# Chapter 9: Multilinear Algebra and Determinants

*Linear Algebra Done Right (4th Edition)*, by Sheldon Axler

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## Contents

# 9A: Bilinear Forms and Quadratic Forms

**Definition 1** (bilinear form)**.** *A **bilinear form** on $V$ is a function $\beta\colon V\times V\to\mathbb{F}$ such that*

$$v\mapsto\beta(v,u)\ \text{and}\ v\mapsto\beta(u,v)$$

*are both linear functionals on $V$ for every $u\in V$.*

**Remark 2.** *A better but less popular terminology is "bilinear functional". If $V$ is real, then the function $(u,v)\mapsto\langle u,v\rangle$ is a bilinear form. If $V$ is complex, then it isn't.*

**Remark 3.** *If $\mathbb{F}=\mathbb{R}$, then a bilinear form differs from an inner product in that it does not require positive definiteness or symmetry.*

**Remark 4.** *A bilinear form $\beta$ on $V$ is a linear map on $V\times V$ only if $\beta=0$.*

**Definition 5** ($V^{(2)}$)**.** *The set of bilinear forms on $V$ is denoted by $V^{(2)}$.*

**Definition 6** (matrix of a bilinear form, $\mathcal{M}(\beta)$)**.** *Suppose $\beta$ is a bilinear form on $V$ and $e_1,\ldots,e_n$ is a basis of $V$. The **matrix** of $\beta$ with respect to this basis is the $n$-by-$n$ matrix matrix $\mathcal{M}(\beta)$ whose entry $\mathcal{M}(\beta)_{j,k}$ in row $j$, column $k$ is given by*

$$\mathcal{M}(\beta)_{j,k}=\beta(e_j,e_k)$$

*If the basis $e_1,\ldots,e_n$ is not clear from the context, then the notation $\mathcal{M}\left(\beta,(e_1,\ldots,e_n)\right)$ is used.*

**Corollary 7** ($\dim V^{(2)}=(\dim V)^2$)**.** *Suppose $e_1,\ldots,e_n$ is a basis of $V$. Then the map $\beta\mapsto\mathcal{M}(\beta)$ is an isomorphism of $V^{(2)}$ onto $\mathbb{F}^{n,n}$. Furthermore, $\dim V^{(2)}=(\dim V)^2$.*

**Lemma 8** (composition of a bilinear form and an operator)**.** *Suppose $\beta$ is a bilinear form on $V$ and $T\in\mathcal{L}(V)$. Define bilinear forms $\alpha$ and $\rho$ on $V$ by*

$$\alpha(u,v)=\beta(u,Tv)\ \text{and}\ \rho(u,v)=\beta(Tu,v)$$

*Let $e_1,\ldots,e_n$ be a basis of $V$. Then*

$$\mathcal{M}(\alpha)=\mathcal{M}(\beta)\mathcal{M}(T)\ \text{and}\ \mathcal{M}(\rho)=\mathcal{M}(T)^\top\mathcal{M}(\beta)$$

**Theorem 9** (change-of-basis formula)**.** *Suppose $\beta\in V^{(2)}$. Suppose $e_1,\ldots,e_n$ and $f_1,\ldots,f_n$ are bases of $V$. Let*

$$A=\mathcal{M}(\beta(e_1,\ldots,e_n))\ \text{and}\ B=\mathcal{M}\left(\beta,(f_1,\ldots,f_n)\right)$$

*and $C=\mathcal{M}\left(I,(e_1,\ldots,e_n),(f_1,\ldots,f_n)\right)$. Then*

$$A=C^\top BC$$

**Definition 10** (symmetric bilinear form, $V^{(2)}_{sym}$)**.** *A bilinear form $\rho \in V^{(2)}$ is called **symmetric** if*

$$\rho(u, w) = \rho(w, u)$$

*for all $u, w \in V$. The set of symmetric bilinear forms on $V$ is denoted by $V^{(2)}_{sym}$.*

**Remark 11.** *For real inner product space, define $\rho(u, w) = \langle u, w \rangle \in V^{(2)}_{sym}$. Additional example include*

$$\rho(u, w) = \langle u, Tw \rangle$$

*where $T$ is self-adjoint and*

$$\rho(S, T) = tr\ (ST)$$

*where here $\rho\colon \mathcal{L}(V) \times \mathcal{L}(V) \to \mathbb{F}$.*

**Definition 12** (symmetric matrix)**.** *A square matrix $A$ is called **symmetric** if it equals its transpose.*

**Theorem 13** (symmetric bilinear forms are diagonalizable)**.** *Suppose $\rho \in V^{(2)}$. Then the following are equivalent.*

*(a) $\rho$ is a symmetric bilinear form on $V$.*

*(b) $\mathcal{M}(\rho, (e_1, \ldots, e_n))$ is a symmetric matrix for every basis $e_1, \ldots, e_n$ of $V$.*

*(c) $\mathcal{M}(\rho, (e_1, \ldots, e_n))$ is a symmetric matrix for some basis $e_1, \ldots, e_n$ of $V$.*

*(d) $\mathcal{M}(\rho, (e_1, \ldots, e_n))$ is a diagonal matrix for some basis $e_1, \ldots, e_n$ of $V$.*

**Theorem 14.** *Suppose $V$ is a real inner product space and $\rho$ is a symmetric bilinear form on $V$. Then $\rho$ has a diagonal matrix with respect to some orthonormal basis of $V$.*

**Definition 15** (alternating bilinear form, $V^{(2)}_{alt}$)**.** *A bilinear form $\alpha \in V^{(2)}$ is called **alternating** if*

$$\alpha(v, v) = 0$$

*for all $v \in V$. The set of alternating bilinear forms on $V$ is denoted by $V^{(2)}_{alt}$.*

**Lemma 16** (characterization of alternating linear forms)**.** *A bilinear form $\alpha$ on $V$ is alternating if and only if*

$$\alpha(u, w) = -\alpha(w, u)$$

*for all $u, w \in V$.*

**Theorem 17.** *The sets $V^{(2)}_{sym}$ and $V^{(2)}_{alt}$ are subspaces of $V^{(2)}$. Furthermore,*

$$V^{(2)} = V^{(2)}_{sym} \oplus V^{(2)}_{alt}$$

**Definition 18** (quadratic form associated with a bilinear form, $q_\beta$)**.** *For $\beta$ a bilinear form on $V$, define a function $q_\beta\colon V \to \mathbb{F}$ by $q_\beta(v) = \beta(v, v)$. A function $q\colon V \to \mathbb{F}$ is called a **quadratic form** on $V$ if there exists a bilinear form $\beta$ on $V$ such that $q = q_\beta$.*

**Corollary 19** (quadratic form on $\mathbb{F}^n$)**.** *Suppose $n$ is a positive integer and $q$ is a function from $\mathbb{F}^n$ to $\mathbb{F}$. Then $q$ is a quadratic form on $\mathbb{F}^n$ if and only if there exist numbers $A_{j,k} \in \mathbb{F}$ for $j, k \in \{1, \ldots, n\}$ such that*

$$q(x_1, \ldots, x_n) = \sum_{k=1}^{n} \sum_{j=1}^{n} A_{j,k} x_j x_k$$

*for all $(x_1, \ldots, x_n) \in \mathbb{F}^n$.*

**Theorem 20** (characterizations of quadratic forms)**.** *Suppose $q\colon V \to \mathbb{F}$ is a function. Then following are equivalent.*

*(a) $q$ is a quadratic form.*

*(b) There exists a unique symmetric bilinear form $\rho$ on $V$ such that $q = q_\rho$.*

*(c) $q(\lambda v) = \lambda^2 q(v)$ for all $\lambda \in \mathbb{F}$ and all $v \in V$, and the function*

$$(u, w) \mapsto q(u + w) - q(u) - q(w)$$

*is a symmetric bilinear form on $V$.*

*(d) $q(2v) = 4q(v)$ for all $v \in V$, and the function*

$$(u, w) \mapsto q(u + w) - q(u) - q(w)$$

*is a symmetric bilinear form on $V$.*

**Theorem 21** (diagonalization of quadratic form)**.** *Suppose $q$ is a quadratic form on $V$.*

*(a) There exist a basis $e_1, \ldots, e_n$ of $V$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that*

$$q(x_1 e_1 + \cdots + x_n e_n) = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2$$

*for all $x_1, \ldots, x_n \in \mathbb{F}$.*

*(b) If $\mathbb{F} = \mathbb{R}$ and $V$ is an inner product space, then the basis in (a) can be chosen to be an orthonormal basis of $V$.*

**Remark 22.** *For each quadratic form we can choose a basis such that the quadratic form looks like a weighted sum of squares of the coordinates.*

**Problem 1**

Prove that if $\beta$ is a bilinear form on $\mathbb{F}$, then there exists $c \in \mathbb{F}$ such that

$$\beta(x, y) = cxy$$

for all $x, y \in \mathbb{F}$.

*Proof.* We note that since the input is taken from $\mathbb{F}$, the basis is naturally 1. So we have that

$$\beta(x, y) = x\beta(1, y) = xy\beta(1, 1) = cxy$$

where we take $c = \beta(1, 1)$. $\square$

**Problem 2**

Let $n = \dim V$. Suppose $\beta$ is a bilinear form on $V$. Prove that there exist $\phi_1, \ldots, \phi_n, \tau_1, \ldots, \tau_n \in V'$ such that

$$\beta(u, v) = \phi_1(u) \cdot \tau_1(v) + \cdots + \phi_n(u) \cdot \tau_n(v)$$

for all $u, v \in V$.

*Proof.*

$$\beta(u, v) = \beta\left(\sum_{i=1}^{n} u_i e_i, \sum_{i=1}^{n} v_j e_j\right) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_i v_j \beta(e_i, e_j)$$

We can now define the linear function $\phi_i(u) = u_i = e_i^*(u)$ and $\tau_j'(v) = v_j = e_j^*(v)$. Then we have that

$$\beta(u, v) = \sum_{i=1}^{n} \phi_i(u) \left(\sum_{j=1}^{n} \beta(e_i, e_j)\tau_j'(v)\right) = \sum_{i=1}^{n} \phi(u)\tau_i(v)$$

$\square$

**Problem 3**

Suppose $\beta\colon V \times V \to \mathbb{F}$ a bilinear form on $V$ and also is a linear functional on $V \times V$. Prove that $\beta = 0$.

*Proof.* First we show that $\beta \in V_{alt}^{(2)}$. Take any $u \in V$, then we have

$$\begin{aligned} \beta((u, u) + (u, u)) &= 2\beta(u, u) \\ \beta(2u, 2u) &= 4\beta(u, u) \end{aligned}$$

this shows that $\beta(u, u) = 0$ for all $u$. Next, we show the off-diagonal terms are 0: first,

$$\beta(u,w) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_i w_j \beta(e_i, e_j) \qquad \text{bilinearity}$$

at the same time,

$$\begin{aligned}
\beta(u,w) &= \beta\left(\sum_{i=1}^{n} u_i e_i, \sum_{j=1}^{n} w_j e_j\right) \\
&= \beta\left(\sum_{i=1}^{n} (u_i e_i, w_i e_i)\right) \\
&= \sum_{i=1}^{n} \beta\left(u_i e_i, w_i e_i\right) && \text{linearity on } V \times V \\
&= \sum_{i=1}^{n} u_i w_i \beta(e_i, e_i)
\end{aligned}$$

This shows that all off-diagonal terms are 0, i.e., $\beta(e_i, e_j) = 0$ for all $i \neq 0$. Therefore, we have $\beta = 0$. □

**Problem 6**

Prove or give a counterexample: If $\rho$ is a symmetric bilinear form on $V$, then

$$\{v \in V : \rho(v,v) = 0\}$$

is a subspace of $V$.

*Proof.* Consider $V = \mathbb{R}^2$ and $\rho(x,y) = x_1y_1 - x_2y_2$. Let $x = (1,1), y = (-1,1)$, then we have that $\rho(x,x) = 1 - 1 = 0, \rho(y,y) = 1 - 1 = 0$, but $\rho(x+y, x+y) = 0 - 4 = -4 \neq 0$. □

**Problem 8**

Find formulas for $\dim V_{sym}^{(2)}$ and $\dim V_{alt}^{(2)}$ in terms of $\dim V$.

*Proof.* Let $\dim V = n$. For $\beta \in V_{sym}^{(2)}$, consider $\mathcal{M}(\beta)$. Its diagonal entries can be chosen arbitrarily. For off-diagonal entries, only half of them can be chosen arbitrarily, therefore the dimension is

$$\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$$

For $\beta \in V_{alt}^{(2)}$, consider $\mathcal{M}(\beta)$. The diagonal entires are all 0 and only half of the off-diagonal entries can be chosen arbitrarily. Therefore, the dimension is $\frac{n(n-1)}{2}$. □

## 9B: Alternating Multilinear Forms

**Definition 23** ($V^m$)**.** *For $m$ a positive integer, define $V^m$ by*

$$V^m = \underbrace{V \times \cdots \times V}_{m \text{ times}}$$

**Definition 24** (m-linear form, $V^{(m)}$, multilinear form)**.** *Below we introduce the definitions.*

- *For $m$ a positive integer, an* ***m-linear form*** *on $V$ is a function $\beta \colon V^m \to \mathbb{F}$ that is linear in each slot when the other slots are held fixed. This means that for each $k \in \{1, \ldots, m\}$ and all $u_1, \ldots, u_m \in V$, the function*

  $$v \mapsto \beta(u_1, \ldots, u_{k-1}, v, u_{k+1}, \ldots, u_m)$$

  *is a linear map from $V$ to $\mathbb{F}$.*

- *The set of $m$-linear forms on $V$ is denoted by $V^{(m)}$.*

- *A function $\beta$ is called a* ***multilinear form*** *on $V$ if it is an $m$-linear form on $V$ for some positive integer $m$.*

**Remark 25.** *A 1-linear form on $V$ is a linear functional on $V$. A 2-linear form on $V$ is a bilinear form on $V$. $V^{(m)}$ is a vector space.*

*Example* (m-linear forms)*.* Suppose $\alpha, \beta \in V^{(2)}$. Define a function $\beta \colon V^4 \to \mathbb{F}$ by

$$\beta(v_1, v_2, v_3, v_4) = \alpha(v_1, v_2)\beta(v_3, v_4)$$

Then $\beta \in V^{(4)}$.

*Example* (m-linear forms)*.* Define $\beta \colon \left(\mathcal{L}\left(V\right)\right)^m \to \mathbb{F}$ by

$$\beta(T_1, \ldots, T_m) = \text{tr}\ (T_1 \cdots T_m)$$

Then $\beta$ is an $m$-linear form on $\mathcal{L}(V)$.

**Definition 26** (alternating forms, $V^{(m)}_{alt}$)**.** *Suppose $m$ is a positive integer.*

- *An $m$-linear form $\alpha$ on $V$ is called* ***alternating*** *if $\alpha(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots, v_m$ is a list of vectors in $V$ with $v_j = v_k$ for some two distinct values of $j$ and $k$ in $\{1, \ldots, m\}$.*

- *$V^{(m)}_{alt} = \{\alpha \in V^{(m)} \colon \alpha$ is an alternating $m$-linear form on $V\}$.*

**Corollary 27.** *Suppose $m$ is a positive integer and $\alpha$ is an alternating $m$-linear form on $V$. If $v_1, \ldots, v_m$ is a linearly dependent list in $V$, then*

$$\alpha(v_1, \ldots, v_m) = 0$$

**Corollary 28.** *Suppose $m > \dim V$. Then 0 is the only alternating $m$-linear form on $V$.*

**Theorem 29** (swapping input vectors in an alternating multilinear form)**.** *Suppose $m$ is a positive integer, $\alpha$ is an alternating $m$-linear form on $V$, and $v_1, \ldots, v_m$ is a list of vectors in $V$. Then swapping the vectors in any two slots of $\alpha(v_1, \ldots, v_m)$ changes the value of $\alpha$ by a factor of $-1$.*

**Remark 30.** *An odd numer of swaps cause the value of $\alpha$ to change by a factor of -1 and it won't change with an even number of swaps.*

**Definition 31** (permutation, perm $m$)**.** *Suppose $m$ is a positive integer.*

- *A **permutation** of $(1, \ldots, m)$ is a list $(j_1, \ldots, j_m)$ that contains each of the number $1, \ldots, m$ exactly once.*
- *The set of permutations of $(1, \ldots, m)$ is denoted by perm $m$.*

**Definition 32** (sign of a permutation)**.** *The **sign** of a permutation $(j_1, \ldots, j_m)$ is defined by*

$$sign(j_1, \ldots, j_m) = (-1)^N$$

*where $N$ is the number of pairs of integers $(k, l)$ with $1 \leq k < l \leq m$ such that $k$ appears after $l$ in the list $(j_1, \ldots, j_m)$.*

**Lemma 33.** *Swapping two entries in a permutation multiplies the sign of the permutation by -1.*

**Lemma 34** (permutation and alternating multilinear form)**.** *Suppose $m$ is a positive integer and $\alpha \in V_{alt}^{(m)}$. Then*

$$\alpha(v_{j_1}, \ldots, v_{j_m}) = \big(sign\,(j_1, \ldots, j_m)\big)\,\alpha\,(v_1, \ldots, v_m)$$

*for every list $v_1, \ldots, v_m$ of vectors in $V$ and all $(j_1, \ldots, j_m) \in$ perm $m$.*

**Theorem 35.** *Let $n = \dim V$. Suppose $e_1, \ldots, e_n$ is a basis of $V$ and $v_1, \ldots, v_n \in V$. For each $k \in \{1, \ldots, n\}$, let $b_{1,k}, \ldots, b_{n,k} \in \mathbb{F}$ be such that*

$$v_k = \sum_{j=1}^{n} b_{j,k} e_j$$

*Then*

$$\alpha(v_1, \ldots, v_n) = \alpha(e_1, \ldots, e_n) \sum_{(j_1, \ldots, j_n) \in perm\ n} \big(sign\,(j_1, \ldots, j_n)\big)\, b_{j_1,1} \cdots b_{j_n,n}$$

*for every alternating $n$-linear form $\alpha$ on $V$.*

**Theorem 36.** *The vector space $V_{alt}^{(dim V)}$ has dimension one.*

**Corollary 37.** *Let $n = \dim V,$. Suppose $\alpha$ is a nonzero alternating $n$-linear form on $V$ and $e_1, \ldots, e_n$ is a list of vectors in $V$. Then*

$$\alpha(e_1, \ldots, e_n) \neq 0$$

*if and only if $e_1, \ldots, e_n$ is linearly independent.*

**Problem 1**

Suppose $m$ is a positive integer. Show that $\dim V^{(m)} = (\dim V)^m$.

*Proof.* Let $\dim V = n$ with basis $e_1, \ldots, e_n$. The basis vector for $V^{(m)}$ can be formed via taking all possible $m$-tuples $b_{j_1}, \ldots, b_{j_m}$ where $b_{j_i}$ is a component of the basis. There are $n$ choices over $m$ positions, so we have that $\dim V^{(m)} = (\dim V)^m$. □

**Problem 3**

Suppose $m$ is a positive integer and $\alpha$ is an $m$-linear form on $V$ such that $\alpha(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots, v_m$ is a list of vectors in $V$ with $v_j = v_{j+1}$ for some $j \in \{1, \ldots, m-1\}$ Prove that $\alpha$ is an alternating $m$-linear form on $V$.

*Proof.* Note that if the list $v_1, \ldots, v_n$ comes with consecutive identical numbers, then by definition the output becomes 0. To prove $\alpha$ to be an alternating $m$-linear form, considers $v_i = v_k$ for $i + 1 < k$. Note that then we can now just swap and gets the same result:

$$\alpha(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k, \ldots, v_n) = -\alpha(v_1, \ldots, v_i, v_k, \ldots, v_{i+1}, \ldots, v_n) = 0$$

□

**Problem 5**

Suppose $m$ is a positive integer and $\beta$ is an $m$-linear form on $V$. Define an $m$-linear form $\alpha$ by

$$\alpha(v_1, \ldots, v_m) = \sum_{(j_1, \ldots, j_m) \in \text{perm } m} \big(\text{sign}\,(j_1, \ldots, j_m)\,\beta(v_{j_1}, \ldots, v_{j_m})\big)$$

for $v_1, \ldots, v_m \in V$. Explain why $\alpha \in V^{(m)}_{alt}$.

*Proof.* If there are two repeating vectors, let's say $v_p = v_q$, then we know that

$$\beta(v_1, \ldots, v_p, \ldots, v_q, \ldots, v_m) = \beta(v_1, \ldots, v_q, \ldots, v_p, \ldots, v_m)$$

However, through swapping, the coefficient differs by (-1), so we have

$$\begin{aligned} &\text{sign}(1, \ldots, p, \ldots, q, \ldots, m)\beta(v_1, \ldots, v_p, \ldots, v_q, \ldots, v_m) \\ = &-\text{sign}(1, \ldots, q, \ldots, p, \ldots, m)\beta(v_1, \ldots, v_q, \ldots, v_p, \ldots, v_m) \end{aligned}$$

This basically shows the main idea of the proof. To make this more rigorous, we claim that for each permutation $\sigma \in$ perm $m$, there is a corresponding permutation $\sigma_{pq} \in$ perm $m$ such that keeps everything unchanged while only swapping the position of $p$ and $q$. This means that for each permutation, there is a corresponding "cancelling" pair permutation. Since we are summing all permutations, the result is finally 0, finishing the proof. □

# 9C: Determinants

**Definition 38** ($\alpha_T$)**.** *Suppose that $m$ is a positive integer and $T \in \mathcal{L}(V)$. For $\alpha \in V_{alt}^{(m)}$, define $\alpha_T \in V_{alt}^{(m)}$ by*

$$\alpha_T(v_1, \ldots, v_m) = \alpha(Tv_1, \ldots, Tv_m)$$

*for each list $v_1, \ldots, v_m$ of vectors in $V$.*

**Remark 39.** *The function $\alpha \mapsto \alpha_T$ is a linear map of $V_{alt}^{(m)}$ to itself. We know that $\dim V_{alt}^{(\dim V)} = 1$, so the linear map is simply a multiplication by some unique scalar. For the linear map $\alpha \mapsto \alpha_T$, we now define $\det T$ to be that scalar.*

**Definition 40** (**determinant** of an operator, $\det T$)**.** *Suppose $T \in \mathcal{L}(V)$. The* ***determinant*** *of $T$, denoted by $\det T$, is defined to be the unique number in $\mathbb{F}$ such that*

$$\alpha_T = (\det T)\alpha$$

*for all $\alpha \in V_{alt}^{(\dim V)}$.*

**Remark 41.** *Let $n = \dim V$.*

- *If $I$ is the identity operator on $V$, then $\alpha_I = \alpha$ for all $\alpha \in V_{alt}^{(n)}$. This gives that $\det I = 1$.*
- *More generally, if $\lambda \in \mathbb{F}$, then $\alpha_{\lambda I} = \lambda^n \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus $\det(\lambda I) = \lambda^n$.*
- *Since $\alpha_{\lambda T} = \lambda^n \alpha_T = \lambda^n (\det T)\alpha$ for all $\alpha \in V_{alt}^{(n)}$, $\det(\lambda T) = \lambda^n \det T$.*
- *Suppose $T \in \mathcal{L}(V)$ and there is a basis $e_1, \ldots, e_n$ of $V$ consisting of eigenvectors of $T$, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. If $\alpha \in V_{alt}^{(n)}$, then*

  $$\alpha_T(e_1, \ldots, e_n) = \alpha(\lambda_1 e_1, \ldots, \lambda_n e_n) = (\lambda_1 \cdots \lambda_n)\alpha(e_1, \ldots, e_n)$$

  *If $\alpha \neq 0$, then $\alpha(e_1, \ldots, e_n) \neq 0$. Thus this means that*

  $$\det T = \lambda_1 \cdots \lambda_n$$

**Definition 42** (determinant of a matrix, det $A$)**.** *Suppose that $n$ is a positive integer and $A$ is an $n$-by-$n$ matrix square matrix with entries in $\mathbb{F}$. Let $T \in \mathcal{L}(\mathbb{F}^n)$ be the operator whose matrix with respect to the standard basis of $\mathbb{F}^n$ equals $A$. The* ***determinant*** *of $A$, denoted by $\det A$, is defined by $\det A = \det T$.*

**Theorem 43** (determinant is an alternating multilinear form)**.** *Suppose that $n$ is a positive integer. The map that takes a list $v_1, \ldots, v_n$ of vectors in $\mathbb{F}^n$ to $\det(v_1 \cdots v_n)$ is an alternating $n$-linear form on $\mathbb{F}^n$.*

**Corollary 44** (formula for determinants of a matrix)**.** *Suppose that $n$ is a positive integer and $A$ is an $n$-by-$n$ matrix square matrix. Then*

$$\det A = \sum_{(j_1,\dots,j_n)\in perm\ n} \big(sign(j_1,\dots,j_n)\big)\, A_{j_1,1}\cdots A_{j_n,n}$$

**Remark 45.** *The sum in the formula above contains $n!$ terms.*

**Corollary 46** (determinant of upper-triangular matrix)**.** *Suppose that $A$ is an upper-triangular matrix with $\lambda_1,\dots,\lambda_n$ on the diagonal. Then $\det A = \lambda_1\cdots\lambda_n$.*

**Theorem 47** (determinant is multiplicative)**.** *We have the following result:*

*(a) Suppose $S, T \in \mathcal{L}(V)$. Then $\det(ST) = \det(S)\det(T)$.*

*(b) Suppose $A$ and $B$ are square matrices of the same size. Then*

$$\det(AB) = \det(A)\det(B)$$

**Corollary 48.** *An operator $T \in \mathcal{L}(V)$ is **invertible** if and only if $\det T \neq 0$. Furthermore, if $T$ is invertible, then $\det(T^{-1}) = \frac{1}{\det T}$.*

**Corollary 49.** *Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. Then $\lambda$ is an eigenvalue of $T$ if and only if $\det(\lambda I - T) = 0$.*

**Corollary 50.** *Suppose $T \in \mathcal{L}(V)$ and $S\colon W \to V$ is an invertible linear map. Then*

$$\det(S^{-1}TS) = \det T$$

**Corollary 51.** *Suppose $T \in \mathcal{L}(V)$ and $e_1,\dots,e_n$ is a basis of $V$. Then*

$$\det T = \det \mathcal{M}\big(T,(e_1,\dots,e_n)\big)$$

**Corollary 52.** *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then $\det T$ equals the product of the eigenvalues of $T$, with each eigenvalue included as many times as its multiplicity.*

**Corollary 53** (determinant of transpose, dual, or adjoint)**.** *We have the following result:*

*(a) Suppose $A$ is a square matrix. Then $\det A^\top = \det A$.*

*(b) Suppose $T \in \mathcal{L}(V)$. Then $\det T' = \det T$.*

*(c) Suppose $V$ is an inner product space and $T \in \mathcal{L}(V)$. Then*

$$\det(T^*) = \overline{\det T}$$

**Corollary 54.** *Helpful results in evaluating the determminants:*

*(a) If either two columns or two rows of a square matrix are equal, then the determinant of the matrix equals 0.*

(b) *Suppose $A$ is a square matrix and $B$ is the matrix obtained from $A$ by swapping either two columns or two rows. Then* $\det A = -\det B$.

(c) *If one column or one row of a square matrix is multiplied by a scalar, then the value of the determinant is multiplied by the same scalar.*

(d) *If a scalar multiple of one column of a square matrix to added to another column, then the value of the determinant is unchanged.*

(e) *If a scalar multiple of one row of a square matrix to added to another row, then the value of the determinant is unchanged.*

**Corollary 55.** *Suppose $V$ is an inner product space and $S \in \mathcal{L}(V)$ an unitary operator. Then* $|\det S| = 1$.

**Corollary 56.** *Suppose $V$ is an inner product space and $T \in \mathcal{L}(V)$ is a positive operator. Then* $\det T \geq 0$.

**Corollary 57.** *Suppose $V$ is an inner product space and $T \in \mathcal{L}(V)$. Then*

$$|\det T| = \sqrt{\det(T^*T)} = \textit{ product of singular values of } T$$

**Lemma 58.** *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of $T$, and let $d_1, \ldots, d_m$ denote their multiplicities. Then*

$$\det(zI - T) = (z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

**Definition 59** (characteristic polynomial)**.** *Suppose $T \in \mathcal{L}(V)$. The polynomial defined by*

$$z \mapsto \det(zI - T)$$

*is called the **characteristic polynomial** of $T$.*

**Theorem 60** (Cayley-Hamilton theorem)**.** *Suppose $T \in \mathcal{L}(V)$ and $q$ is the characteristic polynomial of $T$. Then $q(T) = 0$.*

**Corollary 61** (characteristic polynomial, trace, and determinant)**.** *Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then the characteristic polynomial of $T$ can be written as*

$$z^n - (tr\ T)z^{n-1} + \cdots + (-1)^n(\det T)$$

**Theorem 62** (Hadamard's inequality)**.** *Suppose $A$ is an $n$-by-$n$ matrix matrix. Let $v_1, \ldots, v_n$ denote the columns of $A$. Then*

$$|\det A| \leq \prod_{k=1}^{n} \|v_k\|$$

**Theorem 63** (determinant of Vandermonde matrix)**.** *Suppose $n > 1$ and $\beta_1, \ldots, \beta_n \in \mathbb{F}$. Then*

$$det \begin{pmatrix} 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^{n-1} \\ 1 & \beta_2 & \beta_2^2 & \cdots & \beta_2^{n-1} \\ 1 & \beta_3 & \beta_3^2 & \cdots & \beta_3^{n-1} \\ & & & \ddots & \\ 1 & \beta_n & \beta_n^2 & \cdots & \beta_n^{n-1} \end{pmatrix} = \prod_{1 \le j < k \le n} (\beta_k - \beta_j).$$

**Problem 1**

Prove or give a counterexample: $S, T \in \mathcal{L}(V) \Rightarrow \det(S+T) = \det S + \det T$.

*Proof.* Consider $\mathbb{R}^2$, and that

$$\mathcal{M}(S) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathcal{M}(T) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then clearly $\det S = 1$ and $\det T = 2$. However, we have that

$$\mathcal{M}(S) + \mathcal{M}(T) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

which has $\det(S+T) = 6 \neq \det S + \det T$. $\square$

**Problem 3**

Suppose $T \in \mathcal{L}(V)$ is nilpotent. Prove that $\det(I+T) = 1$.

*Proof.* We know that 0 is the only eigenvalue of $T$ and thus the only eigenvalue of $I+T$ is 1. Hence $\det(I+T) = 1$. $\square$

**Problem 5**

Suppose $A$ is a block triangular matrix

$$A = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where each $A_k$ along the diagonal is a square matrix. Prove that

$$\det A = (\det A_1) \cdots (\det A_m)$$

*Proof.* One can show that $\det A = (\det A_1)(\det A_2)$ through direct proof. We use induction on $m$ for solving this problem. The base case is trivial. We assume the statement holds for $m \leq k-1$. Then for $m = k$, we can partition the matrix into two blocks:

$$\begin{bmatrix} A' & * \\ 0 & A_k \end{bmatrix}$$

where

$$A' = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_{k-1} \end{bmatrix}$$

Then we have that $\det A = (\det A')(\det A_k) = (\det A_1) \cdots (\det A_k)$, finishing the proof. $\square$

**Problem 9**

Suppose that $V$ is a real vector space of even dimension, $T \in \mathcal{L}(V)$, and $\det T < 0$. Prove that $T$ has at least two distinct eigenvalues.

*Proof.* Since $\det(T) \neq 0$, $T$ is invertible and thus have $n$ distinct eigenvalues with $n \geq 2$. Another argument could be that for real cases, there have to be at least one negative and one positive eigenvalue to make the determinant negative; for complex cases, there must be two conjugate pairs. □

**Problem 11**

Prove or give a counter example: If $\mathbb{F} = \mathbb{R}, T \in \mathcal{L}(V)$, and $\det T > 0$, then $T$ has a square root.

*Proof.* Not necessarily. Consider an operator in $\mathbb{R}^2$ with two negative eigenvalues which is clearly non-positive and therefore does not have a square root. □

**Problem 16**

Suppose $T \in \mathcal{L}(V)$. Define $g \colon \mathbb{F} \to \mathbb{F}$ by $g(x) = \det(I + xT)$. Show that $g'(0) = \text{tr } T$.

*Proof.*

$$\begin{aligned} g'(x) &= \frac{d}{dx} \det(I + xT) \\ &= \frac{d}{dx} \prod_{i=1}^{n} (1 + x\lambda_i) \\ &= \sum_{i=1}^{n} \left( \lambda_i \prod_{j \neq i} (1 + x\lambda_i) \right) \end{aligned}$$

Substitute $x = 0$ yields that

$$g'(0) = \sum_{i=1}^{n} \lambda_i = \text{tr } T$$

□

**Problem 19**

Suppose $V$ is an inner product space, $e_1, \ldots, e_n$ is an orthonormal basis of $V$, and $T \in \mathcal{L}(V)$ is a positive operator.

(a) Prove that $\det T \leq \prod_{k=1}^{n} \langle Te_k, e_k \rangle$.

(b) Prove that if $T$ is invertible, then the inequality in (a) is an equality if and only if $e_k$ is an eigenvector of $T$ for each $k = 1, \ldots, n$.

*Proof.* (a) The matrix representation of $T$ wrt. $e_1, \ldots, e_n$ is that $A_{ij} = \langle Te_i, e_j \rangle$. Hence the r.h.s of this inequality is simply the product of all the diagonal terms on the matrix of $T$. We prove this inequality through Cholesky factorization. Note that

$$A = LL^*$$

for lower-triangular matrix $L$, and thus we have

$$\det T = \det A = (\det L)^2 = \left( \prod_{k=1}^{n} l_{kk} \right)^2 = \prod_{k=1}^{n} L_{kk}^2$$

We note that

$$A_{kk} = L_{kk}^2 + \sum_{j=1}^{k-1} L_{kj}^2$$

and thus we have

$$\det A \leq \prod_{k=1}^{n} A_{kk} = \prod_{k=1}^{n} \langle Te_k, e_k \rangle$$

(b) If $e_k$ is an eigenvector of $T$, then $\langle Te_k, e_k \rangle = \lambda_k$, the $k$-th eigenvalue of $T$. Then we know that $\det T$ is the product of all eigenvalues.

Conversely, if (a) is an equality, then we know that $L$ is a diagonal matrix and thus $A$ is also a diagonal matrix. Then the orthonormal basis $e_1, \ldots, e_n$ actually diagonaizes $T$ and hence each of them is an eigenvector of $T$.

□

## 9D: Tensor Products

**Definition 64** (bilinear functional on $V \times W$, the vector space $\mathcal{B}(V, W)$)**.** *A **bilinear functional** on $V \times W$ is a function $\beta\colon V \times W \to \mathbb{F}$ such that $v \mapsto \beta(v, w)$ is a linear functional on $V$ for each $w \in W$ and $w \mapsto \beta(v, w)$ is a linear functional on $W$ for each $v \in V$.*

*The vector space of bilinear functionals on $V \times W$ is denoted by $\mathcal{B}(V, W)$.*

**Remark 65.** *If $V = W$, then a bilinear functional on $V \times W$ is a bilinear form.*

**Corollary 66.** $\dim \mathcal{B}(V, W) = (\dim V)(\dim W)$

**Remark 67.** *We want a* basis-free *definition of the tensor product.*

**Definition 68** (tensor product, $V \otimes W, v \otimes w$)**.** *The **tensor product** $V \otimes W$ is defined to be $\mathcal{B}(V', W')$.*

*For $v \in V$ and $w \in W$, the **tensor product** $v \otimes w$ is the element of $V \otimes W$ defined by*

$$(v \otimes w)(\varphi, \tau) = \varphi(v)\tau(w)$$

*for all $(\varphi, \tau) \in V' \times W'$.*

**Corollary 69.** $\dim(V \otimes W) = (\dim V)(\dim W)$.

**Proposition 70** (bilinearity of tensor product)**.** *Suppose $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$ and $\lambda \in \mathbb{F}$. Then*

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \text{ and } v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

*and*

$$\lambda\, (v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$$

**Theorem 71** (basis of $V \otimes W$)**.** *Suppose $e_1, \ldots, e_m$ is a list of vectors in $V$ and $f_1, \ldots, f_n$ is a list of vectors in $W$.*

*(a) If $e_1, \ldots, e_m$ and $f_1, \ldots, f_n$ are both linearly independent list, then*

$$\{e_j \otimes f_k\}_{j=1,\ldots,m;k=1,\ldots,n}$$

*is a linearly independent list in $V \otimes W$.*

*(b) If $e_1, \ldots, e_m$ is a basis of $V$ and $f_1, \ldots, f_n$ is a basis of $W$, then the list $\{e_j \otimes f_k\}_{j=1,\ldots,m;k=1,\ldots,n}$ is a basis of $V \otimes W$.*

**Definition 72** (bilinear map)**.** *A **bilinear map** from $V \times W$ to a vector space $U$ is a function $\Gamma\colon V \times W \to U$ such that $v \mapsto \Gamma(v, w)$ is a linear map from $V$ to $U$ for each $w \in W$ and $w \mapsto \Gamma(v, w)$ is a linear map from $W$ to $U$ for each $v \in V$.*

**Lemma 73** (converting bilinear maps to linear maps)**.** *Suppose $U$ is a vector space.*

(a) *Suppose* $\Gamma\colon V \times W \to U$ *is a bilinear map. Then there exists a unique linear map* $\tilde{\Gamma}\colon V \otimes W \to U$ *such that*

$$\tilde{\Gamma}(v \otimes w) = \Gamma(v, w)$$

*for all* $(v, w) \in V \times W$.

(b) *Conversely, suppose* $T\colon V \otimes W \to U$ *is a linear map. Then there exists a unique bilinear map* $T^{\#}\colon V \times W \to U$ *such that*

$$T^{\#}(v, w) = T(v \otimes w)$$

*for all* $(v, w) \in V \times W$.

**Theorem 74** (inner product on tensor product of two inner product spaces)**.** *Suppose* $V$ *and* $W$ *are inner product spaces. Then there is a unique inner product on* $V \otimes W$ *such that*

$$\langle v \otimes w, u \otimes x\rangle = \langle v, u\rangle\langle w, x\rangle$$

*for all* $u, v \in V$ *and* $w, x \in W$.

**Remark 75.** *We have that* $\|v \otimes w\| = \|v\|\|w\|$.

**Corollary 76.** *Suppose* $V$ *and* $W$ *are inner product spaces, and* $e_1, \ldots, e_m$ *is an orthonormal basis of* $V$ *and* $f_1, \ldots, f_n$ *is an orthonormal basis of* $W$. *Then*

$$\{e_j \otimes f_k\}_{j=1,\ldots,m;k=1,\ldots,n}$$

*is an orthonormal basis of* $V \otimes W$.

**Definition 77.** *An* ***m-linear*** *functional on* $V_1 \times \cdots \times V_m$ *is a function* $\beta\colon V_1 \times \cdots \times V_m \to \mathbb{F}$ *that is a linear functional in each slot when the other slots are held fixed.*

*The vector space of* $m$*-linear functionals on* $V_1 \times \cdots \times V_m$ *is denoted by* $\mathcal{B}(V_1, \ldots, V_m)$.

**Corollary 78.** $\dim \mathcal{B}(V_1, \ldots, V_m) = (\dim V_1) \times \cdots \times (\dim V_m)$

**Definition 79** (tensor product)**.** *The tensor product* $V_1 \otimes \cdots \otimes V_m$ *is defined to be* $\mathcal{B}(V_1', \ldots, V_m')$.

*For* $v_1 \in V_1, \ldots, v_m \in V_m$, *the* ***tensor product*** $v_1 \otimes \cdots \otimes v_m$ *is the element of* $V_1 \otimes \cdots \otimes V_m$ *defined by*

$$(v_1 \otimes \cdots \otimes v_m)(\varphi_1, \ldots, \varphi_m) = \varphi_1(v_1) \cdots \varphi_m(v_m)$$

*for all* $(\varphi_1, \ldots, \varphi_m) \in V_1' \times \cdots \times V_m'$.

**Corollary 80.** *Suppose* $\dim V_k = n_k$ *and* $e_1^k, \ldots, e_{n_k}^k$ *is a basis of* $V_k$ *for* $k = 1, \ldots, m$. *Then*

$$\{e_{j_1}^1 \otimes \cdots \otimes e_{j_m}^m\}_{j_1=1,\ldots,n_1;\ldots;j_m=1,\ldots,n_m}$$

*is a basis of* $V_1 \otimes \cdots \otimes V_m$.

**Definition 81** (m-linear map). *An $m$-linear map from $V_1 \times \cdots \times V_m$ to a vector space $U$ is a function $\Gamma\colon V_1 \times \cdots \times V_m \to U$ that is a linear map in each slot when the other slots are held fixed.*

**Theorem 82** (converting m-linear map to linear maps). *Suppose $U$ is a vector space.*

*(a) Suppose that $\Gamma\colon V_1 \times \cdots \times V_m \to U$ is an $m$-linear map. Then there exists a unique linear map $\tilde{\Gamma}\colon V_1 \otimes \cdots \otimes V_m \to U$ such that*

$$\tilde{\Gamma}(v_1 \otimes \cdots \otimes v_m) = \Gamma(v_1, \ldots, v_m)$$

*for all $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$.*

*(b) Conversely, suppose $T\colon V_1 \otimes \cdots \otimes V_m \to U$ is a linear map. Then there exists a unique $m$-linear map $T^{\#}\colon V_1 \times \cdots \times V_m \to U$ such that*

$$T^{\#}(v_1, \ldots, v_m) = T(v_1 \otimes \cdots \otimes v_m)$$

*for all $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$.*

**Problem 1**

Suppose $v \in V$ and $w \in W$. Prove that $v \otimes w = 0$ if and only if $v = 0$ or $w = 0$.

*Proof.* By definition, we have for any $(\varphi, \tau) \in V' \times W'$,

$$(v \otimes w)(\varphi, \tau) = \varphi(v)\tau(w)$$

Then this means that $\varphi(v)\tau(w) = 0$ for arbitrary choice of $\varphi, \tau$, meaning that either $v = 0$ or $w = 0$. $\square$

**Problem 3**

Suppose that $v_1, \ldots, v_m$ is a linearly independent list in $V$. Suppose also that $w_1, \ldots, w_m$ is a list in $W$ such that

$$v_1 \otimes w_1 + \cdots + v_m \otimes w_m = 0$$

Prove that $w_1 = \cdots = w_m = 0$.

*Proof.* By the linear map lemma and the linear independence of $v_1, \ldots, v_m$, there exists $\varphi_1, \ldots, \varphi_m \in V'$ such that

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

where $j, k \in \{1, \ldots, m\}$. Applying such $\{\varphi_i\}_{i=1}^m$ to the list

$$\sum_{i=1}^{n} v_i \otimes w_i$$

and take $\tau \in W'$ to be the identity map yields that

$$w_1 = \cdots = w_m = 0$$

$\square$

**Problem 5**

Suppose $m$ and $n$ are positive integers. For $v \in \mathbb{F}^m$ and $w \in \mathbb{F}^n$, identify $v \otimes w$ with an $m$-by-$n$ matrix as in Example 9.76. With that identification, show that the set

$$\{v \otimes w \colon v \in \mathbb{F}^m \text{ and } w \in \mathbb{F}^n\}$$

is the set of $m$-by-$n$ matrix matrices (with entries in $\mathbb{F}$) that have rank at most one.

*Proof.* If one examine the matrices with entries shown on the matrix, it's easy to tell that for row $j$ and row $k$ with $j \neq k$, one can get row $k$ from row $j$ through multiplying $v_k/v_j$. The same applies to arbitrary pairs of columns. Thus the matrix has at most rank one. □

**Problem 8**

Suppose $v_1, \ldots, v_m \in V$ and $w_1, \ldots, w_m \in W$ are such that

$$v_1 \otimes w_1 + \cdots + v_m \otimes w_m = 0$$

Suppose that $U$ is a vector space and $\Gamma \colon V \times W \to U$ is a bilinear map. Show that

$$\Gamma(v_1, w_1) + \cdots + \Gamma(v_m, w_m) = 0$$

*Proof.* We know there exists a unique "converting" linear map $\tilde{\Gamma}$ such that

$$\Gamma(v \tilde{\otimes} w) = \Gamma(v, w)$$

Hence, applying this gives that

$$\begin{aligned}
\sum_{i=1}^{m} \Gamma(v_i, w_i) &= \sum_{i=1}^{m} \tilde{\Gamma}(v_i \otimes w_i) \\
&= \tilde{\Gamma}\left(\sum_{i=1}^{m} v_i \otimes w_i\right) \\
&= 0
\end{aligned}$$

□