Chapter 9: Multilinear Algebra and Determinants

Linear Algebra Done Right, by Sheldon Axler

Contents

9A: Bilinear Forms and Quadratic Forms

Definition 1 (bilinear form)**.** *A **bilinear form** on V is a function $\beta\colon V\times V\to\mathbb{F}$ such that*

$$v\mapsto\beta(v,u)\ \text{and}\ v\mapsto\beta(u,v)$$

are both linear functionals on V for every $u\in V$.

Remark 2. *A better but less popular terminology is "bilinear functional". If V is real, then the function $(u,v)\mapsto\langle u,v\rangle$ is a bilinear form. If V is complex, then it isn't.*

Remark 3. *If $\mathbb{F}=\mathbb{R}$, then a bilinear form differs from an inner product in that it does not require positive definiteness or symmetry.*

Remark 4. *A bilinear form β on V is a linear map on $V\times V$ only if $\beta=0$.*

Definition 5 ($V^{(2)}$)**.** *The set of bilinear forms on V is denoted by $V^{(2)}$.*

Definition 6 (matrix of a bilinear form, $\mathcal{M}(\beta)$)**.** *Suppose β is a bilinear form on V and $e_1,\dots,e_n$ is a basis of V. The **matrix** of β with respect to this basis is the n-by-n matrix $\mathcal{M}(\beta)$ whose entry $\mathcal{M}(\beta)_{j,k}$ in row j, column k is given by*

$$\mathcal{M}(\beta)_{j,k}=\beta(e_j,e_k)$$

If the basis $e_1,\dots,e_n$ is not clear from the context, then the notation $\mathcal{M}\big(\beta,(e_1,\dots,e_n)\big)$ is used.

Corollary 7 ($\dim V^{(2)}=(\dim V)^2$)**.** *Suppose $e_1,\dots,e_n$ is a basis of V. Then the map $\beta\mapsto\mathcal{M}(\beta)$ is an isomorphism of $V^{(2)}$ onto $\mathbb{F}^{n,n}$. Furthermore, $\dim V^{(2)}=(\dim V)^2$.*

Lemma 8 (composition of a bilinear form and an operator)**.** *Suppose β is a bilinear form on V and $T\in\mathcal{L}(V)$. Define bilinear forms α and ρ on V by*

$$\alpha(u,v)=\beta(u,Tv)\ \text{and}\ \rho(u,v)=\beta(Tu,v)$$

Let $e_1,\dots,e_n$ be a basis of V. Then

$$\mathcal{M}(\alpha)=\mathcal{M}(\beta)\mathcal{M}(T)\ \text{and}\ \mathcal{M}(\rho)=\mathcal{M}(T)^\top\mathcal{M}(\beta)$$

Theorem 9 (change-of-basis formula)**.** *Suppose $\beta\in V^{(2)}$. Suppose $e_1,\dots,e_n$ and $f_1,\dots,f_n$ are bases of V. Let*

$$A=\mathcal{M}(\beta(e_1,\dots,e_n))\ \text{and}\ B=\mathcal{M}\big(\beta,(f_1,\dots,f_n)\big)$$

and $C=\mathcal{M}\big(I,(e_1,\dots,e_n),(f_1,\dots,f_n)\big)$. Then

$$A=C^\top BC$$

Definition 10 (symmetric bilinear form, $V^{(2)}_{sym}$)**.** *A bilinear form $\rho \in V^{(2)}$ is called* ***symmetric*** *if*

$$\rho(u, w) = \rho(w, u)$$

for all $u, w \in V$. The set of symmetric bilinear forms on V is denoted by $V^{(2)}_{sym}$.

Remark 11. *For real inner product space, define $\rho(u, w) = \langle u, w\rangle \in V^{(2)}_{sym}$. Additional example include*

$$\rho(u, w) = \langle u, Tw\rangle$$

where T is self-adjoint and

$$\rho(S, T) = tr\ (ST)$$

where here $\rho\colon \mathcal{L}(V) \times \mathcal{L}(V) \to \mathbb{F}$.

Definition 12 (symmetric matrix)**.** *A square matrix A is called* ***symmetric*** *if it equals its transpose.*

Theorem 13 (symmetric bilinear forms are diagonalizable)**.** *Suppose $\rho \in V^{(2)}$. Then the following are equivalent.*

(a) ρ is a symmetric bilinear form on V.

(b) $\mathcal{M}(\rho, (e_1, \ldots, e_n))$ is a symmetric matrix for every basis $e_1, \ldots, e_n$ of V.

(c) $\mathcal{M}(\rho, (e_1, \ldots, e_n))$ is a symmetric matrix for some basis $e_1, \ldots, e_n$ of V.

(d) $\mathcal{M}(\rho, (e_1, \ldots, e_n))$ is a diagonal matrix for some basis $e_1, \ldots, e_n$ of V.

Theorem 14. *Suppose V is a real inner product space and ρ is a symmetric bilinear form on V. Then ρ has a diagonal matrix with respect to some orthonormal basis of V.*

Definition 15 (alternating bilinear form, $V^{(2)}_{alt}$)**.** *A bilinear form $\alpha \in V^{(2)}$ is called* ***alternating*** *if*

$$\alpha(v, v) = 0$$

for all $v \in V$. The set of alternating bilinear forms on V is denoted by $V^{(2)}_{alt}$.

Lemma 16 (characterization of alternating linear forms)**.** *A bilinear form α on V is alternating if and only if*

$$\alpha(u, w) = -\alpha(w, u)$$

for all $u, w \in V$.

Theorem 17. *The sets $V^{(2)}_{sym}$ and $V^{(2)}_{alt}$ are subspaces of $V^{(2)}$. Furthermore,*

$$V^{(2)} = V^{(2)}_{sym} \oplus V^{(2)}_{alt}$$

Definition 18 (quadratic form associated with a bilinear form, q_β)**.** *For β a bilinear form on V, define a function $q_\beta \colon V \to \mathbb{F}$ by $q_\beta(v) = \beta(v, v)$. A function $q \colon V \to \mathbb{F}$ is called a* ***quadratic form*** *on V if there exists a bilinear form β on V such that $q = q_\beta$.*

Corollary 19 (quadratic form on $\mathbb{F}^n$)**.** *Suppose n is a positive integer and q is a function from $\mathbb{F}^n$ to $\mathbb{F}$. Then q is a quadratic form on $\mathbb{F}^n$ if and only if there exist numbers $A_{j,k} \in \mathbb{F}$ for $j, k \in \{1, \ldots, n\}$ such that*

$$q(x_1, \ldots, x_n) = \sum_{k=1}^{n} \sum_{j=1}^{n} A_{j,k} x_j x_k$$

for all $(x_1, \ldots, x_n) \in \mathbb{F}^n$.

Theorem 20 (characterizations of quadratic forms)**.** *Suppose $q \colon V \to \mathbb{F}$ is a function. Then following are equivalent.*

(a) q is a quadratic form.

(b) There exists a unique symmetric bilinear form ρ on V such that $q = q_\rho$.

(c) $q(\lambda v) = \lambda^2 q(v)$ for all $\lambda \in \mathbb{F}$ and all $v \in V$, and the function

$$(u, w) \mapsto q(u + w) - q(u) - q(w)$$

is a symmetric bilinear form on V.

(d) $q(2v) = 4q(v)$ for all $v \in V$, and the function

$$(u, w) \mapsto q(u + w) - q(u) - q(w)$$

is a symmetric bilinear form on V.

Theorem 21 (diagonalization of quadratic form)**.** *Suppose q is a quadratic form on V.*

(a) There exist a basis $e_1, \ldots, e_n$ of V and $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ such that

$$q(x_1 e_1 + \cdots + x_n e_n) = \lambda_1 x_1^2 + \cdots + \lambda_n x_n^2$$

for all $x_1, \ldots, x_n \in \mathbb{F}$.

(b) If $\mathbb{F} = \mathbb{R}$ and V is an inner product space, then the basis in (a) can be chosen to be an orthonormal basis of V.

Remark 22. *For each quadratic form we can choose a basis such that the quadratic form looks like a weighted sum of squares of the coordinates.*

Problem 1

Prove that if β is a bilinear form on $\mathbb{F}$, then there exists $c \in \mathbb{F}$ such that

$$\beta(x, y) = cxy$$

for all $x, y \in \mathbb{F}$.

Proof. We note that since the input is taken from $\mathbb{F}$, the basis is naturally 1. So we have that

$$\beta(x, y) = x\beta(1, y) = xy\beta(1, 1) = cxy$$

where we take $c = \beta(1, 1)$. $\square$

Problem 2

Let $n = \dim V$. Suppose β is a bilinear form on V. Prove that there exist $\phi_1, \ldots, \phi_n, \tau_1, \ldots, \tau_n \in V'$ such that

$$\beta(u, v) = \phi_1(u) \cdot \tau_1(v) + \cdots + \phi_n(u) \cdot \tau_n(v)$$

for all $u, v \in V$.

Proof.

$$\beta(u, v) = \beta(\sum_{i=1}^{n} u_i e_i, \sum_{i=1}^{n} v_j e_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} u_i v_j \beta(e_i, e_j)$$

We can now define the linear function $\phi_i(u) = u_i = e_i^*(u)$ and $\tau_j'(v) = v_j = e_j^*(v)$. Then we have that

$$\beta(u, v) = \sum_{i=1}^{n} \phi_i(u) \left(\sum_{j=1}^{n} \beta(e_i, e_j) \tau_j'(v) \right) = \sum_{i=1}^{n} \phi(u) \tau_i(v)$$

$\square$

Problem 3

Suppose $\beta\colon V \times V \to \mathbb{F}$ a bilinear form on V and also is a linear functional on $V \times V$. Prove that $\beta = 0$.

Proof. First we show that $\beta \in V_{alt}^{(2)}$. Take any $u \in V$, then we have

$$\beta((u, u) + (u, u)) = 2\beta(u, u)$$
$$\beta(2u, 2u) = 4\beta(u, u)$$

this shows that $\beta(u, u) = 0$ for all u. Next we show the off-diagonal terms are 0: first,

$$\beta(u,w) = \sum_{i=1}^{n}\sum_{j=1}^{n} u_i w_j \beta(e_i, e_j) \qquad \text{bilinearity}$$

at the same time,

$$\begin{aligned}
\beta(u,w) &= \beta\left(\sum_{i=1}^{n} u_i e_i, \sum_{j=1}^{n} w_j e_j\right) \\
&= \beta\left(\sum_{i=1}^{n} (u_i e_i, w_i e_i)\right) \\
&= \sum_{i=1}^{n} \beta\left(u_i e_i, w_i e_i\right) && \text{linearity on } V \times V \\
&= \sum_{i=1}^{n} u_i w_i \beta(e_i, e_i)
\end{aligned}$$

This shows that all off-diagonal terms are 0, i.e., $\beta(e_i, e_j) = 0$ for all $i \neq 0$. Therefore, $\beta = 0$. $\square$

Problem 6

Prove or give a counterexample: If ρ is a symmetric bilinear form on V, then

$$\{v \in V : \rho(v,v) = 0\}$$

is a subspace of V.

Proof. Consider $V = \mathbb{R}^2$ and $\rho(x,y) = x_1 y_1 - x_2 y_2$. Let $x = (1,1), y = (-1,1)$, then we have that $\rho(x,x) = 1 - 1 = 0, \rho(y,y) = 1 - 1 = 0$, but $\rho(x+y, x+y) = 0 - 4 = -4 \neq 0$. $\square$

Problem 8

Find formulas for $\dim V^{(2)}_{sym}$ and $\dim V^{(2)}_{alt}$ in terms of $\dim V$.

Proof. Let $\dim V = n$. For $\beta \in V^{(2)}_{sym}$, consider $\mathcal{M}(\beta)$. Its diagonal entries can be chosen arbitrarily. For off-diagonal entries, only half of them can be chosen arbitrarily, therefore the dimension is

$$\frac{n(n-1)}{2} + n = \frac{n(n+1)}{2}$$

For $\beta \in V^{(2)}_{alt}$, consider $\mathcal{M}(\beta)$. The diagonal entires are all 0 and only half of the off-diagonal entries can be chosen arbitrarily. Therefore the dimension is $\frac{n(n-1)}{2}$. $\square$

9B: Alternating Multilinear Forms

Definition 23 (V^m)**.** *For m a positive integer, define V^m by*

$$V^m = \underbrace{V \times \cdots \times V}_{m \text{ times}}$$

Definition 24 (m-linear form, $V^{(m)}$, multilinear form)**.** *Below we introduce the definitions.*

- *For m a positive integer, an **m-linear form** on V is a function $\beta\colon V^m \to \mathbb{F}$ that is linear in each slot when the other slots are held fixed. This means that for each $k \in \{1, \ldots, m\}$ and all $u_1, \ldots, u_m \in V$, the function*

  $$v \mapsto \beta(u_1, \ldots, u_{k-1}, v, u_{k+1}, \ldots, u_m)$$

  is a linear map from V to $\mathbb{F}$.

- *The set of m-linear forms on V is denoted by $V^{(m)}$.*

- *A function β is called a **multilinear form** on V if it is an m-linear form on V for some positive integer m.*

Remark 25. *A 1-linear form on V is a linear functional on V. A 2-linear form on V is a bilinear form on V. $V^{(m)}$ is a vector space.*

Example (m-linear forms)*.* Suppose $\alpha, \beta \in V^{(2)}$. Define a function $\beta\colon V^4 \to \mathbb{F}$ by

$$\beta(v_1, v_2, v_3, v_4) = \alpha(v_1, v_2)\beta(v_3, v_4)$$

Then $\beta \in V^{(4)}$.

Example (m-linear forms)*.* Define $\beta\colon \left(\mathcal{L}(V)\right)^m \to \mathbb{F}$ by

$$\beta(T_1, \ldots, T_m) = \text{tr}\,(T_1 \cdots T_m)$$

Then β is an m-linear form on $\mathcal{L}(V)$.

Definition 26 (alternating forms, $V^{(m)}_{alt}$)**.** *Suppose m is a positive integer.*

- *An m-linear form α on V is called **alternating** if $\alpha(v_1, \ldots, v_m) = 0$ whenever $v_1, \ldots, v_m$ is a list of vectors in V with $v_j = v_k$ for some two distinct values of j and k in $\{1, \ldots, m\}$.*

- *$V^{(m)}_{alt} = \{\alpha \in V^{(m)} \colon \alpha$ is an alternating m-linear form on $V\}$.*

Corollary 27. *Suppose m is a positive integer and α is an alternating m-linear form on V. If $v_1, \ldots, v_m$ is a linearly dependent list in V, then*

$$\alpha(v_1, \ldots, v_m) = 0$$

Corollary 28. *Suppose $m > \dim V$. Then 0 is the only alternating m-linear form on V.*

Theorem 29 (swapping input vectors in an alternating multilinear form)**.** *Suppose m is a positive integer, α is an alternating m-linear form on V, and $v_1, \ldots, v_m$ is a list of vectors in V. Then swapping the vectors in any two slots of $\alpha(v_1, \ldots, v_m)$ changes the value of α by a factor of -1.*

Remark 30. *An odd numer of swaps cause the value of α to change by a factor of -1 and it won't change with an even number of swaps.*

Definition 31 (permutation, perm m)**.** *Suppose m is a positive integer.*

- *A **permutation** of $(1, \ldots, m)$ is a list $(j_1, \ldots, j_m)$ that contains each of the number $1, \ldots, m$ exactly once.*
- *The set of permutations of $(1, \ldots, m)$ is denoted by perm m.*

Definition 32 (sign of a permutation)**.** *The **sign** of a permutation $(j_1, \ldots, j_m)$ is defined by*

$$sign(j_1, \ldots, j_m) = (-1)^N$$

where N is the number of pairs of integers (k, l) with $1 \leq k < l \leq m$ such that k appears after l in the list $(j_1, \ldots, j_m)$.

Lemma 33. *Swapping two entries in a permutation multiplies the sign of the permutation by -1.*

Lemma 34 (permutation and alternating multilinear form)**.** *Suppose m is a positive integer and $\alpha \in V_{alt}^{(m)}$. Then*

$$\alpha(v_{j_1}, \ldots, v_{j_m}) = \big(sign\,(j_1, \ldots, j_m)\big)\, \alpha\,(v_1, \ldots, v_m)$$

for every list $v_1, \ldots, v_m$ of vectors in V and all $(j_1, \ldots, j_m) \in perm\ m$.

Theorem 35. *Let $n = \dim V$. Suppose $e_1, \ldots, e_n$ is a basis of V and $v_1, \ldots, v_n \in V$. For each $k \in \{1, \ldots, n\}$, let $b_{1,k}, \ldots, b_{n,k} \in \mathbb{F}$ be such that*

$$v_k = \sum_{j=1}^{n} b_{j,k} e_j$$

Then

$$\alpha(v_1, \ldots, v_n) = \alpha(e_1, \ldots, e_n) \sum_{(j_1, \ldots, j_n) \in perm\ n} \big(sign\,(j_1, \ldots, j_n)\big)\, b_{j_1,1} \cdots b_{j_n,n}$$

for every alternating n-linear form α on V.

Theorem 36. *The vector space $V_{alt}^{(dimV)}$ has dimension one.*

Corollary 37. *Let $n = \dim V$,. Suppose α is a nonzero alternating n-linear form on V and $e_1, \ldots, e_n$ is a list of vectors in V. Then*

$$\alpha(e_1, \ldots, e_n) \neq 0$$

if and only if $e_1, \ldots, e_n$ is linearly independent.

Problem 1

Suppose m is a positive integer. Show that $\dim V^{(m)} = (\dim V)^m$.

Proof. Let $\dim V = n$ with basis $e_1, \ldots, e_n$. The basis vector for $V^{(m)}$ can be formed via taking all possible m-tuples $b_{j_1}, \ldots, b_{j_m}$ where b_{j_i} is a component of the basis. There are n choices over m positions, so we have that $\dim V^{(m)} = (\dim V)^m$. $\square$

Problem 3

Suppose m is a positive integer and α is an m-linear form on V such that $\alpha(v_1, \ldots, v_m) = 0$ whenver $v_1, \ldots, v_m$ is a list of vectors in V with $v_j = v_{j+1}$ for some $j \in \{1, \ldots, m-1\}$ Prove that α is an alternating m-linear form on V.

Proof. Note that if the list $v_1, \ldots, v_n$ comes with consecutive identical numbers, then by definition the output becomes 0. To prove α to be an alternating m-linear form, considers $v_i = v_k$ for $i + 1 < k$. Note that then we can now just swap and gets the same result:

$$\alpha(v_1, \ldots, v_i, v_{i+1}, \ldots, v_k, \ldots, v_n) = -\alpha(v_1, \ldots, v_i, v_k, \ldots, v_{i+1}, \ldots, v_n) = 0$$

$\square$

Problem 5

Suppose m is a positive integer and β is an m-linear form on V. Define an m-linear form α by

$$\alpha(v_1, \ldots, v_m) = \sum_{(j_1, \ldots, j_m) \in \text{perm } m} \left(\text{sign}\,(j_1, \ldots, j_m)\, \beta(v_{j_1}, \ldots, v_{j_m})\right)$$

for $v_1, \ldots, v_m \in V$. Explain why $\alpha \in V_{alt}^{(m)}$.

Proof. If there are two repeating vectors, let's say $v_p = v_q$, then we know that

$$\beta(v_1, \ldots, v_p, \ldots, v_q, \ldots, v_m) = \beta(v_1, \ldots, v_q, \ldots, v_p, \ldots, v_m)$$

However, through swapping, the coefficient differs by (-1), so we have

$$\begin{aligned} &\text{sign}(1, \ldots, p, \ldots, q, \ldots, m)\beta(v_1, \ldots, v_p, \ldots, v_q, \ldots, v_m) \\ &= -\text{sign}(1, \ldots, q, \ldots, p, \ldots, m)\beta(v_1, \ldots, v_q, \ldots, v_p, \ldots, v_m) \end{aligned}$$

This basically shows the main idea of the proof. To make this more rigorous, we claim that for each permutation $\sigma \in$ perm m, there is a corresponding permutation $\sigma_{pq} \in \operatorname{perm} m$ such that keeps everything unchanged while only swapping the position of p and q. This means that for each permutation, there is a corresponding "cancelling" pair permutation. Since we are summing all permutations, the result is finally 0, finishing the proof. $\square$

9C: Determinants

Definition 38 (α_T)**.** *Suppose that m is a positive integer and $T \in \mathcal{L}(V)$. For $\alpha \in V_{alt}^{(m)}$, define $\alpha_T \in V_{alt}^{(m)}$ by*

$$\alpha_T(v_1, \ldots, v_m) = \alpha(Tv_1, \ldots, Tv_m)$$

for each list $v_1, \ldots, v_m$ of vectors in V.

Remark 39. *The function $\alpha \mapsto \alpha_T$ is a linear map of $V_{alt}^{(m)}$ to itself. We know that $\dim V_{alt}^{(\dim V)} = 1$, so the linear map is simply a multiplication by some unique scalar. For the linear map $\alpha \mapsto \alpha_T$, we now define $\det T$ to be that scalar.*

Definition 40 (**determinant** of an operator, $\det T$)**.** *Suppose $T \in \mathcal{L}(V)$. The **determinant** of T, denoted by $\det T$, is defined to be the unique number in $\mathbb{F}$ such that*

$$\alpha_T = (\det T)\alpha$$

for all $\alpha \in V_{alt}^{(\dim V)}$.

Remark 41. *Let $n = \dim V$.*

- *If I is the identity operator on V, then $\alpha_I = \alpha$ for all $\alpha \in V_{alt}^{(n)}$. This gives that $\det I = 1$.*
- *More generally, if $\lambda \in \mathbb{F}$, then $\alpha_{\lambda I} = \lambda^n \alpha$ for all $\alpha \in V_{alt}^{(n)}$. Thus $\det(\lambda I) = \lambda^n$.*
- *Since $\alpha_{\lambda T} = \lambda^n \alpha_T = \lambda^n (\det T)\alpha$ for all $\alpha \in V_{alt}^{(n)}$, $\det(\lambda T) = \lambda^n \det T$.*
- *Suppose $T \in \mathcal{L}(V)$ and there is a basis $e_1, \ldots, e_n$ of V consisting of eigenvectors of T, with corresponding eigenvalues $\lambda_1, \ldots, \lambda_n$. If $\alpha \in V_{alt}^{(n)}$, then*

  $$\alpha_T(e_1, \ldots, e_n) = \alpha(\lambda_1 e_1, \ldots, \lambda_n e_n) = (\lambda_1 \cdots \lambda_n)\alpha(e_1, \ldots, e_n)$$

  If $\alpha \neq 0$, then $\alpha(e_1, \ldots, e_n) \neq 0$. Thus this means that

  $$\det T = \lambda_1 \cdots \lambda_n$$

Definition 42 (determinant of a matrix, det A)**.** *Suppose that n is a positive integer and A is an n-by-n square matrix with entries in $\mathbb{F}$. Let $T \in \mathcal{L}(\mathbb{F}^n)$ be the operator whose matrix with respect to the standard basis of $\mathbb{F}^n$ equals A. The **determinant** of A, denoted by $\det A$, is defined by $\det A = \det T$.*

Theorem 43 (determinant is an alternating multilinear form)**.** *Suppose that n is a positive integer. The map that takes a list $v_1, \ldots, v_n$ of vectors in $\mathbb{F}^n$ to $\det(v_1 \cdots v_n)$ is an alternating n-linear form on $\mathbb{F}^n$.*

Corollary 44 (formula for determinants of a matrix)**.** *Suppose that n is a positive integer and A is an n-by-n square matrix. Then*

$$\det A = \sum_{(j_1,\ldots,j_n)\in perm\ n} \big(sign(j_1,\ldots,j_n)\big)\, A_{j_1,1}\cdots A_{j_n,n}$$

Remark 45. *The sum in the formula above contains $n!$ terms.*

Corollary 46 (determinant of upper-triangular matrix)**.** *Suppose that A is an upper-triangular matrix with $\lambda_1,\ldots,\lambda_n$ on the diagonal. Then* $\det A = \lambda_1\cdots\lambda_n$.

Theorem 47 (determinant is multiplicative)**.** *We have the following result:*

(a) Suppose $S,T\in\mathcal{L}(V)$. Then $\det(ST)=\det(S)\det(T)$.

(b) Suppose A and B are square matrices of the same size. Then

$$\det(AB)=\det(A)\det(B)$$

Corollary 48. *An operator $T\in\mathcal{L}(V)$ is* ***invertible*** *if and only if* $\det T\neq 0$. *Furthermore, if T is invertible, then* $\det(T^{-1})=\frac{1}{\det T}$.

Corollary 49. *Suppose $T\in\mathcal{L}(V)$ and $\lambda\in\mathbb{F}$. Then λ is an eigenvalue of T if and only if* $\det(\lambda I - T)=0$.

Corollary 50. *Suppose $T\in\mathcal{L}(V)$ and $S\colon W\to V$ is an invertible linear map. Then*

$$\det(S^{-1}TS)=\det T$$

Corollary 51. *Suppose $T\in\mathcal{L}(V)$ and $e_1,\ldots,e_n$ is a basis of V. Then*

$$\det T = \det\mathcal{M}\big(T,(e_1,\ldots,e_n)\big)$$

Corollary 52. *Suppose $\mathbb{F}=\mathbb{C}$ and $T\in\mathcal{L}(V)$. Then $\det T$ equals the product of the eigenvalues of T, wih each eigenvalue included as many times as its multiplicity.*

Corollary 53 (determinant of transpose, dual, or adjoint)**.** *We have the following result:*

(a) Suppose A is a square matrix. Then $\det A^\top = \det A$.

(b) Suppose $T\in\mathcal{L}(V)$. Then $\det T' = \det T$.

(c) Suppose V is an inner product space and $T\in\mathcal{L}(V)$. Then

$$\det(T^*)=\overline{\det T}$$

Corollary 54. *Helpful results in evaluating the determminants:*

(a) If either two columns or two rows of a square matrix are equal, then the determinant of the matrix equals 0.

(b) Suppose A is a square matrix and B is the matrix obtained from A by swapping either two columns or two rows. Then $\det A = -\det B$.

(c) If one column or one row of a square matrix is multiplied by a scalar, then the value of the determinant is multiplied by the same scalar.

(d) If a scalar multiple of one column of a square matrix to added to another column, then the value of the determinant is unchanged.

(e) If a scalar multiple of one row of a square matrix to added to another row, then the value of the determinant is unchanged.

Corollary 55. *Suppose V is an inner product space and $S \in \mathcal{L}(V)$ an unitary operator. Then* $|\det S| = 1$.

Corollary 56. *Suppose V is an inner product space and $T \in \mathcal{L}(V)$ is a positive operator. Then* $\det T \geq 0$.

Corollary 57. *Suppose V is an inner product space and $T \in \mathcal{L}(V)$. Then*

$$|\det T| = \sqrt{\det(T^*T)} = \text{ product of singular values of } T$$

Lemma 58. *Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Let $\lambda_1, \ldots, \lambda_m$ denote the distinct eigenvalues of T, and let $d_1, \ldots, d_m$ denote their multiplicities. Then*

$$\det(zI - T) = (z - \lambda_1)^{d_1} \cdots (z - \lambda_m)^{d_m}$$

Definition 59 (characteristic polynomal)**.** *Suppose $T \in \mathcal{L}(V)$. The polynomial defined by*

$$z \mapsto \det(zI - T)$$

is called the ***characteristic polynomial*** *of T.*

Theorem 60 (Cayley-Hamilton theorem)**.** *Suppose $T \in \mathcal{L}(V)$ and q is the characteristic polynomial of T. Then $q(T) = 0$.*

Corollary 61 (characteristic polynomial, trace, and determinant)**.** *Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$. Then the characteristic polynomial of T can be written as*

$$z^n - (tr\ T)z^{n-1} + \cdots + (-1)^n(\det T)$$

Theorem 62 (Hadamard's inequality)**.** *Suppose A is an n-by-n matrix. Let $v_1, \ldots, v_n$ denote the columns of A. Then*

$$|\det A| \leq \prod_{k=1}^{n} \|v_k\|$$

Theorem 63 (determinant of Vandermonde matrix)**.** *Suppose $n > 1$ and $\beta_1, \ldots, \beta_n \in \mathbb{F}$. Then*

$$
det \begin{pmatrix} 1 & \beta_1 & \beta_1^2 & \cdots & \beta_1^{n-1} \\ 1 & \beta_2 & \beta_2^2 & \cdots & \beta_2^{n-1} \\ 1 & \beta_3 & \beta_3^2 & \cdots & \beta_3^{n-1} \\ & & & \ddots & \\ 1 & \beta_n & \beta_n^2 & \cdots & \beta_n^{n-1} \end{pmatrix} = \prod_{1 \leq j < k \leq n} (\beta_k - \beta_j).
$$

Problem 1

Prove or give a counterexample: $S, T \in \mathcal{L}(V) \Rightarrow \det(S+T) = \det S + \det T$.

Proof. Consider $\mathbb{R}^2$, and that

$$\mathcal{M}(S) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \mathcal{M}(T) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Then clearly $\det S = 1$ and $\det T = 2$. However, we have that

$$\mathcal{M}(S) + \mathcal{M}(T) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

which has $\det(S+T) = 6 \neq \det S + \det T$. $\square$

Problem 3

Suppose $T \in \mathcal{L}(V)$ is nilpotent. Prove that $\det(I+T) = 1$.

Proof. We know that 0 is the only eigenvalue of T and thus the only eigenvalue of $I+T$ is 1. Hence $\det(I+T) = 1$. $\square$

Problem 5

Suppose A is a block triangular matrix

$$A = \begin{pmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where each A_k along the diagonal is a square matrix. Prove that

$$\det A = (\det A_1) \cdots (\det A_m)$$

Proof. One can show that $\det A = (\det A_1)(\det A_2)$ through direct proof. We use induction on m for solving this problem. The base case is trivial. We assume the statement holds for $m \leq k-1$. Then for $m = k$, we can partition the matrix into two blocks:

$$\begin{bmatrix} A' & * \\ 0 & A_k \end{bmatrix}$$

where

$$A' = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_{k-1} \end{bmatrix}$$

Then we have that $\det A = (\det A')(\det A_k) = (\det A_1) \cdots (\det A_k)$, finishing the proof. $\square$

Problem 9

Suppose that V is a real vector space of even dimension, $T \in \mathcal{L}(V)$, and $\det T < 0$. Prove that T has at least two distinct eigenvalues.

Proof. Since $\det(T) \neq 0$, T is invertible and thus have n distinct eigenvalues with $n \geq 2$. Another argument could be that for real cases, there have to be at least one negative and one positive eigenvalue to make the determinant negative; for complex cases, there must be two conjugate pairs. □

Problem 11

Prove or give a counter example: If $\mathbb{F} = \mathbb{R}, T \in \mathcal{L}(V)$, and $\det T > 0$, then T has a square root.

Proof. Not necessarily. Consider an operator in $\mathbb{R}^2$ with two negative eigenvalues which is clearly non-positive and therefore does not have a square root. □

Problem 16

Suppose $T \in \mathcal{L}(V)$. Define $g \colon \mathbb{F} \to \mathbb{F}$ by $g(x) = \det(I + xT)$. Show that $g'(0) = \text{tr } T$.

Proof.

$$\begin{aligned} g'(x) &= \frac{d}{dx} \det(I + xT) \\ &= \frac{d}{dx} \prod_{i=1}^{n} (1 + x\lambda_i) \\ &= \sum_{i=1}^{n} \left(\lambda_i \prod_{j \neq i} (1 + x\lambda_i) \right) \end{aligned}$$

Substitue $x = 0$ yields that

$$g'(0) = \sum_{i=1}^{n} \lambda_i = \text{tr } T$$

□

Problem 19

Suppose V is an inner product space, $e_1, \ldots, e_n$ is an orthonormal basis of V, and $T \in \mathcal{L}(V)$ is a positive operator.

(a) Prove that $\det T \leq \prod_{k=1}^{n} \langle Te_k, e_k \rangle$.

(b) Prove that if T is invertible, then the inequality in (a) is an equality if and only if e_k is an eigenvector of T for each $k = 1, \ldots, n$.

Proof. (a) The matrix representation of T wrt. $e_1, \ldots, e_n$ is that $A_{ij} = \langle Te_i, e_j \rangle$. Hence the r.h.s of this inequality is simply the product of all the diagonal terms on the matrix of T. We prove this inequality through Cholesky decomposition. Note that

$$A = LL^*$$

for lower-triangular matrix L, and thus we have

$$\det T = \det A = (\det L)^2 = \left(\prod_{k=1}^{n} l_{kk} \right)^2 = \prod_{k=1}^{n} L_{kk}^2$$

We note that

$$A_{kk} = L_{kk}^2 + \sum_{j=1}^{k-1} L_{kj}^2$$

and thus we have

$$\det A \leq \prod_{k=1}^{n} A_{kk} = \prod_{k=1}^{n} \langle Te_k, e_k \rangle$$

(b) If e_k is an eigenvector of T, then $\langle Te_k, e_k \rangle = \lambda_k$, the k-th eigenvalue of T. Then we know that $\det T$ is the product of all eigenvalues.

Conversely, if (a) is an equality, then we know that L is a diagonal matrix and thus A is also a diagonal matrix. Then the orthonormal basis $e_1, \ldots, e_n$ actually diagonaizes T and hence each of them is an eigenvector of T.

□

9D: Tensor Products

Definition 64 (bilinear functional on $V \times W$, the vector space $\mathcal{B}(V, W)$)**.** *A* ***bilinear functional*** *on $V \times W$ is a function $\beta\colon V \times W \to \mathbb{F}$ such that $v \mapsto \beta(v, w)$ is a linear functional on V for each $w \in W$ and $w \mapsto \beta(v, w)$ is a linear functional on W for each $v \in V$.*

The vector space of bilinear functionals on $V \times W$ is denoted by $\mathcal{B}(V, W)$.

Remark 65. *If $V = W$, then a bilinear functional on $V \times W$ is a bilinear form.*

Corollary 66. $\dim \mathcal{B}(V, W) = (\dim V)(\dim W)$

Remark 67. *We want a* basis-free *definition of the tensor product.*

Definition 68 (tensor product, $V \otimes W, v \otimes w$)**.** *The* ***tensor product*** *$V \otimes W$ is defined to be $\mathcal{B}(V', W')$.*

For $v \in V$ and $w \in W$, the ***tensor product*** *$v \otimes w$ is the element of $V \otimes W$ defined by*

$$(v \otimes w)(\varphi, \tau) = \varphi(v)\tau(w)$$

for all $(\varphi, \tau) \in V' \times W'$.

Corollary 69. $\dim(V \otimes W) = (\dim V)(\dim W)$.

Proposition 70 (bilinearity of tensor product)**.** *Suppose $v, v_1, v_2 \in V$ and $w, w_1, w_2 \in W$ and $\lambda \in \mathbb{F}$. Then*

$$(v_1 + v_2) \otimes w = v_1 \otimes w + v_2 \otimes w \text{ and } v \otimes (w_1 + w_2) = v \otimes w_1 + v \otimes w_2$$

and

$$\lambda (v \otimes w) = (\lambda v) \otimes w = v \otimes (\lambda w)$$

Theorem 71 (basis of $V \otimes W$)**.** *Suppose $e_1, \dots, e_m$ is a list of vectors in V and $f_1, \dots, f_n$ is a list of vectors in W.*

(a) If $e_1, \dots, e_m$ and $f_1, \dots, f_n$ are both linearly independent list, then

$$\{e_j \otimes f_k\}_{j=1,\dots,m;k=1,\dots,n}$$

is a linearly independent list in $V \otimes W$.

(b) If $e_1, \dots, e_m$ is a basis of V and $f_1, \dots, f_n$ is a basis of W, then the list $\{e_j \otimes f_k\}_{j=1,\dots,m;k=1,\dots,n}$ is a basis of $V \otimes W$.

Definition 72 (bilinear map)**.** *A* ***bilinear map*** *from $V \times W$ to a vector space U is a function $\Gamma\colon V \times W \to U$ such that $v \mapsto \Gamma(v, w)$ is a linear map from V to U for each $w \in W$ and $w \mapsto \Gamma(v, w)$ is a linear map from W to U for each $v \in V$.*

Lemma 73 (converting bilinear maps to linear maps)**.** *Suppose U is a vector space.*

(a) *Suppose* $\Gamma\colon V \times W \to U$ *is a bilinear map. Then there exists a unique linear map* $\tilde{\Gamma}\colon V \otimes W \to U$ *such that*

$$\tilde{\Gamma}(v \otimes w) = \Gamma(v, w)$$

for all $(v, w) \in V \times W$.

(b) *Conversely, suppose* $T\colon V \otimes W \to U$ *is a linear map. Then there exists a unique bilinear map* $T^{\#}\colon V \times W \to U$ *such that*

$$T^{\#}(v, w) = T(v \otimes w)$$

for all $(v, w) \in V \times W$.

Theorem 74 (inner product on tensor product of two inner product spaces)**.** *Suppose* V *and* W *are inner product spaces. Then there is a unique inner product on* $V \otimes W$ *such that*

$$\langle v \otimes w, u \otimes x \rangle = \langle v, u \rangle \langle w, x \rangle$$

for all $u, v \in V$ *and* $w, x \in W$.

Remark 75. *We have that* $\|v \otimes w\| = \|v\| \|w\|$.

Corollary 76. *Suppose* V *and* W *are inner product spaces, and* $e_1, \ldots, e_m$ *is an orthonormal basis of* V *and* $f_1, \ldots, f_n$ *is an orthonormal basis of* W. *Then*

$$\{e_j \otimes f_k\}_{j=1,\ldots,m;k=1,\ldots,n}$$

is an orthonormal basis of $V \otimes W$.

Definition 77. *An* ***m-linear*** *functional on* $V_1 \times \cdots \times V_m$ *is a function* $\beta\colon V_1 \times \cdots \times V_m \to \mathbb{F}$ *that is a linear functional in each slot when the other slots are held fixed.*

The vector space of m-linear functionals on $V_1 \times \cdots \times V_m$ *is denoted by* $\mathcal{B}(V_1, \ldots, V_m)$.

Corollary 78. $\dim \mathcal{B}(V_1, \ldots, V_m) = (\dim V_1) \times \cdots \times (\dim V_m)$

Definition 79 (tensor product)**.** *The tensor product* $V_1 \otimes \cdots \otimes V_m$ *is defined to be* $\mathcal{B}(V_1', \ldots, V_m')$.

For $v_1 \in V_1, \ldots, v_m \in V_m$, *the* ***tensor product*** $v_1 \otimes \cdots \otimes v_m$ *is the element of* $V_1 \otimes \cdots \otimes V_m$ *defined by*

$$(v_1 \otimes \cdots \otimes v_m)(\varphi_1, \ldots, \varphi_m) = \varphi_1(v_1) \cdots \varphi_m(v_m)$$

for all $(\varphi_1, \ldots, \varphi_m) \in V_1' \times \cdots \times V_m'$.

Corollary 80. *Suppose* $\dim V_k = n_k$ *and* $e_1^k, \ldots, e_{n_k}^k$ *is a basis of* V_k *for* $k = 1, \ldots, m$. *Then*

$$\{e_{j_1}^1 \otimes \cdots \otimes e_{j_m}^m\}_{j_1=1,\ldots,n_1;\ldots;j_m=1,\ldots,n_m}$$

is a basis of $V_1 \otimes \cdots \otimes V_m$.

Definition 81 (m-linear map)**.** *An m-linear map from $V_1 \times \cdots \times V_m$ to a vector space U is a function $\Gamma\colon V_1 \times \cdots \times V_m \to U$ that is a linear map in each slot when the other slots are held fixed.*

Theorem 82 (converting m-linear map to linear maps)**.** *Suppose U is a vector space.*

(a) Suppose that $\Gamma\colon V_1 \times \cdots \times V_m \to U$ is an m-linear map. Then there exists a unique linear map $\tilde{\Gamma}\colon V_1 \otimes \cdots \otimes V_m \to U$ such that

$$\tilde{\Gamma}(v_1 \otimes \cdots \otimes v_m) = \Gamma(v_1, \ldots, v_m)$$

for all $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$.

(b) Conversely, suppose $T\colon V_1 \otimes \cdots \otimes V_m \to U$ is a linear map. Then there exists a unique m-linear map $T^{\#}\colon V_1 \times \cdots \times V_m \to U$ such that

$$T^{\#}(v_1, \ldots, v_m) = T(v_1 \otimes \cdots \otimes v_m)$$

for all $(v_1, \ldots, v_m) \in V_1 \times \cdots \times V_m$.

Problem 1

Suppose $v \in V$ and $w \in W$. Prove that $v \otimes w = 0$ if and only if $v = 0$ or $w = 0$.

Proof. By definition, we have for any $(\varphi, \tau) \in V' \times W'$,

$$(v \otimes w)(\varphi, \tau) = \varphi(v)\tau(w)$$

Then this means that $\varphi(v)\tau(w) = 0$ for arbitrary choice of φ, τ, meaning that either $v = 0$ or $w = 0$. $\square$

Problem 3

Suppose that $v_1, \ldots, v_m$ is a linearly independent list in V. Suppose also that $w_1, \ldots, w_m$ is a list in W such that

$$v_1 \otimes w_1 + \cdots + v_m \otimes w_m = 0$$

Prove that $w_1 = \cdots = w_m = 0$.

Proof. By the linear map lemma and the linear independence of $v_1, \ldots, v_m$, there exists $\varphi_1, \ldots, \varphi_m \in V'$ such that

$$\varphi_j(v_k) = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

where $j, k \in \{1, \ldots, m\}$. Applying such $\{\varphi_i\}_{i=1}^m$ to the list

$$\sum_{i=1}^{n} v_i \otimes w_i$$

and take $\tau \in W'$ to be the identity map yields that

$$w_1 = \cdots = w_m = 0$$

$\square$

Problem 5

Suppose m and n are positive integers. For $v \in \mathbb{F}^m$ and $w \in \mathbb{F}^n$, identify $v \otimes w$ with an m-by-n matrix as in Example 9.76. With that identification, show that the set

$$\{v \otimes w \colon v \in \mathbb{F}^m \text{ and } w \in \mathbb{F}^n\}$$

is the set of m-by-n matrices (with entrices in $\mathbb{F}$) that have rank at most one.

Proof. If one examine the matrices with entries shown on the matrix, it's easy to tell that for row j and row k with $j \neq k$, one can get row k from row j through multiplying v_k/v_j. The same applies to arbitrary pairs of columns. Thus the matrix has at most rank one. $\square$

Problem 8

Suppose $v_1, \ldots, v_m \in V$ and $w_1, \ldots, w_m \in W$ are such that

$$v_1 \otimes w_1 + \cdots + v_m \otimes w_m = 0$$

Suppose that U is a vector space and $\Gamma\colon V \times W \to U$ is a bilinear map. Show that

$$\Gamma(v_1, w_1) + \cdots + \Gamma(v_m, w_m) = 0$$

Proof. We know there exists a unique "converting" linear map $\tilde{\Gamma}$ such that

$$\Gamma(v \tilde{\otimes} w) = \Gamma(v, w)$$

Hence applying this gives that

$$\begin{aligned} \sum_{i=1}^{m} \Gamma(v_i, w_i) &= \sum_{i=1}^{m} \tilde{\Gamma}(v_i \otimes w_i) \\ &= \tilde{\Gamma}(\sum_{i=1}^{m} v_i \otimes w_i) \\ &= 0 \end{aligned}$$

$\square$