Chapter 1: Vector Spaces

Linear Algebra Done Right, by Sheldon Axler

1A: $\mathbb{R}^n$ and $\mathbb{C}^n$

Problem 1

Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \mathbb{C}$.

Proof. Let $\alpha = a + bi, \beta = c + di$. Then we have that

$$\begin{aligned} \alpha + \beta &= (a + bi) + (c + di) \\ &= (c + di) + (a + bi) \\ &= \beta + \alpha \end{aligned}$$

□

Problem 3

Show that $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ for all $\alpha, \beta, \gamma \in \mathbb{C}$.

Proof. Choose arbitrary $\alpha, \beta, \gamma \in \mathbb{C}$. Denote $\alpha = a + bi, \beta = c + di, \gamma = e + fi$. Then we have that

$$\begin{aligned} (\alpha\beta)\gamma &= ((ac - bd) + (ad + bc)i)(e + fi) \\ &= (ace - bde - adf - bcf) + (ade + bce + acf - bdf)i \end{aligned}$$

At the same time we have

$$\begin{aligned} \alpha(\beta\gamma) &= (a + bi)((ce - df) + (cf + de)i) \\ &= (ace - adf - bcf - bde) + (ade + acf + bce - bdf)i \end{aligned}$$

Hence $(\alpha\beta)\gamma = \alpha(\beta\gamma)$.

□

Problem 5

Show that for any $\alpha \in \mathbb{C}$, there exists a unique $\beta \in \mathbb{C}$ such that $\alpha + \beta = 0$.

Proof. Denote $\alpha = a + bi$. By property of filed, we know there exists unique $c = -a$ and $d = -b$ such that $\beta = c + di$ and $\alpha + \beta = 0$. Suppose for the sake of contradiction that β is not unique, then there exists $c' + d'i$ such that $(a + c') + (b + d')i = 0$ while $c' \neq a$ or $d' \neq d$, contradicting the uniqueness of additive inverse property. □

Problem 8

Find two distinct squared roots of i.

Proof. Suppose $\alpha = a + bi$'s square equals one.

$$(a+bi)^2 = a^2 - b^2 + 2abi = 1$$

Then $|a| = |b|, 2ab = 1$. So we get the solution to be

$$\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

□

Problem 10

Show that $(\mathbf{x}+\mathbf{y})+\mathbf{z} = \mathbf{x}+(\mathbf{y}+\mathbf{z})\ \forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{F}^n$.

Proof.

$$\begin{aligned}(\mathbf{x}+\mathbf{y})+\mathbf{z} &= (x_1+y_1, \ldots, x_n+y_n) + (z_1, \ldots, z_n) \\ &= (x_1, \ldots, x_n) + (y_1+z_1, \ldots, y_n+z_n) \\ &= \mathbf{x} + (\mathbf{y}+\mathbf{z})\end{aligned}$$

□

Problem 14

Show that $\gamma(\mathbf{x}+\mathbf{y}) = \gamma\mathbf{x} + \gamma\mathbf{y}\ \forall \gamma \in \mathbb{F}, \mathbf{x}, \mathbf{y} \in \mathbb{F}^n$.

Proof.

$$\begin{aligned}\gamma(\mathbf{x}+\mathbf{y}) &= \gamma(x_1+y_1, \ldots, x_n+y_n) \\ &= (\gamma(x_1+y_1), \ldots, \gamma(x_n+y_n)) \\ &= (\gamma x_1 + \gamma y_1, \ldots, \gamma x_n + \gamma y_n) \\ &= \gamma(x_1, \ldots, x_n) + \gamma(y_1, \ldots, y_n) \\ &= \gamma\mathbf{x} + \gamma\mathbf{y}\end{aligned}$$

□

1B: Definition of the Vector Space

Theorem 1. *A vector space is a set that is closed under* vector addition *and* scalar multiplication*. It also has the following properties:*

- *commutativity*
- *associativity*
- *additive identity*
- *additive inverse*
- *multiplicative identity*
- *(scalar) distributive*

Notation: $\mathbb{F}^S$.

Explanation: If S is a set, then $\mathbb{F}^S$ denotes the set of functions from S to $\mathbb{F}$ (scalar function). e.g. $f \in \mathbb{F}^S$.

Comment: $\mathbb{F}^S$ is a vector space; One can think of $f \in \mathbb{F}^n$ as $f : \{1, 2, \ldots, n\} \to \mathbb{F}$.

Problem 1

Prove that $-(-\mathbf{v}) = \mathbf{v}$ for all $\mathbf{v} \in V$.

Proof. We know $-(-\mathbf{v})$ is the unique additive inverse of $-\mathbf{v}$. At same time by definition, $\mathbf{v} + (-\mathbf{v}) = 0$ and thus by commutativity $(-\mathbf{v}) + \mathbf{v} = 0$. This shows that $\mathbf{v}$ is the unique additive inverse of $(-\mathbf{v})$, and such that $-(-\mathbf{v}) = \mathbf{v}$. □

Problem 2

Suppose $a \in \mathbb{F}, \mathbf{v} \in V$, and $a\mathbf{v} = 0$. Prove that $a = 0$ or $\mathbf{v} = 0$.

Proof. Suppose for the sake of contradiction that $a \neq 0$ and $\mathbf{v} \neq 0$ but $a\mathbf{v} = 0$.

$$\mathbf{v} = 1\mathbf{v}$$
$$\mathbf{v} = \frac{1}{a} \cdot a\mathbf{v}$$
$$\mathbf{v} = \frac{1}{a}0$$

This forms a contradiction. □

Problem 3

Suppose $\mathbf{v}, \mathbf{w} \in V$. Explain why there exists a unique $\mathbf{x} \in V$ such that $\mathbf{v} + 3\mathbf{x} = \mathbf{w}$.

Proof. Suppose there exists $\mathbf{x}'$ which also satisfies the condition. Then we have

$$\mathbf{v} + 3\mathbf{x} = \mathbf{w} \qquad \mathbf{v} + 3\mathbf{x}' = \mathbf{w} \tag{1}$$

This gives that $\mathbf{x} = (\mathbf{w} - \mathbf{v})/3 = \mathbf{x}'$ which shows $\mathbf{x}$ is unique. $\square$

Problem 4

The empty set is not a vector space, why?

Proof. There is no additive identity in the empty set. $\square$

Problem 7

Suppose S is a nonempty set. Let V^S denotes the set of functions from S to V. Define a natural addition and scalar multiplication on V^S, and show that V^S is a vector space with these definitions.

Proof. Let $f, g \in V^S \colon S \to V$. Define the addition and multiplication to be that

$$f + g(x) = f(x) + g(x)$$

We have that

- commutativity: $f + g(x) = f(x) + g(x) = g(x) + f(x) = g + f(x)$
- associativity: $(f + g) + h(x) = f(x) + g(x) + h(x) = f + (g + h)(x)$
- additive identity: Define $0 \colon S \to 0 \in V$, then $f + 0(x) = 0 + f(x) = f(x)$
- additive inverse: for every $f \in V^S$, define $g(x) = -f(x)$ which exists by the property of vector space and thus we have that $g + f = 0$ for every x and thus that it exists.
- multiplicative identity: same as above
- (scalar) distributive: $a(f + g)(x) = a(f(x) + g(x)) = af(x) + g(x)$

$\square$

1C: Subspaces

Definition 2 (subspace)**.** *A subset $\mathcal{U}$ of V is called* subspace *of V if $\mathcal{U}$ is also a vector space with the same additive identity, addition, and scalar multiplication as on V.*

Remark 3. *The set $\{0\}$ is the smallest subspace of V, and V itself is the largest subspace of V.*

Remark 4. *The subspace of $\mathbb{R}^2$ are precisely $\{0\}$, all lines in $\mathbb{R}^2$ containing the origin, $\mathbb{R}^2$.*

Definition 5 (Sum of subspace)**.** *Suppose $V_1, \cdots, V_m$ are subspaces. The* sum *of them, denoted by $V_1 + \cdots + V_m$, is the set of all possible sums of element of $V_1, \cdots, V_m$. Specifically,*

$$V_1 + \cdots + V_m = \{v_1 + \cdots + v_m \colon v_1 \in V_1, \cdots, v_m \in V_m\}$$

Lemma 6. *Suppose $V_1, \cdots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is the smallest subspace of V containing $V_1, \cdots, V_m$.*

Definition 7 (Direct Sum)**.** *Suppose $V_1, \cdots, V_m$ are subspaces of V.*

- *The sum $V_1 + \cdots + V_m$ is called a* direct sum *if each element of $V_1 + \cdots + V_m$ can be written only as a sum of $v_1, \cdots, v_m$, where each $v_k \in V_k$.*
- *If $V_1 + \cdots + V_m$ is a direct sum, then $V_1 \oplus \cdots \oplus V_m$ denotes $V_1 + \cdots + V_m$, with $\oplus$ serving as the indication that this is a direct sum.*

Example. Suppose V_k is a subspace of $\mathbb{F}^n$ of those vectors whose coordinates are all zero but k-th coordinate. Then we have

$$\mathbb{F}^n = V_1 \oplus \cdots + V_m$$

Example (Sum that is not a direct sum). Suppose

$$\begin{aligned} V_1 &= \{(x, y, 0) \in \mathbb{F}^3 \colon x, y \in \mathbb{F}\} \\ V_2 &= \{(0, 0, z) \in \mathbb{F}^3 \colon z \in \mathbb{F}\} \\ V_3 &= \{(0, y, y) \in \mathbb{F}^3 \colon y \in \mathbb{F}\} \end{aligned}$$

Then $F^3 = V_1 + V_2 + V_3$ because for every $(x, y, z) \in \mathbb{F}^3$,

$$(x, y, z) = (x, y, 0) + (0, 0, z) + (0, 0, 0)$$

However, $F^3 \neq V_1 \oplus V_2 \oplus V_3$ since

$$\begin{aligned} (0, 0, 0) &= (0, -1, 0) + (0, 0, -1) + (0, 1, 1) \\ &= (0, 0, 0) + (0, 0, 0) + (0, 0, 0) \end{aligned}$$

Theorem 8. *Suppose $V_1, \dots, V_m$ are subspaces of V. Then $V_1 + \cdots + V_m$ is a direct sum if and only if the only way to write 0 as a sum of $v_1 + \cdots + v_m$, where $v_k \in V_k$, is by taking each v_k to equal 0.*

Theorem 9. *Suppose that $\mathcal{U}$ and $\mathcal{W}$ are subspaces of V. Then*

$$\mathcal{U} + \mathcal{W} \textit{ is a direct sum} \Leftrightarrow \mathcal{U} \cap \mathcal{W} = \{0\}$$

Problem 1

Verify the following examples to be valid subspaces:

1. If $b \in \mathbb{F}$, then
$$\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 \colon x_3 = 5x_4 + b\}$$
is a subspace of $\mathbb{F}^4$ if and only if $b = 0$.
2. The set of continuous real-valued functions on the interval $[0, 1]$ is a subspace of $\mathbb{R}^{[0,1]}$.
3. The set of differential real-valued functions on $\mathbb{R}$ is a subspace of $\mathbb{R}^{\mathbb{R}}$.
4. The set of differentiable real-valued functions f on the interval $(0, 3)$ such that $f'(2) = b$ is a subspace of $\mathbb{R}^{(0,3)}$ if and only if $b = 0$.
5. The set of all sequences of complex numbers with limit 0 is a subspace of $\mathbb{C}^{\infty}$.

Proof.

1. $\Rightarrow$ $(0, 0, 0, 0)$ is an element and thus $0 = 0 + b$, b = 0 $\Leftarrow$ Easy to verify.
2. 0 function is cts; cts functions are closed under addition and scalar multiplication.
3. 0 function is differentiable; differentiable functions are closed under addition and scalar multiplication.
4. For this to be closed under addition, one needs to restrict that $f'(2)+g'(2) = b + b = b$ and thus $b = 0$.
5. $\lim_{n\to\infty} a(S_1 + S_2) = \lim_{n\to\infty} aS_1 + \lim_{n\to\infty} aS_2 = 0 + 0 = 0$. At the same time, the 0 sequence has limit 0.

$\square$

Problem 4

Suppose $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

Proof. $\Rightarrow \int_0^1 f + g = \int_0^1 f + \int_0^1 g = 2b = b$ so $b = 0$.
$\Leftarrow$ 0 is in the set; closed under addition/multiplication. □

Problem 5

Prove that $\mathbb{R}^2$ is not a subspace of $\mathbb{C}^2$ over the field $\mathbb{C}$.

Proof. This does not hold for scalar multiplication since we've defined scalar to be complex numbers. To see this, Let $a = (x + yi) \in \mathbb{C}$ and $\mathbf{z} = (z_1, z_2) \in \mathbb{R}^2$. We have that $a\mathbf{z} = (z_1(x + yi), z_2(x + yi)) \notin \mathbb{R}^2$.

□

Problem 7

Prove or disprove: If $\mathcal{U}$ is a nonempty subset of $\mathbb{R}^2$ such that $\mathcal{U}$ is closed under addition and under taking additive inverse ($-u \in \mathcal{U}$), then $\mathcal{U}$ is a subspace in $\mathbb{R}^2$.

Proof. No. $\mathcal{U} = \{(x_1, x_2) \colon x_1, x_2 \in \mathbb{Z}\}$. Then $\frac{1}{2}(1, 1) \notin \mathcal{U}$. □

Problem 9

A function $f \colon \mathbb{R} \to \mathbb{R}$ is called *periodic* if there exists a positive number p s.t. $f(x) = f(x + p)$ for all $x \in \mathbb{R}$. Is the set of periodic functions from $\mathbb{R}$ to $\mathbb{R}$ a subspace of $\mathbb{R}^{\mathbb{R}}$?

Proof. No, problem occurs with the addition. Suppose we have $f(x) = f(x + p)$ and $g(x) = g(x + q)$. Then $(f + g)(x) = f(x) + g(x) = f(x + p) + g(x + q) \neq (f + g)(x + l)$ for some fixed l for all p, q.

□

Problem 11

Prove that the intersection of every collection of subspaces of V is a subspace of V.

Proof. Let $\bigcap_i V_i$ denote the collection of subspaces of V. Then we know $0 \in$ $bigcup_i V_i$. Let $a \in \mathbb{F}, \mathbf{x}, \mathbf{y} \in \bigcap_i V_i$. We have that $a(\mathbf{x} + \mathbf{y}) \in \bigcap_i V_i$ and thus finish the proof. □

Problem 12

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Proof. Let V_1, V_2 be two subspaces of V.
$\Rightarrow$ Suppose for the sake of contradiction that there exists $v_1 \in V_1$ s.t. $v_1 \notin V_2$ and $v_2 \in V_2$ s.t. $v_2 \notin V_1$. Then by assumption we have that $v_1 + v_2 \in V_1 \cup V_2$. Here

we can also show that $v_1 + v_2 \notin V_1$ because if it does, $v_1 + v_2 + (-v_1) = v_2 \in V_1$. Similarily, $v_1 + v_2 \notin V_2$. Thus we've reached a contradiction.

$\Leftarrow$ This direction is trivial. $\square$

Problem 14

Suppose

$$\mathcal{U} = \{(x, -x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\} \text{ and } \mathcal{W} = \{(x, x, 2x) \in \mathbb{F}^3 : x \in \mathbb{F}\}$$

Describe $\mathcal{U} + \mathcal{W}$.

Proof.

$$(x, -x, 2x) + (y, y, 2y) = (x + y, -x + y, 2(x + y))$$

One can think of this as $\mathcal{U} + \mathcal{W} = \{(a, b, 2a) : a, b \in \mathbb{F}\}$. $\square$

Problem 15

Suppose $\mathcal{U}$ is a subspace of V, what is $\mathcal{U} + \mathcal{U}$?

Proof. $\mathcal{U} + \mathcal{U} = \mathcal{U}$.

Take $\mathbf{u}_1 + \mathbf{u}_2 \in \mathcal{U} + \mathcal{U}$, then $\mathbf{u}_1 + \mathbf{u}_2 \in \mathcal{U}$. Conversely, take $\mathbf{u} \in \mathcal{U}$, then $\mathbf{u} = \mathbf{u} + 0 \in \mathcal{U} + \mathcal{U}$. $\square$

Problem 16

Is the operation of addition on the subspaces of V commutative ($\mathcal{U}+\mathcal{W} = \mathcal{W} + \mathcal{U}$)?

Proof. Take $u \in \mathcal{U}, w \in \mathcal{W}$, then $u + w = w + u$, implying the conclusion. $\square$

Problem 18

Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Proof. Yes, the zero subspace *i.e.* $\{0\}$ is the additive identity. The subspace that have additive inverses is only $\{0\}$. $\square$

Problem 19

Prove or disprove: If $V_1, V_2, \mathcal{U}$ are subspaces of V such that

$$V_1 + \mathcal{U} = V_2 + \mathcal{U}$$

then $V_1 = V_2$.

Proof. Counterexample: Consider when $\mathcal{U} = V_1 \bigsqcup V_2$, then the relation holds while $V_1 \neq V_2$. $\square$

Problem 20

Suppose

$$\mathcal{U} = \{(x, x, y, y) \in \mathbb{F}^4 \colon x, y \in \mathbb{F}\}$$

Find a subspace $\mathcal{W} \in \mathbb{F}^4$ st. $\mathbb{F}^4 = \mathcal{U} \oplus \mathcal{W}$.

Proof. Define $\mathcal{W} = \{(0, a, b, 0) \in \mathbb{F}^4 \colon a, b \in \mathbb{F}\}$ to be a subspace of $\mathbb{F}^4$.

Then first $\mathcal{W} + \mathcal{U} \subseteq \mathbb{F}^4$. Take $(q, w, e, r) \in \mathbb{F}^4$, we can have $(q, w, e, r) = (q, q, r, r) + (0, w - q, e - r, r) \in \mathcal{U} + \mathcal{W}$. We have $\mathbb{F}^4 = \mathcal{U} + \mathcal{W}$. Furthermore, take $(x, x, y, y) \in \mathcal{U}, (0, a, b, 0) \in \mathcal{W}$. For the element to be in the intersection, we need to have $(x, x, y, y) = (0, a, b, 0)$ which implies that $x = y = a = b = 0$ and thus $\mathcal{W} \cap \mathcal{U} = \{0\}$. □

Problem 21

Suppose

$$\mathcal{U} = \{x, y, x + y, x - y, 2x\} \in \mathbb{F}^5 \colon x, y \in \mathbb{F}$$

Find a subspace $\mathcal{W} \in \mathbb{F}^5$ s.t. $\mathbb{F}^5 = \mathcal{U} \oplus \mathcal{W}$.

Proof. Define

$$\mathcal{W} = \{(0, 0, m, n, z) \colon m, n, z \in \mathbb{F}\}$$

Then $(a, b, c, d, e) = (a, b, a + b, a - b, 2a) + (0, 0, c - (a + b), d - (a - b), e - 2a)$. The rest follows exactly as in P20. □

Problem 23

Porve or disprove: If $V_1, V_2, \mathcal{U}$ are subspaces of V s.t.

$$V = V_1 \oplus \mathcal{U} \text{ and } V = V_2 \oplus \mathcal{U}$$

,
then $V_1 = V_2$

Proof. Counterexample: Let $\mathcal{U} = \{(x, x) \colon x \in \mathbb{F}\}, V_1 = \{(x, 0) \colon x \in \mathbb{F}\}, V_2 = \{(0, x) \colon x \in \mathbb{F}\}$. □

Problem 24

A function $f \colon \mathbb{R} \to \mathbb{R}$ is called *even* if $f(-x) = f(x)$ and *odd* if $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Let V_e denote the set of real-valued even functions on $\mathbb{R}$ and let V_o denote the set of real-valued odd functions on $\mathbb{R}$. Show that $\mathbb{R}^{\mathbb{R}} = V_e \oplus V_o$.

Proof. $\Leftarrow$ Trivial direction.

$\Rightarrow f(x) = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$ □