Topology Note

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Contents

Definitions

Chapter 1

Topology Spaces and Continuous Function

1.1 Basic Definition of Topology

Definition 1.1.1 (topology)**.** *A **topology** on a set $\mathbb{X}$ is a collection $\mathbb{T}$ of subsets of $\mathbb{X}$ having the following properties:*

- *$\emptyset$ and $\mathbb{X}$ are in $\mathbb{T}$*
- *The union of the elements of any sub collection of $\mathbb{T}$ is in $\mathbb{T}$*
- *The intersection of the elements of any **finite** sub collection of $\mathbb{T}$ is in $\mathbb{T}$*

Definition 1.1.2 (topology space)**.** *A **topological space** is a set $\mathbb{X}$ for which a topology $\mathbb{T}$ has been specified.*

Definition 1.1.3 (open set)**.** *A **open set** $\mathbb{U}$ is a subset of $\mathbb{X}$ that belongs to a topology $\mathbb{T}$ of $\mathbb{X}$.*

Definition 1.1.4 (open sets)**.** *A topology can also be called a **open sets***

Definition 1.1.5 (discrete topology)**.** *The set of all subsets of a set $\mathbb{X}$ formed a topology called **discrete topology***

Definition 1.1.6 (trivial topology)**.** *The set consisting the set $\mathbb{X}$ and $\emptyset$ only formed a topology of $\mathbb{X}$ called **trivial topology***

Definition 1.1.7 (finite complement topology)**.** *Let $\mathbb{X}$ be a set. Let $\mathbb{T}_f$ be the collection of all subsets $\mathbb{U}$ of $\mathbb{X}$ such that $\mathbb{X} - \mathbb{U}$ either if a **finite** [1] of is all of $\mathbb{X}$. Then $\mathbb{T}_f$ is a topology on $\mathbb{X}$, called the .*

Definition 1.1.8 (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable)**.** *Let $\mathbb{T}$ and $\mathbb{T}'$ be two topology on a given set $\mathbb{X}$. If $\mathbb{T}$ is a subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **finer** or **larger** than $\mathbb{T}$. If $\mathbb{T}$ is a proper subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **strictly finer** or **strictly larger** than $\mathbb{T}$. We also say that $\mathbb{T}$ is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than $\mathbb{T}'$. We say that $\mathbb{T}$ and $\mathbb{T}'$ is **comparable** if either $\mathbb{T}$ is a subset of $\mathbb{T}'$ or $\mathbb{T}'$ is a subset of $\mathbb{T}$.*

[1] The set $\mathbb{U}$ can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, $\mathbb{U}$ will not be a topology.

1.2 Basis for a Topology

Definition 1.2.1 (basis). *If $\mathbb{X}$ is a set, a **basis** for a topology on $\mathbb{X}$ is a collection $\mathbb{B}$ of subsets of $\mathbb{X}$ (called **basis elements**) such that:*

- *For each $x \in \mathbb{X}$, there is at least one basis element B containing x*
- *If x belongs to the intersection of two basis elements B_1 and B_2, then there is another element $x \in B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$*

Definition 1.2.2 (topology generated by basis). *Let $\mathbb{B}$ be a basis on $\mathbb{X}$. Let $\mathbb{U}$ be a set containing all subsets U of $\mathbb{X}$ such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such $\mathbb{U}$ formed a topology on $\mathbb{X}$, called **topology $\mathbb{T}$ generated by** $\mathbb{B}$*

Lemma 1.2.1. *Let $\mathbb{X}$ be a set. Let $\mathbb{B}$ be a basis for a topology $\mathbb{T}$ on $\mathbb{X}$. Then $\mathbb{T}$ equals to the set of all possible unions of elements of $\mathbb{B}$.*

Proof. Let set $\mathbb{U}$ be the set of all possible unions of elements of $\mathbb{B}$. For any $U \in \mathbb{U}$. $U = \cup B$ [2] for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in U$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, $\mathbb{U}$ equals to $\mathbb{T}$. □

Lemma 1.2.2. [3] *Let $\mathbb{X}$ be a topological space. Suppose that $\mathbb{C}$ is a collection of open sets of $\mathbb{X}$ such that for each open set U of $\mathbb{X}$ and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq C$. Then $\mathbb{C}$ is a basis for the topology of $\mathbb{X}$.*

Lemma 1.2.3. [4] *Let $\mathbb{B}$ and $\mathbb{B}'$ be basis for the topologies $\mathbb{T}$ and $\mathbb{T}'$, respectively, on $\mathbb{X}$. Then the following are equivalent:*

- *$\mathbb{T}'$ is finer than $\mathbb{T}$*
- *For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing X, there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.*

Definition 1.2.3 (standard topology on the real line). *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **standard topology on the real line*** [5] .

Definition 1.2.4 (lower limit topology on the real line). *Let be $\mathbb{B} = \{B | B = \{x | a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **lower limit topology on the real line**. When $\mathbb{R}$ is given this topology,we denote it by $\mathbb{R}_l$.*

Definition 1.2.5 (K-topology on the real line). *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$. $\mathbb{B} \cup \{B - K | B \in \mathbb{B}\}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **K-topology on the real line**. When $\mathbb{R}$ is given this topology,we denote it by $\mathbb{R}_\mathbb{K}$.*

Lemma 1.2.4. [6] *The topologies $\mathbb{R}_l$ and $\mathbb{R}_\mathbb{K}$ is strictly finer than the standard topology on $\mathbb{R}$.*

[2] Note that this expression may not be unique.
[3] We omit the proof of this lemma as it is obvious.
[4] We omit the proof of this lemma as it is obvious.
[5] Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.
[6] We omit the proof of this lemma as it is obvious.

Lemma 1.2.5. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is not comparable.*

Proof. Let $\mathbb{T}_l$ and $\mathbb{T}_{\mathbb{K}}$ be topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ respectively. Let $K = \{x|x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that $\mathbb{T}_l$ is not finer than $\mathbb{T}_{\mathbb{K}}$. Let $U = \{x| -1 < x < 1\} - K, x = 0$. If there exist $B = \{x|a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus B is not a subset of U.

Then we proof that $\mathbb{T}_{\mathbb{K}}$ is not finer than $\mathbb{T}_l$. Let $U' = \{x|a' \leq x < b'\}$. If there exist $B' = \{x|a'' < x < b''\} or \{x|a'' < x < b''\} - K$ such that $a' \in B \subseteq U$. Thus $a'' < a < b''$. Thus there exist c that $a'' < x < a, x \in B, x \notin U'$. Thus $B' \nsubseteq U'$.

Thus the topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is not comparable. □

Definition 1.2.6 (subbasis)**.** *A* ***subbasis*** $\mathbb{S}$ *for a topology on* $\mathbb{X}$ *is a collection of subsets of* $\mathbb{X}$ *whose union equals* $\mathbb{X}$*. The* ***topology generated by the subbasis*** $\mathbb{S}$ *is defined to be the collection* $\mathbb{T}$ [7] *of all unions of finite intersections of elements of* $\mathbb{S}$*.*

1.2.1 Exercise

1. Show that if $\mathbb{A}$ is a basis for a topology on $\mathbb{X}$, then the topology generated by $\mathbb{A}$ equals the intersection of all topologies on $\mathbb{X}$ that contain $\mathbb{A}$. Prove the same if $\mathbb{A}$ is a subbasis.

   Proof. As a subbasis is also a basis, we will directly prove the case of subbasis here.

   Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain $\mathbb{A}$. Let $\mathbb{T}$ be the topology that $\mathbb{A}$ generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.[8]

   First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

   Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

   Thus, $\mathbb{T} = \mathbb{T}'$ □

1.3 The Order Topology

Definition 1.3.1 (interval)**.** *Let* $\mathbb{X}$ *is a set having a simple order relation* $<$*. Given elements* a *and* b *of* $\mathbb{X}$ *such that* $a < b$*, there are four subsets of* $\mathbb{X}$ *that are called* ***intervals*** *determined by* a *and* b*:*

- $(a, b) = \{x|a < x < b\}$
- $(a, b] = \{x|a < x \leq b\}$
- $[a, b) = \{x|a \leq x < b\}$
- $[a, b] = \{x|a \leq x \leq b\}$

(a, b) *is called an* ***open interval*** *on* $\mathbb{X}$*.* $[a, b]$ *is called an* ***closed interval*** *on* $\mathbb{X}$*.* $(a, b]$ *and* $[a, b)$ *is called* ***half-open intervals****.*

Definition 1.3.2 (order topology)**.** [9] *Let* $\mathbb{X}$ *be a set with a simple order relation; assume* $\mathbb{X}$ *has more than one element. Let* $\mathbb{B}$ *be the collection of all sets of the following types:*

- *All open intervals* (a, b) *in* $\mathbb{X}$*.*

[7] It is obvious that $\mathbb{T}$ is a topology, we just omit the proof here.
[8] It is obvious that $\mathbb{T}'$ is also a topology, we just omit the proof here.
[9] The standard topology on $\mathbb{R}$ is an order topology derived from the usual order on $\mathbb{R}$.

- *All intervals of the form* $[a_0, b)$, *where* a_0 *is the smallest element(if exist) of* $\mathbb{X}$.
- *All intervals of the form* $(a, b_0]$, *where* b_0 *is the largest element(if exist) of* $\mathbb{X}$.

The collection $\mathbb{B}$ *formed a basis for a topology on* $\mathbb{X}$, *which is called the order topology.*

Definition 1.3.3 (ray)**.** [10][11] *If* $\mathbb{X}$ *is an ordered set, and* a *is an element of* $\mathbb{X}$, *there are four subsets of* $\mathbb{X}$ *that are called* ***rays*** *determined by* a:

- $(a, +\infty) = \{x|x > a\}$
- $(-\infty, a) = \{x|x < a\}$
- $[a, +\infty) = \{x|x \geq a\}$
- $(-\infty, a] = \{x|x \leq a\}$

$(a, +\infty)$ *and* $(-\infty, a)$ *are called* ***open rays***. $[a, +\infty)$ *and* $(-\infty, a]$ *are called* ***closed rays***.

1.4 The Product Topology

Definition 1.4.1 (product topology)**.** *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be topological spaces. The* ***product topology*** *on* $\mathbb{X} \times \mathbb{Y}$ *having a basis* $\mathbb{B}$ *containing all sets of the form* $U \times V$, *where* U *and* V *is open sets of* $\mathbb{X}$ *and* $\mathbb{Y}$ *respectively.*

Theorem 1.4.1. [12] *If* $\mathbb{B}$ *and* $\mathbb{C}$ *is basis for the topology of* $\mathbb{X}$ *and* $\mathbb{Y}$ *respectively, then the collection*

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} and C \in \mathbb{C}\}$$

is a basis for the topology of $\mathbb{X} \times \mathbb{Y}$

Definition 1.4.2 (projection)**.** *Let* $\pi_1 : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ *be defined by the equation:*

$$\pi_1(x, y) = x$$

Let $\pi_2 : \mathbb{X} \times \mathbb{Y} \to \mathbb{Y}$ *be defined by the equation:*

$$\pi_1(x, y) = y$$

The maps π_1 *and* π_2 *are called the* ***projections*** *of* $\mathbb{X} \times \mathbb{Y}$ *onto its first and second factors, respectively.*

Theorem 1.4.2. [13] *The collection*

$$\mathbb{S} = \{\pi_1^{-1}(U) | U open in \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V open in \mathbb{Y}\}$$

is a subbasis for the product topology on $\mathbb{X} \times \mathbb{Y}$.

[10] open rays are always open sets in the order topology
[11] the open rays also formed a subbasis of the order topology
[12] We omit the proof of this lemma as it is obvious.
[13] We omit the proof of this lemma as it is obvious.

1.5 The Subspace Topology

Definition 1.5.1 (subspace topology). *Let $\mathbb{X}$ be a topological space with topology $\mathbb{T}$. If Y is a subset of $\mathbb{X}$, the collection $\mathbb{T}_Y = \{Y \cap U | U \in \mathbb{T}\}$ is a topology on Y, called the **subspace topology**.*

*Y is also called a **subspace** of $\mathbb{X}$*

Lemma 1.5.1. [14]*If $\mathbb{B}$ is basis for the topology of $\mathbb{X}$, Y is a subset of $\mathbb{X}$ then the collection*

$$\mathbb{B}_Y = \{B \cap Y | B \in \mathbb{B}\}$$

is a basis for the subspace topology on Y

Lemma 1.5.2. [15]*Let Y be a subspace of $\mathbb{X}$. If U is open in Y and Y is open in $\mathbb{X}$, then U is open in $\mathbb{X}$.*

Theorem 1.5.1. [16] *If A is a subspace of $\mathbb{X}$ and B is a subspace of $\mathbb{Y}$, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$*

Proof. Let $\mathbb{B}_{\mathbb{X}}$ and $\mathbb{B}_{\mathbb{Y}}$ and $\mathbb{B}_{\mathbb{X}\mathbb{Y}}$ be basis of topology of $\mathbb{X}$ and $\mathbb{Y}$ and $\mathbb{X} \times \mathbb{Y}$ respectively. Let $\mathbb{B}'_{\mathbb{X}}$ and $\mathbb{B}'_{\mathbb{Y}}$ and $\mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ be basis of topology of A and A and $A \times B$ respectively. We will show that $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}} = \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$. Thus, the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$.

First, every element in $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ can be represented by $B_A \cap A \times B_B \cap B = B_A \times B_B \cap A \times B \in \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ where $B_A \in \mathbb{B}'_{\mathbb{X}}, B_B \in \mathbb{B}'_{\mathbb{Y}}$. Thus $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}} \subseteq \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$.

Next, we show that $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$. For any open set U in $\mathbb{X} \times \mathbb{Y}$, and $\forall x \in U \cap A \times B, \exists B_{\mathbb{X}} \times B_{\mathbb{Y}} \in \mathbb{B}_{\mathbb{X}\mathbb{Y}}, x \in B_{\mathbb{X}} \times B_{\mathbb{Y}} \subseteq \mathbb{X} \times \mathbb{Y}$. Thus $x \in B_{\mathbb{X}} \times B_{\mathbb{Y}} \cap A \times B \subseteq A \times B, B_{\mathbb{X}} \times B_{\mathbb{Y}} \cap A \times B \in \mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$. Thus $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$.gi ☐

Definition 1.5.2 (ordered square). *Let $I = [0, 1]$. The set $I \times I$ in the dictionary order* [17] *topology will be called **ordered square**, and denoted by I_o^2*

Definition 1.5.3 (convex). *Given an ordered set $\mathbb{X}$, let us say that a subset $\mathbb{Y}$ of $\mathbb{X}$ is **convex** in $\mathbb{X}$ if for each pair of points $a < b$ of $\mathbb{Y}$, the entire interval (a, b) of points of $\mathbb{X}$ lies in $\mathbb{Y}$*

Theorem 1.5.2. [18] *Let $\mathbb{X}$ be an ordered set in the order topology. Let $\mathbb{Y}$ be a subset of $\mathbb{X}$ that is convex in $\mathbb{X}$. Then the order topology on $\mathbb{Y}$ is the same as the topology $\mathbb{Y}$ inherits as a subspace of $\mathbb{X}$.*

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[14]We omit the proof of this lemma as it is obvious.

[15]We omit the proof of this lemma as it is obvious.

[16]If $\mathbb{X}$ is an ordered set in the order topology, and $\mathbb{Y}$ is a subset of $\mathbb{X}$. The order relation, when restricted to $\mathbb{Y}$, makes $\mathbb{Y}$ into and ordered set. However, *the resulting order topology on $\mathbb{Y}$ need not be the same as the topology that $\mathbb{Y}$ inherits as a subspace of $\mathbb{X}$.*

[17]the dictionary means for $X_1, X_2 \in \mathbb{Y} = \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \dots$ which:

$$\begin{aligned} X_1 &= (x_1, x_2, x_3 \dots) \\ X_2 &= (x'_1, x'_2, x'_3 \dots) \end{aligned}$$

$X_1 > X_2$ only when

$$\begin{aligned} &\exists k \in \mathbb{Z}_+, \forall i \in \mathbb{Z}_+, 0 < i < k \\ &x_i = x'_i \\ &x_k > x'_k \end{aligned}$$

[18]Given $\mathbb{X}$ is an ordered set in the order topology and $\mathbb{Y}$ is a subset of $\mathbb{X}$, we shall assume that $\mathbb{Y}$ *is given the subspace topology* unless we specifically state otherwise.

Proof. Consider the ray $(a, +\infty)$ in $\mathbb{X}$. If $a \in \mathbb{Y}$, then

$$(a, +\infty) \cap \mathbb{Y} = \{x | x \in \mathbb{Y} and x > a\}$$

This is an open ray of the ordered set of $\mathbb{Y}$. if $a \notin Y$, then a is either a lower bound on $\mathbb{Y}$ or an upper bound on $\mathbb{Y}$, since $\mathbb{Y}$ is convex. In the former case, the set $(a, +\infty) \cap \mathbb{Y}$ equals all of $\mathbb{Y}$, in the latter case, it is empty.

A similar remark shows that the intersection of the rat $(-\infty, a)$ with $\mathbb{Y}$ is either an open ray of $\mathbb{Y}$, or $\mathbb{Y}$ itself, or empty. Since the sets $(a, +\infty)\mathbb{Y}$ and $(-\infty, a) \cap \mathbb{Y}$ form a subbasis for the subspace topology on $\mathbb{Y}$, and since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of $\mathbb{Y}$ equals the intersection of an open ray of $\mathbb{X}$ with $\mathbb{Y}$, so it is open in the subspace topology on $\mathbb{Y}$. Since the open rays of $\mathbb{Y}$ are a subbasis for the order topology on $\mathbb{Y}$, this topology is contained in the subspace topology. $\square$