Topology Note

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Contents

1 Basic Definition of Topology

Definition 1.1 (topology)**.** *A **topology** on a set $\mathbb{X}$ is a collection $\mathbb{T}$ of subsets of $\mathbb{X}$ having the following properties:*

- $\emptyset$ *and* $\mathbb{X}$ *are in* $\mathbb{T}$
- *The union of the elements of any sub collection of* $\mathbb{T}$ *is in* $\mathbb{T}$
- *The intersection of the elements of any **finite** sub collection of* $\mathbb{T}$ *is in* $\mathbb{T}$

Definition 1.2 (topology space)**.** *A **topological space** is a set $\mathbb{X}$ for which a topology $\mathbb{T}$ has been specified.*

Definition 1.3 (open set)**.** *A **open set** $\mathbb{U}$ is a subset of $\mathbb{X}$ that belongs to a topology $\mathbb{T}$ of $\mathbb{X}$.*

Definition 1.4 (open sets)**.** *A topology can also be called a **open sets***

Definition 1.5 (discrete topology)**.** *The set of all subsets of a set $\mathbb{X}$ formed a topology called **discrete topology***

Definition 1.6 (trivial topology)**.** *The set consisting the set $\mathbb{X}$ and $\emptyset$ only formed a topology of $\mathbb{X}$ called **trivial topology***

Definition 1.7 (finite complement topology)**.** *Let $\mathbb{X}$ be a set. Let $\mathbb{T}_f$ be the collection of all subsets $\mathbb{U}$ of $\mathbb{X}$ such that $\mathbb{X} - \mathbb{U}$ either if a **finite** of is all of $\mathbb{X}$. Then $\mathbb{T}_f$ is a topology on $\mathbb{X}$, called the .*

The set $\mathbb{U}$ can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, $\mathbb{U}$ will not be a topology.

Definition 1.8 (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable)**.** *Let $\mathbb{T}$ and $\mathbb{T}'$ be two topology on a given set $\mathbb{X}$. If $\mathbb{T}$ is a subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **finer** or **larger** than $\mathbb{T}$. If $\mathbb{T}$ is a proper subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **strictly finer** or **strictly larger** than $\mathbb{T}$. We also say that $\mathbb{T}$ is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than $\mathbb{T}'$. We say that $\mathbb{T}$ and $\mathbb{T}'$ is **comparable** if either $\mathbb{T}$ is a subset of $\mathbb{T}'$ or $\mathbb{T}'$ is a subset of $\mathbb{T}$.*

2 Basis for a Topology

Definition 2.1 (basis)**.** *If $\mathbb{X}$ is a set, a **basis** for a topology on $\mathbb{X}$ is a collection $\mathbb{B}$ of subsets of $\mathbb{X}$ (called **basis elements**) such that:*

- *For each $x \in \mathbb{X}$, there is at least one basis element B containing x*
- *If x belongs to the intersection of two basis elements B_1 and B_2, then there is another element $x \in B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$*

Definition 2.2 (topology generated by basis)**.** *Let $\mathbb{B}$ be a basis on $\mathbb{X}$. Let $\mathbb{U}$ be a set containing all subsets U of $\mathbb{X}$ such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such $\mathbb{U}$ formed a topology on $\mathbb{X}$, called **topology** $\mathbb{T}$ **generated by** $\mathbb{B}$*

Lemma 2.1. *Let $\mathbb{X}$ be a set. Let $\mathbb{B}$ be a basis for a topology $\mathbb{T}$ on $\mathbb{X}$. Then $\mathbb{T}$ equals to the set of all possible unions of elements of $\mathbb{B}$.*

Proof. Let set $\mathbb{U}$ be the set of all possible unions of elements of $\mathbb{B}$. For any $U \in \mathbb{U}$. $U = \cup B$ for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Note that the expression may not be unique.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in U$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, $\mathbb{U}$ equals to $\mathbb{T}$. □

Lemma 2.2. [1] *Let $\mathbb{X}$ be a topological space. Suppose that $\mathbb{C}$ is a collection of open sets of $\mathbb{X}$ such that for each open set U of $\mathbb{X}$ and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq C$. Then $\mathbb{C}$ is a basis for the topology of $\mathbb{X}$.*

Lemma 2.3. [2] *Let $\mathbb{B}$ and $\mathbb{B}'$ be basis for the topologies $\mathbb{T}$ and $\mathbb{T}'$, respectively, on $\mathbb{X}$. Then the following are equivalent:*

- *$\mathbb{T}'$ is finer than $\mathbb{T}$*
- *For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing X, there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.*

Definition 2.3 (standard topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **standard topology on the real line** .*

Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.

Definition 2.4 (lower limit topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **lower limit topology on the real line**. When $\mathbb{R}$ is given this topology,we denote it by $\mathbb{R}_l$.*

Definition 2.5 (K-topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$. $\mathbb{B} \cup \{B - K | B \in \mathbb{B}\}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **K-topology on the real line**. When $\mathbb{R}$ is given this topology,we denote it by $\mathbb{R}_{\mathbb{K}}$.*

Lemma 2.4. [3] *The topologies $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is strictly finer than the standard topology on $\mathbb{R}$.*

Lemma 2.5. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is not comparable.*

Proof. Let $\mathbb{T}_l$ and $\mathbb{T}_{\mathbb{K}}$ be topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ respectively. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that $\mathbb{T}_l$ is not finer than $\mathbb{T}_{\mathbb{K}}$. Let $U = \{x | -1 < x < 1\} - K, x = 0$. If there exist $B = \{x | a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus B is not a subset of U.

Then we proof that $\mathbb{T}_{\mathbb{K}}$ is not finer than $\mathbb{T}_l$. Let $U' = \{x | a' \leq x < b'\}$. If there exist $B' = \{x | a'' < x < b''\} or \{x | a'' < x < b''\} - K$ such that $a' \in B \subseteq U$. Thus $a'' < a < b''$. Thus there exist c that $a'' < x < a, x \in B, x \notin U'$. Thus $B' \nsubseteq U'$.

Thus the topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is not comparable. □

Definition 2.6 (subbasis)**.** *A **subbasis** $\mathbb{S}$ for a topology on $\mathbb{X}$ is a collection of subsets of $\mathbb{X}$ whose union equals $\mathbb{X}$. The **topology generated by the subbasis** $\mathbb{S}$ is defined to be the collection $\mathbb{T}$ of all unions of finite intersections of elements of $\mathbb{S}$.*

It is obvious that $\mathbb{T}$ is a topology, we just omit the proof here.

2.1 Exercise

[1] We omit the proof of this lemma as it is obvious.
[2] We omit the proof of this lemma as it is obvious.
[3] We omit the proof of this lemma as it is obvious.