# Topology Note

Alex

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# Chapter 1

# Topology Spaces and Continuous Function

## 1.1 Basic Definition of Topology

**Definition 1.1.1** (topology). *A **topology** on a set $\mathbb{X}$ is a collection $\mathbb{T}$ of subsets of $\mathbb{X}$ having the following properties:*

- *$\emptyset$ and $\mathbb{X}$ are in $\mathbb{T}$*
- *The union of the elements of any sub collection of $\mathbb{T}$ is in $\mathbb{T}$*
- *The intersection of the elements of any **finite** sub collection of $\mathbb{T}$ is in $\mathbb{T}$*

**Definition 1.1.2** (topology space). *A **topological space** is a set $\mathbb{X}$ for which a topology $\mathbb{T}$ has been specified.*

**Definition 1.1.3** (open set). *A **open set** $\mathbb{U}$ is a subset of $\mathbb{X}$ that belongs to a topology $\mathbb{T}$ of $\mathbb{X}$.*

**Definition 1.1.4** (open sets). *A topology can also be called a **open sets***

**Definition 1.1.5** (discrete topology). *The set of all subsets of a set $\mathbb{X}$ formed a topology called **discrete topology***

**Definition 1.1.6** (trivial topology). *The set consisting the set $\mathbb{X}$ and $\emptyset$ only formed a topology of $\mathbb{X}$ called **trivial topology***

**Definition 1.1.7** (finite complement topology). *Let $\mathbb{X}$ be a set. Let $\mathbb{T}_f$ be the collection of all subsets $\mathbb{U}$ of $\mathbb{X}$ such that $\mathbb{X} - \mathbb{U}$ either if a **finite** [1] of is all of $\mathbb{X}$. Then $\mathbb{T}_f$ is a topology on $\mathbb{X}$, called the **finite complement topology**.*

[1] The set $\mathbb{U}$ can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, $\mathbb{U}$ will not be a topology.

**Definition 1.1.8** (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable)**.** *Let $\mathbb{T}$ and $\mathbb{T}'$ be two topology on a given set $\mathbb{X}$. If $\mathbb{T}$ is a subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **finer** or **larger** than $\mathbb{T}$. If $\mathbb{T}$ is a proper subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **strictly finer** or **strictly larger** than $\mathbb{T}$. We also say that $\mathbb{T}$ is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than $\mathbb{T}'$. We say that $\mathbb{T}$ and $\mathbb{T}'$ is **comparable** if either $\mathbb{T}$ is a subset of $\mathbb{T}'$ or $\mathbb{T}'$ is a subset of $\mathbb{T}$.*

## 1.2 Basis for a Topology

**Definition 1.2.1** (basis)**.** *If $\mathbb{X}$ is a set, a **basis** for a topology on $\mathbb{X}$ is a collection $\mathbb{B}$ of subsets of $\mathbb{X}$ (called **basis elements**) such that:*

- *For each $x \in \mathbb{X}$, there is at least one basis element $B$ containing $x$*
- *If $x$ belongs to the intersection of two basis elements $B_1$ and $B_2$, then there is another element $x \in B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$*

**Definition 1.2.2** (topology generated by basis)**.** *Let $\mathbb{B}$ be a basis on $\mathbb{X}$ . Let $\mathbb{U}$ be a set containing all subsets $U$ of $\mathbb{X}$ such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such $\mathbb{U}$ formed a topology on $\mathbb{X}$ , called **topology** $\mathbb{T}$ **generated by** $\mathbb{B}$*

**Lemma 1.2.1.** *Let $\mathbb{X}$ be a set. Let $\mathbb{B}$ be a basis for a topology $\mathbb{T}$ on $\mathbb{X}$ . Then $\mathbb{T}$ equals to the set of all possible unions of elements of $\mathbb{B}$ .*

*Proof.* Let set $\mathbb{U}$ be the set of all possible unions of elements of $\mathbb{B}$ . For any $U \in \mathbb{U}$. $U = \cup B$ [2] for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in U$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, $\mathbb{U}$ equals to $\mathbb{T}$ . □

**Lemma 1.2.2.** [3] *Let $\mathbb{X}$ be a topological space. Suppose that $\mathbb{C}$ is a collection of open sets of $\mathbb{X}$ such that for each open set $U$ of $\mathbb{X}$ and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq C$. Then $\mathbb{C}$ is a basis for the topology of $\mathbb{X}$ .*

**Lemma 1.2.3.** [4] *Let $\mathbb{B}$ and $\mathbb{B}'$ be basis for the topologies $\mathbb{T}$ and $\mathbb{T}'$ , respectively, on $\mathbb{X}$ . Then the following are equivalent:*

- $\mathbb{T}'$ *is finer than* $\mathbb{T}$
- *For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing X, there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.*

[2] Note that this expression may not be unique.
[3] We omit the proof of this lemma as it is obvious.
[4] We omit the proof of this lemma as it is obvious.

**Definition 1.2.3** (standard topology on the real line)**.** *Let be* $\mathbb{B} = \{B|B = \{x|a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$*.* $\mathbb{B}$ *formed a basis on real line. The topology generated by* $\mathbb{B}$ *is called the* ***standard topology on the real line*** [5] *.*

**Definition 1.2.4** (lower limit topology on the real line)**.** *Let be* $\mathbb{B} = \{B|B = \{x|a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$*.* $\mathbb{B}$ *formed a basis on real line. The topology generated by* $\mathbb{B}$ *is called the* ***lower limit topology on the real line****. When* $\mathbb{R}$ *is given this topology,we denote it by* $\mathbb{R}_l$*.*

**Definition 1.2.5** (K-topology on the real line)**.** *Let be* $\mathbb{B} = \{B|B = \{x|a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$*. Let* $K = \{x|x = \frac{1}{n}, n \in \mathbb{Z}_+\}$*.* $\mathbb{B} \cup \{B - K|B \in \mathbb{B}\}$ *formed a basis on real line. The topology generated by* $\mathbb{B}$ *is called the* ***K-topology on the real line****. When* $\mathbb{R}$ *is given this topology,we denote it by* $\mathbb{R}_{\mathbb{K}}$*.*

**Lemma 1.2.4.** [6]*The topologies* $\mathbb{R}_l$ *and* $\mathbb{R}_{\mathbb{K}}$ *is strictly finer than the standard topology on* $\mathbb{R}$ *.*

**Lemma 1.2.5.** *The topologies of* $\mathbb{R}_l$ *and* $\mathbb{R}_{\mathbb{K}}$ *is not comparable.*

*Proof.* Let $\mathbb{T}_l$ and $\mathbb{T}_{\mathbb{K}}$ be topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ respectively. Let $K = \{x|x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that $\mathbb{T}_l$ is not finer than $\mathbb{T}_{\mathbb{K}}$ . Let $U = \{x|-1 < x < 1\} - K, x = 0$. If there exist $B = \{x|a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus $B$ is not a subset of $U$.

Then we proof that $\mathbb{T}_{\mathbb{K}}$ is not finer than $\mathbb{T}_l$. Let $U' = \{x|a' \leq x < b'\}$. If there exist $B' = \{x|a'' < x < b''\} or \{x|a'' < x < b''\} - K$ such that $a' \in B \subseteq U$. Thus $a'' < a < b''$. Thus there exist $c$ that $a'' < x < a, x \in B, x \notin U'$. Thus $B' \nsubseteq U'$.

Thus the topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is not comparable. □

**Definition 1.2.6** (subbasis)**.** *A* ***subbasis*** $\mathbb{S}$ *for a topology on* $\mathbb{X}$ *is a collection of subsets of* $\mathbb{X}$ *whose union equals* $\mathbb{X}$ *. The* ***topology generated by the subbasis*** $\mathbb{S}$ *is defined to be the collection* $\mathbb{T}$ [7] *of all unions of finite intersections of elements of* $\mathbb{S}$ *.*

### 1.2.1 Exercise

1. Show that if $\mathbb{A}$ is a basis for a topology on $\mathbb{X}$ , then the topology generated by $\mathbb{A}$ equals the intersection of all topologies on $\mathbb{X}$ that contain $\mathbb{A}$ . Prove the same if $\mathbb{A}$ is a subbasis.

   *Proof.* As a subbasis is also a basis, we will directly prove the case of subbasis here.

[5]Whenever we consider $\mathbb{R}$ , we shall suppose it is given this topology unless we specifically state otherwise.
[6]We omit the proof of this lemma as it is obvious.
[7]It is obvious that $\mathbb{T}$ is a topology, we just omit the proof here.

Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain $\mathbb{A}$ . Let $\mathbb{T}$ be the topology that $\mathbb{A}$ generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.[8]

First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

Thus, $\mathbb{T} = \mathbb{T}'$ □

## 1.3 The Order Topology

**Definition 1.3.1** (interval)**.** *Let $\mathbb{X}$ is a set having a simple order relation $<$. Given elements $a$ and $b$ of $\mathbb{X}$ such that $a < b$, there are four subsets of $\mathbb{X}$ that are called **intervals** determined by $a$ and $b$:*

- $(a, b) = \{x | a < x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $[a, b] = \{x | a \leq x \leq b\}$

*$(a, b)$ is called an **open interval** on $\mathbb{X}$ . $[a, b]$ is called an **closed interval** on $\mathbb{X}$ . $(a, b]$ and $[a, b)$ is called **half-open intervals**.*

**Definition 1.3.2** (order topology)**.** [9] *Let $\mathbb{X}$ be a set with a simple order relation; assume $\mathbb{X}$ has more than one element. Let $\mathbb{B}$ be the collection of all sets of the following types:*

- *All open intervals $(a, b)$ in $\mathbb{X}$ .*
- *All intervals of the form $[a_0, b)$, where $a_0$ is the smallest element(if exist) of $\mathbb{X}$ .*
- *All intervals of the form $(a, b_0]$, where $b_0$ is the largest element(if exist) of $\mathbb{X}$ .*

*The collection $\mathbb{B}$ formed a basis for a topology on $\mathbb{X}$ , which is called the order topology.*

**Definition 1.3.3** (ray)**.** [10][11] *If $\mathbb{X}$ is an ordered set, and $a$ is an element of $\mathbb{X}$ , there are four subsets of $\mathbb{X}$ that are called **rays** determined by $a$:*

- $(a, +\infty) = \{x | x > a\}$
- $(-\infty, a) = \{x | x < a\}$

[8] It is obvious that $\mathbb{T}'$ is also a topology, we just omit the proof here.

[9] The standard topology on $\mathbb{R}$ is an order topology derived from the usual order on $\mathbb{R}$ .

[10] open rays are always open sets in the order topology

[11] the open rays also formed a subbasis of the order topology

- $[a, +\infty) = \{x | x \geq a\}$
- $(-\infty, a] = \{x | x \leq a\}$

*$(a, +\infty)$ and $(-\infty, a)$ are called **open rays**. $[a, +\infty)$ and $(-\infty, a]$ are called **closed rays**.*

## 1.4 The Product Topology

**Definition 1.4.1** (product topology)**.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces. The **product topology** on $\mathbb{X} \times \mathbb{Y}$ having a basis $\mathbb{B}$ containing all sets of the form $U \times V$, where $U$ and $V$ is open sets of $\mathbb{X}$ and $\mathbb{Y}$ respectively.*

**Theorem 1.4.1.** [12]*If $\mathbb{B}$ and $\mathbb{C}$ is basis for the topology of $\mathbb{X}$ and $\mathbb{Y}$ respectively, then the collection*

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} and C \in \mathbb{C}\}$$

*is a basis for the topology of $\mathbb{X} \times \mathbb{Y}$*

**Definition 1.4.2** (projection)**.** *Let $\pi_1 : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ be defined by the equation:*

$$\pi_1(x, y) = x$$

*Let $\pi_2 : \mathbb{X} \times \mathbb{Y} \to \mathbb{Y}$ be defined by the equation:*

$$\pi_1(x, y) = y$$

*The maps $\pi_1$ and $\pi_2$ are called the **projections** of $\mathbb{X} \times \mathbb{Y}$ onto its first and second factors, respectively.*

**Theorem 1.4.2.** [13]*The collection*

$$\mathbb{S} = \{\pi_1^{-1}(U) | U open in \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V open in \mathbb{Y}\}$$

*is a subbasis for the product topology on $\mathbb{X} \times \mathbb{Y}$.*

**Definition 1.4.3** (box topology)**.** *Let,*

$$\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n or \mathbb{X}_1 \times \mathbb{X}_2 \times \ldots$$

*In the first case, all the sets of the form $U_1 \times \cdots \times U_n$ where $U_i$ is a open set of $\mathbb{X}_i$ form a basis.*

*In the second case, all the sets of the form $U_1 \times U_2 \times \ldots$ where $U_i$ is a open set of $\mathbb{X}_i$ also form a basis.*

*Topology defined in this way was called a **box topology**.*

[12]We omit the proof of this theorem as it is obvious.

[13]We omit the proof of this theorem as it is obvious.

**Definition 1.4.4** (product topology)**.** [14] *Let,*

$$\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n or \mathbb{X}_1 \times \mathbb{X}_2 \times \dots$$

*Let* $\pi_i$ *be the projection function*[15] *that*

$$\pi_i : \mathbb{X} \to \mathbb{X}_i$$

*And if* $x \in \mathbb{X}$

$$\pi_i(x) = x_i$$

*All the set of the form* $\pi_i^{-1}(U_i)$ *where* $i$ *is arbitrary and* $U_i$ *is an open set of* $\mathbb{X}_i$ *, form a subbasis of* $\mathbb{X}$ *. The topology generated by this subbasis is called* ***product topology****. And* $\mathbb{X}$ *is called a* ***product space****.*

**Definition 1.4.5** (J-tuple)**.** *Let* $J$ *be an index set*[16]*. Give a set* $\mathbb{X}$ *, a* ***J-tuple*** *is defined as a function* $x : J \to \mathbb{X}$ *. If* $\alpha$ *is an element of* $J$ *,* $x(\alpha)$ *is often denoted by* $x_\alpha$ *and is called the* $\alpha$*th* ***coordinate*** *of* $x$ *. And the function* $x$ *itself is often denoted by the symbol*

$$(x_\alpha)_{\alpha \in J}$$

*The set of all J-tuples of elements of* $\mathbb{X}$ *is often denoted by* $\mathbb{X}^J$ *.*

**Definition 1.4.6** (cartesian product)**.** *Let* $\{A_\alpha\}_{\alpha \in J}$ *be an indexed family of sets; let* $\mathbb{X} = \bigcup_{\alpha \in J} A_\alpha$ *. The* ***cartesian product*** *of this indexed family is denoted by*

$$\prod_{\alpha \in J} A_\alpha$$

*And is defined to be the set of all J-tuples* $(x_\alpha)_{\alpha \in J}$ *of elements of* $\mathbb{X}$ *such that* $x_\alpha \in A_\alpha$ *for each* $\alpha \in J$ *. That is, it is the set of all functions*

$$x : J \to \bigcup_{\alpha \in J} A_\alpha$$

*such that* $x(\alpha) \in A_\alpha$ *for each* $\alpha \in J$ *.*

**Theorem 1.4.3** (Comparison of the box and product topologies)**.** [17] *The box topology on* $\prod \mathbb{X}_\alpha$ *has a basis all sets of the form* $\prod U_\alpha$ *where* $U_\alpha$ *is open in* $X_\alpha$ *for each* $\alpha$ *. The product topology on* $\prod \mathbb{X}_\alpha$ *has a basis all sets of the form* $\prod U_\alpha$ *where* $U_\alpha$ *is open in* $X_\alpha$ *for each* $\alpha$ *and* $U_\alpha$ *equals* $\mathbb{X}_\alpha$ *except for finitely many values of* $\alpha$ *.*

[14] In the finite case, the product topology and box topology are the same, however they differ when $\mathbb{X}$ is a infinite cartesian product.

[15] This is also called a ***projection mapping*** in a cartesian product.

[16] A index set was the set $\{1, \dots, n\}$ or the set $\mathbb{Z}_+$ .

[17] It is assumed that it is given product topology when considering $\prod X_\alpha$ unless it state specifically.

**Theorem 1.4.4.** [18]*Suppose the topology on each space* $\mathbb{X}_\alpha$ *is given by a basis* $\mathbb{X}_\alpha$ *. The collection of all sets of the form*

$$\prod_{\alpha\in J} B_\alpha$$

*where* $B_\alpha \in \mathbb{B}_\alpha$ *form a basis for the box topology on* $\prod_{\alpha\in J}\mathbb{X}_\alpha$ *.*

*The collection of all sets of the same form, where* $B_\alpha \in \mathbb{B}_\alpha$ *for finitely many indices* $\alpha$ *and* $B_\alpha = \mathbb{X}_\alpha$ *for all the remaining indices, will form a basis for the product topology* $\prod_{\alpha\in J}\mathbb{X}_\alpha$ *.*

**Theorem 1.4.5.** [19]*Let* $A_\alpha$ *be a subspace of* $\mathbb{X}_\alpha$ *, for each* $\alpha \in J$ *. Then* $\prod A_\alpha$ *is a subspace of* $\prod \mathbb{X}_\alpha$ *if both products are given the box topology, or if both products are given the product topology.*

**Theorem 1.4.6.** [20]*If each space* $\mathbb{X}_\alpha$ *is a Hausdorff space, then* $\prod \mathbb{X}_\alpha$ *is a Hausdorff space in both the box and product topologies.*

**Theorem 1.4.7.** *Let* $\{\mathbb{X}_\alpha\}$ *be an indexed family of spaces; let* $A_\alpha \subseteq \mathbb{X}_\alpha$ *for each* $\alpha$ *. If* $\prod \mathbb{X}_\alpha$ *is given either the product or the box topology, then*

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$$

*Proof.* Let $\pi_\alpha$ represent the projection mapping.

Let $x$ be an element of $\prod \mathbb{X}_\alpha$ . Let $V$ be an open set in $\prod \mathbb{X}_\alpha$ that containing $x$ .

If $x \in \prod \overline{A_\alpha}$ , then $\pi_\alpha(V)$ is a open set in $\mathbb{X}_\alpha$ that containing $x_\alpha$ . Thus $\pi_\alpha(V)$ intersect with $A_\alpha$ . Thus $V$ intersect with $\prod A_\alpha$ . Thus $x \in \overline{\prod A_\alpha}$ .

If $x \in \overline{\prod A_\alpha}$ . Let $U_\alpha$ be an open set of $A_\alpha$ that contain $x_\alpha$ . Let $V = \prod U_\beta$ such that $U_\beta = \begin{cases} \mathbb{X}_\beta, & \beta \neq \alpha \\ U_\alpha, & \beta = \alpha \end{cases}$ . It is obvious that $V$ is an open set that contain $x$ . Thus $V$ intersect with $\prod A_\alpha$ . Thus $U_\alpha$ intersect with $A_\alpha$ . Thus $x \in \prod \overline{A_\alpha}$ . □

**Theorem 1.4.8.** *Let* $f : A \to \prod_{\alpha\in J}\mathbb{X}_\alpha$ *be given by the equation*

$$f(a) = (f_\alpha(a))_{\alpha\in J}$$

*where* $f_\alpha : A \to \mathbb{X}_\alpha$ *for each* $\alpha$ *. Let* $\prod \mathbb{X}_\alpha$ *have the product topology. Then the function* $f$ *is continuous if and only if each function* $f_\alpha$ *is continuous.*

[18]We omit the proof of this theorem as it is obvious.
[19]We omit the proof of this theorem as it is obvious.
[20]We omit the proof of this theorem as it is obvious.

*Proof.* Let $\pi_\alpha$ be the projection mapping

It is obvious that

$$f^{-1}(U) = \bigcap_{\alpha \in J} f_\alpha^{-1}(\pi_\alpha(U))$$

If $f_\alpha$ is continuous. Let $V$ be a closed set of $\prod_{\alpha \in J} \mathbb{X}_\alpha$ . Then $\pi_\alpha(V)$ is closed. Then $f^{-1}(V)$ is intersect of closed set. Thus $\pi_\alpha(V)$ is closed. So $f$ is continuous.

If $f$ is continuous. Let $U_\alpha$ be an open set of $\mathbb{X}_\alpha$ . Let $U_\beta = \mathbb{X}_\beta$ if $\beta \neq \alpha$ . Let $V = \prod_{\beta \in J} U_\beta$ . It is obvious that $V$ is an open set of $\prod \mathbb{X}_\alpha$ . And

$$\begin{aligned} f^{-1}V &= \bigcap_{\alpha \in J} f_\alpha^{-1}(\pi_\alpha(U)) \\ &= f_\alpha^{-1}(U_\alpha) \end{aligned}$$

which is an open set in $A$ . Thus $f_\alpha$ is continuous. □

## 1.5 The Subspace Topology

**Definition 1.5.1** (subspace topology)**.** *Let $\mathbb{X}$ be a topological space with topology $\mathbb{T}$ . If $Y$ is a subset of $\mathbb{X}$ , the collection $\mathbb{T}_Y = \{Y \cap U | U \in \mathbb{T}\}$ is a topology on $Y$ , called the* ***subspace topology****.*

*$Y$ is also called a* ***subspace*** *of $\mathbb{X}$*

**Lemma 1.5.1.** [21]*If $\mathbb{B}$ is basis for the topology of $\mathbb{X}$ , $Y$ is a subset of $\mathbb{X}$ then the collection*

$$\mathbb{B}_Y = \{B \cap Y | B \in \mathbb{B}\}$$

*is a basis for the subspace topology on $Y$*

**Lemma 1.5.2.** [22]*Let $Y$ be a subspace of $\mathbb{X}$ . If $U$ is open in $Y$ and $Y$ is open in $\mathbb{X}$ , then $U$ is open in $\mathbb{X}$ .*

**Theorem 1.5.1.** [23] *If $A$ is a subspace of $\mathbb{X}$ and $B$ is a subspace of $\mathbb{Y}$ , then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$*

*Proof.* Let $\mathbb{B}_\mathbb{X}$ and $\mathbb{B}_\mathbb{Y}$ and $\mathbb{B}_{\mathbb{X}\mathbb{Y}}$ be basis of topology of $\mathbb{X}$ and $\mathbb{Y}$ and $\mathbb{X} \times \mathbb{Y}$ respectively. Let $\mathbb{B}'_\mathbb{X}$ and $\mathbb{B}'_\mathbb{Y}$ and $\mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ be basis of topology of $A$ and $A$ and $A \times B$ respectively. We will show that $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} = \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$. Thus, the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$ .

[21]We omit the proof of this lemma as it is obvious.

[22]We omit the proof of this lemma as it is obvious.

[23]If $\mathbb{X}$ is an ordered set in the order topology, and $\mathbb{Y}$ is a subset of $\mathbb{X}$ . The order relation, when restricted to $\mathbb{Y}$ , makes $\mathbb{Y}$ into and ordered set. However, *the resulting order topology on $\mathbb{Y}$ need not be the same as the topology that $\mathbb{Y}$ inherits as a subspace of $\mathbb{X}$ .*

First, every element in $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ can be represented by $B_A \cap A \times B_B \cap B = B_A \times B_B \cap A \times B \in \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ where $B_A \in \mathbb{B}'_{\mathbb{X}}, B_B \in \mathbb{B}'_{\mathbb{Y}}$. Thus $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}} \subseteq \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$.

Next, we show that $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$ . For any open set $U$ in $\mathbb{X} \times \mathbb{Y}$ , and $\forall x \in U \cap A \times B, \exists B_{\mathbb{X}} \times B_{\mathbb{Y}} \in \mathbb{B}_{\mathbb{X}\mathbb{Y}}, x \in B_{\mathbb{X}} \times B_{\mathbb{Y}} \subseteq \mathbb{X} \times \mathbb{Y}$. Thus $x \in B_{\mathbb{X}} \times B_{\mathbb{Y}} \cap A \times B \subseteq A \times B, B_{\mathbb{X}} \times B_{\mathbb{Y}} \cap A \times B \in \mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$. Thus $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$ .gi □

**Definition 1.5.2** (ordered square). *Let $I = [0, 1]$. The set $I \times I$ in the dictionary order* [24] *topology will be called* ***ordered square****, and denoted by $I_o^2$*

**Definition 1.5.3** (convex). *Given an ordered set $\mathbb{X}$ , let us say that a subset $\mathbb{Y}$ of $\mathbb{X}$ is* ***convex*** *in $\mathbb{X}$ if for each pair of points $a < b$ of $\mathbb{Y}$ , the entire interval $(a, b)$ of points of $\mathbb{X}$ lies in $\mathbb{Y}$*

**Theorem 1.5.2.** [25] *Let $\mathbb{X}$ be an ordered set in the order topology. Let $\mathbb{Y}$ be a subset of $\mathbb{X}$ that is convex in $\mathbb{X}$ . Then the order topology on $\mathbb{Y}$ is the same as the topology $\mathbb{Y}$ inherits as a subspace of $\mathbb{X}$ .*

*Proof.* Consider the ray $(a, +\infty)$ in $\mathbb{X}$ . If $a \in \mathbb{Y}$, then

$$(a, +\infty) \cap \mathbb{Y} = \{x | x \in \mathbb{Y} and x > a\}$$

This is an open ray of the ordered set of $\mathbb{Y}$ . if $a \notin Y$, then $a$ is either a lower bound on $\mathbb{Y}$ or an upper bound on $\mathbb{Y}$ , since $\mathbb{Y}$ is convex. In the former case, the set $(a, +\infty) \cap \mathbb{Y}$ equals all of $\mathbb{Y}$ , in the latter case, it is empty.

A similar remark shows that the intersection of the rat $(-\infty, a)$ with $\mathbb{Y}$ is either an open ray of $\mathbb{Y}$ , or $\mathbb{Y}$ itself, or empty. Since the sets $(a, +\infty)\mathbb{Y}$ and $(-\infty, a) \cap \mathbb{Y}$ form a subbasis for the subspace topology on $\mathbb{Y}$ , and since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of $\mathbb{Y}$ equals the intersection of an open ray of $\mathbb{X}$ with $\mathbb{Y}$ , so it is open in the subspace topology on $\mathbb{Y}$ . Since the open rays of $\mathbb{Y}$ are a subbasis for the order topology on $\mathbb{Y}$ , this topology is contained in the subspace topology. □

[24] the dictionary means for $X_1, X_2 \in \mathbb{Y} = \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \ldots$ which:

$$\begin{aligned} X_1 &= (x_1, x_2, x_3 \ldots) \\ X_2 &= (x'_1, x'_2, x'_3 \ldots) \end{aligned}$$

$X_1 > X_2$ only when

$$\begin{aligned} &\exists k \in \mathbb{Z}_+, \forall i \in \mathbb{Z}_+, 0 < i < k \\ &x_i = x'_i \\ &x_k > x'_k \end{aligned}$$

[25] Given $\mathbb{X}$ is an ordered set in the order topology and $\mathbb{Y}$ is a subset of $\mathbb{X}$ , we shall assume that $\mathbb{Y}$ *is given the subspace topology* unless we specifically state otherwise.

**Exercise**

1. A map $f : \mathbb{X} \to \mathbb{Y}$ is said to be a ***open map*** if for every open set $U \subseteq \mathbb{X}$, the set $f(U)$ is open in $\mathbb{Y}$ . Show that $\pi : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ is open map.

   *Proof.* An open set in $\mathbb{X} \times \mathbb{Y}$ can be represented by
   $$\cup(U_i \times U_i')$$
   where $U_i, U_i'$ are open sets in $\mathbb{X}$ , $\mathbb{Y}$ , respectively.
   Also,
   $$\cup(U_i \times U_i') = \cup(U_i) \times \cup(U_i')$$
   Thus,
   $$\pi(\cup(U_i \times U_i')) = \cup(U_i)$$
   Thus, $\pi(U)$ is open in $\mathbb{X}$ . □

2. Let $\mathbb{X}$ and $\mathbb{X}'$ denote a single set in the topologies $\mathbb{T}$ and $\mathbb{T}'$ , respectively; let $\mathbb{Y}$ and $\mathbb{Y}'$ denote a single set in the topologies $\mathbb{U}$ and $\mathbb{U}'$ , respectively. [26] Assume these sets are nonempty.
   (a) Show that if $\mathbb{T}' \supseteq \mathbb{T}$ and $\mathbb{U}' \supseteq \mathbb{U}$ , then the product topologies $\mathbb{X}' \times \mathbb{Y}'$ is finer than the product topology on $\mathbb{X} \times \mathbb{Y}$ .
   (b) Does the converse of the previous statement hold?

3. Show that the countable collection[27]
   $$\{(a, b) \times (c, d)|a < b, c < d, a \in \mathbb{Q}, b \in \mathbb{Q}, c \in \mathbb{Q}, d \in \mathbb{Q}\}$$
   is a basis for $\mathbb{R}^2$

   *Proof.* This is obvious if you prove that $(a, b) \times (c, d)$ is a rectangle in the $\mathbb{R}^2$ plane. □

4. Let $\mathbb{X}$ be an ordered set. If $\mathbb{Y}$ is a proper subset of $\mathbb{X}$ that is convex in $\mathbb{X}$ prove that $\mathbb{Y}$ may not be an interval or a ray in $\mathbb{X}$ .

   *Proof.* Let $\mathbb{X} = \mathbb{R}^2$ with dictionary order. Then $Y = \{(x, y)| -1 \leq x \leq 1\}$ is convex in $\mathbb{X}$ , however it is not an interval or a ray. □

   There is a false prove given by myself.

[26] what does $\mathbb{X}$ , $\mathbb{X}'$ , $\mathbb{Y}$ , $\mathbb{Y}'$ really mean here?? I do not know, so I just put the exercise here without a proof.
[27] The prove of this set is countable is typically similar to Cantor's enumeration of a countable collection of countable sets.

*Proof.* Let $\mathbb{S}$ be a set that contain all intervals and rays of $\mathbb{Y}$ . We define a partial order on $\mathbb{S}$ by inclusion. So if there is a chain in $\mathbb{S}$ :

$$S_1 \subseteq S_2 \subseteq S_3 \dots$$

Let

$$S = S_1 \cup S_2 \cup S_3 \cup \dots$$

Thus, $S$ is an upper bound of the chain.

Thus, by Zorn's Lemma, there is a maximal element of $\mathbb{S}$ , say $U$ , then we prove that $U = \mathbb{Y}$ .

If $U \neq \mathbb{Y}$ , then $\exists x, x \in \mathbb{Y} - U$ .

If $U$ is a ray say $(a, +\infty)$ . Then $x < a$ , thus $U \subseteq (x, +\infty) \subseteq \mathbb{B}$ , then there is contradiction with the maximal element.

If $U$ is an interval, the circumstance is similar with the proof of $U$ is a ray.

Thus $\mathbb{Y}$ is a ray or an interval. □

However, there is issue with this proof, the set $S$ does exists. However, it may not be an interval or ray, so it may not be contained in $\mathbb{S}$

## 1.6 Closed Sets and Limit Points

**Definition 1.6.1** (closed)**.** [28] *A subset $A$ of a topological space is said to be closed if the set $\mathbb{X} - A$ is open.*

**Theorem 1.6.1.** [29]*Let $\mathbb{X}$ be a topological space. Then the following conditions hold*

1. *$\emptyset$ and $\mathbb{X}$ are closed.*
2. *Arbitrary intersections of closed sets are closed*
3. *Finite unions of closed sets are closed*

**Definition 1.6.2** (closed in)**.** *Let $\mathbb{X}$ be a topological space; let $\mathbb{Y}$ be a subspace of $\mathbb{X}$ . We say that a set $A$ is **closed in** $\mathbb{Y}$ if $A$ is a subset of $\mathbb{Y}$ and $A$ is closed in the subspace topology of $\mathbb{Y}$*

**Theorem 1.6.2.** *Let $\mathbb{Y}$ be a subspace of $\mathbb{X}$ . Then a set $A$ is closed in $\mathbb{Y}$ if and only if it equals the intersection of a closed set of $\mathbb{X}$ with $\mathbb{Y}$*

[28]A set can be open, or closed, or both, or neither
[29]We omit the proof of this theorem as it is obvious.

*Proof.* First we proof that if $A$ is closed in $\mathbb{Y}$, then $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. As the origin topology form a surjective map to its subspace topology, there exists a $B$ closed in $\mathbb{X}$ that $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $B \cap \mathbb{Y} = A$

Conversely, if $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. Then, $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $\mathbb{X} - B$ is open in $\mathbb{Y}$, $\mathbb{Y} - A$ is open in $\mathbb{Y}$. Then $A$ is closed in $\mathbb{Y}$ □

**Theorem 1.6.3.** [30] *Let $\mathbb{Y}$ be a subspace of $\mathbb{X}$. If $A$ is closed in $\mathbb{Y}$ and $\mathbb{Y}$ is closed in $\mathbb{X}$, then $A$ is closed in $\mathbb{X}$.*

**Definition 1.6.3** (interior)**.** *Given a subset $A$ of a topological space $\mathbb{X}$, the **interior** of $A$ is defined as the union of all open sets contained in $A$. Denoted by $Int(A)$.*

**Definition 1.6.4** (closure)**.** *Given a subset $A$ of a topological space $\mathbb{X}$, the **closure** of $A$ is defined as the intersection of all closed sets containing $A$. Denoted by $Cl(A)$ or $\overline{A}$*

**Theorem 1.6.4.** [31][32] *Let $\mathbb{Y}$ be a subspace of a topological space $\mathbb{X}$; let $A$ be a subset of $\mathbb{X}$. Let $\overline{A}$ denote the closure of $A$ in $\mathbb{X}$. Then the closure of $A$ in $\mathbb{Y}$ equals $\overline{A} \cap \mathbb{Y}$*

**Definition 1.6.5** (intersect)**.** *We say that a set $A$ **intersects** $B$ if $A \cap B$ is not empty.*

**Theorem 1.6.5.** *Let $A$ be a subset of the topological space $\mathbb{X}$*

1. *The $x \in \overline{A}$ if and only if every open set $U$ containing $x$ intersect $A$.*
2. *Supposing the topology of $\mathbb{X}$ is given by a basis, then $x \in \overline{A}$ if and only if every basis element $B$ containing $x$ intersects $A$*

*Proof.* There are only two types of closed set $U$ in $\mathbb{X}$:

1. $U \supseteq \overline{A}$
2. $U \cap A \neq A$

Thus, there are only two types of open set $U$ in $\mathbb{X}$ respectively.

1. $U$ does not intersects $A$.
2. $U \cap \overline{A} \neq \emptyset$

1. If $x \in \overline{A}$, then every open set containing $x$ is the open set of second type, thus every open set containing $x$ intersects $A$

   If every open set containing $x$ intersect $\mathbb{A}$, suppose $x \notin \overline{A}$. Then $\mathbb{X} - \overline{A}$ is a open set containing x, however, it does not intersects $A$. Thus, $x \in \overline{A}$.

[30] As the proof is similar to the case in the open set, so we omit the proof here.
[31] We omit the proof of this theorem as it is obvious.
[32] As the closure of $A$ in $\mathbb{X}$ and the closure $A$ in $\mathbb{Y}$ will sometimes be different. We always use $\overline{A}$ to denote the closure of $A$ in $\mathbb{X}$

2. If $x \in \overline{A}$ , as every basis element of $\mathbb{X}$ is a open set, thus every basis element containing $x$ intersects $\mathbb{A}$

   If every open set containing $x$ intersect $\mathbb{A}$ , suppose $x \notin \overline{A}$ .

   As every open sets can be represented by union of basis. Let

   $$\mathbb{X} - \overline{A} = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_1' \cup B_2' \cup B_3' \cup \ldots$$

   where $B$ are bases containing $x$ , and $B'$ are bases that does not contain $x$ .

   Thus,

   $$x \in B_1 \cup B_2 \cup B_3 \cup \cdots \subseteq \mathbb{X} - \overline{A}$$

   Then $B_1 \cup B_2 \cup B_3 \cup \ldots$ that is a open set can be generated by all the bases containing $x$ , however, that does not intersects $A$ . So, $x \in \overline{A}$ .

□

**Definition 1.6.6** (neighbourhood). [33] *If we say $U$ is a neighbourhood of $x$ in $\mathbb{X}$ , then $U$ is an open set in $\mathbb{X}$ containing $x$*

**Definition 1.6.7** (limit point, point of accumulation, cluster point). [34] *If $A$ is a subset of topological space $\mathbb{X}$ . We say that $x$ is a limit point of $A$ if and only if every open sets containing $x$ intersects A with some points other than $x$ .*

*This condition is also equivalent to the condition that if $x$ is a limit point of $A$ if and only if $x \in \overline{A - \{x\}}$*

**Theorem 1.6.6.** [35] *Let $A$ be a subset of topological space $\mathbb{X}$ ; let $A'$ be the set of all limit points of $A$ . Then*

$$\overline{A} = A \cup A'$$

**Corollary 1.6.1.** [36] *A subset of a topological space is closed if and only if it contains all its limit point.*

**Definition 1.6.8** (converge). [37] *We say that a sequence of $x_1, x_2, x_3 \ldots$ **converge** to $x$ . When for every neighbourhood $U$ of $x$ , there exists a positive integer $N$ , such that for all $n > N$ , $x_n \in U$ .*

**Definition 1.6.9** (Hausdorff space). *A topological space is called a **Hausdorff space**, if for every distinct $x_1$ , $x_2$ in $\mathbb{X}$ , there exists disjoint neighbourhood of $U_1$ , $U_2$ of $x_1$ , $x_2$ in $\mathbb{X}$ .*

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[33] Some other mathematicians use neighbourhood to say that $U$ merely contains an open set containing $x$ . The book does not give a formal definition for the word merely, and I am not sure either.

[34] Note that, $x$ may belong to $A$ or not, this does not matter.

[35] We omit the proof of this theorem as it is obvious.

[36] We omit the proof of this corollary as it is obvious.

[37] In real line, a sequence can not converge to multiple points, but for an arbitrary topological space, this is possible.

**Theorem 1.6.7.** [38][39] *Every finite point set in a Hausdorff space $\mathbb{X}$ is closed.*

*Proof.* Let $A$ be a finite point set in a Hausdorff space $\mathbb{X}$ .

Suppose $A$ only have one element. Then for every $x \in \mathbb{X} - A$ , there exists a neighbourhood of $x$ that does not intersect with $A$ . So $A$ is closed.

Suppose $A$ is a closed finite point set. We take $x_0 \in \mathbb{X} - A$ . As finite union of closed set is closed, $A \cup \{x_0\}$ is closed.

Then, from induction, all finite point set in a Hausdorff space is closed. □

**Theorem 1.6.8.** *If $\mathbb{X}$ is a Hausdorff space, then a sequence of points in $\mathbb{X}$ converges to at most one point.*

*Proof.* Suppose that the following sequence

$$x_1, x_2, x_3 \dots$$

Converge to more than one points say

$$y_1, y_2, y_3 \cdots$$

Then there exists

$$n_1, n_2, n_3 \dots, U_1, U_2, U_3 \dots$$

Such that for $n > n_i$

$$x_n \in U_i, y_i \in U_i$$

If we take disjoint $U_1, U_2$ which is possible as this is a Hausdorff space.

Then the previews condition does not stand. So, every sequence of points in a Hausdorff space can only converge to at most one point. □

**Definition 1.6.10** (limit)**.** *If a sequence $x_n$ of points in Hausdorff space converge to the point $x$ , we denote this by $x_n \to x$ and we say the **limit** of $x_n$ is $x$ .*

**Definition 1.6.11** ( $T_1$ axiom)**.** *The condition that all finite point set of a topological space is closed is called $T_1$ **axiom**.*

**Theorem 1.6.9.** *Let $\mathbb{X}$ be a space satisfying the $T_1$ axiom; let $A$ be a subset of $\mathbb{X}$ . Then the point $x$ is a limit point of $A$ if and only if every neighbourhood of $x$ contains infinitely many points of $A$ .*

*Proof.* If every neighbourhood of $x$ contains infinitely many point of $A$ . Than every neighbourhood of $x$ intersect with $A$ with infinite element other than $x$ , then $x$ is a limit point of $A$ .

[38]This implies that a sequence in a Hausdorff space cannot converge to multiple points. The following theorem prove this.

[39]The condition every finite point set is closed is weaker than the Hausdorff space condition. For instance, the finite complement topology of $\mathbb{R}$ met the condition of finite point set. However it is not a Hausdorff space.

If $x$ is a limit point of $A$ . Suppose that there exists a open set $U$ containing $x$ and intersect with $A$ for finite many points. Let

$$U' = U \cap (A - x)$$

Then, $x \notin U'$ . Let

$$U'' = U - U'$$

Then $U''$ is open as $U'$ is a finite point set and

$$U'' = U - U' = U \cap (\mathbb{X} - U')$$

Also, $x \in U''$ . Thus, $U''$ is a open set containing $x$ that only intersect $A$ with $x$ or do not intersect $A$ . This is a contradiction of x is a limit point. Thus there does not exists a open set $U$ containing $x$ and intersect with $A$ for finite many points. □

**Theorem 1.6.10.** [40] *Every simply ordered set is a Hausdorff space in order topology.*

**Theorem 1.6.11.** [41] *The product of two Hausdorff space is a Hausdorff space.*

**Theorem 1.6.12.** [42] *A subspace of a Hausdorff space is a Hausdorff space.*

### 1.6.1 Exercise

1. Give an counter example why $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$ dose not hold.

   *Proof.* Consider the X be the K-topology on the real line.
   Let

$$\begin{aligned} A_n &= (\frac{1}{n+1}, \frac{1}{n}), n \in \mathbb{Z}_+ \\ A &= \cup A_n \end{aligned}$$

   Then

$$\begin{aligned} \overline{A_n} &= [\frac{1}{n+1}, \frac{1}{n}] \\ \cup \overline{A_n} &= (0, 1] \end{aligned}$$

   However, as every neighbourhood of $0$ intersect $\cup A_\alpha$ . $0 \in \overline{\cup A_\alpha}$ .
   Thus, $\overline{\cup A_\alpha} \neq \cup \overline{A_\alpha}$ □

[40] We omit the proof of this theorem as it is obvious.
[41] We omit the proof of this theorem as it is obvious.
[42] We omit the proof of this theorem as it is obvious.

2. Prove that
$$\overline{A-B} \supseteq \overline{A} - \overline{B}$$

   *Proof.* If $x \in \overline{A} - \overline{B}$ . Then
$$x \in \overline{A}, x \notin \overline{B}$$
   .

   Thus for open set $U$ containing $x$
$$\begin{array}{ll} \exists & U_1 \cap B = \emptyset \\ \forall & U \cap A \neq \emptyset \end{array}$$

   Suppose that $x \notin \overline{A-B}$ . Then
$$\exists U_0 \cap (A-B) = \emptyset$$

   Thus,
$$U_0 \cap A \subseteq B$$

   Thus,
$$\begin{aligned} U_1 \cap B &= \emptyset \\ U_1 \cap U_0 \cap A &= \emptyset \end{aligned}$$

   As $U_1 \cap U_0$ is an open set containing $x$ , so there is contradiction with $x \in \overline{A}$ . Thus $x \in \overline{A-B}$ . □

3. A ***diagonal*** is a subset $\Delta = \{x \times x | x \in \mathbb{X}\}$ of the product topology $\mathbb{X} \times \mathbb{X}$ where $\mathbb{X}$ is a topological space. Show that the diagonal is closed in $\mathbb{X} \times \mathbb{X}$ if and only if $\mathbb{X}$ is a Hausdorff space.

   *Proof.* If $\mathbb{X}$ is a Hausdorff space. For every element $x \times y$ of $\mathbb{X} \times \mathbb{X}$ that not in $\Delta$ . We take disjoint set $U_x, U_y$ where $x \in U_x, y \in U_y$ . Then $\mathbb{X} \times \mathbb{X} - \Delta = \cup_{x \neq y} U_x \times U_y$ . Where $\cup_{x \neq y} U_x \times U_y$ is an open set. Thus $\Delta$ is a closed set.

   Conversely, if $\Delta$ is a closed set, suppose that $\mathbb{X}$ is not a Hausdorff space. Then there exists distinct $x, y$ such that every neighbourhood of $x$ and $y$ intersect. Let $\mathbb{B}$ be a basis of topology of $\mathbb{X}$ . Then $x \times y \in \mathbb{X} \times \mathbb{X} - \Delta$ . However we cannot find $B_1, B_2 \in \mathbb{B}, x \times y \in B_1 \times B_2 \subset \mathbb{X} \times \mathbb{X} - \Delta$ . Then $\Delta$ is not a closed set. So there is a contradiction, then $\mathbb{X}$ must be a Hausdorff space. □

4. Prove that $T_1$ axiom is equivalent to the condition such that for every distinct pair $x, y$ of $\mathbb{X}$, there exists neighbourhood of $x$ does not contain $y$.

   *Proof.* First if $T_1$ axiom hold, then for every pair $x, y$, the neighbourhood $\mathbb{X} - \{y\}$ of $x$ does not contain $y$, so the second condition hold.

   Conversely, if the second condition hold. Suppose that we can find a finite points set say $\{x_1, x_2, x_3 \dots\}$, then there must exists $x \in \{x_1, x_2, x_3 \dots\}$ such that the set $\{x\}$ is not closed. Then $\overline{\{x\}} - \{x\} \neq \emptyset$. Let $y \in \overline{\{x\}} - \{x\}$, then every neighbourhood of y must contain $x$, this is a contradiction to the second condition, so the $T_1$ axiom must hold. $\square$

5. If $A \subseteq \mathbb{X}$, we define the ***boundary*** of $A$ by the equation

$$\mathrm{Bd}A = \overline{A} \cap \overline{\mathbb{X} - A}$$

   (a) Show that $\mathrm{Int}A$ and $\mathrm{Bd}A$ are disjoint and $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$.

   *Proof.* For every $x \in \mathrm{Bd}A$, every open set contain $x$ must intersect $A$ and $\mathbb{X} - A$ so, there is no open set $U$ contain $x$, $U \subseteq A$.
   For every $x' \in \mathrm{Int}A$, there exists $U' \subseteq A$, so $\mathrm{Bd}A$ and $\mathrm{Int}A$ are disjoint sets.
   For every $x \in \overline{A}$, $x \in \mathrm{Bd}A$ or $x \notin \mathrm{Bd}A$. We discuss the condition that $x \notin \mathrm{Bd}A$.
   Then $x \notin \overline{\mathbb{X} - A}$, then there exists a open set $U$ containing $x$, that does not intersect with $\mathbb{X} - A$. Thus $U \subseteq A$, thus $x \in \mathrm{Int}A$. So $\overline{A} \subseteq \mathrm{Int}A \cup \mathrm{Bd}A$.
   Then, $\mathrm{Bd}A \subseteq \overline{A}$, $\mathrm{Int}A \subseteq A \subseteq \overline{A}$. Thus, $\overline{A} \supseteq \mathrm{Int}A \cup \mathrm{Bd}A$
   So, $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$ $\square$

   (b) Show that $\mathrm{Bd}A = \emptyset$ if and only if $A$ is both open and closed.

   *Proof.* So, $\mathrm{Int}A = \overline{A}$, then $\mathrm{Bd}A = \emptyset$ follows directly from $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$. $\square$

   (c) Show that $U$ is open if and only if $\mathrm{Bd}U = \overline{U} - U$.

   *Proof.* Suppose U is open. Then $\overline{\mathbb{X} - U} = \mathbb{X} - U$. Then for every $x \in U$, $x \notin \mathbb{X} - U, x \notin \overline{\mathbb{X} - U}$. Thus $\overline{U} \cap \overline{\mathbb{X} - U} = \overline{U} - U$.
   Conversely, suppose $\mathrm{Bd}U = \overline{U} - U$. Then for every $x \in U$, $x \notin \mathrm{Bd}U$. Then as $\overline{U} = \mathrm{Int}U \cup \mathrm{Bd}U$, $x \in \mathrm{Int}U$. So $\mathrm{Int}U \supseteq U$. Thus $U = \mathrm{Int}U$. Thus, $U$ is open. $\square$

## 1.7 Continuous Function

**Definition 1.7.1** (continuous). [43] *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces. A function $f : \mathbb{X} \to \mathbb{Y}$ is said to be **continuous** if for each open subset $V$ of $\mathbb{Y}$, the set $f^{-1}(V)$ is an open subset of $\mathbb{X}$.*

**Theorem 1.7.1.** *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces; let $f : \mathbb{X} \to \mathbb{Y}$. Then the following are equivalent.*

1. *$f$ is continuous.*
2. *For every subset $A$ of $X$, one has $f(\overline{A}) \subseteq \overline{f(A)}$.*
3. *For every closed set $B$ of $\mathbb{Y}$, the set $f^{-1}(B)$ is closed in $\mathbb{X}$.*
4. *For each $x \in \mathbb{X}$ and each neighbourhood of $V$ of $f(x)$, there is a neighbourhood $U$ of $x$ such that $f(U) \subseteq V$.*

*Proof.*
$1 \Rightarrow 3$:
Let $A$ be a open set in $\mathbb{Y}$. $f^{-1}(\mathbb{Y} - A) = \mathbb{X} - f^{-1}(A)$.

$3 \Rightarrow 1$:
Let $A$ be a closed set in $\mathbb{Y}$. $f^{-1}(\mathbb{Y} - A) = \mathbb{X} - f^{-1}(A)$.

$1 \Rightarrow 2$:
For $x \in \overline{A}$, we take a open set $f(x) \in U \subseteq \mathbb{Y}$. Thus $x \in f^{-1}(U) \cap A \neq \emptyset$. Thus $U \cap f(A) \neq \emptyset$. So $f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$.

$2 \Rightarrow 3$:
Suppose $f$ is not continuous. Then there must exists $V$, such that $f^{-1}(V) = U$ is not closed. Thus $\overline{U} \supset B = f^{-1}(A)$. Thus $f\overline{B} \supset A$. However $f(\overline{B}) \subseteq \overline{f(B)} = A$. There is a contradiction. So $f$ must be continuous.

$1 \Rightarrow 4$:
For every neighbourhood $V$ of $f(x)$, $f^{-1}(V)$ is a neighbourhood of $x$ that $f(f^{-1}(V)) \subseteq V$.

$4 \Rightarrow 1$:
We take a open set $V$ of $\mathbb{Y}$. Let $S$ be the collection of all open set $U$ in $\mathbb{X}$ such that $f(U) \subseteq V$. The set cannot be empty unless $f^{-1}(V) = \emptyset$. Let $U_0$ denote the union of all the element in $S$. We prove that $U_0 = f^{-1}(V)$.
For all element $x \in U_0$, $f(x) \in V$. Thus $U_0 \subseteq f^{-1}(V)$.

[43] As the continuity of a function is different as the topological spaces are different. So if we want to emphasis this fact, we say that $f$ is continuous ***relative*** to specific topologies on $\mathbb{X}$ and $\mathbb{Y}$.

For all element $x \in f^{-1}(V)$ . There is a $U'$ such that $x \in U', f(U') \subseteq V$ . This follows from the condition 4. Thus $U' \in S$ . Thus $x \in U_0$ . Thus $U_0 \subseteq f^{-1}(V)$ . As $U_0$ is union of open set, $U_0$ is also open. Thus, $f^{-1}(V)$ is also open.

Thus $f$ is continuous. □

**Definition 1.7.2** (homeomorphism)**.** [44] *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be topological space; let* $f : \mathbb{X} \to \mathbb{Y}$ *be a bijection. If both the function* $f$ *and the inverse function*

$$f^{-1} : \mathbb{Y} \to \mathbb{X}$$

*are continuous, then f is called a* ***homeomorphism***

**Definition 1.7.3** (topological imbedding)**.** *Suppose that* $f : \mathbb{X} \to \mathbb{Y}$ *is an injective continuous map, where* $\mathbb{X}$ *and* $\mathbb{Y}$ *are topological spaces. Let* $\mathbb{Z}$ *be the image set* $f(\mathbb{X})$ *, considered as a subspace of* $\mathbb{Y}$ *; then the function* $f' : \mathbb{X} \to \mathbb{Z}$ *obtained by restricting the range of* $f$ *is bijective. If* $f'$ *happens to be a homeomorphism of* $\mathbb{X}$ *with* $\mathbb{Z}$ *, we say that the map* $f : \mathbb{X} \to \mathbb{Y}$ *is a* ***topological imbedding****, or simply an* ***imbedding****, of* $\mathbb{X}$ *in* $\mathbb{Y}$ *.*

**Theorem 1.7.2** (Rules for constructing continuous functions)**.** *Let* $\mathbb{X}$ *,* $\mathbb{Y}$ *, and* $\mathbb{Z}$ *be topological spaces.*

1. *(Constant function) If* $f : \mathbb{X} \to \mathbb{Y}$ *maps all of* $\mathbb{X}$ *into the single point* $y_0$ *of* $\mathbb{Y}$ *, then* $f$ *is continuous.*

2. *(Inclusion) If* $A$ *is a subspace of* $\mathbb{X}$ *, the inclusion function* $j : A \to \mathbb{X}$ *is continuous.*

3. *(Composites) If* $f : \mathbb{X} \to \mathbb{Y}$ *and* $g : \mathbb{Y} \to \mathbb{Z}$ *are continuous, then the map* $g \circ f : \mathbb{X} \to \mathbb{Z}$ *is continuous.*

4. *(Restricting the domain) If* $f : \mathbb{X} \to \mathbb{Y}$ *is continuous, and if* $A$ *is a subspace of* $\mathbb{X}$ *, then the restriction function* $f|A : A \to \mathbb{Y}$ *is continuous.*

5. *(Restricting or expanding the range) Let* $f : \mathbb{X} \to \mathbb{Y}$ *is continuous. Let* $\mathbb{Z}$ *be a subspace of* $\mathbb{Y}$ *containing the image* $f(\mathbb{X})$ *, the function* $h : \mathbb{X} \to \mathbb{Z}$ *obtained by restricting the range of* $f$ *is continuous. If* $\mathbb{Z}$ *is a space having* $\mathbb{Y}$ *as a subspace, then the function* $h : \mathbb{X} \to \mathbb{Y}$ *obtained by expanding the range of* $f$ *is continuous.*

6. *(Local formulation of continuity) The map* $f : \mathbb{X} \to \mathbb{Y}$ *is continuous if* $\mathbb{X}$ *can be written as the union of open sets* $U_\alpha$ *such set* $f|U_\alpha$ *is continuous for each* $\alpha$

*Proof.*

[44] A equivalent way to define homeomorphism, is that for any open subset $U$ of $\mathbb{X}$ , $f(U)$ is open if and only if $U$ is open.

1. $f^{-1}(U)$ of any open set $U$ is $\mathbb{X}$ , thus $f$ is continuous.
2. For every open subset $U$ of $\mathbb{X}$ , $j^{-1}(U) = U \cap A$ is continuous in $A$ . Thus $j$ is a continuous function.
3. For every open subset $U$ of $\mathbb{Z}$ , $f^{-1}(U)$ is open in $\mathbb{Y}$ , and $g^{-1}(f^{-1}(U))$ is open in $\mathbb{X}$ . Thus, $g \circ f$ is continuous
4. For every open subset $U$ of $\mathbb{Y}$ , $f^{-1}(U)$ is open in $\mathbb{X}$ , thus $f^{-1}(U) \cap A$ is open in $A$ . Thus the function $f|A$ is continuous.
5. If $\mathbb{Z}$ is a subspace of $\mathbb{Y}$ , then every open subset of $\mathbb{Z}$ can be represented as $U \cap \mathbb{Z}$ , where $U$ is a open subset of $\mathbb{Y}$ . Thus $h^{-1}(U \cap \mathbb{Z}) = g^{-1}(\mathbb{Z}) \cap g^{-1}(U) = \mathbb{X} \cap g^{-1}(U)$ which is a open subset of $X$ , thus $h$ is continuous.

   If $\mathbb{Y}$ is a subspace of $\mathbb{Z}$ . Then we take a open subset $U$ of $\mathbb{Z}$ . $h^{-1}(U) = g^(-1)(U \cap \mathbb{Y})$ which is open in $\mathbb{X}$ , thus $h$ is continuous.
6. if $f|U_\alpha$ is continuous for each $\alpha$ . For every open subset $U$ of $\mathbb{Y}$ .

$$U = \cup_\alpha (U_\alpha \cap U)$$

   where $U_\alpha \cap U$ is open both in $U_\alpha$ and in $\mathbb{Y}$ .

   Thus,

$$\begin{aligned} f^{-1}(U) &= f^{-1}(\cup_\alpha (U_\alpha \cap U)) \\ &= \cup_\alpha ((f|U_\alpha)^{-1}(U_\alpha \cap U)) \end{aligned}$$

   and each $(f|U_\alpha)^{-1}(U_\alpha \cap U)$ is open, thus $f^{-1}(U)$ is open.

□

**Theorem 1.7.3** (The pasting lemma)**.** [45] *Let $\mathbb{X} = A \cup B$ , where $A, B$ are closed in $\mathbb{X}$ . Let $f : A \to \mathbb{Y}$ and $g : B \to \mathbb{Y}$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$ , then $f, g$ combine to give a continuous function $h : \mathbb{X} \to \mathbb{Y}$ , defined by setting $h(x) = f(x), x \in A$ and $h(x) = g(x), x \in B$ .*

**Theorem 1.7.4** (Maps into products)**.** [46] *Let $f : A \to \mathbb{X} \times \mathbb{Y}$ be given by the equation*

$$f(a) = (f_1(a), f_2(a))$$

*Then, the function $f$ is continuous if and only if the functions*

$$f_1 : A \to \mathbb{X}, f_2 : A \to \mathbb{Y}$$

*are continuous.*

[45] The proof of this theorem is similar to the "Local formulation of continuity" condition of "Rules for constructing continuous functions", so we omit the proof here.

[46] The map $f_1, f_2$ are called the ***coordinate functions*** of $f$

*Proof.* Let $\pi_1, \pi_2$ be the projection function

$$\begin{aligned} \pi_1 &: \mathbb{X} \times \mathbb{Y} \to \mathbb{X} \\ \pi_2 &: \mathbb{X} \times \mathbb{Y} \to \mathbb{Y} \end{aligned}$$

We first proof that if $U$ is an open subset of $\mathbb{X} \times \mathbb{Y}$ ,

$$f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$$

Let $x \times y \in U$ , $f^{-1}(x \times y)$ contains all $a$ such that $f(a) = x \times y$ .
Then for any $a \in f^{-1}(x \times y)$ , $a \in f_1^{-1}(\pi_1(x \times y)), a \in f_2^{-1}(\pi_2(x \times y))$ .
Thus, $f^{-1}(x \times y) \subseteq f_1^{-1}(\pi_1(x \times y)) \cap f_2^{-1}(\pi_2(x \times y))$ .
Thus $f^{-1}(U) \subseteq f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$ .

Also, if $a \in f_1^{-1}(\pi_1(x \times y)), a \in f_2^{-1}(\pi_2(x \times y))$ , $f_1(a) = x, f_2(a) = y$ .
Thus $f(a) = x \times y$ . Thus $a \in f^{-1}(x \times y)$ .
Thus $f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$

Let $U$ be any open subset of $\mathbb{X} \times \mathbb{Y}$

$$f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$$

Where $f_1^{-1}(\pi_1(U))$ and $f_2^{-1}(\pi_2(U))$ are both open set. Thus $f^{-1}(U)$ is open. □

### 1.7.1 Exercise

1. Let $\mathbb{Y}$ be an ordered set in the order topology. Let $f, g : \mathbb{X} \to \mathbb{Y}$ be continuous, show that the set $A \quad \{x | f(x) \leq g(x)\}$ is closed in $\mathbb{X}$ .

   *Proof.* We only need to proof $\mathbb{X} - A$ is open in $\mathbb{X}$ . We take $x \in \mathbb{X} - A$ . Thus $f(x) > g(x)$ .

   Let $U_1, U2$ be the open set in $\mathbb{Y}$ that met the following demand

   $$\begin{aligned} &\forall y_1 \in U_1, y_2 \in U_2, y_1 > y_2 \\ &f(x) \in U_1, g_x \in U_2 \end{aligned}$$

   As $\mathbb{Y}$ is an ordered set, $U_1, U2$ must exist.

   Let $U = f^{-1}(U_1) \cap g^{-1}(U_2)$ . It is obvious that $U$ is a open set, and $x \in U$ .

   Also, for any $y \in U$ . $f(y) > g(y)$ . Thus $U \subseteq A$ . Thus $A$ is an open set. □

2. Let $\{A_\alpha\}$ be a collection of subsets of $\mathbb{X}$ ; let $\mathbb{X} = \cup_\alpha A_\alpha$ . Lef $f : \mathbb{X} \to \mathbb{Y}$ ; suppose that $f|A_\alpha$ is continuous for each $\alpha$ . An indexed family of sets $\{A_\alpha\}$ is said to be ***locally finite*** if each point $x$ of $\mathbb{X}$ has a neighbourhood that intersect $A_\alpha$ for only finitely main values of $\alpha$ . Show that if the family $\{A_\alpha\}$ is locally finite and each $A_\alpha$ is closed, then $f$ is continuous.

   *Proof.* For any closed subset $U$ of $\mathbb{Y}$ . Let

   $$V = \cup f|A_\alpha(U)$$

   We prove that $V$ is closed, so, $f$ is continuous.

   To prove that $V$ is closed, we prove that $\overline{V} = V$ . That is for any $x \in \overline{V}$ , we prove $x \in V$ . For any neighbourhood $B$ if $x$ , let $C_B$ denote the set that contain all $\alpha$ , such that $f|A_{\alpha(U)}$ intersect with $B$ . As $B$ intersect with $V$ , $C_B$ can not be empty.

   Let

   $$\mathbb{C} = \{C_B | B \text{ be a neighbourhood of } x\}$$

   As $\{A_\alpha\}$ is locally definite, $\mathbb{C}$ contain at least one element with finite elements.

   Also

   $$C_{B_1 \cap B_2} \subseteq C_{B_1} \cap C_{B_2}$$

   Let $\leq$ be a partial order on the $\mathbb{C}$ . If $C_{B_1} \subseteq C_{B_2}$ , we say that $C_{B_1} \geq C_{B_2}$ .

   If there is chain in $\mathbb{C}$

   $$C_{B_1} \leq C_{B_2} \ldots$$

   Let $C_{B_0}$ be a element of $\mathbb{C}$ with finite element. If $C_{B_0} \subseteq C_{B_1}, C_{B_0} \subseteq C_{B_2} \ldots$ . Then $C_{B_0}$ is a upper bound of the chain.

   If $C$ is not a subset of all element of the chain. Then we construct a new set say

   $$D = \{C_{B_0 \cap B_1}, C_{B_0 \cap B_2} \ldots\}$$

   Let

   $$\mathbb{D} = \{C_{D_1 \cap D_2 \cap \ldots} | C_{D_1}, C_{D_2} \cdots \in D\}$$

   As $C_{B_0}$ is a finite set, $D$ is a finite set, $\mathbb{D}$ is also a finite set. Thus there must be a maximal element $E \in \mathbb{D}$ that is the subset of all element of $\mathbb{D}$ . Then $E$ is a subset of all element of the chain. Thus $E$ is a upper bound of the chain.

   Thus, there must be a maximal element $C_F$ of $\mathbb{C}$ , that is a subset of all element of $\mathbb{C}$ .

Let $G$ be the set be the union of all element of $C_F$ .

As $C_F$ is finite, $G$ is closed. And all neighbourhood of $x$ intersect with $G$ . Thus $x \in G$

As $G$ is a subset of $V$ , $x \in V$ . So $V$ is closed. And $f$ is a continuous function on $\mathbb{X}$ .

□

3. Let $A$ be a subset of topological space $\mathbb{X}$ , let $\mathbb{Y}$ be a Hausdorff space. Let $f : A \to \mathbb{Y}$ be a continuous function. Let $g : \overline{A} \to \mathbb{Y}$ also be a continuous function where $g(x) = f(x), x \in A$ . Prove that $g$ us uniquely determined by $f$ .[47]

*Proof.* Say $g$ and $h$ are two distinct function that met the demand.

So there exist $x_0$ such that $g(x_0) \neq h(x_0)$ .

As $\mathbb{Y}$ is a Hausdorff space, so there exist adjoint open subset $g(x_0) \in U$ and $h(x_0) \in V$ .

Then $g^{-1}(U)$ and $h^{-1}(V)$ are both open subset of $\mathbb{X}$ that contain $x_0$ .

If $g^{-1}(U) \cap h^{-1}(V) \cap A \neq \emptyset$ . Then there exist $x_1 \in g^{-1}(U) \cap h^{-1}(V) \cap A$ such that $g(x_1) \in U$ and $h(x_1) \in V$ and $g(x_1) = h(x_1)$ . However $U$ and $V$ are disjoint. So there is a contradiction.

As ${}^{-1}(U) \cap h^{-1}(V)$ is a open subset contain $x_0$ . So ${}^{-1}(U) \cap h^{-1}(V)$ must intersect with $A$ . So it is impossible that $g^{-1}(U) \cap h^{-1}(V) \cap A = \emptyset$ .

So $g = h$ . □

## 1.8 Metric Topology

**Definition 1.8.1** (metric)**.** *A* ***metric*** *on a set* $\mathbb{X}$ *is a function*

$$d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$$

*having the following properties:*

1. $d(x, y) \geq 0$ *for all* $x, y \in \mathbb{X}$ *; equality hold if and only if* $x = y$
2. $d(x, y) = d(y, x), \forall x, y \in \mathbb{X}$

[47] It is possible that $g$ does not exist.
Let $\mathbb{X}$ be the real line with order topology. Let $\mathbb{Y}$ be $\{0, 1\}$ .
Let $A = \mathbb{X} - \{0\}$ .
Let,

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

So, it is obvious that $f$ is a continuous function on $\mathbb{X}$ . However $g$ does not exist in this case.

*3. (Triangle Inequality)* $d(x,y) + d(y,z) \geq d(x,z), \forall x,y,z \in \mathbb{X}$

*Given a metric* $d$ *on* $\mathbb{X}$ *, the number* $d(x,y)$ *is often called the* ***distance*** *between* $x$ *and* $y$ *in the metric* $d$ *.*

**Definition 1.8.2** ( $\epsilon$ -ball centered at $x$ )**.** [48] *Given metric* $d$ *on a set* $\mathbb{X}$ *and* $\epsilon > 0$ *. The set*

$$B_d(x,\epsilon) = \{y | d(x,y) < \epsilon\}$$

*is called* $\epsilon$ *-ball centered at* $x$ *.*

**Definition 1.8.3** (metric topology)**.** *If* $d$ *is a metric on the set* $\mathbb{X}$ *, then the collection of all* $\epsilon$ *-balls* $B_d(x,\epsilon)$ *, such that* $x \in \mathbb{X}$ *and* $\epsilon > 0$ *, is a basis for a topology on* $\mathbb{X}$ *, called the* ***metric topology*** *induced by* $d$ *.*

**Definition 1.8.4** (metrizable)**.** *If* $\mathbb{X}$ *is topological space,* $\mathbb{X}$ *is said to be* ***metrizable*** *if there exists a metric* $d$ *on the set* $\mathbb{X}$ *that induces the topology of* $\mathbb{X}$ *. A* ***metric space*** *is a metrizable space* $\mathbb{X}$ *together with a specific metric* $d$ *that gives the topology of* $\mathbb{X}$ *.*

**Definition 1.8.5** (bounded)**.** *Let* $\mathbb{X}$ *be a metric space with metric* $d$ *. A subset* $A$ *of* $\mathbb{X}$ *is said to be* ***bounded*** *if there is some number* $M$ *such that*

$$d(a_1, a_2) \leq M$$

*for every pair* $a_1$ *and* $a_2$ *if points of* $A$ *.*

**Definition 1.8.6** (diameter)**.** *Let* $\mathbb{X}$ *be a metric space with metric* $d$ *. Let* $A$ *be a bounded subset of* $\mathbb{X}$ *. Then* ***diameter*** *is defined to be*

$$\mathrm{diam} A = \sup\{d(a_1,a_2) | a_1, a_2 \in A\}$$

**Theorem 1.8.1.** *Let* $\mathbb{X}$ *be a metric space with metric* $d$ *. Define* $\overline{d} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ *by the equation*

$$\overline{d}(x,y) = min\{d(x,y), 1\}$$

*Then* $\overline{d}$ *is a metric that induces the same topology as* $d$ *.*

*The metric* $\overline{d}$ *is called the* ***standard bounded metric*** *corresponding to* $d$

*Proof.* It is obvious that $\overline{d}$ is a metric.

To prove that $d$ and $\overline{d}$ induces the same topology, it is suffice to prove that for all $a \in X$ and $\epsilon > 0$ there exists $\{a_\alpha\}$ and $\{\epsilon_\alpha\}$ where $\epsilon_\alpha \leq 1$ such that

$$B_d(a,\epsilon) = \bigcup B_{\overline{d}}(a_\alpha, \epsilon_\alpha)$$

For every $x \in B_d(a,\epsilon)$ take $a_x = x$ and $\epsilon_x < min(\epsilon - d(a,x), 1)$ . Then

$$B_d(a,\epsilon) \supseteq B_{\overline{d}}(a_x, \epsilon_x)$$

[48] When no confusion will arise, the metric $d$ may be omit in $B_d(x,\epsilon)$

as for all $y \in B_{\overline{d}}(a_x, \epsilon_x)$

$$\begin{aligned} d(a,y) &\leq d(a,a_x) + d(a_x,y) \\ &< min(\epsilon - d(a,x), 1) + d(a,a_x) \\ &\leq \epsilon \end{aligned}$$

Thus
$$B_d(a,\epsilon) = \bigcup_{x \in B_d(a,\epsilon)} B_{\overline{d}}(a_x, \epsilon_x)$$

□

**Definition 1.8.7** (norm). *Given $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. The **norm** of $x$ is defined by the equation*
$$||x|| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$$

**Definition 1.8.8** (euclidean metric). *The euclidean metric $d$ on $\mathbb{R}^n$ is defined by*
$$d(x,y) = ||x - y||$$

**Definition 1.8.9** (square metric). *The square metric $\rho$ on $\mathbb{R}^n$ is defined by*

$$\rho(x,y) = max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$$

**Lemma 1.8.1.** *Let $d$ and $d'$ be two metrics on the set $\mathbb{X}$; let $\mathbb{T}$ and $\mathbb{T}'$ be the topology induced by $d$ and $d'$ respectively. Then $\mathbb{T}'$ is finer than $T$ if and only if for all $x \in \mathbb{X}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$$

*Proof.* If $\mathbb{T}'$ is finer than $\mathbb{T}$. Then for all $B_d(x,\epsilon)$ there exists a open set $U$ that containing $x$ such that $U \subseteq B_d(x,\epsilon)$. As $\{B_{d'}(x,\delta)\}$ is a basis of $T'$, then there exists $B_{d'}(x,\delta) \subseteq U$ that containing $x$.

If for all $B_d(x,\epsilon)$, there exists $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$. Then as $\{B_{d'}(x,\epsilon)\}$ and $\{B_d(x,\epsilon)\}$ are both basis, then $\mathbb{T}'$ is finer than $T$. □

**Theorem 1.8.2.** [49] *The topologies on $\mathbb{R}^n$ induced by the euclidean metric $d$ and the square metric $\rho$ are the same as the product topology on $\mathbb{R}^n$.*

**Definition 1.8.10** (uniform metric, uniform topology). *Given an index set $J$, and given points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric $\overline{\rho}$ on $\mathbb{R}^J$ by the equation*

$$\overline{\rho}(x,y) = \sup\{\overline{d}(x_\alpha, y_\alpha) | \alpha \in J\}$$

*where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$. $\overline{\rho}$ is called the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology***

[49] We omit the proof of this theorem as it is obvious.

**Theorem 1.8.3.** [50] *The uniform topology on* $\mathbb{R}^J$ *is finer than the product topology and is coarser than the box topology.*

**Theorem 1.8.4.** *Let* $\overline{d}(a,b) = \min\{|a-b|, 1\}$ *be the standard bounded metric on* $\mathbb{R}$ *. If* $x$ *nad* $y$ *are two points of* $\mathbb{R}^\omega$ *, define*

$$D(x,y) = \sup\left\{\frac{\overline{d}(x_i,y_i)}{i}\right\}$$

*Then* $D$ *is a metric that induces the product topology on* $\mathbb{R}^\omega$

*Proof.* The properties of a metric are satisfied trivially except for the triangle inequality, which is proved by noting that for all $i$ ,

$$\begin{aligned}\frac{\overline{d}(x_i,z_i)}{i} &\leq \frac{\overline{d}(x_i,y_i)}{i} + \frac{\overline{d}(y_i,z_i)}{i} \\ &\leq D(x,y) + D(y,z)\end{aligned}$$

so that

$$\sup\left\{\frac{\overline{d}(x_i,z_i)}{i}\right\} \leq D(x,y) + D(y,z)$$

The fact that $D$ gives the product topology requires a little more work. First, let $U$ be open in the metric topology and let $x \in U$ ; we find an open set $V$ in the product topology such that $x \in V \supseteq U$ . Choose an $\epsilon - ball$ $B_D(x,\epsilon)$ lying in $U$ . Then choose $N$ large enough that $\frac{1}{N} < \epsilon$ . Finally, let $V$ be the basis element for the product topology

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times R \times R \times \ldots$$

We assert that $V \in B_D(x,\epsilon)$ : Given any $y$ in $\mathbb{R}^\omega$

$$\frac{\overline{d}(x_i,y_i)}{i} \leq \frac{1}{N}, \forall i \geq N$$

Therefore,

$$D(x,y) \leq \max\left\{\frac{\overline{d}(x_1,y_1)}{1}, \ldots, \frac{\overline{d}(x_N,y_N)}{N}, \frac{1}{N}\right\}$$

If $y$ is in $V$ , this expression is less than $\epsilon$ , so that $V \subseteq B_D(x,\epsilon)$ , as desired. Conversely, consider a basis element

$$U = \prod_{i \in \mathbb{Z}_+} U_i$$

[50] We omit the proof of this theorem as it is obvious.

for the product topology, where $U_i$ is open in $\mathbb{R}$ for $i = \alpha_1, \dots, \alpha_n$ and $U_i = \mathbb{R}$ for all other indices $i$ . Given $x \in U$ , we find an open set $V$ of the metric topology such that $x \in V \supseteq U$ . Choose an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ in $\mathbb{R}$ centered about $x_i$ and lying in $U_i$ for $i = \alpha_1, \dots, \alpha_n$ ; choose each $\epsilon_i \leq 1$ . Then define

$$\epsilon = \min\left\{\frac{\epsilon_i}{i} | i = \alpha_1, \dots, \alpha_n\right\}$$

We assert that

$$x \in B_D(x, \epsilon) \subseteq U$$

Let $y$ be a point of $B_D(x, \epsilon)$ . Then for all $i$

$$\frac{\overline{d}(x_i, y_i)}{i} \leq D(x, y) < \epsilon$$

Now if $i = \alpha_1, \dots, \alpha_n$ , then $\epsilon \leq \frac{\epsilon_i}{i}$ , so that $\overline{d}(x_i, y_i) < \epsilon_i \leq 1$ ; it follows that $|x_i - y_i| < \epsilon_i$ . Therefore $y \in \prod U_i$ , as desired.

□

**Definition 1.8.11** (Hilbert Cube)**.** *The set*

$$H = \prod_{n \in \mathbb{Z}_+} [0, \frac{1}{n}]$$

*is called* ***Hilbert cube***

**Definition 1.8.12** ($l^2$-topology)**.** *Let* $\mathbb{X}$ *be the subset of* $\mathbb{R}^\omega$ *consisting of all sequences* $x$ *such that* $\sum x_i^2$ *converges.*

*Then the formula*

$$d(x, y) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{\frac{1}{2}}$$

*defines a metric on* $\mathbb{X}$ *. The topology induced by* $d$ *is called the* $l^2$*-topology.*

**Definition 1.8.13** (countable basis at point $x$ )**.** *A space is said to be have* ***countable basis at point*** $x$ *if there is a countable collection* $\{U_n\}_{n \in \mathbb{Z}_+}$ *of neighbourhoods of* $x$ *such that any neighbourhood* $U$ *of* $x$ *contains at least on of the sets* $U_n$ *. A space* $\mathbb{X}$ *that has a countable basis at each of its point is said to satisfy the* ***first countability axiom***

**Theorem 1.8.5.** *Let* $f : \mathbf{X} \to \mathbf{Y}$ *be metrizable with metric* $d_{\mathbf{X}}$ *and* $d_{\mathbf{Y}}$ *, respectively. Then continuity of* $f$ *is equivalent to the requirement that given* $x \in \mathbb{X}$ *and given* $\epsilon > 0$ *, there exists* $\delta > 0$ *such that*

$$d_{\mathbf{X}}(x, y) < \delta \implies d_{\mathbb{Y}}(f(x), f(y)) < \epsilon$$

*Proof.* Suppose $f$ is continuous. Given $x$ and $\epsilon$, consider the set

$$f^{-1}(B(f(x),\epsilon))$$

which is open in $\mathbb{X}$ and contains the point $x$. It contains some $\delta$-ball $B(x,\delta)$ centered at $x$. If $y$ is in this $\delta$-ball, then $f(y)$ is in this $\delta$-ball as desired.

Conversely, suppose that the $\epsilon-\delta$ condition is satisfied. Let $V$ be open in $\mathbb{Y}$; we show that $f^{-1}(V)$ is open in $\mathbb{X}$. Let $x$ be a point of the set $f^{-1}(V)$. Since $f(x)\in V$ there is an $\epsilon$-ball $B(f(x),\epsilon)$ centered at $f(x)$ and contained in $V$. By the $\epsilon-\delta$ condition, there exists a $\delta$-ball centered at $x$ such that $f(B(x,\delta))\subseteq B(f(x),\epsilon)$. Then $B(x,\delta)$ is a neighbourhood of $x$ contained in $f^{-1}(V)$, so that $f^{-1}(V)$ is open, as desired. $\square$

**Lemma 1.8.2** (The sequence lemma)**.** [51]*Let $\mathbb{X}$ be a topological space; let $A\subseteq\mathbb{X}$ If there is a sequence of points of $A$ converging to $x$, then $x\in\overline{A}$, the converse holds if $\mathbb{X}$ is metrizable.*

**Theorem 1.8.6.** [52]*Let $f:\mathbb{X}\to\mathbb{Y}$. If the function $f$ is continuous, then for every convergent sequence $x_n\to x$, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if $\mathbb{X}$ is metrizable.*

**Lemma 1.8.3.** [53]*The addition, subtraction, and multiplication operations are continuous functions from $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$; and the quotient operation is continuous function from $\mathbb{R}\times(\mathbb{R}-\{0\})$ into $\mathbb{R}$.*

**Theorem 1.8.7.** [54]*If $\mathbb{X}$ is a topological space, and if $f,g:\mathbb{X}\to\mathbb{R}$ are continuous functions, then $f+g$, $f-g$ and $f\cdot g$ are continuous. If $g(x)\neq 0$ for all $x$, then $\frac{f}{g}$ is continuous.*

**Definition 1.8.14** (converge uniformly)**.** *Let $f_n:\mathbb{X}\to\mathbb{Y}$ be a sequence of functions from the set $\mathbb{X}$ to the metric space $\mathbb{Y}$. Let $d$ be the metric for $\mathbb{Y}$. We say that the sequence $(f_n)$ **converges uniformly** to the function $f:\mathbb{X}\to\mathbb{Y}$ if given $\epsilon>0$, there exists an integer $N$ such that*

$$d(f_n(x),f(x))<\epsilon$$

*for all $n>N$ and all $x\in\mathbb{X}$*

**Theorem 1.8.8** (Uniform limit theorem)**.** *Let $f_n:\mathbb{X}\to\mathbb{Y}$ be a sequence of continuous functions from the topological space $\mathbb{X}$ to the metric space $\mathbb{Y}$. If $(f_n)$ converges uniformly to $f$, then $f$ is continuous.*

[51] We omit the proof of this lemma as it is obvious.
[52] We omit the proof of this theorem as it is obvious.
[53] We omit the proof of this lemma as it is obvious.
[54] We omit the proof of this theorem as it is obvious.

**Definition 1.8.15** (isometric imbedding)**.** *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be metric spaces with metric* $d_{\mathbb{X}}$ *and* $d_{\mathbb{Y}}$ *, respectively. Let* $f : \mathbb{X} \to \mathbb{Y}$ *have the property that for every pair of points* $x_1$ *,* $x_2$ *of* $\mathbb{X}$ *, and*

$$d_{\mathbb{Y}}(f(x_1), f(x_2)) = d_{\mathbb{X}}(x_1, x_2)$$

$f$ *is an topological imbedding and is called an* ***isometric imbedding*** *of* $\mathbb{X}$ *in* $\mathbb{Y}$