# Topology Note

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## Definitions

# Chapter 1

# Topology Spaces and Continuous Function

## 1.1 Basic Definition of Topology

**Definition 1.1.1** (topology)**.** *A **topology** on a set $\mathbb{X}$ is a collection $\mathbb{T}$ of subsets of $\mathbb{X}$ having the following properties:*

- *$\emptyset$ and $\mathbb{X}$ are in $\mathbb{T}$*
- *The union of the elements of any sub collection of $\mathbb{T}$ is in $\mathbb{T}$*
- *The intersection of the elements of any **finite** sub collection of $\mathbb{T}$ is in $\mathbb{T}$*

**Definition 1.1.2** (topology space)**.** *A **topological space** is a set $\mathbb{X}$ for which a topology $\mathbb{T}$ has been specified.*

**Definition 1.1.3** (open set)**.** *A **open set** $\mathbb{U}$ is a subset of $\mathbb{X}$ that belongs to a topology $\mathbb{T}$ of $\mathbb{X}$.*

**Definition 1.1.4** (open sets)**.** *A topology can also be called a **open sets***

**Definition 1.1.5** (discrete topology)**.** *The set of all subsets of a set $\mathbb{X}$ formed a topology called **discrete topology***

**Definition 1.1.6** (trivial topology)**.** *The set consisting the set $\mathbb{X}$ and $\emptyset$ only formed a topology of $\mathbb{X}$ called **trivial topology***

**Definition 1.1.7** (finite complement topology)**.** *Let $\mathbb{X}$ be a set. Let $\mathbb{T}_f$ be the collection of all subsets $\mathbb{U}$ of $\mathbb{X}$ such that $\mathbb{X} - \mathbb{U}$ either if a **finite** [1] of is all of $\mathbb{X}$. Then $\mathbb{T}_f$ is a topology on $\mathbb{X}$, called the .*

**Definition 1.1.8** (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable)**.** *Let $\mathbb{T}$ and $\mathbb{T}'$ be two topology on a given set $\mathbb{X}$. If $\mathbb{T}$ is a subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **finer** or **larger** than $\mathbb{T}$. If $\mathbb{T}$ is a proper subset of $\mathbb{T}'$, we say that $\mathbb{T}'$ is **strictly finer** or **strictly larger** than $\mathbb{T}$. We also say that $\mathbb{T}$ is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than $\mathbb{T}'$. We say that $\mathbb{T}$ and $\mathbb{T}'$ is **comparable** if either $\mathbb{T}$ is a subset of $\mathbb{T}'$ or $\mathbb{T}'$ is a subset of $\mathbb{T}$.*

[1] The set $\mathbb{U}$ can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, $\mathbb{U}$ will not be a topology.

## 1.2 Basis for a Topology

**Definition 1.2.1** (basis). *If $\mathbb{X}$ is a set, a **basis** for a topology on $\mathbb{X}$ is a collection $\mathbb{B}$ of subsets of $\mathbb{X}$ (called **basis elements**) such that:*

- *For each $x \in \mathbb{X}$, there is at least one basis element $B$ containing $x$*
- *If $x$ belongs to the intersection of two basis elements $B_1$ and $B_2$, then there is another element $x \in B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$*

**Definition 1.2.2** (topology generated by basis). *Let $\mathbb{B}$ be a basis on $\mathbb{X}$. Let $\mathbb{U}$ be a set containing all subsets $U$ of $\mathbb{X}$ such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such $\mathbb{U}$ formed a topology on $\mathbb{X}$, called **topology $\mathbb{T}$ generated by** $\mathbb{B}$*

**Lemma 1.2.1.** *Let $\mathbb{X}$ be a set. Let $\mathbb{B}$ be a basis for a topology $\mathbb{T}$ on $\mathbb{X}$. Then $\mathbb{T}$ equals to the set of all possible unions of elements of $\mathbb{B}$.*

*Proof.* Let set $\mathbb{U}$ be the set of all possible unions of elements of $\mathbb{B}$. For any $U \in \mathbb{U}$. $U = \cup B$ [2] for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in U$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, $\mathbb{U}$ equals to $\mathbb{T}$. □

**Lemma 1.2.2.** [3] *Let $\mathbb{X}$ be a topological space. Suppose that $\mathbb{C}$ is a collection of open sets of $\mathbb{X}$ such that for each open set $U$ of $\mathbb{X}$ and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq C$. Then $\mathbb{C}$ is a basis for the topology of $\mathbb{X}$.*

**Lemma 1.2.3.** [4] *Let $\mathbb{B}$ and $\mathbb{B}'$ be basis for the topologies $\mathbb{T}$ and $\mathbb{T}'$, respectively, on $\mathbb{X}$. Then the following are equivalent:*

- *$\mathbb{T}'$ is finer than $\mathbb{T}$*
- *For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing $X$, there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.*

**Definition 1.2.3** (standard topology on the real line). *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **standard topology on the real line** [5] .*

**Definition 1.2.4** (lower limit topology on the real line). *Let be $\mathbb{B} = \{B | B = \{x | a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. $\mathbb{B}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **lower limit topology on the real line**. When $\mathbb{R}$ is given this topology,we denote it by $\mathbb{R}_l$.*

**Definition 1.2.5** (K-topology on the real line). *Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$. $\mathbb{B} \cup \{B - K | B \in \mathbb{B}\}$ formed a basis on real line. The topology generated by $\mathbb{B}$ is called the **K-topology on the real line**. When $\mathbb{R}$ is given this topology,we denote it by $\mathbb{R}_{\mathbb{K}}$.*

**Lemma 1.2.4.** [6] *The topologies $\mathbb{R}_l$ and $\mathbb{R}_{\mathbb{K}}$ is strictly finer than the standard topology on $\mathbb{R}$.*

[2] Note that this expression may not be unique.
[3] We omit the proof of this lemma as it is obvious.
[4] We omit the proof of this lemma as it is obvious.
[5] Whenever we consider $\mathbb{R}$, we shall suppose it is given this topology unless we specifically state otherwise.
[6] We omit the proof of this lemma as it is obvious.

**Lemma 1.2.5.** *The topologies of $\mathbb{R}_l$ and $\mathbb{R}_\mathbb{K}$ is not comparable.*

*Proof.* Let $\mathbb{T}_l$ and $\mathbb{T}_\mathbb{K}$ be topologies of $\mathbb{R}_l$ and $\mathbb{R}_\mathbb{K}$ respectively. Let $K = \{x|x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that $\mathbb{T}_l$ is not finer than $\mathbb{T}_\mathbb{K}$. Let $U = \{x| - 1 < x < 1\} - K, x = 0$. If there exist $B = \{x|a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus $B$ is not a subset of $U$.

Then we proof that $\mathbb{T}_\mathbb{K}$ is not finer than $\mathbb{T}_l$. Let $U' = \{x|a' \leq x < b'\}$. If there exist $B' = \{x|a'' < x < b''\} or \{x|a'' < x < b''\} - K$ such that $a' \in B \subseteq U$. Thus $a'' < a < b''$. Thus there exist $c$ that $a'' < x < a, x \in B, x \notin U'$. Thus $B' \nsubseteq U'$.

Thus the topologies of $\mathbb{R}_l$ and $\mathbb{R}_\mathbb{K}$ is not comparable. □

**Definition 1.2.6** (subbasis)**.** *A* ***subbasis*** *$\mathbb{S}$ for a topology on $\mathbb{X}$ is a collection of subsets of $\mathbb{X}$ whose union equals $\mathbb{X}$. The* ***topology generated by the subbasis*** *$\mathbb{S}$ is defined to be the collection $\mathbb{T}$* [7] *of all unions of finite intersections of elements of $\mathbb{S}$.*

### 1.2.1 Exercise

1. Show that if $\mathbb{A}$ is a basis for a topology on $\mathbb{X}$, then the topology generated by $\mathbb{A}$ equals the intersection of all topologies on $\mathbb{X}$ that contain $\mathbb{A}$. Prove the same if $\mathbb{A}$ is a subbasis.

   *Proof.* As a subbasis is also a basis, we will directly prove the case of subbasis here.

   Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain $\mathbb{A}$. Let $\mathbb{T}$ be the topology that $\mathbb{A}$ generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.[8]

   First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

   Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

   Thus, $\mathbb{T} = \mathbb{T}'$ □

## 1.3 The Order Topology

**Definition 1.3.1** (interval)**.** *Let $\mathbb{X}$ is a set having a simple order relation $<$. Given elements $a$ and $b$ of $\mathbb{X}$ such that $a < b$, there are four subsets of $\mathbb{X}$ that are called* ***intervals*** *determined by $a$ and $b$:*

- $(a, b) = \{x|a < x < b\}$
- $(a, b] = \{x|a < x \leq b\}$
- $[a, b) = \{x|a \leq x < b\}$
- $[a, b] = \{x|a \leq x \leq b\}$

*$(a, b)$ is called an* ***open interval*** *on $\mathbb{X}$. $[a, b]$ is called an* ***closed interval*** *on $\mathbb{X}$. $(a, b]$ and $[a, b)$ is called* ***half-open intervals****.*

**Definition 1.3.2** (order topology)**.** [9] *Let $\mathbb{X}$ be a set with a simple order relation; assume $\mathbb{X}$ has more than one element. Let $\mathbb{B}$ be the collection of all sets of the following types:*

- *All open intervals $(a, b)$ in $\mathbb{X}$.*

[7] It is obvious that $\mathbb{T}$ is a topology, we just omit the proof here.
[8] It is obvious that $\mathbb{T}'$ is also a topology, we just omit the proof here.
[9] The standard topology on $\mathbb{R}$ is an order topology derived from the usual order on $\mathbb{R}$.

- *All intervals of the form* $[a_0, b)$*, where* $a_0$ *is the smallest element(if exist) of* $\mathbb{X}$*.*
- *All intervals of the form* $(a, b_0]$*, where* $b_0$ *is the largest element(if exist) of* $\mathbb{X}$*.*

*The collection* $\mathbb{B}$ *formed a basis for a topology on* $\mathbb{X}$*, which is called the order topology.*

**Definition 1.3.3** (ray)**.** [10][11] *If* $\mathbb{X}$ *is an ordered set, and* $a$ *is an element of* $\mathbb{X}$*, there are four subsets of* $\mathbb{X}$ *that are called* ***rays*** *determined by* $a$*:*

- $(a, +\infty) = \{x|x > a\}$
- $(-\infty, a) = \{x|x < a\}$
- $[a, +\infty) = \{x|x \geq a\}$
- $(-\infty, a] = \{x|x \leq a\}$

$(a, +\infty)$ *and* $(-\infty, a)$ *are called* ***open rays****.* $[a, +\infty)$ *and* $(-\infty, a]$ *are called* ***closed rays****.*

## 1.4 The Product Topology

**Definition 1.4.1** (product topology)**.** *Let* $\mathbb{X}$ *and* $\mathbb{Y}$ *be topological spaces. The* ***product topology*** *on* $\mathbb{X} \times \mathbb{Y}$ *having a basis* $\mathbb{B}$ *containing all sets of the form* $U \times V$*, where* $U$ *and* $V$ *is open sets of* $\mathbb{X}$ *and* $\mathbb{Y}$ *respectively.*

**Theorem 1.4.1.** [12] *If* $\mathbb{B}$ *and* $\mathbb{C}$ *is basis for the topology of* $\mathbb{X}$ *and* $\mathbb{Y}$ *respectively, then the collection*

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} and C \in \mathbb{C}\}$$

*is a basis for the topology of* $\mathbb{X} \times \mathbb{Y}$

**Definition 1.4.2** (projection)**.** *Let* $\pi_1 : \mathbb{X} \times \mathbb{Y} \to \mathbb{X}$ *be defined by the equation:*

$$\pi_1(x, y) = x$$

*Let* $\pi_2 : \mathbb{X} \times \mathbb{Y} \to \mathbb{Y}$ *be defined by the equation:*

$$\pi_1(x, y) = y$$

*The maps* $\pi_1$ *and* $\pi_2$ *are called the* ***projections*** *of* $\mathbb{X} \times \mathbb{Y}$ *onto its first and second factors, respectively.*

**Theorem 1.4.2.** [13] *The collection*

$$\mathbb{S} = \{\pi_1^{-1}(U) | U open in \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V open in \mathbb{Y}\}$$

*is a subbasis for the product topology on* $\mathbb{X} \times \mathbb{Y}$*.*

[10] open rays are always open sets in the order topology
[11] the open rays also formed a subbasis of the order topology
[12] We omit the proof of this lemma as it is obvious.
[13] We omit the proof of this lemma as it is obvious.