# Topology Note

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# Definitions

# Chapter 1

# Topology Spaces and Continuous Function

## 1.1 Basic Definition of Topology

**Definition 1.1.1** (topology). *A **topology** on a set $\mathbb{X}$ is a collection $\mathbb{T}$ of subsets of $\mathbb{X}$ having the following properties:*

- *$\emptyset$ and $\mathbb{X}$ are in $\mathbb{T}$*

- *The union of the elements of any sub collection of $\mathbb{T}$ is in $\mathbb{T}$*

- *The intersection of the elements of any **finite** sub collection of $\mathbb{T}$ is in $\mathbb{T}$*

**Definition 1.1.2** (topology space). *A **topological space** is a set $\mathbb{X}$ for which a topology $\mathbb{T}$ has been specified.*

**Definition 1.1.3** (open set). *A **open set** $\mathbb{U}$ is a subset of $\mathbb{X}$ that belongs to a topology $\mathbb{T}$ of $\mathbb{X}$.*

**Definition 1.1.4** (open sets). *A topology can also be called a **open sets***

**Definition 1.1.5** (discrete topology). *The set of all subsets of a set $\mathbb{X}$ formed a topology called **discrete topology***

**Definition 1.1.6** (trivial topology). *The set consisting the set $\mathbb{X}$ and $\emptyset$ only formed a topology of $\mathbb{X}$ called **trivial topology***

**Definition 1.1.7** (finite complement topology). *Let $\mathbb{X}$ be a set. Let $\mathbb{T}_f$ be the collection of all subsets $\mathbb{U}$ of $\mathbb{X}$ such that $\mathbb{X} - \mathbb{U}$ either if a **finite** [1] of is all of $\mathbb{X}$. Then $\mathbb{T}_f$ is a topology on $\mathbb{X}$, called the .*

**Definition 1.1.8** (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable). *Let $\mathbb{T}$ and $\mathbb{T}'$ be two topology on a given set $\mathbb{X}$. If $\mathbb{T}$ is a subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **finer** or **larger** than $\mathbb{T}$. If $\mathbb{T}$ is a proper subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **strictly finer** or **strictly larger** than $\mathbb{T}$. We also say that $\mathbb{T}$ is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than $\mathbb{T}'$. We say that $\mathbb{T}$ and $\mathbb{T}'$ is **comparable** if either $\mathbb{T}$ is a subset of $\mathbb{T}'$ or $\mathbb{T}'$ is a subset of $\mathbb{T}$.*

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[1]The set $\mathbb{U}$ can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, $\mathbb{U}$ will not be a topology.

## 1.2   Basis for a Topology

**Definition 1.2.1** (basis)**.** *If $\mathbb{X}$ is a set, a **basis** for a topology on $\mathbb{X}$ is a collection $\mathbb{B}$ of subsets of $\mathbb{X}$ (called **basis elements**) such that:*

- *For each $x \in \mathbb{X}$, there is at least one basis element $B$ containing $x$*

- *If $x$ belongs to the intersection of two basis elements $B_1$ and $B_2$, then there is another element $x \in B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$*

**Definition 1.2.2** (topology generated by basis)**.** *Let $\mathbb{B}$ be a basis on $\mathbb{X}$. Let $\mathbb{U}$ be a set containing all subsets $U$ of $\mathbb{X}$ such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such $\mathbb{U}$ formed a topology on $\mathbb{X}$, called **topology** $\mathbb{T}$ **generated by** $\mathbb{B}$*

**Lemma 1.2.1.** *Let $\mathbb{X}$ be a set. Let $\mathbb{B}$ be a basis for a topology $\mathbb{T}$ on $\mathbb{X}$. Then $\mathbb{T}$ equals to the set of all possible unions of elements of $\mathbb{B}$.*

*Proof.* Let set $\mathbb{U}$ be the set of all possible unions of elements of $\mathbb{B}$. For any $U \in \mathbb{U}$. $U = \cup B$ [2] for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in U$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, $\mathbb{U}$ equals to $\mathbb{T}$.                                                                            $\square$

**Lemma 1.2.2.** [3] *Let $\mathbb{X}$ be a topological space. Suppose that $\mathbb{C}$ is a collection of open sets of $\mathbb{X}$ such that for each open set $U$ of $\mathbb{X}$ and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq C$. Then $\mathbb{C}$ is a basis for the topology of $\mathbb{X}$.*

**Lemma 1.2.3.** [4] *Let $\mathbb{B}$ and $\mathbb{B}'$ be basis for the topologies $\mathbb{T}$ and $\mathbb{T}'$, respectively, on $\mathbb{X}$. Then the following are equivalent:*

- *$\mathbb{T}'$ is finer than $\mathbb{T}$*

- *For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing $X$, there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.*

**Definition 1.2.3** (standard topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **standard topology on the real line*** [5] .

**Definition 1.2.4** (lower limit topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **lower limit topology on the real line**. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_l$.*

**Definition 1.2.5** (K-topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$. $\mathbb{B} \cup \{B - K | B \in \mathbb{B}\}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **K-topology on the real line**. When $\mathbb{R}$ is given this topology, we denote it by $\mathbb{R}_{\mathbb{K}}$.*

**Lemma 1.2.4.** [6] *The topologies $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is strictly finer than the standard topology on $\mathbb{R}$.*

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[2]Note that this expression may not be unique.

[3]We omit the proof of this lemma as it is obvious.

[4]We omit the proof of this lemma as it is obvious.

[5]Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.

[6]We omit the proof of this lemma as it is obvious.

**Lemma 1.2.5.** *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_\mathbb{K}$ is not comparable.*

*Proof.* Let $\mathbb{T}_l$ and $\mathbb{T}_\mathbb{K}$ be topologies of $\mathbb{R}_l$ and $\mathbb{R}_\mathbb{K}$ respectively. Let $K = \{x|x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that $\mathbb{T}_l$ is not finer than $\mathbb{T}_\mathbb{K}$. Let $U = \{x| -1 < x < 1\} - K, x = 0$. If there exist $B = \{x|a \le x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus $B$ is not a subset of $U$.

Then we proof that $\mathbb{T}_\mathbb{K}$ is not finer than $\mathbb{T}_l$. Let $U' = \{x|a' \le x < b'\}$. If there exist $B' = \{x|a'' < x < b''\}or\{x|a'' < x < b''\} - K$ such that $a' \in B \subseteq U$. Thus $a'' < a < b''$. Thus there exist $c$ that $a'' < x < a, x \in B, x \notin U'$. Thus $B' \nsubseteq U'$.

Thus the topologies of $\mathbb{R}_l$ and $\mathbb{R}_\mathbb{K}$ is not comparable. $\qquad\square$

**Definition 1.2.6** (subbasis). *A **subbasis** $\mathbb{S}$ for a topology on $\mathbb{X}$ is a collection of subsets of $\mathbb{X}$ whose union equals $\mathbb{X}$. The **topology generated by the subbasis** $\mathbb{S}$ is defined to be the collection $\mathbb{T}$ [7] of all unions of finite intersections of elements of $\mathbb{S}$.*

### 1.2.1 Exercise

1. Show that if $\mathbb{A}$ is a basis for a topology on $\mathbb{X}$, then the topology generated by $\mathbb{A}$ equals the intersection of all topologies on $\mathbb{X}$ that contain $\mathbb{A}$. Prove the same if $\mathbb{A}$ is a subbasis.

   *Proof.* As a subbasis is also a basis, we will directly prove the case of subbasis here.

   Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain $\mathbb{A}$. Let $\mathbb{T}$ be the topology that $\mathbb{A}$ generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.[8]

   First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

   Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

   Thus, $\mathbb{T} = \mathbb{T}'$ $\qquad\square$

## 1.3 The Order Topology

**Definition 1.3.1** (interval). *Let $\mathbb{X}$ is a set having a simple order relation $<$. Given elements $a$ and $b$ of $\mathbb{X}$ such that $a < b$, there are four subsets of $\mathbb{X}$ that are called **intervals** determined by $a$ and $b$:*

- $(a, b) = \{x|a < x < b\}$

- $(a, b] = \{x|a < x \le b\}$

- $[a, b) = \{x|a \le x < b\}$

- $[a, b] = \{x|a \le x \le b\}$

*$(a, b)$ is called an **open interval** on $\mathbb{X}$. $[a, b]$ is called an **closed interval** on $\mathbb{X}$. $(a, b]$ and $[a, b)$ is called **half-open intervals**.*

**Definition 1.3.2** (order topology). [9] *Let $\mathbb{X}$ be a set with a simple order relation; assume $\mathbb{X}$ has more than one element. Let $\mathbb{B}$ be the collection of all sets of the following types:*

- *All open intervals $(a, b)$ in $\mathbb{X}$.*

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[7] It is obvious that $\mathbb{T}$ is a topology, we just omit the proof here.

[8] It is obvious that $\mathbb{T}'$ is also a topology, we just omit the proof here.

[9] The standard topology on $\mathbb{R}$ is an order topology derived from the usual order on $\mathbb{R}$.

- *All intervals of the form $[a_0, b)$, where $a_0$ is the smallest element(if exist) of $\mathbb{X}$.*

- *All intervals of the form $(a, b_0]$, where $b_0$ is the largest element(if exist) of $\mathbb{X}$.*

*The collection $\mathbb{B}$ formed a basis for a topology on $\mathbb{X}$, which is called the order topology.*

**Definition 1.3.3** (ray). [10][11] *If $\mathbb{X}$ is an ordered set, and a is an element of $\mathbb{X}$, there are four subsets of $\mathbb{X}$ that are called **rays** determined by a:*

- $(a, +\infty) = \{x | x > a\}$

- $(-\infty, a) = \{x | x < a\}$

- $[a, +\infty) = \{x | x \geq a\}$

- $(-\infty, a] = \{x | x \leq a\}$

$(a, +\infty)$ *and* $(-\infty, a)$ *are called **open rays**.* $[a, +\infty)$ *and* $(-\infty, a]$ *are called **closed rays**.*

## 1.4   The Product Topology

**Definition 1.4.1** (product topology). *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces. The **product topology** on $\mathbb{X} \times \mathbb{Y}$ having a basis $\mathbb{B}$ containing all sets of the form $U \times V$, where $U$ and $V$ is open sets of $\mathbb{X}$ and $\mathbb{Y}$ respectively.*

**Theorem 1.4.1.** [12] *If $\mathbb{B}$ and $\mathbb{C}$ is basis for the topology of $\mathbb{X}$ and $\mathbb{Y}$ respectively, then the collection*

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} and C \in \mathbb{C}\}$$

*is a basis for the topology of $\mathbb{X} \times \mathbb{Y}$*

**Definition 1.4.2** (projection). *Let $\pi_1 : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ be defined by the equation:*

$$\pi_1(x, y) = x$$

*Let $\pi_2 : \mathbb{X} \times \mathbb{Y} \to \mathbb{Y}$ be defined by the equation:*

$$\pi_1(x, y) = y$$

*The maps $\pi_1$ and $\pi_2$ are called the **projections** of $\mathbb{X} \times \mathbb{Y}$ onto its first and second factors, respectively.*

**Theorem 1.4.2.** [13] *The collection*

$$\mathbb{S} = \{\pi_1^{-1}(U) | U open in \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V open in \mathbb{Y}\}$$

*is a subbasis for the product topology on $\mathbb{X} \times \mathbb{Y}$.*

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[10]open rays are always open sets in the order topology
[11]the open rays also formed a subbasis of the order topology
[12]We omit the proof of this lemma as it is obvious.
[13]We omit the proof of this lemma as it is obvious.