Topology Note

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Contents

Definitions

Chapter 1

Topology Spaces and Continuous Function

1.1 Basic Definition of Topology

Definition 1.1.1 (topology). *A **topology** on a set $\mathbb{X}$ is a collection $\mathbb{T}$ of subsets of $\mathbb{X}$ having the following properties:*

- *$\emptyset$ and $\mathbb{X}$ are in $\mathbb{T}$*
- *The union of the elements of any sub collection of $\mathbb{T}$ is in $\mathbb{T}$*
- *The intersection of the elements of any **finite** sub collection of $\mathbb{T}$ is in $\mathbb{T}$*

Definition 1.1.2 (topology space). *A **topological space** is a set $\mathbb{X}$ for which a topology $\mathbb{T}$ has been specified.*

Definition 1.1.3 (open set). *A **open set** $\mathbb{U}$ is a subset of $\mathbb{X}$ that belongs to a topology $\mathbb{T}$ of $\mathbb{X}$.*

Definition 1.1.4 (open sets). *A topology can also be called a **open sets***

Definition 1.1.5 (discrete topology). *The set of all subsets of a set $\mathbb{X}$ formed a topology called **discrete topology***

Definition 1.1.6 (trivial topology). *The set consisting the set $\mathbb{X}$ and $\emptyset$ only formed a topology of $\mathbb{X}$ called **trivial topology***

Definition 1.1.7 (finite complement topology). *Let $\mathbb{X}$ be a set. Let $\mathbb{T}_f$ be the collection of all subsets $\mathbb{U}$ of $\mathbb{X}$ such that $\mathbb{X} - \mathbb{U}$ either if a **finite** [1] of is all of $\mathbb{X}$. Then $\mathbb{T}_f$ is a topology on $\mathbb{X}$, called the **finite complement topology**.*

Definition 1.1.8 (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable). *Let $\mathbb{T}$ and $\mathbb{T}'$ be two topology on a given set $\mathbb{X}$. If $\mathbb{T}$ is a subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **finer** or **larger** than $\mathbb{T}$. If $\mathbb{T}$ is a proper subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **strictly finer** or **strictly larger** than $\mathbb{T}$. We also say that $\mathbb{T}$ is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than $\mathbb{T}'$. We say that $\mathbb{T}$ and $\mathbb{T}'$ is **comparable** if either $\mathbb{T}$ is a subset of $\mathbb{T}'$ or $\mathbb{T}'$ is a subset of $\mathbb{T}$.*

[1] The set $\mathbb{U}$ can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, $\mathbb{U}$ will not be a topology.

1.2 Basis for a Topology

Definition 1.2.1 (basis)**.** *If $\mathbb{X}$ is a set, a* ***basis*** *for a topology on $\mathbb{X}$ is a collection $\mathbb{B}$ of subsets of $\mathbb{X}$ (called* ***basis elements****) such that:*

- *For each $x \in \mathbb{X}$, there is at least one basis element B containing x*
- *If x belongs to the intersection of two basis elements B_1 and B_2, then there is another element $x \in B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$*

Definition 1.2.2 (topology generated by basis)**.** *Let $\mathbb{B}$ be a basis on $\mathbb{X}$. Let $\mathbb{U}$ be a set containing all subsets U of $\mathbb{X}$ such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such $\mathbb{U}$ formed a topology on $\mathbb{X}$, called* ***topology*** *$\mathbb{T}$* ***generated by*** *$\mathbb{B}$*

Lemma 1.2.1. *Let $\mathbb{X}$ be a set. Let $\mathbb{B}$ be a basis for a topology $\mathbb{T}$ on $\mathbb{X}$. Then $\mathbb{T}$ equals to the set of all possible unions of elements of $\mathbb{B}$.*

Proof. Let set $\mathbb{U}$ be the set of all possible unions of elements of $\mathbb{B}$. For any $U \in \mathbb{U}$. $U = \cup B$ [2] for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in U$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, $\mathbb{U}$ equals to $\mathbb{T}$. □

Lemma 1.2.2. [3] *Let $\mathbb{X}$ be a topological space. Suppose that $\mathbb{C}$ is a collection of open sets of $\mathbb{X}$ such that for each open set U of $\mathbb{X}$ and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq C$. Then $\mathbb{C}$ is a basis for the topology of $\mathbb{X}$.*

Lemma 1.2.3. [4] *Let $\mathbb{B}$ and $\mathbb{B}'$ be basis for the topologies $\mathbb{T}$ and $\mathbb{T}'$, respectively, on $\mathbb{X}$. Then the following are equivalent:*

- *$\mathbb{T}'$ is finer than $\mathbb{T}$*
- *For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing X, there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.*

Definition 1.2.3 (standard topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the* ***standard topology on the real line*** [5] *.*

Definition 1.2.4 (lower limit topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the* ***lower limit topology on the real line****. When $\mathbb{R}$ is given this topology,we denote it by $\mathbb{R}_l$.*

Definition 1.2.5 (K-topology on the real line)**.** *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$. $\mathbb{B} \cup \{B - K | B \in \mathbb{B}\}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the* ***K-topology on the real line****. When $\mathbb{R}$ is given this topology,we denote it by $\mathbb{R}_{\mathbb{K}}$.*

Lemma 1.2.4. [6] *The topologies $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is strictly finer than the standard topology on $\mathbb{R}$.*

[2] Note that this expression may not be unique.
[3] We omit the proof of this lemma as it is obvious.
[4] We omit the proof of this lemma as it is obvious.
[5] Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.
[6] We omit the proof of this lemma as it is obvious.

Lemma 1.2.5. *The topologies of* $\mathbb{R}_l$ *and* $\mathbb{R}_{\mathbb{K}}$ *is not comparable.*

Proof. Let $\mathbb{T}_l$ and $\mathbb{T}_{\mathbb{K}}$ be topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ respectively. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that $\mathbb{T}_l$ is not finer than $\mathbb{T}_{\mathbb{K}}$. Let $U = \{x | -1 < x < 1\} - K, x = 0$. If there exist $B = \{x | a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus B is not a subset of U.

Then we proof that $\mathbb{T}_{\mathbb{K}}$ is not finer than $\mathbb{T}_l$. Let $U' = \{x | a' \leq x < b'\}$. If there exist $B' = \{x | a'' < x < b''\} or \{x | a'' < x < b''\} - K$ such that $a' \in B \subseteq U$. Thus $a'' < a < b''$. Thus there exist c that $a'' < x < a, x \in B, x \notin U'$. Thus $B' \nsubseteq U'$.

Thus the topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is not comparable. □

Definition 1.2.6 (subbasis)**.** *A* ***subbasis*** $\mathbb{S}$ *for a topology on* $\mathbb{X}$ *is a collection of subsets of* $\mathbb{X}$ *whose union equals* $\mathbb{X}$. *The* ***topology generated by the subbasis*** $\mathbb{S}$ *is defined to be the collection* $\mathbb{T}$ [7] *of all unions of finite intersections of elements of* $\mathbb{S}$.

1.2.1 Exercise

1. Show that if $\mathbb{A}$ is a basis for a topology on $\mathbb{X}$, then the topology generated by $\mathbb{A}$ equals the intersection of all topologies on $\mathbb{X}$ that contain $\mathbb{A}$. Prove the same if $\mathbb{A}$ is a subbasis.

   Proof. As a subbasis is also a basis, we will directly prove the case of subbasis here.

   Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain $\mathbb{A}$. Let $\mathbb{T}$ be the topology that $\mathbb{A}$ generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.[8]

   First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

   Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

   Thus, $\mathbb{T} = \mathbb{T}'$ □

1.3 The Order Topology

Definition 1.3.1 (interval)**.** *Let* $\mathbb{X}$ *is a set having a simple order relation* $<$. *Given elements* a *and* b *of* $\mathbb{X}$ *such that* $a < b$, *there are four subsets of* $\mathbb{X}$ *that are called* ***intervals*** *determined by* a *and* b:

- $(a, b) = \{x | a < x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $[a, b] = \{x | a \leq x \leq b\}$

(a, b) *is called an* ***open interval*** *on* $\mathbb{X}$. $[a, b]$ *is called an* ***closed interval*** *on* $\mathbb{X}$. $(a, b]$ *and* $[a, b)$ *is called* ***half-open intervals***.

Definition 1.3.2 (order topology)**.** [9] *Let* $\mathbb{X}$ *be a set with a simple order relation; assume* $\mathbb{X}$ *has more than one element. Let* $\mathbb{B}$ *be the collection of all sets of the following types:*

- *All open intervals* (a, b) *in* $\mathbb{X}$.

[7] It is obvious that $\mathbb{T}$ is a topology, we just omit the proof here.
[8] It is obvious that $\mathbb{T}'$ is also a topology, we just omit the proof here.
[9] The standard topology on $\mathbb{R}$ is an order topology derived from the usual order on $\mathbb{R}$.

- *All intervals of the form* $[a_0, b)$*, where* a_0 *is the smallest element(if exist) of* $\mathbb{X}$ *.*
- *All intervals of the form* $(a, b_0]$*, where* b_0 *is the largest element(if exist) of* $\mathbb{X}$ *.*

The collection $\mathbb{B}$ *formed a basis for a topology on* $\mathbb{X}$ *, which is called the order topology.*

Definition 1.3.3 (ray)**.** [10][11] *If* $\mathbb{X}$ *is an ordered set, and a is an element of* $\mathbb{X}$ *, there are four subsets of* $\mathbb{X}$ *that are called* ***rays*** *determined by a:*

- $(a, +\infty) = \{x|x > a\}$
- $(-\infty, a) = \{x|x < a\}$
- $[a, +\infty) = \{x|x \geq a\}$
- $(-\infty, a] = \{x|x \leq a\}$

$(a, +\infty)$ *and* $(-\infty, a)$ *are called* ***open rays****.* $[a, +\infty)$ *and* $(-\infty, a]$ *are called* ***closed rays****.*

1.4 The Product Topology

Definition 1.4.1 (product topology)**.** *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be topological spaces. The* ***product topology*** *on* $\mathbb{X} \times \mathbb{Y}$ *having a basis* $\mathbb{B}$ *containing all sets of the form* $U \times V$*, where* U *and* V *is open sets of* $\mathbb{X}$ *and* $\mathbb{Y}$ *respectively.*

Theorem 1.4.1. [12]*If* $\mathbb{B}$ *and* $\mathbb{C}$ *is basis for the topology of* $\mathbb{X}$ *and* $\mathbb{Y}$ *respectively, then the collection*

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} and C \in \mathbb{C}\}$$

is a basis for the topology of $\mathbb{X} \times \mathbb{Y}$

Definition 1.4.2 (projection)**.** *Let* $\pi_1 : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ *be defined by the equation:*

$$\pi_1(x, y) = x$$

Let $\pi_2 : \mathbb{X} \times \mathbb{Y} \to \mathbb{Y}$ *be defined by the equation:*

$$\pi_1(x, y) = y$$

The maps π_1 *and* π_2 *are called the* ***projections*** *of* $\mathbb{X} \times \mathbb{Y}$ *onto its first and second factors, respectively.*

Theorem 1.4.2. [13]*The collection*

$$\mathbb{S} = \{\pi_1^{-1}(U) | U open in \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V open in \mathbb{Y}\}$$

is a subbasis for the product topology on $\mathbb{X} \times \mathbb{Y}$*.*

[10]open rays are always open sets in the order topology
[11]the open rays also formed a subbasis of the order topology
[12]We omit the proof of this lemma as it is obvious.
[13]We omit the proof of this lemma as it is obvious.

1.5 The Subspace Topology

Definition 1.5.1 (subspace topology)**.** *Let $\mathbb{X}$ be a topological space with topology $\mathbb{T}$. If Y is a subset of $\mathbb{X}$, the collection $\mathbb{T}_Y = \{Y \cap U | U \in \mathbb{T}\}$ is a topology on Y , called the* ***subspace topology****.*

Y is also called a ***subspace*** *of $\mathbb{X}$*

Lemma 1.5.1. [14]*If $\mathbb{B}$ is basis for the topology of $\mathbb{X}$, Y is a subset of $\mathbb{X}$ then the collection*

$$\mathbb{B}_Y = \{B \cap Y | B \in \mathbb{B}\}$$

is a basis for the subspace topology on Y

Lemma 1.5.2. [15]*Let Y be a subspace of $\mathbb{X}$. If U is open in Y and Y is open in $\mathbb{X}$, then U is open in $\mathbb{X}$.*

Theorem 1.5.1. [16] *If A is a subspace of $\mathbb{X}$ and B is a subspace of $\mathbb{Y}$, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$*

Proof. Let $\mathbb{B}_\mathbb{X}$ and $\mathbb{B}_\mathbb{Y}$ and $\mathbb{B}_{\mathbb{X}\mathbb{Y}}$ be basis of topology of $\mathbb{X}$ and $\mathbb{Y}$ and $\mathbb{X} \times \mathbb{Y}$ respectively. Let $\mathbb{B}'_\mathbb{X}$ and $\mathbb{B}'_\mathbb{Y}$ and $\mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ be basis of topology of A and A and $A \times B$ respectively. We will show that $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} = \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$. Thus, the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$.

First, every element in $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$ can be represented by $B_A \cap A \times B_B \cap B = B_A \times B_B \cap A \times B \in \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ where $B_A \in \mathbb{B}'_\mathbb{X}, B_B \in \mathbb{B}'_\mathbb{Y}$. Thus $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} \subseteq \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$.

Next, we show that $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$. For any open set U in $\mathbb{X} \times \mathbb{Y}$, and $\forall x \in U \cap A \times B, \exists B_\mathbb{X} \times B_\mathbb{Y} \in \mathbb{B}_{\mathbb{X}\mathbb{Y}}, x \in B_\mathbb{X} \times B_\mathbb{Y} \subseteq \mathbb{X} \times \mathbb{Y}$. Thus $x \in B_\mathbb{X} \times B_\mathbb{Y} \cap A \times B \subseteq A \times B, B_\mathbb{X} \times B_\mathbb{Y} \cap A \times B \in \mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$. Thus $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$ .gi □

Definition 1.5.2 (ordered square)**.** *Let $I = [0, 1]$. The set $I \times I$ in the dictionary order* [17] *topology will be called* ***ordered square****, and denoted by I_o^2*

Definition 1.5.3 (convex)**.** *Given an ordered set $\mathbb{X}$, let us say that a subset $\mathbb{Y}$ of $\mathbb{X}$ is* ***convex*** *in $\mathbb{X}$ if for each pair of points $a < b$ of $\mathbb{Y}$, the entire interval (a, b) of points of $\mathbb{X}$ lies in $\mathbb{Y}$*

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[14]We omit the proof of this lemma as it is obvious.

[15]We omit the proof of this lemma as it is obvious.

[16]If $\mathbb{X}$ is an ordered set in the order topology, and $\mathbb{Y}$ is a subset of $\mathbb{X}$. The order relation, when restricted to $\mathbb{Y}$, makes $\mathbb{Y}$ into and ordered set. However, *the resulting order topology on $\mathbb{Y}$ need not be the same as the topology that $\mathbb{Y}$ inherits as a subspace of $\mathbb{X}$.*

[17]the dictionary means for $X_1, X_2 \in \mathbb{Y} = \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \ldots$ which:

$$\begin{aligned} X_1 &= (x_1, x_2, x_3 \ldots) \\ X_2 &= (x'_1, x'_2, x'_3 \ldots) \end{aligned}$$

$X_1 > X_2$ only when

$$\begin{aligned} &\exists k \in \mathbb{Z}_+, \forall i \in \mathbb{Z}_+, 0 < i < k \\ &x_i = x'_i \\ &x_k > x'_k \end{aligned}$$

Theorem 1.5.2. [18] *Let $\mathbb{X}$ be an ordered set in the order topology. Let $\mathbb{Y}$ be a subset of $\mathbb{X}$ that is convex in $\mathbb{X}$. Then the order topology on $\mathbb{Y}$ is the same as the topology $\mathbb{Y}$ inherits as a subspace of $\mathbb{X}$.*

Proof. Consider the ray $(a, +\infty)$ in $\mathbb{X}$. If $a \in \mathbb{Y}$, then

$$(a, +\infty) \cap \mathbb{Y} = \{x | x \in \mathbb{Y} and x > a\}$$

This is an open ray of the ordered set of $\mathbb{Y}$. if $a \notin Y$, then a is either a lower bound on $\mathbb{Y}$ or an upper bound on $\mathbb{Y}$, since $\mathbb{Y}$ is convex. In the former case, the set $(a, +\infty) \cap \mathbb{Y}$ equals all of $\mathbb{Y}$, in the latter case, it is empty.

A similar remark shows that the intersection of the rat $(-\infty, a)$ with $\mathbb{Y}$ is either an open ray of $\mathbb{Y}$, or $\mathbb{Y}$ itself, or empty. Since the sets $(a, +\infty)\mathbb{Y}$ and $(-\infty, a) \cap \mathbb{Y}$ form a subbasis for the subspace topology on $\mathbb{Y}$, and since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of $\mathbb{Y}$ equals the intersection of an open ray of $\mathbb{X}$ with $\mathbb{Y}$, so it is open in the subspace topology on $\mathbb{Y}$. Since the open rays of $\mathbb{Y}$ are a subbasis for the order topology on $\mathbb{Y}$, this topology is contained in the subspace topology. □

Exercise

1. A map $f : \mathbb{X} \to \mathbb{Y}$ is said to be a ***open map*** if for every open set $U \subseteq \mathbb{X}$, the set $f(U)$ is open in $\mathbb{Y}$. Show that $\pi : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ is open map.

   Proof. An open set in $\mathbb{X} \times \mathbb{Y}$ can be represented by

   $$\cup (U_i \times U_i')$$

   where U_i, U_i' are open sets in $\mathbb{X}$, $\mathbb{Y}$, respectively.

   Also,

   $$\cup (U_i \times U_i') = \cup (U_i) \times \cup (U_i')$$

   Thus,

   $$\pi(\cup (U_i \times U_i')) = \cup (U_i)$$

   Thus, $\pi(U)$ is open in $\mathbb{X}$. □

2. Let $\mathbb{X}$ and $\mathbb{X}'$ denote a single set in the topologies $\mathbb{T}$ and $\mathbb{T}'$, respectively; let $\mathbb{Y}$ and $\mathbb{Y}'$ denote a single set in the topologies $\mathbb{U}$ and $\mathbb{U}'$, respectively. [19] Assume these sets are nonempty.

   (a) Show that if $\mathbb{T}' \supseteq \mathbb{T}$ and $\mathbb{U}' \supseteq \mathbb{U}$, then the product topologies $\mathbb{X}' \times \mathbb{Y}'$ is finer than the product topology on $\mathbb{X} \times \mathbb{Y}$.

   (b) Does the converse of the previous statement hold?

[18] Given $\mathbb{X}$ is an ordered set in the order topology and $\mathbb{Y}$ is a subset of $\mathbb{X}$, we shall assume that $\mathbb{Y}$ *is given the subspace topology* unless we specifically state otherwise.

[19] what does $\mathbb{X}$, $\mathbb{X}'$, $\mathbb{Y}$, $\mathbb{Y}'$ really mean here?? I do not know, so I just put the exercise here without a proof.

3. Show that the countable collection[20]

$$\{(a,b)\times(c,d)|a<b,c<d,a\in\mathbb{Q},b\in\mathbb{Q},c\in\mathbb{Q},d\in\mathbb{Q}\}$$

is a basis for $\mathbb{R}^2$

Proof. This is obvious if you prove that $(a,b)\times(c,d)$ is a rectangle in the $\mathbb{R}^2$ plane. □

4. Let $\mathbb{X}$ be an ordered set. If $\mathbb{Y}$ is a proper subset of $\mathbb{X}$ that is convex in $\mathbb{X}$ prove that $\mathbb{Y}$ may not be an interval or a ray in $\mathbb{X}$.

Proof. Let $\mathbb{X}=\mathbb{R}^2$ with dictionary order. Then $Y=\{(x,y)|-1\leq x\leq 1\}$ is convex in $\mathbb{X}$, however it is not an interval or a ray. □

There is a false prove given by myself.

Proof. Let $\mathbb{S}$ be a set that contain all intervals and rays of $\mathbb{Y}$. We define a partial order on $\mathbb{S}$ by inclusion. So if there is a chain in $\mathbb{S}$:

$$S_1\subseteq S_2\subseteq S_3\ldots$$

Let

$$S=S_1\cup S_2\cup S_3\cup\ldots$$

Thus, S is an upper bound of the chain.

Thus, by Zorn's Lemma, there is a maximal element of $\mathbb{S}$, say U , then we prove that $U=\mathbb{Y}$.

If $U\neq\mathbb{Y}$, then $\exists x,x\in\mathbb{Y}-U$.

If U is a ray say $(a,+\infty)$. Then $x<a$, thus $U\subseteq(x,+\infty)\subseteq\mathbb{B}$, then there is contradiction with the maximal element.

If U is an interval, the circumstance is similar with the proof of U is a ray.

Thus $\mathbb{Y}$ is a ray or an interval. □

However, there is issue with this proof, the set S does exists. However, it may not be an interval or ray, so it may not be contained in $\mathbb{S}$

1.6 Closed Sets and Limit Points

Definition 1.6.1 (closed). [21] *A subset A of a topological space is said to be closed if the set $\mathbb{X}-A$ is open.*

Theorem 1.6.1. [22]*Let $\mathbb{X}$ be a topological space. Then the following conditions hold*

1. $\emptyset$ and $\mathbb{X}$ are closed.

[20]The prove of this set is countable is typically similar to Cantor's enumeration of a countable collection of countable sets.

[21]A set can be open, or closed, or both, or neither

[22]We omit the proof of this lemma as it is obvious.

2. *Arbitrary intersections of closed sets are closed*

3. *Finite unions of closed sets are closed*

Definition 1.6.2 (closed in)**.** *Let* $\mathbb{X}$ *be a topological space; let* $\mathbb{Y}$ *be a subspace of* $\mathbb{X}$ *. We say that a set* A *is* ***closed in*** $\mathbb{Y}$ *if* A *is a subset of* $\mathbb{Y}$ *and* A *is closed in the subspace topology of* $\mathbb{Y}$

Theorem 1.6.2. *Let* $\mathbb{Y}$ *be a subspace of* $\mathbb{X}$ *. Then a set* A *is closed in* $\mathbb{Y}$ *if and only if it equals the intersection of a closed set of* $\mathbb{X}$ *with* $\mathbb{Y}$

Proof. First we proof that if A is closed in $\mathbb{Y}$, then $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. As the origin topology form a surjective map to its subspace topology, there exists a B closed in $\mathbb{X}$ that $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $B \cap \mathbb{Y} = A$

Conversely, if $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. Then, $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $\mathbb{X} - B$ is open in $\mathbb{Y}$, $\mathbb{Y} - A$ is open in $\mathbb{Y}$. Then A is closed in $\mathbb{Y}$ ☐

Theorem 1.6.3. [23] *Let* $\mathbb{Y}$ *be a subspace of* $\mathbb{X}$ *. If* A *is closed in* $\mathbb{Y}$ *and* $\mathbb{Y}$ *is closed in* $\mathbb{X}$ *, then* A *is closed in* $\mathbb{X}$ *.*

Definition 1.6.3 (interior)**.** *Given a subset* A *of a topological space* $\mathbb{X}$ *, the* ***interior*** *of* A *is defined as the union of all open sets contained in* A *. Denoted by* $Int(A)$*.*

Definition 1.6.4 (closure)**.** *Given a subset* A *of a topological space* $\mathbb{X}$ *, the* ***closure*** *of* A *is defined as the intersection of all closed sets containing* A *. Denoted by* $Cl(A)$ *or* $\overline{A}$

Theorem 1.6.4. [24][25] *Let* $\mathbb{Y}$ *be a subspace of a topological space* $\mathbb{X}$ *; let* A *be a subset of* $\mathbb{X}$ *. Let* $\overline{A}$ *denote the closure of* A *in* $\mathbb{X}$ *. Then the closure of* A *in* $\mathbb{Y}$ *equals* $\overline{A} \cap \mathbb{Y}$

Definition 1.6.5 (intersect)**.** *We say that a set* A ***intersects*** B *if* $A \cap B$ *is not empty.*

Theorem 1.6.5. *Let* A *be a subset of the topological space* $\mathbb{X}$

1. *The* $x \in \overline{A}$ *if and only if every open set* U *containing* x *intersect* A *.*

2. *Supposing the topology of* $\mathbb{X}$ *is given by a basis, then* $x \in \overline{A}$ *if and only if every basis element* B *containing* x *intersects* A

Proof. There are only two types of closed set U in $\mathbb{X}$:

1. $U \supseteq \overline{A}$

2. $U \cap A \neq A$

Thus, there are only two types of open set U in $\mathbb{X}$ respectively.

1. U does not intersects A .

2. $U \cap \overline{A} \neq \emptyset$

[23] As the proof is similar to the case in the open set, so we omit the proof here.

[24] We omit the proof of this lemma as it is obvious.

[25] As the closure of A in $\mathbb{X}$ and the closure A in $\mathbb{Y}$ will sometimes be different. We always use $\overline{A}$ to denote the closure of A in $\mathbb{X}$

1. If $x \in \overline{A}$, then every open set containing x is the open set of second type, thus every open set containing x intersects A

   If every open set containing x intersect $\mathbb{A}$, suppose $x \notin \overline{A}$. Then $\mathbb{X} - \overline{A}$ is a open set containing x, however, it does not intersects A . Thus, $x \in \overline{A}$.

2. If $x \in \overline{A}$, as every basis element of $\mathbb{X}$ is a open set, thus every basis element containing x intersects $\mathbb{A}$

   If every open set containing x intersect $\mathbb{A}$, suppose $x \notin \overline{A}$.

   As every open sets can be represented by union of basis. Let

   $$\mathbb{X} - \overline{A} = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_1' \cup B_2' \cup B_3' \cup \ldots$$

   where B are bases containing x , and B' are bases that does not contain x .

   Thus,

   $$x \in B_1 \cup B_2 \cup B_3 \cup \cdots \subseteq \mathbb{X} - \overline{A}$$

   Then $B_1 \cup B_2 \cup B_3 \cup \ldots$ that is a open set can be generated by all the bases containing x , however, that does not intersects A . So, $x \in \overline{A}$.

□

Definition 1.6.6 (neighbourhood). [26] *If we say U is a neighbourhood of x in $\mathbb{X}$, then U is an open set in $\mathbb{X}$ containing x*

Definition 1.6.7 (limit point, point of accumulation, cluster point). [27] *If A is a subset of topological space $\mathbb{X}$ .We say that x is a limit point of A if and only if every open sets containing x intersects A with some points other than x .*

This condition is also equivalent to the condition that if x is a limit point of A if and only if $x \in \overline{A - \{x\}}$

Theorem 1.6.6. [28]*Let A be a subset of topological space $\mathbb{X}$; let A' be the set of all limit points of A . Then*

$$\overline{A} = A \cup A'$$

Corollary 1.6.1. [29]*A subset of a topological space is closed if and only if it contains all its limit point.*

Definition 1.6.8 (converge). [30] *We say that a sequence of $x_1, x_2, x_3 \ldots$ **converge** to x . When for every neighbourhood U of x , there exists a positive integer N , such that for all $n > N$, $x_n \in U$.*

Definition 1.6.9 (Hausdorff space). *A topological space is called a **Hausdorff space**, if for every distinct x_1 , x_2 in $\mathbb{X}$, there exists disjoint neighbourhood of U_1 , U_2 of x_1 , x_2 in $\mathbb{X}$.*

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[26]Some other mathematicians use neighbourhood to say that U merely contains an open set containing x . The book does not give a formal definition for the word merely, and I am not sure either.

[27]Note that, x may belong to A or not, this does not matter.

[28]We omit the proof of this lemma as it is obvious.

[29]We omit the proof of this lemma as it is obvious.

[30]In real line, a sequence can not converge to multiple points, but for an arbitrary topological space, this is possible.

Theorem 1.6.7. [31][32] *Every finite point set in a Hausdorff space $\mathbb{X}$ is closed.*

Proof. Let A be a finite point set in a Hausdorff space $\mathbb{X}$.

Suppose A only have one element. Then for every $x \in \mathbb{X} - A$, there exists a neighbourhood of x that does not intersect with A. So A is closed.

Suppose A is a closed finite point set. We take $x_0 \in \mathbb{X} - A$. As finite union of closed set is closed, $A \cup \{x_0\}$ is closed.

Then, from induction, all finite point set in a Hausdorff space is closed. □

Theorem 1.6.8. *If $\mathbb{X}$ is a Hausdorff space, then a sequence of points in $\mathbb{X}$ converges to at most one point.*

Proof. Suppose that the following sequence

$$x_1, x_2, x_3 \dots$$

Converge to more than one points say

$$y_1, y_2, y_3 \cdots$$

Then there exists

$$n_1, n_2, n_3 \dots, U_1, U_2, U_3 \dots$$

Such that for $n > n_i$

$$x_n \in U_i, y_i \in U_i$$

If we take disjoint U_1, U_2 which is possible as this is a Hausdorff space.

Then the previews condition does not stand. So, every sequence of points in a Hausdorff space can only converge to at most one point. □

Definition 1.6.10 (limit)**.** *If a sequence x_n of points in Hausdorff space converge to the point x, we denote this by $x_n \to x$ and we say the* ***limit*** *of x_n is x.*

Definition 1.6.11 (T_1 axiom)**.** *The condition that all finite point set of a topological space is closed is called T_1* ***axiom****.*

Theorem 1.6.9. *Let $\mathbb{X}$ be a space satisfying the T_1 axiom; let A be a subset of $\mathbb{X}$. Then the point x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.*

Proof. If every neighbourhood of x contains infinitely many point of A. Than every neighbourhood of x intersect with A with infinite element other than x, then x is a limit point of A.

If x is a limit point of A. Suppose that there exists a open set U containing x and intersect with A for finite many points. Let

$$U' = U \cap (A - x)$$

Then, $x \notin U'$. Let

$$U'' = U - U'$$

[31]This implies that a sequence in a Hausdorff space cannot converge to multiple points. The following theorem prove this.

[32]The condition every finite point set is closed is weaker than the Hausdorff space condition. For instance, the finite complement topology of $\mathbb{R}$ met the condition of finite point set. However it is not a Hausdorff space.

Then U'' is open as U' is a finite point set and

$$U'' = U - U' = U \cap (\mathbb{X} - U')$$

Also, $x \in U''$. Thus, U'' is a open set containing x that only intersect A with x or do not intersect A. This is a contradiction of x is a limit point. Thus there does not exists a open set U containing x and intersect with A for finite many points. □

Theorem 1.6.10. [33] *Every simply ordered set is a Hausdorff space in order topology.*

Theorem 1.6.11. [34] *The product of two Hausdorff space is a Hausdorff space.*

Theorem 1.6.12. [35] *A subspace of a Hausdorff space is a Hausdorff space.*

1.6.1 Exercise

1. Give an counter example why $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$ dose not hold.

   Proof. Consider the X be the K-topology on the real line.
   Let

$$\begin{aligned} A_n &= (\frac{1}{n+1}, \frac{1}{n}), n \in \mathbb{Z}_+ \\ A &= \cup A_n \end{aligned}$$

   Then

$$\begin{aligned} \overline{A_n} &= [\frac{1}{n+1}, \frac{1}{n}] \\ \cup \overline{A_n} &= (0, 1] \end{aligned}$$

   However, as every neighbourhood of 0 intersect $\cup A_\alpha$. $0 \in \overline{\cup A_\alpha}$.
   Thus, $\overline{\cup A_\alpha} \neq \cup \overline{A_\alpha}$ □

2. Prove that

$$\overline{A - B} \supseteq \overline{A} - \overline{B}$$

   Proof. If $x \in \overline{A} - \overline{B}$. Then

$$x \in \overline{A}, x \notin \overline{B}$$

   .

   Thus for open set U containing x

$$\begin{aligned} \exists \quad & U_1 \cap B = \emptyset \\ \forall \quad & U \cap A \neq \emptyset \end{aligned}$$

[33] We omit the proof of this lemma as it is obvious.
[34] We omit the proof of this lemma as it is obvious.
[35] We omit the proof of this lemma as it is obvious.

Suppose that $x \notin \overline{A-B}$. Then

$$\exists U_0 \cap (A-B) = \emptyset$$

Thus,

$$U_0 \cap A \subseteq B$$

Thus,

$$\begin{aligned} U_1 \cap B &= \emptyset \\ U_1 \cap U_0 \cap A &= \emptyset \end{aligned}$$

As $U_1 \cap U_0$ is an open set containing x , so there is contradiction with $x \in \overline{A}$. Thus $x \in \overline{A-B}$. □

3. A ***diagonal*** is a subset $\Delta = \{x \times x | x \in \mathbb{X}\}$ of the product topology $\mathbb{X} \times \mathbb{X}$ where $\mathbb{X}$ is a topological space. Show that the diagonal is closed in $\mathbb{X} \times \mathbb{X}$ if and only if $\mathbb{X}$ is a Hausdorff space.

   Proof. If $\mathbb{X}$ is a Hausdorff space. For every element $x \times y$ of $\mathbb{X} \times \mathbb{X}$ that not in Δ . We take disjoint set U_x, U_y where $x \in U_x, y \in U_y$. Then $\mathbb{X} \times \mathbb{X} - \Delta = \cup_{x \neq y} U_x \times U_y$. Where $\cup_{x \neq y} U_x \times U_y$ is an open set. Thus Δ is a closed set.

   Conversely, if Δ is a closed set, suppose that $\mathbb{X}$ is not a Hausdorff space. Then there exists distinct x, y such that every neighbourhood of x and y intersect. Let $\mathbb{B}$ be a basis of topology of $\mathbb{X}$. Then $x \times y \in \mathbb{X} \times \mathbb{X} - \Delta$. However we cannot find $B_1, B_2 \in \mathbb{B}, x \times y \in B_1 \times B_2 \subset \mathbb{X} \times \mathbb{X} - \Delta$. Then Δ is not a closed set. So there is a contradiction, then $\mathbb{X}$ must be a Hausdorff space. □

4. Prove that T_1 axiom is equivalent to the condition such that for every distinct pair x, y of $\mathbb{X}$, there exists neighbourhood of x does not contain y .

   Proof. First if T_1 axiom hold, then for every pair x, y , the neighbourhood $\mathbb{X} - \{y\}$ of x does not contain y , so the second condition hold.

   Conversely, if the second condition hold. Suppose that we can find a finite points set say $\{x_1, x_2, x_3 \dots\}$, then there must exists $x \in \{x_1, x_2, x_3 \dots\}$ such that the set $\{x\}$ is not closed. Then $\overline{\{x\}} - \{x\} \neq \emptyset$. Let $y \in \overline{\{x\}} - \{x\}$, then every neighbourhood of y must contain x , this is a contradiction to the second condition, so the T_1 axiom must hold. □

5. If $A \subseteq \mathbb{X}$, we define the ***boundary*** of A by the equation

   $$\mathrm{Bd}A = \overline{A} \cap \overline{\mathbb{X} - A}$$

   (a) Show that $\mathrm{Int}A$ and $\mathrm{Bd}A$ are disjoint and $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$.

   Proof. For every $x \in \mathrm{Bd}A$, every open set contain x must intersect A and $\mathbb{X} - A$ so, there is no open set U contain x , $U \subseteq A$.
   For every $x' \in \mathrm{Int}A$, there exists $U' \subseteq A$, so $\mathrm{Bd}A$ and $\mathrm{Int}A$ are disjoint sets.
   For every $x \in \overline{A}$, $x \in \mathrm{Bd}A$ or $x \notin \mathrm{Bd}A$. We discuss the condition that $x \notin \mathrm{Bd}A$.

Then $x \notin \overline{\mathbb{X} - A}$, then there exists a open set U containing x , that does not intersect with $\mathbb{X} - A$. Thus $U \subseteq A$, thus $x \in \mathrm{Int}A$. So $\overline{A} \subseteq \mathrm{Int}A \cup \mathrm{Bd}A$.
Then, $\mathrm{Bd}A \subseteq \overline{A}$, $\mathrm{Int}A \subseteq A \subseteq \overline{A}$. Thus, $\overline{A} \supseteq \mathrm{Int}A \cup \mathrm{Bd}A$
So, $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$ □

(b) Show that $\mathrm{Bd}A = \emptyset$ if and only if A is both open and closed.

Proof. So, $\mathrm{Int}A = \overline{A}$, then $\mathrm{Bd}A = \emptyset$ follows directly from $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$. □

(c) Show that U is open if and only if $\mathrm{Bd}U = \overline{U} - U$.

Proof. Suppose U is open. Then $\overline{\mathbb{X} - U} = \mathbb{X} - U$. Then for every $x \in U$, $x \notin \mathbb{X} - U, x \notin \overline{\mathbb{X} - U}$. Thus $\overline{U} \cap \overline{\mathbb{X} - U} = \overline{U} - U$.
Conversely, suppose $\mathrm{Bd}U = \overline{U} - U$. Then for every $x \in U$, $x \notin \mathrm{Bd}U$. Then as $\overline{U} = \mathrm{Int}U \cup \mathrm{Bd}U$, $x \in \mathrm{Int}U$. So $\mathrm{Int}U \supseteq U$. Thus $U = \mathrm{Int}U$. Thus, U is open. □