Chapter 4: Polynomials

Linear Algebra Done Right, by Sheldon Axler

Definition 1 (Re z, Im z)**.** *Suppose $z = a + bi$, where a and b are real numbers.*

- *The **real part** of z, denoted by Rez, is defined by $Rez = a$.*
- *The **imaginary part** of z, denoted by Imz, is defined by $Imz = a$.*

Definition 2 (complex conjugate, $\overline{z}$, absolute value, $|z|$)**.** *Suppose $z \in \mathbb{C}$.*

- *The **complex conjugate** of $z \in \mathbb{C}$, denoted by $\overline{z}$, is defined by*

$$\overline{z} = Rez - Imz$$

- *The **absolute value** of a complex number z, denoted by $|z|$, is defined by*

$$|z| = \sqrt{(Re(z))^2 + (Im(z))^2}$$

A function $p\colon \mathbb{F} \to \mathbb{F}$ is called a polynomial of degree m if there exists $a_0, \ldots, a_m \in \mathbb{F}$ with $a_m \neq 0$ such that

$$p(z) = a_0 + a_1 z + \cdots + a_m z^m$$

for all $z \in \mathbb{F}$.

Definition 3 (zero of a polynomial)**.** *A number $\lambda \in \mathbb{F}$ is called a **zero** (or **root**) of a polynomial $p \in \mathcal{P}(\mathbb{F})$ if*

$$p(\lambda) = 0.$$

Lemma 4 (each zero of a polynomial corresponds to a degree-one factor)**.** *Suppose m is a positive integer and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial of degree m. Suppose $\lambda \in \mathbb{F}$. Then $p(\lambda) = 0$ if and only if there exists a polynomial $q \in \mathcal{P}(\mathbb{F})$ of degree $m - 1$ such that*

$$p(z) = (z - \lambda)q(z)$$

for every $z \in \mathbb{F}$.

Theorem 5 (degree m implies at most m zeros)**.** *Suppose m is a positive integer and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial of degree m. Then p has at most m zeros in $\mathbb{F}$.*

Theorem 6 (division algorithm for polynomials)**.** *Suppose that $p, s \in \mathcal{P}(\mathbb{F})$ with $s \neq 0$. Then there exist unique polynomials $q, r \in \mathcal{P}(\mathbb{F})$ such that*

$$p = sq + r$$

and $\deg r < \deg s$.

Theorem 7 (fundamental theorem of algebra, first version)**.** *Every nonconstant polynomial with complex coefficients has a zero in* $\mathbb{C}$*.*

Theorem 8 (fundamental theorem of algebra, second version)**.** *If* $p \in \mathcal{P}(\mathbb{C})$ *is a nonconstant polynomial, then* p *has a unique factorization (except for the order of the factors) of the form*

$$p(z) = c(z - \lambda_1) \cdots (z - \lambda_m),$$

where $c, \lambda_1, \ldots, \lambda_m \in \mathbb{C}$*.*

Lemma 9 (polynomials with real coefficients have nonreal zeros in pairs)**.** *Suppose* $p \in \mathcal{P}(\mathbb{C})$ *is a polynomial with real coefficients. If* $\lambda \in \mathbb{C}$ *is a zero of* p*, then so is* $\overline{\lambda}$*.*

Theorem 10 (factorization of a quadratic polynomial)**.** *Suppose* $b, c \in \mathbb{R}$*. Then there is a polynomial factorization of the form*

$$x^2 + bx + c = (x - \lambda_1)(x - \lambda_2)$$

with $\lambda_1, \lambda_2 \in \mathbb{R}$ *if and only if* $b^2 \geq 4c$*.*

Theorem 11 (factorization of a polynomial over $\mathbb{R}$)**.** *Suppose* $p \in \mathcal{P}(\mathbb{R})$ *is a nonconstant polynomial. Then* p *has a unique factorization (except for the order of the factors) of the form*

$$p(x) = c(x - \lambda_1) \cdots (x - \lambda_m)(x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M)$$

where $c, \lambda_1, \ldots, \lambda_m, c_1, \ldots, c_M \in \mathbb{R}$*, with* $b_k^2 < 4c_k$ *for each* k*.*

Problem 1

Suppose $w, z \in \mathbb{C}$. Verify the following equalities and inequalities.

(a) $z + \overline{z} = 2\Re z$

(b) $z - \overline{z} = 2(\Im z)i$

(c) $z\overline{z} = |z|^2$

(d) $\overline{w+z} = \overline{w} + \overline{z}$ and $\overline{wz} = \overline{w}\overline{z}$

(e) $\overline{\overline{z}} = z$

(f) $|\Re z| \leq |z|$ and $|\Im z| =\leq |z|$

(g) $|\overline{z}| = |z|$

(h) $|wz| = |w||z|$

Proof. Suppose $z = a + bi, w = c + di$. We verify these as follows:

(a) $z + \overline{z} = (a + bi) + (a - bi) = 2a = 2\Re z$

(b) $z - \overline{z} = (a + bi) - (a - bi) = 2bi = 2(\Im z)i$

(c) $z\overline{z} = (a + bi)(a - bi) = a^2 + b^2 = |z|^2$

(d) $\overline{w+z} = \overline{(a+c) + (b+d)i} = (a+c) - (b+d)i = (a - bi) + (c - di) = \overline{z} + \overline{w}$

(e) $\overline{\overline{z}} = \overline{a - bi} = a + bi = z$

(f) $|\Re z| = a^2 \leq a^2 + b^2 = |z|^2$; $\quad |\Im z| = b^2 \leq a^2 + b^2 = |z|^2$

(g) $|\overline{z}| = |a - bi| = |a + bi| = |z|$

(h) $|wz| = |(c+di)(a+bi)| = |(ca-db)+(cb+da)i| = (ca-db)^2+(cd+ba)^2$ This further equals that $|wz| = c^2a^2 + d^2b^2 + c^2d^2 + b^2a^2$. At the same time $|w||z| = (c^2 + d^2)(a^2 + b^2) = c^2a^2 + c^2b^2 + d^2a^2 + d^2b^2$. Therefore they equal each other.

□

Problem 2

Prove the **reverse triangle inequality**: if $w, z \in \mathbb{C}$, then $||w| - |z|| \leq |w - z|$.

Proof. Suppose $z = a + bi, w = c + di$. We have that

$$\begin{aligned}
|w - z|^2 &= (w - z)(\overline{w} - \overline{z}) \\
&= w\overline{w} + z\overline{z} - w\overline{z} - z\overline{w} \\
&= |w|^2 + |z|^2 - w\overline{z} - \overline{w\overline{z}} \\
&= |w|^2 + |z|^2 - 2\Re(w\overline{z}) \\
&\leq |w|^2 + |z|^2 - 2|w\overline{z}| \\
&= |w|^2 + |z|^2 - 2|w||z| \\
&= (|w| - |z|)^2
\end{aligned}$$

Taking the square root gets the desired equality. □

Problem 3

Suppose V is a complex vector space and $\varphi \in V'$. Define $\sigma \colon V \to \mathbb{R}$ by $\sigma(v) = \Re\varphi(v)$ for each $v \in V$. Show that

$$\varphi(v) = \sigma(v) - i\sigma(iv)$$

for all $v \in V$.

Proof.

$$\begin{aligned}
\varphi(v) &= \Re\varphi(v) + \Im\varphi(v) \\
&= \sigma(v) + i\Re(-iiv) \\
&= \sigma(v) - i\sigma(iv)
\end{aligned}$$

□

Problem 6

Suppose that m and n are positive integers wih $m \leq n$, and suppose $\lambda_1, \ldots, \lambda_m \in \mathbb{F}$. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbb{F})$ with $\deg p = n$ such that $0 = p(\lambda_1) = \cdots = p(\lambda_m)$ and such that p has no other zeros.

Proof. We can first construct $p(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_m)$ for $x \in \mathbb{F}$. This is a polynomial with degree m and has exactly m zeros. To increase the degree without introducing additional roots, we can make that $p(x) = (x - \lambda_1) \cdots (x - \lambda_m)^{n-m+1}$. This polynomial doesn't have additional roots, as the proof procedure follows from the second version of fundamental theorem of algebra. □

Problem 7

Suppose that m is a nonnegative integer, $z_1, \ldots, z_{m+1}$ are distinct elements of $\mathbb{F}$, and $w_1, \ldots, w_{m+1} \in \mathbb{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbb{F})$ such that

$$p(z_k) = w_k$$

for each $k = 1, \ldots, m+1$.

Proof. Define a linear map $T\colon \mathcal{P}_m(\mathbb{F}) \to \mathbb{F}^{m+1}$ by

$$T(p) = (p(z_1), \ldots, p(z_{m+1}))$$

We can tell that T is injective because if there exists nonzero $p \in \operatorname{null} T$, then this means there are $m+1$ roots for the m-deg polynomial, forming a contradiction. Therefore, T is an isomorphism as the dimension matches.

This implies the existence of the unique solution as the question asks. □

Problem 9

Prove that every polynomial of odd degree with real coefficients has a real zero.

Proof. Suppose for the sake of contradiction that there is no real root. Then the factorization would become

$$p(x) = (x^2 + b_1 x + c_1) \cdots (x^2 + b_M x + c_M)$$

but this contradicts that the degree is odd. □

Problem 11

Suppose $p \in \mathcal{P}(\mathbb{C})$. Define $q\colon \mathbb{C} \to \mathbb{C}$ by

$$q(z) = p(z)\overline{p(\overline{z})}$$

Prove that q is a polynomial with real coefficients.

Proof. Suppose $p(z) = \sum_{i=0}^{n} a_i z^i$ for $a_i \in \mathbb{C}, z \in \mathbb{C}$. Then we can find that

$$p(\overline{z}) = \sum_{i=0}^{n} a_i \overline{z}^i$$

and thus

$$\overline{p(\overline{z})} = \overline{\sum_{i=0}^{n} a_i \overline{z}^i} = \sum_{i=0}^{n} \overline{a}_i z^i$$

Finally we can get that

$$q(z) = p(z)\overline{p(\overline{z})} = (\sum_{i=0}^{n} a_i z^i)(\sum_{i=0}^{n} \overline{a}_i z^i)$$

We can consider the general form of a coefficient of z^k:

$$c_k = \sum_{i+j=k} a_i \overline{a}_j = \sum_{j+i=k} \overline{a}_j a_i = \overline{c_k}$$

Thus the coefficients are real.

$\square$