

Topology Note

Alex

July 17, 2022

Contents

1	Topology Spaces and Continuous Function	7
1.1	Basic Definition of Topology	7
1.2	Basis for a Topology	8
1.2.1	Exercise	9
1.3	The Order Topology	10
1.4	The Product Topology	11
1.5	The Subspace Topology	14
1.6	Closed Sets and Limit Points	17
1.6.1	Exercise	21
1.7	Continuous Function	24
1.7.1	Exercise	27

Definitions

B

basis, 8
boundary, 23
box topology, 11

C

cartesian product, 12
closed, 17
 closed in, 17
closure, 18
coarser, 8
 strictly coarser, 8
finer, 8
 strictly finer, 8
larger, 8
 strictly larger, 8
smaller, 8
 strictly smaller, 8
continuous, 24
 continuous relative to, 24
converge, 19
convex, 15
coordinate functions, 26

D

diagonal, 22
discrete topology, 7

F

finite complement topology, 7

H

Hausdorff space, 19
homeomorphism, 25

I

interior, 18
intersect, 18
interval, 10
 closed interval, 10
 half-open interval, 10

open interval, 10

J

J-tuple, 12

K

K-topology on \mathbb{R} , 9

L

limit, 20
cluster point, 19
limit point, 19
point of accumulation, 19
locally finite, 28
lower limit topology on \mathbb{R} , 9

N

neighbourhood, 19

O

open map, 16
open set, 7
open sets, 7
ordered square, 15
order topology, 10

P

product topology, 11
product space, 12
product topology, 12
projection, 11
projection mapping, 12

R

ray, 10
 closed ray, 10
 open ray, 10

S

standard topology on \mathbb{R} , 9
subbasis, 9

subspace, 14
subspace topology, 14

T

T_1 axiom, 20

topological imbedding, 25
topology, 7
topology generated by basis, 8
topology space, 7
trivial topology, 7

Theorems**M**

Maps into products, 26

R

Rules for constructing continuous functions, 25

T

The pasting lemma, 26

Chapter 1

Topology Spaces and Continuous Function

1.1 Basic Definition of Topology

Definition 1.1.1 (topology). A **topology** on a set \mathbb{X} is a collection \mathbb{T} of subsets of \mathbb{X} having the following properties:

- \emptyset and \mathbb{X} are in \mathbb{T}
- The union of the elements of any sub collection of \mathbb{T} is in \mathbb{T}
- The intersection of the elements of any **finite** sub collection of \mathbb{T} is in \mathbb{T}

Definition 1.1.2 (topology space). A **topological space** is a set \mathbb{X} for which a topology \mathbb{T} has been specified.

Definition 1.1.3 (open set). A **open set** \mathbb{U} is a subset of \mathbb{X} that belongs to a topology \mathbb{T} of \mathbb{X} .

Definition 1.1.4 (open sets). A topology can also be called a **open sets**

Definition 1.1.5 (discrete topology). The set of all subsets of a set \mathbb{X} formed a topology called **discrete topology**

Definition 1.1.6 (trivial topology). The set consisting the set \mathbb{X} and \emptyset only formed a topology of \mathbb{X} called **trivial topology**

Definition 1.1.7 (finite complement topology). Let \mathbb{X} be a set. Let \mathbb{T}_f be the collection of all subsets \mathbb{U} of \mathbb{X} such that $\mathbb{X} - \mathbb{U}$ either if a **finite**¹ or is all of \mathbb{X} . Then \mathbb{T}_f is a topology on \mathbb{X} , called the **finite complement topology**.

¹The set \mathbb{U} can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, \mathbb{U} will not be a topology.

Definition 1.1.8 (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable). Let \mathbb{T} and \mathbb{T}' be two topology on a given set \mathbb{X} . If \mathbb{T} is a subset of \mathbb{T}' , we say that \mathbb{T}' is **finer** or **larger** than \mathbb{T} . If \mathbb{T} is a proper subset of \mathbb{T}' , we say that \mathbb{T}' is **strictly finer** or **strictly larger** than \mathbb{T} . We also say that \mathbb{T} is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than \mathbb{T}' . We say that \mathbb{T} and \mathbb{T}' is **comparable** if either \mathbb{T} is a subset of \mathbb{T}' or \mathbb{T}' is a subset of \mathbb{T} .

1.2 Basis for a Topology

Definition 1.2.1 (basis). If \mathbb{X} is a set, a **basis** for a topology on \mathbb{X} is a collection \mathbb{B} of subsets of \mathbb{X} (called **basis elements**) such that:

- For each $x \in \mathbb{X}$, there is at least one basis element B containing x
- If x belongs to the intersection of two basis elements B_1 and B_2 , then there is another element $B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$

Definition 1.2.2 (topology generated by basis). Let \mathbb{B} be a basis on \mathbb{X} . Let \mathbb{U} be a set containing all subsets U of \mathbb{X} such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such \mathbb{U} formed a topology on \mathbb{X} , called **topology \mathbb{T} generated by \mathbb{B}**

Lemma 1.2.1. Let \mathbb{X} be a set. Let \mathbb{B} be a basis for a topology \mathbb{T} on \mathbb{X} . Then \mathbb{T} equals to the set of all possible unions of elements of \mathbb{B} .

Proof. Let set \mathbb{U} be the set of all possible unions of elements of \mathbb{B} . For any $U \in \mathbb{U}$. $U = \cup B$ ² for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in \mathbb{B}$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, \mathbb{U} equals to \mathbb{T} . □

Lemma 1.2.2.³ Let \mathbb{X} be a topological space. Suppose that \mathbb{C} is a collection of open sets of \mathbb{X} such that for each open set U of \mathbb{X} and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq U$. Then \mathbb{C} is a basis for the topology of \mathbb{X} .

Lemma 1.2.3.⁴ Let \mathbb{B} and \mathbb{B}' be basis for the topologies \mathbb{T} and \mathbb{T}' , respectively, on \mathbb{X} . Then the following are equivalent:

- \mathbb{T}' is finer than \mathbb{T}
- For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing x , there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.

²Note that this expression may not be unique.

³We omit the proof of this lemma as it is obvious.

⁴We omit the proof of this lemma as it is obvious.

Definition 1.2.3 (standard topology on the real line). Let $\mathbb{B} = \{B \mid B = \{x \mid a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. \mathbb{B} formed a basis on real line. The topology generated by \mathbb{B} is called the **standard topology on the real line**⁵.

Definition 1.2.4 (lower limit topology on the real line). Let $\mathbb{B} = \{B \mid B = \{x \mid a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. \mathbb{B} formed a basis on real line. The topology generated by \mathbb{B} is called the **lower limit topology on the real line**. When \mathbb{R} is given this topology, we denote it by \mathbb{R}_l .

Definition 1.2.5 (K-topology on the real line). Let $\mathbb{B} = \{B \mid B = \{x \mid a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. Let $K = \{x \mid x = \frac{1}{n}, n \in \mathbb{Z}_+\}$. $\mathbb{B} \cup \{B - K \mid B \in \mathbb{B}\}$ formed a basis on real line. The topology generated by \mathbb{B} is called the **K-topology on the real line**. When \mathbb{R} is given this topology, we denote it by \mathbb{R}_K .

Lemma 1.2.4.⁶ The topologies \mathbb{R}_l and \mathbb{R}_K is strictly finer than the standard topology on \mathbb{R} .

Lemma 1.2.5. The topologies of \mathbb{R}_l and \mathbb{R}_K is not comparable.

Proof. Let \mathbb{T}_l and \mathbb{T}_K be topologies of \mathbb{R}_l and \mathbb{R}_K respectively. Let $K = \{x \mid x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that \mathbb{T}_l is not finer than \mathbb{T}_K . Let $U = \{x \mid -1 < x < 1\} - K, x = 0$. If there exist $B = \{x \mid a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus B is not a subset of U .

Then we proof that \mathbb{T}_K is not finer than \mathbb{T}_l . Let $U' = \{x \mid a' \leq x < b'\}$. If there exist $B' = \{x \mid a'' < x < b''\}$ or $\{x \mid a'' < x < b''\} - K$ such that $a' \in B' \subseteq U'$. Thus $a'' < a' < b''$. Thus there exist c that $a'' < c < a', c \in B', c \notin U'$. Thus $B' \not\subseteq U'$.

Thus the topologies of \mathbb{R}_l and \mathbb{R}_K is not comparable. \square

Definition 1.2.6 (subbasis). A **subbasis** \mathbb{S} for a topology on \mathbb{X} is a collection of subsets of \mathbb{X} whose union equals \mathbb{X} . The **topology generated by the subbasis** \mathbb{S} is defined to be the collection \mathbb{T} ⁷ of all unions of finite intersections of elements of \mathbb{S} .

1.2.1 Exercise

1. Show that if \mathbb{A} is a basis for a topology on \mathbb{X} , then the topology generated by \mathbb{A} equals the intersection of all topologies on \mathbb{X} that contain \mathbb{A} . Prove the same if \mathbb{A} is a subbasis.

Proof. As a subbasis is also a basis, we will directly prove the case of subbasis here.

⁵Whenever we consider \mathbb{R} , we shall suppose it is given this topology unless we specifically state otherwise.

⁶We omit the proof of this lemma as it is obvious.

⁷It is obvious that \mathbb{T} is a topology, we just omit the proof here.

Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain \mathbb{A} . Let \mathbb{T} be the topology that \mathbb{A} generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.⁸

First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

Thus, $\mathbb{T} = \mathbb{T}'$ □

1.3 The Order Topology

Definition 1.3.1 (interval). Let \mathbb{X} is a set having a simple order relation $<$. Given elements a and b of \mathbb{X} such that $a < b$, there are four subsets of \mathbb{X} that are called **intervals** determined by a and b :

- $(a, b) = \{x | a < x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $[a, b] = \{x | a \leq x \leq b\}$

(a, b) is called an **open interval** on \mathbb{X} . $[a, b]$ is called an **closed interval** on \mathbb{X} . $(a, b]$ and $[a, b)$ is called **half-open intervals**.

Definition 1.3.2 (order topology).⁹ Let \mathbb{X} be a set with a simple order relation; assume \mathbb{X} has more than one element. Let \mathbb{B} be the collection of all sets of the following types:

- All open intervals (a, b) in \mathbb{X} .
- All intervals of the form $[a_0, b)$, where a_0 is the smallest element(if exist) of \mathbb{X} .
- All intervals of the form $(a, b_0]$, where b_0 is the largest element(if exist) of \mathbb{X} .

The collection \mathbb{B} formed a basis for a topology on \mathbb{X} , which is called the order topology.

Definition 1.3.3 (ray).¹⁰¹¹ If \mathbb{X} is an ordered set, and a is an element of \mathbb{X} , there are four subsets of \mathbb{X} that are called **rays** determined by a :

- $(a, +\infty) = \{x | x > a\}$
- $(-\infty, a) = \{x | x < a\}$

⁸It is obvious that \mathbb{T}' is also a topology, we just omit the proof here.

⁹The standard topology on \mathbb{R} is an order topology derived from the usual order on \mathbb{R} .

¹⁰open rays are always open sets in the order topology

¹¹the open rays also formed a subbasis of the order topology

- $[a, +\infty) = \{x | x \geq a\}$
- $(-\infty, a] = \{x | x \leq a\}$

$(a, +\infty)$ and $(-\infty, a)$ are called **open rays**. $[a, +\infty)$ and $(-\infty, a]$ are called **closed rays**.

1.4 The Product Topology

Definition 1.4.1 (product topology). Let \mathbb{X} and \mathbb{Y} be topological spaces. The **product topology** on $\mathbb{X} \times \mathbb{Y}$ having a basis \mathbb{B} containing all sets of the form $U \times V$, where U and V is open sets of \mathbb{X} and \mathbb{Y} respectively.

Theorem 1.4.1.¹² If \mathbb{B} and \mathbb{C} is basis for the topology of \mathbb{X} and \mathbb{Y} respectively, then the collection

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} \text{ and } C \in \mathbb{C}\}$$

is a basis for the topology of $\mathbb{X} \times \mathbb{Y}$

Definition 1.4.2 (projection). Let $\pi_1 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$ be defined by the equation:

$$\pi_1(x, y) = x$$

Let $\pi_2 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}$ be defined by the equation:

$$\pi_2(x, y) = y$$

The maps π_1 and π_2 are called the **projections** of $\mathbb{X} \times \mathbb{Y}$ onto its first and second factors, respectively.

Theorem 1.4.2.¹³ The collection

$$\mathbb{S} = \{\pi_1^{-1}(U) | U \text{ open in } \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V \text{ open in } \mathbb{Y}\}$$

is a subbasis for the product topology on $\mathbb{X} \times \mathbb{Y}$.

Definition 1.4.3 (box topology). Let,

$$\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n \text{ or } \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots$$

In the first case, all the sets of the form $U_1 \times \cdots \times U_n$ where U_i is a open set of \mathbb{X}_i form a basis.

In the second case, all the sets of the form $U_1 \times U_2 \times \cdots$ where U_i is a open set of \mathbb{X}_i also form a basis.

Topology defined in this way was called a **box topology**.

¹²We omit the proof of this theorem as it is obvious.

¹³We omit the proof of this theorem as it is obvious.

Definition 1.4.4 (product topology).¹⁴ Let,

$$\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n \text{ or } \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots$$

Let π_i be the projection function¹⁵ that

$$\pi_i : \mathbb{X} \rightarrow \mathbb{X}_i$$

And if $x \in \mathbb{X}$

$$\pi_i(x) = x_i$$

All the set of the form $\pi_i^{-1}(U_i)$ where i is arbitrary and U_i is an open set of \mathbb{X}_i , form a subbasis of \mathbb{X} . The topology generated by this subbasis is called **product topology**. And \mathbb{X} is called a **product space**.

Definition 1.4.5 (J-tuple). Let J be an index set¹⁶. Give a set \mathbb{X} , a **J-tuple** is defined as a function $x : J \rightarrow \mathbb{X}$. If α is an element of J , $x(\alpha)$ is often denoted by x_α and is called the α th **coordinate** of x . And the function x itself is often denoted by the symbol

$$(x_\alpha)_{\alpha \in J}$$

The set of all J-tuples of elements of \mathbb{X} is often denoted by \mathbb{X}^J .

Definition 1.4.6 (cartesian product). Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $\mathbb{X} = \bigcup_{\alpha \in J} A_\alpha$. The **cartesian product** of this indexed family is denoted by

$$\prod_{\alpha \in J} A_\alpha$$

And is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of \mathbb{X} such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \rightarrow \bigcup_{\alpha \in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Theorem 1.4.3 (Comparison of the box and product topologies).¹⁷ The box topology on $\prod \mathbb{X}_\alpha$ has a basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α . The product topology on $\prod \mathbb{X}_\alpha$ has a basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α and U_α equals \mathbb{X}_α except for finitely many values of α .

¹⁴In the finite case, the product topology and box topology are the same, however they differ when \mathbb{X} is a infinite cartesian product.

¹⁵This is also called a **projection mapping** in a cartesian product.

¹⁶A index set was the set $\{1, \dots, n\}$ or the set \mathbb{Z}_+ .

¹⁷It is assumed that it is given product topology when considering $\prod X_\alpha$ unless it state specifically.

Theorem 1.4.4. ¹⁸Suppose the topology on each space \mathbb{X}_α is given by a basis \mathbb{B}_α . The collection of all sets of the form

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathbb{B}_\alpha$ form a basis for the box topology on $\prod_{\alpha \in J} \mathbb{X}_\alpha$.

The collection of all sets of the same form, where $B_\alpha \in \mathbb{B}_\alpha$ for finitely many indices α and $B_\alpha = \mathbb{X}_\alpha$ for all the remaining indices, will form a basis for the product topology $\prod_{\alpha \in J} \mathbb{X}_\alpha$.

Theorem 1.4.5. ¹⁹Let A_α be a subspace of \mathbb{X}_α , for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod \mathbb{X}_\alpha$ if both products are given the box topology, or if both products are given the product topology.

Theorem 1.4.6. ²⁰If each space \mathbb{X}_α is a Hausdorff space, then $\prod \mathbb{X}_\alpha$ is a Hausdorff space in both the box and product topologies.

Theorem 1.4.7. Let $\{\mathbb{X}_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subseteq \mathbb{X}_\alpha$ for each α . If $\prod \mathbb{X}_\alpha$ is given either the product or the box topology, then

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$$

Proof. Let π_α represent the projection mapping.

Let x be an element of $\prod \mathbb{X}_\alpha$. Let V be an open set in $\prod \mathbb{X}_\alpha$ that containing x .

If $x \in \prod \overline{A_\alpha}$, then $\pi_\alpha(V)$ is a open set in \mathbb{X}_α that containing x_α . Thus $\pi_\alpha(V)$ intersect with A_α . Thus V intersect with $\prod A_\alpha$. Thus $x \in \overline{\prod A_\alpha}$.

If $x \in \overline{\prod A_\alpha}$. Let U_α be an open set of A_α that contain x_α . Let $V = \prod U_\beta$ such that $U_\beta = \begin{cases} \mathbb{X}_\beta, & \beta \neq \alpha \\ U_\alpha, & \beta = \alpha \end{cases}$. It is obvious that V is an open set that contain x . Thus V intersect with $\prod A_\alpha$. Thus U_α intersect with A_α . Thus $x \in \prod \overline{A_\alpha}$. \square

Theorem 1.4.8. Let $f : A \rightarrow \prod_{\alpha \in J} \mathbb{X}_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

where $f_\alpha : A \rightarrow \mathbb{X}_\alpha$ for each α . Let $\prod \mathbb{X}_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

¹⁸We omit the proof of this theorem as it is obvious.

¹⁹We omit the proof of this theorem as it is obvious.

²⁰We omit the proof of this theorem as it is obvious.

Proof. Let π_α be the projection mapping

It is obvious that

$$f^{-1}(U) = \bigcap_{\alpha \in J} f_\alpha^{-1}(\pi_\alpha(U))$$

If f_α is continuous. Let V be a closed set of $\prod_{\alpha \in J} \mathbb{X}_\alpha$. Then $\pi_\alpha(V)$ is closed. Then $f^{-1}(V)$ is intersect of closed set. Thus $\pi_\alpha(V)$ is closed. So f is continuous.

If f is continuous. Let U_α be an open set of \mathbb{X}_α . Let $U_\beta = \mathbb{X}_\beta$ if $\beta \neq \alpha$. Let $V = \prod_{\beta \in J} U_\beta$. It is obvious that V is an open set of $\prod \mathbb{X}_\alpha$. And

$$\begin{aligned} f^{-1}V &= \bigcap_{\alpha \in J} f_\alpha^{-1}(\pi_\alpha(U)) \\ &= f_\alpha^{-1}(U_\alpha) \end{aligned}$$

which is an open set in A . Thus f_α is continuous. \square

1.5 The Subspace Topology

Definition 1.5.1 (subspace topology). Let \mathbb{X} be a topological space with topology \mathbb{T} . If Y is a subset of \mathbb{X} , the collection $\mathbb{T}_Y = \{Y \cap U \mid U \in \mathbb{T}\}$ is a topology on Y , called the **subspace topology**.

Y is also called a **subspace** of \mathbb{X}

Lemma 1.5.1. ²¹If \mathbb{B} is basis for the topology of \mathbb{X} , Y is a subset of \mathbb{X} then the collection

$$\mathbb{B}_Y = \{B \cap Y \mid B \in \mathbb{B}\}$$

is a basis for the subspace topology on Y

Lemma 1.5.2. ²²Let Y be a subspace of \mathbb{X} . If U is open in Y and Y is open in \mathbb{X} , then U is open in \mathbb{X} .

Theorem 1.5.1. ²³If A is a subspace of \mathbb{X} and B is a subspace of \mathbb{Y} , then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$

Proof. Let $\mathbb{B}_\mathbb{X}$ and $\mathbb{B}_\mathbb{Y}$ and $\mathbb{B}_{\mathbb{X} \times \mathbb{Y}}$ be basis of topology of \mathbb{X} and \mathbb{Y} and $\mathbb{X} \times \mathbb{Y}$ respectively. Let $\mathbb{B}'_\mathbb{X}$ and $\mathbb{B}'_\mathbb{Y}$ and $\mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$ be basis of topology of A and A and $A \times B$ respectively. We will show that $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} = \mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$. Thus, the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$.

²¹We omit the proof of this lemma as it is obvious.

²²We omit the proof of this lemma as it is obvious.

²³If \mathbb{X} is an ordered set in the order topology, and \mathbb{Y} is a subset of \mathbb{X} . The order relation, when restricted to \mathbb{Y} , makes \mathbb{Y} into an ordered set. However, the resulting order topology on \mathbb{Y} need not be the same as the topology that \mathbb{Y} inherits as a subspace of \mathbb{X} .

First, every element in $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ can be represented by $B_A \cap A \times B_B \cap B = B_A \times B_B \cap A \times B \in \mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$ where $B_A \in \mathbb{B}'_{\mathbb{X}}, B_B \in \mathbb{B}'_{\mathbb{Y}}$. Thus $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}} \subseteq \mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$.

Next, we show that $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$. For any open set U in $\mathbb{X} \times \mathbb{Y}$, and $\forall x \in U \cap A \times B, \exists B_{\mathbb{X}} \times B_{\mathbb{Y}} \in \mathbb{B}_{\mathbb{X} \times \mathbb{Y}}, x \in B_{\mathbb{X}} \times B_{\mathbb{Y}} \subseteq \mathbb{X} \times \mathbb{Y}$. Thus $x \in B_{\mathbb{X}} \times B_{\mathbb{Y}} \cap A \times B \subseteq A \times B, B_{\mathbb{X}} \times B_{\mathbb{Y}} \cap A \times B \in \mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$. Thus $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$. \square

Definition 1.5.2 (ordered square). *Let $I = [0, 1]$. The set $I \times I$ in the dictionary order ²⁴ topology will be called **ordered square**, and denoted by I_o^2*

Definition 1.5.3 (convex). *Given an ordered set \mathbb{X} , let us say that a subset \mathbb{Y} of \mathbb{X} is **convex** in \mathbb{X} if for each pair of points $a < b$ of \mathbb{Y} , the entire interval (a, b) of points of \mathbb{X} lies in \mathbb{Y}*

Theorem 1.5.2. ²⁵ *Let \mathbb{X} be an ordered set in the order topology. Let \mathbb{Y} be a subset of \mathbb{X} that is convex in \mathbb{X} . Then the order topology on \mathbb{Y} is the same as the topology \mathbb{Y} inherits as a subspace of \mathbb{X} .*

Proof. Consider the ray $(a, +\infty)$ in \mathbb{X} . If $a \in \mathbb{Y}$, then

$$(a, +\infty) \cap \mathbb{Y} = \{x | x \in \mathbb{Y} \text{ and } x > a\}$$

This is an open ray of the ordered set of \mathbb{Y} . if $a \notin \mathbb{Y}$, then a is either a lower bound on \mathbb{Y} or an upper bound on \mathbb{Y} , since \mathbb{Y} is convex. In the former case, the set $(a, +\infty) \cap \mathbb{Y}$ equals all of \mathbb{Y} , in the latter case, it is empty.

A similar remark shows that the intersection of the ray $(-\infty, a)$ with \mathbb{Y} is either an open ray of \mathbb{Y} , or \mathbb{Y} itself, or empty. Since the sets $(a, +\infty) \cap \mathbb{Y}$ and $(-\infty, a) \cap \mathbb{Y}$ form a subbasis for the subspace topology on \mathbb{Y} , and since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of \mathbb{Y} equals the intersection of an open ray of \mathbb{X} with \mathbb{Y} , so it is open in the subspace topology on \mathbb{Y} . Since the open rays of \mathbb{Y} are a subbasis for the order topology on \mathbb{Y} , this topology is contained in the subspace topology. \square

²⁴the dictionary means for $X_1, X_2 \in \mathbb{Y} = \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \dots$ which:

$$\begin{aligned} X_1 &= (x_1, x_2, x_3 \dots) \\ X_2 &= (x'_1, x'_2, x'_3 \dots) \end{aligned}$$

$X_1 > X_2$ only when

$$\begin{aligned} \exists k \in \mathbb{Z}_+, \forall i \in \mathbb{Z}_+, 0 < i < k \\ x_i &= x'_i \\ x_k &> x'_k \end{aligned}$$

²⁵Given \mathbb{X} is an ordered set in the order topology and \mathbb{Y} is a subset of \mathbb{X} , we shall assume that \mathbb{Y} is given the subspace topology unless we specifically state otherwise.

Exercise

1. A map $f : \mathbb{X} \rightarrow \mathbb{Y}$ is said to be a **open map** if for every open set $U \subseteq \mathbb{X}$, the set $f(U)$ is open in \mathbb{Y} . Show that $\pi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$ is open map.

Proof. An open set in $\mathbb{X} \times \mathbb{Y}$ can be represented by

$$\cup(U_i \times U'_i)$$

where U_i, U'_i are open sets in \mathbb{X}, \mathbb{Y} , respectively.

Also,

$$\cup(U_i \times U'_i) = \cup(U_i) \times \cup(U'_i)$$

Thus,

$$\pi(\cup(U_i \times U'_i)) = \cup(U_i)$$

Thus, $\pi(U)$ is open in \mathbb{X} . □

2. Let \mathbb{X} and \mathbb{X}' denote a single set in the topologies \mathbb{T} and \mathbb{T}' , respectively; let \mathbb{Y} and \mathbb{Y}' denote a single set in the topologies \mathbb{U} and \mathbb{U}' , respectively.
²⁶ Assume these sets are nonempty.

- (a) Show that if $\mathbb{T}' \supseteq \mathbb{T}$ and $\mathbb{U}' \supseteq \mathbb{U}$, then the product topologies $\mathbb{X}' \times \mathbb{Y}'$ is finer than the product topology on $\mathbb{X} \times \mathbb{Y}$.
 (b) Does the converse of the previous statement hold?

3. Show that the countable collection²⁷

$$\{(a, b) \times (c, d) | a < b, c < d, a \in \mathbb{Q}, b \in \mathbb{Q}, c \in \mathbb{Q}, d \in \mathbb{Q}\}$$

is a basis for \mathbb{R}^2

Proof. This is obvious if you prove that $(a, b) \times (c, d)$ is a rectangle in the \mathbb{R}^2 plane. □

4. Let \mathbb{X} be an ordered set. If \mathbb{Y} is a proper subset of \mathbb{X} that is convex in \mathbb{X} prove that \mathbb{Y} may not be an interval or a ray in \mathbb{X} .

Proof. Let $\mathbb{X} = \mathbb{R}^2$ with dictionary order. Then $Y = \{(x, y) | -1 \leq x \leq 1\}$ is convex in \mathbb{X} , however it is not an interval or a ray. □

There is a false prove given by myself.

²⁶what does $\mathbb{X}, \mathbb{X}', \mathbb{Y}, \mathbb{Y}'$ really mean here?? I do not know, so I just put the exercise here without a proof.

²⁷The prove of this set is countable is typically similar to Cantor's enumeration of a countable collection of countable sets.

Proof. Let \mathbb{S} be a set that contain all intervals and rays of \mathbb{Y} . We define a partial order on \mathbb{S} by inclusion. So if there is a chain in \mathbb{S} :

$$S_1 \subseteq S_2 \subseteq S_3 \dots$$

Let

$$S = S_1 \cup S_2 \cup S_3 \cup \dots$$

Thus, S is an upper bound of the chain.

Thus, by Zorn's Lemma, there is a maximal element of \mathbb{S} , say U , then we prove that $U = \mathbb{Y}$.

If $U \neq \mathbb{Y}$, then $\exists x, x \in \mathbb{Y} - U$.

If U is a ray say $(a, +\infty)$. Then $x < a$, thus $U \subseteq (x, +\infty) \subseteq \mathbb{B}$, then there is contradiction with the maximal element.

If U is an interval, the circumstance is similar with the proof of U is a ray.

Thus \mathbb{Y} is a ray or an interval. \square

However, there is issue with this proof, the set S does exists. However, it may not be an interval or ray, so it may not be contained in \mathbb{S}

1.6 Closed Sets and Limit Points

Definition 1.6.1 (closed).²⁸ A subset A of a topological space is said to be closed if the set $\mathbb{X} - A$ is open.

Theorem 1.6.1.²⁹ Let \mathbb{X} be a topological space. Then the following conditions hold

1. \emptyset and \mathbb{X} are closed.
2. Arbitrary intersections of closed sets are closed
3. Finite unions of closed sets are closed

Definition 1.6.2 (closed in). Let \mathbb{X} be a topological space; let \mathbb{Y} be a subspace of \mathbb{X} . We say that a set A is **closed in** \mathbb{Y} if A is a subset of \mathbb{Y} and A is closed in the subspace topology of \mathbb{Y}

Theorem 1.6.2. Let \mathbb{Y} be a subspace of \mathbb{X} . Then a set A is closed in \mathbb{Y} if and only if it equals the intersection of a closed set of \mathbb{X} with \mathbb{Y}

²⁸A set can be open, or closed, or both, or neither

²⁹We omit the proof of this theorem as it is obvious.

Proof. First we proof that if A is closed in \mathbb{Y} , then $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. As the origin topology form a surjective map to its subspace topology, there exists a B closed in \mathbb{X} that $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $B \cap \mathbb{Y} = A$

Conversely, if $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. Then, $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $\mathbb{X} - B$ is open in \mathbb{Y} , $\mathbb{Y} - A$ is open in \mathbb{Y} . Then A is closed in \mathbb{Y} \square

Theorem 1.6.3. ³⁰ Let \mathbb{Y} be a subspace of \mathbb{X} . If A is closed in \mathbb{Y} and \mathbb{Y} is closed in \mathbb{X} , then A is closed in \mathbb{X} .

Definition 1.6.3 (interior). Given a subset A of a topological space \mathbb{X} , the **interior** of A is defined as the union of all open sets contained in A . Denoted by $Int(A)$.

Definition 1.6.4 (closure). Given a subset A of a topological space \mathbb{X} , the **closure** of A is defined as the intersection of all closed sets containing A . Denoted by $Cl(A)$ or \bar{A}

Theorem 1.6.4. ³¹³² Let \mathbb{Y} be a subspace of a topological space \mathbb{X} ; let A be a subset of \mathbb{X} . Let \bar{A} denote the closure of A in \mathbb{X} . Then the closure of A in \mathbb{Y} equals $\bar{A} \cap \mathbb{Y}$

Definition 1.6.5 (intersect). We say that a set A **intersects** B if $A \cap B$ is not empty.

Theorem 1.6.5. Let A be a subset of the topological space \mathbb{X}

1. The $x \in \bar{A}$ if and only if every open set U containing x intersect A .
2. Supposing the topology of \mathbb{X} is given by a basis, then $x \in \bar{A}$ if and only if every basis element B containing x intersects A

Proof. There are only two types of closed set U in \mathbb{X} :

1. $U \supseteq \bar{A}$
2. $U \cap A \neq A$

Thus, there are only two types of open set U in \mathbb{X} respectively.

1. U does not intersects A .
 2. $U \cap \bar{A} \neq \emptyset$
1. If $x \in \bar{A}$, then every open set containing x is the open set of second type, thus every open set containing x intersects A

If every open set containing x intersect A , suppose $x \notin \bar{A}$. Then $\mathbb{X} - \bar{A}$ is a open set containing x , however, it does not intersects A . Thus, $x \in \bar{A}$.

³⁰As the proof is similar to the case in the open set, so we omit the proof here.

³¹We omit the proof of this theorem as it is obvious.

³²As the closure of A in \mathbb{X} and the closure A in \mathbb{Y} will sometimes be different. We always use \bar{A} to denote the closure of A in \mathbb{X}

2. If $x \in \overline{A}$, as every basis element of \mathbb{X} is a open set, thus every basis element containing x intersects A

If every open set containing x intersect A , suppose $x \notin \overline{A}$.

As every open sets can be represented by union of basis. Let

$$\mathbb{X} - \overline{A} = B_1 \cup B_2 \cup B_3 \cup \dots \cup B'_1 \cup B'_2 \cup B'_3 \cup \dots$$

where B are bases containing x , and B' are bases that does not contain x .

Thus,

$$x \in B_1 \cup B_2 \cup B_3 \cup \dots \subseteq \mathbb{X} - \overline{A}$$

Then $B_1 \cup B_2 \cup B_3 \cup \dots$ that is a open set can be generated by all the bases containing x , however, that does not intersects A . So, $x \in \overline{A}$.

□

Definition 1.6.6 (neighbourhood).³³ If we say U is a neighbourhood of x in \mathbb{X} , then U is an open set in \mathbb{X} containing x

Definition 1.6.7 (limit point, point of accumulation, cluster point).³⁴ If A is a subset of topological space \mathbb{X} . We say that x is a limit point of A if and only if every open sets containing x intersects A with some points other than x .

This condition is also equivalent to the condition that if x is a limit point of A if and only if $x \in A - \{x\}$

Theorem 1.6.6.³⁵ Let A be a subset of topological space \mathbb{X} ; let A' be the set of all limit points of A . Then

$$\overline{A} = A \cup A'$$

Corollary 1.6.1.³⁶ A subset of a topological space is closed if and only if it contains all its limit point.

Definition 1.6.8 (converge).³⁷ We say that a sequence of $x_1, x_2, x_3 \dots$ converge to x . When for every neighbourhood U of x , there exists a positive integer N , such that for all $n > N$, $x_n \in U$.

Definition 1.6.9 (Hausdorff space). A topological space is called a **Hausdorff space**, if for every distinct x_1, x_2 in \mathbb{X} , there exists disjoint neighbourhood of U_1, U_2 of x_1, x_2 in \mathbb{X} .

³³Some other mathematicians use neighbourhood to say that U merely contains an open set containing x . The book does not give a formal definition for the word merely, and I am not sure either.

³⁴Note that, x may belong to A or not, this does not matter.

³⁵We omit the proof of this theorem as it is obvious.

³⁶We omit the proof of this corollary as it is obvious.

³⁷In real line, a sequence can not converge to multiple points, but for an arbitrary topological space, this is possible.

Theorem 1.6.7. ³⁸³⁹ *Every finite point set in a Hausdorff space \mathbb{X} is closed.*

Proof. Let A be a finite point set in a Hausdorff space \mathbb{X} .

Suppose A only have one element. Then for every $x \in \mathbb{X} - A$, there exists a neighbourhood of x that does not intersect with A . So A is closed.

Suppose A is a closed finite point set. We take $x_0 \in \mathbb{X} - A$. As finite union of closed set is closed, $A \cup \{x_0\}$ is closed.

Then, from induction, all finite point set in a Hausdorff space is closed. \square

Theorem 1.6.8. *If \mathbb{X} is a Hausdorff space, then a sequence of points in \mathbb{X} converges to at most one point.*

Proof. Suppose that the following sequence

$$x_1, x_2, x_3 \dots$$

Converge to more than one points say

$$y_1, y_2, y_3 \dots$$

Then there exists

$$n_1, n_2, n_3 \dots, U_1, U_2, U_3 \dots$$

Such that for $n > n_i$

$$x_n \in U_i, y_i \in U_i$$

If we take disjoint U_1, U_2 which is possible as this is a Hausdorff space.

Then the previous condition does not stand. So, every sequence of points in a Hausdorff space can only converge to at most one point. \square

Definition 1.6.10 (limit). *If a sequence x_n of points in Hausdorff space converge to the point x , we denote this by $x_n \rightarrow x$ and we say the **limit** of x_n is x .*

Definition 1.6.11 (T_1 axiom). *The condition that all finite point set of a topological space is closed is called T_1 **axiom**.*

Theorem 1.6.9. *Let \mathbb{X} be a space satisfying the T_1 axiom; let A be a subset of \mathbb{X} . Then the point x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A .*

Proof. If every neighbourhood of x contains infinitely many point of A . Then every neighbourhood of x intersect with A with infinite element other than x , then x is a limit point of A .

³⁸This implies that a sequence in a Hausdorff space cannot converge to multiple points. The following theorem prove this.

³⁹The condition every finite point set is closed is weaker than the Hausdorff space condition. For instance, the finite complement topology of \mathbb{R} met the condition of finite point set. However it is not a Hausdorff space.

If x is a limit point of A . Suppose that there exists a open set U containing x and intersect with A for finite many points. Let

$$U' = U \cap (A - x)$$

Then, $x \notin U'$. Let

$$U'' = U - U'$$

Then U'' is open as U' is a finite point set and

$$U'' = U - U' = U \cap (\mathbb{X} - U')$$

Also, $x \in U''$. Thus, U'' is a open set containing x that only intersect A with x or do not intersect A . This is a contradiction of x is a limit point. Thus there does not exists a open set U containing x and intersect with A for finite many points. \square

Theorem 1.6.10. ⁴⁰Every simply ordered set is a Hausdorff space in order topology.

Theorem 1.6.11. ⁴¹The product of two Hausdorff space is a Hausdorff space.

Theorem 1.6.12. ⁴²A subspace of a Hausdorff space is a Hausdorff space.

1.6.1 Exercise

1. Give an counter example why $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$ dose not hold.

Proof. Consider the X be the K-topology on the real line.

Let

$$\begin{aligned} A_n &= \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{Z}_+ \\ A &= \cup A_n \end{aligned}$$

Then

$$\begin{aligned} \overline{A_n} &= \left[\frac{1}{n+1}, \frac{1}{n}\right] \\ \cup \overline{A_n} &= (0, 1] \end{aligned}$$

However, as every neighbourhood of 0 intersect $\cup A_\alpha$. $0 \in \overline{\cup A_\alpha}$.

Thus, $\overline{\cup A_\alpha} \neq \cup \overline{A_\alpha}$ \square

⁴⁰We omit the proof of this theorem as it is obvious.

⁴¹We omit the proof of this theorem as it is obvious.

⁴²We omit the proof of this theorem as it is obvious.

2. Prove that

$$\overline{A - B} \supseteq \overline{A} - \overline{B}$$

Proof. If $x \in \overline{A} - \overline{B}$. Then

$$x \in \overline{A}, x \notin \overline{B}$$

.

Thus for open set U containing x

$$\exists U_1 \cap B = \emptyset$$

$$\forall U \cap A \neq \emptyset$$

Suppose that $x \notin \overline{A - B}$. Then

$$\exists U_0 \cap (A - B) = \emptyset$$

Thus,

$$U_0 \cap A \subseteq B$$

Thus,

$$U_1 \cap B = \emptyset$$

$$U_1 \cap U_0 \cap A = \emptyset$$

As $U_1 \cap U_0$ is an open set containing x , so there is contradiction with $x \in \overline{A}$. Thus $x \in \overline{A - B}$. \square

3. A **diagonal** is a subset $\Delta = \{x \times x | x \in \mathbb{X}\}$ of the product topology $\mathbb{X} \times \mathbb{X}$ where \mathbb{X} is a topological space. Show that the diagonal is closed in $\mathbb{X} \times \mathbb{X}$ if and only if \mathbb{X} is a Hausdorff space.

Proof. If \mathbb{X} is a Hausdorff space. For every element $x \times y$ of $\mathbb{X} \times \mathbb{X}$ that not in Δ . We take disjoint set U_x, U_y where $x \in U_x, y \in U_y$. Then $\mathbb{X} \times \mathbb{X} - \Delta = \cup_{x \neq y} U_x \times U_y$. Where $\cup_{x \neq y} U_x \times U_y$ is an open set. Thus Δ is a closed set.

Conversely, if Δ is a closed set, suppose that \mathbb{X} is not a Hausdorff space. Then there exists distinct x, y such that every neighbourhood of x and y intersect. Let \mathbb{B} be a basis of topology of \mathbb{X} . Then $x \times y \in \mathbb{X} \times \mathbb{X} - \Delta$. However we cannot find $B_1, B_2 \in \mathbb{B}, x \times y \in B_1 \times B_2 \subset \mathbb{X} \times \mathbb{X} - \Delta$. Then Δ is not a closed set. So there is a contradiction, then \mathbb{X} must be a Hausdorff space. \square

4. Prove that T_1 axiom is equivalent to the condition such that for every distinct pair x, y of \mathbb{X} , there exists neighbourhood of x does not contain y .

Proof. First if T_1 axiom hold, then for every pair x, y , the neighbourhood $\mathbb{X} - \{y\}$ of x does not contain y , so the second condition hold.

Conversely, if the second condition hold. Suppose that we can find a finite points set say $\{x_1, x_2, x_3 \dots\}$, then there must exists $x \in \{x_1, x_2, x_3 \dots\}$ such that the set $\{x\}$ is not closed. Then $\overline{\{x\}} - \{x\} \neq \emptyset$. Let $y \in \overline{\{x\}} - \{x\}$, then every neighbourhood of y must contain x , this is a contradiction to the second condition, so the T_1 axiom must hold. \square

5. If $A \subseteq \mathbb{X}$, we define the **boundary** of A by the equation

$$\text{Bd}A = \overline{A} \cap \overline{\mathbb{X} - A}$$

- (a) Show that $\text{Int}A$ and $\text{Bd}A$ are disjoint and $\overline{A} = \text{Int}A \cup \text{Bd}A$.

Proof. For every $x \in \text{Bd}A$, every open set contain x must intersect A and $\mathbb{X} - A$ so, there is no open set U contain x , $U \subseteq A$.

For every $x' \in \text{Int}A$, there exists $U' \subseteq A$, so $\text{Bd}A$ and $\text{Int}A$ are disjoint sets.

For every $x \in \overline{A}$, $x \in \text{Bd}A$ or $x \notin \text{Bd}A$. We discuss the condition that $x \notin \text{Bd}A$.

Then $x \notin \overline{\mathbb{X} - A}$, then there exists a open set U containing x , that does not intersect with $\mathbb{X} - A$. Thus $U \subseteq A$, thus $x \in \text{Int}A$. So $\overline{A} \subseteq \text{Int}A \cup \text{Bd}A$.

Then, $\text{Bd}A \subseteq \overline{A}$, $\text{Int}A \subseteq A \subseteq \overline{A}$. Thus, $\overline{A} \supseteq \text{Int}A \cup \text{Bd}A$

So, $\overline{A} = \text{Int}A \cup \text{Bd}A$ \square

- (b) Show that $\text{Bd}A = \emptyset$ if and only if A is both open and closed.

Proof. So, $\text{Int}A = \overline{A}$, then $\text{Bd}A = \emptyset$ follows directly from $\overline{A} = \text{Int}A \cup \text{Bd}A$. \square

- (c) Show that U is open if and only if $\text{Bd}U = \overline{U} - U$.

Proof. Suppose U is open. Then $\overline{\mathbb{X} - U} = \mathbb{X} - U$. Then for every $x \in U$, $x \notin \mathbb{X} - U$, $x \notin \overline{\mathbb{X} - U}$. Thus $\overline{U} \cap \overline{\mathbb{X} - U} = \overline{U} - U$.

Conversely, suppose $\text{Bd}U = \overline{U} - U$. Then for every $x \in U$, $x \notin \text{Bd}U$. Then as $\overline{U} = \text{Int}U \cup \text{Bd}U$, $x \in \text{Int}U$. So $\text{Int}U \supseteq U$. Thus $U = \text{Int}U$. Thus, U is open. \square

1.7 Continuous Function

Definition 1.7.1 (continuous).⁴³ Let \mathbb{X} and \mathbb{Y} be topological spaces. A function $f : \mathbb{X} \rightarrow \mathbb{Y}$ is said to be **continuous** if for each open subset V of \mathbb{Y} , the set $f^{-1}(V)$ is an open subset of \mathbb{X} .

Theorem 1.7.1. Let \mathbb{X} and \mathbb{Y} be topological spaces; let $f : \mathbb{X} \rightarrow \mathbb{Y}$. Then the following are equivalent.

1. f is continuous.
2. For every subset A of \mathbb{X} , one has $f(\overline{A}) \subseteq \overline{f(A)}$.
3. For every closed set B of \mathbb{Y} , the set $f^{-1}(B)$ is closed in \mathbb{X} .
4. For each $x \in \mathbb{X}$ and each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subseteq V$.

Proof.

1 \Rightarrow 3:

Let A be a open set in \mathbb{Y} . $f^{-1}(\mathbb{Y} - A) = \mathbb{X} - f^{-1}(A)$.

3 \Rightarrow 1:

Let A be a closed set in \mathbb{Y} . $f^{-1}(\mathbb{Y} - A) = \mathbb{X} - f^{-1}(A)$.

1 \Rightarrow 2:

For $x \in \overline{A}$, we take a open set $f(x) \in U \subseteq \mathbb{Y}$. Thus $x \in f^{-1}(U) \cap A \neq \emptyset$. Thus $U \cap f(A) \neq \emptyset$. So $f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$.

2 \Rightarrow 3:

Suppose f is not continuous. Then there must exists V , such that $f^{-1}(V) = U$ is not closed. Thus $\overline{U} \supset B = f^{-1}(A)$. Thus $f\overline{B} \supset A$. However $f(\overline{B}) \subseteq \overline{f(B)} = A$. There is a contradiction. So f must be continuous.

1 \Rightarrow 4:

For every neighbourhood V of $f(x)$, $f^{-1}(V)$ is a neighbourhood of x that $f(f^{-1}(V)) \subseteq V$.

4 \Rightarrow 1:

We take a open set V of \mathbb{Y} . Let S be the collection of all open set U in \mathbb{X} such that $f(U) \subseteq V$. The set cannot be empty unless $f^{-1}(V) = \emptyset$. Let U_0 denote the union of all the element in S . We prove that $U_0 = f^{-1}(V)$.

For all element $x \in U_0$, $f(x) \in V$. Thus $U_0 \subseteq f^{-1}(V)$.

⁴³As the continuity of a function is different as the topological spaces are different. So if we want to emphasis this fact, we say that f is continuous **relative** to specific topologies on \mathbb{X} and \mathbb{Y} .

For all element $x \in f^{-1}(V)$. There is a U' such that $x \in U'$, $f(U') \subseteq V$. This follows from the condition 4. Thus $U' \in S$. Thus $x \in U_0$. Thus $U_0 \subseteq f^{-1}(V)$. As U_0 is union of open set, U_0 is also open. Thus, $f^{-1}(V)$ is also open.

Thus f is continuous. \square

Definition 1.7.2 (homeomorphism).⁴⁴ Let \mathbb{X} and \mathbb{Y} be topological space; let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a bijection. If both the function f and the inverse function

$$f^{-1} : \mathbb{Y} \rightarrow \mathbb{X}$$

are continuous, then f is called a **homeomorphism**

Definition 1.7.3 (topological imbedding). Suppose that $f : \mathbb{X} \rightarrow \mathbb{Y}$ is an injective continuous map, where \mathbb{X} and \mathbb{Y} are topological spaces. Let \mathbb{Z} be the image set $f(\mathbb{X})$, considered as a subspace of \mathbb{Y} ; then the function $f' : \mathbb{X} \rightarrow \mathbb{Z}$ obtained by restricting the range of f is bijective. If f' happens to be a homeomorphism of \mathbb{X} with \mathbb{Z} , we say that the map $f : \mathbb{X} \rightarrow \mathbb{Y}$ is a **topological imbedding**, or simply an **imbedding**, of \mathbb{X} in \mathbb{Y} .

Theorem 1.7.2 (Rules for constructing continuous functions). Let \mathbb{X} , \mathbb{Y} , and \mathbb{Z} be topological spaces.

1. (Constant function) If $f : \mathbb{X} \rightarrow \mathbb{Y}$ maps all of \mathbb{X} into the single point y_0 of \mathbb{Y} , then f is continuous.
2. (Inclusion) If A is a subspace of \mathbb{X} , the inclusion function $j : A \rightarrow \mathbb{X}$ is continuous.
3. (Composites) If $f : \mathbb{X} \rightarrow \mathbb{Y}$ and $g : \mathbb{Y} \rightarrow \mathbb{Z}$ are continuous, then the map $g \circ f : \mathbb{X} \rightarrow \mathbb{Z}$ is continuous.
4. (Restricting the domain) If $f : \mathbb{X} \rightarrow \mathbb{Y}$ is continuous, and if A is a subspace of \mathbb{X} , then the restriction function $f|_A : A \rightarrow \mathbb{Y}$ is continuous.
5. (Restricting or expanding the range) Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ is continuous. Let \mathbb{Z} be a subspace of \mathbb{Y} containing the image $f(\mathbb{X})$, the function $h : \mathbb{X} \rightarrow \mathbb{Z}$ obtained by restricting the range of f is continuous. If \mathbb{Z} is a space having \mathbb{Y} as a subspace, then the function $h : \mathbb{X} \rightarrow \mathbb{Y}$ obtained by expanding the range of f is continuous.
6. (Local formulation of continuity) The map $f : \mathbb{X} \rightarrow \mathbb{Y}$ is continuous if \mathbb{X} can be written as the union of open sets U_α such set $f|_{U_\alpha}$ is continuous for each α .

Proof.

⁴⁴A equivalent way to define homeomorphism, is that for any open subset U of \mathbb{X} , $f(U)$ is open if and only if U is open.

1. $f^{-1}(U)$ of any open set U is \mathbb{X} , thus f is continuous.
2. For every open subset U of \mathbb{X} , $j^{-1}(U) = U \cap A$ is continuous in A . Thus j is a continuous function.
3. For every open subset U of \mathbb{Z} , $f^{-1}(U)$ is open in \mathbb{Y} , and $g^{-1}(f^{-1}(U))$ is open in \mathbb{X} . Thus, $g \circ f$ is continuous.
4. For every open subset U of \mathbb{Y} , $f^{-1}(U)$ is open in \mathbb{X} , thus $f^{-1}(U) \cap A$ is open in A . Thus the function $f|_A$ is continuous.
5. If \mathbb{Z} is a subspace of \mathbb{Y} , then every open subset of \mathbb{Z} can be represented as $U \cap \mathbb{Z}$, where U is a open subset of \mathbb{Y} . Thus $h^{-1}(U \cap \mathbb{Z}) = g^{-1}(\mathbb{Z}) \cap g^{-1}(U) = \mathbb{X} \cap g^{-1}(U)$ which is a open subset of X , thus h is continuous.
If \mathbb{Y} is a subspace of \mathbb{Z} . Then we take a open subset U of \mathbb{Z} . $h^{-1}(U) = g^{-1}(U) \cap \mathbb{X}$ which is open in \mathbb{X} , thus h is continuous.
6. if $f|_{U_\alpha}$ is continuous for each α . For every open subset U of \mathbb{Y} .

$$U = \cup_\alpha (U_\alpha \cap U)$$

where $U_\alpha \cap U$ is open both in U_α and in \mathbb{Y} .

Thus,

$$\begin{aligned} f^{-1}(U) &= f^{-1}(\cup_\alpha (U_\alpha \cap U)) \\ &= \cup_\alpha ((f|_{U_\alpha})^{-1}(U_\alpha \cap U)) \end{aligned}$$

and each $(f|_{U_\alpha})^{-1}(U_\alpha \cap U)$ is open, thus $f^{-1}(U)$ is open.

□

Theorem 1.7.3 (The pasting lemma).⁴⁵ Let $\mathbb{X} = A \cup B$, where A, B are closed in \mathbb{X} . Let $f : A \rightarrow \mathbb{Y}$ and $g : B \rightarrow \mathbb{Y}$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f, g combine to give a continuous function $h : \mathbb{X} \rightarrow \mathbb{Y}$, defined by setting $h(x) = f(x), x \in A$ and $h(x) = g(x), x \in B$.

Theorem 1.7.4 (Maps into products).⁴⁶ Let $f : A \rightarrow \mathbb{X} \times \mathbb{Y}$ be given by the equation

$$f(a) = (f_1(a), f_2(a))$$

Then, the function f is continuous if and only if the functions

$$f_1 : A \rightarrow \mathbb{X}, f_2 : A \rightarrow \mathbb{Y}$$

are continuous.

⁴⁵The proof of this theorem is similar to the "Local formulation of continuity" condition of "Rules for constructing continuous functions", so we omit the proof here.

⁴⁶The map f_1, f_2 are called the **coordinate functions** of f

Proof. Let π_1, π_2 be the projection function

$$\begin{aligned}\pi_1 &: \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \\ \pi_2 &: \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}\end{aligned}$$

We first proof that if U is an open subset of $\mathbb{X} \times \mathbb{Y}$,

$$f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$$

Let $x \times y \in U$, $f^{-1}(x \times y)$ contains all a such that $f(a) = x \times y$.
Then for any $a \in f^{-1}(x \times y)$, $a \in f_1^{-1}(\pi_1(x \times y)), a \in f_2^{-1}(\pi_2(x \times y))$.
Thus, $f^{-1}(x \times y) \subseteq f_1^{-1}(\pi_1(x \times y)) \cap f_2^{-1}(\pi_2(x \times y))$.
Thus $f^{-1}(U) \subseteq f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$.

Also, if $a \in f_1^{-1}(\pi_1(x \times y)), a \in f_2^{-1}(\pi_2(x \times y))$, $f_1(a) = x, f_2(a) = y$.
Thus $f(a) = x \times y$. Thus $a \in f^{-1}(x \times y)$.
Thus $f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$

Let U be any open subset of $\mathbb{X} \times \mathbb{Y}$

$$f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$$

Where $f_1^{-1}(\pi_1(U))$ and $f_2^{-1}(\pi_2(U))$ are both open set. Thus $f^{-1}(U)$ is open. \square

1.7.1 Exercise

1. Let \mathbb{Y} be an ordered set in the order topology. Let $f, g : \mathbb{X} \rightarrow \mathbb{Y}$ be continuous, show that the set $A = \{x | f(x) \leq g(x)\}$ is closed in \mathbb{X} .

Proof. We only need to proof $\mathbb{X} - A$ is open in \mathbb{X} . We take $x \in \mathbb{X} - A$.
Thus $f(x) > g(x)$.

Let U_1, U_2 be the open set in \mathbb{Y} that met the following demand

$$\begin{aligned}\forall y_1 \in U_1, y_2 \in U_2, y_1 > y_2 \\ f(x) \in U_1, g(x) \in U_2\end{aligned}$$

As \mathbb{Y} is an ordered set, U_1, U_2 must exist.

Let $U = f^{-1}(U_1) \cap g^{-1}(U_2)$. It is obvious that U is a open set, and $x \in U$.

Also, for any $y \in U$. $f(y) > g(y)$. Thus $U \subseteq A$. Thus A is an open set. \square

2. Let $\{A_\alpha\}$ be a collection of subsets of \mathbb{X} ; let $\mathbb{X} = \cup_\alpha A_\alpha$. Let $f : \mathbb{X} \rightarrow \mathbb{Y}$; suppose that $f|_{A_\alpha}$ is continuous for each α . An indexed family of sets $\{A_\alpha\}$ is said to be **locally finite** if each point x of \mathbb{X} has a neighbourhood that intersect A_α for only finitely many values of α . Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

Proof. For any closed subset U of \mathbb{Y} . Let

$$V = \cup f|_{A_\alpha}(U)$$

We prove that V is closed, so, f is continuous.

To prove that V is closed, we prove that $\bar{V} = V$. That is for any $x \in \bar{V}$, we prove $x \in V$. For any neighbourhood B of x , let C_B denote the set that contain all α , such that $f|_{A_\alpha(U)}$ intersect with B . As B intersect with V , C_B can not be empty.

Let

$$\mathbb{C} = \{C_B | B \text{ be a neighbourhood of } x\}$$

As $\{A_\alpha\}$ is locally finite, \mathbb{C} contain at least one element with finite elements.

Also

$$C_{B_1 \cap B_2} \subseteq C_{B_1} \cap C_{B_2}$$

Let \leq be a partial order on the \mathbb{C} . If $C_{B_1} \subseteq C_{B_2}$, we say that $C_{B_1} \geq C_{B_2}$.

If there is chain in \mathbb{C}

$$C_{B_1} \leq C_{B_2} \dots$$

Let C_{B_0} be a element of \mathbb{C} with finite element. If $C_{B_0} \subseteq C_{B_1}, C_{B_0} \subseteq C_{B_2} \dots$. Then C_{B_0} is a upper bound of the chain.

If C is not a subset of all element of the chain. Then we construct a new set say

$$D = \{C_{B_0 \cap B_1}, C_{B_0 \cap B_2} \dots\}$$

Let

$$\mathbb{D} = \{C_{D_1 \cap D_2 \cap \dots} | C_{D_1}, C_{D_2} \dots \in D\}$$

As C_{B_0} is a finite set, D is a finite set, \mathbb{D} is also a finite set. Thus there must be a maximal element $E \in \mathbb{D}$ that is the subset of all element of \mathbb{D} . Then E is a subset of all element of the chain. Thus E is a upper bound of the chain.

Thus, there must be a maximal element C_F of \mathbb{C} , that is a subset of all element of \mathbb{C} .

Let G be the set be the union of all element of C_F .

As C_F is finite, G is closed. And all neighbourhood of x intersect with G . Thus $x \in G$.

As G is a subset of V , $x \in V$. So V is closed. And f is a continuous function on \mathbb{X} .

□

3. Let A be a subset of topological space \mathbb{X} , let \mathbb{Y} be a Hausdorff space. Let $f : A \rightarrow \mathbb{Y}$ be a continuous function. Let $g : \overline{A} \rightarrow \mathbb{Y}$ also be a continuous function where $g(x) = f(x), x \in A$. Prove that g is uniquely determined by f .⁴⁷

Proof. Say g and h are two distinct function that met the demand.

So there exist x_0 such that $g(x_0) \neq h(x_0)$.

As \mathbb{Y} is a Hausdorff space, so there exist disjoint open subset $g(x_0) \in U$ and $h(x_0) \in V$.

Then $g^{-1}(U)$ and $h^{-1}(V)$ are both open subset of \mathbb{X} that contain x_0 .

If $g^{-1}(U) \cap h^{-1}(V) \cap A \neq \emptyset$. Then there exist $x_1 \in g^{-1}(U) \cap h^{-1}(V) \cap A$ such that $g(x_1) \in U$ and $h(x_1) \in V$ and $g(x_1) = h(x_1)$. However U and V are disjoint. So there is a contradiction.

As $g^{-1}(U) \cap h^{-1}(V)$ is a open subset contain x_0 . So $g^{-1}(U) \cap h^{-1}(V)$ must intersect with A . So it is impossible that $g^{-1}(U) \cap h^{-1}(V) \cap A = \emptyset$.

So $g = h$.

□

⁴⁷It is possible that g does not exist.

Let \mathbb{X} be the real line with order topology. Let \mathbb{Y} be $\{0, 1\}$.

Let $A = \mathbb{X} - \{0\}$.

Let,

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

So, it is obvious that f is a continuous function on \mathbb{X} . However g does not exist in this case.