

# Chapter 1: Linear Equations in linear algebra - The matrix Equation

## Summary

Adding the matrix equation to our epic linear algebra arsenal is the final piece of the puzzle for representing a linear system. We now have the matrix equation, a linear combination, the augmented matrix, and the original formulation. All these are exactly the same.

## Terminology

TFAE

- the following statements are equivalent
- either all true or all false

## The matrix Equation

$$Ax = b$$

### Four ways to represent a linear system

let  $A$  be an  $m \times n$  matrix with columns  $[a_1 \dots a_n]$

- 1) the matrix equation  $Ax = b$
- 2) a linear combination  $x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b$
- 3 the augmented matrix  $[A | b]$
- 4) the original formulation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

## Property

Let  $A$  be an  $m \times n$  matrix,  $u$  and  $v$  two vectors in  $\mathbb{R}^n$ , and  $c$  a scalar

$$A(u+v) = Au + Av$$

$$A(cu) = cAu$$

## Main Theorem

Theorem. Let  $A$  be an  $m \times n$  matrix. The following statements are equivalent (TFAE: for a given  $A$  they are either all four true or all four false):

- (a) For each  $b$  in  $\mathbb{R}^m$ , the equation  $Ax = b$  has a solution.
- (b) Each  $b$  in  $\mathbb{R}^m$  is a linear combination of the columns of  $A$ .
- (c) The columns of  $A$  span  $\mathbb{R}^m$ .
- (d)  $A$  has a pivot position in every row.

*Note  $A$  is the coefficient matrix not the augmented matrix*

The 4 ways of looking at systems tells us that c) is equivalent to saying that the system with augmented matrix  $[A|b]$  is consistent for all  $b$

Write  $[u|d]$  for the Reduced Echelon form