

Chapter 1: Linear Equations in linear algebra - The matrix Equation

Summary

Adding the matrix equation to our epic linear algebra arsenal is the final piece of the puzzle for representing a linear system. We now have the matrix equation, a linear combination, the augmented matrix, and the original formulation. All these are exactly the same.

Terminology

TFAE

- the following statements are equivalent
- either all true or all false

The matrix Equation

$$Ax = b$$

Four ways to represent a linear system

let A be an $m \times n$ matrix with columns $[a_1 \dots a_n]$

- 1) the matrix equation $Ax = b$
- 2) a linear combination $x_1 a_1 + x_2 a_2 + \dots + x_n a_n = b$
- 3 the augmented matrix $[A | b]$
- 4) the original formulation

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ &\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

Property

Let A be an $m \times n$ matrix, u and v two vectors in \mathbb{R}^n , and c a scalar

$$A(u+v) = Au + Av$$

$$A(cu) = cAu$$

Main Theorem

Theorem. Let A be an $m \times n$ matrix. The following statements are equivalent (TFAE: for a given A they are either all four true or all four false):

- (a) For each b in \mathbb{R}^m , the equation $Ax = b$ has a solution.
- (b) Each b in \mathbb{R}^m is a linear combination of the columns of A .
- (c) The columns of A span \mathbb{R}^m .
- (d) A has a pivot position in every row.

Note A is the coefficient matrix not the augmented matrix

The 4 ways of looking at systems tells us tht c) is equivalent to saying that the system with augmented matrix $[A|b]$ is consistent for all b

Write $[u|d]$ for the Reduced Echelon form