

# Topology Note

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# Chapter 1

## Topology Spaces and Continuous Function

### 1.1 Basic Definition of Topology

**Definition 1.1.1** (topology). A **topology** on a set  $\mathbb{X}$  is a collection  $\mathbb{T}$  of subsets of  $\mathbb{X}$  having the following properties:

- $\emptyset$  and  $\mathbb{X}$  are in  $\mathbb{T}$
- The union of the elements of any sub collection of  $\mathbb{T}$  is in  $\mathbb{T}$
- The intersection of the elements of any **finite** sub collection of  $\mathbb{T}$  is in  $\mathbb{T}$

**Definition 1.1.2** (topology space). A **topological space** is a set  $\mathbb{X}$  for which a topology  $\mathbb{T}$  has been specified.

**Definition 1.1.3** (open set). A **open set**  $\mathbb{U}$  is a subset of  $\mathbb{X}$  that belongs to a topology  $\mathbb{T}$  of  $\mathbb{X}$ .

**Definition 1.1.4** (open sets). A topology can also be called a **open sets**

**Definition 1.1.5** (discrete topology). The set of all subsets of a set  $\mathbb{X}$  formed a topology called **discrete topology**

**Definition 1.1.6** (trivial topology). The set consisting the set  $\mathbb{X}$  and  $\emptyset$  only formed a topology of  $\mathbb{X}$  called **trivial topology**

**Definition 1.1.7** (finite complement topology). Let  $\mathbb{X}$  be a set. Let  $\mathbb{T}_f$  be the collection of all subsets  $\mathbb{U}$  of  $\mathbb{X}$  such that  $\mathbb{X} - \mathbb{U}$  either if a **finite**<sup>1</sup> of is all of  $\mathbb{X}$ . Then  $\mathbb{T}_f$  is a topology on  $\mathbb{X}$ , called the .

**Definition 1.1.8** (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable). Let  $\mathbb{T}$  and  $\mathbb{T}'$  be two topology on a given set  $\mathbb{X}$ . If  $\mathbb{T}$  is a subset of  $\mathbb{T}'$ , we say that  $\mathbb{T}'$  is **finer** or **larger** than  $\mathbb{T}$ . If  $\mathbb{T}$  is a proper subset of  $\mathbb{T}'$ , we say that  $\mathbb{T}'$  is **strictly finer** or **strictly larger** than  $\mathbb{T}$ . We also say that  $\mathbb{T}$  is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than  $\mathbb{T}'$ . We say that  $\mathbb{T}$  and  $\mathbb{T}'$  is **comparable** if either  $\mathbb{T}$  is a subset of  $\mathbb{T}'$  or  $\mathbb{T}'$  is a subset of  $\mathbb{T}$ .

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<sup>1</sup>The set  $\mathbb{U}$  can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection,  $\mathbb{U}$  will not be a topology.

## 1.2 Basis for a Topology

**Definition 1.2.1** (basis). If  $\mathbb{X}$  is a set, a **basis** for a topology on  $\mathbb{X}$  is a collection  $\mathbb{B}$  of subsets of  $\mathbb{X}$  (called **basis elements**) such that:

- For each  $x \in \mathbb{X}$ , there is at least one basis element  $B$  containing  $x$
- If  $x$  belongs to the intersection of two basis elements  $B_1$  and  $B_2$ , then there is another element  $x \in B_3 \in \mathbb{B}$  such that  $B_3 \subseteq B_1 \cap B_2$

**Definition 1.2.2** (topology generated by basis). Let  $\mathbb{B}$  be a basis on  $\mathbb{X}$ . Let  $\mathbb{U}$  be a set containing all subsets  $U$  of  $\mathbb{X}$  such that for each element  $x \in U$ , there is  $B \in \mathbb{B}$  that  $x \in B \subseteq U$ . Such  $\mathbb{U}$  formed a topology on  $\mathbb{X}$ , called **topology  $\mathbb{T}$  generated by  $\mathbb{B}$**

**Lemma 1.2.1.** Let  $\mathbb{X}$  be a set. Let  $\mathbb{B}$  be a basis for a topology  $\mathbb{T}$  on  $\mathbb{X}$ . Then  $\mathbb{T}$  equals to the set of all possible unions of elements of  $\mathbb{B}$ .

*Proof.* Let set  $\mathbb{U}$  be the set of all possible unions of elements of  $\mathbb{B}$ . For any  $U \in \mathbb{U}$ .  $U = \cup B$  <sup>2</sup> for some  $B \in \mathbb{B}$ . Thus, for every  $x \in U$ , there exist a  $B' \in \mathbb{B}$  that  $x \in B' \subseteq U$ . Thus,  $U \in \mathbb{T}$ .

Conversely, for any  $U \in \mathbb{T}$ . For any  $x \in U$ , let  $x \in B_x \in \mathbb{B}$ . Then,  $U = \cup_{x \in U} B_x$ . Thus,  $U \in \mathbb{U}$ .

Therefore,  $\mathbb{U}$  equals to  $\mathbb{T}$ . □

**Lemma 1.2.2.** <sup>3</sup> Let  $\mathbb{X}$  be a topological space. Suppose that  $\mathbb{C}$  is a collection of open sets of  $\mathbb{X}$  such that for each open set  $U$  of  $\mathbb{X}$  and each  $x \in U$ , there is an element  $C \in \mathbb{C}$  such that  $x \in C \subseteq U$ . Then  $\mathbb{C}$  is a basis for the topology of  $\mathbb{X}$ .

**Lemma 1.2.3.** <sup>4</sup> Let  $\mathbb{B}$  and  $\mathbb{B}'$  be basis for the topologies  $\mathbb{T}$  and  $\mathbb{T}'$ , respectively, on  $\mathbb{X}$ . Then the following are equivalent:

- $\mathbb{T}'$  is finer than  $\mathbb{T}$
- For each  $x \in \mathbb{X}$  and each basis element  $B \in \mathbb{B}$  containing  $x$ , there is a basis element  $B' \in \mathbb{B}'$  such that  $x \in B' \subseteq B$ .

**Definition 1.2.3** (standard topology on the real line). Let be  $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$ .  $\mathbb{B}$  formed a basis on real line. The topology generated by  $\mathbb{B}$  is called the **standard topology on the real line** <sup>5</sup>.

**Definition 1.2.4** (lower limit topology on the real line). Let be  $\mathbb{B} = \{B | B = \{x | a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$ .  $\mathbb{B}$  formed a basis on real line. The topology generated by  $\mathbb{B}$  is called the **lower limit topology on the real line**. When  $\mathbb{R}$  is given this topology, we denote it by  $\mathbb{R}_l$ .

**Definition 1.2.5** (K-topology on the real line). Let be  $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$ . Let  $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$ .  $\mathbb{B} \cup \{B - K | B \in \mathbb{B}\}$  formed a basis on real line. The topology generated by  $\mathbb{B}$  is called the **K-topology on the real line**. When  $\mathbb{R}$  is given this topology, we denote it by  $\mathbb{R}_K$ .

**Lemma 1.2.4.** <sup>6</sup> The topologies  $\mathbb{R}_l$  and  $\mathbb{R}_K$  is strictly finer than the standard topology on  $\mathbb{R}$ .

<sup>2</sup>Note that this expression may not be unique.

<sup>3</sup>We omit the proof of this lemma as it is obvious.

<sup>4</sup>We omit the proof of this lemma as it is obvious.

<sup>5</sup>Whenever we consider  $\mathbb{R}$ , we shall suppose it is given this topology unless we specifically state otherwise.

<sup>6</sup>We omit the proof of this lemma as it is obvious.

**Lemma 1.2.5.** *The topologies of  $\mathbb{R}_l$  and  $\mathbb{R}_\mathbb{K}$  is not comparable.*

*Proof.* Let  $\mathbb{T}_l$  and  $\mathbb{T}_\mathbb{K}$  be topologies of  $\mathbb{R}_l$  and  $\mathbb{R}_\mathbb{K}$  respectively. Let  $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$ .

We first proof that  $\mathbb{T}_l$  is not finer than  $\mathbb{T}_\mathbb{K}$ . Let  $U = \{x | -1 < x < 1\} - K, x = 0$ . If there exist  $B = \{x | a \leq x < b\} \in \mathbb{T}_l$  such that  $x \in B \subseteq U$ , then  $0 < b < 1$ . Thus, there exist  $n \in \mathbb{Z}_+$  that  $0 < \frac{1}{n} < b$ . Thus  $B$  is not a subset of  $U$ .

Then we proof that  $\mathbb{T}_\mathbb{K}$  is not finer than  $\mathbb{T}_l$ . Let  $U' = \{x | a' \leq x < b'\}$ . If there exist  $B' = \{x | a'' < x < b''\} \text{ or } \{x | a'' < x < b''\} - K$  such that  $a' \in B \subseteq U$ . Thus  $a'' < a < b''$ . Thus there exist  $c$  that  $a'' < x < a, x \in B, x \notin U'$ . Thus  $B' \not\subseteq U'$ .

Thus the topologies of  $\mathbb{R}_l$  and  $\mathbb{R}_\mathbb{K}$  is not comparable.  $\square$

**Definition 1.2.6** (subbasis). A **subbasis**  $\mathbb{S}$  for a topology on  $\mathbb{X}$  is a collection of subsets of  $\mathbb{X}$  whose union equals  $\mathbb{X}$ . The **topology generated by the subbasis**  $\mathbb{S}$  is defined to be the collection  $\mathbb{T}$ <sup>7</sup> of all unions of finite intersections of elements of  $\mathbb{S}$ .

### 1.2.1 Exercise

1. Show that if  $\mathbb{A}$  is a basis for a topology on  $\mathbb{X}$ , then the topology generated by  $\mathbb{A}$  equals the intersection of all topologies on  $\mathbb{X}$  that contain  $\mathbb{A}$ . Prove the same if  $\mathbb{A}$  is a subbasis.

*Proof.* As a subbasis is also a basis, we will directly prove the case of subbasis here.

Let  $\mathbb{S} = \{\mathbb{T}_\alpha\}$  be set contain all the topologies that contain  $\mathbb{A}$ . Let  $\mathbb{T}$  be the topology that  $\mathbb{A}$  generated. Let  $\mathbb{T}' = \cap \mathbb{T}_\alpha$ .<sup>8</sup>

First,  $\mathbb{A} \subseteq \mathbb{T}_\alpha$ . Thus,  $\mathbb{T} \subseteq \mathbb{T}_\alpha$ . Thus,  $\mathbb{T} \subseteq \mathbb{T}'$ .

Also,  $\mathbb{A} \subseteq \mathbb{T}$ . Thus,  $\mathbb{T} \in \mathbb{S}$ . Thus,  $\mathbb{T}' \subseteq \mathbb{T}$ .

Thus,  $\mathbb{T} = \mathbb{T}'$   $\square$

## 1.3 The Order Topology

**Definition 1.3.1** (interval). Let  $\mathbb{X}$  is a set having a simple order relation  $<$ . Given elements  $a$  and  $b$  of  $\mathbb{X}$  such that  $a < b$ , there are four subsets of  $\mathbb{X}$  that are called **intervals** determined by  $a$  and  $b$ :

- $(a, b) = \{x | a < x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $[a, b] = \{x | a \leq x \leq b\}$

$(a, b)$  is called an **open interval** on  $\mathbb{X}$ .  $[a, b]$  is called an **closed interval** on  $\mathbb{X}$ .  $(a, b]$  and  $[a, b)$  is called **half-open intervals**.

**Definition 1.3.2** (order topology).<sup>9</sup> Let  $\mathbb{X}$  be a set with a simple order relation; assume  $\mathbb{X}$  has more than one element. Let  $\mathbb{B}$  be the collection of all sets of the following types:

- All open intervals  $(a, b)$  in  $\mathbb{X}$ .

<sup>7</sup>It is obvious that  $\mathbb{T}$  is a topology, we just omit the proof here.

<sup>8</sup>It is obvious that  $\mathbb{T}'$  is also a topology, we just omit the proof here.

<sup>9</sup>The standard topology on  $\mathbb{R}$  is an order topology derived from the usual order on  $\mathbb{R}$ .

- All intervals of the form  $[a_0, b)$ , where  $a_0$  is the smallest element(if exist) of  $\mathbb{X}$ .
- All intervals of the form  $(a, b_0]$ , where  $b_0$  is the largest element(if exist) of  $\mathbb{X}$ .

The collection  $\mathbb{B}$  formed a basis for a topology on  $\mathbb{X}$ , which is called the order topology.

**Definition 1.3.3** (ray).<sup>1011</sup> If  $\mathbb{X}$  is an ordered set, and  $a$  is an element of  $\mathbb{X}$ , there are four subsets of  $\mathbb{X}$  that are called **rays** determined by  $a$ :

- $(a, +\infty) = \{x | x > a\}$
- $(-\infty, a) = \{x | x < a\}$
- $[a, +\infty) = \{x | x \geq a\}$
- $(-\infty, a] = \{x | x \leq a\}$

$(a, +\infty)$  and  $(-\infty, a)$  are called **open rays**.  $[a, +\infty)$  and  $(-\infty, a]$  are called **closed rays**.

## 1.4 The Product Topology

**Definition 1.4.1** (product topology). Let  $\mathbb{X}$  and  $\mathbb{Y}$  be topological spaces. The **product topology** on  $\mathbb{X} \times \mathbb{Y}$  having a basis  $\mathbb{B}$  containing all sets of the form  $U \times V$ , where  $U$  and  $V$  is open sets of  $\mathbb{X}$  and  $\mathbb{Y}$  respectively.

**Theorem 1.4.1.**<sup>12</sup> If  $\mathbb{B}$  and  $\mathbb{C}$  is basis for the topology of  $\mathbb{X}$  and  $\mathbb{Y}$  respectively, then the collection

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} \text{ and } C \in \mathbb{C}\}$$

is a basis for the topology of  $\mathbb{X} \times \mathbb{Y}$

**Definition 1.4.2** (projection). Let  $\pi_1 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$  be defined by the equation:

$$\pi_1(x, y) = x$$

Let  $\pi_2 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}$  be defined by the equation:

$$\pi_2(x, y) = y$$

The maps  $\pi_1$  and  $\pi_2$  are called the **projections** of  $\mathbb{X} \times \mathbb{Y}$  onto its first and second factors, respectively.

**Theorem 1.4.2.**<sup>13</sup> The collection

$$\mathbb{S} = \{\pi_1^{-1}(U) | U \text{ open in } \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V \text{ open in } \mathbb{Y}\}$$

is a subbasis for the product topology on  $\mathbb{X} \times \mathbb{Y}$ .

<sup>10</sup>open rays are always open sets in the order topology

<sup>11</sup>the open rays also formed a subbasis of the order topology

<sup>12</sup>We omit the proof of this lemma as it is obvious.

<sup>13</sup>We omit the proof of this lemma as it is obvious.



## 1.5 The Subspace Topology

**Definition 1.5.1** (subspace topology). *Let  $\mathbb{X}$  be a topological space with topology  $\mathbb{T}$ . If  $Y$  is a subset of  $\mathbb{X}$ , the collection  $\mathbb{T}_Y = \{Y \cap U | U \in \mathbb{T}\}$  is a topology on  $Y$ , called the **subspace topology**.*

*$Y$  is also called a **subspace** of  $\mathbb{X}$*

**Lemma 1.5.1.** <sup>14</sup>*If  $\mathbb{B}$  is basis for the topology of  $\mathbb{X}$ ,  $Y$  is a subset of  $\mathbb{X}$  then the collection*

$$\mathbb{B}_Y = \{B \cap Y | B \in \mathbb{B}\}$$

*is a basis for the subspace topology on  $Y$*

**Lemma 1.5.2.** <sup>15</sup>*Let  $Y$  be a subspace of  $\mathbb{X}$ . If  $U$  is open in  $Y$  and  $Y$  is open in  $\mathbb{X}$ , then  $U$  is open in  $\mathbb{X}$ .*

**Theorem 1.5.1.** <sup>16</sup>*If  $A$  is a subspace of  $\mathbb{X}$  and  $B$  is a subspace of  $\mathbb{Y}$ , then the product topology on  $A \times B$  is the same as the topology  $A \times B$  inherits as a subspace of  $\mathbb{X} \times \mathbb{Y}$*

*Proof.* Let  $\mathbb{B}_\mathbb{X}$  and  $\mathbb{B}_\mathbb{Y}$  and  $\mathbb{B}_{\mathbb{X}\mathbb{Y}}$  be basis of topology of  $\mathbb{X}$  and  $\mathbb{Y}$  and  $\mathbb{X} \times \mathbb{Y}$  respectively. Let  $\mathbb{B}'_\mathbb{X}$  and  $\mathbb{B}'_\mathbb{Y}$  and  $\mathbb{B}'_{\mathbb{X}\mathbb{Y}}$  be basis of topology of  $A$  and  $A$  and  $A \times B$  respectively. We will show that  $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} = \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ . Thus, the product topology on  $A \times B$  is the same as the topology  $A \times B$  inherits as a subspace of  $\mathbb{X} \times \mathbb{Y}$ .

First, every element in  $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$  can be represented by  $B_A \cap A \times B_B \cap B = B_A \times B_B \cap A \times B \in \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$  where  $B_A \in \mathbb{B}'_\mathbb{X}$ ,  $B_B \in \mathbb{B}'_\mathbb{Y}$ . Thus  $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} \subseteq \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ .

Next, we show that  $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$  generate the topology  $A \times B$  inherits as a subspace of  $\mathbb{X} \times \mathbb{Y}$ . For any open set  $U$  in  $\mathbb{X} \times \mathbb{Y}$ , and  $\forall x \in U \cap A \times B$ ,  $\exists B_\mathbb{X} \times B_\mathbb{Y} \in \mathbb{B}_{\mathbb{X}\mathbb{Y}}$ ,  $x \in B_\mathbb{X} \times B_\mathbb{Y} \subseteq \mathbb{X} \times \mathbb{Y}$ . Thus  $x \in B_\mathbb{X} \times B_\mathbb{Y} \cap A \times B \subseteq A \times B$ ,  $B_\mathbb{X} \times B_\mathbb{Y} \cap A \times B \in \mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$ . Thus  $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$  generate the topology  $A \times B$  inherits as a subspace of  $\mathbb{X} \times \mathbb{Y}$ .  $\square$

**Definition 1.5.2** (ordered square). *Let  $I = [0, 1]$ . The set  $I \times I$  in the dictionary order <sup>17</sup> topology will be called **ordered square**, and denoted by  $I_o^2$*

**Definition 1.5.3** (convex). *Given an ordered set  $\mathbb{X}$ , let us say that a subset  $\mathbb{Y}$  of  $\mathbb{X}$  is **convex** in  $\mathbb{X}$  if for each pair of points  $a < b$  of  $\mathbb{Y}$ , the entire interval  $(a, b)$  of points of  $\mathbb{X}$  lies in  $\mathbb{Y}$*

**Theorem 1.5.2.** <sup>18</sup>*Let  $\mathbb{X}$  be an ordered set in the order topology. Let  $\mathbb{Y}$  be a subset of  $\mathbb{X}$  that is convex in  $\mathbb{X}$ . Then the order topology on  $\mathbb{Y}$  is the same as the topology  $\mathbb{Y}$  inherits as a subspace of  $\mathbb{X}$ .*

<sup>14</sup>We omit the proof of this lemma as it is obvious.

<sup>15</sup>We omit the proof of this lemma as it is obvious.

<sup>16</sup>If  $\mathbb{X}$  is an ordered set in the order topology, and  $\mathbb{Y}$  is a subset of  $\mathbb{X}$ . The order relation, when restricted to  $\mathbb{Y}$ , makes  $\mathbb{Y}$  into an ordered set. However, the resulting order topology on  $\mathbb{Y}$  need not be the same as the topology that  $\mathbb{Y}$  inherits as a subspace of  $\mathbb{X}$ .

<sup>17</sup>the dictionary means for  $X_1, X_2 \in \mathbb{Y} = \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \dots$  which:

$$\begin{aligned} X_1 &= (x_1, x_2, x_3 \dots) \\ X_2 &= (x'_1, x'_2, x'_3 \dots) \end{aligned}$$

$X_1 > X_2$  only when

$$\begin{aligned} \exists k \in \mathbb{Z}_+, \forall i \in \mathbb{Z}_+, 0 < i < k \\ x_i &= x'_i \\ x_k &> x'_k \end{aligned}$$

<sup>18</sup>Given  $\mathbb{X}$  is an ordered set in the order topology and  $\mathbb{Y}$  is a subset of  $\mathbb{X}$ , we shall assume that  $\mathbb{Y}$  is given the subspace topology unless we specifically state otherwise.

*Proof.* Consider the ray  $(a, +\infty)$  in  $\mathbb{X}$ . If  $a \in \mathbb{Y}$ , then

$$(a, +\infty) \cap \mathbb{Y} = \{x \mid x \in \mathbb{Y} \text{ and } x > a\}$$

This is an open ray of the ordered set of  $\mathbb{Y}$ . If  $a \notin \mathbb{Y}$ , then  $a$  is either a lower bound on  $\mathbb{Y}$  or an upper bound on  $\mathbb{Y}$ , since  $\mathbb{Y}$  is convex. In the former case, the set  $(a, +\infty) \cap \mathbb{Y}$  equals all of  $\mathbb{Y}$ , in the latter case, it is empty.

A similar remark shows that the intersection of the ray  $(-\infty, a)$  with  $\mathbb{Y}$  is either an open ray of  $\mathbb{Y}$ , or  $\mathbb{Y}$  itself, or empty. Since the sets  $(a, +\infty) \cap \mathbb{Y}$  and  $(-\infty, a) \cap \mathbb{Y}$  form a subbasis for the subspace topology on  $\mathbb{Y}$ , and since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of  $\mathbb{Y}$  equals the intersection of an open ray of  $\mathbb{X}$  with  $\mathbb{Y}$ , so it is open in the subspace topology on  $\mathbb{Y}$ . Since the open rays of  $\mathbb{Y}$  are a subbasis for the order topology on  $\mathbb{Y}$ , this topology is contained in the subspace topology.  $\square$

### Exercise

1. A map  $f : \mathbb{X} \rightarrow \mathbb{Y}$  is said to be a **open map** if for every open set  $U \subseteq \mathbb{X}$ , the set  $f(U)$  is open in  $\mathbb{Y}$ . Show that  $\pi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$  is open map.

*Proof.* An open set in  $\mathbb{X} \times \mathbb{Y}$  can be represented by

$$\cup(U_i \times U'_i)$$

where  $U_i, U'_i$  are open sets in  $\mathbb{X}, \mathbb{Y}$ , respectively.

Also,

$$\cup(U_i \times U'_i) = \cup(U_i) \times \cup(U'_i)$$

Thus,

$$\pi(\cup(U_i \times U'_i)) = \cup(U_i)$$

Thus,  $\pi(U)$  is open in  $\mathbb{X}$ .  $\square$

2. Let  $\mathbb{X}$  and  $\mathbb{X}'$  denote a single set in the topologies  $\mathbb{T}$  and  $\mathbb{T}'$ , respectively; let  $\mathbb{Y}$  and  $\mathbb{Y}'$  denote a single set in the topologies  $\mathbb{U}$  and  $\mathbb{U}'$ , respectively.<sup>19</sup> Assume these sets are nonempty.

- (a) Show that if  $\mathbb{T}' \supseteq \mathbb{T}$  and  $\mathbb{U}' \supseteq \mathbb{U}$ , then the product topologies  $\mathbb{X}' \times \mathbb{Y}'$  is finer than the product topology on  $\mathbb{X} \times \mathbb{Y}$ .
- (b) Does the converse of the previous statement hold?

3. Show that the countable collection<sup>20</sup>

$$\{(a, b) \times (c, d) \mid a < b, c < d, a \in \mathbb{Q}, b \in \mathbb{Q}, c \in \mathbb{Q}, d \in \mathbb{Q}\}$$

is a basis for  $\mathbb{R}^2$

*Proof.* This is obvious if you prove that  $(a, b) \times (c, d)$  is a rectangle in the  $\mathbb{R}^2$  plane.  $\square$

<sup>19</sup>what does  $\mathbb{X}, \mathbb{X}', \mathbb{Y}, \mathbb{Y}'$  really mean here?? I do not know, so I just put the exercise here without a proof.

<sup>20</sup>The prove of this set is countable is typically similar to Cantor's enumeration of a countable collection of countable sets.

4. Let  $\mathbb{X}$  be an ordered set. If  $\mathbb{Y}$  is a proper subset of  $\mathbb{X}$  that is convex in  $\mathbb{X}$  prove that  $\mathbb{Y}$  may not be an interval or a ray in  $\mathbb{X}$ .

*Proof.* Let  $\mathbb{X} = \mathbb{R}^2$  with dictionary order. Then  $Y = \{(x, y) \mid -1 \leq x \leq 1\}$  is convex in  $\mathbb{X}$ , however it is not an interval or a ray.  $\square$

There is a false prove given by myself.

*Proof.* Let  $\mathbb{S}$  be a set that contain all intervals and rays of  $\mathbb{Y}$ . We define a partial order on  $\mathbb{S}$  by inclusion. So if there is a chain in  $\mathbb{S}$ :

$$S_1 \subseteq S_2 \subseteq S_3 \dots$$

Let

$$S = S_1 \cup S_2 \cup S_3 \cup \dots$$

Thus,  $S$  is an upper bound of the chain.

Thus, by Zorn's Lemma, there is a maximal element of  $\mathbb{S}$ , say  $U$ , then we prove that  $U = \mathbb{Y}$ .

If  $U \neq \mathbb{Y}$ , then  $\exists x, x \in \mathbb{Y} - U$ .

If  $U$  is a ray say  $(a, +\infty)$ . Then  $x < a$ , thus  $U \subseteq (x, +\infty) \subseteq \mathbb{B}$ , then there is contradiction with the maximal element.

If  $U$  is an interval, the circumstance is similar with the proof of  $U$  is a ray.

Thus  $\mathbb{Y}$  is a ray or an interval.  $\square$

However, there is issue with this proof, the set  $S$  does exists. However, it may not be an interval or ray, so it may not be contained in  $\mathbb{S}$