdots = \varphi(v_m) = 0$. So for all $v \in V$, $\varphi(v) = \sum_{i=1}^{m} a_i \varphi(v_i) = 0$. Thus φ is the zero map and thus Γ is injective.

$\Leftarrow$ Suppose for the sake of contradiction that $v_1, \ldots, v_m$ does not span V, then this means there exists some subspace W such that $\text{span}(v_1, \ldots, v_m) \oplus W = V$. Then there exists nonzero $\varphi \in V'$ such that $\varphi(v_k) = 0$ for all k (one can set the basis in W to be nonzero mapping). Then this means that $\varphi \in \text{null } \Gamma \neq 0$, contradicting that Γ is injective.

(b) $\Rightarrow$ Let K be the subspace spanned by the linearly independent list of vectors $v_1, \ldots, v_m$. Then by the linear map lemma, for all $(a_1, \ldots, a_m) \in \mathbb{F}^m$, there exists a unique mapping $T' \colon K \to \mathbb{F}$ such that $T' v_i = a_i$. Extending T' to V ensures that Γ is surjective.

$\Leftarrow$ Suppose for the sake of contradiction that $v_1, \ldots, v_m$ are not linearly independent, then we know there exists nonzero $a_i's$ such that $a_1 v_1 + \cdots + a_m v_m = 0$. Applying any linear function $\varphi \in V'$ to this linear combination yields that

$$\varphi\left(\sum_{i=1}^{m} a_i v_i\right) = \varphi(0) = 0$$

This equals that

$$\sum_{i=1}^{m} a_i \varphi(v_i) = \sum_{i=1}^{m} a_i \Gamma(\varphi)_i = 0$$

Since there is nonzero a_i's, this means that the image of Γ is a strict subspace of $\mathbb{F}^m$, therefore contradicting the hypothesis that it's surjective.

□

Problem 25

Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_m \in V'$. Define a linear map $\Gamma \colon V \to \mathbb{F}^m$ by $\Gamma(v) = (\varphi_1(v), \ldots, \varphi_m(v))$.
(a) Prove that $\varphi_1, \ldots, \varphi_m$ spans V' if and only if Γ is injective.
(b) Prove that $\varphi_1, \ldots, \varphi_m$ is linearly independent if and only if Γ is surjective.

Proof. (a) $\Rightarrow$ Take any $\varphi \in V'$, then $\varphi = \sum_{i=1}^{m} a_i \varphi_i$. Take any $v \in V$ and let $\Gamma(v) = 0$. Then this means that

$$\Gamma(v) = (\varphi_1(v), \ldots, \varphi_m(v)) = 0$$

So we have that $\varphi_i(v) = 0$ for all i. By the spanning assumption, we have that for any $\varphi \in V', \varphi(v) = \sum_{i=1}^{m} a_i \varphi_i(v) = 0$. Thus $v = 0$ and then null $(\Gamma) = \{0\}$.

$\Leftarrow$ Suppose $\varphi_1, \ldots, \varphi_m$ does not span V', then this means there exists some subspace W such that $\text{span}(\varphi_1, \ldots, \varphi_m) \oplus W = V'$. Let $\varphi_1, \ldots, \varphi_n$ be the basis of $\text{span}(\varphi_1, \ldots, \varphi_m)$, $\varphi_{n+1}, \ldots, \varphi_l$ to be the basis of W. Let $v_1, \ldots, v_n, v_{n+1}, \ldots, v_l$ be the corresponding basis for V. Then there exists nonzero $v \in V$ such that

$\varphi_k(v) = 0$ for all k as $v = \sum_{i=1}^{n} \varphi_i(v)v_i + \sum_{j=n+1}^{l} \varphi_j(v)v_j$, where one can obtain nonzero $\varphi_j(v)$ for some $n+1 \leq j \leq l$ but zero $\varphi_i(v)$ for all $1 \leq i \leq n$. Such $v \in \operatorname{null} \Gamma$, contradicting the hypothesis that Γ is injective.

(b) $\Rightarrow$ Let K be the subspace spanned by the linearly independent list of vectors $\varphi_1, \ldots, \varphi_m$. Then for all $(a_1, \ldots, a_m) \in \mathbb{F}^m$, take $v = \sum_{i=1}^{m} a_i v_i$ for the corresponding basis $v_1, \ldots, v_m$. Then we have that $\varphi_i(v) = a_i$ so Γ is surjective.

$\Leftarrow$ Suppose for the sake of contradiction that $\varphi_1, \ldots, \varphi_m$ are not linearly independent, then we know there exists nonzero $a_i's$ such that $a_1\varphi_1 + \cdots + a_m\varphi_m = 0$. Applying any vector $v \in V$ to this linear functional yields that

$$\sum_{i=1}^{m} a_i \varphi_i(v) = 0(v) = 0$$

This equals that

$$\sum_{i=1}^{m} a_i \varphi_i(v) = \sum_{i=1}^{m} a_i \Gamma(v)_i = 0$$

Since there is nonzero a_i's, this means that the image of Γ is a strict subspace of $\mathbb{F}^m$, therefore contradicting the hypothesis that it's surjective. $\square$

Problem 26

Suppose V is finite-dimensional and Ω is a subspace of V'. Prove that

$$\Omega = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}^0$$

Proof. Denote $U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}$. So we have that

$$U^0 = \{\psi \in V' : \psi(v) = 0 \text{ for every } v \in U\}$$

$\Rightarrow$ Take $\varphi \in \Omega$ and $v \in U$, then by definition $\varphi(v) = 0$ and thus $\varphi \in U^0$.

$\Leftarrow$ Take $\psi \in U^0$, suppose $\psi \notin \Omega$. Take $v \in U$, then we know that for every $\varphi \in \Omega, \varphi(v) = 0$. Since $\psi \notin \Omega$, there exists v, s.t. $\psi(v) \neq 0$. However, this contradicts that $\psi(v) = 0$ for all $v \in U$, the proof is completed. $\square$

Problem 28

Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_m$ is a linearly independent list in V'. Prove that

$$\dim((\operatorname{null} \varphi_i) \cap \cdots \cap (\operatorname{null} \varphi_m)) = (\dim V) - m$$

Proof. Note that $(\bigcap_{i=1}^{m} \operatorname{null} \varphi_i)^0 = \operatorname{span}(\varphi_1, \ldots, \varphi_m)$ by P23. Then we have that $\dim \bigcap_{i=1}^{m} \operatorname{null} \varphi_i = \dim V - \operatorname{span}(\varphi_1, \ldots, \varphi_m) = \dim V - m$. $\square$

Problem 30

Suppose V is finite-dimensional and $\varphi_1, \ldots, \varphi_n$ is a basis of V'. Show that there exists a basis of V whose dual basis is $\varphi_1, \ldots, \varphi_n$.

Proof. We know that $\dim V' = n$ and that $\dim \operatorname{null} \varphi_i + 1(\dim \operatorname{range} \varphi_i) = n$. Thus we can construct that $V = \operatorname{null} \varphi_i \oplus U_i$ for all $1 \leq i \leq n$. Here $\dim U_i = 1$. Here we extract all $v_i's$ from those U_i. By linear map lemma, there exists $v_i \in U_i$ such that $\varphi_i(v_i) = 1$. We claim that $v_1, \ldots, v_n$ is the corresponding basis of V.

We first verify that it is the "corresponding basis". Consider $\varphi_j(v_i)$ for $j \neq i$. Suppose for contradiction that there exists nonzero $w \in U_i \cap U_k$. Since we know all U_i are 1-d, $U_i = U_k$. Here we have that $\operatorname{null} \varphi_i = \operatorname{null} \varphi_k$, then this means that $\varphi_i = c\varphi_k$ for some $c \in \mathbb{F}$, but this contradicts the assumption that $\varphi_1, \ldots, \varphi_n$ is the basis of V'.

Next we verify it is basis. It only suffices to verify that the list is linearly independent. To see this, $0 = \sum_{i=1}^m a_i v_i$. Apply φ_i on both side yield that $0 = \varphi_i(0) = \varphi(\sum_{i=1}^m a_i v_i) = a_i$. Thus all coefficients are zero. ◻

Problem 32

The *double dual* space of V, denoted by V'', is defined to be the dual space of V'. In other words, $V'' = (V')'$. Define $\Lambda \colon V \to V''$ by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each $v \in V$ and $\varphi \in V'$.
(a) Show that Λ is a linear map from V to V''.
(b) Show that if $T \in \mathcal{L}(V)$, then $T'' \circ \Lambda = \Lambda \circ T$, where $T'' = (T')'$.
(c) Show that if V is finite-dimensional, then Λ is an isomorphism from V onto V''.

Proof. (a) First, $\Lambda(0)(\varphi) = \varphi(0) = 0$. Next, take $v_1, v_2 \in V$, then $\Lambda(\lambda v_1 + v_2)(\varphi) = \lambda\varphi(v_1) + \varphi(v_2) = \lambda\Lambda(v_1) + \Lambda(v_2)$.

(b) By definition, the double dual map $T'' \colon V'' \to W''$ is defined as $T''(\psi) = \psi(T'(\varphi))$ for $\psi \in V''$ and for all $\varphi \in W'$. This further equals that $T''(\psi) = \psi\varphi \circ T(v)$ for $v \in V$.

$$\begin{aligned} T'' \circ (\Lambda v)(\varphi) &= (\Lambda v) \circ T'(\varphi) \\ &= (\Lambda v)\varphi \circ T \\ &= \Lambda \circ T \end{aligned}$$

(c) First we have that $\dim V = \dim V' = \dim V''$. Note that if we take $v \in \operatorname{null} \Lambda$, then this means for all linear functional $\varphi \in V'$, we have that $\Lambda v(\varphi) = \varphi(v) = 0$. So $v = 0$ and thus Λ is injective and therefore an isomorphism. ◻