

# Chapter 3: Linear Maps

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### 3A: Vector Space of Linear Maps

**Definition 1** (Linear Map). *A linear map from  $V$  to  $W$  is a function  $T: V \rightarrow W$  with the following properties:*

- **Additivity:**  $T(u + v) = Tu + Tv$  for all  $u, v \in V$ .
- **Homogeneity:**  $T(\lambda v) = \lambda T(v)$  for all  $\lambda \in \mathbb{F}$  and  $v \in V$ .

Notation:  $\mathcal{L}(V, W), \mathcal{L}(V)$

- The set of linear maps from  $V$  to  $W$  is denoted by  $\mathcal{L}(V, W)$ .
- The set of linear maps from  $V$  to  $V$  is denoted by  $\mathcal{L}(V)$ .

**Lemma 2** (linear map basis lemma). *Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_n \in W$ . Then there exists a unique linear map  $T: V \rightarrow W$  such that*

$$Tv_k = w_k$$

*for each  $k = 1, \dots, n$ .*

**Definition 3** (additional and scalar multiplication on  $\mathcal{L}(V, W)$ ). *Suppose  $S, T \in \mathcal{L}(V, W)$  and  $\lambda \in \mathbb{F}$ . The sum  $S + T$  and the product  $\lambda T$  are the linear maps from  $V$  to  $W$  defined by*

$$(S + T)(v) = Sv + Tv \text{ and } (\lambda T)(v) = \lambda(Tv)$$

*for all  $v \in V$ .*

**Remark 4.**  $\mathcal{L}(V, W)$  is a vector space.

**Definition 5** (product of linear maps). *If  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$ , then the product  $ST \in \mathcal{L}(U, W)$  is defined by*

$$(ST)(u) = S(Tu)$$

*for all  $u \in U$ .*

**Remark 6** (algebraic properties of product of linear maps). *We have associativity, identity and distributive properties whenever such properties are defined.*

**Theorem 7** (linear maps take 0 to 0). *Suppose  $T$  is a linear map from  $V$  to  $W$ . Then  $T(0) = 0$ .*

**Problem 1**

Suppose  $b, c \in \mathbb{R}$ . Define  $T: \mathbb{R}^3 \rightarrow \mathbb{R}^2$  by

$$T(x, y, z) = (2x - 4y + 3z + b, 6x + cxyz)$$

Show that  $T$  is linear if and only if  $b = c = 0$ .

*Proof.*  $T$  is linear  $\Leftrightarrow T(x, y, z = 0) \Leftrightarrow b = 0$

In addition, let's only consider the second coordinate, then we have

$$T(x_1 + x_2, y_1 + y_2, z_1 + z_2)_2 = 6(x_1 + x_2) + c(x_1 + x_2)(y_1 + y_2)(z_1 + z_2)$$

which only equals  $T(x_1, y_1, z_1) + T(x_2, y_2, z_2)$  if  $c = 0$ .  $\square$

**Problem 3**

Suppose that  $T \in \mathcal{L}(\mathbb{F}^n, \mathbb{F}^m)$ . Show that there exist scalars  $A_{j,k} \in \mathbb{F}$  for  $j = 1, \dots, m$  and  $k = 1, \dots, n$  such that

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$$

for every  $(x_1, \dots, x_n) \in \mathbb{F}^n$ .

*Proof.* Let  $\{u_1, \dots, u_n\}$  denote the standard basis of  $\mathbb{F}^n$ . We have that

$$Tu_i = (A_{1,i}, \dots, A_{m,i})$$

Take arbitrary  $(x_1, \dots, x_n) \in \mathbb{F}^n$ , we have that

$$Tx_i u_i = x_i(A_{1,i}, \dots, A_{m,i})$$

thus we have that

$$T(x_1, \dots, x_n) = (A_{1,1}x_1 + \dots + A_{1,n}x_n, \dots, A_{m,1}x_1 + \dots + A_{m,n}x_n)$$

$\square$

**Problem 4**

Suppose  $T \in \mathcal{L}(V, W)$  and  $v_1, \dots, v_m$  is a list of vectors in  $V$  such that  $Tv_1, \dots, Tv_m$  is a linearly independent list in  $W$ . Prove that  $v_1, \dots, v_m$  is linearly independent.

*Proof.* This means that the only solution to  $\sum_{i=1}^m a_i T(v_i) = \sum_{i=1}^m T(a_i v_i) = 0$  is all  $a_i = 0$ . However, as we know  $T(0) = 0$  so  $a_i v_i = 0$  and thus we've proved the claim.  $\square$

**Problem 7**

Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if  $\dim V = 1$  and  $T \in \mathcal{L}(V)$ , then there exists  $\lambda \in \mathbb{F}$  such that  $Tv = \lambda v$  for all  $v \in V$ .

*Proof.* Since  $\dim V = 1$ , every  $v \in V$  can be expressed as  $\lambda v'$  for some other  $v' \in V$ . As  $Tv \in V$ , we have  $Tv = \beta v$ . Then  $Tv = T\lambda v' = \beta\lambda v' = \beta v$   $\square$

**Problem 8**

Give an example of a function  $\phi: \mathbb{R}^2 \rightarrow \mathbb{R}$  such that

$$\phi(av) = a\phi(v)$$

for all  $a \in \mathbb{R}$  and all  $v \in \mathbb{R}^2$  but  $\phi$  is not linear.

*Proof.* Consider  $f(x_1, x_2) = x_1^2/x_2$  if  $x_2 \neq 0$  o.w. 0, then  $f(a(x_1, x_2)) = ax_1^2 = af(x_1, x_2)$ . However,  $f((x_1 + y_1, x_2 + y_2)) = (x_1 + y_1)^2/(x_2 + y_2) \neq f(x_1, x_2) + f(y_1, y_2)$ .  $\square$

**Problem 11**

Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Prove that  $T$  is a scalar multiple of the identity if and only if  $ST = TS$  for all  $S \in \mathcal{L}(V)$ .

*Proof.*  $\Rightarrow$  We can express  $T = \lambda I$ . Then  $S(\lambda I) = \lambda S = \lambda IS = TS$ .

$\Leftarrow$  Let  $v_1, \dots, v_m$  be a basis of  $V$ . Pick  $S_i$  such that  $S_i(\sum_{i=1}^m a_i v_i) = a_i v_i$ , which is clearly a linear operator. Then we have that

$$\begin{aligned} S_i T(v) &= T S_i(v) \\ S_i \sum_{j=1}^m b_j v_j &= T(a_i v_i) \\ b_i v_i &= a_i T(v_i) \end{aligned}$$

This shows that for all  $v_i$ , there exists  $\lambda_i$  such that  $T(v_i) = \lambda_i v_i$ . Next we show that such  $\lambda_i$  does not depend on  $i$ . Construct  $S_{ij}$  subtly such that  $S_{ij} \sum_{k=1}^n a_k v_k = a_j v_i + a_i v_j$ .

Then we have that

$$\begin{aligned} S_{ij} T v &= T S_{ij} v \\ S_{ij} \left( \sum_{k=1}^n \lambda_k a_k v_k \right) &= T(a_j v_i + a_i v_j) \\ \lambda_j a_j v_i + \lambda_i a_i v_j &= \lambda_i a_j v_i + \lambda_j a_i v_j \end{aligned}$$

This shows that  $\lambda_i = \lambda_j$  for all  $i, j$  and thus we've shown that  $T = \lambda I$  for some  $\lambda$ .

**Problem 12**

Suppose  $U$  is a subspace of  $V$  with  $U \neq V$ . Suppose  $S \in \mathcal{L}(U, W)$  and  $S \neq 0$ . Define  $T: V \rightarrow W$  by

$$Tv = \begin{cases} Sv & \text{if } v \in U \\ 0 & \text{if } v \in V \text{ and } v \notin U \end{cases}$$

Prove that  $T$  is not a linear map on  $V$ .

*Proof.* Take  $u \in U$  such that  $u \notin \text{Null}(S)$ . Take  $v \in V \setminus U$ , then  $u + v \in V \setminus U$ . This means that

$$T(u + v) = S(u + v) = 0 \neq T(u) + T(v) = Su$$

□

**Problem 13**

Suppose  $V$  is finite-dimensional. Prove that every linear map on a subspace of  $V$  can be extended to a linear map on  $V$ . In other words, show that if  $U$  is a subspace of  $V$  and  $S \in \mathcal{L}(U, W)$ , then there exists  $T \in \mathcal{L}(V, W)$  such that  $Tu = Su$  for all  $u \in U$ .

*Proof.* Note that there exists subspace  $P$  s.t.  $V = P \oplus U$ . For all  $v \in V$ ,  $v = p + u$  for some  $p, u$ . Then define  $T(v) = Su + p$ . Clearly we have that  $T(u) = Su$  for all  $u \in U$ . It now only suffices to prove  $T$  is a linear map. Homogeneity is trivial to show. For additivity,  $T(v_1 + v_2) = T(u_1 + u_2 + p_1 + p_2) = S(u_1 + u_2) + p_1 + p_2 = (Su_1 + p_1) + (Su_2 + p_2) = Tv_1 + Tv_2$ .

□

**Problem 14**

Suppose  $V$  is finite-dimensional with  $\dim V > 0$ , and suppose  $W$  is infinite-dimensional. Prove that  $\mathcal{L}(V, W)$  is infinite-dimensional.

*Proof.* Recall the definition of infinite-dimension from ch2 P17:

$V$  is infinite-dimensional if and only if there is a sequence  $v_1, v_2, \dots$  of vectors in  $V$  such that  $v_1, \dots, v_m$  is linearly independent for every positive integer  $m$ .

$W$  being infinite-dimensional implies this. Denote the sequence by  $w_1, w_2, \dots$ . Let  $v_1, \dots, v_m$  be the basis for  $V$ . Define a sequence of linear operators as follows:  $T_k \in \mathcal{L}(V, W)$  such that  $T_k(v) = v_k$ . Then we have that for every positive integer  $m$ , the solution for the following equation is all  $a_i = 0$ .

$$a_1 T_1(v) + \dots + a_m T_m(v) = a_1 v_1 + \dots + a_m v_m = 0$$

This shows that  $\mathcal{L}(V, W)$  is infinite-dimensional. □

**Problem 15**

Suppose  $v_1, \dots, v_m$  is a linearly dependent list of vectors in  $V$ . Suppose also that  $W \neq \{0\}$ . Prove that there exist  $w_1, \dots, w_m \in W$  such that no  $T \in \mathcal{L}(V, W)$  satisfies  $Tv_k = w_k$  for each  $k = 1, \dots, m$ .

*Proof.* Take  $w_1, \dots, w_m$  to be linearly independent list of vectors in  $W$ . Then we have that

$$Ta_1v_1 + \dots + Ta_mv_m = a_1w_1 + \dots + a_mw_m = 0$$

The only solution is that  $a_i = 0$  for all  $i$ , but this contradicts that  $v_1, \dots, v_m$  is linearly dependent. □

**Problem 16**

Suppose  $V$  is finite-dimensional with  $\dim V > 1$ . Prove that there exist  $S, T \in \mathcal{L}(V)$  such that  $ST \neq TS$ .

*Proof.* Let  $v_1, \dots, v_m$  be the basis of  $V$ . Define  $S(v) = S(\sum_{i=1}^m a_i v_i) = \sum_{i=1}^m a_{m-i} v_i$  and  $T(v) = a_1 v_1$ . Then we have that

$$STv = Sa_1v_1 = a_1v_1$$

but

$$TSv = T \sum_{i=1}^m a_{m-i} v_i = a_m v_1$$

□

**Problem 17**

Suppose  $V$  is finite-dimensional. Show that the only two-sided ideals of  $\mathcal{L}(V)$  are  $\{0\}$  and  $\mathcal{L}(V)$ , where we define that a subspace  $\mathcal{E}$  of  $\mathcal{L}(V)$  is called two-sided ideal if  $TE \in \mathcal{E}$  and  $ET \in \mathcal{E}$  for all  $E \in \mathcal{E}$  and all  $T \in \mathcal{L}(V)$ .

*Proof.* It's easy to verify that  $\{0\}$  and  $\mathcal{L}(V)$  are two-sided ideal of  $\mathcal{L}(V)$ . Suppose for the sake of contradiction that such  $\mathcal{E}$  exists. Let  $e_1, \dots, e_m$  be its basis and let  $e_1, \dots, e_m, e_{m+1}, \dots, e_n$  be the basis for  $V$ . Define  $T(v) = T(\sum_{i=1}^n a_i e_i) = \sum_{i=1}^n a_i e_{n-i}$  to be a linear map in  $V$  (ez to verify), and  $E_j(u) = E_j(\sum_{k=1}^m b_k e_k) = b_j e_j$ . We have reached the contradicting example such that  $TE_j(v) = a_j e_{n-j} \in V \setminus \mathcal{E}$ . □

□

### 3B: Null Spaces and Ranges

**Definition 8** (null space). For  $T \in \mathcal{L}(V, W)$ , the null space of  $T$ , denoted by  $\text{null } T$ , is the subset of  $V$  consisting of those vectors that  $T$  maps to 0:

$$\text{null } T = \{v \in V : Tv = 0\}$$

**Corollary 9.** Suppose  $T \in \mathcal{L}(V, W)$ , then  $\text{null } T$  is a subspace of  $V$ .

**Definition 10** (injective). A function  $T: V \rightarrow W$  is called **injective** if  $Tu = Tv$  implies  $u = v$ .

**Theorem 11.** Let  $T \in \mathcal{L}(V, W)$ . Then  $T$  is injective if and only if  $\text{null } T = \{0\}$ .

**Definition 12** (range). For  $T \in \mathcal{L}(V, W)$ , the **range** of  $T$  is the subset of  $W$  consisting of those vectors that are equal to  $Tv$  for some  $v \in V$ :

$$\text{range } T = \{Tv : v \in V\}$$

**Corollary 13.** Suppose  $T \in \mathcal{L}(V, W)$ , then  $\text{range } T$  is a subspace of  $W$ .

**Definition 14** (surjective). A function  $T: V \rightarrow W$  is called **surjective** if its range equals  $W$ .

**Theorem 15** (fundamental theorem of linear map). Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then  $\text{range } T$  is finite-dimensional and

$$\dim V = \dim \text{null } T + \dim \text{range } T$$

*Proof.* Let  $u_1, \dots, u_m$  be a basis of  $\text{null } T$  and let  $u_1, \dots, u_m, v_1, \dots, v_n$  be a basis of  $V$ . It now suffices to prove  $\dim \text{range } T = n$ . Let  $v \in T$ , then

$$v = \sum_{i=1}^m a_i u_i + \sum_{j=1}^n b_j v_j$$

We apply  $T$  on both sides to get that

$$Tv = \sum_{j=1}^n b_j Tv_j$$

which shows that  $Tv_1, \dots, Tv_n$  spans  $\text{range } T$  and thus it's finite-dimensional. To show they are linearly independent, we have

$$\sum_{j=1}^n b_j Tv_j = T \sum_{j=1}^n b_j v_j = 0$$

The only solution is that all  $b_j = 0$ .

□

**Corollary 16** (linear map to a lower-dimen space is not injective). *Suppose  $V$  and  $W$  are finite-dimensional vector spaces such that  $\dim V > \dim W$ . Then no linear map from  $V$  to  $W$  is injective.*

**Corollary 17** (linear map to a higher-dimen space is not surjective). *Suppose  $V$  and  $W$  are finite-dimensional vector spaces such that  $\dim V < \dim W$ . Then no linear map from  $V$  to  $W$  is surjective.*

**Application** Consider the system of linear equation defined by the map  $T: \mathbb{F}^n \rightarrow \mathbb{F}^m$ :

$$T(x_1, \dots, x_n) = \left( \sum_{k=1}^n A_{1,k} x_k, \dots, \sum_{k=1}^n A_{m,k} x_k \right)$$

**Corollary 18** (homogeneous systems of linear equations). *A homogeneous system of linear equations with more variables than equations has nonzero solutions.*

**Corollary 19** (inhomogeneous system fo linear equations). *An inhomogeneous system of linear equations with more equations than variables has no solution for some choice of the constant terms.*

**Problem 1**

Give an example of a linear map  $T$  with  $\dim \text{null } T = 3$  and  $\dim \text{range } T = 2$ .

*Proof.* Consider  $T(x_1, x_2, x_3, x_4, x_5) = (0, 0, 0, x_4, x_5)$ .

It's easy to verify that  $\dim \text{null } T = 3$  and applying the theorem of linear map solves the problem.  $\square$

**Problem 2**

Suppose  $S, T \in \mathcal{L}(V)$  are such that  $\text{range } S \subseteq \text{null } T$ . Prove that  $(ST)^2 = 0$ .

*Proof.* This means that take  $x \in V$ ,  $T(S(x)) = 0$ . We have  $(ST)^2 = STST = 0$ .  $\square$

**Problem 3**

Suppose  $v_1, \dots, v_m$  is a list of vectors in  $V$ . Define  $T \in \mathcal{L}(\mathbb{F}^m, V)$  by

$$T(z_1, \dots, z_m) = z_1 v_1 + \dots + z_m v_m$$

- (a) what property of  $T$  corresponds to  $v_1, \dots, v_m$  spanning  $V$ ?
- (b) what property of  $T$  corresponds to  $v_1, \dots, v_m$  being linearly independent?



*Proof.* (a) surjective (range =  $V$ )

(b) injective ( $z_1, \dots, z_m$  is identically zero if and only if  $z_1 v_1 + \dots + z_m v_m = 0$ )  $\square$

#### Problem 4

Show that  $\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \text{null } T > 2\}$  is not a subspace of  $\mathcal{L}(\mathbb{R}^5, \mathbb{R}^4)$ .

*Proof.* Try to come up with a counterexample. Consider  $f(e_1, e_2, e_3, e_4, e_5) = (e_1, 0, 0, e_4)$  and  $g(e_1, e_2, e_3, e_4, e_5) = (0, e_2, e_3, 0)$ . Then we have that  $\text{null } f = \{0, e_2, e_3, 0, e_5\}$  and  $\text{null } g = \{e_1, 0, 0, e_4, e_5\}$  so both in  $\{T \in \mathcal{L}(\mathbb{R}^5, \mathbb{R}^4) : \dim \text{null } T > 2\}$ . However,  $f + g(e_1, e_2, e_3, e_4, e_5) = \{(e_1, e_2, e_3, e_4)\}$  and this means their  $\dim \text{null} = 5 - 4 = 1 \nmid 3$ .  $\square$

#### Problem 7

Suppose  $V$  and  $W$  are finite-dimensional and  $2 \leq \dim V \leq \dim W$ . Show that  $\{T \in \mathcal{L}(V, W) : T \text{ is not injective}\}$  is not a subspace of  $\mathcal{L}(V, W)$ .

*Proof.*  $T$  is not injective means that  $\dim V > \dim W$ , which contradicts the assumption in the question.  $\square$

#### Problem 9

Suppose  $T \in \mathcal{L}(V, W)$  is injective and  $v_1, \dots, v_n$  is linearly independent in  $V$ . Prove that  $Tv_1, \dots, Tv_n$  is linearly independent in  $W$ .

*Proof.*

$$\sum_{i=1}^n z_i v_i = 0 \Leftrightarrow \sum_{i=1}^n z_i T(v_i) = 0$$

$\square$

#### Problem 10

Suppose  $v_1, \dots, v_n$  spans  $V$  and  $T \in \mathcal{L}(V, W)$ . Show that  $Tv_1, \dots, Tv_n$  spans  $\text{range } T$ .

*Proof.* Take  $w$  in  $\text{range } T$ , then we know  $w = \sum_{i=1}^n a_i T(v_i)$ . Take  $w$  in  $\text{range } T$ . Then we know there exists  $v \in V$  such that  $w = T(v)$  and thus  $w = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i T(v_i)$ .  $\square$

**Problem 11**

Suppose that  $V$  is finite-dimensional and that  $T \in \mathcal{L}(V, W)$ . Prove that there exists a subspace  $\mathcal{U}$  of  $V$  such that

$$\mathcal{U} \cap \text{null } T = \{0\} \text{ and } \text{range } T = \{Tu : u \in \mathcal{U}\}$$

*Proof.* Since we know  $\text{null } T$  is a subspace of  $V$ , then there exists  $\mathcal{U}$  such that  $\mathcal{U} \oplus \text{null } T = V$ . We can define  $\mathcal{U}$  to be such case. To finish the proof, let  $v = u + t, u \in \mathcal{U}, t \in \text{null } T$ , then  $\text{range } T = \{T(u + t) = Tu : u \in \mathcal{U}\}$ .  $\square$

**Problem 16**

Suppose  $V$  and  $W$  are both finite-dimensional. Prove that there exists an injective linear map from  $V$  to  $W$  if and only if  $\dim V \leq \dim W$ .

*Proof.*  $\Rightarrow T \in \mathcal{L}(V, W)$  is injective. Then this means that  $\dim V = \dim \text{range } T \leq \dim W$ .

$\Leftarrow$  By assumption, we can define a linear map  $T$  such that  $Tv_i = w_i$  where  $v_i, w_i$  are the respective basis of  $V$  and  $W$ . Then we have that

$$\sum_{i=1}^n a_i Tv_i = \sum_{i=1}^n a_i w_i = 0 \Leftrightarrow a_i \text{ are identically zero}$$

This means that  $\text{null } T$  is  $\{0\}$  and thus it is injective.  $\square$

**Problem 17**

Suppose  $V$  and  $W$  are both finite-dimensional. Prove that there exists a surjective linear map from  $V$  onto  $W$  if and only if  $\dim V \geq \dim W$ .

*Proof.*  $\Rightarrow \dim V \geq \dim \text{range } T = \dim W$ .

$\Leftarrow$  By assumption we can define a linear map  $T$  such that  $Tv_i = w_i$  for  $1 \leq i \leq \dim W$  and  $Tv_i = 0$  for  $\dim W < i \leq \dim V$ . This means that take  $w \in W$ , then  $w = \sum_{i=1}^{\dim W} a_i w_i$ . At the same time, we know that for all  $w_i$ , there exists  $v_i$  s.t.  $Tv_i = w_i$ , therefore we have  $w = \sum_{i=1}^{\dim W} a_i Tv_i$  and thus  $T$  is surjective.  $\square$

**Problem 18**

Suppose  $V$  and  $W$  are finite-dimensional and that  $\mathcal{U}$  is a subspace of  $V$ . Prove that there exists  $T \in \mathcal{L}(V, W)$  such that  $\text{null } T = \mathcal{U}$  if and only if  $\dim \mathcal{U} \geq \dim V - \dim W$ .

*Proof.* We know  $\dim V = \dim \text{null } T + \dim \text{range } T$  and  $\dim \text{range } T \leq \dim W$ . Therefore,  $\dim \mathcal{U} = \dim \text{null } T \geq \dim V - \dim W$ .  $\square$

**Problem 19**

Suppose  $W$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Prove that  $T$  is injective if and only if there exists  $S \in \mathcal{L}(W, V)$  such that  $ST$  is the identity operator on  $V$ .

*Proof.*  $T$  is injective  $\Leftrightarrow T(v)$  is unique for  $v \in V$ .  $\Leftrightarrow$  We can define  $S: STv = v$ .  $\square$

**Problem 21**

Suppose  $V$  is finite-dimensional,  $T \in \mathcal{L}(V, W)$ , and  $\mathcal{U}$  is a subspace of  $W$ . Prove that  $\mathcal{T} = \{v \in V: Tv \in \mathcal{U}\}$  is a subspace of  $V$  and

$$\dim \mathcal{T} = \dim \text{null } T + \dim(\mathcal{U} \cap \text{range } T)$$

*Proof.* To prove subspace, we can simply follow by definition. We can define  $S \in \mathcal{L}(\mathcal{T}, \mathcal{U})$  such that  $Sv = Tv$  for all  $v \in \mathcal{T}$ . Then we have that  $\text{range}(S) \in \mathcal{U} \cap \text{range}(T)$  and that  $\text{null } T = \text{null } S$ . Hence we have proved the claim.  $\square$

**Problem 22**

Suppose  $\mathcal{U}$  and  $V$  are finite-dimensional vector spaces and  $S \in \mathcal{L}(V, W)$  and  $T \in \mathcal{L}(\mathcal{U}, V)$ . Prove that

$$\dim \text{null } ST \leq \dim \text{null } S + \dim \text{null } T$$

*Proof.* We can write that

$$\text{null } ST = \{u \in \mathcal{U}: T(u) \in \text{null } (S)\}$$

By the previous question we know that

$$\dim \text{null } ST = \dim \text{null } T + \dim \text{null } S \cap \text{range } T \leq \dim \text{null } T + \dim \text{null } S$$

$\square$

**Problem 23**

Suppose  $\mathcal{U}$  and  $V$  are finite-dimensional vector spaces and  $S \in \mathcal{L}(V, W)$  and  $T \in \mathcal{L}(\mathcal{U}, V)$ . Prove that

$$\dim \text{range } ST \leq \min\{\dim \text{range } S, \dim \text{range } T\}$$

*Proof.* First note that  $\dim \text{range } ST \leq \dim \text{range } S$ . This because  $\text{range } ST \subseteq \text{range } S$ .

To prove  $\dim \text{range } ST \leq \dim \text{range } T$ . We have that  $\dim \mathcal{U} = \dim \text{null } T + \dim \text{range } T = \dim \text{null } ST + \dim \text{range } ST$ . Since we know  $\dim \text{null } T \geq \dim \text{null } ST$ ,  $\dim \text{range } ST \leq \dim \text{range } T$ .  $\square$

**Problem 24**

- (a) Suppose  $\dim V = 5$  and  $S, T \in \mathcal{L}(V)$  are such that  $ST = 0$ . Prove that  $\dim \text{range } TS \leq 2$ .
- (b) Give an example of  $S, T \in \mathcal{L}(\mathbb{F}^5)$  with  $ST = 0$  and  $\dim \text{range } TS = 2$ .

*Proof.* (a)  $\dim \text{null } ST = 5 \leq \dim \text{null } T + \dim \text{null } S$

We also know that  $5 = \dim \text{null } T + \dim \text{range } T = \dim \text{range } S + \dim \text{null } S$  and thus we have that  $\dim \text{range } T + \dim \text{range } S \leq 5$ . This implies that  $\min\{\dim \text{range } T, \dim \text{range } S\} \leq 2$ . Hence by applying P23 we finish the proof.

(b) Consider  $T(x_1, x_2, x_3, x_4, x_5) = (x_3, x_4, 0, 0, 0)$ ,  $S(x_1, x_2, x_3, x_4, x_5) = (0, 0, x_3, x_4, x_5)$ . Then we have that  $ST(x_1, x_2, x_3, x_4, x_5) = S(x_3, x_4, 0, 0, 0) = (0, 0, 0, 0, 0)$  while  $TS(x_1, x_2, x_3, x_4, x_5) = T(0, 0, x_3, x_4, x_5) = (x_3, x_4, 0, 0, 0)$ .  $\square$

**Problem 25**

Suppose that  $W$  is finite-dimensional and  $S, T \in \mathcal{L}(V, W)$ . Prove that  $\text{null } S \subseteq \text{null } T$  if and only if there exists  $E \in \mathcal{L}(W)$  such that  $T = ES$ .

*Proof.*  $\text{null } S \subseteq \text{null } T \Rightarrow \dim \text{range } T \leq \dim \text{range } S \Rightarrow$  let  $s_1, \dots, s_n$  be basis of range  $S$  and  $t_1, \dots, t_m$  be basis of range  $T$  where  $m \leq n$ . We can always define  $E \in \mathcal{L}(W)$  such that  $E(s_i) = t_i$  for all  $1 \leq i \leq m$  and 0 otherwise.

The other direction is trivial.  $\square$

**Problem 26**

Suppose that  $V$  is finite-dimensional and  $S, T \in \mathcal{L}(V, W)$ . Prove that  $\text{range } S \subseteq \text{range } T$  if and only if there exists  $E \in \mathcal{L}(V)$  such that  $S = TE$ .

*Proof.*  $\text{range } S \subseteq \text{range } T \Rightarrow$  Take  $v_1, \dots, v_m$  to be the basis of  $V$ , define the linear map  $Sv_i = s_i$  for  $s_i \in \text{range } S \subseteq \text{range } T$ . Then there exists  $u_1, \dots, u_m$  such that  $Tu_i = s_i$  for all  $i$ . Then we can define the linear map  $Ev_i = u_i$  for all  $i$  such that  $S = TE$ .

The other direction is trivial.  $\square$

**Problem 27**

Suppose  $P \in \mathcal{L}(V)$  and  $P^2 = P$ . Prove that  $V = \text{null } P \oplus \text{range } P$ .

*Proof.* Take  $v \in \text{null } (P) \cap \text{range } (P)$ . Then this means that  $P(v) = 0$  and there exists  $u$  s.t.  $v = P(u)$ . At the same time,  $P(v) = P(P(u)) = P(u) = 0 = v$ . So we have that  $\text{null } P \cap \text{range } P = \{0\}$ . Since  $P$  is defined on  $\mathcal{L}(P)$ , we have that  $V = \text{null } P \oplus \text{range } P$ .  $\square$

### 3C: Matrices

**Definition 20** (matrix,  $\mathbf{A}_{j,k}$ ). Suppose  $m$  and  $n$  are nonnegative integers. An  $m$ -by- $n$  matrix  $\mathbf{A}$  is a rectangular array of elements of  $\mathbb{F}$  with  $m$  rows and  $n$  columns:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{1,1} & \cdots & \mathbf{A}_{1,n} \\ \vdots & & \vdots \\ \mathbf{A}_{m,1} & \cdots & \mathbf{A}_{m,n} \end{pmatrix}$$

where the notation  $\mathbf{A}_{j,k}$  denotes the entry in row  $j$  and column  $k$ .

**Definition 21** (matrix of a linear map,  $\mathcal{M}(T)$ ). Suppose  $T \in \mathcal{L}(V, W)$  and  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . The matrix of  $T$  with respect to these bases is the  $m$ -by- $n$  matrix  $\mathcal{M}(T)$  whose entries  $\mathbf{A}_{j,k}$  are defined by

$$Tv_k = \mathbf{A}_{1,k}w_1 + \cdots + \mathbf{A}_{m,k}w_m$$

If the bases  $v_1, \dots, v_n$  and  $w_1, \dots, w_m$  are not clear from the context, then the notation  $\mathcal{M}(T, (v_1, \dots, v_n), (w_1, \dots, w_m))$  is used.

**Remark 22.** The  $k$ -th column of  $\mathcal{M}(T)$  consists of the scalars needed to write  $Tv_k$  as a linear combination of  $w_1, \dots, w_m$ :

$$Tv_k = \sum_{j=1}^m \mathbf{A}_{j,k}w_j$$

**Remark 23.** If  $T$  is a linear map from  $n$ -dimensional vector space to an  $m$ -dimensional vector space, then  $\mathcal{M}(T)$  is an  $m$ -by- $n$  matrix.

**Corollary 24** (Matrix addition and scalar multiplication). Suppose  $S, T \in \mathcal{L}(V, W)$  and  $\lambda \in \mathbb{F}$ . Then  $\mathcal{M}(\lambda S + T) = \lambda \mathcal{M}(S) + \mathcal{M}(T)$ .

For  $m$  and  $n$  positive integers, the set of all  $m$ -by- $n$  matrices with entries in  $F$  is denoted by  $\mathbb{F}^{m,n}$ .

**Theorem 25.** Suppose  $m$  and  $n$  are positive integers.  $\mathbb{F}^{m,n}$  is a vector space of dimension  $mn$ .

**Definition 26** (matrix multiplication). Suppose  $\mathbf{A}$  is an  $m$ -by- $n$  matrix and  $\mathbf{B}$  is an  $n$ -by- $p$  matrix. Then  $\mathbf{AB}$  is defined to be the  $m$ -by- $p$  matrix whose entry in row  $j$ , column  $k$ , is given by the equation

$$(\mathbf{AB})_{j,k} = \sum_{r=1}^n \mathbf{A}_{j,r} \mathbf{B}_{r,k}$$

In words, the entry in row  $j$ , column  $k$  of  $\mathbf{AB}$  is computed by taking row  $j$  of  $\mathbf{A}$  and column  $k$  of  $\mathbf{B}$ .

**Motivations.** Let  $v_1, \dots, v_n$  to be the basis of  $V$ ,  $w_1, \dots, w_m$  to be the basis of  $W$ , and  $u_1, \dots, u_p$  to be the basis of  $U$ . Consider linear maps  $T: U \rightarrow V$  and  $S: V \rightarrow W$ . Suppose  $\mathcal{M}(T) = \mathbf{A}$  and  $\mathcal{M}(S) = \mathbf{B}$ . For  $1 \leq k \leq p$ , we have

$$\begin{aligned} (ST)u_k &= S\left(\sum_{r=1}^n \mathbf{B}_{r,k}v_r\right) \\ &= \sum_{r=1}^n \mathbf{B}_{r,k}(Sv_r) \\ &= \sum_{r=1}^n \mathbf{B}_{r,k} \sum_{j=1}^m \mathbf{A}_{j,r}w_j \\ &= \sum_{j=1}^m \left(\sum_{r=1}^n \mathbf{A}_{j,r}\mathbf{B}_{r,k}\right)w_j \end{aligned}$$

**Theorem 27** (matrix of product of linear maps). *If  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$ , then  $\mathcal{M}(ST) = \mathcal{M}(S)\mathcal{M}(T)$ .*

**Definition 28** ( $\mathbf{A}_{j,\cdot}$ ,  $\mathbf{A}_{\cdot,k}$ ). *Suppose  $\mathbf{A}$  is an  $m$ -by- $n$  matrix*

- *If  $1 \leq j \leq m$ , then  $\mathbf{A}_{j,\cdot}$  denotes the 1-by- $n$  matrix consisting of row  $j$  of  $\mathbf{A}$ .*
- *If  $1 \leq k \leq n$ , then  $\mathbf{A}_{\cdot,k}$  denotes the  $m$ -by-1 matrix consisting of column  $k$  of  $\mathbf{A}$ .*

**Corollary 29** (entry of matrix product equals row times column). *Suppose  $\mathbf{A}$  is an  $m$ -by- $n$  matrix and  $\mathbf{B}$  is an  $n$ -by- $p$  matrix. Then*

$$(\mathbf{AB})_{j,k} = \mathbf{A}_{j,\cdot}\mathbf{B}_{\cdot,k}$$

*if  $1 \leq j \leq m$  and  $1 \leq k \leq p$ . In other words, the entry in row  $j$ , column  $k$ , of  $\mathbf{AB}$  equals (row  $j$  of  $\mathbf{A}$ ) times (column  $k$  of  $\mathbf{B}$ ).*

**Corollary 30** (column of matrix product equals matrix times column). *Suppose  $\mathbf{A}$  is an  $m$ -by- $n$  matrix and  $\mathbf{B}$  is an  $n$ -by- $p$  matrix. Then*

$$(\mathbf{AB})_{\cdot,k} = \mathbf{A}\mathbf{B}_{\cdot,k}$$

*if  $1 \leq k \leq p$ . In other words, column  $k$  of  $\mathbf{AB}$  equals  $\mathbf{A}$  times column  $k$  of  $\mathbf{B}$ .*

**Corollary 31** (linear combination of columns). *Suppose  $\mathbf{A}$  is an  $m$ -by- $n$  matrix*

*and  $\mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$  is an  $n$ -by-1 matrix. Then*

$$\mathbf{Ab} = \mathbf{b}_1\mathbf{A}_{\cdot,1} + \dots + \mathbf{b}_n\mathbf{A}_{\cdot,n}.$$

*In other words,  $\mathbf{Ab}$  is a linear combination of the columns of  $\mathbf{A}$ , with the scalars that multiply the columns coming from  $\mathbf{b}$ .*

**Theorem 32** (matrix multiplication as linear combination of columns). *Suppose  $C$  is an  $m$ -by- $c$  matrix and  $R$  is a  $c$ -by- $n$  matrix.*

- *If  $k \in \{1, \dots, n\}$ , then column  $k$  of  $CR$  is a linear combination of the columns of  $C$ , with the coefficients of this linear combination coming from column  $k$  of  $R$ .*
- *If  $j \in \{1, \dots, m\}$ , then row  $j$  of  $CR$  is a linear combination of the rows of  $R$ , with the coefficients of this linear combination coming from row  $j$  of  $C$ .*

**Definition 33** (column rank, row rank). *Suppose  $\mathbf{A}$  is an  $m$ -by- $n$  matrix with entries in  $\mathbb{F}$ .*

- *The column rank of  $\mathbf{A}$  is the dimension of the span of the columns of  $\mathbf{A}$  in  $\mathbb{F}^{m,1}$ .*
- *The row rank of  $\mathbf{A}$  is the dimension of the span of the rows of  $\mathbf{A}$  in  $\mathbb{F}^{1,n}$ .*

**Definition 34** (transpose,  $\mathbf{A}^\top$ ). *The transpose of a matrix  $\mathbf{A}$ , denoted by  $\mathbf{A}^\top$ , is the matrix obtained from  $\mathbf{A}$  by interchanging rows and columns. Specifically, if  $\mathbf{A}$  is an  $m$ -by- $n$  matrix, then  $\mathbf{A}^\top$  is the  $n$ -by- $m$  matrix whose entries are given by the equation*

$$(\mathbf{A}^\top)_{k,j} = \mathbf{A}_{j,k}$$

**Lemma 35** (column-row factorization). *Suppose  $\mathbf{A}$  is an  $m$ -by- $n$  matrix with entries in  $\mathbb{F}$  and column rank  $c \geq 1$ . Then there exists an  $m$ -by- $c$  matrix  $\mathbf{C}$  and a  $c$ -by- $n$  matrix  $\mathbf{R}$ , both with entries in  $\mathbb{F}$ , such that  $\mathbf{A} = \mathbf{CR}$ .*

*Proof.* Each column of  $\mathbf{A}$  is an  $m$ -by-1 matrix. The list  $\mathbf{A}_{\cdot,1}, \dots, \mathbf{A}_{\cdot,n}$  of columns of  $\mathbf{A}$  can be reduced to a basis of the span of the columns of  $\mathbf{A}$ . This basis has length  $c$ . The  $c$  columns in this basis can be put together to form an  $m$ -by- $c$  matrix  $\mathbf{C}$ .

If  $k \in \{1, \dots, n\}$ , then column  $k$  of  $\mathbf{A}$  is a linear combination of the columns of  $\mathbf{C}$ . Make the coefficients of this linear combination into column  $k$  of a  $c$ -by- $n$  matrix that we call  $\mathbf{R}$ . Then  $\mathbf{A} = \mathbf{CR}$ .  $\square$

**Theorem 36** (column rank equals row rank). *Suppose  $\mathbf{A} \in \mathbb{F}^{m,n}$ . Then the column rank of  $\mathbf{A}$  equals the row rank of  $\mathbf{A}$ .*

*Proof.* Let  $c$  denote the column rank of  $\mathbf{A}$ . Let  $\mathbf{A} = \mathbf{CR}$  be the column-row factorization of  $\mathbf{A}$  given by the proof before, where  $\mathbf{C}$  is  $m$ -by- $c$  and  $\mathbf{R}$  is  $c$ -by- $n$ . Then the column-row factorization lemma tells us that every row of  $\mathbf{A}$  is a linear combination of the rows of  $\mathbf{R}$ . Because  $\mathbf{R}$  has  $c$  rows, this implies that the row of  $\mathbf{A}$  is less than or equal to the column rank  $c$  of  $\mathbf{A}$ .

To prove the other direction, we can do the same thing to  $\mathbf{A}^\top$  and then we can get that column rank of  $\mathbf{A} = \text{row rank of } \mathbf{A}^\top \leq \text{column rank of } \mathbf{A}^\top = \text{row rank of } \mathbf{A}$  which we proved above.  $\square$



**Problem 1**

Suppose  $T \in \mathcal{L}(V, W)$ . Show that with respect to each choice of bases of  $V$  and  $W$ , the matrix of  $T$  has at least  $\dim \text{range } T$  nonzero entries.

*Proof.* Let  $v_1, \dots, v_n$  be the basis of  $V$  and  $w_1, \dots, w_m$  be the basis of  $W$ . Suppose there exists a matrix  $\mathbf{A} = \mathcal{M}(T)$  of  $T$  has less than  $\dim \text{range } T$  nonzero entries. This means that  $\mathbf{A}$  has at most  $\dim \text{range } T - 1$  nonzero columns so this implies that  $\dim \text{range } T < \dim \text{range } T - 1$  which forms a contradiction.  $\square$

**Problem 2**

Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Prove that  $\dim \text{range } T = 1$  if and only if there exists a basis of  $V$  and a basis of  $W$  such that with respect to these bases, all entries of  $\mathcal{M}(T)$  equal 1.

*Proof.*  $\Rightarrow$  We solve this problem through careful construction of the basis. That is, we will construct basis  $v_1, \dots, v_n$  of  $V$  and  $w_1, \dots, w_m$  of  $W$  such that  $Tv_i = w_1 + \dots + w_m$  for all  $i$ . We can achieve this again mainly because  $\dim \text{range } T = 1$ . let  $u_1, \dots, u_m$  be a set of arbitrary basis of  $W$ . Take  $w \in \text{range } T$  so we have  $w = \sum_{i=1}^m a_i u_i$ . We consider all the coefficients  $a_i$  as follows: take an index set  $I$  such that for all  $i \in I, a_i \neq 0$  and we have  $|I| = c$ . Take arbitrary  $j \in I$  and we construct the basis as follows: let  $w_j = (a_j - (m - r))u_j, w_k = a_k u_k$  for all  $k \in I, k \neq j$ , and  $w_k = u_k + u_j$  for all  $k \notin I$ . Then we have that  $w = \sum_{i=1}^m w_i$ . Since we know  $w \in \text{range } T$ , then there exists  $v_1 \in V$  s.t.  $T(v_1) = w$ . Let  $v_2, \dots, v_n$  be the basis of  $\text{null } T$ . Since we know  $\dim V = \dim \text{null } T + 1$  so  $v_1, v_2, \dots, v_n$  constitutes a basis of  $V$ . In this way, we successfully constructs the basis such that all entries of  $\mathcal{M}(T)$  equal 1.

$\Leftarrow$  There exists basis  $v_1, \dots, v_n$  of  $V$  and  $w_1, \dots, w_m$  of  $W$  such that all entries of  $\mathcal{M}(T)$  equal 1. This means that  $Tv_1 = \dots = Tv_n = w_1 + \dots + w_m$ . So we first have that  $\dim \text{range } T \neq 0$ . To prove  $\dim \text{range } T = 1$ , take arbitrary  $u_1, u_2 \in \text{range } T$ . Then we know that  $u_1 = T(\sum_{i=1}^n a_i v_i) = \sum_{i=1}^n a_i (w_1 + \dots + w_m)$  and similarly  $u_2 = \sum_{i=1}^n b_i (w_1 + \dots + w_m)$ . Here we have that

$$u_1 = \left( \sum_i^n a_i \right) / \left( \sum_i^n b_i \right) u_2$$

and thus  $\dim \text{range } T$  has to be 1-dimensional since every two arbitrary vector is simply a scalar multiple of each other.  $\square$

**Problem 3**

Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . Show that if  $S, T \in \mathcal{L}(V, W)$  and  $\lambda \in \mathbb{F}$ , then  $\mathcal{M}(\lambda S + T) = \lambda \mathcal{M}(S) + \mathcal{M}(T)$ .

*Proof.*

$$\begin{aligned} (\lambda S + T)v_k &= \lambda S(v_k) + T(v_k) \\ &= \lambda \sum_{i=1}^m \mathbf{A}_{i,k} w_i + \sum_{i=1}^m \mathbf{B}_{i,k} w_i \end{aligned}$$

□

**Problem 5**

Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Prove that there exists a basis of  $V$  and a basis of  $W$  such that with respect to these bases, all entries of  $\mathcal{M}(T)$  are 0 except that the entries in row  $k$ , column  $k$  equal 1 if  $1 \leq k \leq \dim \text{range } T$ .

*Proof.* Let  $\dim V = n, \dim W = m, \dim \text{range } T = k$ . The main idea of the proof is to construct basis  $v_1, \dots, v_n$  and  $w_1, \dots, w_m$  such that  $Tv_i = w_i$  for all  $1 \leq i \leq k$  and zero otherwise. (So the constructed  $T$  satisfies the requirement).

We proceed with obtaining the basis  $w_1, \dots, w_k$  of range  $T$  and extend this  $w_1, \dots, w_k, \dots, w_m$  to the basis of  $W$ . We know for all  $1 \leq i \leq k$ , there exists  $v_i \in V$  s.t.  $Tv_i = w_i$ . We claim that  $v_1, \dots, v_k$  are linearly independent. To prove this, see  $\sum_{j=1}^k a_j v_j = 0 \Rightarrow \sum_{j=1}^k a_j T(v_j) = \sum_{j=1}^k a_j w_j = 0$ . Similar to the proof in the fundamental theorem of linear map, we extend the basis to  $V$  through considering the null space. We further claim that (let  $K = \text{span}(v_1, \dots, v_k)$ )  $V = K \oplus \text{null } T$  (which is easy to prove). Hence, extending the basis from  $K$  with the basis from null  $T$  completes the proof. □

**Problem 6**

Suppose  $v_1, \dots, v_m$  is a basis of  $V$  and  $W$  is finite-dimensional. Suppose  $T \in \mathcal{L}(V, W)$ . Prove that there exists a basis  $w_1, \dots, w_n$  of  $W$  such that all entries in the first column of  $\mathcal{M}(T)$  [with respect to these bases] are 0 except for possibly a 1 in the first row, first column.

*Proof.* Let  $w_1 = Tv_1$  and extend  $w_1$  to basis of  $W$ . Then we automatically obtains  $T$  that gets the desired property. □

**Problem 7**

Suppose  $w_1, \dots, w_n$  is a basis of  $W$  and  $V$  is finite-dimensional. Suppose  $T \in \mathcal{L}(V, W)$ . Prove that there exists a basis  $v_1, \dots, v_m$  of  $V$  such that all entries in the first of  $\mathcal{M}(T)$  [wrt. these bases] are 0 except for possibly a 1 in the first row, first column.

*Proof.* Take arbitrary  $u_1 \in V$  and we have that  $Tu_1 = \sum_{i=1}^n a_{i,1} w_i$ . Take  $v_1 = u_1/a_{1,1}$ , then  $Tv_1 = w_1 + \sum_{j=2}^n b_{j,1} w_j$ . We can extend  $v_1, u_2, \dots, u_m$  to be

the basis of  $V$ . Then consider

$$Tu_j = \sum_{k=1}^n b_{k,j} w_k$$

Consider to let  $b_{1,j} = 0$  for all  $2 \leq j \leq n$ . To do this, let  $v_j = u_j - b_{1,j}v_1$ , then we have that  $T(v_j) = T(u_j - b_{1,j}v_1) = \sum_{k=2}^n c_{k,j} w_k$ . It now left to verify that  $v_1, v_2, \dots, v_m$  is the basis.

$$\begin{aligned} q_1 v_1 + \dots + q_m v_m &= 0 \\ q_1 v_1 + q_2(u_2 - b_{1,2}v_1) + \dots + q_m(u_m - b_{1,m}v_1) &= 0 \\ (q_1 - (q_2 b_{1,2} + \dots + q_m b_{1,m}))v_1 + q_2 u_2 + \dots + q_m u_m &= 0 \end{aligned}$$

The only solution is that  $q_1 - (q_2 b_{1,2} + \dots + q_m b_{1,m}) = q_2 = \dots = q_m = 0$  and thus we have completed the proof.  $\square$

#### Problem 8

Suppose  $\mathbf{A}$  is an  $m$ -by- $n$  matrix and  $\mathbf{B}$  is an  $n$ -by- $p$  matrix. Prove that

$$(\mathbf{AB})_{j,\cdot} = \mathbf{A}_{j,\cdot} \mathbf{B}$$

for each  $1 \leq j \leq m$ . In other words, show that row  $j$  of  $\mathbf{AB}$  equals (row  $j$  of  $\mathbf{A}$ ) times  $\mathbf{B}$ .

*Proof.* We know that

$$(\mathbf{AB})_{j,k} = \sum_{i=1}^n \mathbf{A}_{j,i} \mathbf{B}_{i,k}$$

For instance  $(\mathbf{AB})_{j,1} = \sum_{i=1}^n \mathbf{A}_{j,i} \mathbf{B}_{i,1} = \mathbf{A}_{j,\cdot} \mathbf{B}_{\cdot,1}$ . Similarly,  $(\mathbf{AB})_{j,k} = \mathbf{A}_{j,\cdot} \mathbf{B}_{\cdot,k}$  and thus by treating this we have

$$(\mathbf{AB})_{j,\cdot} = \mathbf{A}_{j,\cdot} \mathbf{B}$$

$\square$

#### Problem 9

Suppose  $\mathbf{a} = (a_1, \dots, a_n)$  is a 1-by- $n$  matrix and  $\mathbf{B}$  is an  $n$ -by- $p$  matrix. Prove that

$$\mathbf{aB} = a_1 \mathbf{B}_{1,\cdot} + \dots + a_n \mathbf{B}_{n,\cdot}$$

In other words, show that  $\mathbf{aB}$  is a linear combination of the rows of  $\mathbf{B}$ , with the scalars that multiply the rows coming from  $\mathbf{a}$ .

*Proof.* By definition of matrix multiplication, the entry in the column  $k$  of the l.h.s. equals

$$(\mathbf{aB})_{1,k} = a_1 \mathbf{B}_{1,k} + \dots + a_n \mathbf{B}_{n,k}$$

which equals the right side.  $\square$

**Problem 11**

Prove that the distributive property holds for matrix addition and multiplication. In other words, suppose  $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}$  are matrices whose sizes are such that  $\mathbf{A}(\mathbf{B} + \mathbf{C})$  and  $(\mathbf{D} + \mathbf{E})\mathbf{F}$  make sense. Explain this and prove that

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC} \quad (\mathbf{D} + \mathbf{E})\mathbf{F} = \mathbf{DF} + \mathbf{EF}$$

*Proof.* We let  $\mathbf{A}, \mathbf{D}, \mathbf{E}$  to be  $m$ -by- $n$  matrix and  $\mathbf{B}, \mathbf{C}, \mathbf{F}$  to be  $n$ -by- $p$  matrix. Then all the matrix multiplication make sense. We proceed to prove the equality:

$$\begin{aligned} (\mathbf{A}(\mathbf{B} + \mathbf{C}))_{j,k} &= \sum_{i=1}^n \mathbf{A}_{j,i}(\mathbf{B} + \mathbf{C})_{i,k} \\ &= \sum_{i=1}^n \mathbf{A}_{j,i}(\mathbf{B}_{i,k} + \mathbf{C}_{i,k}) \\ &= \sum_{i=1}^n (\mathbf{A}_{j,i}\mathbf{B}_{i,k}) + (\mathbf{A}_{j,i}\mathbf{C}_{i,k}) \\ &= \mathbf{AB} + \mathbf{AC} \end{aligned}$$

For the other one, we have that

$$\begin{aligned} ((\mathbf{D} + \mathbf{E})\mathbf{F})_{j,k} &= \sum_{i=1}^n (\mathbf{D} + \mathbf{E})_{j,i}\mathbf{F}_{i,k} \\ &= \sum_{i=1}^n (\mathbf{D}_{j,i}\mathbf{F}_{i,k}) + (\mathbf{E}_{j,i}\mathbf{F}_{i,k}) \\ &= \mathbf{DF} + \mathbf{EF} \end{aligned}$$

□

**Problem 13**

Suppose  $\mathbf{A}$  is an  $n$ -by- $n$  matrix and  $1 \leq j, k \leq n$ . Show that the entry in row  $j$ , column  $k$  of  $\mathbf{A}^3$  (which is defined to mean  $\mathbf{AAA}$ ) is

$$\sum_{p=1}^n \sum_{r=1}^n \mathbf{A}_{j,p} \mathbf{A}_{p,r} \mathbf{A}_{r,k}$$

*Proof.*

$$\begin{aligned}
 (\mathbf{A}(\mathbf{A}\mathbf{A}))_{j,k} &= \sum_p^n \mathbf{A}_{j,p}(\mathbf{A}\mathbf{A})_{p,k} \\
 &= \sum_p^n \mathbf{A}_{j,p} \sum_{r=1}^n \mathbf{A}_{p,r} \mathbf{A}_{r,k} \\
 &= \sum_p^n \sum_r^n \mathbf{A}_{j,p} \mathbf{A}_{p,r} \mathbf{A}_{r,k}
 \end{aligned}$$

□

**Problem 14**

Suppose  $m$  and  $n$  are postiive integers. Prove that the function  $\mathbf{A} \mapsto \mathbf{A}^\top$  is a linear map from  $\mathbf{F}^{m,n}$  to  $\mathbf{F}^{n,m}$ .

*Proof.* Let  $T$  denote the linear map such that  $T(\mathbf{A}) = \mathbf{A}^\top$ . Then we have that

$$T(\lambda\mathbf{A} + \mathbf{B}) = (\lambda\mathbf{A} + \mathbf{B})^\top = \lambda\mathbf{A}^\top + \mathbf{B}^\top = \lambda T(\mathbf{A}) + T(\mathbf{B})$$

□

**Problem 15**

Prove that if  $\mathbf{A}$  is an  $m$ -by- $n$  matrix and  $\mathbf{C}$  is an  $n$ -by- $p$  matrix, then

$$(\mathbf{AC})^\top = \mathbf{C}^\top \mathbf{A}^\top$$

*Proof.*

$$\begin{aligned}
 ((\mathbf{AC})^\top)_{j,k} &= (\mathbf{AC})_{k,j} \\
 &= \sum_{i=1}^n \mathbf{A}_{k,i} \mathbf{C}_{i,j}
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 (\mathbf{C}^\top \mathbf{A}^\top)_{j,k} &= \sum_{i=1}^n \mathbf{C}_{j,i}^\top \mathbf{A}_{i,k}^\top \\
 &= \sum_{i=1}^n \mathbf{A}_{k,i} \mathbf{C}_{i,j}
 \end{aligned}$$

□

### Problem 16

Suppose  $\mathbf{A}$  is an  $m$ -by- $n$  matrix with  $\mathbf{A} \neq 0$ . Prove that the rank of  $\mathbf{A}$  is 1 if and only if there exist  $(c_1, \dots, c_m) \in \mathbb{F}^m$  and  $(d_1, \dots, d_n) \in \mathbb{F}^n$  such that  $\mathbf{A}_{j,k} = c_j d_k$  for every  $j = 1, \dots, m$  and every  $k = 1, \dots, n$ .

*Proof.*  $\Rightarrow$  If rank of  $\mathbf{A}$  is 1, then this means that the span of the columns of  $\mathbf{A}$  is 1-dimensional, take  $\mathbf{A}_{\cdot,1} \in \mathbb{F}^m$  from the span. (Note that here we assume  $\mathbf{A}_{\cdot,1} \neq 0$  ow. take the first nonzero one) Then we can let  $(c_1, \dots, c_m) = \text{vec}(\mathbf{A}_{\cdot,1})$ . and let For every other columns, they are the scalar multiple of the the first one, so there exists  $d_2, \dots, d_n$  s.t.  $\mathbf{A}_{j,k} = c_j d_k$  (we take  $d_1 = 1$ ).

$\Leftarrow$  If we denote  $\mathbf{c} = (c_1, \dots, c_m)$ , then

$$\mathbf{A} = (d_1 \mathbf{c} \quad \cdots \quad d_n \mathbf{c})$$

so each column is a scalar multiple of the other and thus  $\mathbf{A}$  has rank 1.  $\square$

### Problem 17

Suppose  $T \in \mathcal{L}(V)$ , and  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  are bases of  $V$ . Prove that the following are equivalent:

- (a)  $T$  is injective.
  - (b) The column of  $\mathcal{M}(T)$  are linearly independent in  $\mathbb{F}^{n,1}$ .
  - (c) The columns of  $\mathcal{M}(T)$  span  $\mathbb{F}^{n,1}$ .
  - (d) The rows of  $\mathcal{M}(T)$  span  $\mathbb{F}^{1,n}$ .
  - (e) The rows of  $\mathcal{M}(T)$  are linearly independent in  $\mathbb{F}^{1,n}$ .
- Here  $\mathcal{M}(T)$  means  $\mathcal{M}(T, (u_1, \dots, u_n), (v_1, \dots, v_n))$ .

*Proof.* Let  $\mathbf{A}$  represents the matrix of  $\mathcal{M}(T)$ .

(a)  $\Rightarrow$  (b):  $a_1 \mathbf{A}_{\cdot,1} + \dots + a_n \mathbf{A}_{\cdot,n} = 0$ . We also know that  $Tv_i = \sum_{j=1}^n \mathbf{A}_{j,i} u_j$ . Then we have that

$$T(a_1 v_1 + \dots + a_n v_n) = \sum_{i=1}^n a_i T(v_i) = \sum_{i=1}^n a_i \sum_{j=1}^n \mathbf{A}_{j,i} u_j = \sum_{j=1}^n \left( \sum_{i=1}^n a_i \mathbf{A}_{j,i} \right) u_j = 0$$

To prove all  $a_i$  is zero we note that  $a_1 v_1 + \dots + a_n v_n \in \text{null } T$  and since  $T$  is injective, the only possible case is that all  $a_i = 0$  and thus we prove the claim.

(b)  $\Rightarrow$  (c) The columns of  $\mathcal{M}(T)$  forms a basis and thus span the space.

(c)  $\Rightarrow$  (d) Column rank of  $\mathcal{M}(T)$  is  $n$  and thus row rank is also  $n$  and thus the rows span  $\mathbb{F}^{1,n}$ .

(d)  $\Rightarrow$  (e) Same as above.

(e)  $\Rightarrow$  (a) The only solution to  $a_1 \mathbf{A}_{1,\cdot} + \dots + a_n \mathbf{A}_{n,\cdot} = 0$  is all  $a_i = 0$ . Suppose for the sake of contradiction that  $T$  is not injective so there exists nonzero  $v \in \text{null } T$ . We know that  $v = \sum_{i=1}^n a_i v_i$  and that  $Tv_i = \sum_{j=1}^n \mathbf{A}_{j,i} u_j$ . Then

$$Tv = \sum_{i=1}^n a_i T(v_i) = \sum_{i=1}^n a_i \sum_{j=1}^n \mathbf{A}_{j,i} u_j = \sum_{j=1}^n \left( \sum_{i=1}^n a_i \mathbf{A}_{j,i} \right) u_j = 0$$

and thus  $\sum_{j=1}^n a_i \mathbf{A}_{j,i} = 0$  for all  $j$ . We know not all  $a_i = 0$  since  $v$  is non-trivial and thus the solution contradicts that the rows are all linearly independent. Proof completed. □

### 3D: Invertibility and Isomorphisms

**Definition 37** (invertible, inverse). • A linear map  $T \in \mathcal{L}(V, W)$  is called invertible if there exists a linear map  $S \in \mathcal{L}(W, V)$  such that  $ST$  equals the identity operator on  $V$  and  $TS$  equals the identity operator on  $W$ .

- A linear map  $S \in \mathcal{L}(W, V)$  satisfying  $ST = I$  and  $TS = I$  is called an inverse of  $T$ .

**Theorem 38.** An invertible linear map has a unique inverse.

If  $T$  is invertible, then its inverse is denoted by  $T^{-1}$ .

**Theorem 39.** A linear map is invertible if and only if it is injective and surjective.

*Proof.*  $\Rightarrow$  We have an invertible linear map  $T$  that  $Tu = Tv$ . Then we have

$$u = T^{-1}Tu = T^{-1}Tv = v$$

To show  $T$  is surjective, we have that for any  $w \in W, w = T(T^{-1}w)$ .

$\Leftarrow$  Define  $S$  such that for each  $w \in W$ ,  $S(w)$  is the unique element s.t.  $T(S(w)) = w$ . (we can do this due to injectivity and surjectivity). Then we have that  $T(ST)v = (TS)Tv = Tv$  and thus  $STv = v$  and thus  $ST = I$ . So we suffice the identity operator condition. It's easy to show that  $S$  is a linear map.  $\square$

**Theorem 40.** Suppose that  $V$  and  $W$  are finite-dimensional vector spaces,  $\dim V = \dim W$ , and  $T \in \mathcal{L}(V, W)$ . Then

$$T \text{ is invertible} \Leftrightarrow T \text{ is injective} \Leftrightarrow T \text{ is surjective.}$$

**Corollary 41.** Suppose  $V$  and  $W$  are finite-dimensional vector spaces of the same dimension,  $S \in \mathcal{L}(W, V)$ , and  $T \in \mathcal{L}(V, W)$ . Then  $ST = I$  if and only if  $TS = I$ .

**Definition 42** (Isomorphism). An **isomorphism** is an invertible linear map. Two vector spaces are called **isomorphic** if there is an isomorphism from one vector space onto the other one.

**Theorem 43** (dimension shows whether vector spaces are isomorphic). Two finite-dimensional vector spaces over  $\mathbb{F}$  are isomorphic if and only if they have the same dimension.

**Theorem 44.** Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . Then  $\mathcal{M}$  is an isomorphism between  $\mathcal{L}(V, W)$  and  $\mathbb{F}^{m,n}$ .

**Corollary 45.** Suppose  $V$  and  $W$  are finite-dimensional. Then  $\mathcal{L}(V, W)$  is finite-dimensional and

$$\dim \mathcal{L}(V, W) = (\dim V)(\dim W)$$



**Definition 46** (matrix of a vector,  $\mathcal{M}(v)$ ). Suppose  $v \in V$  and  $v_1, \dots, v_n$  is a basis of  $V$ . Then **matrix of  $v$**  with respect to the basis is the  $n$ -by-1 matrix

$$\mathcal{M}(v) = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

where  $b_1, \dots, b_n$  are the scalars such that

$$v = b_1 v_1 + \dots + b_n v_n$$

**Remark 47.** The matrix  $\mathcal{M}(v)$  of a vector  $v \in V$  depends on the basis  $v_1, \dots, v_n$  and  $v$ . We can think of elements of  $V$  as relabeled to be  $n$ -by-1 matrices, i.e.  $V \mapsto \mathbb{F}^{n,1}$ .

**Corollary 48.** Suppose  $T \in \mathcal{L}(V, W)$  and  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . Let  $1 \leq k \leq n$ . Then  $k$ th column of  $\mathcal{M}(T)$ , which is denoted by  $\mathcal{M}(T)_{\cdot, k}$ , equals  $\mathcal{M}(Tv_k)$

**Theorem 49** (linear maps act like matrix multiplication). Suppose  $T \in \mathcal{L}(V, W)$  and  $v \in V$ . Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $w_1, \dots, w_m$  is a basis of  $W$ . Then

$$\mathcal{M}(Tv) = \mathcal{M}(T)\mathcal{M}(v)$$

*Proof.*

$$\begin{aligned} \mathcal{M}(Tv) &= b_1 \mathcal{M}(Tv_1) + \dots + b_n \mathcal{M}(Tv_n) \\ &= b_1 \mathcal{M}(T)_{\cdot, 1} + \dots + b_n \mathcal{M}(T)_{\cdot, n} \\ &= \mathcal{M}(T)\mathcal{M}(v) \end{aligned}$$

□

**Remark 50.** Each  $m$ -by- $n$  matrix  $\mathbf{A}$  induces a linear map from  $\mathbb{F}^{n,1}$  to  $\mathbb{F}^{m,1}$ , namely the matrix multiplication function that takes  $x \in \mathbb{F}^{n,1}$  to  $\mathbf{A}x \in \mathbb{F}^{m,1}$ . We can think of every linear map (from a finite vector space to another) as a matrix multiplication map after suitable relabeling via the isomorphisms given by  $\mathcal{M}$ . Specifically, if  $T \in \mathcal{L}(V, W)$  and we identify  $v \in V$  with  $\mathcal{M}(v) \in \mathbb{F}^{n,1}$ , then the result above says that we can identify  $Tv$  with  $\mathcal{M}(T)\mathcal{M}(v)$ .

**Theorem 51** (dimension of range  $T$  equals column rank of  $\mathcal{M}(T)$ ). Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then  $\dim \text{range } T$  equals the column rank of  $\mathcal{M}(T)$ .

**Theorem 52** (change-of-basis-formula). Suppose  $T \in \mathcal{L}(V)$ . Suppose  $u_1, \dots, u_n$  and  $v_1, \dots, v_n$  are bases of  $V$ . Let

$$\mathbf{A} = \mathcal{M}(T, (u_1, \dots, u_n)) \text{ and } \mathbf{B} = \mathcal{M}(T, (v_1, \dots, v_n))$$

and  $\mathbf{C} = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n))$ . Then

$$\mathbf{A} = \mathbf{C}^{-1}\mathbf{B}\mathbf{C}.$$

*Proof.*

□

**Problem 1**

Suppose  $T \in \mathcal{L}(V, W)$  is invertible. Show that  $T^{-1}$  is invertible and

$$(T^{-1})^{-1} = T.$$

*Proof.*  $T^{-1}$  is invertible because there exists  $T$  such that  $TT^{-1} = T^{-1}T = I$ .

$$T^{-1}T = TT^{-1} = I$$

so  $(T^{-1})^{-1} = T$ .

□

**Problem 2**

Suppose  $T \in \mathcal{L}(U, V)$  and  $S \in \mathcal{L}(V, W)$  are both invertible linear maps. Prove that  $ST \in \mathcal{L}(U, W)$  is invertible and that  $(ST)^{-1} = T^{-1}S^{-1}$ .

*Proof.*  $(ST)(T^{-1}S^{-1}) = S(TT^{-1})S^{-1} = I = T^{-1}S^{-1}ST$ .

□

**Problem 3**

Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Prove that the following are equivalent.

- (a)  $T$  is invertible.
- (b)  $Tv_1, \dots, Tv_n$  is a basis of  $V$  for every basis  $v_1, \dots, v_n$  of  $V$ .
- (c)  $Tv_1, \dots, Tv_n$  is a basis of  $V$  for some basis  $v_1, \dots, v_n$  of  $V$ .

*Proof.* (a)  $\Rightarrow$  (b) It only suffice to prove linear independence. We can show this

$$a_1Tv_1 + \dots + a_nTv_n = 0 \iff a_1v_1 + \dots + a_nv_n = 0$$

since  $T$  is injective and thus the only solution is all  $a_i$  are identically zero.

(b)  $\Rightarrow$  (c) Trivial.

(c)  $\Rightarrow$  (a) By the linear map lemma, there exists  $S \in \mathcal{L}(V)$  such that  $S(Tv_i) = v_i$  for all  $i$ . Such  $S$  is the inverse of  $T$  (one can verify) and thus  $T$  is invertible. □

**Problem 5**

Suppose  $V$  is finite-dimensional,  $U$  is a subspace of  $V$ , and  $S \in \mathcal{L}(U, V)$ . Prove that there exists an invertible linear map  $T$  from  $V$  to itself such that  $Tu = Su$  for every  $u \in U$  if and only if  $S$  is injective.

*Proof.*  $\Rightarrow$  Since  $T$  is invertible, we can define  $T^{-1}$  restricted to  $U$ , where  $S^{-1}u = T^{-1}u$  for all  $u \in U$ . Then  $S$  has an inverse and thus is injective.

$\Leftarrow$  Let  $v_1, \dots, v_m$  be the basis of  $U$  and  $v_1, \dots, v_m, x_1, \dots, x_n$  be the extended basis of  $V$ . Define  $Tv_i = Sv_i$  and  $Tx_i = x_i$ . Then  $T$  is injective and thus invertible.  $\square$

### Problem 6

Suppose that  $W$  is finite-dimensional and  $S, T \in \mathcal{L}(V, W)$ . Prove that  $\text{null } S = \text{null } T$  if and only if there exists an invertible  $E \in \mathcal{L}(W)$  such that  $S = ET$ .

*Proof.*  $\Rightarrow$  Suppose  $V$  is finite-dimensional for simplicity (one can do more work for relaxing this assumption). Let  $V = \text{null } S \oplus C = \text{null } T \oplus C$  for some  $C$ . Then we know that  $T|_C: C \rightarrow \text{range } (T(C))$  and  $S|_C: C \rightarrow \text{range } (S(C))$  is invertible. Then there exists an invertible map  $\hat{E}: \text{range } (T(C)) \rightarrow \text{range } (S(C))$  as  $\dim \text{range } T(C) = \text{range } (S(C)) = \dim V$ . Extending this map to  $W$  solves the problem (take bases from  $\text{range } T(C)$  and extend to  $W$  with “identity” map).

$\Leftarrow$  Take  $v \in \text{null } T$ , then  $S(v) = E(Tv) = E(0) = 0$  and thus  $v \in \text{null } S$ . Conversely, take  $v \in \text{null } S$ , then  $T(v) = E^{-1}S(v) = 0$  so  $v \in \text{null } T$ . Thus  $\text{null } S = \text{null } T$ .  $\square$

### Problem 7

Suppose that  $V$  is finite-dimensional and  $S, T \in \mathcal{L}(V, W)$ . Prove that  $\text{range } S = \text{range } T$  if and only if there exists an invertible  $E \in \mathcal{L}(V)$  such that  $S = TE$ .

*Proof.*  $\Rightarrow$  By 3B P26, we know that there exists  $\tilde{E} \in \mathcal{L}(V)$  s.t.  $S = T\tilde{E}$  as  $\text{range } S \subseteq \text{range } T$ .  $\text{range } S = \text{range } T$  implies that  $\dim \text{null } S = \dim \text{range } T$ . Then we can define an invertible map  $\bar{E}$  from  $\text{null } S$  to  $\text{null } T$ . We can define the map  $E$  as follows: Denote  $V = \text{null } S \oplus P$ , then  $E(v) = \tilde{E}(v_1) + \bar{E}(v_2)$  for  $v = v_1 + v_2$ , where  $v_1 \in P, v_2 \in \text{null } S$  and now  $\tilde{E} = \tilde{E}|_P$ . Then we have that  $TE(v) = T(\tilde{E}(v_1) + \bar{E}(v_2)) = S(v_1) + S(v_2) = S(v)$ . Now it left to verify that our proposed  $E$  is invertible.

To show this, we mainly aim at proving  $\text{range } \tilde{E} \cap \text{range } \bar{E} = \{0\}$ , as if we have this, we know that  $E(v) = \tilde{E}(v_1) + \bar{E}(v_2) = 0 \Rightarrow \tilde{E}(v_1) = \bar{E}(v_2) = 0$ . We know  $\bar{E}$  is invertible so  $v_2 = 0$ . To see  $\tilde{E}$  is injective, we can see that for any  $v_1 \in \text{null } \tilde{E}$ , we have  $Sv_1 = T\tilde{E}v_1 = T0 = 0$  and thus  $v_1 \in \text{null } S \cap P = 0$ , so  $v_1 = 0$  and thus  $\text{null } (\tilde{E}) = \{0\}$ ,  $E$  is injective and therefore invertible.

To complete the proof, let  $u \in \text{range } \tilde{E} \cap \text{range } \bar{E}$ , then we know there exists  $v_1 \in P, v_2 \in \text{null } S$  s.t.  $u = \tilde{E}(v_1) = \bar{E}(v_2)$ . We also know that  $u \in \text{null } T$  as  $\text{range } \bar{E} = \text{null } T$ . Now we have that  $S(v_1) = T\tilde{E}(v_1) = T\bar{E}(v_2) = 0$  and thus  $v_1 \in \text{null } S$ . So  $v_1 \in P \cap \text{null } S = \{0\}$  and thus we have that  $u = \{0\}$ , finishing the proof.

$\Leftarrow$  Take any  $s \in \text{range } S$ , then there exists  $E(s)$  s.t.  $T(E(s)) = s$ . Conversely, take any  $t \in \text{range } T$ , then there exists  $v \in V$  s.t.  $t = Tv = TEE^{-1}v = S(E^{-1}v)$ .

Thus  $\text{range } S = \text{range } T$ . □

**Problem 8**

Suppose  $V$  and  $W$  are finite-dimensional and  $S, T \in \mathcal{L}(V, W)$ . Prove that there exist invertible  $E_1 \in \mathcal{L}(V)$  and  $E_2 \in \mathcal{L}(W)$  such that  $S = E_2 T E_1$  if and only if  $\dim \text{null } S = \dim \text{null } T$ .

*Proof.*  $\Rightarrow$  We know from 3B exercises that  $\dim \text{range } ST \leq \min(\dim \text{range } S, \dim \text{range } T)$ . Applying here gets that  $\dim \text{range } S \leq \dim \text{range } T$ . Consider that  $T = E_2^{-1} S E_1^{-1}$ , then we can also derive  $\dim \text{range } T \leq \dim \text{range } S$  and thus  $\dim \text{range } S = \dim \text{range } T$ . By the fundamental theorem of linear map, we can get that  $\dim \text{null } S = \dim \text{null } T$ .

$\Leftarrow$  We know  $\dim \text{null } S = \dim \text{null } T$ , so there exists an isomorphism  $\tilde{E}_1: \text{null } S \rightarrow \text{null } T$ . Extending  $\tilde{E}_1: \text{null } S \rightarrow V$  still preserves its injectivity. Therefore by P5 we have an invertible  $E_1: V \rightarrow V$  such that  $E_1 u = \tilde{E}_1 u$  for all  $u \in \text{null } S$ . Now we intend to show that  $\text{null } S = \text{null } T E_1$ , as proving so would imply there exists an invertible  $E_2 \in \mathcal{L}(W)$  such that  $S = E_2 T E_1$  by P6. To see this, take  $s \in \text{null } S$ , then  $T E_1(s) = T(\tilde{E}_1 s) = 0$  and thus  $s \in \text{null } T E_1$ . To see the other direction, take  $t \in \text{null } T E_1$ , then we know  $E_1 t \in \text{null } T = \text{range } \tilde{E}_1$ . Thus there exists  $v \in \text{null } S$  s.t.  $E_1 t = \tilde{E}_1 v = E_1 v$ . As  $E_1$  is invertible,  $t = v \in \text{null } S$  and thus we have shown  $\text{null } S = \text{null } T E_1$ , completing the proof. □

**Problem 9**

Suppose  $V$  is finite-dimensional and  $T: V \rightarrow W$  is a surjective linear map of  $V$  onto  $W$ . Prove that there is a subspace  $U$  of  $V$  such that  $T|_U$  is an isomorphism of  $U$  onto  $W$ .

*Proof.*  $T$  being surjective means that  $\dim V \geq \dim W$ . Take any subspace of  $V$  that has  $\dim W$ , then there is an isomorphism between this subspace and  $W$ . □

**Problem 10**

Suppose  $V$  and  $W$  are finite-dimensional and  $U$  is a subspace of  $V$ . Let

$$\mathcal{E} = \{T \in \mathcal{L}(V, W) : U \subseteq \text{null } T\}.$$

- (a) Show that  $\mathcal{E}$  is a subspace of  $\mathcal{L}(V, W)$ .
- (b) Find a formula for  $\dim \mathcal{E}$  in terms of  $\dim V$ ,  $\dim W$ , and  $\dim U$ .

*Proof.* (a) Take  $T_1, T_2 \in \mathcal{E}$  and  $\lambda \in \mathbb{F}$ , then consider  $\text{null } (\lambda T_1 + T_2)$ . Let  $u \in U$ , then  $\lambda T_1 + T_2(u) = \lambda T_1(u) + T_2(u) = 0 + 0 = 0$  and thus we have  $\lambda T_1 + T_2 \in \mathcal{E}$  and thus  $\mathcal{E}$  is a subspace of  $\mathcal{L}(V, W)$ .

(b) Following the hint, define  $\Phi: \mathcal{L}(V, W) \rightarrow \mathcal{L}(U, W)$  by  $\Phi(T) = T|_U$ . Then  $\Phi \in \mathcal{L}(\mathcal{L}(V, W), \mathcal{L}(U, W))$ . We try to show the range and null space of  $\Phi$ .

We first claim  $\text{null } \Phi = \mathcal{E}$ . To see this, take  $T \in \text{null } \Phi$ , then we know  $\Phi(T) = T|_U = 0 \in \mathcal{L}(U, W)$ , meaning that  $U$  is the subset of null  $T$ , so  $\text{null } \Phi \subseteq \mathcal{E}$ . Conversely, take  $T \in \mathcal{E}$ , then we have that for all  $u \in U$ ,  $Tu = 0$ , then we have that  $T|_U = 0$ . This shows that  $\mathcal{E} \subseteq \text{null } \Phi$ .

Next we claim  $\text{range } \Phi = \mathcal{L}(U, W)$ . Take any  $S \in \mathcal{L}(U, W)$ , then we can naturally always extend  $S$  to  $T \in \mathcal{L}(V, W)$  by setting  $Tu = Su$  for all  $u \in U$ . The converse direction is trivial.

Thus we have that  $\dim \mathcal{E} = \dim \text{null } \Phi$  and  $\dim \text{range } \Phi = \dim \mathcal{L}(U, W) = \dim U \dim W$  and  $\dim \mathcal{L}(V, W) = \dim V \dim W = \dim \text{null } \Phi + \dim \text{range } \Phi$ , thus we derive that

$$\dim V \dim W = \dim \mathcal{E} + \dim U \dim W.$$

□

#### Problem 14

Prove or give a counterexample: If  $V$  is finite-dimensional and  $R, S, T \in \mathcal{L}(V)$  are such that  $RST$  is surjective, then  $S$  is injective.

*Proof.* We know that

$$\dim \text{range } RST = \dim \text{range } V \leq \min\{\dim \text{range } R, \dim \text{range } S, \dim \text{range } T\}$$

This means that  $\dim RST \leq \dim \text{range } S$ . If  $\dim \text{null } S > 0$ , then  $\dim \text{range } S < \dim V$  and thus contradicts the assumption that  $RST$  is surjective. □

#### Problem 15

Suppose  $T \in \mathcal{L}(V)$  and  $v_1, \dots, v_m$  is a list in  $V$  such that  $Tv_1, \dots, Tv_m$  spans  $V$ . Prove that  $v_1, \dots, v_m$  spans  $V$ .

*Proof.* We can reduce the list to basis and use previous conclusions. Since  $Tv_1, \dots, Tv_m$  spans  $V$ , we can reduce the list to  $Tv_1, \dots, Tv_n$  such that the list is the basis of  $V$ . By previous results,  $v_1, \dots, v_n$  is also linearly independent and its dimension equals the  $\dim V$  and thus is the basis of  $V$ . It thus spans  $V$ . Extending the list to  $v_1, \dots, v_m$  also spans  $V$ . □

#### Problem 16

Prove that every linear map from  $\mathbb{F}^{n,1}$  to  $\mathbb{F}^{m,1}$  is given by a matrix multiplication. In other words, prove that if  $T \in \mathcal{L}(\mathbb{F}^{n,1}, \mathbb{F}^{m,1})$ , then there exists an  $m$ -by- $n$  matrix  $\mathbf{A}$  such that  $Tx = \mathbf{A}x$  for every  $x \in \mathbb{F}^{n,1}$ .

*Proof.* We can simply consider the standard basis of  $\mathbb{F}^{n,1}$  and  $\mathbb{F}^{m,1}$ . Then Let  $\mathbf{A}$  be the matrix of  $T$  wrt these bases. Then we have that by definition,

$$Tx = \mathcal{M}(Tx) = \mathcal{M}(T)\mathcal{M}(x) = \mathbf{A}x$$

□

**Problem 17**

Suppose  $V$  is finite-dimensional and  $S \in \mathcal{L}(V)$ . Define  $\mathcal{A} \in \mathcal{L}(\mathcal{L}(V))$  by

$$\mathcal{A}(T) = ST$$

for  $T \in \mathcal{L}(V)$ .

- (a) Show that  $\dim \text{null } \mathcal{A} = \dim V \dim \text{null } S$ .
- (b) Show that  $\dim \text{range } \mathcal{A} = \dim V \dim \text{range } S$ .

*Proof.* (a) We prove this by showing  $\text{null } \mathcal{A} = \mathcal{L}(V, \text{null } S)$ . Take  $A \in \text{null } \mathcal{A}$ , then we know that  $\mathcal{A}(A) = SA = 0$  and thus  $Av \in \text{null } S$  for all  $v \in V$ . For the other direction, take  $A \in \mathcal{L}(V, \text{null } S)$ , then  $SA(v) = \mathcal{A}(v) = 0$  and thus  $A \in \text{null } \mathcal{A}$ .

(b) We know  $\dim(\mathcal{L}(V)) = \dim \text{null } \mathcal{A} + \dim \text{range } \mathcal{A}$ . Hence

$$\dim \text{range } \mathcal{A} = \dim V (\dim V - \dim \text{null } S) = \dim V \dim \text{range } S.$$

□

**Problem 18**

Show that  $V$  and  $\mathcal{L}(\mathbb{F}, V)$  are isomorphic vector spaces.

*Proof.* For simplicity, assume  $V$  is finite-dimensional, then  $\dim V = \dim \mathbb{F} \dim V = \dim \mathcal{L}(\mathbb{F}, V)$  and thus they are isomorphic. One might do a more careful construction for infinite-dimensional  $V$ . □

**Problem 19**

Suppose  $V$  is finite-dimensional and  $T \in \mathcal{L}(V)$ . Prove that  $T$  has the same matrix with respect to every basis of  $V$  if and only if  $T$  is a scalar multiple of the identity operator.

*Proof.*  $\Rightarrow$  We know  $Tv_k = \sum_{j=1}^n A_{j,k}v_j = \sum_{j=1}^n 2A_{j,k}v_j = T(2v_k)$ . This means that  $A_{j,k} = 0$  for  $j \neq k$ . When  $j = k$ , we have that  $Tv_k = A_{k,k}v_k$  for some arbitrary ordering of the same basis. Then this means that  $A_{i,i} = A_{j,j}$  for all  $i, j$ . Thus,  $T$  is a scalar multiple of the identity operator.

$\Leftarrow$  This follows by the definition of the identity operator. □

### 3E: Products and Quotients of Vector Spaces

**Definition 53** (product of vector spaces). *Suppose  $V_1, \dots, V_m$  are vector spaces over  $\mathbb{F}$ .*

- The **product**  $V_1 \times \dots \times V_m$  is defined by

$$V_1 \times \dots \times V_m = \{(v_1, \dots, v_m) : v_1 \in V_1, \dots, v_m \in V_m\}.$$

- Addition on  $V_1 \times \dots \times V_m$  is defined by

$$(u_1, \dots, u_m) + (v_1, \dots, v_m) = (u_1 + v_1, \dots, u_m + v_m).$$

- Scalar multiplication on  $V_1 \times \dots \times V_m$  is defined by

$$\lambda(v_1, \dots, v_m) = (\lambda v_1, \dots, \lambda v_m).$$

**Theorem 54** (product of vector spaces is a vector space). *Suppose  $V_1, \dots, V_m$  are vector spaces over  $\mathbb{F}$ . Then  $V_1 \times \dots \times V_m$  is a vector space over  $\mathbb{F}$ .*

**Theorem 55** (dimension of a product is the sum of dimensions). *Suppose  $V_1, \dots, V_m$  are finite-dimensional vector spaces. Then  $V_1 \times \dots \times V_m$  is finite-dimensional and*

$$\dim(V_1 \times \dots \times V_m) = \dim V_1 + \dots + \dim V_m.$$

**Lemma 56** (products and direct sums). *Suppose that  $V_1, \dots, V_m$  are subspaces of  $V$ . Define a linear map  $\Gamma : V_1 \times \dots \times V_m \rightarrow V_1 + \dots + V_m$  by*

$$\Gamma(v_1, \dots, v_m) = v_1 + \dots + v_m.$$

*Then  $V_1 + \dots + V_m$  is a direct sum if and only if  $\Gamma$  is injective.*

**Theorem 57** (a sum is a direct sum if and only if dimensions add up). *Suppose  $V$  is finite-dimensional and  $V_1, \dots, V_m$  are subspaces of  $V$ . Then  $V_1 + \dots + V_m$  is a direct sum if and only if*

$$\dim(V_1 + \dots + V_m) = \dim V_1 + \dots + \dim V_m.$$

**Definition 58** ( $v + U$ , translation). *Suppose  $v \in V$  and  $U \subset V$ . Then  $v + U$  is the subset of  $V$  defined by*

$$v + U = \{v + u : u \in U\}.$$

*The set  $v + U$  is said to be a **translate** of  $U$ .*

**Definition 59** (quotient space,  $V/U$ ). *Suppose  $U$  is a subspace of  $V$ . Then the **quotient space**  $V/U$  is the set of all translates of  $U$ . Thus*

$$V/U = \{v + U : v \in V\}.$$

**Remark 60.** If  $U = \{(x, 2x) \in \mathbb{R}^2 : x \in R\}$ , then  $\mathbb{R}^2/U$  is the set of all lines in  $\mathbb{R}^2$  that have slope 2.

**Lemma 61** (two translates of a subspace are equal or disjoint). Suppose  $U$  is a subspace of  $V$  and  $v, w \in V$ . Then

$$v - w \in U \Leftrightarrow v + U = w + U \Leftrightarrow (v + U) \cap (w + U) = \emptyset$$

**Definition 62** (addition and scalar multiplication on  $V/U$ ). Suppose  $U$  is a subspace of  $V$ . Then **addition** and **scalar multiplication** are defined on  $V/U$  by

$$\begin{aligned}(v + U) + (w + U) &= (v + w) + U \\ \lambda(v + U) &= (\lambda v) + U\end{aligned}$$

for all  $v, w \in V$  and all  $\lambda \in \mathbb{F}$ .

**Theorem 63** (quotient space is a vector space). Suppose  $U$  is a subspace of  $V$ . Then  $V/U$ , with the operations of addition and scalar multiplication as defined above, is a vector space.

**Definition 64** (quotient map). Suppose  $U$  is a subspace of  $V$ . The **quotient map**  $\pi: V \rightarrow V/U$  is the linear map defined by

$$\pi(v) = v + U$$

for each  $v \in V$ .

**Theorem 65** (dimension of quotient space). Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then

$$\dim V/U = \dim V - \dim U$$

**Definition 66.** Suppose  $T \in \mathcal{L}(V, W)$ . Define  $\tilde{T}: V/(\text{null } T) \rightarrow W$  by

$$\tilde{T}(v + \text{null } T) = Tv.$$

Another interpretation of quotient space is on **congruence of subspaces**. The definition goes as follows:

If  $Y$  is a subspace of  $X$ , then two vectors  $x_1, x_2 \in X$  are **congruent modulo  $Y$** , denoted  $x_1 \cong x_2 \pmod{Y}$  if  $x_1 - x_2 \in Y$ .

The quotient space  $X/Y$  denotes the set of equivalence classes in  $X$ , modulo  $Y$  by defining that

$$\{x\} + \{z\} := \{x + z\} \quad a\{x\} := \{ax\}$$

So here one can think of the zero vector of  $v + U$  defined as the equivalence class that contains that zero vector  $0 \in V$ :

$$0 + U = \{0 + u : u \in U\} = U.$$

This means that the zero element in the quotient space  $V/U$  is simply the subspace  $U$ , and this equivalence class contains all vectors differ from 0 by an element of  $U$ , meaning it is precisely  $U$  itself, so  $\pi(v) = 0$  if and only if  $v \in U$ .



**Remark 67.** *The output of  $\pi(v)$  is the equivalence class  $v + U$ , which is the set of all vectors in  $V$  that are equivalent to  $v$  under the subspace  $U$ .*

For  $\tilde{T}$ , the input to it is an equivalent class  $v + \text{null } (T)$  in the quotient space  $V/\text{null } T$ . It represents all vectors in  $V$  that differ from  $v$  by a vector in  $\text{null } T$ . Its output is  $Tv$ , which is an element in  $W$ .

The null space is

$$\text{null } (\tilde{T}) = \{v + \text{null } (T) : \tilde{T}(v + \text{null } (T)) = 0\}.$$

Since we know that  $\tilde{T}(v + \text{null } T) = Tv$ , we have that

$$\tilde{T}(v + \text{null } T) = 0 \Leftrightarrow Tv = 0.$$

so  $v \in \text{null } T$  and thus  $\text{null } \tilde{T} = \{\text{null } T\}$ . We have that  $\text{range } \tilde{T} = \text{range } T$ .

**Theorem 68.** *Suppose  $T \in \mathcal{L}(V, W)$ . Then*

- (a)  $\tilde{T} \circ \pi = T$ , where  $\pi$  is the quotient map of  $V$  onto  $V/\text{null } T$ ;
- (b)  $\tilde{T}$  is injective;
- (c)  $\text{range } \tilde{T} = \text{range } T$ ;
- (d)  $V/\text{null } T$  and  $\text{range } T$  are isomorphic vector spaces.

**Remark 69.** *One might think of the domain of  $\tilde{T}$  as the non-trivial domain of  $T$  and the range as the normal image of  $T$ .*

**Problem 1**

Suppose  $T$  is function from  $V$  to  $W$ . The *graph* of  $T$  is the subset of  $V \times W$  defined by

$$\text{graph of } T = \{(v, Tv) \in V \times W : v \in V\}.$$

Prove that  $T$  is a linear map if and only if the graph of  $T$  is a subspace of  $V \times W$ .

*Proof.* The two properties closely relate to each other:

$$(v_1, Tv_1) + (v_2, Tv_2) \in V \times W : v \in V \Leftrightarrow Tv_1 + Tv_2 = T(v_1 + v_2)$$

and

$$(\lambda v, T\lambda v) \in V \times W : v \in V \Leftrightarrow T\lambda v = \lambda Tv$$

□

**Problem 2**

Suppose that  $V_1, \dots, V_m$  are vector spaces such that  $V_1 \times \dots \times V_m$  is finite-dimensional. Prove that  $V_k$  is finite-dimensional for each  $k = 1, \dots, m$ .

*Proof.* Recall the definition of infinite-dimension from ch2 P17:

$V$  is infinite-dimensional if and only if there is a sequence  $v_1, v_2, \dots$  of vectors in  $V$  such that  $v_1, \dots, v_m$  is linearly independent for every positive integer  $m$ .

Suppose for the sake of contradiction that there exists  $V_k$  to be infinite-dimensional. Then this means that there is a sequence  $v_{k,1}, v_{k,2}, \dots$  of vectors in  $V_k$  such that  $v_{k,1}, \dots, v_{k,m}$  is linearly independent for every positive integer  $m$ . We can build such sequence for the product space as well. More specifically, consider the sequence  $0, \dots, v_{k,j}, \dots, 0$  where we fill 0 for each of other  $V_j$ 's and only leave the sequence from  $V_k$ . In this case, we can see that the product space is infinite-dimensional, contradicting the claim. □

**Problem 3**

Suppose  $V_1, \dots, V_m$  are vector spaces. Prove that  $\mathcal{L}(V_1 \times \dots \times V_m, W)$  and  $\mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W)$  are isomorphic vector spaces.

*Proof.* We define a permutation invariant set function  $S: \mathcal{L}(V_1, W) \times \dots \times \mathcal{L}(V_m, W) \rightarrow \mathcal{L}(V_1 \times \dots \times V_m, W)$  such that

$$S(T_1, \dots, T_m) = T_1 + \dots + T_m$$

We will show that  $S$  is an isomorphism (it's linear). For injectivity, null  $S$  means that for any arbitrary input  $T_1 + \dots + T_m = 0$ , which implies that  $T_i = 0$  for all  $i$ . For surjectivity, take  $\phi \in \mathcal{L}(V_1 \times \dots \times V_m, W)$ , we can define

$T_i(v_i) = \phi((0, \dots, v_i, \dots, 0))$  where  $i$  means for the space  $V_i$ . Then we can naturally get that  $\phi((v_1, \dots, v_m)) = \phi(\sum_{i=1}^m e_i^\top v_i) = \sum_{i=1}^m T_i(v_i)$  and thus we finish the proof.  $\square$

#### Problem 4

Suppose  $W_1, \dots, W_m$  are vector spaces. Prove that  $\mathcal{L}(V, W_1 \times \dots \times W_m)$  and  $\mathcal{L}(V, W_1) \times \dots \times \mathcal{L}(V, W_m)$  are isomorphic vector spaces.

We can simply define a concatenate operator  $S: \mathcal{L}(V, W_1) \times \dots \times \mathcal{L}(V, W_m) \rightarrow \mathcal{L}(V, W_1 \times \dots \times W_m)$  such that

$$S(T_1, \dots, T_m) = (T_1, \dots, T_m)$$

We prove this linear map is an isomorphism. The injective property can be proved using the same logic as above. For surjectivity, take  $\phi \in \mathcal{L}(V, W_1 \times \dots \times W_m)$ , then we can naturally obtain  $T_k = \pi_k \phi$  where  $\pi_k$  means projecting the  $k$ -th component of the input. Note that  $T_k \in \mathcal{L}(V, W_k)$  and therefore we finish the proof.

#### Problem 5

For  $m$  a positive integer, define  $V^m$  by

$$V^m = \underbrace{V \times \dots \times V}_{m \text{ times}}.$$

Prove that  $V^m$  and  $\mathcal{L}(\mathbb{F}^m, V)$  are isomorphic vector spaces.

*Proof.*

$$\dim(V^m) = m \dim V = \dim(\mathcal{L}(\mathbb{F}^m, V)).$$

$\square$

#### Problem 6

Suppose that  $v, x$  are vectors in  $V$  and that  $U, W$  are subspaces of  $V$  such that  $v + U = x + W$ . Prove that  $U = W$ .

*Proof.* Note that  $v = x + w$  for some  $w \in W$  if we take  $u = 0 \in U$  and similarly  $x = v + u$  for some  $u \in U$ . Hence  $v - x \in W$  and  $x - v \in U$ . Then we have that take  $u \in U, u = (x + w) - v = (x - v) + w \in W$ . Similarly, take  $w \in W, w = (v + u) - x = (v - x) + u \in U$ . Hence  $W = U$ .  $\square$

#### Problem 9

Prove that a nonempty subset  $A$  of  $V$  is a translate of some subspace of  $V$  if and only if  $\lambda v + (1 - \lambda)w \in A$  for all  $v, w \in A$  and all  $\lambda \in \mathbb{F}$ .

*Proof.*  $\Rightarrow$  Suppose  $A = x + U$  for some subspace  $U \subseteq V$ . Then take  $a_1 = x + u_1 \in A$  and  $a_2 = x + u_2 \in A$ . We have that

$$\lambda a_1 + (1 - \lambda)a_2 = \lambda x + \lambda u_1 + (1 - \lambda)x + (1 - \lambda)u_2 = x + (\lambda u_1 + (1 - \lambda)u_2) \in A$$

$\Leftarrow$  take any  $x \in A$  and define the subspace such that

$$U := (-x) + A$$

first  $0 = -x + x \in U$ . Next, take  $v_1 - x$  and  $v_2 - x \in U$ . We have that  $v_1 - x + v_2 - x = (-x) + (2v_1 - x)/2 + (2v_2 - x)/2 \in A$  as  $2v_i - x \in A$  by taking  $\lambda = 2$ . Finally,  $\lambda(-x + v) = (-x) + (\lambda v + (1 - \lambda)x) \in A$ .  $\square$

#### Problem 10

Suppose  $A_1 = v + U_1$  and  $A_2 = w + U_2$  for some  $v, w \in V$  and some subspaces  $U_1, U_2$  of  $V$ . Prove that the intersection  $A_1 \cap A_2$  is either a translate of some subspace of  $V$  or is the empty set.

*Proof.* If  $A_1 \cap A_2$  is not empty, then  $S = A_1 \cap A_2$  is a subspace such that  $v + U_1|_S = w + U_2|_S$ , so we know that  $U_1|_S = U_2|_S$  by Problem 6. Hence it is a translate of some subspace of  $V$ . Otherwise, it is the empty set.  $\square$

#### Problem 11

Suppose  $U = \{(x_1, x_2, \dots) \in \mathbb{F}^\infty : x_k \neq 0 \text{ for only finitely many } k\}$ .  
(a) Show that  $U$  is a subspace of  $\mathbb{F}^\infty$ .  
(b) Prove that  $\mathbb{F}^\infty/U$  is infinite-dimensional.

*Proof.* (a) all zero, where  $x_k \neq 0$  for zero  $k$ 's (finitely many) is an element of  $U$ . Next, take  $(x_1, x_2, \dots)$  and  $(y_1, y_2, \dots) \in U$ . Then  $(x_1 + y_1, x_2 + y_2, \dots)$  will have finitely many nonzero since each of them has only finitely many nonzero entries. Same holds for  $\lambda(x_1, x_2, \dots)$ .

(b) Let's consider a "standard basis"  $\{v_i\}_{i=1}^\infty$  that only has 0 on  $i$ -th spot and all 1 otherwise. We can see that for all  $m$ ,  $v_1, \dots, v_m$  is linearly independent. And so does  $v_1 + U, \dots, v_m + U$  since each  $v_i \notin U$ ,  $\square$

#### Problem 12

Suppose  $v_1, \dots, v_m \in V$ . Let

$$A = \{\lambda_1 v_1 + \dots + \lambda_m v_m : \lambda_1, \dots, \lambda_m \in F \text{ and } \lambda_1 + \dots + \lambda_m = 1\}.$$

- (a) Prove that  $A$  is a translate of some subspace of  $V$ .
- (b) Prove that if  $B$  is a translate of some subspace of  $V$  and  $\{v_1, \dots, v_m\} \subseteq B$ , then  $A \subseteq B$ .
- (c) Prove that  $A$  is a translate of some subspace of  $V$  of dimension less than  $m$ .

*Proof.* (a) We try to show this through using the conclusion from Problem 9. Let  $a_1 = \sum_{i=1}^m \alpha_i v_i$  and  $a_2 = \sum_{i=1}^m \gamma_i v_i$  s.t.  $\sum_{i=1}^m \alpha_i = \sum_{i=1}^m \gamma_i = 1$ . We have that

$$\lambda a_1 + (1 - \lambda) a_2 = \sum_{i=1}^m (\lambda \alpha_i + (1 - \lambda) \gamma_i) v_i$$

We can show that

$$\sum_{i=1}^m (\lambda \alpha_i + (1 - \lambda) \gamma_i) = \lambda \sum_{i=1}^m \alpha_i + (1 - \lambda) \sum_{i=1}^m \gamma_i = 1$$

Thus  $\lambda a_1 + (1 - \lambda) a_2 \in A$  and thus  $A$  is a translate of some subspace of  $V$ .

(b) Let  $B = w + Y$  for some subspace  $Y \subseteq V$ . Then we have that  $v_k = w + y_k$ . Take  $x \in A$ , then we know that

$$x = \sum_{i=1}^m \lambda_i v_i = \sum_{i=1}^m \lambda_i (w + y_i) = w + \sum_{i=1}^m \lambda_i y_i \in B$$

(c) Write  $A = w + U$ . Denote  $B = \text{span}\{v_1, \dots, v_m\}$ . Then by (b) we know that  $A \subseteq B$ . We will show that  $\dim U < m$ . The statement is trivial if  $v_1, \dots, v_m$  is not linearly independent. Let's consider the case that it's indeed linearly independent. Here we claim that  $w \notin U$  and thus  $\dim U < m$ . To see this, the only way to write  $\sum_{i=1}^m \lambda_i v_i = 0$  is to let all  $\lambda_i = 0$ , but in this case the vectors will not be in  $A$ , so  $0 \notin A$ . Suppose for the sake of contradiction that  $w \in U$ . This implies that  $0 = w + (-w) \in w + U = A$ , forming a contradiction. Hence we finish the proof.  $\square$

### Problem 13

Suppose  $U$  is a subspace of  $V$  such that  $V/U$  is finite-dimensional. Prove that  $V$  is isomorphic to  $U \times (V/U)$ .

*Proof.*

$$\dim V = \dim U + (\dim V - \dim U) = \dim U + \dim(V/U).$$

$\square$

### Problem 14

Suppose  $U$  and  $W$  are two subspaces of  $V$  and  $V = U \oplus W$ . Suppose  $w_1, \dots, w_m$  is a basis of  $W$ . Prove that  $w_1 + U, \dots, w_m + U$  is a basis of  $V/U$ .

*Proof.* First we show that the list is linearly independent. We consider the following system:

$$0 + U = \lambda_1(w_1 + U) + \cdots + \lambda_m(w_m + U) = (\lambda_1 w_1 + \cdots + \lambda_m w_m) + U$$

This means that  $\lambda_1 w_1 + \cdots + \lambda_m w_m \in U$  and since we know that  $w_1, \dots, w_m \notin U$ , we have that the only solution is  $\lambda_i = 0$  for all  $i$ .

Next we show that the list spans  $V/U$ . Take arbitrary  $v + U \in V/U$ , then we have that  $v = u + w$  for some  $w = \sum_{i=1}^m \lambda_i w_i$ . Then we have that

$$v + U = \sum_{i=1}^m \lambda_i w_i + U = \sum_{i=1}^m \lambda_i (w_i + U).$$

Thus the list spans the quotient space. □

### Problem 15

Suppose  $U$  is a subspace of  $V$  and  $v_1 + U, \dots, v_m + U$  is a basis of  $V/U$  and  $u_1, \dots, u_n$  is a basis of  $U$ . Prove that  $v_1, \dots, v_m, u_1, \dots, u_n$  is a basis of  $V$ .

*Proof.* We first know that

$$\dim V = \dim V/U + \dim U.$$

So it only suffices to prove the list either spans  $V$  or is linearly independent. Take any  $v \in V$ . Then we have that

$$v + U = \lambda_1 v_1 + \cdots + \lambda_m v_m + U.$$

Thus  $v - \sum_{i=1}^m \lambda_i v_i \in U$ . So

$$v = \sum_{i=1}^m \lambda_i v_i + \sum_{j=1}^n \alpha_j u_j.$$

Hence we finish the proof. □

### Problem 16

Suppose  $\varphi \in \mathcal{L}(V, \mathbb{F})$  and  $\varphi \neq 0$ . Prove that  $\dim V/(\text{null } \varphi) = 1$ .

*Proof.* We know that  $\dim \text{range } \varphi = 1$  and thus

$$\dim V/\text{null } \varphi = \dim V - \dim \text{null } \varphi = \dim \text{range } \varphi = 1.$$

□

### Problem 17

Suppose that  $U$  is a subspace of  $V$  such that  $\dim V/U = 1$ . Prove that there exists  $\varphi \in \mathcal{L}(V, \mathbb{F})$  such that  $\text{null } \varphi = U$ .

*Proof.* We make construction as follows: (1) there exists a natural isomorphism  $T: V/U \rightarrow \mathbb{F}$ . We further define  $S: V \rightarrow V/U$  by  $S(v) = v + U$ . Then we show the map

$$\varphi = TS$$

satisfies the requirement.

First take  $u \in U$ . Then we have that

$$\varphi(u) = TS(u) = T(u + U) = T(0) = 0$$

and thus  $u \in \text{null } \varphi$ .

Conversely take  $u \in \text{null } \varphi$ , then

$$0 = \varphi(u) = T(S(u)) = T^{-1}T(S(u)) = S(u).$$

Thus  $u \in \text{null } S = U$ . We've finished the proof.  $\square$

### Problem 18

Suppose  $U$  is a subspace of  $V$  such that  $V/U$  is finite-dimensional.

(a) Show that if  $W$  is a finite-dimensional subspace of  $V$  and  $V = U + W$ , then  $\dim W \geq \dim V/U$ .

(b) Prove that there exists a finite-dimensional subspace  $W$  of  $V$  such that  $\dim W = \dim V/U$  and  $V = U \oplus W$ .

*Proof.* (a) We know that

$$\dim W = \dim V + \dim(U \cap W) - \dim U \geq \dim V - \dim U = \dim V/U.$$

(b) We will construct the space  $W$  through some manipulation with the basis. Since we are given  $V/U$ , we start with the basis  $w_1 + U, \dots, w_m + U$  of that. Note that  $\{w_1, \dots, w_m\}$  is linearly independent as the only solution is all-zero. We can now define

$$W = \text{span}\{w_1, \dots, w_m\}.$$

Then we have that  $\dim W = \dim V/U$  and now it suffices to verify  $V = U \oplus W$ . To see this, one only needs to show  $W \cap U = \{0\}$  as we've proved that  $V = W + U$  in previous questions (of similar construction). Take arbitrary  $v \in W \cap U$ . Then we have that

$$v = \sum_{i=1}^m \lambda_i w_i$$

and that

$$v + U = \sum_{i=1}^m \lambda_i w_i + U = 0 + U$$

The only solution is all  $\lambda_i = 0$ .  $\square$

**Problem 19**

Suppose  $T \in \mathcal{L}(V, W)$  and  $U$  is a subspace of  $V$ . Let  $\pi$  denote the quotient map from  $V$  onto  $V/U$ . Prove that there exists  $S \in \mathcal{L}(V/U, W)$  such that  $T = S \circ \pi$  if and only if  $U \subseteq \text{null } T$ .

*Proof.*  $\Rightarrow$  Take  $u \in U$ , then we have that

$$T(u) = S\pi(u) = S(u + U) = S(0) = 0$$

and thus  $u \in \text{null } T$ .

$\Leftarrow$  We define  $S: V/U \rightarrow W$  by

$$S(v + U) = T(v)$$

It now only suffices to show that this map is valid for the equivalence mapping. Take  $v_1 + U = v_2 + U$ . Then we have that  $v_1 - v_2 \in U \subseteq \text{null } T$  and thus  $T(v_1 - v_2) = 0 = Tv_1 - Tv_2$  and thus  $S(v_1 + U) = Tv_1 = Tv_2 = S(v_2 + U)$ . This map is clearly linear and thus we have  $T = S \circ \pi$ .  $\square$



### 3F: Duality

**Definition 70** (linear functional). A **linear functional** on  $V$  is a linear map from  $V$  to  $\mathbb{F}$ . In other words, a linear functional is an element of  $\mathcal{L}(V, \mathbb{F})$ .

**Definition 71** (dual space,  $V'$ ). The **dual space** of  $V$ , denoted by  $V'$ , is the vector space of all linear functionals on  $V$ . In other words,  $V' = \mathcal{L}(V, \mathbb{F})$ .

**Lemma 72.** Suppose  $V$  is finite-dimensional. Then  $V'$  is also finite-dimensional and

$$\dim V' = \dim V.$$

**Definition 73** (dual basis). If  $v_1, \dots, v_n$  is a basis of  $V$ , then the **dual basis** of  $v_1, \dots, v_n$  is the list  $\varphi_1, \dots, \varphi_n$  of elements of  $V'$ , where each  $\varphi_j$  is the linear functional on  $V$  such that

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j. \end{cases}$$

**Theorem 74** (dual basis gives coefficients for linear combination). Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $\varphi_1, \dots, \varphi_n$  is the dual basis. Then

$$v = \varphi_1(v)v_1 + \dots + \varphi_n(v)v_n$$

for each  $v \in V$ .

**Theorem 75** (dual basis is a basis of the dual space). Suppose  $V$  is finite-dimensional. Then the dual basis of a basis of  $V$  is a basis of  $V'$ .

**Definition 76** (dual map,  $T'$ ). Suppose  $T \in \mathcal{L}(V, W)$ . The **dual map** of  $T$  is the linear map  $T' \in \mathcal{L}(W', V')$  defined for each  $\varphi \in W'$  by

$$T'(\varphi) = \varphi \circ T.$$

**Corollary 77** (algebraic properties of dual map). Suppose  $T \in \mathcal{L}(V, W)$ . Then

- (a)  $(S + T)' = S' + T'$  for all  $S \in \mathcal{L}(V, W)$ ;
- (b)  $(\lambda T)' = \lambda T'$  for all  $\lambda \in \mathbb{F}$ ;
- (c)  $(ST)' = T'S'$  for all  $S \in \mathcal{L}(W, U)$ .

The goal of this section is to describe null  $T'$  and range  $T'$  in terms of range  $T$  and null  $T$ .

**Definition 78** (annihilator,  $U^0$ ). For  $U \subseteq V$ , the **annihilator** of  $U$ , denoted by  $U^0$ , is defined by

$$U^0 = \{\varphi \in V' : \varphi(u) = 0 \text{ for all } u \in U\}.$$

**Remark 79.**  $U^0$  is a subspace of  $V'$ .

**Theorem 80** (dimension of the annihilator). *Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then*

$$\dim U^0 = \dim V - \dim U.$$

**Lemma 81.** *Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Then*

- (a)  $U^0 = 0 \Leftrightarrow U = V$ ;
- (b)  $U^0 = V' \Leftrightarrow U = \{0\}$ .

**Lemma 82** (the null space of  $T'$ ). *Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then*

- (a)  $\text{null } T' = (\text{range } T)^0$ ;
- (b)  $\dim \text{null } T' = \dim \text{null } T + \dim W - \dim V$ .

**Theorem 83** ( $T$  surjective is equivalent to  $T'$  injective). *Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then*

$$T \text{ is surjective} \Leftrightarrow T' \text{ is injective}.$$

**Lemma 84** (the range of  $T'$ ). *Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then*

- (a)  $\dim \text{range } T' = \dim \text{range } T$ ;
- (b)  $\text{range } T' = (\text{null } T)^0$ .

**Theorem 85** ( $T$  surjective is equivalent to  $T'$  injective). *Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then*

$$T \text{ is surjective} \Leftrightarrow T' \text{ is injective}.$$

**Theorem 86** (matrix of  $T'$  is transpose of matrix of  $T$ ). *Suppose  $V$  and  $W$  are finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Then*

$$\mathcal{M}(T') = (\mathcal{M}(T))^{\top}$$

*Proof.* Let  $v_1, \dots, v_n$  be basis of  $V$  and  $w_1, \dots, w_m$  be basis of  $W$ . Let  $\varphi_1, \dots, \varphi_n$  be the dual basis of  $V'$  and  $\psi_1, \dots, \psi_m$  be the dual basis of  $W'$ .

Let  $A = \mathcal{M}(T)$  and  $C = \mathcal{M}(T')$ . Suppose  $1 \leq j \leq m$  and  $1 \leq k \leq n$ . From the definition of  $\mathcal{M}(T')$  we have

$$T'(\psi_j) = \sum_{r=1}^n C_{r,j} \varphi_r$$

Then applying both sides of the equation above to  $v_k$  gives that

$$\begin{aligned} (\psi_j \circ T)(v_k) &= \sum_{r=1}^n C_{r,j} \varphi_r(v_k) \\ &= C_{k,j} \end{aligned}$$

At the same time, we have that

$$\begin{aligned}
(\psi_j \circ T)(v_k) &= \psi_j(Tv_k) \\
&= \psi_j\left(\sum_{r=1}^m A_{r,k} w_r\right) \\
&= \sum_{r=1}^m A_{r,k} \psi_j(w_r) \\
&= A_{j,k}
\end{aligned}$$

Here we have that  $C_{k,j} = A_{j,k}$  and thus  $C = A^\top$ . Hence,  $\mathcal{M}(T') = (\mathcal{M}(T))^\top$ , as desired. □

We can use duality to provide an alternative proof for the rank of matrix:

**Theorem 87** (column rank equals row rank). *Suppose  $\mathbf{A} \in \mathbb{F}^{m,n}$ . Then the column rank of  $\mathbf{A}$  equals the row rank of  $\mathbf{A}$ .*

**Problem 3**

Suppose  $V$  is finite-dimensional and  $v \in V$  with  $v \neq 0$ . Prove that there exists  $\varphi \in V'$  such that  $\varphi(v) = 1$ .

*Proof.* The main requirement is to ensure that the constructed  $\varphi$  is valid. Note that there exists subspace  $W \subseteq V$  such that  $V = W \oplus \text{span}\{v\}$ . Thus we can define  $\varphi(u) = i$  where  $u = w + iv$  for  $u \in V, w \in W, i \in \mathbb{F}$ . Here  $\varphi$  is a valid linear map and  $\varphi(v) = 1$ .  $\square$

**Problem 6**

Suppose  $\varphi, \beta \in V'$ . Prove that  $\text{null } \varphi \subseteq \text{null } \beta$  if and only if there exists  $c \in \mathbb{F}$  such that  $\beta = c\varphi$ .

*Proof.*  $\Rightarrow$  If  $\varphi = 0$ , then it trivially holds. If  $\varphi \neq 0$ , then we can first take  $v_0 \in V$  s.t.  $\varphi(v_0) \neq 0$ , then define  $c = \beta(v_0)/\varphi(v_0)$ , then we have that  $\beta(v) = \beta(v_0 + u) = \beta(v_0) = c\varphi(v_0) = c\varphi(v)$  for some  $u \in \text{null } \varphi$ .

$\Leftarrow$  Take  $v \in \text{null } \varphi$ , then  $\beta(v) = c\varphi(v) = 0$ . Thus  $\text{null } \varphi \subseteq \text{null } \beta$ .  $\square$

**Problem 8**

Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $\varphi_1, \dots, \varphi_n$  is the dual basis of  $V'$ . Define  $\Gamma: V \rightarrow \mathbb{F}^n$  and  $\Lambda: \mathbb{F}^n \rightarrow V$  by

$$\Gamma(v) = (\varphi_1(v), \dots, \varphi_n(v)) \text{ and } \Lambda(a_1, \dots, a_n) = a_1v_1 + \dots + a_nv_n.$$

Explain why  $\Gamma$  and  $\Lambda$  are inverses of each other.

*Proof.* Take  $a_1, \dots, a_n \in \mathbb{F}^n$ , then

$$\begin{aligned} \Gamma(\Lambda(a_1, \dots, a_n)) &= \Gamma(a_1v_1 + \dots + a_nv_n) \\ &= (\varphi_1(a_1v_1), \dots, \varphi_n(a_nv_n)) \\ &= (a_1, \dots, a_n) \end{aligned}$$

Similarly, take  $v = a_1v_1 + \dots + a_nv_n$ , we can get that

$$\Lambda(\Gamma(v)) = v$$

$\square$

**Problem 9**

Suppose  $m$  is a positive integer. Show that the dual basis of the basis  $1, x, \dots, x^m$  of  $\mathcal{P}_m(\mathbb{R})$  is  $\varphi_0, \varphi_1, \dots, \varphi_m$ , where

$$\varphi_k(p) = \frac{p^{(k)}(0)}{k!}.$$

*Proof.* We consider different cases. If  $j = k$ , then

$$\varphi_k(x^j) = \frac{(x^j)^{(k)}}{k!} = \frac{k!}{k!} = 1$$

If  $j < k$ , then

$$\varphi_k(x^j) = \frac{(x^j)^{(k)}}{k!} = \frac{0^{j-k} j! / (j-k)!}{k!} = 0$$

If  $j > k$ , then

$$\varphi_k(x^j) = \frac{(x^j)^{(k)}}{k!} = \frac{0}{k!} = 0$$

Hence, we have that

$$\varphi_k(x_j) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{o.w.} \end{cases}$$

□

**Problem 11**

Suppose  $v_1, \dots, v_n$  is a basis of  $V$  and  $\varphi_1, \dots, \varphi_n$  is the corresponding dual basis of  $V'$ . Prove that

$$\psi = \psi(v_1)\varphi_1 + \dots + \psi(v_n)\varphi_n.$$

*Proof.* By property of dual basis, we have that

$$\psi = a_1\varphi_1 + \dots + a_n\varphi_n$$

Apply  $v_k$  on both sides give that

$$\begin{aligned} \psi(v_k) &= a_1\varphi_1(v_k) + \dots + a_n\varphi_n(v_k) \\ &= a_k \end{aligned}$$

Substitue this back gives the desired equality.

□

**Problem 13**

Show that the dual map of the identity operator on  $V$  is the identity operator on  $V'$ .

*Proof.* Take arbitrary  $f \in V'$ , then we have that

$$I'(f)(v) = f \circ I(v) = f(v)$$

for all  $v \in V$ .

□

**Problem 16**

Suppose  $W$  is finite-dimensional and  $T \in \mathcal{L}(V, W)$ . Prove that

$$T' = 0 \Leftrightarrow T = 0.$$

*Proof.* This is obvious from  $\dim \text{range } T' = \dim \text{range } T$ .  $\square$

**Problem 19**

Suppose  $U \subseteq V$ . Explain why

$$U^0 = \{\varphi \in V' : U \subseteq \text{null } \varphi\}$$

*Proof.* This follows from definition.  $\square$

**Problem 20**

Suppose  $V$  is finite-dimensional and  $U$  is a subspace of  $V$ . Show that

$$U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}.$$

*Proof.* Denote  $K = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in U^0\}$ .

Take  $u \in U$  and  $\varphi \in U^0$ , then by definition  $\varphi(u) = 0$ , and thus  $u \in K$ .

Conversely, take  $\varphi \in U^0$ . Suppose for the sake of contradiction that there exist  $v \notin U$  but  $v \in K$ , then  $\varphi(v) = 0$ , contradicting that  $v \in K$ , thus completing the proof.  $\square$

**Problem 21**

Suppose  $V$  is finite-dimensional and  $U$  and  $W$  are subspaces of  $V$ .

- (a) Prove that  $W^0 \subseteq U^0$  if and only if  $U \subseteq W$ .
- (b) Prove that  $W^0 = U^0$  if and only if  $U = W$ .

*Proof.* (a)  $\Rightarrow$  We know that there exists  $\psi \in V'$  such that  $\text{null } \psi = W$ . Then  $\psi \in W^0$  and thus  $\psi \in U^0$ . This means that  $U \subseteq W = \text{null } \psi$ .

$\Leftarrow$  Take  $\psi \in W^0$ . So we know  $\psi(w) = 0$  for all  $w \in W$ . Suppose for the sake of contradiction that  $\psi \notin U^0$ , then there exists  $u \in U$  s.t.  $\psi(u) \neq 0$ . However, as  $u \in W$ , we have reached a contradiction.

(b)  $W^0 = U^0 \Leftrightarrow W_0 \subseteq U_0$  and  $U_0 \subseteq W_0 \Leftrightarrow U \subseteq W$  and  $W \subseteq U \Leftrightarrow U = W$ .  $\square$

**Problem 22**

Suppose  $V$  is finite-dimensional and  $U$  and  $W$  are subspaces of  $V$ .

- (a) Show that  $(U + W)^0 = U^0 \cap W^0$ .
- (b) Show that  $(U \cap W)^0 = U^0 + W^0$ .

*Proof.* Note that we have

$$\begin{aligned}(U + W)^0 &= \{\varphi \in V' : \varphi(v) = 0 \text{ for every } v \in U + W\} \\ (U \cap W)^0 &= \{\varphi \in V' : \varphi(v) = 0 \text{ for every } v \in U \cap W\}\end{aligned}$$

(a) Take  $\varphi \in (U + W)^0$ , then we know that  $\varphi(u) = 0$  and  $\varphi(w) = 0$  for all  $u \in U, w \in W$ . Thus  $(U + W)^0 \subseteq U^0 \cap W^0$ . Conversely, take  $\varphi \in U^0 \cap W^0$ , then we know that for every  $v = u + w, \varphi(v) = \varphi(u) + \varphi(w) = 0$ , therefore  $U^0 \cap W^0 \subseteq (U + W)^0$ .

(b) Take  $\varphi_1 \in U^0, \varphi_2 \in W^0$ , then we have that  $\varphi_1 + \varphi_2(v) = \varphi_1(v) + \varphi_2(v) = 0$  for  $v \in U \cap W$ , therefore  $U^0 + W^0 \subseteq (U \cap W)^0$ . It suffices now to show that the dimension equal each other.

$$\dim((U \cap W)^0) = \dim(V) - \dim(U \cap W)$$

Note that

$$\begin{aligned}\dim(U^0 + W^0) &= \dim(U^0) + \dim(W^0) - \dim(U^0 \cap W^0) \\ &= (\dim(V) - \dim(U)) + (\dim(V) - \dim(W)) - \dim((U + W)^0) \\ &= 2 \dim(V) - (\dim(U) + \dim(W)) - (\dim(V) - \dim(U + W)) \\ &= \dim(V) - (\dim(U) + \dim(W) - \dim(U + W)) \\ &= \dim(V) - \dim(U \cap W)\end{aligned}$$

Thus we have completed the proof.  $\square$

### Problem 23

Suppose  $V$  is finite-dimensional and  $\varphi_1, \dots, \varphi_m \in V'$ . Prove that the following three sets are equal to each other.

- (a)  $\text{span}(\varphi_1, \dots, \varphi_m)$ .
- (b)  $((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m))^0 := A$
- (c)  $\{\varphi \in V' : (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m) \subseteq \text{null } \varphi\} := B$

*Proof.* (a)  $\Rightarrow$  (b) take  $\varphi = \sum_{i=1}^m a_i \varphi_i \in \text{span}(\varphi_1, \dots, \varphi_m)$ . Then we know that for every  $v \in (\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)$ ,  $\varphi(v) = \sum_{i=1}^m a_i \varphi_i(v) = 0$ .

(b)  $\Rightarrow$  (c) Take  $\varphi \in A$ , then by P19 this holds.

(c)  $\Rightarrow$  (a) Take  $\varphi \in B$ , then we know by P3 there exists  $c \in \mathbb{F}$  such that  $\varphi = c\varphi_i$  for all  $1 \leq i \leq m$ . This means that  $\varphi \in \text{span}(\varphi_1, \dots, \varphi_m)$ .  $\square$

### Problem 24

Suppose  $V$  is finite-dimensional and  $v_1, \dots, v_m \in V$ . Define a linear map  $\Gamma: V' \rightarrow \mathbb{F}^m$  by  $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m))$ .

- (a) Prove that  $v_1, \dots, v_m$  spans  $V$  if and only if  $\Gamma$  is injective.
- (b) Prove that  $v_1, \dots, v_m$  is linearly independent if and only if  $\Gamma$  is surjective.

*Proof.* (a)  $\Rightarrow$  Take  $v = \sum_{i=1}^m a_i v_i$  and  $\varphi \in \text{null } \Gamma$ . We aim to prove that  $\varphi = 0$ . We have that  $\Gamma(\varphi) = (\varphi(v_1), \dots, \varphi(v_m)) = 0$  so  $\varphi(v_1) = \dots = \varphi(v_m) = 0$ . So for all  $v \in V$ ,  $\varphi(v) = \sum_{i=1}^m a_i \varphi(v_i) = 0$ . Thus  $\varphi$  is the zero map and thus  $\Gamma$  is injective.

$\Leftarrow$  Suppose for the sake of contradiction that  $v_1, \dots, v_m$  does not span  $V$ , then this means there exists some subspace  $W$  such that  $\text{span}(v_1, \dots, v_m) \oplus W = V$ . Then there exists nonzero  $\varphi \in V'$  such that  $\varphi(v_k) = 0$  for all  $k$  (one can set the basis in  $W$  to be nonzero mapping). Then this means that  $\varphi \in \text{null } \Gamma \neq 0$ , contradicting that  $\Gamma$  is injective.

(b)  $\Rightarrow$  Let  $K$  be the subspace spanned by the linearly independent list of vectors  $v_1, \dots, v_m$ . Then by the linear map lemma, for all  $(a_1, \dots, a_m) \in \mathbb{F}^m$ , there exists a unique mapping  $T': K \rightarrow \mathbb{F}$  such that  $T'v_i = a_i$ . Extending  $T'$  to  $V$  ensures that  $\Gamma$  is surjective.

$\Leftarrow$  Suppose for the sake of contradiction that  $v_1, \dots, v_m$  are not linearly independent, then we know there exists nonzero  $a'_i$ 's such that  $a_1 v_1 + \dots + a_m v_m = 0$ . Applying any linear function  $\varphi \in V'$  to this linear combination yields that

$$\varphi\left(\sum_{i=1}^m a_i v_i\right) = \varphi(0) = 0$$

This equals that

$$\sum_{i=1}^m a_i \varphi(v_i) = \sum_{i=1}^m a_i \Gamma(\varphi)_i = 0$$

Since there is nonzero  $a_i$ 's, this means that the image of  $\Gamma$  is a strict subspace of  $\mathbb{F}^m$ , therefore contradicting the hypothesis that it's surjective.  $\square$

### Problem 25

Suppose  $V$  is finite-dimensional and  $\varphi_1, \dots, \varphi_m \in V'$ . Define a linear map  $\Gamma: V \rightarrow \mathbb{F}^m$  by  $\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v))$ .

(a) Prove that  $\varphi_1, \dots, \varphi_m$  spans  $V'$  if and only if  $\Gamma$  is injective.

(b) Prove that  $\varphi_1, \dots, \varphi_m$  is linearly independent if and only if  $\Gamma$  is surjective.

*Proof.* (a)  $\Rightarrow$  Take any  $\varphi \in V'$ , then  $\varphi = \sum_{i=1}^m a_i \varphi_i$ . Take any  $v \in V$  and let  $\Gamma(v) = 0$ . Then this means that

$$\Gamma(v) = (\varphi_1(v), \dots, \varphi_m(v)) = 0$$

So we have that  $\varphi_i(v) = 0$  for all  $i$ . By the spanning assumption, we have that for any  $\varphi \in V'$ ,  $\varphi(v) = \sum_{i=1}^m a_i \varphi_i(v) = 0$ . Thus  $v = 0$  and then  $\text{null } (\Gamma) = \{0\}$ .

$\Leftarrow$  Suppose  $\varphi_1, \dots, \varphi_m$  does not span  $V'$ , then this means there exists some subspace  $W$  such that  $\text{span}(\varphi_1, \dots, \varphi_m) \oplus W = V'$ . Let  $\varphi_1, \dots, \varphi_n$  be the basis of  $\text{span}(\varphi_1, \dots, \varphi_m)$ ,  $\varphi_{n+1}, \dots, \varphi_l$  to be the basis of  $W$ . Let  $v_1, \dots, v_n, v_{n+1}, \dots, v_l$  be the corresponding basis for  $V$ . Then there exists nonzero  $v \in V$  such that



$\varphi_k(v) = 0$  for all  $k$  as  $v = \sum_{i=1}^n \varphi_i(v)v_i + \sum_{j=n+1}^l \varphi_j(v)v_j$ , where one can obtain nonzero  $\varphi_j(v)$  for some  $n+1 \leq j \leq l$  but zero  $\varphi_i(v)$  for all  $1 \leq i \leq n$ . Such  $v \in \text{null } \Gamma$ , contradicting the hypothesis that  $\Gamma$  is injective.

(b)  $\Rightarrow$  Let  $K$  be the subspace spanned by the linearly independent list of vectors  $\varphi_1, \dots, \varphi_m$ . Then for all  $(a_1, \dots, a_m) \in \mathbb{F}^m$ , take  $v = \sum_{i=1}^m a_i v_i$  for the corresponding basis  $v_1, \dots, v_m$ . Then we have that  $\varphi_i(v) = a_i$  so  $\Gamma$  is surjective.

$\Leftarrow$  Suppose for the sake of contradiction that  $\varphi_1, \dots, \varphi_m$  are not linearly independent, then we know there exists nonzero  $a_i$ 's such that  $a_1\varphi_1 + \dots + a_m\varphi_m = 0$ . Applying any vector  $v \in V$  to this linear functional yields that

$$\sum_{i=1}^m a_i \varphi_i(v) = 0(v) = 0$$

This equals that

$$\sum_{i=1}^m a_i \varphi_i(v) = \sum_{i=1}^m a_i \Gamma(v)_i = 0$$

Since there is nonzero  $a_i$ 's, this means that the image of  $\Gamma$  is a strict subspace of  $\mathbb{F}^m$ , therefore contradicting the hypothesis that it's surjective.  $\square$

#### Problem 26

Suppose  $V$  is finite-dimensional and  $\Omega$  is a subspace of  $V'$ . Prove that

$$\Omega = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}^0$$

*Proof.* Denote  $U = \{v \in V : \varphi(v) = 0 \text{ for every } \varphi \in \Omega\}$ . So we have that

$$U^0 = \{\psi \in V' : \psi(v) = 0 \text{ for every } v \in U\}$$

$\Rightarrow$  Take  $\varphi \in \Omega$  and  $v \in U$ , then by definition  $\varphi(v) = 0$  and thus  $\varphi \in U^0$ .

$\Leftarrow$  Take  $\psi \in U^0$ , suppose  $\psi \notin \Omega$ . Take  $v \in U$ , then we know that for every  $\varphi \in \Omega$ ,  $\varphi(v) = 0$ . Since  $\psi \notin \Omega$ , there exists  $v$ , s.t.  $\psi(v) \neq 0$ . However, this contradicts that  $\psi(v) = 0$  for all  $v \in U$ , the proof is completed.  $\square$

#### Problem 28

Suppose  $V$  is finite-dimensional and  $\varphi_1, \dots, \varphi_m$  is a linearly independent list in  $V'$ . Prove that

$$\dim((\text{null } \varphi_1) \cap \dots \cap (\text{null } \varphi_m)) = (\dim V) - m$$

*Proof.* Note that  $(\bigcap_{i=1}^m \text{null } \varphi_i)^0 = \text{span}(\varphi_1, \dots, \varphi_m)$  by P23. Then we have that  $\dim \bigcap_{i=1}^m \text{null } \varphi_i = \dim V - \dim \text{span}(\varphi_1, \dots, \varphi_m) = \dim V - m$ .  $\square$

**Problem 30**

Suppose  $V$  is finite-dimensional and  $\varphi_1, \dots, \varphi_n$  is a basis of  $V'$ . Show that there exists a basis of  $V$  whose dual basis is  $\varphi_1, \dots, \varphi_n$ .

*Proof.* We know that  $\dim V' = n$  and that  $\dim \text{null } \varphi_i + 1(\dim \text{range } \varphi_i) = n$ . Thus we can construct that  $V = \text{null } \varphi_i \oplus U_i$  for all  $1 \leq i \leq n$ . Here  $\dim U_i = 1$ . Here we extract all  $v_i$ 's from those  $U_i$ . By linear map lemma, there exists  $v_i \in U_i$  such that  $\varphi_i(v_i) = 1$ . We claim that  $v_1, \dots, v_n$  is the corresponding basis of  $V$ .

We first verify that it is the “corresponding basis”. Consider  $\varphi_j(v_i)$  for  $j \neq i$ . Suppose for contradiction that there exists nonzero  $w \in U_i \cap U_k$ . Since we know all  $U_i$  are 1-d,  $U_i = U_k$ . Here we have that  $\text{null } \varphi_i = \text{null } \varphi_k$ , then this means that  $\varphi_i = c\varphi_k$  for some  $c \in \mathbb{F}$ , but this contradicts the assumption that  $\varphi_1, \dots, \varphi_n$  is the basis of  $V'$ .

Next we verify it is basis. It only suffices to verify that the list is linearly independent. To see this,  $0 = \sum_{i=1}^m a_i v_i$ . Apply  $\varphi_i$  on both side yield that  $0 = \varphi_i(0) = \varphi(\sum_{i=1}^m a_i v_i) = a_i$ . Thus all coefficients are zero.  $\square$

**Problem 32**

The *double dual* space of  $V$ , denoted by  $V''$ , is defined to be the dual space of  $V'$ . In other words,  $V'' = (V')'$ . Define  $\Lambda: V \rightarrow V''$  by

$$(\Lambda v)(\varphi) = \varphi(v)$$

for each  $v \in V$  and  $\varphi \in V'$ .

- (a) Show that  $\Lambda$  is a linear map from  $V$  to  $V''$ .
- (b) Show that if  $T \in \mathcal{L}(V)$ , then  $T'' \circ \Lambda = \Lambda \circ T$ , where  $T'' = (T')'$ .
- (c) Show that if  $V$  is finite-dimensional, then  $\Lambda$  is an isomorphism from  $V$  onto  $V''$ .

*Proof.* (a) First,  $\Lambda(0)(\varphi) = \varphi(0) = 0$ . Next, take  $v_1, v_2 \in V$ , then  $\Lambda(\lambda v_1 + v_2)(\varphi) = \lambda\varphi(v_1) + \varphi(v_2) = \lambda\Lambda(v_1) + \Lambda(v_2)$ .

(b) By definition, the double dual map  $T'': V'' \rightarrow W''$  is defined as  $T''(\psi) = \psi(T'(\varphi))$  for  $\psi \in V''$  and for all  $\varphi \in W'$ . This further equals that  $T''(\psi) = \psi\varphi \circ T(v)$  for  $v \in V$ .

$$\begin{aligned} T'' \circ (\Lambda v)(\varphi) &= (\Lambda v) \circ T'(\varphi) \\ &= (\Lambda v)\varphi \circ T \\ &= \Lambda \circ T \end{aligned}$$

(c) First we have that  $\dim V = \dim V' = \dim V''$ . Note that if we take  $v \in \text{null } \Lambda$ , then this means for all linear functional  $\varphi \in V'$ , we have that  $\Lambda v(\varphi) = \varphi(v) = 0$ . So  $v = 0$  and thus  $\Lambda$  is injective and therefore an isomorphism.  $\square$