Topology Note

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Chapter 1

Topology Spaces and Continuous Function

1.1 Basic Definition of Topology

Definition 1.1.1 (topology). *A **topology** on a set $\mathbb{X}$ is a collection $\mathbb{T}$ of subsets of $\mathbb{X}$ having the following properties:*

- *$\emptyset$ and $\mathbb{X}$ are in $\mathbb{T}$*
- *The union of the elements of any sub collection of $\mathbb{T}$ is in $\mathbb{T}$*
- *The intersection of the elements of any **finite** sub collection of $\mathbb{T}$ is in $\mathbb{T}$*

Definition 1.1.2 (topology space). *A **topological space** is a set $\mathbb{X}$ for which a topology $\mathbb{T}$ has been specified.*

Definition 1.1.3 (open set). *A **open set** $\mathbb{U}$ is a subset of $\mathbb{X}$ that belongs to a topology $\mathbb{T}$ of $\mathbb{X}$.*

Definition 1.1.4 (open sets). *A topology can also be called a **open sets***

Definition 1.1.5 (discrete topology). *The set of all subsets of a set $\mathbb{X}$ formed a topology called **discrete topology***

Definition 1.1.6 (trivial topology). *The set consisting the set $\mathbb{X}$ and $\emptyset$ only formed a topology of $\mathbb{X}$ called **trivial topology***

Definition 1.1.7 (finite complement topology). *Let $\mathbb{X}$ be a set. Let $\mathbb{T}_f$ be the collection of all subsets $\mathbb{U}$ of $\mathbb{X}$ such that $\mathbb{X} - \mathbb{U}$ either if a **finite** [1] of is all of $\mathbb{X}$. Then $\mathbb{T}_f$ is a topology on $\mathbb{X}$, called the **finite complement topology**.*

[1] The set $\mathbb{U}$ can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, $\mathbb{U}$ will not be a topology.

Definition 1.1.8 (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable)**.** *Let $\mathbb{T}$ and $\mathbb{T}'$ be two topology on a given set $\mathbb{X}$. If $\mathbb{T}$ is a subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **finer** or **larger** than $\mathbb{T}$. If $\mathbb{T}$ is a proper subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **strictly finer** or **strictly larger** than $\mathbb{T}$. We also say that $\mathbb{T}$ is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than $\mathbb{T}'$. We say that $\mathbb{T}$ and $\mathbb{T}'$ is **comparable** if either $\mathbb{T}$ is a subset of $\mathbb{T}'$ or $\mathbb{T}'$ is a subset of $\mathbb{T}$.*

1.2 Basis for a Topology

Definition 1.2.1 (basis)**.** *If $\mathbb{X}$ is a set, a **basis** for a topology on $\mathbb{X}$ is a collection $\mathbb{B}$ of subsets of $\mathbb{X}$ (called **basis elements**) such that:*

- *For each $x \in \mathbb{X}$, there is at least one basis element B containing x*
- *If x belongs to the intersection of two basis elements B_1 and B_2, then there is another element $x \in B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$*

Definition 1.2.2 (topology generated by basis)**.** *Let $\mathbb{B}$ be a basis on $\mathbb{X}$. Let $\mathbb{U}$ be a set containing all subsets U of $\mathbb{X}$ such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such $\mathbb{U}$ formed a topology on $\mathbb{X}$, called **topology** $\mathbb{T}$ **generated by** $\mathbb{B}$*

Lemma 1.2.1. *Let $\mathbb{X}$ be a set. Let $\mathbb{B}$ be a basis for a topology $\mathbb{T}$ on $\mathbb{X}$. Then $\mathbb{T}$ equals to the set of all possible unions of elements of $\mathbb{B}$.*

Proof. Let set $\mathbb{U}$ be the set of all possible unions of elements of $\mathbb{B}$. For any $U \in \mathbb{U}$. $U = \cup B$ [2] for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in U$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, $\mathbb{U}$ equals to $\mathbb{T}$. □

Lemma 1.2.2. [3] *Let $\mathbb{X}$ be a topological space. Suppose that $\mathbb{C}$ is a collection of open sets of $\mathbb{X}$ such that for each open set U of $\mathbb{X}$ and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq C$. Then $\mathbb{C}$ is a basis for the topology of $\mathbb{X}$.*

Lemma 1.2.3. [4] *Let $\mathbb{B}$ and $\mathbb{B}'$ be basis for the topologies $\mathbb{T}$ and $\mathbb{T}'$, respectively, on $\mathbb{X}$. Then the following are equivalent:*

- *$\mathbb{T}'$ is finer than $\mathbb{T}$*
- *For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing X, there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.*

[2] Note that this expression may not be unique.
[3] We omit the proof of this lemma as it is obvious.
[4] We omit the proof of this lemma as it is obvious.

Definition 1.2.3 (standard topology on the real line)**.** *Let be* $\mathbb{B} = \{B|B = \{x|a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$*.* $\mathbb{B}$ *formed a basis on real line. The topology generated by* $\mathbb{B}$ *is called the **standard topology on the real line*** [5] *.*

Definition 1.2.4 (lower limit topology on the real line)**.** *Let be* $\mathbb{B} = \{B|B = \{x|a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$*.* $\mathbb{B}$ *formed a basis on real line. The topology generated by* $\mathbb{B}$ *is called the **lower limit topology on the real line**. When* $\mathbb{R}$ *is given this topology,we denote it by* $\mathbb{R}_l$*.*

Definition 1.2.5 (K-topology on the real line)**.** *Let be* $\mathbb{B} = \{B|B = \{x|a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$*. Let* $K = \{x|x = \frac{1}{n}, n \in \mathbb{Z}_+\}$*.* $\mathbb{B} \cup \{B - K|B \in \mathbb{B}\}$ *formed a basis on real line. The topology generated by* $\mathbb{B}$ *is called the **K-topology on the real line**. When* $\mathbb{R}$ *is given this topology,we denote it by* $\mathbb{R}_{\mathbb{K}}$*.*

Lemma 1.2.4. [6] *The topologies* $\mathbb{R}_l$ *and* $\mathbb{R}_{\mathbb{K}}$ *is strictly finer than the standard topology on* $\mathbb{R}$ *.*

Lemma 1.2.5. *The topologies of* $\mathbb{R}_l$ *and* $\mathbb{R}_{\mathbb{K}}$ *is not comparable.*

Proof. Let $\mathbb{T}_l$ and $\mathbb{T}_{\mathbb{K}}$ be topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ respectively. Let $K = \{x|x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that $\mathbb{T}_l$ is not finer than $\mathbb{T}_{\mathbb{K}}$. Let $U = \{x|-1 < x < 1\} - K, x = 0$. If there exist $B = \{x|a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus B is not a subset of U.

Then we proof that $\mathbb{T}_{\mathbb{K}}$ is not finer than $\mathbb{T}_l$. Let $U' = \{x|a' \leq x < b'\}$. If there exist $B' = \{x|a'' < x < b''\} or \{x|a'' < x < b''\} - K$ such that $a' \in B \subseteq U$. Thus $a'' < a < b''$. Thus there exist c that $a'' < x < a, x \in B, x \notin U'$. Thus $B' \nsubseteq U'$.

Thus the topologies of $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is not comparable. □

Definition 1.2.6 (subbasis)**.** *A **subbasis*** $\mathbb{S}$ *for a topology on* $\mathbb{X}$ *is a collection of subsets of* $\mathbb{X}$ *whose union equals* $\mathbb{X}$ *. The **topology generated by the subbasis*** $\mathbb{S}$ *is defined to be the collection* $\mathbb{T}$ [7] *of all unions of finite intersections of elements of* $\mathbb{S}$ *.*

1.2.1 Exercise

1. Show that if $\mathbb{A}$ is a basis for a topology on $\mathbb{X}$, then the topology generated by $\mathbb{A}$ equals the intersection of all topologies on $\mathbb{X}$ that contain $\mathbb{A}$. Prove the same if $\mathbb{A}$ is a subbasis.

   Proof. As a subbasis is also a basis, we will directly prove the case of subbasis here.

[5] Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.
[6] We omit the proof of this lemma as it is obvious.
[7] It is obvious that $\mathbb{T}$ is a topology, we just omit the proof here.

Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain $\mathbb{A}$. Let $\mathbb{T}$ be the topology that $\mathbb{A}$ generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.[8]

First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

Thus, $\mathbb{T} = \mathbb{T}'$ □

1.3 The Order Topology

Definition 1.3.1 (interval)**.** *Let $\mathbb{X}$ is a set having a simple order relation $<$. Given elements a and b of $\mathbb{X}$ such that $a < b$, there are four subsets of $\mathbb{X}$ that are called **intervals** determined by a and b:*

- $(a, b) = \{x | a < x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $[a, b] = \{x | a \leq x \leq b\}$

*(a, b) is called an **open interval** on $\mathbb{X}$. $[a, b]$ is called an **closed interval** on $\mathbb{X}$. $(a, b]$ and $[a, b)$ is called **half-open intervals**.*

Definition 1.3.2 (order topology)**.** [9] *Let $\mathbb{X}$ be a set with a simple order relation; assume $\mathbb{X}$ has more than one element. Let $\mathbb{B}$ be the collection of all sets of the following types:*

- *All open intervals (a, b) in $\mathbb{X}$.*
- *All intervals of the form $[a_0, b)$, where a_0 is the smallest element(if exist) of $\mathbb{X}$.*
- *All intervals of the form $(a, b_0]$, where b_0 is the largest element(if exist) of $\mathbb{X}$.*

The collection $\mathbb{B}$ formed a basis for a topology on $\mathbb{X}$, which is called the order topology.

Definition 1.3.3 (ray)**.** [10][11] *If $\mathbb{X}$ is an ordered set, and a is an element of $\mathbb{X}$, there are four subsets of $\mathbb{X}$ that are called **rays** determined by a:*

- $(a, +\infty) = \{x | x > a\}$
- $(-\infty, a) = \{x | x < a\}$

[8] It is obvious that $\mathbb{T}'$ is also a topology, we just omit the proof here.

[9] The standard topology on $\mathbb{R}$ is an order topology derived from the usual order on $\mathbb{R}$.

[10] open rays are always open sets in the order topology

[11] the open rays also formed a subbasis of the order topology

- $[a, +\infty) = \{x|x \geq a\}$
- $(-\infty, a] = \{x|x \leq a\}$

*$(a, +\infty)$ and $(-\infty, a)$ are called **open rays**. $[a, +\infty)$ and $(-\infty, a]$ are called **closed rays**.*

1.4 The Product Topology

Definition 1.4.1 (product topology). *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces. The **product topology** on $\mathbb{X} \times \mathbb{Y}$ having a basis $\mathbb{B}$ containing all sets of the form $U \times V$, where U and V is open sets of $\mathbb{X}$ and $\mathbb{Y}$ respectively.*

Theorem 1.4.1. [12]*If $\mathbb{B}$ and $\mathbb{C}$ is basis for the topology of $\mathbb{X}$ and $\mathbb{Y}$ respectively, then the collection*

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} and C \in \mathbb{C}\}$$

is a basis for the topology of $\mathbb{X} \times \mathbb{Y}$

Definition 1.4.2 (projection). *Let $\pi_1 : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ be defined by the equation:*

$$\pi_1(x, y) = x$$

Let $\pi_2 : \mathbb{X} \times \mathbb{Y} \to \mathbb{Y}$ be defined by the equation:

$$\pi_1(x, y) = y$$

*The maps π_1 and π_2 are called the **projections** of $\mathbb{X} \times \mathbb{Y}$ onto its first and second factors, respectively.*

Theorem 1.4.2. [13]*The collection*

$$\mathbb{S} = \{\pi_1^{-1}(U) | U open in \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V open in \mathbb{Y}\}$$

is a subbasis for the product topology on $\mathbb{X} \times \mathbb{Y}$.

Definition 1.4.3 (box topology). *Let,*

$$\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n or \mathbb{X}_1 \times \mathbb{X}_2 \times \dots$$

In the first case, all the sets of the form $U_1 \times \cdots \times U_n$ where U_i is a open set of $\mathbb{X}_i$ form a basis.

In the second case, all the sets of the form $U_1 \times U_2 \times \dots$ where U_i is a open set of $\mathbb{X}_i$ also form a basis.

*Topology defined in this way was called a **box topology**.*

[12]We omit the proof of this theorem as it is obvious.

[13]We omit the proof of this theorem as it is obvious.

Definition 1.4.4 (product topology). [14] *Let,*

$$\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n or \mathbb{X}_1 \times \mathbb{X}_2 \times \ldots$$

Let π_i *be the projection function*[15] *that*

$$\pi_i : \mathbb{X} \to \mathbb{X}_i$$

And if $x \in \mathbb{X}$

$$\pi_i(x) = x_i$$

All the set of the form $\pi_i^{-1}(U_i)$ *where* i *is arbitrary and* U_i *is an open set of* $\mathbb{X}_i$ *, form a subbasis of* $\mathbb{X}$ *. The topology generated by this subbasis is called* ***product topology****. And* $\mathbb{X}$ *is called a* ***product space****.*

Definition 1.4.5 (J-tuple). *Let* J *be an index set*[16]*. Give a set* $\mathbb{X}$ *, a* ***J-tuple*** *is defined as a function* $x : J \to \mathbb{X}$ *. If* α *is an element of* J *,* $x(\alpha)$ *is often denoted by* x_α *and is called the* α*th* ***coordinate*** *of* x *. And the function* x *itself is often denoted by the symbol*

$$(x_\alpha)_{\alpha \in J}$$

The set of all J-tuples of elements of $\mathbb{X}$ *is often denoted by* $\mathbb{X}^J$ *.*

Definition 1.4.6 (cartesian product). *Let* $\{A_\alpha\}_{\alpha \in J}$ *be an indexed family of sets; let* $\mathbb{X} = \bigcup_{\alpha \in J} A_\alpha$ *. The* ***cartesian product*** *of this indexed family is denoted by*

$$\prod_{\alpha \in J} A_\alpha$$

And is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ *of elements of* $\mathbb{X}$ *such that* $x_\alpha \in A_\alpha$ *for each* $\alpha \in J$ *. That is, it is the set of all functions*

$$x : J \to \bigcup_{\alpha \in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ *for each* $\alpha \in J$ *.*

Theorem 1.4.3 (Comparison of the box and product topologies). [17] *The box topology on* $\prod \mathbb{X}_\alpha$ *has a basis all sets of the form* $\prod U_\alpha$ *where* U_α *is open in* X_α *for each* α *. The product topology on* $\prod \mathbb{X}_\alpha$ *has a basis all sets of the form* $\prod U_\alpha$ *where* U_α *is open in* X_α *for each* α *and* U_α *equals* $\mathbb{X}_\alpha$ *except for finitely many values of* α *.*

[14]In the finite case, the product topology and box topology are the same, however they differ when $\mathbb{X}$ is a infinite cartesian product.

[15]This is also called a ***projection mapping*** in a cartesian product.

[16]A index set was the set $\{1, \ldots, n\}$ or the set $\mathbb{Z}_+$.

[17]It is assumed that it is given product topology when considering $\prod X_\alpha$ unless it state specifically.

Theorem 1.4.4. [18]*Suppose the topology on each space* $\mathbb{X}_\alpha$ *is given by a basis* $\mathbb{X}_\alpha$ *. The collection of all sets of the form*

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathbb{B}_\alpha$ *form a basis for the box topology on* $\prod_{\alpha\in J} \mathbb{X}_\alpha$ *.*

The collection of all sets of the same form, where $B_\alpha \in \mathbb{B}_\alpha$ *for finitely many indices* α *and* $B_\alpha = \mathbb{X}_\alpha$ *for all the remaining indices, will form a basis for the product topology* $\prod_{\alpha\in J} \mathbb{X}_\alpha$ *.*

Theorem 1.4.5. [19]*Let* A_α *be a subspace of* $\mathbb{X}_\alpha$ *, for each* $\alpha \in J$ *. Then* $\prod A_\alpha$ *is a subspace of* $\prod \mathbb{X}_\alpha$ *if both products are given the box topology, or if both products are given the product topology.*

Theorem 1.4.6. [20]*If each space* $\mathbb{X}_\alpha$ *is a Hausdorff space, then* $\prod \mathbb{X}_\alpha$ *is a Hausdorff space in both the box and product topologies.*

Theorem 1.4.7. *Let* $\{\mathbb{X}_\alpha\}$ *be an indexed family of spaces; let* $A_\alpha \subseteq \mathbb{X}_\alpha$ *for each* α *. If* $\prod \mathbb{X}_\alpha$ *is given either the product or the box topology, then*

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$$

Proof. Let π_α represent the projection mapping.

Let x be an element of $\prod \mathbb{X}_\alpha$. Let V be an open set in $\prod \mathbb{X}_\alpha$ that containing x .

If $x \in \prod \overline{A_\alpha}$, then $\pi_\alpha(V)$ is a open set in $\mathbb{X}_\alpha$ that containing x_α . Thus $\pi_\alpha(V)$ intersect with A_α . Thus V intersect with $\prod A_\alpha$. Thus $x \in \overline{\prod A_\alpha}$.

If $x \in \overline{\prod A_\alpha}$. Let U_α be an open set of A_α that contain x_α . Let $V = \prod U_\beta$ such that $U_\beta = \begin{cases} \mathbb{X}_\beta, & \beta \neq \alpha \\ U_\alpha, & \beta = \alpha \end{cases}$. It is obvious that V is an open set that contain x . Thus V intersect with $\prod A_\alpha$. Thus U_α intersect with A_α . Thus $x \in \prod \overline{A_\alpha}$. □

Theorem 1.4.8. *Let* $f : A \to \prod_{\alpha\in J} \mathbb{X}_\alpha$ *be given by the equation*

$$f(a) = (f_\alpha(a))_{\alpha\in J}$$

where $f_\alpha : A \to \mathbb{X}_\alpha$ *for each* α *. Let* $\prod \mathbb{X}_\alpha$ *have the product topology. Then the function* f *is continuous if and only if each function* f_α *is continuous.*

[18]We omit the proof of this theorem as it is obvious.
[19]We omit the proof of this theorem as it is obvious.
[20]We omit the proof of this theorem as it is obvious.

Proof. Let π_α be the projection mapping

It is obvious that

$$f^{-1}(U) = \bigcap_{\alpha \in J} f_\alpha^{-1}(\pi_\alpha(U))$$

If f_α is continuous. Let V be a closed set of $\prod_{\alpha \in J} \mathbb{X}_\alpha$. Then $\pi_\alpha(V)$ is closed. Then $f^{-1}(V)$ is intersect of closed set. Thus $\pi_\alpha(V)$ is closed. So f is continuous.

If f is continuous. Let U_α be an open set of $\mathbb{X}_\alpha$. Let $U_\beta = \mathbb{X}_\beta$ if $\beta \neq \alpha$. Let $V = \prod_{\beta \in J} U_\beta$. It is obvious that V is an open set of $\prod \mathbb{X}_\alpha$. And

$$\begin{aligned} f^{-1}V &= \bigcap_{\alpha \in J} f_\alpha^{-1}(\pi_\alpha(U)) \\ &= f_\alpha^{-1}(U_\alpha) \end{aligned}$$

which is an open set in A . Thus f_α is continuous. □

1.5 The Subspace Topology

Definition 1.5.1 (subspace topology)**.** *Let $\mathbb{X}$ be a topological space with topology $\mathbb{T}$. If Y is a subset of $\mathbb{X}$, the collection $\mathbb{T}_Y = \{Y \cap U | U \in \mathbb{T}\}$ is a topology on Y , called the* ***subspace topology****.*

Y is also called a ***subspace*** *of $\mathbb{X}$*

Lemma 1.5.1. [21]*If $\mathbb{B}$ is basis for the topology of $\mathbb{X}$, Y is a subset of $\mathbb{X}$ then the collection*

$$\mathbb{B}_Y = \{B \cap Y | B \in \mathbb{B}\}$$

is a basis for the subspace topology on Y

Lemma 1.5.2. [22]*Let Y be a subspace of $\mathbb{X}$. If U is open in Y and Y is open in $\mathbb{X}$, then U is open in $\mathbb{X}$.*

Theorem 1.5.1. [23] *If A is a subspace of $\mathbb{X}$ and B is a subspace of $\mathbb{Y}$, then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$*

Proof. Let $\mathbb{B}_\mathbb{X}$ and $\mathbb{B}_\mathbb{Y}$ and $\mathbb{B}_{\mathbb{X}\mathbb{Y}}$ be basis of topology of $\mathbb{X}$ and $\mathbb{Y}$ and $\mathbb{X} \times \mathbb{Y}$ respectively. Let $\mathbb{B}'_\mathbb{X}$ and $\mathbb{B}'_\mathbb{Y}$ and $\mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ be basis of topology of A and A and $A \times B$ respectively. We will show that $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} = \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$. Thus, the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$.

[21] We omit the proof of this lemma as it is obvious.

[22] We omit the proof of this lemma as it is obvious.

[23] If $\mathbb{X}$ is an ordered set in the order topology, and $\mathbb{Y}$ is a subset of $\mathbb{X}$. The order relation, when restricted to $\mathbb{Y}$, makes $\mathbb{Y}$ into and ordered set. However, *the resulting order topology on $\mathbb{Y}$ need not be the same as the topology that $\mathbb{Y}$ inherits as a subspace of $\mathbb{X}$.*

First, every element in $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ can be represented by $B_A \cap A \times B_B \cap B = B_A \times B_B \cap A \times B \in \mathbb{B}'_{\mathbb{XY}}$ where $B_A \in \mathbb{B}'_{\mathbb{X}}, B_B \in \mathbb{B}'_{\mathbb{Y}}$. Thus $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}} \subseteq \mathbb{B}'_{\mathbb{XY}}$.

Next, we show that $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$. For any open set U in $\mathbb{X} \times \mathbb{Y}$, and $\forall x \in U \cap A \times B, \exists B_{\mathbb{X}} \times B_{\mathbb{Y}} \in \mathbb{B}_{\mathbb{XY}}, x \in B_{\mathbb{X}} \times B_{\mathbb{Y}} \subseteq \mathbb{X} \times \mathbb{Y}$. Thus $x \in B_{\mathbb{X}} \times B_{\mathbb{Y}} \cap A \times B \subseteq A \times B, B_{\mathbb{X}} \times B_{\mathbb{Y}} \cap A \times B \in \mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$. Thus $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$ .gi □

Definition 1.5.2 (ordered square)**.** *Let $I = [0, 1]$. The set $I \times I$ in the dictionary order* [24] *topology will be called* ***ordered square****, and denoted by I_o^2*

Definition 1.5.3 (convex)**.** *Given an ordered set $\mathbb{X}$, let us say that a subset $\mathbb{Y}$ of $\mathbb{X}$ is* ***convex*** *in $\mathbb{X}$ if for each pair of points $a < b$ of $\mathbb{Y}$, the entire interval (a, b) of points of $\mathbb{X}$ lies in $\mathbb{Y}$*

Theorem 1.5.2. [25] *Let $\mathbb{X}$ be an ordered set in the order topology. Let $\mathbb{Y}$ be a subset of $\mathbb{X}$ that is convex in $\mathbb{X}$. Then the order topology on $\mathbb{Y}$ is the same as the topology $\mathbb{Y}$ inherits as a subspace of $\mathbb{X}$.*

Proof. Consider the ray $(a, +\infty)$ in $\mathbb{X}$. If $a \in \mathbb{Y}$, then

$$(a, +\infty) \cap \mathbb{Y} = \{x | x \in \mathbb{Y} and x > a\}$$

This is an open ray of the ordered set of $\mathbb{Y}$. if $a \notin Y$, then a is either a lower bound on $\mathbb{Y}$ or an upper bound on $\mathbb{Y}$, since $\mathbb{Y}$ is convex. In the former case, the set $(a, +\infty) \cap \mathbb{Y}$ equals all of $\mathbb{Y}$, in the latter case, it is empty.

A similar remark shows that the intersection of the rat $(-\infty, a)$ with $\mathbb{Y}$ is either an open ray of $\mathbb{Y}$, or $\mathbb{Y}$ itself, or empty. Since the sets $(a, +\infty)\mathbb{Y}$ and $(-\infty, a) \cap \mathbb{Y}$ form a subbasis for the subspace topology on $\mathbb{Y}$, and since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of $\mathbb{Y}$ equals the intersection of an open ray of $\mathbb{X}$ with $\mathbb{Y}$, so it is open in the subspace topology on $\mathbb{Y}$. Since the open rays of $\mathbb{Y}$ are a subbasis for the order topology on $\mathbb{Y}$, this topology is contained in the subspace topology. □

[24]the dictionary means for $X_1, X_2 \in \mathbb{Y} = \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \ldots$ which:

$$\begin{aligned} X_1 &= (x_1, x_2, x_3 \ldots) \\ X_2 &= (x'_1, x'_2, x'_3 \ldots) \end{aligned}$$

$X_1 > X_2$ only when

$$\begin{aligned} &\exists k \in \mathbb{Z}_+, \forall i \in \mathbb{Z}_+, 0 < i < k \\ &x_i = x'_i \\ &x_k > x'_k \end{aligned}$$

[25]Given $\mathbb{X}$ is an ordered set in the order topology and $\mathbb{Y}$ is a subset of $\mathbb{X}$, we shall assume that $\mathbb{Y}$ *is given the subspace topology* unless we specifically state otherwise.

Exercise

1. A map $f : \mathbb{X} \to \mathbb{Y}$ is said to be a ***open map*** if for every open set $U \subseteq \mathbb{X}$, the set $f(U)$ is open in $\mathbb{Y}$. Show that $\pi : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ is open map.

   Proof. An open set in $\mathbb{X} \times \mathbb{Y}$ can be represented by
   $$\cup(U_i \times U_i')$$
   where U_i, U_i' are open sets in $\mathbb{X}$, $\mathbb{Y}$, respectively.
   Also,
   $$\cup(U_i \times U_i') = \cup(U_i) \times \cup(U_i')$$
   Thus,
   $$\pi(\cup(U_i \times U_i')) = \cup(U_i)$$
   Thus, $\pi(U)$ is open in $\mathbb{X}$. □

2. Let $\mathbb{X}$ and $\mathbb{X}'$ denote a single set in the topologies $\mathbb{T}$ and $\mathbb{T}'$, respectively; let $\mathbb{Y}$ and $\mathbb{Y}'$ denote a single set in the topologies $\mathbb{U}$ and $\mathbb{U}'$, respectively. [26] Assume these sets are nonempty.
   (a) Show that if $\mathbb{T}' \supseteq \mathbb{T}$ and $\mathbb{U}' \supseteq \mathbb{U}$, then the product topologies $\mathbb{X}' \times \mathbb{Y}'$ is finer than the product topology on $\mathbb{X} \times \mathbb{Y}$.
   (b) Does the converse of the previous statement hold?

3. Show that the countable collection[27]
   $$\{(a, b) \times (c, d) | a < b, c < d, a \in \mathbb{Q}, b \in \mathbb{Q}, c \in \mathbb{Q}, d \in \mathbb{Q}\}$$
   is a basis for $\mathbb{R}^2$

   Proof. This is obvious if you prove that $(a, b) \times (c, d)$ is a rectangle in the $\mathbb{R}^2$ plane. □

4. Let $\mathbb{X}$ be an ordered set. If $\mathbb{Y}$ is a proper subset of $\mathbb{X}$ that is convex in $\mathbb{X}$ prove that $\mathbb{Y}$ may not be an interval or a ray in $\mathbb{X}$.

   Proof. Let $\mathbb{X} = \mathbb{R}^2$ with dictionary order. Then $Y = \{(x, y)| -1 \leq x \leq 1\}$ is convex in $\mathbb{X}$, however it is not an interval or a ray. □

   There is a false prove given by myself.

[26] what does $\mathbb{X}$, $\mathbb{X}'$, $\mathbb{Y}$, $\mathbb{Y}'$ really mean here?? I do not know, so I just put the exercise here without a proof.

[27] The prove of this set is countable is typically similar to Cantor's enumeration of a countable collection of countable sets.

Proof. Let $\mathbb{S}$ be a set that contain all intervals and rays of $\mathbb{Y}$. We define a partial order on $\mathbb{S}$ by inclusion. So if there is a chain in $\mathbb{S}$:

$$S_1 \subseteq S_2 \subseteq S_3 \dots$$

Let

$$S = S_1 \cup S_2 \cup S_3 \cup \dots$$

Thus, S is an upper bound of the chain.

Thus, by Zorn's Lemma, there is a maximal element of $\mathbb{S}$, say U , then we prove that $U = \mathbb{Y}$.

If $U \neq \mathbb{Y}$, then $\exists x, x \in \mathbb{Y} - U$.

If U is a ray say $(a, +\infty)$. Then $x < a$, thus $U \subseteq (x, +\infty) \subseteq \mathbb{B}$, then there is contradiction with the maximal element.

If U is an interval, the circumstance is similar with the proof of U is a ray.

Thus $\mathbb{Y}$ is a ray or an interval. □

However, there is issue with this proof, the set S does exists. However, it may not be an interval or ray, so it may not be contained in $\mathbb{S}$

1.6 Closed Sets and Limit Points

Definition 1.6.1 (closed)**.** [28] *A subset A of a topological space is said to be closed if the set $\mathbb{X} - A$ is open.*

Theorem 1.6.1. [29]*Let $\mathbb{X}$ be a topological space. Then the following conditions hold*

1. *$\emptyset$ and $\mathbb{X}$ are closed.*

2. *Arbitrary intersections of closed sets are closed*

3. *Finite unions of closed sets are closed*

Definition 1.6.2 (closed in)**.** *Let $\mathbb{X}$ be a topological space; let $\mathbb{Y}$ be a subspace of $\mathbb{X}$. We say that a set A is **closed in** $\mathbb{Y}$ if A is a subset of $\mathbb{Y}$ and A is closed in the subspace topology of $\mathbb{Y}$*

Theorem 1.6.2. *Let $\mathbb{Y}$ be a subspace of $\mathbb{X}$. Then a set A is closed in $\mathbb{Y}$ if and only if it equals the intersection of a closed set of $\mathbb{X}$ with $\mathbb{Y}$*

[28]A set can be open, or closed, or both, or neither

[29]We omit the proof of this theorem as it is obvious.

Proof. First we proof that if A is closed in $\mathbb{Y}$, then $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. As the origin topology form a surjective map to its subspace topology, there exists a B closed in $\mathbb{X}$ that $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $B \cap \mathbb{Y} = A$

Conversely, if $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. Then, $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $\mathbb{X} - B$ is open in $\mathbb{Y}$, $\mathbb{Y} - A$ is open in $\mathbb{Y}$. Then A is closed in $\mathbb{Y}$ □

Theorem 1.6.3. [30] *Let $\mathbb{Y}$ be a subspace of $\mathbb{X}$. If A is closed in $\mathbb{Y}$ and $\mathbb{Y}$ is closed in $\mathbb{X}$, then A is closed in $\mathbb{X}$.*

Definition 1.6.3 (interior)**.** *Given a subset A of a topological space $\mathbb{X}$, the **interior** of A is defined as the union of all open sets contained in A. Denoted by $Int(A)$.*

Definition 1.6.4 (closure)**.** *Given a subset A of a topological space $\mathbb{X}$, the **closure** of A is defined as the intersection of all closed sets containing A. Denoted by $Cl(A)$ or $\overline{A}$*

Theorem 1.6.4. [31][32] *Let $\mathbb{Y}$ be a subspace of a topological space $\mathbb{X}$; let A be a subset of $\mathbb{X}$. Let $\overline{A}$ denote the closure of A in $\mathbb{X}$. Then the closure of A in $\mathbb{Y}$ equals $\overline{A} \cap \mathbb{Y}$*

Definition 1.6.5 (intersect)**.** *We say that a set A **intersects** B if $A \cap B$ is not empty.*

Theorem 1.6.5. *Let A be a subset of the topological space $\mathbb{X}$*

1. *The $x \in \overline{A}$ if and only if every open set U containing x intersect A.*
2. *Supposing the topology of $\mathbb{X}$ is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A*

Proof. There are only two types of closed set U in $\mathbb{X}$:

1. $U \supseteq \overline{A}$
2. $U \cap A \neq A$

Thus, there are only two types of open set U in $\mathbb{X}$ respectively.

1. U does not intersects A.
2. $U \cap \overline{A} \neq \emptyset$

1. If $x \in \overline{A}$, then every open set containing x is the open set of second type, thus every open set containing x intersects A

   If every open set containing x intersect $\mathbb{A}$, suppose $x \notin \overline{A}$. Then $\mathbb{X} - \overline{A}$ is a open set containing x, however, it does not intersects A. Thus, $x \in \overline{A}$.

[30] As the proof is similar to the case in the open set, so we omit the proof here.

[31] We omit the proof of this theorem as it is obvious.

[32] As the closure of A in $\mathbb{X}$ and the closure A in $\mathbb{Y}$ will sometimes be different. We always use $\overline{A}$ to denote the closure of A in $\mathbb{X}$

2. If $x \in \overline{A}$, as every basis element of $\mathbb{X}$ is a open set, thus every basis element containing x intersects $\mathbb{A}$

   If every open set containing x intersect $\mathbb{A}$, suppose $x \notin \overline{A}$.

   As every open sets can be represented by union of basis. Let

   $$\mathbb{X} - \overline{A} = B_1 \cup B_2 \cup B_3 \cup \cdots \cup B_1' \cup B_2' \cup B_3' \cup \ldots$$

   where B are bases containing x , and B' are bases that does not contain x .

   Thus,

   $$x \in B_1 \cup B_2 \cup B_3 \cup \cdots \subseteq \mathbb{X} - \overline{A}$$

   Then $B_1 \cup B_2 \cup B_3 \cup \ldots$ that is a open set can be generated by all the bases containing x , however, that does not intersects A . So, $x \in \overline{A}$.

□

Definition 1.6.6 (neighbourhood)**.** [33] *If we say U is a neighbourhood of x in $\mathbb{X}$, then U is an open set in $\mathbb{X}$ containing x*

Definition 1.6.7 (limit point, point of accumulation, cluster point)**.** [34] *If A is a subset of topological space $\mathbb{X}$ .We say that x is a limit point of A if and only if every open sets containing x intersects A with some points other than x .*

This condition is also equivalent to the condition that if x is a limit point of A if and only if $x \in \overline{A - \{x\}}$

Theorem 1.6.6. [35]*Let A be a subset of topological space $\mathbb{X}$; let A' be the set of all limit points of A . Then*

$$\overline{A} = A \cup A'$$

Corollary 1.6.1. [36]*A subset of a topological space is closed if and only if it contains all its limit point.*

Definition 1.6.8 (converge)**.** [37] *We say that a sequence of $x_1, x_2, x_3 \ldots$ **converge** to x . When for every neighbourhood U of x , there exists a positive integer N , such that for all $n > N$, $x_n \in U$.*

Definition 1.6.9 (Hausdorff space)**.** *A topological space is called a **Hausdorff space**, if for every distinct x_1 , x_2 in $\mathbb{X}$, there exists disjoint neighbourhood of U_1 , U_2 of x_1 , x_2 in $\mathbb{X}$.*

[33]Some other mathematicians use neighbourhood to say that U merely contains an open set containing x . The book does not give a formal definition for the word merely, and I am not sure either.

[34]Note that, x may belong to A or not, this does not matter.

[35]We omit the proof of this theorem as it is obvious.

[36]We omit the proof of this corollary as it is obvious.

[37]In real line, a sequence can not converge to multiple points, but for an arbitrary topological space, this is possible.

Theorem 1.6.7. [38][39] *Every finite point set in a Hausdorff space $\mathbb{X}$ is closed.*

Proof. Let A be a finite point set in a Hausdorff space $\mathbb{X}$.

Suppose A only have one element. Then for every $x \in \mathbb{X} - A$, there exists a neighbourhood of x that does not intersect with A. So A is closed.

Suppose A is a closed finite point set. We take $x_0 \in \mathbb{X} - A$. As finite union of closed set is closed, $A \cup \{x_0\}$ is closed.

Then, from induction, all finite point set in a Hausdorff space is closed. □

Theorem 1.6.8. *If $\mathbb{X}$ is a Hausdorff space, then a sequence of points in $\mathbb{X}$ converges to at most one point.*

Proof. Suppose that the following sequence

$$x_1, x_2, x_3 \dots$$

Converge to more than one points say

$$y_1, y_2, y_3 \cdots$$

Then there exists

$$n_1, n_2, n_3 \dots, U_1, U_2, U_3 \dots$$

Such that for $n > n_i$

$$x_n \in U_i, y_i \in U_i$$

If we take disjoint U_1, U_2 which is possible as this is a Hausdorff space.

Then the previews condition does not stand. So, every sequence of points in a Hausdorff space can only converge to at most one point. □

Definition 1.6.10 (limit)**.** *If a sequence x_n of points in Hausdorff space converge to the point x, we denote this by $x_n \to x$ and we say the* ***limit*** *of x_n is x.*

Definition 1.6.11 (T_1 axiom)**.** *The condition that all finite point set of a topological space is closed is called T_1* ***axiom****.*

Theorem 1.6.9. *Let $\mathbb{X}$ be a space satisfying the T_1 axiom; let A be a subset of $\mathbb{X}$. Then the point x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A.*

Proof. If every neighbourhood of x contains infinitely many point of A. Than every neighbourhood of x intersect with A with infinite element other than x, then x is a limit point of A.

[38] This implies that a sequence in a Hausdorff space cannot converge to multiple points. The following theorem prove this.

[39] The condition every finite point set is closed is weaker than the Hausdorff space condition. For instance, the finite complement topology of $\mathbb{R}$ met the condition of finite point set. However it is not a Hausdorff space.

If x is a limit point of A . Suppose that there exists a open set U containing x and intersect with A for finite many points. Let

$$U' = U \cap (A - x)$$

Then, $x \notin U'$. Let

$$U'' = U - U'$$

Then U'' is open as U' is a finite point set and

$$U'' = U - U' = U \cap (\mathbb{X} - U')$$

Also, $x \in U''$. Thus, U'' is a open set containing x that only intersect A with x or do not intersect A . This is a contradiction of x is a limit point. Thus there does not exists a open set U containing x and intersect with A for finite many points. □

Theorem 1.6.10. [40] *Every simply ordered set is a Hausdorff space in order topology.*

Theorem 1.6.11. [41] *The product of two Hausdorff space is a Hausdorff space.*

Theorem 1.6.12. [42] *A subspace of a Hausdorff space is a Hausdorff space.*

1.6.1 Exercise

1. Give an counter example why $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$ dose not hold.

   Proof. Consider the X be the K-topology on the real line.
   Let

$$\begin{aligned} A_n &= (\frac{1}{n+1}, \frac{1}{n}), n \in \mathbb{Z}_+ \\ A &= \cup A_n \end{aligned}$$

   Then

$$\begin{aligned} \overline{A_n} &= [\frac{1}{n+1}, \frac{1}{n}] \\ \cup \overline{A_n} &= (0, 1] \end{aligned}$$

   However, as every neighbourhood of 0 intersect $\cup A_\alpha$. $0 \in \overline{\cup A_\alpha}$.
   Thus, $\overline{\cup A_\alpha} \neq \cup \overline{A_\alpha}$ □

[40] We omit the proof of this theorem as it is obvious.
[41] We omit the proof of this theorem as it is obvious.
[42] We omit the proof of this theorem as it is obvious.

2. Prove that
$$\overline{A-B} \supseteq \overline{A} - \overline{B}$$

   Proof. If $x \in \overline{A} - \overline{B}$. Then
$$x \in \overline{A}, x \notin \overline{B}$$
   .

   Thus for open set U containing x
$$\begin{aligned} \exists \quad U_1 \cap B = \emptyset \\ \forall \quad U \cap A \neq \emptyset \end{aligned}$$

   Suppose that $x \notin \overline{A-B}$. Then
$$\exists U_0 \cap (A-B) = \emptyset$$

   Thus,
$$U_0 \cap A \subseteq B$$

   Thus,
$$\begin{aligned} U_1 \cap B &= \emptyset \\ U_1 \cap U_0 \cap A &= \emptyset \end{aligned}$$

   As $U_1 \cap U_0$ is an open set containing x , so there is contradiction with $x \in \overline{A}$. Thus $x \in \overline{A-B}$. □

3. A ***diagonal*** is a subset $\Delta = \{x \times x | x \in \mathbb{X}\}$ of the product topology $\mathbb{X} \times \mathbb{X}$ where $\mathbb{X}$ is a topological space. Show that the diagonal is closed in $\mathbb{X} \times \mathbb{X}$ if and only if $\mathbb{X}$ is a Hausdorff space.

   Proof. If $\mathbb{X}$ is a Hausdorff space. For every element $x \times y$ of $\mathbb{X} \times \mathbb{X}$ that not in Δ . We take disjoint set U_x, U_y where $x \in U_x, y \in U_y$. Then $\mathbb{X} \times \mathbb{X} - \Delta = \cup_{x \neq y} U_x \times U_y$. Where $\cup_{x \neq y} U_x \times U_y$ is an open set. Thus Δ is a closed set.

   Conversely, if Δ is a closed set, suppose that $\mathbb{X}$ is not a Hausdorff space. Then there exists distinct x, y such that every neighbourhood of x and y intersect. Let $\mathbb{B}$ be a basis of topology of $\mathbb{X}$. Then $x \times y \in \mathbb{X} \times \mathbb{X} - \Delta$. However we cannot find $B_1, B_2 \in \mathbb{B}, x \times y \in B_1 \times B_2 \subset \mathbb{X} \times \mathbb{X} - \Delta$. Then Δ is not a closed set. So there is a contradiction, then $\mathbb{X}$ must be a Hausdorff space. □

4. Prove that T_1 axiom is equivalent to the condition such that for every distinct pair x, y of $\mathbb{X}$, there exists neighbourhood of x does not contain y .

   Proof. First if T_1 axiom hold, then for every pair x, y , the neighbourhood $\mathbb{X} - \{y\}$ of x does not contain y , so the second condition hold.

   Conversely, if the second condition hold. Suppose that we can find a finite points set say $\{x_1, x_2, x_3 \dots\}$, then there must exists $x \in \{x_1, x_2, x_3 \dots\}$ such that the set $\{x\}$ is not closed. Then $\overline{\{x\}} - \{x\} \neq \emptyset$. Let $y \in \overline{\{x\}} - \{x\}$, then every neighbourhood of y must contain x , this is a contradiction to the second condition, so the T_1 axiom must hold. □

5. If $A \subseteq \mathbb{X}$, we define the ***boundary*** of A by the equation

$$\mathrm{Bd}A = \overline{A} \cap \overline{\mathbb{X} - A}$$

   (a) Show that $\mathrm{Int}A$ and $\mathrm{Bd}A$ are disjoint and $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$.

   Proof. For every $x \in \mathrm{Bd}A$, every open set contain x must intersect A and $\mathbb{X} - A$ so, there is no open set U contain x , $U \subseteq A$.
   For every $x' \in \mathrm{Int}A$, there exists $U' \subseteq A$, so $\mathrm{Bd}A$ and $\mathrm{Int}A$ are disjoint sets.
   For every $x \in \overline{A}$, $x \in \mathrm{Bd}A$ or $x \notin \mathrm{Bd}A$. We discuss the condition that $x \notin \mathrm{Bd}A$.
   Then $x \notin \overline{\mathbb{X} - A}$, then there exists a open set U containing x , that does not intersect with $\mathbb{X} - A$. Thus $U \subseteq A$, thus $x \in \mathrm{Int}A$. So $\overline{A} \subseteq \mathrm{Int}A \cup \mathrm{Bd}A$.
   Then, $\mathrm{Bd}A \subseteq \overline{A}$, $\mathrm{Int}A \subseteq A \subseteq \overline{A}$. Thus, $\overline{A} \supseteq \mathrm{Int}A \cup \mathrm{Bd}A$
   So, $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$ □

   (b) Show that $\mathrm{Bd}A = \emptyset$ if and only if A is both open and closed.

   Proof. So, $\mathrm{Int}A = \overline{A}$, then $\mathrm{Bd}A = \emptyset$ follows directly from $\overline{A} = \mathrm{Int}A \cup \mathrm{Bd}A$. □

   (c) Show that U is open if and only if $\mathrm{Bd}U = \overline{U} - U$.

   Proof. Suppose U is open. Then $\overline{\mathbb{X} - U} = \mathbb{X} - U$. Then for every $x \in U$, $x \notin \mathbb{X} - U, x \notin \overline{\mathbb{X} - U}$. Thus $\overline{U} \cap \overline{\mathbb{X} - U} = \overline{U} - U$.
   Conversely, suppose $\mathrm{Bd}U = \overline{U} - U$. Then for every $x \in U$, $x \notin \mathrm{Bd}U$. Then as $\overline{U} = \mathrm{Int}U \cup \mathrm{Bd}U$, $x \in \mathrm{Int}U$. So $\mathrm{Int}U \supseteq U$. Thus $U = \mathrm{Int}U$. Thus, U is open. □

1.7 Continuous Function

Definition 1.7.1 (continuous). [43] *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces. A function $f : \mathbb{X} \to \mathbb{Y}$ is said to be* ***continuous*** *if for each open subset V of $\mathbb{Y}$, the set $f^{-1}(V)$ is an open subset of $\mathbb{X}$.*

Theorem 1.7.1. *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological spaces; let $f : \mathbb{X} \to \mathbb{Y}$. Then the following are equivalent.*

1. *f is continuous.*
2. *For every subset A of X, one has $f(\overline{A}) \subseteq \overline{f(A)}$.*
3. *For every closed set B of $\mathbb{Y}$, the set $f^{-1}(B)$ is closed in $\mathbb{X}$.*
4. *For each $x \in \mathbb{X}$ and each neighbourhood of V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subseteq V$.*

Proof.

$1 \Rightarrow 3$:
Let A be a open set in $\mathbb{Y}$. $f^{-1}(\mathbb{Y} - A) = \mathbb{X} - f^{-1}(A)$.

$3 \Rightarrow 1$:
Let A be a closed set in $\mathbb{Y}$. $f^{-1}(\mathbb{Y} - A) = \mathbb{X} - f^{-1}(A)$.

$1 \Rightarrow 2$:
For $x \in \overline{A}$, we take a open set $f(x) \in U \subseteq \mathbb{Y}$. Thus $x \in f^{-1}(U) \cap A \neq \emptyset$. Thus $U \cap f(A) \neq \emptyset$. So $f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$.

$2 \Rightarrow 3$:
Suppose f is not continuous. Then there must exists V, such that $f^{-1}(V) = U$ is not closed. Thus $\overline{U} \supset B = f^{-1}(A)$. Thus $f\overline{B} \supset A$. However $f(\overline{B}) \subseteq \overline{f(B)} = A$. There is a contradiction. So f must be continuous.

$1 \Rightarrow 4$:
For every neighbourhood V of $f(x)$, $f^{-1}(V)$ is a neighbourhood of x that $f(f^{-1}(V)) \subseteq V$.

$4 \Rightarrow 1$:
We take a open set V of $\mathbb{Y}$. Let S be the collection of all open set U in $\mathbb{X}$ such that $f(U) \subseteq V$. The set cannot be empty unless $f^{-1}(V) = \emptyset$. Let U_0 denote the union of all the element in S. We prove that $U_0 = f^{-1}(V)$.
For all element $x \in U_0$, $f(x) \in V$. Thus $U_0 \subseteq f^{-1}(V)$.

[43] As the continuity of a function is different as the topological spaces are different. So if we want to emphasis this fact, we say that f is continuous ***relative*** to specific topologies on $\mathbb{X}$ and $\mathbb{Y}$.

For all element $x \in f^{-1}(V)$. There is a U' such that $x \in U', f(U') \subseteq V$. This follows from the condition 4. Thus $U' \in S$. Thus $x \in U_0$. Thus $U_0 \subseteq f^{-1}(V)$. As U_0 is union of open set, U_0 is also open. Thus, $f^{-1}(V)$ is also open.

Thus f is continuous. □

Definition 1.7.2 (homeomorphism)**.** [44] *Let $\mathbb{X}$ and $\mathbb{Y}$ be topological space; let $f : \mathbb{X} \to \mathbb{Y}$ be a bijection. If both the function f and the inverse function*

$$f^{-1} : \mathbb{Y} \to \mathbb{X}$$

*are continuous, then f is called a **homeomorphism***

Definition 1.7.3 (topological imbedding)**.** *Suppose that $f : \mathbb{X} \to \mathbb{Y}$ is an injective continuous map, where $\mathbb{X}$ and $\mathbb{Y}$ are topological spaces. Let $\mathbb{Z}$ be the image set $f(\mathbb{X})$, considered as a subspace of $\mathbb{Y}$; then the function $f' : \mathbb{X} \to \mathbb{Z}$ obtained by restricting the range of f is bijective. If f' happens to be a homeomorphism of $\mathbb{X}$ with $\mathbb{Z}$, we say that the map $f : \mathbb{X} \to \mathbb{Y}$ is a **topological imbedding**, or simply an **imbedding**, of $\mathbb{X}$ in $\mathbb{Y}$.*

Theorem 1.7.2 (Rules for constructing continuous functions)**.** *Let $\mathbb{X}$, $\mathbb{Y}$, and $\mathbb{Z}$ be topological spaces.*

1. *(Constant function) If $f : \mathbb{X} \to \mathbb{Y}$ maps all of $\mathbb{X}$ into the single point y_0 of $\mathbb{Y}$, then f is continuous.*

2. *(Inclusion) If A is a subspace of $\mathbb{X}$, the inclusion function $j : A \to \mathbb{X}$ is continuous.*

3. *(Composites) If $f : \mathbb{X} \to \mathbb{Y}$ and $g : \mathbb{Y} \to \mathbb{Z}$ are continuous, then the map $g \circ f : \mathbb{X} \to \mathbb{Z}$ is continuous.*

4. *(Restricting the domain) If $f : \mathbb{X} \to \mathbb{Y}$ is continuous, and if A is a subspace of $\mathbb{X}$, then the restriction function $f|A : A \to \mathbb{Y}$ is continuous.*

5. *(Restricting or expanding the range) Let $f : \mathbb{X} \to \mathbb{Y}$ is continuous. Let $\mathbb{Z}$ be a subspace of $\mathbb{Y}$ containing the image $f(\mathbb{X})$, the function $h : \mathbb{X} \to \mathbb{Z}$ obtained by restricting the range of f is continuous. If $\mathbb{Z}$ is a space having $\mathbb{Y}$ as a subspace, then the function $h : \mathbb{X} \to \mathbb{Y}$ obtained by expanding the range of f is continuous.*

6. *(Local formulation of continuity) The map $f : \mathbb{X} \to \mathbb{Y}$ is continuous if $\mathbb{X}$ can be written as the union of open sets U_α such set $f|U_\alpha$ is continuous for each α*

Proof.

[44] A equivalent way to define homeomorphism, is that for any open subset U of $\mathbb{X}$, $f(U)$ is open if and only if U is open.

1. $f^{-1}(U)$ of any open set U is $\mathbb{X}$, thus f is continuous.
2. For every open subset U of $\mathbb{X}$, $j^{-1}(U) = U \cap A$ is continuous in A . Thus j is a continuous function.
3. For every open subset U of $\mathbb{Z}$, $f^{-1}(U)$ is open in $\mathbb{Y}$, and $g^{-1}(f^{-1}(U))$ is open in $\mathbb{X}$. Thus, $g \circ f$ is continuous
4. For every open subset U of $\mathbb{Y}$, $f^{-1}(U)$ is open in $\mathbb{X}$, thus $f^{-1}(U) \cap A$ is open in A . Thus the function $f|A$ is continuous.
5. If $\mathbb{Z}$ is a subspace of $\mathbb{Y}$, then every open subset of $\mathbb{Z}$ can be represented as $U \cap \mathbb{Z}$, where U is a open subset of $\mathbb{Y}$. Thus $h^{-1}(U \cap \mathbb{Z}) = g^{-1}(\mathbb{Z}) \cap g^{-1}(U) = \mathbb{X} \cap g^{-1}(U)$ which is a open subset of X , thus h is continuous.

   If $\mathbb{Y}$ is a subspace of $\mathbb{Z}$. Then we take a open subset U of $\mathbb{Z}$. $h^{-1}(U) = g^(-1)(U \cap \mathbb{Y})$ which is open in $\mathbb{X}$, thus h is continuous.
6. if $f|U_\alpha$ is continuous for each α . For every open subset U of $\mathbb{Y}$.

   $$U = \cup_\alpha (U_\alpha \cap U)$$

   where $U_\alpha \cap U$ is open both in U_α and in $\mathbb{Y}$.

   Thus,

   $$\begin{aligned} f^{-1}(U) &= f^{-1}(\cup_\alpha (U_\alpha \cap U)) \\ &= \cup_\alpha ((f|U_\alpha)^{-1}(U_\alpha \cap U)) \end{aligned}$$

   and each $(f|U_\alpha)^{-1}(U_\alpha \cap U)$ is open, thus $f^{-1}(U)$ is open.

$\square$

Theorem 1.7.3 (The pasting lemma)**.** [45] *Let $\mathbb{X} = A \cup B$, where A, B are closed in $\mathbb{X}$. Let $f : A \to \mathbb{Y}$ and $g : B \to \mathbb{Y}$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f, g combine to give a continuous function $h : \mathbb{X} \to \mathbb{Y}$, defined by setting $h(x) = f(x), x \in A$ and $h(x) = g(x), x \in B$.*

Theorem 1.7.4 (Maps into products)**.** [46] *Let $f : A \to \mathbb{X} \times \mathbb{Y}$ be given by the equation*

$$f(a) = (f_1(a), f_2(a))$$

Then, the function f is continuous if and only if the functions

$$f_1 : A \to \mathbb{X}, f_2 : A \to \mathbb{Y}$$

are continuous.

[45] The proof of this theorem is similar to the "Local formulation of continuity" condition of "Rules for constructing continuous functions", so we omit the proof here.

[46] The map f_1, f_2 are called the ***coordinate functions*** of f

Proof. Let π_1, π_2 be the projection function

$$\begin{aligned} \pi_1 &: \mathbb{X} \times \mathbb{Y} \to \mathbb{X} \\ \pi_2 &: \mathbb{X} \times \mathbb{Y} \to \mathbb{Y} \end{aligned}$$

We first proof that if U is an open subset of $\mathbb{X} \times \mathbb{Y}$,

$$f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$$

Let $x \times y \in U$, $f^{-1}(x \times y)$ contains all a such that $f(a) = x \times y$.
Then for any $a \in f^{-1}(x \times y)$, $a \in f_1^{-1}(\pi_1(x \times y)), a \in f_2^{-1}(\pi_2(x \times y))$.
Thus, $f^{-1}(x \times y) \subseteq f_1^{-1}(\pi_1(x \times y)) \cap f_2^{-1}(\pi_2(x \times y))$.
Thus $f^{-1}(U) \subseteq f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$.

Also, if $a \in f_1^{-1}(\pi_1(x \times y)), a \in f_2^{-1}(\pi_2(x \times y))$, $f_1(a) = x, f_2(a) = y$.
Thus $f(a) = x \times y$. Thus $a \in f^{-1}(x \times y)$.
Thus $f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$

Let U be any open subset of $\mathbb{X} \times \mathbb{Y}$

$$f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$$

Where $f_1^{-1}(\pi_1(U))$ and $f_2^{-1}(\pi_2(U))$ are both open set. Thus $f^{-1}(U)$ is open. □

1.7.1 Exercise

1. Let $\mathbb{Y}$ be an ordered set in the order topology. Let $f, g : \mathbb{X} \to \mathbb{Y}$ be continuous, show that the set $A \quad \{x | f(x) \le g(x)\}$ is closed in $\mathbb{X}$.

   Proof. We only need to proof $\mathbb{X} - A$ is open in $\mathbb{X}$. We take $x \in \mathbb{X} - A$. Thus $f(x) > g(x)$.

   Let $U_1, U2$ be the open set in $\mathbb{Y}$ that met the following demand

   $$\begin{aligned} &\forall y_1 \in U_1, y_2 \in U_2, y_1 > y_2 \\ &f(x) \in U_1, g_x \in U_2 \end{aligned}$$

   As $\mathbb{Y}$ is an ordered set, $U_1, U2$ must exist.

   Let $U = f^{-1}(U_1) \cap g^{-1}(U_2)$. It is obvious that U is a open set, and $x \in U$.

   Also, for any $y \in U$. $f(y) > g(y)$. Thus $U \subseteq A$. Thus A is an open set. □

2. Let $\{A_\alpha\}$ be a collection of subsets of $\mathbb{X}$; let $\mathbb{X} = \cup_\alpha A_\alpha$. Lef $f : \mathbb{X} \to \mathbb{Y}$; suppose that $f|A_\alpha$ is continuous for each α . An indexed family of sets $\{A_\alpha\}$ is said to be ***locally finite*** if each point x of $\mathbb{X}$ has a neighbourhood that intersect A_α for only finitely main values of α . Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

   Proof. For any closed subset U of $\mathbb{Y}$. Let

$$V = \cup f|A_\alpha(U)$$

   We prove that V is closed, so, f is continuous.

   To prove that V is closed, we prove that $\overline{V} = V$. That is for any $x \in \overline{V}$, we prove $x \in V$. For any neighbourhood B if x , let C_B denote the set that contain all α , such that $f|A_{\alpha(U)}$ intersect with B . As B intersect with V , C_B can not be empty.

   Let

$$\mathbb{C} = \{C_B | B \text{ be a neighbourhood of } x\}$$

   As $\{A_\alpha\}$ is locally definite, $\mathbb{C}$ contain at least one element with finite elements.

   Also

$$C_{B_1 \cap B_2} \subseteq C_{B_1} \cap C_{B_2}$$

   Let $\leq$ be a partial order on the $\mathbb{C}$. If $C_{B_1} \subseteq C_{B_2}$, we say that $C_{B_1} \geq C_{B_2}$.

   If there is chain in $\mathbb{C}$

$$C_{B_1} \leq C_{B_2} \dots$$

   Let C_{B_0} be a element of $\mathbb{C}$ with finite element. If $C_{B_0} \subseteq C_{B_1}, C_{B_0} \subseteq C_{B_2} \dots$. Then C_{B_0} is a upper bound of the chain.

   If C is not a subset of all element of the chain. Then we construct a new set say

$$D = \{C_{B_0 \cap B_1}, C_{B_0 \cap B_2} \dots\}$$

   Let

$$\mathbb{D} = \{C_{D_1 \cap D_2 \cap \dots} | C_{D_1}, C_{D_2} \dots \in D\}$$

   As C_{B_0} is a finite set, D is a finite set, $\mathbb{D}$ is also a finite set. Thus there must be a maximal element $E \in \mathbb{D}$ that is the subset of all element of $\mathbb{D}$. Then E is a subset of all element of the chain. Thus E is a upper bound of the chain.

   Thus, there must be a maximal element C_F of $\mathbb{C}$, that is a subset of all element of $\mathbb{C}$.

Let G be the set be the union of all element of C_F .

As C_F is finite, G is closed. And all neighbourhood of x intersect with G . Thus $x \in G$

As G is a subset of V , $x \in V$. So V is closed. And f is a continuous function on $\mathbb{X}$.

□

3. Let A be a subset of topological space $\mathbb{X}$, let $\mathbb{Y}$ be a Hausdorff space. Let $f : A \to \mathbb{Y}$ be a continuous function. Let $g : \overline{A} \to \mathbb{Y}$ also be a continuous function where $g(x) = f(x), x \in A$. Prove that g us uniquely determined by f .[47]

   Proof. Say g and h are two distinct function that met the demand.

   So there exist x_0 such that $g(x_0) \neq h(x_0)$.

   As $\mathbb{Y}$ is a Hausdorff space, so there exist adjoint open subset $g(x_0) \in U$ and $h(x_0) \in V$.

   Then $g^{-1}(U)$ and $h^{-1}(V)$ are both open subset of $\mathbb{X}$ that contain x_0 .

   If $g^{-1}(U) \cap h^{-1}(V) \cap A \neq \emptyset$. Then there exist $x_1 \in g^{-1}(U) \cap h^{-1}(V) \cap A$ such that $g(x_1) \in U$ and $h(x_1) \in V$ and $g(x_1) = h(x_1)$. However U and V are disjoint. So there is a contradiction.

   As $^{-1}(U) \cap h^{-1}(V)$ is a open subset contain x_0 . So $^{-1}(U) \cap h^{-1}(V)$ must intersect with A . So it is impossible that $g^{-1}(U) \cap h^{-1}(V) \cap A = \emptyset$.

   So $g = h$. □

1.8 Metric Topology

Definition 1.8.1 (metric)**.** *A* ***metric*** *on a set* $\mathbb{X}$ *is a function*

$$d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$$

having the following properties:

1. $d(x, y) \geq 0$ *for all* $x, y \in \mathbb{X}$ *; equality hold if and only if* $x = y$
2. $d(x, y) = d(y, x), \forall x, y \in \mathbb{X}$

[47] It is possible that g does not exist.
Let $\mathbb{X}$ be the real line with order topology. Let $\mathbb{Y}$ be $\{0, 1\}$.
Let $A = \mathbb{X} - \{0\}$.
Let,

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

So, it is obvious that f is a continuous function on $\mathbb{X}$. However g does not exist in this case.

3. *(Triangle Inequality)* $d(x,y)+d(y,z) \geq d(x,z), \forall x,y,z \in \mathbb{X}$

Given a metric d *on* $\mathbb{X}$ *, the number* $d(x,y)$ *is often called the* ***distance*** *between* x *and* y *in the metric* d *.*

Definition 1.8.2 (ϵ -ball centered at x)**.** [48] *Given metric* d *on a set* $\mathbb{X}$ *and* $\epsilon > 0$ *. The set*

$$B_d(x,\epsilon) = \{y|d(x,y) < \epsilon\}$$

is called ϵ *-ball centered at* x *.*

Definition 1.8.3 (metric topology)**.** *If* d *is a metric on the set* $\mathbb{X}$ *, then the collection of all* ϵ *-balls* $B_d(x,\epsilon)$ *, such that* $x \in \mathbb{X}$ *and* $\epsilon > 0$ *, is a basis for a topology on* $\mathbb{X}$ *, called the* ***metric topology*** *induced by* d *.*

Definition 1.8.4 (metrizable)**.** *If* $\mathbb{X}$ *is topological space,* $\mathbb{X}$ *is said to be* ***metrizable*** *if there exists a metric* d *on the set* $\mathbb{X}$ *that induces the topology of* $\mathbb{X}$ *. A* ***metric space*** *is a metrizable space* $\mathbb{X}$ *together with a specific metric* d *that gives the topology of* $\mathbb{X}$ *.*

Definition 1.8.5 (bounded)**.** *Let* $\mathbb{X}$ *be a metric space with metric* d *. A subset* A *of* $\mathbb{X}$ *is said to be* ***bounded*** *if there is some number* M *such that*

$$d(a_1,a_2) \leq M$$

for every pair a_1 *and* a_2 *if points of* A *.*

Definition 1.8.6 (diameter)**.** *Let* $\mathbb{X}$ *be a metric space with metric* d *. Let* A *be a bounded subset of* $\mathbb{X}$ *. Then* ***diameter*** *is defined to be*

$$\mathrm{diam}A = \sup\{d(a_1,a_2)|a_1,a_2 \in A\}$$

Theorem 1.8.1. *Let* $\mathbb{X}$ *be a metric space with metric* d *. Define* $\overline{d} : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ *by the equation*

$$\overline{d}(x,y) = min\{d(x,y),1\}$$

Then $\overline{d}$ *is a metric that induces the same topology as* d *.*

The metric $\overline{d}$ *is called the* ***standard bounded metric*** *corresponding to* d

Proof. It is obvious that $\overline{d}$ is a metric.

To prove that d and $\overline{d}$ induces the same topology, it is suffice to prove that for all $a \in X$ and $\epsilon > 0$ there exists $\{a_\alpha\}$ and $\{\epsilon_\alpha\}$ where $\epsilon_\alpha \leq 1$ such that

$$B_d(a,\epsilon) = \bigcup B_{\overline{d}}(a_\alpha,\epsilon_\alpha)$$

For every $x \in B_d(a,\epsilon)$ take $a_x = x$ and $\epsilon_x < min(\epsilon - d(a,x), 1)$. Then

$$B_d(a,\epsilon) \supseteq B_{\overline{d}}(a_x,\epsilon_x)$$

[48] When no confusion will arise, the metric d may be omit in $B_d(x,\epsilon)$

as for all $y \in B_{\overline{d}}(a_x, \epsilon_x)$

$$\begin{aligned} d(a,y) &\leq d(a,a_x) + d(a_x,y) \\ &< min(\epsilon - d(a,x), 1) + d(a,a_x) \\ &\leq \epsilon \end{aligned}$$

Thus

$$B_d(a,\epsilon) = \bigcup_{x \in B_d(a,\epsilon)} B_{\overline{d}}(a_x, \epsilon_x)$$

□

Definition 1.8.7 (norm). *Given $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$. The **norm** of x is defined by the equation*

$$||x|| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$$

Definition 1.8.8 (euclidean metric). *The euclidean metric d on $\mathbb{R}^n$ is defined by*

$$d(x,y) = ||x-y||$$

Definition 1.8.9 (square metric). *The square metric ρ on $\mathbb{R}^n$ is defined by*

$$\rho(x,y) = max\{|x_1 - y_1|, \ldots, |x_n - y_n|\}$$

Lemma 1.8.1. *Let d and d' be two metrics on the set $\mathbb{X}$; let $\mathbb{T}$ and $\mathbb{T}'$ be the topology induced by d and d' respectively. Then $\mathbb{T}'$ is finer than T if and only if for all $x \in \mathbb{X}$ and $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$$

Proof. If $\mathbb{T}'$ is finer than $\mathbb{T}$. Then for all $B_d(x,\epsilon)$ there exists a open set U that containing x such that $U \subseteq B_d(x,\epsilon)$. As $\{B_{d'}(x,\delta)\}$ is a basis of T', then there exists $B_{d'}(x,\delta) \subseteq U$ that containing x.

If for all $B_d(x,\epsilon)$, there exists $B_{d'}(x,\delta) \subseteq B_d(x,\epsilon)$. Then as $\{B_{d'}(x,\epsilon)\}$ and $\{B_d(x,\epsilon)\}$ are both basis, then $\mathbb{T}'$ is finer than T. □

Theorem 1.8.2. [49] *The topologies on $\mathbb{R}^n$ induced by the euclidean metric d and the square metric ρ are the same as the product topology on $\mathbb{R}^n$.*

Definition 1.8.10 (uniform metric, uniform topology). *Given an index set J, and given points $x = (x_\alpha)_{\alpha \in J}$ and $y = (y_\alpha)_{\alpha \in J}$ of $\mathbb{R}^J$, let us define a metric $\overline{\rho}$ on $\mathbb{R}^J$ by the equation*

$$\overline{\rho}(x,y) = \sup\{\overline{d}(x_\alpha, y_\alpha) | \alpha \in J\}$$

*where $\overline{d}$ is the standard bounded metric on $\mathbb{R}$. $\overline{\rho}$ is called the **uniform metric** on $\mathbb{R}^J$, and the topology it induces is called the **uniform topology***

[49] We omit the proof of this theorem as it is obvious.

Theorem 1.8.3. [50] *The uniform topology on $\mathbb{R}^J$ is finer than the product topology and is coarser than the box topology.*

Theorem 1.8.4. *Let $\overline{d}(a,b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$. If x nad y are two points of $\mathbb{R}^\omega$, define*

$$D(x,y) = \sup\left\{\frac{\overline{d}(x_i,y_i)}{i}\right\}$$

Then D is a metric that induces the product topology on $\mathbb{R}^\omega$

Proof. The properties of a metric are satisfied trivially except for the triangle inequality, which is proved by noting that for all i ,

$$\begin{aligned}\frac{\overline{d}(x_i,z_i)}{i} &\leq \frac{\overline{d}(x_i,y_i)}{i} + \frac{\overline{d}(y_i,z_i)}{i} \\ &\leq D(x,y) + D(y,z)\end{aligned}$$

so that

$$\sup\left\{\frac{\overline{d}(x_i,z_i)}{i}\right\} \leq D(x,y) + D(y,z)$$

The fact that D gives the product topology requires a little more work. First, let U be open in the metric topology and let $x \in U$; we find an open set V in the product topology such that $x \in V \supseteq U$. Choose an $\epsilon - ball$ $B_D(x,\epsilon)$ lying in U . Then choose N large enough that $\frac{1}{N} < \epsilon$. Finally, let V be the basis element for the product topology

$$V = (x_1 - \epsilon, x_1 + \epsilon) \times \cdots \times (x_N - \epsilon, x_N + \epsilon) \times R \times R \times \ldots$$

We assert that $V \in B_D(x,\epsilon)$: Given any y in $\mathbb{R}^\omega$

$$\frac{\overline{d}(x_i,y_i)}{i} \leq \frac{1}{N}, \forall i \geq N$$

Therefore,

$$D(x,y) \leq \max\left\{\frac{\overline{d}(x_1,y_1)}{1}, \ldots, \frac{\overline{d}(x_N,y_N)}{N}, \frac{1}{N}\right\}$$

If y is in V , this expression is less than ϵ , so that $V \subseteq B_D(x,\epsilon)$, as desired. Conversely, consider a basis element

$$U = \prod_{i \in \mathbb{Z}_+} U_i$$

[50] We omit the proof of this theorem as it is obvious.

for the product topology, where U_i is open in $\mathbb{R}$ for $i = \alpha_1, \ldots, \alpha_n$ and $U_i = \mathbb{R}$ for all other indices i . Given $x \in U$, we find an open set V of the metric topology such that $x \in V \supseteq U$. Choose an interval $(x_i - \epsilon_i, x_i + \epsilon_i)$ in $\mathbb{R}$ centered about x_i and lying in U_i for $i = \alpha_1, \ldots, \alpha_n$; choose each $\epsilon_i \leq 1$. Then define

$$\epsilon = \min\left\{\frac{\epsilon_i}{i} | i = \alpha_1, \ldots, \alpha_n\right\}$$

We assert that

$$x \in B_D(x, \epsilon) \subseteq U$$

Let y be a point of $B_D(x, \epsilon)$. Then for all i

$$\frac{\overline{d}(x_i, y_i)}{i} \leq D(x, y) < \epsilon$$

Now if $i = \alpha_1, \ldots, \alpha_n$, then $\epsilon \leq \frac{\epsilon_i}{i}$, so that $\overline{d}(x_i, y_i) < \epsilon_i \leq 1$; it follows that $|x_i - y_i| < \epsilon_i$. Therefore $y \in \prod U_i$, as desired.

□

Definition 1.8.11 (Hilbert Cube)**.** *The set*

$$H = \prod_{n \in \mathbb{Z}_+} [0, \frac{1}{n}]$$

is called ***Hilbert cube***

Definition 1.8.12 (l^2-topology)**.** *Let $\mathbb{X}$ be the subset of $\mathbb{R}^\omega$ consisting of all sequences x such that $\sum x_i^2$ converges.*

Then the formula

$$d(x, y) = \left[\sum_{i=1}^{\infty} (x_i - y_i)^2\right]^{\frac{1}{2}}$$

defines a metric on $\mathbb{X}$. The topology induced by d is called the l^2-topology.

Definition 1.8.13 (countable basis at point x)**.** *A space is said to be have* ***countable basis at point*** *x if there is a countable collection $\{U_n\}_{n \in \mathbb{Z}_+}$ of neighbourhoods of x such that any neighbourhood U of x contains at least on of the sets U_n . A space $\mathbb{X}$ that has a countable basis at each of its point is said to satisfy the* ***first countability axiom***

Theorem 1.8.5. *Let $f : \mathbf{X} \to \mathbf{Y}$ be metrizable with metric $d_{\mathbf{X}}$ and $d_{\mathbf{Y}}$, respectively. Then continuity of f is equivalent to the requirement that given $x \in \mathbb{X}$ and given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$d_{\mathbf{X}}(x, y) < \delta \implies d_{\mathbb{Y}}(f(x), f(y)) < \epsilon$$

Proof. Suppose f is continuous. Given x and ϵ , consider the set

$$f^{-1}(B(f(x),\epsilon))$$

which is open in $\mathbb{X}$ and contains the point x . It contains some δ -ball $B(x,\delta)$ centered at x . If y is in this δ -ball, then $f(y)$ is in this δ -ball as desired.

Conversely, suppose that the $\epsilon-\delta$ condition is satisfied. Let V be open in $\mathbb{Y}$; we show that $f^{-1}(V)$ is open in $\mathbb{X}$. Let x be a point of the set $f^{-1}(V)$. Since $f(x)\in V$ there is an ϵ -ball $B(f(x),\epsilon)$ centered at $f(x)$ and contained in V . By the $\epsilon-\delta$ condition, there exists a δ -ball centered at x such that $f(B(x,\delta))\subseteq B(f(x),\epsilon)$. Then $B(x,\delta)$ is a neighbourhood of x contained in $f^{-1}(V)$, so that $f^{-1}(V)$ is open, as desired. □

Lemma 1.8.2 (The sequence lemma)**.** [51]*Let $\mathbb{X}$ be a topological space; let $A\subseteq\mathbb{X}$ If there is a sequence of points of A converging to x , then $x\in\overline{A}$, the converse holds if $\mathbb{X}$ is metrizable.*

Theorem 1.8.6. [52]*Let $f:\mathbb{X}\to\mathbb{Y}$. If the function f is continuous, then for every convergent sequence $x_n\to x$, the sequence $f(x_n)$ converges to $f(x)$. The converse holds if $\mathbb{X}$ is metrizable.*

Lemma 1.8.3. [53]*The addition, subtraction, and multiplication operations are continuous functions from $\mathbb{R}\times\mathbb{R}$ into $\mathbb{R}$; and the quotient operation is continuous function from $\mathbb{R}\times(\mathbb{R}-\{0\})$ into $\mathbb{R}$.*

Theorem 1.8.7. [54]*If $\mathbb{X}$ is a topological space, and if $f,g:\mathbb{X}\to\mathbb{R}$ are continuous functions, then $f+g$, $f-g$ and $f\cdot g$ are continuous. If $g(x)\neq 0$ for all x , then $\frac{f}{g}$ is continuous.*

Definition 1.8.14 (converge uniformly)**.** *Let $f_n:\mathbb{X}\to\mathbb{Y}$ be a sequence of functions from the set $\mathbb{X}$ to the metric space $\mathbb{Y}$. Let d be the metric for $\mathbb{Y}$. We say that the sequence (f_n) **converges uniformly** to the function $f:\mathbb{X}\to\mathbb{Y}$ if given $\epsilon>0$, there exists an integer N such that*

$$d(f_n(x),f(x))<\epsilon$$

for all $n>N$ and all $x\in\mathbb{X}$

Theorem 1.8.8 (Uniform limit theorem)**.** *Let $f_n:\mathbb{X}\to\mathbb{Y}$ be a sequence of continuous functions from the topological space $\mathbb{X}$ to the metric space $\mathbb{Y}$. If (f_n) converges uniformly to f , then f is continuous.*

[51]We omit the proof of this lemma as it is obvious.
[52]We omit the proof of this theorem as it is obvious.
[53]We omit the proof of this lemma as it is obvious.
[54]We omit the proof of this theorem as it is obvious.

Definition 1.8.15 (isometric imbedding)**.** *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be metric spaces with metric* $d_{\mathbb{X}}$ *and* $d_{\mathbb{Y}}$ *, respectively. Let* $f : \mathbb{X} \to \mathbb{Y}$ *have the property that for every pair of points* x_1 *,* x_2 *of* $\mathbb{X}$ *, and*

$$d_{\mathbb{Y}}(f(x_1), f(x_2)) = d_{\mathbb{X}}(x_1, x_2)$$

f *is an topological imbedding and is called an* ***isometric imbedding*** *of* $\mathbb{X}$ *in* $\mathbb{Y}$