Topology Note

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Contents

1 Basic Definition of Topology

Definition 1.1 (topology)**.** *A* ***topology*** *on a set* $\mathbb{X}$ *is a collection* $\mathbb{T}$ *of subsets of* $\mathbb{X}$ *having the following properties:*

- $\emptyset$ *and* $\mathbb{X}$ *are in* $\mathbb{T}$
- *The union of the elements of any sub collection of* $\mathbb{T}$ *is in* $\mathbb{T}$
- *The intersection of the elements of any* ***finite*** *sub collection of* $\mathbb{T}$ *is in* $\mathbb{T}$

Definition 1.2 (topology space)**.** *A* ***topological space*** *is a set* $\mathbb{X}$ *for which a topology* $\mathbb{T}$ *has been specified.*

Definition 1.3 (open set)**.** *A* ***open set*** $\mathbb{U}$ *is a subset of* $\mathbb{X}$ *that belongs to a topology* $\mathbb{T}$ *of* $\mathbb{X}$*.*

Definition 1.4 (open sets)**.** *A topology can also be called a* ***open sets***

Definition 1.5 (discrete topology)**.** *The set of all subsets of a set* $\mathbb{X}$ *formed a topology called* ***discrete topology***

Definition 1.6 (trivial topology)**.** *The set consisting the set* $\mathbb{X}$ *and* $\emptyset$ *only formed a topology of* $\mathbb{X}$ *called* ***trivial topology***

Definition 1.7 (finite complement topology)**.** *Let* $\mathbb{X}$ *be a set. Let* $\mathbb{T}_f$ *be the collection of all subsets* $\mathbb{U}$ *of* $\mathbb{X}$ *such that* $\mathbb{X} - \mathbb{U}$ *either if a* ***finite*** *of is all of* $\mathbb{X}$*. Then* $\mathbb{T}_f$ *is a topology on* $\mathbb{X}$*, called the .*

The set $\mathbb{U}$ can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, $\mathbb{U}$ will not be a topology.

Definition 1.8 (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable)**.** *Let* $\mathbb{T}$ *and* $\mathbb{T}'$ *be two topology on a given set* $\mathbb{X}$*. If* $\mathbb{T}$ *is a subset of* $\mathbb{T}'$*, we say that* $\mathbb{T}'$ *is* ***finer*** *or* **larger** *than* $\mathbb{T}$*. If* $\mathbb{T}$ *is a proper subset of* $\mathbb{T}'$*, we say that* $\mathbb{T}'$ *is* ***strictly finer*** *or* ***strictly larger*** *than* $\mathbb{T}$*. We also say that* $\mathbb{T}$ *is* **coarser** *or* ***smaller*** *or* ***strictly coarser*** *or* ***strictly smaller*** *than* $\mathbb{T}'$*. We say that* $\mathbb{T}$ *and* $\mathbb{T}'$ *is* **comparable** *if either* $\mathbb{T}$ *is a subset of* $\mathbb{T}'$ *or* $\mathbb{T}'$ *is a subset of* $\mathbb{T}$*.*

2 Basis for a Topology

Definition 2.1 (basis)**.** *If* $\mathbb{X}$ *is a set, a* **basis** *for a topology on* $\mathbb{X}$ *is a collection* $\mathbb{B}$ *of subsets of* $\mathbb{X}$ *(called* ***basis elements****) such that:*

- *For each* $x \in \mathbb{X}$*, there is at least one basis element* B *containing* x
- *If* x *belongs to the intersection of two basis elements* B_1 *and* B_2*, then there is another element* $x \in B_3 \in \mathbb{B}$ *such that* $B_3 \subseteq B_1 \cap B_2$

Definition 2.2 (topology generated by basis)**.** *Let* $\mathbb{B}$ *be a basis on* $\mathbb{X}$*. Let* $\mathbb{U}$ *be a set containing all subsets* U *of* $\mathbb{X}$ *such that for each element* $x \in U$*, there is* $B \in \mathbb{B}$ *that* $x \in B \subseteq U$*. Such* $\mathbb{U}$ *formed a topology on* $\mathbb{X}$*, called* **topology** $\mathbb{T}$ ***generated by*** $\mathbb{B}$

Lemma 2.1. *Let* $\mathbb{X}$ *be a set. Let* $\mathbb{B}$ *be a basis for a topology* $\mathbb{T}$ *on* $\mathbb{X}$*. Then* $\mathbb{T}$ *equals to the set of all possible unions of elements of* $\mathbb{B}$*.*

Proof. Let set $\mathbb{U}$ be the set of all possible unions of elements of $\mathbb{B}$. For any $U \in \mathbb{U}$. $U = \cup B$ for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Note that t expression n not be unique

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in U$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, $\mathbb{U}$ equals to $\mathbb{T}$. $\square$

Lemma 2.2. [1] *Let $\mathbb{X}$ be a topological space. Suppose that $\mathbb{C}$ is a collection of open sets of $\mathbb{X}$ such that for each open set U of $\mathbb{X}$ and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq C$. Then $\mathbb{C}$ is a basis for the topology of $\mathbb{X}$.*

Lemma 2.3. [2] *Let $\mathbb{B}$ and $\mathbb{B}'$ be basis for the topologies $\mathbb{T}$ and $\mathbb{T}'$, respectively, on $\mathbb{X}$. Then the following are equivalent:*

- *$\mathbb{T}'$ is finer than $\mathbb{T}$*
- *For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing X, there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.*

Definition 2.3 (standard topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the* ***standard topology on the real line*** *.*

Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.

Definition 2.4 (lower limit topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the* ***lower limit topology on the real line****. When $\mathbb{R}$ is given this topology,we denote it by $\mathbb{R}_l$.*

Definition 2.5 (K-topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$. $\mathbb{B} \cup \{B - K | B \in \mathbb{B}\}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the* ***K-topology on the real line****. When $\mathbb{R}$ is given this topology,we denote it by $\mathbb{R}_{\mathbb{K}}$.*

Lemma 2.4. [3] *The topologies $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is strictly finer than the standard topology on $\mathbb{R}$.*

Lemma 2.5. *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is not comparable.*

Proof. Let $\mathbb{T}_l$ and $\mathbb{T}_{\mathbb{K}}$ be topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ respectively. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that $\mathbb{T}_l$ is not finer than $\mathbb{T}_{\mathbb{K}}$. Let $U = \{x | -1 < x < 1\} - K, x = 0$. If there exist $B = \{x | a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus B is not a subset of U.

Then we proof that $\mathbb{T}_{\mathbb{K}}$ is not finer than $\mathbb{T}_l$. Let $U' = \{x | a' \leq x < b'\}$. If there exist $B' = \{x | a'' < x < b''\} or \{x | a'' < x < b''\} - K$ such that $a' \in B \subseteq U$. Thus $a'' < a < b''$. Thus there exist c that $a'' < x < a, x \in B, x \notin U'$. Thus $B' \nsubseteq U'$.

Thus the topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is not comparable. $\square$

Definition 2.6 (subbasis)**.** *A* ***subbasis*** *$\mathbb{S}$ for a topology on $\mathbb{X}$ is a collection of subsets of $\mathbb{X}$ whose union equals $\mathbb{X}$. The* ***topology generated by the subbasis*** *$\mathbb{S}$ is defined to be the collection $\mathbb{T}$ of all unions of finite intersections of elements of $\mathbb{S}$.*

It is obvious that $\mathbb{T}$ is a topology, we just omit the proof here.

[1] We omit the proof of this lemma as it is obvious.
[2] We omit the proof of this lemma as it is obvious.
[3] We omit the proof of this lemma as it is obvious.

2.1 Exercise

1. Show that if $\mathbb{A}$ is a basis for a topology on $\mathbb{X}$, then the topology generated by $\mathbb{A}$ equals the intersection of all topologies on $\mathbb{X}$ that contain $\mathbb{A}$. Prove the same if $\mathbb{A}$ is a subbasis.

   Proof. As a subbasis is also a basis, we will directly prove the case of subbasis here.

   Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain $\mathbb{A}$. Let $\mathbb{T}$ be the topology that $\mathbb{A}$ generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.

   It is obvious that $\mathbb{T}'$ is also a topology, we just omit the proof here.

   First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

   Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

   Thus, $\mathbb{T} = \mathbb{T}'$ $\square$