

# Topology Note

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# Chapter 1

## Topology Spaces and Continuous Function

### 1.1 Basic Definition of Topology

**Definition 1.1.1** (topology). A **topology** on a set  $\mathbb{X}$  is a collection  $\mathbb{T}$  of subsets of  $\mathbb{X}$  having the following properties:

- $\emptyset$  and  $\mathbb{X}$  are in  $\mathbb{T}$
- The union of the elements of any sub collection of  $\mathbb{T}$  is in  $\mathbb{T}$
- The intersection of the elements of any **finite** sub collection of  $\mathbb{T}$  is in  $\mathbb{T}$

**Definition 1.1.2** (topology space). A **topological space** is a set  $\mathbb{X}$  for which a topology  $\mathbb{T}$  has been specified.

**Definition 1.1.3** (open set). A **open set**  $\mathbb{U}$  is a subset of  $\mathbb{X}$  that belongs to a topology  $\mathbb{T}$  of  $\mathbb{X}$ .

**Definition 1.1.4** (open sets). A topology can also be called a **open sets**

**Definition 1.1.5** (discrete topology). The set of all subsets of a set  $\mathbb{X}$  formed a topology called **discrete topology**

**Definition 1.1.6** (trivial topology). The set consisting the set  $\mathbb{X}$  and  $\emptyset$  only formed a topology of  $\mathbb{X}$  called **trivial topology**

**Definition 1.1.7** (finite complement topology). Let  $\mathbb{X}$  be a set. Let  $\mathbb{T}_f$  be the collection of all subsets  $\mathbb{U}$  of  $\mathbb{X}$  such that  $\mathbb{X} - \mathbb{U}$  either if a **finite**<sup>1</sup> or is all of  $\mathbb{X}$ . Then  $\mathbb{T}_f$  is a topology on  $\mathbb{X}$ , called the **finite complement topology**.

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<sup>1</sup>The set  $\mathbb{U}$  can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection,  $\mathbb{U}$  will not be a topology.

**Definition 1.1.8** (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable). Let  $\mathbb{T}$  and  $\mathbb{T}'$  be two topology on a given set  $\mathbb{X}$ . If  $\mathbb{T}$  is a subset of  $\mathbb{T}'$ , we say that  $\mathbb{T}'$  is **finer** or **larger** than  $\mathbb{T}$ . If  $\mathbb{T}$  is a proper subset of  $\mathbb{T}'$ , we say that  $\mathbb{T}'$  is **strictly finer** or **strictly larger** than  $\mathbb{T}$ . We also say that  $\mathbb{T}$  is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than  $\mathbb{T}'$ . We say that  $\mathbb{T}$  and  $\mathbb{T}'$  is **comparable** if either  $\mathbb{T}$  is a subset of  $\mathbb{T}'$  or  $\mathbb{T}'$  is a subset of  $\mathbb{T}$ .

## 1.2 Basis for a Topology

**Definition 1.2.1** (basis). If  $\mathbb{X}$  is a set, a **basis** for a topology on  $\mathbb{X}$  is a collection  $\mathbb{B}$  of subsets of  $\mathbb{X}$  (called **basis elements**) such that:

- For each  $x \in \mathbb{X}$ , there is at least one basis element  $B$  containing  $x$
- If  $x$  belongs to the intersection of two basis elements  $B_1$  and  $B_2$ , then there is another element  $B_3 \in \mathbb{B}$  such that  $B_3 \subseteq B_1 \cap B_2$

**Definition 1.2.2** (topology generated by basis). Let  $\mathbb{B}$  be a basis on  $\mathbb{X}$ . Let  $\mathbb{U}$  be a set containing all subsets  $U$  of  $\mathbb{X}$  such that for each element  $x \in U$ , there is  $B \in \mathbb{B}$  that  $x \in B \subseteq U$ . Such  $\mathbb{U}$  formed a topology on  $\mathbb{X}$ , called **topology  $\mathbb{T}$  generated by  $\mathbb{B}$**

**Lemma 1.2.1.** Let  $\mathbb{X}$  be a set. Let  $\mathbb{B}$  be a basis for a topology  $\mathbb{T}$  on  $\mathbb{X}$ . Then  $\mathbb{T}$  equals to the set of all possible unions of elements of  $\mathbb{B}$ .

*Proof.* Let set  $\mathbb{U}$  be the set of all possible unions of elements of  $\mathbb{B}$ . For any  $U \in \mathbb{U}$ .  $U = \cup B$ <sup>2</sup> for some  $B \in \mathbb{B}$ . Thus, for every  $x \in U$ , there exist a  $B' \in \mathbb{B}$  that  $x \in B' \subseteq U$ . Thus,  $U \in \mathbb{T}$ .

Conversely, for any  $U \in \mathbb{T}$ . For any  $x \in U$ , let  $x \in B_x \in \mathbb{B}$ . Then,  $U = \cup_{x \in U} B_x$ . Thus,  $U \in \mathbb{U}$ .

Therefore,  $\mathbb{U}$  equals to  $\mathbb{T}$ . □

**Lemma 1.2.2.**<sup>3</sup> Let  $\mathbb{X}$  be a topological space. Suppose that  $\mathbb{C}$  is a collection of open sets of  $\mathbb{X}$  such that for each open set  $U$  of  $\mathbb{X}$  and each  $x \in U$ , there is an element  $C \in \mathbb{C}$  such that  $x \in C \subseteq U$ . Then  $\mathbb{C}$  is a basis for the topology of  $\mathbb{X}$ .

**Lemma 1.2.3.**<sup>4</sup> Let  $\mathbb{B}$  and  $\mathbb{B}'$  be basis for the topologies  $\mathbb{T}$  and  $\mathbb{T}'$ , respectively, on  $\mathbb{X}$ . Then the following are equivalent:

- $\mathbb{T}'$  is finer than  $\mathbb{T}$
- For each  $x \in \mathbb{X}$  and each basis element  $B \in \mathbb{B}$  containing  $x$ , there is a basis element  $B' \in \mathbb{B}'$  such that  $x \in B' \subseteq B$ .

<sup>2</sup>Note that this expression may not be unique.

<sup>3</sup>We omit the proof of this lemma as it is obvious.

<sup>4</sup>We omit the proof of this lemma as it is obvious.



**Definition 1.2.3** (standard topology on the real line). Let  $\mathbb{B} = \{B \mid B = \{x \mid a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$ .  $\mathbb{B}$  formed a basis on real line. The topology generated by  $\mathbb{B}$  is called the **standard topology on the real line**<sup>5</sup>.

**Definition 1.2.4** (lower limit topology on the real line). Let  $\mathbb{B} = \{B \mid B = \{x \mid a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$ .  $\mathbb{B}$  formed a basis on real line. The topology generated by  $\mathbb{B}$  is called the **lower limit topology on the real line**. When  $\mathbb{R}$  is given this topology, we denote it by  $\mathbb{R}_l$ .

**Definition 1.2.5** (K-topology on the real line). Let  $\mathbb{B} = \{B \mid B = \{x \mid a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$ . Let  $K = \{x \mid x = \frac{1}{n}, n \in \mathbb{Z}_+\}$ .  $\mathbb{B} \cup \{B - K \mid B \in \mathbb{B}\}$  formed a basis on real line. The topology generated by  $\mathbb{B}$  is called the **K-topology on the real line**. When  $\mathbb{R}$  is given this topology, we denote it by  $\mathbb{R}_K$ .

**Lemma 1.2.4.**<sup>6</sup> The topologies  $\mathbb{R}_l$  and  $\mathbb{R}_K$  is strictly finer than the standard topology on  $\mathbb{R}$ .

**Lemma 1.2.5.** The topologies of  $\mathbb{R}_l$  and  $\mathbb{R}_K$  is not comparable.

*Proof.* Let  $\mathbb{T}_l$  and  $\mathbb{T}_K$  be topologies of  $\mathbb{R}_l$  and  $\mathbb{R}_K$  respectively. Let  $K = \{x \mid x = \frac{1}{n}, n \in \mathbb{Z}_+\}$ .

We first proof that  $\mathbb{T}_l$  is not finer than  $\mathbb{T}_K$ . Let  $U = \{x \mid -1 < x < 1\} - K, x = 0$ . If there exist  $B = \{x \mid a \leq x < b\} \in \mathbb{T}_l$  such that  $x \in B \subseteq U$ , then  $0 < b < 1$ . Thus, there exist  $n \in \mathbb{Z}_+$  that  $0 < \frac{1}{n} < b$ . Thus  $B$  is not a subset of  $U$ .

Then we proof that  $\mathbb{T}_K$  is not finer than  $\mathbb{T}_l$ . Let  $U' = \{x \mid a' \leq x < b'\}$ . If there exist  $B' = \{x \mid a'' < x < b''\}$  or  $\{x \mid a'' < x < b''\} - K$  such that  $a' \in B' \subseteq U'$ . Thus  $a'' < a' < b''$ . Thus there exist  $c$  that  $a'' < c < a', c \in B', c \notin U'$ . Thus  $B' \not\subseteq U'$ .

Thus the topologies of  $\mathbb{R}_l$  and  $\mathbb{R}_K$  is not comparable.  $\square$

**Definition 1.2.6** (subbasis). A **subbasis**  $\mathbb{S}$  for a topology on  $\mathbb{X}$  is a collection of subsets of  $\mathbb{X}$  whose union equals  $\mathbb{X}$ . The **topology generated by the subbasis**  $\mathbb{S}$  is defined to be the collection  $\mathbb{T}$ <sup>7</sup> of all unions of finite intersections of elements of  $\mathbb{S}$ .

### 1.2.1 Exercise

1. Show that if  $\mathbb{A}$  is a basis for a topology on  $\mathbb{X}$ , then the topology generated by  $\mathbb{A}$  equals the intersection of all topologies on  $\mathbb{X}$  that contain  $\mathbb{A}$ . Prove the same if  $\mathbb{A}$  is a subbasis.

*Proof.* As a subbasis is also a basis, we will directly prove the case of subbasis here.

<sup>5</sup>Whenever we consider  $\mathbb{R}$ , we shall suppose it is given this topology unless we specifically state otherwise.

<sup>6</sup>We omit the proof of this lemma as it is obvious.

<sup>7</sup>It is obvious that  $\mathbb{T}$  is a topology, we just omit the proof here.

Let  $\mathbb{S} = \{\mathbb{T}_\alpha\}$  be set contain all the topologies that contain  $\mathbb{A}$ . Let  $\mathbb{T}$  be the topology that  $\mathbb{A}$  generated. Let  $\mathbb{T}' = \cap \mathbb{T}_\alpha$ .<sup>8</sup>

First,  $\mathbb{A} \subseteq \mathbb{T}_\alpha$ . Thus,  $\mathbb{T} \subseteq \mathbb{T}_\alpha$ . Thus,  $\mathbb{T} \subseteq \mathbb{T}'$ .

Also,  $\mathbb{A} \subseteq \mathbb{T}$ . Thus,  $\mathbb{T} \in \mathbb{S}$ . Thus,  $\mathbb{T}' \subseteq \mathbb{T}$ .

Thus,  $\mathbb{T} = \mathbb{T}'$  □

### 1.3 The Order Topology

**Definition 1.3.1** (interval). Let  $\mathbb{X}$  is a set having a simple order relation  $<$ . Given elements  $a$  and  $b$  of  $\mathbb{X}$  such that  $a < b$ , there are four subsets of  $\mathbb{X}$  that are called **intervals** determined by  $a$  and  $b$ :

- $(a, b) = \{x | a < x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $[a, b] = \{x | a \leq x \leq b\}$

$(a, b)$  is called an **open interval** on  $\mathbb{X}$ .  $[a, b]$  is called an **closed interval** on  $\mathbb{X}$ .  $(a, b]$  and  $[a, b)$  is called **half-open intervals**.

**Definition 1.3.2** (order topology).<sup>9</sup> Let  $\mathbb{X}$  be a set with a simple order relation; assume  $\mathbb{X}$  has more than one element. Let  $\mathbb{B}$  be the collection of all sets of the following types:

- All open intervals  $(a, b)$  in  $\mathbb{X}$ .
- All intervals of the form  $[a_0, b)$ , where  $a_0$  is the smallest element(if exist) of  $\mathbb{X}$ .
- All intervals of the form  $(a, b_0]$ , where  $b_0$  is the largest element(if exist) of  $\mathbb{X}$ .

The collection  $\mathbb{B}$  formed a basis for a topology on  $\mathbb{X}$ , which is called the order topology.

**Definition 1.3.3** (ray).<sup>1011</sup> If  $\mathbb{X}$  is an ordered set, and  $a$  is an element of  $\mathbb{X}$ , there are four subsets of  $\mathbb{X}$  that are called **rays** determined by  $a$ :

- $(a, +\infty) = \{x | x > a\}$
- $(-\infty, a) = \{x | x < a\}$

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<sup>8</sup>It is obvious that  $\mathbb{T}'$  is also a topology, we just omit the proof here.

<sup>9</sup>The standard topology on  $\mathbb{R}$  is an order topology derived from the usual order on  $\mathbb{R}$ .

<sup>10</sup>open rays are always open sets in the order topology

<sup>11</sup>the open rays also formed a subbasis of the order topology

- $[a, +\infty) = \{x | x \geq a\}$
- $(-\infty, a] = \{x | x \leq a\}$

$(a, +\infty)$  and  $(-\infty, a)$  are called **open rays**.  $[a, +\infty)$  and  $(-\infty, a]$  are called **closed rays**.

## 1.4 The Product Topology

**Definition 1.4.1** (product topology). Let  $\mathbb{X}$  and  $\mathbb{Y}$  be topological spaces. The **product topology** on  $\mathbb{X} \times \mathbb{Y}$  having a basis  $\mathbb{B}$  containing all sets of the form  $U \times V$ , where  $U$  and  $V$  is open sets of  $\mathbb{X}$  and  $\mathbb{Y}$  respectively.

**Theorem 1.4.1.** <sup>12</sup>If  $\mathbb{B}$  and  $\mathbb{C}$  is basis for the topology of  $\mathbb{X}$  and  $\mathbb{Y}$  respectively, then the collection

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} \text{ and } C \in \mathbb{C}\}$$

is a basis for the topology of  $\mathbb{X} \times \mathbb{Y}$

**Definition 1.4.2** (projection). Let  $\pi_1 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$  be defined by the equation:

$$\pi_1(x, y) = x$$

Let  $\pi_2 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}$  be defined by the equation:

$$\pi_2(x, y) = y$$

The maps  $\pi_1$  and  $\pi_2$  are called the **projections** of  $\mathbb{X} \times \mathbb{Y}$  onto its first and second factors, respectively.

**Theorem 1.4.2.** <sup>13</sup>The collection

$$\mathbb{S} = \{\pi_1^{-1}(U) | U \text{ open in } \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V \text{ open in } \mathbb{Y}\}$$

is a subbasis for the product topology on  $\mathbb{X} \times \mathbb{Y}$ .

## 1.5 The Subspace Topology

**Definition 1.5.1** (subspace topology). Let  $\mathbb{X}$  be a topological space with topology  $\mathbb{T}$ . If  $Y$  is a subset of  $\mathbb{X}$ , the collection  $\mathbb{T}_Y = \{Y \cap U | U \in \mathbb{T}\}$  is a topology on  $Y$ , called the **subspace topology**.

$Y$  is also called a **subspace** of  $\mathbb{X}$

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<sup>12</sup>We omit the proof of this lemma as it is obvious.

<sup>13</sup>We omit the proof of this lemma as it is obvious.

**Lemma 1.5.1.** <sup>14</sup>If  $\mathbb{B}$  is basis for the topology of  $\mathbb{X}$ ,  $Y$  is a subset of  $\mathbb{X}$  then the collection

$$\mathbb{B}_Y = \{B \cap Y | B \in \mathbb{B}\}$$

is a basis for the subspace topology on  $Y$

**Lemma 1.5.2.** <sup>15</sup>Let  $Y$  be a subspace of  $\mathbb{X}$ . If  $U$  is open in  $Y$  and  $Y$  is open in  $\mathbb{X}$ , then  $U$  is open in  $\mathbb{X}$ .

**Theorem 1.5.1.** <sup>16</sup>If  $A$  is a subspace of  $\mathbb{X}$  and  $B$  is a subspace of  $\mathbb{Y}$ , then the product topology on  $A \times B$  is the same as the topology  $A \times B$  inherits as a subspace of  $\mathbb{X} \times \mathbb{Y}$

*Proof.* Let  $\mathbb{B}_\mathbb{X}$  and  $\mathbb{B}_\mathbb{Y}$  and  $\mathbb{B}_{\mathbb{X} \times \mathbb{Y}}$  be basis of topology of  $\mathbb{X}$  and  $\mathbb{Y}$  and  $\mathbb{X} \times \mathbb{Y}$  respectively. Let  $\mathbb{B}'_\mathbb{X}$  and  $\mathbb{B}'_\mathbb{Y}$  and  $\mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$  be basis of topology of  $A$  and  $A$  and  $A \times B$  respectively. We will show that  $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} = \mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$ . Thus, the product topology on  $A \times B$  is the same as the topology  $A \times B$  inherits as a subspace of  $\mathbb{X} \times \mathbb{Y}$ .

First, every element in  $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$  can be represented by  $B_A \cap A \times B_B \cap B = B_A \times B_B \cap A \times B \in \mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$  where  $B_A \in \mathbb{B}'_\mathbb{X}$ ,  $B_B \in \mathbb{B}'_\mathbb{Y}$ . Thus  $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} \subseteq \mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$ .

Next, we show that  $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$  generate the topology  $A \times B$  inherits as a subspace of  $\mathbb{X} \times \mathbb{Y}$ . For any open set  $U$  in  $\mathbb{X} \times \mathbb{Y}$ , and  $\forall x \in U \cap A \times B$ ,  $\exists B_\mathbb{X} \times B_\mathbb{Y} \in \mathbb{B}_{\mathbb{X} \times \mathbb{Y}}$ ,  $x \in B_\mathbb{X} \times B_\mathbb{Y} \subseteq \mathbb{X} \times \mathbb{Y}$ . Thus  $x \in B_\mathbb{X} \times B_\mathbb{Y} \cap A \times B \subseteq A \times B$ ,  $B_\mathbb{X} \times B_\mathbb{Y} \cap A \times B \in \mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$ . Thus  $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$  generate the topology  $A \times B$  inherits as a subspace of  $\mathbb{X} \times \mathbb{Y}$ .  $\square$

**Definition 1.5.2** (ordered square). Let  $I = [0, 1]$ . The set  $I \times I$  in the dictionary order <sup>17</sup> topology will be called **ordered square**, and denoted by  $I_o^2$

**Definition 1.5.3** (convex). Given an ordered set  $\mathbb{X}$ , let us say that a subset  $\mathbb{Y}$  of  $\mathbb{X}$  is **convex** in  $\mathbb{X}$  if for each pair of points  $a < b$  of  $\mathbb{Y}$ , the entire interval  $(a, b)$  of points of  $\mathbb{X}$  lies in  $\mathbb{Y}$

<sup>14</sup>We omit the proof of this lemma as it is obvious.

<sup>15</sup>We omit the proof of this lemma as it is obvious.

<sup>16</sup>If  $\mathbb{X}$  is an ordered set in the order topology, and  $\mathbb{Y}$  is a subset of  $\mathbb{X}$ . The order relation, when restricted to  $\mathbb{Y}$ , makes  $\mathbb{Y}$  into an ordered set. However, the resulting order topology on  $\mathbb{Y}$  need not be the same as the topology that  $\mathbb{Y}$  inherits as a subspace of  $\mathbb{X}$ .

<sup>17</sup>the dictionary means for  $X_1, X_2 \in \mathbb{Y} = \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \dots$  which:

$$\begin{aligned} X_1 &= (x_1, x_2, x_3 \dots) \\ X_2 &= (x'_1, x'_2, x'_3 \dots) \end{aligned}$$

$X_1 > X_2$  only when

$$\begin{aligned} \exists k \in \mathbb{Z}_+, \forall i \in \mathbb{Z}_+, 0 < i < k \\ x_i &= x'_i \\ x_k &> x'_k \end{aligned}$$

**Theorem 1.5.2.**<sup>18</sup> Let  $\mathbb{X}$  be an ordered set in the order topology. Let  $\mathbb{Y}$  be a subset of  $\mathbb{X}$  that is convex in  $\mathbb{X}$ . Then the order topology on  $\mathbb{Y}$  is the same as the topology  $\mathbb{Y}$  inherits as a subspace of  $\mathbb{X}$ .

*Proof.* Consider the ray  $(a, +\infty)$  in  $\mathbb{X}$ . If  $a \in \mathbb{Y}$ , then

$$(a, +\infty) \cap \mathbb{Y} = \{x \mid x \in \mathbb{Y} \text{ and } x > a\}$$

This is an open ray of the ordered set of  $\mathbb{Y}$ . if  $a \notin \mathbb{Y}$ , then  $a$  is either a lower bound on  $\mathbb{Y}$  or an upper bound on  $\mathbb{Y}$ , since  $\mathbb{Y}$  is convex. In the former case, the set  $(a, +\infty) \cap \mathbb{Y}$  equals all of  $\mathbb{Y}$ , in the latter case, it is empty.

A similar remark shows that the intersection of the ray  $(-\infty, a)$  with  $\mathbb{Y}$  is either an open ray of  $\mathbb{Y}$ , or  $\mathbb{Y}$  itself, or empty. Since the sets  $(a, +\infty) \cap \mathbb{Y}$  and  $(-\infty, a) \cap \mathbb{Y}$  form a subbasis for the subspace topology on  $\mathbb{Y}$ , and since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of  $\mathbb{Y}$  equals the intersection of an open ray of  $\mathbb{X}$  with  $\mathbb{Y}$ , so it is open in the subspace topology on  $\mathbb{Y}$ . Since the open rays of  $\mathbb{Y}$  are a subbasis for the order topology on  $\mathbb{Y}$ , this topology is contained in the subspace topology.  $\square$

### Exercise

1. A map  $f : \mathbb{X} \rightarrow \mathbb{Y}$  is said to be a **open map** if for every open set  $U \subseteq \mathbb{X}$ , the set  $f(U)$  is open in  $\mathbb{Y}$ . Show that  $\pi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$  is open map.

*Proof.* An open set in  $\mathbb{X} \times \mathbb{Y}$  can be represented by

$$\cup(U_i \times U'_i)$$

where  $U_i, U'_i$  are open sets in  $\mathbb{X}, \mathbb{Y}$ , respectively.

Also,

$$\cup(U_i \times U'_i) = \cup(U_i) \times \cup(U'_i)$$

Thus,

$$\pi(\cup(U_i \times U'_i)) = \cup(U_i)$$

Thus,  $\pi(U)$  is open in  $\mathbb{X}$ .  $\square$

2. Let  $\mathbb{X}$  and  $\mathbb{X}'$  denote a single set in the topologies  $\mathbb{T}$  and  $\mathbb{T}'$ , respectively; let  $\mathbb{Y}$  and  $\mathbb{Y}'$  denote a single set in the topologies  $\mathbb{U}$  and  $\mathbb{U}'$ , respectively.

<sup>19</sup> Assume these sets are nonempty.

<sup>18</sup>Given  $\mathbb{X}$  is an ordered set in the order topology and  $\mathbb{Y}$  is a subset of  $\mathbb{X}$ , we shall assume that  $\mathbb{Y}$  is given the subspace topology unless we specifically state otherwise.

<sup>19</sup>what does  $\mathbb{X}, \mathbb{X}', \mathbb{Y}, \mathbb{Y}'$  really mean here?? I do not know, so I just put the exercise here without a proof.

- (a) Show that if  $\mathbb{T}' \supseteq \mathbb{T}$  and  $\mathbb{U}' \supseteq \mathbb{U}$ , then the product topologies  $\mathbb{X}' \times \mathbb{Y}'$  is finer than the product topology on  $\mathbb{X} \times \mathbb{Y}$ .
- (b) Does the converse of the previous statement hold?
3. Show that the countable collection<sup>20</sup>

$$\{(a, b) \times (c, d) | a < b, c < d, a \in \mathbb{Q}, b \in \mathbb{Q}, c \in \mathbb{Q}, d \in \mathbb{Q}\}$$

is a basis for  $\mathbb{R}^2$

*Proof.* This is obvious if you prove that  $(a, b) \times (c, d)$  is a rectangle in the  $\mathbb{R}^2$  plane.  $\square$

4. Let  $\mathbb{X}$  be an ordered set. If  $\mathbb{Y}$  is a proper subset of  $\mathbb{X}$  that is convex in  $\mathbb{X}$  prove that  $\mathbb{Y}$  may not be an interval or a ray in  $\mathbb{X}$ .

*Proof.* Let  $\mathbb{X} = \mathbb{R}^2$  with dictionary order. Then  $Y = \{(x, y) | -1 \leq x \leq 1\}$  is convex in  $\mathbb{X}$ , however it is not an interval or a ray.  $\square$

There is a false prove given by myself.

*Proof.* Let  $\mathbb{S}$  be a set that contain all intervals and rays of  $\mathbb{Y}$ . We define a partial order on  $\mathbb{S}$  by inclusion. So if there is a chain in  $\mathbb{S}$ :

$$S_1 \subseteq S_2 \subseteq S_3 \dots$$

Let

$$S = S_1 \cup S_2 \cup S_3 \cup \dots$$

Thus,  $S$  is an upper bound of the chain.

Thus, by Zorn's Lemma, there is a maximal element of  $\mathbb{S}$ , say  $U$ , then we prove that  $U = \mathbb{Y}$ .

If  $U \neq \mathbb{Y}$ , then  $\exists x, x \in \mathbb{Y} - U$ .

If  $U$  is a ray say  $(a, +\infty)$ . Then  $x < a$ , thus  $U \subseteq (x, +\infty) \subseteq \mathbb{B}$ , then there is contradiction with the maximal element.

If  $U$  is an interval, the circumstance is similar with the proof of  $U$  is a ray.

Thus  $\mathbb{Y}$  is a ray or an interval.  $\square$

However, there is issue with this proof, the set  $S$  does exists. However, it may not be an interval or ray, so it may not be contained in  $\mathbb{S}$

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<sup>20</sup>The prove of this set is countable is typically similar to Cantor's enumeration of a countable collection of countable sets.

## 1.6 Closed Sets and Limit Points

**Definition 1.6.1** (closed).<sup>21</sup> A subset  $A$  of a topological space is said to be closed if the set  $\mathbb{X} - A$  is open.

**Theorem 1.6.1.**<sup>22</sup> Let  $\mathbb{X}$  be a topological space. Then the following conditions hold

1.  $\emptyset$  and  $\mathbb{X}$  are closed.
2. Arbitrary intersections of closed sets are closed
3. Finite unions of closed sets are closed

**Definition 1.6.2** (closed in). Let  $\mathbb{X}$  be a topological space; let  $\mathbb{Y}$  be a subspace of  $\mathbb{X}$ . We say that a set  $A$  is **closed in**  $\mathbb{Y}$  if  $A$  is a subset of  $\mathbb{Y}$  and  $A$  is closed in the subspace topology of  $\mathbb{Y}$ .

**Theorem 1.6.2.** Let  $\mathbb{Y}$  be a subspace of  $\mathbb{X}$ . Then a set  $A$  is closed in  $\mathbb{Y}$  if and only if it equals the intersection of a closed set of  $\mathbb{X}$  with  $\mathbb{Y}$ .

*Proof.* First we prove that if  $A$  is closed in  $\mathbb{Y}$ , then  $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$ . As the origin topology forms a surjective map to its subspace topology, there exists a  $B$  closed in  $\mathbb{X}$  that  $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$ . Then  $B \cap \mathbb{Y} = A$ .

Conversely, if  $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$ . Then,  $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$ . Then  $\mathbb{X} - B$  is open in  $\mathbb{Y}$ ,  $\mathbb{Y} - A$  is open in  $\mathbb{Y}$ . Then  $A$  is closed in  $\mathbb{Y}$ .  $\square$

**Theorem 1.6.3.**<sup>23</sup> Let  $\mathbb{Y}$  be a subspace of  $\mathbb{X}$ . If  $A$  is closed in  $\mathbb{Y}$  and  $\mathbb{Y}$  is closed in  $\mathbb{X}$ , then  $A$  is closed in  $\mathbb{X}$ .

**Definition 1.6.3** (interior). Given a subset  $A$  of a topological space  $\mathbb{X}$ , the **interior** of  $A$  is defined as the union of all open sets contained in  $A$ . Denoted by  $\text{Int}(A)$ .

**Definition 1.6.4** (closure). Given a subset  $A$  of a topological space  $\mathbb{X}$ , the **closure** of  $A$  is defined as the intersection of all closed sets containing  $A$ . Denoted by  $\text{Cl}(A)$  or  $\overline{A}$ .

**Theorem 1.6.4.**<sup>24,25</sup> Let  $\mathbb{Y}$  be a subspace of a topological space  $\mathbb{X}$ ; let  $A$  be a subset of  $\mathbb{X}$ . Let  $\overline{A}$  denote the closure of  $A$  in  $\mathbb{X}$ . Then the closure of  $A$  in  $\mathbb{Y}$  equals  $\overline{A} \cap \mathbb{Y}$ .

**Definition 1.6.5** (intersect). We say that a set  $A$  **intersects**  $B$  if  $A \cap B$  is not empty.

<sup>21</sup>A set can be open, or closed, or both, or neither

<sup>22</sup>We omit the proof of this lemma as it is obvious.

<sup>23</sup>As the proof is similar to the case in the open set, so we omit the proof here.

<sup>24</sup>We omit the proof of this lemma as it is obvious.

<sup>25</sup>As the closure of  $A$  in  $\mathbb{X}$  and the closure  $A$  in  $\mathbb{Y}$  will sometimes be different. We always use  $\overline{A}$  to denote the closure of  $A$  in  $\mathbb{X}$ .

**Theorem 1.6.5.** *Let  $A$  be a subset of the topological space  $\mathbb{X}$*

1. *The  $x \in \overline{A}$  if and only if every open set  $U$  containing  $x$  intersect  $A$ .*
2. *Supposing the topology of  $\mathbb{X}$  is given by a basis, then  $x \in \overline{A}$  if and only if every basis element  $B$  containing  $x$  intersects  $A$*

*Proof.* There are only two types of closed set  $U$  in  $\mathbb{X}$  :

1.  $U \supseteq \overline{A}$
2.  $U \cap A \neq A$

Thus, there are only two types of open set  $U$  in  $\mathbb{X}$  respectively.

1.  $U$  does not intersects  $A$ .
2.  $U \cap \overline{A} \neq \emptyset$

1. If  $x \in \overline{A}$ , then every open set containing  $x$  is the open set of second type, thus every open set containing  $x$  intersects  $A$

If every open set containing  $x$  intersect  $\mathbb{A}$ , suppose  $x \notin \overline{A}$ . Then  $\mathbb{X} - \overline{A}$  is a open set containing  $x$ , however, it does not intersects  $A$ . Thus,  $x \in \overline{A}$ .

2. If  $x \in \overline{A}$ , as every basis element of  $\mathbb{X}$  is a open set, thus every basis element containing  $x$  intersects  $\mathbb{A}$

If every open set containing  $x$  intersect  $\mathbb{A}$ , suppose  $x \notin \overline{A}$ .

As every open sets can be represented by union of basis. Let

$$\mathbb{X} - \overline{A} = B_1 \cup B_2 \cup B_3 \cup \dots \cup B'_1 \cup B'_2 \cup B'_3 \cup \dots$$

where  $B$  are bases containing  $x$ , and  $B'$  are bases that does not contain  $x$ .

Thus,

$$x \in B_1 \cup B_2 \cup B_3 \cup \dots \subseteq \mathbb{X} - \overline{A}$$

Then  $B_1 \cup B_2 \cup B_3 \cup \dots$  that is a open set can be generated by all the bases containing  $x$ , however, that does not intersects  $A$ . So,  $x \in \overline{A}$ .

□

**Definition 1.6.6** (neighbourhood). <sup>26</sup> *If we say  $U$  is a neighbourhood of  $x$  in  $\mathbb{X}$ , then  $U$  is an open set in  $\mathbb{X}$  containing  $x$*

<sup>26</sup>Some other mathematicians use neighbourhood to say that  $U$  merely contains an open set containing  $x$ . The book does not give a formal definition for the word merely, and I am not sure either.



**Definition 1.6.7** (limit point, point of accumulation, cluster point).<sup>27</sup> If  $A$  is a subset of topological space  $\mathbb{X}$ . We say that  $x$  is a limit point of  $A$  if and only if every open sets containing  $x$  intersects  $A$  with some points other than  $x$ .

This condition is also equivalent to the condition that if  $x$  is a limit point of  $A$  if and only if  $x \in A - \{x\}$

**Theorem 1.6.6.**<sup>28</sup> Let  $A$  be a subset of topological space  $\mathbb{X}$ ; let  $A'$  be the set of all limit points of  $A$ . Then

$$\overline{A} = A \cup A'$$

**Corollary 1.6.1.**<sup>29</sup> A subset of a topological space is closed if and only if it contains all its limit point.

**Definition 1.6.8** (converge).<sup>30</sup> We say that a sequence of  $x_1, x_2, x_3 \dots$  **converge** to  $x$ . When for every neighbourhood  $U$  of  $x$ , there exists a positive integer  $N$ , such that for all  $n > N$ ,  $x_n \in U$ .

**Definition 1.6.9** (Hausdorff space). A topological space is called a **Hausdorff space**, if for every distinct  $x_1, x_2$  in  $\mathbb{X}$ , there exists disjoint neighbourhood of  $U_1, U_2$  of  $x_1, x_2$  in  $\mathbb{X}$ .

**Theorem 1.6.7.**<sup>31,32</sup> Every finite point set in a Hausdorff space  $\mathbb{X}$  is closed.

*Proof.* Let  $A$  be a finite point set in a Hausdorff space  $\mathbb{X}$ .

Suppose  $A$  only have one element. Then for every  $x \in \mathbb{X} - A$ , there exists a neighbourhood of  $x$  that does not intersect with  $A$ . So  $A$  is closed.

Suppose  $A$  is a closed finite point set. We take  $x_0 \in \mathbb{X} - A$ . As finite union of closed set is closed,  $A \cup \{x_0\}$  is closed.

Then, from induction, all finite point set in a Hausdorff space is closed.  $\square$

**Theorem 1.6.8.** If  $\mathbb{X}$  is a Hausdorff space, then a sequence of points in  $\mathbb{X}$  converges to at most one point.

*Proof.* Suppose that the following sequence

$$x_1, x_2, x_3 \dots$$

Converge to more than one points say

$$y_1, y_2, y_3 \dots$$

<sup>27</sup>Note that,  $x$  may belong to  $A$  or not, this does not matter.

<sup>28</sup>We omit the proof of this lemma as it is obvious.

<sup>29</sup>We omit the proof of this lemma as it is obvious.

<sup>30</sup>In real line, a sequence can not converge to multiple points, but for an arbitrary topological space, this is possible.

<sup>31</sup>This implies that a sequence in a Hausdorff space cannot converge to multiple points. The following theorem prove this.

<sup>32</sup>The condition every finite point set is closed is weaker than the Hausdorff space condition. For instance, the finite complement topology of  $\mathbb{R}$  met the condition of finite point set. However it is not a Hausdorff space.

Then there exists

$$n_1, n_2, n_3 \dots, U_1, U_2, U_3 \dots$$

Such that for  $n > n_i$

$$x_n \in U_i, y_i \in U_i$$

If we take disjoint  $U_1, U_2$  which is possible as this is a Hausdorff space.

Then the previous condition does not stand. So, every sequence of points in a Hausdorff space can only converge to at most one point.  $\square$

**Definition 1.6.10** (limit). *If a sequence  $x_n$  of points in Hausdorff space converge to the point  $x$ , we denote this by  $x_n \rightarrow x$  and we say the **limit** of  $x_n$  is  $x$ .*

**Definition 1.6.11** ( $T_1$  axiom). *The condition that all finite point set of a topological space is closed is called  $T_1$  **axiom**.*

**Theorem 1.6.9.** *Let  $\mathbb{X}$  be a space satisfying the  $T_1$  axiom; let  $A$  be a subset of  $\mathbb{X}$ . Then the point  $x$  is a limit point of  $A$  if and only if every neighbourhood of  $x$  contains infinitely many points of  $A$ .*

*Proof.* If every neighbourhood of  $x$  contains infinitely many point of  $A$ . Then every neighbourhood of  $x$  intersect with  $A$  with infinite element other than  $x$ , then  $x$  is a limit point of  $A$ .

If  $x$  is a limit point of  $A$ . Suppose that there exists a open set  $U$  containing  $x$  and intersect with  $A$  for finite many points. Let

$$U' = U \cap (A - x)$$

Then,  $x \notin U'$ . Let

$$U'' = U - U'$$

Then  $U''$  is open as  $U'$  is a finite point set and

$$U'' = U - U' = U \cap (\mathbb{X} - U')$$

Also,  $x \in U''$ . Thus,  $U''$  is a open set containing  $x$  that only intersect  $A$  with  $x$  or do not intersect  $A$ . This is a contradiction of  $x$  is a limit point. Thus there does not exists a open set  $U$  containing  $x$  and intersect with  $A$  for finite many points.  $\square$

**Theorem 1.6.10.** <sup>33</sup>*Every simply ordered set is a Hausdorff space in order topology.*

**Theorem 1.6.11.** <sup>34</sup>*The product of two Hausdorff space is a Hausdorff space.*

**Theorem 1.6.12.** <sup>35</sup>*A subspace of a Hausdorff space is a Hausdorff space.*

<sup>33</sup>We omit the proof of this lemma as it is obvious.

<sup>34</sup>We omit the proof of this lemma as it is obvious.

<sup>35</sup>We omit the proof of this lemma as it is obvious.

**1.6.1 Exercise**

1. Give an counter example why  $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$  dose not hold.

*Proof.* Consider the X be the K-topology on the real line.

Let

$$\begin{aligned} A_n &= \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{Z}_+ \\ A &= \cup A_n \end{aligned}$$

Then

$$\begin{aligned} \overline{A_n} &= \left[\frac{1}{n+1}, \frac{1}{n}\right] \\ \cup \overline{A_n} &= (0, 1] \end{aligned}$$

However, as every neighbourhood of 0 intersect  $\cup A_\alpha$  .  $0 \in \overline{\cup A_\alpha}$  .

Thus,  $\overline{\cup A_\alpha} \neq \cup \overline{A_\alpha}$  □

2. Prove that

$$\overline{A - B} \supseteq \overline{A} - \overline{B}$$

*Proof.* If  $x \in \overline{A} - \overline{B}$  . Then

$$x \in \overline{A}, x \notin \overline{B}$$

Thus for open set  $U$  containing  $x$

$$\begin{aligned} \exists U_1 \cap B &= \emptyset \\ \forall U \cap A &\neq \emptyset \end{aligned}$$

Suppose that  $x \notin \overline{A - B}$  . Then

$$\exists U_0 \cap (A - B) = \emptyset$$

Thus,

$$U_0 \cap A \subseteq B$$

Thus,

$$\begin{aligned} U_1 \cap B &= \emptyset \\ U_1 \cap U_0 \cap A &= \emptyset \end{aligned}$$

As  $U_1 \cap U_0$  is an open set containing  $x$  , so there is contradiction with  $x \in \overline{A}$  . Thus  $x \in \overline{A - B}$  . □

3. A **diagonal** is a subset  $\Delta = \{x \times x | x \in \mathbb{X}\}$  of the product topology  $\mathbb{X} \times \mathbb{X}$  where  $\mathbb{X}$  is a topological space. Show that the diagonal is closed in  $\mathbb{X} \times \mathbb{X}$  if and only if  $\mathbb{X}$  is a Hausdorff space.

*Proof.* If  $\mathbb{X}$  is a Hausdorff space. For every element  $x \times y$  of  $\mathbb{X} \times \mathbb{X}$  that not in  $\Delta$ . We take disjoint set  $U_x, U_y$  where  $x \in U_x, y \in U_y$ . Then  $\mathbb{X} \times \mathbb{X} - \Delta = \cup_{x \neq y} U_x \times U_y$ . Where  $\cup_{x \neq y} U_x \times U_y$  is an open set. Thus  $\Delta$  is a closed set.

Conversely, if  $\Delta$  is a closed set, suppose that  $\mathbb{X}$  is not a Hausdorff space. Then there exists distinct  $x, y$  such that every neighbourhood of  $x$  and  $y$  intersect. Let  $\mathbb{B}$  be a basis of topology of  $\mathbb{X}$ . Then  $x \times y \in \mathbb{X} \times \mathbb{X} - \Delta$ . However we cannot find  $B_1, B_2 \in \mathbb{B}, x \times y \in B_1 \times B_2 \subset \mathbb{X} \times \mathbb{X} - \Delta$ . Then  $\Delta$  is not a closed set. So there is a contradiction, then  $\mathbb{X}$  must be a Hausdorff space.  $\square$

4. Prove that  $T_1$  axiom is equivalent to the condition such that for every distinct pair  $x, y$  of  $\mathbb{X}$ , there exists neighbourhood of  $x$  does not contain  $y$ .

*Proof.* First if  $T_1$  axiom hold, then for every pair  $x, y$ , the neighbourhood  $\mathbb{X} - \{y\}$  of  $x$  does not contain  $y$ , so the second condition hold.

Conversely, if the second condition hold. Suppose that we can find a finite points set say  $\{x_1, x_2, x_3 \dots\}$ , then there must exists  $x \in \{x_1, x_2, x_3 \dots\}$  such that the set  $\{x\}$  is not closed. Then  $\overline{\{x\}} - \{x\} \neq \emptyset$ . Let  $y \in \overline{\{x\}} - \{x\}$ , then every neighbourhood of  $y$  must contain  $x$ , this is a contradiction to the second condition, so the  $T_1$  axiom must hold.  $\square$

5. If  $A \subseteq \mathbb{X}$ , we define the **boundary** of  $A$  by the equation

$$\text{Bd}A = \overline{A} \cap \overline{\mathbb{X} - A}$$

- (a) Show that  $\text{Int}A$  and  $\text{Bd}A$  are disjoint and  $\overline{A} = \text{Int}A \cup \text{Bd}A$ .

*Proof.* For every  $x \in \text{Bd}A$ , every open set contain  $x$  must intersect  $A$  and  $\mathbb{X} - A$  so, there is no open set  $U$  contain  $x$ ,  $U \subseteq A$ .

For every  $x' \in \text{Int}A$ , there exists  $U' \subseteq A$ , so  $\text{Bd}A$  and  $\text{Int}A$  are disjoint sets.

For every  $x \in \overline{A}$ ,  $x \in \text{Bd}A$  or  $x \notin \text{Bd}A$ . We discuss the condition that  $x \notin \text{Bd}A$ .

Then  $x \notin \overline{\mathbb{X} - A}$ , then there exists a open set  $U$  containing  $x$ , that does not intersect with  $\mathbb{X} - A$ . Thus  $U \subseteq A$ , thus  $x \in \text{Int}A$ . So  $\overline{A} \subseteq \text{Int}A \cup \text{Bd}A$ .

Then,  $\text{Bd}A \subseteq \overline{A}$ ,  $\text{Int}A \subseteq A \subseteq \overline{A}$ . Thus,  $\overline{A} \supseteq \text{Int}A \cup \text{Bd}A$

So,  $\overline{A} = \text{Int}A \cup \text{Bd}A$   $\square$

(b) Show that  $\text{Bd}A = \emptyset$  if and only if  $A$  is both open and closed.

*Proof.* So,  $\text{Int}A = \overline{A}$ , then  $\text{Bd}A = \emptyset$  follows directly from  $\overline{A} = \text{Int}A \cup \text{Bd}A$ .  $\square$

(c) Show that  $U$  is open if and only if  $\text{Bd}U = \overline{U} - U$ .

*Proof.* Suppose  $U$  is open. Then  $\overline{\mathbb{X} - U} = \mathbb{X} - U$ . Then for every  $x \in U$ ,  $x \notin \mathbb{X} - U$ ,  $x \notin \overline{\mathbb{X} - U}$ . Thus  $\overline{U} \cap \overline{\mathbb{X} - U} = \overline{U} - U$ .

Conversely, suppose  $\text{Bd}U = \overline{U} - U$ . Then for every  $x \in U$ ,  $x \notin \text{Bd}U$ . Then as  $\overline{U} = \text{Int}U \cup \text{Bd}U$ ,  $x \in \text{Int}U$ . So  $\text{Int}U \supseteq U$ . Thus  $U = \text{Int}U$ . Thus,  $U$  is open.  $\square$

## 1.7 Continuous Function

**Definition 1.7.1** (continuous).<sup>36</sup> Let  $\mathbb{X}$  and  $\mathbb{Y}$  be topological spaces. A function  $f : \mathbb{X} \rightarrow \mathbb{Y}$  is said to be **continuous** if for each open subset  $V$  of  $\mathbb{Y}$ , the set  $f^{-1}(V)$  is an open subset of  $\mathbb{X}$ .

**Theorem 1.7.1.** Let  $\mathbb{X}$  and  $\mathbb{Y}$  be topological spaces; let  $f : \mathbb{X} \rightarrow \mathbb{Y}$ . Then the following are equivalent.

1.  $f$  is continuous.
2. For every subset  $A$  of  $X$ , one has  $f(\overline{A}) \subseteq \overline{f(A)}$ .
3. For every closed set  $B$  of  $\mathbb{Y}$ , the set  $f^{-1}(B)$  is closed in  $\mathbb{X}$ .
4. For each  $x \in \mathbb{X}$  and each neighbourhood  $V$  of  $f(x)$ , there is a neighbourhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .

*Proof.*

1  $\Rightarrow$  3:

Let  $A$  be a open set in  $\mathbb{Y}$ .  $f^{-1}(\mathbb{Y} - A) = \mathbb{X} - f^{-1}(A)$ .

3  $\Rightarrow$  1:

Let  $A$  be a closed set in  $\mathbb{Y}$ .  $f^{-1}(\mathbb{Y} - A) = \mathbb{X} - f^{-1}(A)$ .

1  $\Rightarrow$  2:

For  $x \in \overline{A}$ , we take a open set  $f(x) \in U \subseteq \mathbb{Y}$ . Thus  $x \in f^{-1}(U) \cap A \neq \emptyset$ . Thus  $U \cap f(A) \neq \emptyset$ . So  $f(x) \in \overline{f(A)}$ . Thus  $f(\overline{A}) \subseteq \overline{f(A)}$ .

2  $\Rightarrow$  3:

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<sup>36</sup>As the continuity of a function is different as the topological spaces are different. So if we want to emphasis this fact, we say that  $f$  is continuous **relative** to specific topologies on  $\mathbb{X}$  and  $\mathbb{Y}$ .

Suppose  $f$  is not continuous. Then there must exist  $V$ , such that  $f^{-1}(V) = U$  is not closed. Thus  $\bar{U} \supset B = f^{-1}(A)$ . Thus  $f\bar{B} \supset A$ . However  $f(\bar{B}) \subseteq \overline{f(B)} = A$ . There is a contradiction. So  $f$  must be continuous.

1  $\Rightarrow$  4:

For every neighbourhood  $V$  of  $f(x)$ ,  $f^{-1}(V)$  is a neighbourhood of  $x$  that  $f(f^{-1}(V)) \subseteq V$ .

4  $\Rightarrow$  1:

We take an open set  $V$  of  $\mathbb{Y}$ . Let  $S$  be the collection of all open set  $U$  in  $\mathbb{X}$  such that  $f(U) \subseteq V$ . The set cannot be empty unless  $f^{-1}(V) = \emptyset$ . Let  $U_0$  denote the union of all the element in  $S$ . We prove that  $U_0 = f^{-1}(V)$ .

For all element  $x \in U_0$ ,  $f(x) \in V$ . Thus  $U_0 \subseteq f^{-1}(V)$ .

For all element  $x \in f^{-1}(V)$ . There is a  $U'$  such that  $x \in U'$ ,  $f(U') \subseteq V$ . This follows from the condition 4. Thus  $U' \in S$ . Thus  $x \in U_0$ . Thus  $U_0 \subseteq f^{-1}(V)$ . As  $U_0$  is union of open set,  $U_0$  is also open. Thus,  $f^{-1}(V)$  is also open.

Thus  $f$  is continuous.  $\square$

**Definition 1.7.2** (homeomorphism).<sup>37</sup> Let  $\mathbb{X}$  and  $\mathbb{Y}$  be topological space; let  $f : \mathbb{X} \rightarrow \mathbb{Y}$  be a bijection. If both the function  $f$  and the inverse function

$$f^{-1} : \mathbb{Y} \rightarrow \mathbb{X}$$

are continuous, then  $f$  is called a **homeomorphism**

**Definition 1.7.3** (topological imbedding). Suppose that  $f : \mathbb{X} \rightarrow \mathbb{Y}$  is an injective continuous map, where  $\mathbb{X}$  and  $\mathbb{Y}$  are topological spaces. Let  $\mathbb{Z}$  be the image set  $f(\mathbb{X})$ , considered as a subspace of  $\mathbb{Y}$ ; then the function  $f' : \mathbb{X} \rightarrow \mathbb{Z}$  obtained by restricting the range of  $f$  is bijective. If  $f'$  happens to be a homeomorphism of  $\mathbb{X}$  with  $\mathbb{Z}$ , we say that the map  $f : \mathbb{X} \rightarrow \mathbb{Y}$  is a **topological imbedding**, or simply an **imbedding**, of  $\mathbb{X}$  in  $\mathbb{Y}$ .

**Theorem 1.7.2** (Rules for constructing continuous functions). Let  $\mathbb{X}$ ,  $\mathbb{Y}$ , and  $\mathbb{Z}$  be topological spaces.

1. (Constant function) If  $f : \mathbb{X} \rightarrow \mathbb{Y}$  maps all of  $\mathbb{X}$  into the single point  $y_0$  of  $\mathbb{Y}$ , then  $f$  is continuous.
2. (Inclusion) If  $A$  is a subspace of  $\mathbb{X}$ , the inclusion function  $j : A \rightarrow \mathbb{X}$  is continuous.
3. (Composites) If  $f : \mathbb{X} \rightarrow \mathbb{Y}$  and  $g : \mathbb{Y} \rightarrow \mathbb{Z}$  are continuous, then the map  $g \circ f : \mathbb{X} \rightarrow \mathbb{Z}$  is continuous.

<sup>37</sup>A equivalent way to define homeomorphism, is that for any open subset  $U$  of  $\mathbb{X}$ ,  $f(U)$  is open if and only if  $U$  is open.

4. (Restricting the domain) If  $f : \mathbb{X} \rightarrow \mathbb{Y}$  is continuous, and if  $A$  is a subspace of  $\mathbb{X}$ , then the restriction function  $f|_A : A \rightarrow \mathbb{Y}$  is continuous.
5. (Restricting or expanding the range) Let  $f : \mathbb{X} \rightarrow \mathbb{Y}$  is continuous. Let  $\mathbb{Z}$  be a subspace of  $\mathbb{Y}$  containing the image  $f(\mathbb{X})$ , the function  $h : \mathbb{X} \rightarrow \mathbb{Z}$  obtained by restricting the range of  $f$  is continuous. If  $\mathbb{Z}$  is a space having  $\mathbb{Y}$  as a subspace, then the function  $h : \mathbb{X} \rightarrow \mathbb{Y}$  obtained by expanding the range of  $f$  is continuous.
6. (Local formulation of continuity) The map  $f : \mathbb{X} \rightarrow \mathbb{Y}$  is continuous if  $\mathbb{X}$  can be written as the union of open sets  $U_\alpha$  such set  $f|_{U_\alpha}$  is continuous for each  $\alpha$ .

*Proof.*

1.  $f^{-1}(U)$  of any open set  $U$  is  $\mathbb{X}$ , thus  $f$  is continuous.
2. For every open subset  $U$  of  $\mathbb{X}$ ,  $j^{-1}(U) = U \cap A$  is continuous in  $A$ . Thus  $j$  is a continuous function.
3. For every open subset  $U$  of  $\mathbb{Z}$ ,  $f^{-1}(U)$  is open in  $\mathbb{Y}$ , and  $g^{-1}(f^{-1}(U))$  is open in  $\mathbb{X}$ . Thus,  $g \circ f$  is continuous
4. For every open subset  $U$  of  $\mathbb{Y}$ ,  $f^{-1}(U)$  is open in  $\mathbb{X}$ , thus  $f^{-1}(U) \cap A$  is open in  $A$ . Thus the function  $f|_A$  is continuous.
5. If  $\mathbb{Z}$  is a subspace of  $\mathbb{Y}$ , then every open subset of  $\mathbb{Z}$  can be represented as  $U \cap \mathbb{Z}$ , where  $U$  is a open subset of  $\mathbb{Y}$ . Thus  $h^{-1}(U \cap \mathbb{Z}) = g^{-1}(\mathbb{Z}) \cap g^{-1}(U) = \mathbb{X} \cap g^{-1}(U)$  which is a open subset of  $X$ , thus  $h$  is continuous.  
If  $\mathbb{Y}$  is a subspace of  $\mathbb{Z}$ . Then we take a open subset  $U$  of  $\mathbb{Z}$ .  $h^{-1}(U) = g^{-1}(U \cap \mathbb{Y})$  which is open in  $\mathbb{X}$ , thus  $h$  is continuous.
6. if  $f|_{U_\alpha}$  is continuous for each  $\alpha$ . For every open subset  $U$  of  $\mathbb{Y}$ .

$$U = \cup_\alpha (U_\alpha \cap U)$$

where  $U_\alpha \cap U$  is open both in  $U_\alpha$  and in  $\mathbb{Y}$ .

Thus,

$$\begin{aligned} f^{-1}(U) &= f^{-1}(\cup_\alpha (U_\alpha \cap U)) \\ &= \cup_\alpha ((f|_{U_\alpha})^{-1}(U_\alpha \cap U)) \end{aligned}$$

and each  $(f|_{U_\alpha})^{-1}(U_\alpha \cap U)$  is open, thus  $f^{-1}(U)$  is open.

□

**Theorem 1.7.3** (The pasting lemma).<sup>38</sup> Let  $\mathbb{X} = A \cup B$ , where  $A, B$  are closed in  $\mathbb{X}$ . Let  $f : A \rightarrow \mathbb{Y}$  and  $g : B \rightarrow \mathbb{Y}$  be continuous. If  $f(x) = g(x)$  for every  $x \in A \cap B$ , then  $f, g$  combine to give a continuous function  $h : \mathbb{X} \rightarrow \mathbb{Y}$ , defined by setting  $h(x) = f(x), x \in A$  and  $h(x) = g(x), x \in B$ .

**Theorem 1.7.4** (Maps into products).<sup>39</sup> Let  $f : A \rightarrow \mathbb{X} \times \mathbb{Y}$  be given by the equation

$$f(a) = (f_1(a), f_2(a))$$

Then, the function  $f$  is continuous if and only if the functions

$$f_1 : A \rightarrow \mathbb{X}, f_2 : A \rightarrow \mathbb{Y}$$

are continuous.

*Proof.* Let  $\pi_1, \pi_2$  be the projection function

$$\begin{aligned}\pi_1 & : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \\ \pi_2 & : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}\end{aligned}$$

We first proof that if  $U$  is an open subset of  $\mathbb{X} \times \mathbb{Y}$ ,

$$f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$$

Let  $x \times y \in U$ ,  $f^{-1}(x \times y)$  contains all  $a$  such that  $f(a) = x \times y$ . Then for any  $a \in f^{-1}(x \times y)$ ,  $a \in f_1^{-1}(\pi_1(x \times y)), a \in f_2^{-1}(\pi_2(x \times y))$ . Thus,  $f^{-1}(x \times y) \subseteq f_1^{-1}(\pi_1(x \times y)) \cap f_2^{-1}(\pi_2(x \times y))$ . Thus  $f^{-1}(U) \subseteq f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$ .

Also, if  $a \in f_1^{-1}(\pi_1(x \times y)), a \in f_2^{-1}(\pi_2(x \times y))$ ,  $f_1(a) = x, f_2(a) = y$ . Thus  $f(a) = x \times y$ . Thus  $a \in f^{-1}(x \times y)$ . Thus  $f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$

Let  $U$  be any open subset of  $\mathbb{X} \times \mathbb{Y}$

$$f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$$

Where  $f_1^{-1}(\pi_1(U))$  and  $f_2^{-1}(\pi_2(U))$  are both open set. Thus  $f^{-1}(U)$  is open.  $\square$

<sup>38</sup>The proof of this theorem is similar to the "Local formulation of continuity" condition of "Rules for constructing continuous functions", so we omit the proof here.

<sup>39</sup>The map  $f_1, f_2$  are called the **coordinate functions** of  $f$



## 1.7.1 Exercise

1. Let  $\mathbb{Y}$  be an ordered set in the order topology. Let  $f, g : \mathbb{X} \rightarrow \mathbb{Y}$  be continuous, show that the set  $A = \{x | f(x) \leq g(x)\}$  is closed in  $\mathbb{X}$ .

*Proof.* We only need to prove  $\mathbb{X} - A$  is open in  $\mathbb{X}$ . We take  $x \in \mathbb{X} - A$ . Thus  $f(x) > g(x)$ .

Let  $U_1, U_2$  be the open set in  $\mathbb{Y}$  that met the following demand

$$\begin{aligned} \forall y_1 \in U_1, y_2 \in U_2, y_1 > y_2 \\ f(x) \in U_1, g(x) \in U_2 \end{aligned}$$

As  $\mathbb{Y}$  is an ordered set,  $U_1, U_2$  must exist.

Let  $U = f^{-1}(U_1) \cap g^{-1}(U_2)$ . It is obvious that  $U$  is a open set, and  $x \in U$ .

Also, for any  $y \in U$ ,  $f(y) > g(y)$ . Thus  $U \subseteq A$ . Thus  $A$  is an open set.  $\square$

2. Let  $\{A_\alpha\}$  be a collection of subsets of  $\mathbb{X}$ ; let  $\mathbb{X} = \cup_\alpha A_\alpha$ . Let  $f : \mathbb{X} \rightarrow \mathbb{Y}$ ; suppose that  $f|_{A_\alpha}$  is continuous for each  $\alpha$ . An indexed family of sets  $\{A_\alpha\}$  is said to be **locally finite** if each point  $x$  of  $\mathbb{X}$  has a neighbourhood that intersect  $A_\alpha$  for only finitely many values of  $\alpha$ . Show that if the family  $\{A_\alpha\}$  is locally finite and each  $A_\alpha$  is closed, then  $f$  is continuous.

*Proof.* For any closed subset  $U$  of  $\mathbb{Y}$ . Let

$$V = \cup f|_{A_\alpha}(U)$$

We prove that  $V$  is closed, so,  $f$  is continuous.

To prove that  $V$  is closed, we prove that  $\bar{V} = V$ . That is for any  $x \in \bar{V}$ , we prove  $x \in V$ . For any neighbourhood  $B$  of  $x$ , let  $C_B$  denote the set that contain all  $\alpha$ , such that  $f|_{A_\alpha(U)}$  intersect with  $B$ . As  $B$  intersect with  $V$ ,  $C_B$  can not be empty.

Let

$$\mathbb{C} = \{C_B | B \text{ be a neighbourhood of } x\}$$

As  $\{A_\alpha\}$  is locally finite,  $\mathbb{C}$  contain at least one element with finite elements.

Also

$$C_{B_1 \cap B_2} \subseteq C_{B_1} \cap C_{B_2}$$

Let  $\leq$  be a partial order on the  $\mathbb{C}$ . If  $C_{B_1} \subseteq C_{B_2}$ , we say that  $C_{B_1} \geq C_{B_2}$ .

If there is chain in  $\mathbb{C}$

$$C_{B_1} \leq C_{B_2} \dots$$

Let  $C_{B_0}$  be a element of  $\mathbb{C}$  with finite element. If  $C_{B_0} \subseteq C_{B_1}, C_{B_0} \subseteq C_{B_2} \dots$ . Then  $C_{B_0}$  is a upper bound of the chain.

If  $C$  is not a subset of all element of the chain. Then we construct a new set say

$$D = \{C_{B_0 \cap B_1}, C_{B_0 \cap B_2} \dots\}$$

Let

$$\mathbb{D} = \{C_{D_1 \cap D_2 \cap \dots} | C_{D_1}, C_{D_2} \dots \in D\}$$

As  $C_{B_0}$  is a finite set,  $D$  is a finite set,  $\mathbb{D}$  is also a finite set. Thus there must be a maximal element  $E \in \mathbb{D}$  that is the subset of all element of  $\mathbb{D}$ . Then  $E$  is a subset of all element of the chain. Thus  $E$  is a upper bound of the chain.

Thus, there must be a maximal element  $C_F$  of  $\mathbb{C}$ , that is a subset of all element of  $\mathbb{C}$ .

Let  $G$  be the set be the union of all element of  $C_F$ .

As  $C_F$  is finite,  $G$  is closed. And all neighbourhood of  $x$  intersect with  $G$ . Thus  $x \in G$ .

As  $G$  is a subset of  $V$ ,  $x \in V$ . So  $V$  is closed. And  $f$  is a continuous function on  $\mathbb{X}$ .

□

3. Let  $A$  be a subset of topological space  $\mathbb{X}$ , let  $\mathbb{Y}$  be a Hausdorff space. Let  $f : A \rightarrow \mathbb{Y}$  be a continuous function. Let  $g : \bar{A} \rightarrow \mathbb{Y}$  also be a continuous function where  $g(x) = f(x), x \in A$ . Prove that  $g$  is uniquely determined by  $f$ .<sup>40</sup>

*Proof.* Say  $g$  and  $h$  are two distinct function that met the demand.

So there exist  $x_0$  such that  $g(x_0) \neq h(x_0)$ .

As  $\mathbb{Y}$  is a Hausdorff space, so there exist disjoint open subset  $g(x_0) \in U$  and  $h(x_0) \in V$ .

Then  $g^{-1}(U)$  and  $h^{-1}(V)$  are both open subset of  $\mathbb{X}$  that contain  $x_0$ .

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<sup>40</sup>It is possible that  $g$  does not exist.

Let  $\mathbb{X}$  be the real line with order topology. Let  $\mathbb{Y}$  be  $\{0, 1\}$ .

Let  $A = \mathbb{X} - \{0\}$ .

Let,

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

So, it is obvious that  $f$  is a continuous function on  $\mathbb{X}$ . However  $g$  does not exist in this case.

If  $g^{-1}(U) \cap h^{-1}(V) \cap A \neq \emptyset$ . Then there exist  $x_1 \in g^{-1}(U) \cap h^{-1}(V) \cap A$  such that  $g(x_1) \in U$  and  $h(x_1) \in V$  and  $g(x_1) = h(x_1)$ . However  $U$  and  $V$  are disjoint. So there is a contradiction.

As  $g^{-1}(U) \cap h^{-1}(V)$  is a open subset contain  $x_0$ . So  $g^{-1}(U) \cap h^{-1}(V)$  must intersect with  $A$ . So it is impossible that  $g^{-1}(U) \cap h^{-1}(V) \cap A = \emptyset$ .

So  $g = h$ .

□