

Topology Note

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Chapter 1

Topology Spaces and Continuous Function

1.1 Basic Definition of Topology

Definition 1.1.1 (topology). A **topology** on a set \mathbb{X} is a collection \mathbb{T} of subsets of \mathbb{X} having the following properties:

- \emptyset and \mathbb{X} are in \mathbb{T}
- The union of the elements of any sub collection of \mathbb{T} is in \mathbb{T}
- The intersection of the elements of any **finite** sub collection of \mathbb{T} is in \mathbb{T}

Definition 1.1.2 (topology space). A **topological space** is a set \mathbb{X} for which a topology \mathbb{T} has been specified.

Definition 1.1.3 (open set). A **open set** \mathbb{U} is a subset of \mathbb{X} that belongs to a topology \mathbb{T} of \mathbb{X} .

Definition 1.1.4 (open sets). A topology can also be called a **open sets**

Definition 1.1.5 (discrete topology). The set of all subsets of a set \mathbb{X} formed a topology called **discrete topology**

Definition 1.1.6 (trivial topology). The set consisting the set \mathbb{X} and \emptyset only formed a topology of \mathbb{X} called **trivial topology**

Definition 1.1.7 (finite complement topology). Let \mathbb{X} be a set. Let \mathbb{T}_f be the collection of all subsets \mathbb{U} of \mathbb{X} such that $\mathbb{X} - \mathbb{U}$ either if a **finite**¹ or is all of \mathbb{X} . Then \mathbb{T}_f is a topology on \mathbb{X} , called the **finite complement topology**.

¹The set \mathbb{U} can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, \mathbb{U} will not be a topology.

Definition 1.1.8 (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable). Let \mathbb{T} and \mathbb{T}' be two topology on a given set \mathbb{X} . If \mathbb{T} is a subset of \mathbb{T}' , we say that \mathbb{T}' is **finer** or **larger** than \mathbb{T} . If \mathbb{T} is a proper subset of \mathbb{T}' , we say that \mathbb{T}' is **strictly finer** or **strictly larger** than \mathbb{T} . We also say that \mathbb{T} is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than \mathbb{T}' . We say that \mathbb{T} and \mathbb{T}' is **comparable** if either \mathbb{T} is a subset of \mathbb{T}' or \mathbb{T}' is a subset of \mathbb{T} .

1.2 Basis for a Topology

Definition 1.2.1 (basis). If \mathbb{X} is a set, a **basis** for a topology on \mathbb{X} is a collection \mathbb{B} of subsets of \mathbb{X} (called **basis elements**) such that:

- For each $x \in \mathbb{X}$, there is at least one basis element B containing x
- If x belongs to the intersection of two basis elements B_1 and B_2 , then there is another element $B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$

Definition 1.2.2 (topology generated by basis). Let \mathbb{B} be a basis on \mathbb{X} . Let \mathbb{U} be a set containing all subsets U of \mathbb{X} such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such \mathbb{U} formed a topology on \mathbb{X} , called **topology \mathbb{T} generated by \mathbb{B}**

Lemma 1.2.1. Let \mathbb{X} be a set. Let \mathbb{B} be a basis for a topology \mathbb{T} on \mathbb{X} . Then \mathbb{T} equals to the set of all possible unions of elements of \mathbb{B} .

Proof. Let set \mathbb{U} be the set of all possible unions of elements of \mathbb{B} . For any $U \in \mathbb{U}$. $U = \cup B$ ² for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in \mathbb{B}$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, \mathbb{U} equals to \mathbb{T} . □

Lemma 1.2.2.³ Let \mathbb{X} be a topological space. Suppose that \mathbb{C} is a collection of open sets of \mathbb{X} such that for each open set U of \mathbb{X} and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq U$. Then \mathbb{C} is a basis for the topology of \mathbb{X} .

Lemma 1.2.3.⁴ Let \mathbb{B} and \mathbb{B}' be basis for the topologies \mathbb{T} and \mathbb{T}' , respectively, on \mathbb{X} . Then the following are equivalent:

- \mathbb{T}' is finer than \mathbb{T}
- For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing x , there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.

²Note that this expression may not be unique.

³We omit the proof of this lemma as it is obvious.

⁴We omit the proof of this lemma as it is obvious.

Definition 1.2.3 (standard topology on the real line). Let $\mathbb{B} = \{B \mid B = \{x \mid a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. \mathbb{B} formed a basis on real line. The topology generated by \mathbb{B} is called the **standard topology on the real line**⁵.

Definition 1.2.4 (lower limit topology on the real line). Let $\mathbb{B} = \{B \mid B = \{x \mid a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. \mathbb{B} formed a basis on real line. The topology generated by \mathbb{B} is called the **lower limit topology on the real line**. When \mathbb{R} is given this topology, we denote it by \mathbb{R}_l .

Definition 1.2.5 (K-topology on the real line). Let $\mathbb{B} = \{B \mid B = \{x \mid a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. Let $K = \{x \mid x = \frac{1}{n}, n \in \mathbb{Z}_+\}$. $\mathbb{B} \cup \{B - K \mid B \in \mathbb{B}\}$ formed a basis on real line. The topology generated by \mathbb{B} is called the **K-topology on the real line**. When \mathbb{R} is given this topology, we denote it by \mathbb{R}_K .

Lemma 1.2.4.⁶ The topologies \mathbb{R}_l and \mathbb{R}_K is strictly finer than the standard topology on \mathbb{R} .

Lemma 1.2.5. The topologies of \mathbb{R}_l and \mathbb{R}_K is not comparable.

Proof. Let \mathbb{T}_l and \mathbb{T}_K be topologies of \mathbb{R}_l and \mathbb{R}_K respectively. Let $K = \{x \mid x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that \mathbb{T}_l is not finer than \mathbb{T}_K . Let $U = \{x \mid -1 < x < 1\} - K, x = 0$. If there exist $B = \{x \mid a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus B is not a subset of U .

Then we proof that \mathbb{T}_K is not finer than \mathbb{T}_l . Let $U' = \{x \mid a' \leq x < b'\}$. If there exist $B' = \{x \mid a'' < x < b''\}$ or $\{x \mid a'' < x < b''\} - K$ such that $a' \in B' \subseteq U'$. Thus $a'' < a' < b''$. Thus there exist c that $a'' < c < a', c \in B', c \notin U'$. Thus $B' \not\subseteq U'$.

Thus the topologies of \mathbb{R}_l and \mathbb{R}_K is not comparable. \square

Definition 1.2.6 (subbasis). A **subbasis** \mathbb{S} for a topology on \mathbb{X} is a collection of subsets of \mathbb{X} whose union equals \mathbb{X} . The **topology generated by the subbasis** \mathbb{S} is defined to be the collection \mathbb{T} ⁷ of all unions of finite intersections of elements of \mathbb{S} .

1.2.1 Exercise

1. Show that if \mathbb{A} is a basis for a topology on \mathbb{X} , then the topology generated by \mathbb{A} equals the intersection of all topologies on \mathbb{X} that contain \mathbb{A} . Prove the same if \mathbb{A} is a subbasis.

Proof. As a subbasis is also a basis, we will directly prove the case of subbasis here.

⁵Whenever we consider \mathbb{R} , we shall suppose it is given this topology unless we specifically state otherwise.

⁶We omit the proof of this lemma as it is obvious.

⁷It is obvious that \mathbb{T} is a topology, we just omit the proof here.

Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain \mathbb{A} . Let \mathbb{T} be the topology that \mathbb{A} generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.⁸

First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

Thus, $\mathbb{T} = \mathbb{T}'$ □

1.3 The Order Topology

Definition 1.3.1 (interval). Let \mathbb{X} is a set having a simple order relation $<$. Given elements a and b of \mathbb{X} such that $a < b$, there are four subsets of \mathbb{X} that are called **intervals** determined by a and b :

- $(a, b) = \{x | a < x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $[a, b] = \{x | a \leq x \leq b\}$

(a, b) is called an **open interval** on \mathbb{X} . $[a, b]$ is called an **closed interval** on \mathbb{X} . $(a, b]$ and $[a, b)$ is called **half-open intervals**.

Definition 1.3.2 (order topology).⁹ Let \mathbb{X} be a set with a simple order relation; assume \mathbb{X} has more than one element. Let \mathbb{B} be the collection of all sets of the following types:

- All open intervals (a, b) in \mathbb{X} .
- All intervals of the form $[a_0, b)$, where a_0 is the smallest element(if exist) of \mathbb{X} .
- All intervals of the form $(a, b_0]$, where b_0 is the largest element(if exist) of \mathbb{X} .

The collection \mathbb{B} formed a basis for a topology on \mathbb{X} , which is called the order topology.

Definition 1.3.3 (ray).¹⁰¹¹ If \mathbb{X} is an ordered set, and a is an element of \mathbb{X} , there are four subsets of \mathbb{X} that are called **rays** determined by a :

- $(a, +\infty) = \{x | x > a\}$
- $(-\infty, a) = \{x | x < a\}$

⁸It is obvious that \mathbb{T}' is also a topology, we just omit the proof here.

⁹The standard topology on \mathbb{R} is an order topology derived from the usual order on \mathbb{R} .

¹⁰open rays are always open sets in the order topology

¹¹the open rays also formed a subbasis of the order topology

- $[a, +\infty) = \{x | x \geq a\}$
- $(-\infty, a] = \{x | x \leq a\}$

$(a, +\infty)$ and $(-\infty, a)$ are called **open rays**. $[a, +\infty)$ and $(-\infty, a]$ are called **closed rays**.

1.4 The Product Topology

Definition 1.4.1 (product topology). Let \mathbb{X} and \mathbb{Y} be topological spaces. The **product topology** on $\mathbb{X} \times \mathbb{Y}$ having a basis \mathbb{B} containing all sets of the form $U \times V$, where U and V is open sets of \mathbb{X} and \mathbb{Y} respectively.

Theorem 1.4.1.¹² If \mathbb{B} and \mathbb{C} is basis for the topology of \mathbb{X} and \mathbb{Y} respectively, then the collection

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} \text{ and } C \in \mathbb{C}\}$$

is a basis for the topology of $\mathbb{X} \times \mathbb{Y}$

Definition 1.4.2 (projection). Let $\pi_1 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$ be defined by the equation:

$$\pi_1(x, y) = x$$

Let $\pi_2 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}$ be defined by the equation:

$$\pi_2(x, y) = y$$

The maps π_1 and π_2 are called the **projections** of $\mathbb{X} \times \mathbb{Y}$ onto its first and second factors, respectively.

Theorem 1.4.2.¹³ The collection

$$\mathbb{S} = \{\pi_1^{-1}(U) | U \text{ open in } \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V \text{ open in } \mathbb{Y}\}$$

is a subbasis for the product topology on $\mathbb{X} \times \mathbb{Y}$.

Definition 1.4.3 (box topology). Let,

$$\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n \text{ or } \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots$$

In the first case, all the sets of the form $U_1 \times \cdots \times U_n$ where U_i is a open set of \mathbb{X}_i form a basis.

In the second case, all the sets of the form $U_1 \times U_2 \times \cdots$ where U_i is a open set of \mathbb{X}_i also form a basis.

Topology defined in this way was called a **box topology**.

¹²We omit the proof of this theorem as it is obvious.

¹³We omit the proof of this theorem as it is obvious.

Definition 1.4.4 (product topology).¹⁴ Let,

$$\mathbb{X} = \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots \times \mathbb{X}_n \text{ or } \mathbb{X}_1 \times \mathbb{X}_2 \times \cdots$$

Let π_i be the projection function¹⁵ that

$$\pi_i : \mathbb{X} \rightarrow \mathbb{X}_i$$

And if $x \in \mathbb{X}$

$$\pi_i(x) = x_i$$

All the set of the form $\pi_i^{-1}(U_i)$ where i is arbitrary and U_i is an open set of \mathbb{X}_i , form a subbasis of \mathbb{X} . The topology generated by this subbasis is called **product topology**. And \mathbb{X} is called a **product space**.

Definition 1.4.5 (J-tuple). Let J be an index set¹⁶. Give a set \mathbb{X} , a **J-tuple** is defined as a function $x : J \rightarrow \mathbb{X}$. If α is an element of J , $x(\alpha)$ is often denoted by x_α and is called the α th **coordinate** of x . And the function x itself is often denoted by the symbol

$$(x_\alpha)_{\alpha \in J}$$

The set of all J-tuples of elements of \mathbb{X} is often denoted by \mathbb{X}^J .

Definition 1.4.6 (cartesian product). Let $\{A_\alpha\}_{\alpha \in J}$ be an indexed family of sets; let $\mathbb{X} = \bigcup_{\alpha \in J} A_\alpha$. The **cartesian product** of this indexed family is denoted by

$$\prod_{\alpha \in J} A_\alpha$$

And is defined to be the set of all J-tuples $(x_\alpha)_{\alpha \in J}$ of elements of \mathbb{X} such that $x_\alpha \in A_\alpha$ for each $\alpha \in J$. That is, it is the set of all functions

$$x : J \rightarrow \bigcup_{\alpha \in J} A_\alpha$$

such that $x(\alpha) \in A_\alpha$ for each $\alpha \in J$.

Theorem 1.4.3 (Comparison of the box and product topologies).¹⁷ The box topology on $\prod \mathbb{X}_\alpha$ has a basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α . The product topology on $\prod \mathbb{X}_\alpha$ has a basis all sets of the form $\prod U_\alpha$ where U_α is open in X_α for each α and U_α equals \mathbb{X}_α except for finitely many values of α .

¹⁴In the finite case, the product topology and box topology are the same, however they differ when \mathbb{X} is a infinite cartesian product.

¹⁵This is also called a **projection mapping** in a cartesian product.

¹⁶A index set was the set $\{1, \dots, n\}$ or the set \mathbb{Z}_+ .

¹⁷It is assumed that it is given product topology when considering $\prod X_\alpha$ unless it state specifically.

Theorem 1.4.4. ¹⁸Suppose the topology on each space \mathbb{X}_α is given by a basis \mathbb{B}_α . The collection of all sets of the form

$$\prod_{\alpha \in J} B_\alpha$$

where $B_\alpha \in \mathbb{B}_\alpha$ form a basis for the box topology on $\prod_{\alpha \in J} \mathbb{X}_\alpha$.

The collection of all sets of the same form, where $B_\alpha \in \mathbb{B}_\alpha$ for finitely many indices α and $B_\alpha = \mathbb{X}_\alpha$ for all the remaining indices, will form a basis for the product topology $\prod_{\alpha \in J} \mathbb{X}_\alpha$.

Theorem 1.4.5. ¹⁹Let A_α be a subspace of \mathbb{X}_α , for each $\alpha \in J$. Then $\prod A_\alpha$ is a subspace of $\prod \mathbb{X}_\alpha$ if both products are given the box topology, or if both products are given the product topology.

Theorem 1.4.6. ²⁰If each space \mathbb{X}_α is a Hausdorff space, then $\prod \mathbb{X}_\alpha$ is a Hausdorff space in both the box and product topologies.

Theorem 1.4.7. Let $\{\mathbb{X}_\alpha\}$ be an indexed family of spaces; let $A_\alpha \subseteq \mathbb{X}_\alpha$ for each α . If $\prod \mathbb{X}_\alpha$ is given either the product or the box topology, then

$$\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$$

Proof. Let π_α represent the projection mapping.

Let x be an element of $\prod \mathbb{X}_\alpha$. Let V be an open set in $\prod \mathbb{X}_\alpha$ that containing x .

If $x \in \prod \overline{A_\alpha}$, then $\pi_\alpha(V)$ is a open set in \mathbb{X}_α that containing x_α . Thus $\pi_\alpha(V)$ intersect with A_α . Thus V intersect with $\prod A_\alpha$. Thus $x \in \overline{\prod A_\alpha}$.

If $x \in \overline{\prod A_\alpha}$. Let U_α be an open set of A_α that contain x_α . Let $V = \prod U_\beta$ such that $U_\beta = \begin{cases} \mathbb{X}_\beta, & \beta \neq \alpha \\ U_\alpha, & \beta = \alpha \end{cases}$. It is obvious that V is an open set that contain x . Thus V intersect with $\prod A_\alpha$. Thus U_α intersect with A_α . Thus $x \in \prod \overline{A_\alpha}$. \square

Theorem 1.4.8. Let $f : A \rightarrow \prod_{\alpha \in J} \mathbb{X}_\alpha$ be given by the equation

$$f(a) = (f_\alpha(a))_{\alpha \in J}$$

where $f_\alpha : A \rightarrow \mathbb{X}_\alpha$ for each α . Let $\prod \mathbb{X}_\alpha$ have the product topology. Then the function f is continuous if and only if each function f_α is continuous.

¹⁸We omit the proof of this theorem as it is obvious.

¹⁹We omit the proof of this theorem as it is obvious.

²⁰We omit the proof of this theorem as it is obvious.

Proof. Let π_α be the projection mapping

It is obvious that

$$f^{-1}(U) = \bigcap_{\alpha \in J} f_\alpha^{-1}(\pi_\alpha(U))$$

If f_α is continuous. Let V be a closed set of $\prod_{\alpha \in J} \mathbb{X}_\alpha$. Then $\pi_\alpha(V)$ is closed. Then $f^{-1}(V)$ is intersect of closed set. Thus $\pi_\alpha(V)$ is closed. So f is continuous.

If f is continuous. Let U_α be an open set of \mathbb{X}_α . Let $U_\beta = \mathbb{X}_\beta$ if $\beta \neq \alpha$. Let $V = \prod_{\beta \in J} U_\beta$. It is obvious that V is an open set of $\prod \mathbb{X}_\alpha$. And

$$\begin{aligned} f^{-1}V &= \bigcap_{\alpha \in J} f_\alpha^{-1}(\pi_\alpha(U)) \\ &= f_\alpha^{-1}(U_\alpha) \end{aligned}$$

which is an open set in A . Thus f_α is continuous. \square

1.5 The Subspace Topology

Definition 1.5.1 (subspace topology). Let \mathbb{X} be a topological space with topology \mathbb{T} . If Y is a subset of \mathbb{X} , the collection $\mathbb{T}_Y = \{Y \cap U | U \in \mathbb{T}\}$ is a topology on Y , called the **subspace topology**.

Y is also called a **subspace** of \mathbb{X}

Lemma 1.5.1. ²¹If \mathbb{B} is basis for the topology of \mathbb{X} , Y is a subset of \mathbb{X} then the collection

$$\mathbb{B}_Y = \{B \cap Y | B \in \mathbb{B}\}$$

is a basis for the subspace topology on Y

Lemma 1.5.2. ²²Let Y be a subspace of \mathbb{X} . If U is open in Y and Y is open in \mathbb{X} , then U is open in \mathbb{X} .

Theorem 1.5.1. ²³If A is a subspace of \mathbb{X} and B is a subspace of \mathbb{Y} , then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$

Proof. Let $\mathbb{B}_\mathbb{X}$ and $\mathbb{B}_\mathbb{Y}$ and $\mathbb{B}_{\mathbb{X} \times \mathbb{Y}}$ be basis of topology of \mathbb{X} and \mathbb{Y} and $\mathbb{X} \times \mathbb{Y}$ respectively. Let $\mathbb{B}'_\mathbb{X}$ and $\mathbb{B}'_\mathbb{Y}$ and $\mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$ be basis of topology of A and A and $A \times B$ respectively. We will show that $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} = \mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$. Thus, the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$.

²¹We omit the proof of this lemma as it is obvious.

²²We omit the proof of this lemma as it is obvious.

²³If \mathbb{X} is an ordered set in the order topology, and \mathbb{Y} is a subset of \mathbb{X} . The order relation, when restricted to \mathbb{Y} , makes \mathbb{Y} into an ordered set. However, the resulting order topology on \mathbb{Y} need not be the same as the topology that \mathbb{Y} inherits as a subspace of \mathbb{X} .

First, every element in $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ can be represented by $B_A \cap A \times B_B \cap B = B_A \times B_B \cap A \times B \in \mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$ where $B_A \in \mathbb{B}'_{\mathbb{X}}, B_B \in \mathbb{B}'_{\mathbb{Y}}$. Thus $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}} \subseteq \mathbb{B}'_{\mathbb{X} \times \mathbb{Y}}$.

Next, we show that $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$. For any open set U in $\mathbb{X} \times \mathbb{Y}$, and $\forall x \in U \cap A \times B, \exists B_{\mathbb{X}} \times B_{\mathbb{Y}} \in \mathbb{B}_{\mathbb{X} \times \mathbb{Y}}, x \in B_{\mathbb{X}} \times B_{\mathbb{Y}} \subseteq \mathbb{X} \times \mathbb{Y}$. Thus $x \in B_{\mathbb{X}} \times B_{\mathbb{Y}} \cap A \times B \subseteq A \times B, B_{\mathbb{X}} \times B_{\mathbb{Y}} \cap A \times B \in \mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$. Thus $\mathbb{B}'_{\mathbb{X}} \times \mathbb{B}'_{\mathbb{Y}}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$. \square

Definition 1.5.2 (ordered square). *Let $I = [0, 1]$. The set $I \times I$ in the dictionary order ²⁴ topology will be called **ordered square**, and denoted by I_o^2*

Definition 1.5.3 (convex). *Given an ordered set \mathbb{X} , let us say that a subset \mathbb{Y} of \mathbb{X} is **convex** in \mathbb{X} if for each pair of points $a < b$ of \mathbb{Y} , the entire interval (a, b) of points of \mathbb{X} lies in \mathbb{Y}*

Theorem 1.5.2. ²⁵ *Let \mathbb{X} be an ordered set in the order topology. Let \mathbb{Y} be a subset of \mathbb{X} that is convex in \mathbb{X} . Then the order topology on \mathbb{Y} is the same as the topology \mathbb{Y} inherits as a subspace of \mathbb{X} .*

Proof. Consider the ray $(a, +\infty)$ in \mathbb{X} . If $a \in \mathbb{Y}$, then

$$(a, +\infty) \cap \mathbb{Y} = \{x | x \in \mathbb{Y} \text{ and } x > a\}$$

This is an open ray of the ordered set of \mathbb{Y} . if $a \notin \mathbb{Y}$, then a is either a lower bound on \mathbb{Y} or an upper bound on \mathbb{Y} , since \mathbb{Y} is convex. In the former case, the set $(a, +\infty) \cap \mathbb{Y}$ equals all of \mathbb{Y} , in the latter case, it is empty.

A similar remark shows that the intersection of the ray $(-\infty, a)$ with \mathbb{Y} is either an open ray of \mathbb{Y} , or \mathbb{Y} itself, or empty. Since the sets $(a, +\infty) \cap \mathbb{Y}$ and $(-\infty, a) \cap \mathbb{Y}$ form a subbasis for the subspace topology on \mathbb{Y} , and since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of \mathbb{Y} equals the intersection of an open ray of \mathbb{X} with \mathbb{Y} , so it is open in the subspace topology on \mathbb{Y} . Since the open rays of \mathbb{Y} are a subbasis for the order topology on \mathbb{Y} , this topology is contained in the subspace topology. \square

²⁴the dictionary means for $X_1, X_2 \in \mathbb{Y} = \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \dots$ which:

$$\begin{aligned} X_1 &= (x_1, x_2, x_3 \dots) \\ X_2 &= (x'_1, x'_2, x'_3 \dots) \end{aligned}$$

$X_1 > X_2$ only when

$$\begin{aligned} \exists k \in \mathbb{Z}_+, \forall i \in \mathbb{Z}_+, 0 < i < k \\ x_i &= x'_i \\ x_k &> x'_k \end{aligned}$$

²⁵Given \mathbb{X} is an ordered set in the order topology and \mathbb{Y} is a subset of \mathbb{X} , we shall assume that \mathbb{Y} is given the subspace topology unless we specifically state otherwise.

Exercise

1. A map $f : \mathbb{X} \rightarrow \mathbb{Y}$ is said to be a **open map** if for every open set $U \subseteq \mathbb{X}$, the set $f(U)$ is open in \mathbb{Y} . Show that $\pi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$ is open map.

Proof. An open set in $\mathbb{X} \times \mathbb{Y}$ can be represented by

$$\cup(U_i \times U'_i)$$

where U_i, U'_i are open sets in \mathbb{X}, \mathbb{Y} , respectively.

Also,

$$\cup(U_i \times U'_i) = \cup(U_i) \times \cup(U'_i)$$

Thus,

$$\pi(\cup(U_i \times U'_i)) = \cup(U_i)$$

Thus, $\pi(U)$ is open in \mathbb{X} . □

2. Let \mathbb{X} and \mathbb{X}' denote a single set in the topologies \mathbb{T} and \mathbb{T}' , respectively; let \mathbb{Y} and \mathbb{Y}' denote a single set in the topologies \mathbb{U} and \mathbb{U}' , respectively.
²⁶ Assume these sets are nonempty.

- (a) Show that if $\mathbb{T}' \supseteq \mathbb{T}$ and $\mathbb{U}' \supseteq \mathbb{U}$, then the product topologies $\mathbb{X}' \times \mathbb{Y}'$ is finer than the product topology on $\mathbb{X} \times \mathbb{Y}$.
 (b) Does the converse of the previous statement hold?

3. Show that the countable collection²⁷

$$\{(a, b) \times (c, d) | a < b, c < d, a \in \mathbb{Q}, b \in \mathbb{Q}, c \in \mathbb{Q}, d \in \mathbb{Q}\}$$

is a basis for \mathbb{R}^2

Proof. This is obvious if you prove that $(a, b) \times (c, d)$ is a rectangle in the \mathbb{R}^2 plane. □

4. Let \mathbb{X} be an ordered set. If \mathbb{Y} is a proper subset of \mathbb{X} that is convex in \mathbb{X} prove that \mathbb{Y} may not be an interval or a ray in \mathbb{X} .

Proof. Let $\mathbb{X} = \mathbb{R}^2$ with dictionary order. Then $Y = \{(x, y) | -1 \leq x \leq 1\}$ is convex in \mathbb{X} , however it is not an interval or a ray. □

There is a false prove given by myself.

²⁶what does $\mathbb{X}, \mathbb{X}', \mathbb{Y}, \mathbb{Y}'$ really mean here?? I do not know, so I just put the exercise here without a proof.

²⁷The prove of this set is countable is typically similar to Cantor's enumeration of a countable collection of countable sets.

Proof. Let \mathbb{S} be a set that contain all intervals and rays of \mathbb{Y} . We define a partial order on \mathbb{S} by inclusion. So if there is a chain in \mathbb{S} :

$$S_1 \subseteq S_2 \subseteq S_3 \dots$$

Let

$$S = S_1 \cup S_2 \cup S_3 \cup \dots$$

Thus, S is an upper bound of the chain.

Thus, by Zorn's Lemma, there is a maximal element of \mathbb{S} , say U , then we prove that $U = \mathbb{Y}$.

If $U \neq \mathbb{Y}$, then $\exists x, x \in \mathbb{Y} - U$.

If U is a ray say $(a, +\infty)$. Then $x < a$, thus $U \subseteq (x, +\infty) \subseteq \mathbb{B}$, then there is contradiction with the maximal element.

If U is an interval, the circumstance is similar with the proof of U is a ray.

Thus \mathbb{Y} is a ray or an interval. \square

However, there is issue with this proof, the set S does exists. However, it may not be an interval or ray, so it may not be contained in \mathbb{S}

1.6 Closed Sets and Limit Points

Definition 1.6.1 (closed).²⁸ A subset A of a topological space is said to be closed if the set $\mathbb{X} - A$ is open.

Theorem 1.6.1.²⁹ Let \mathbb{X} be a topological space. Then the following conditions hold

1. \emptyset and \mathbb{X} are closed.
2. Arbitrary intersections of closed sets are closed
3. Finite unions of closed sets are closed

Definition 1.6.2 (closed in). Let \mathbb{X} be a topological space; let \mathbb{Y} be a subspace of \mathbb{X} . We say that a set A is **closed in** \mathbb{Y} if A is a subset of \mathbb{Y} and A is closed in the subspace topology of \mathbb{Y}

Theorem 1.6.2. Let \mathbb{Y} be a subspace of \mathbb{X} . Then a set A is closed in \mathbb{Y} if and only if it equals the intersection of a closed set of \mathbb{X} with \mathbb{Y}

²⁸A set can be open, or closed, or both, or neither

²⁹We omit the proof of this theorem as it is obvious.

Proof. First we proof that if A is closed in \mathbb{Y} , then $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. As the origin topology form a surjective map to its subspace topology, there exists a B closed in \mathbb{X} that $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $B \cap \mathbb{Y} = A$

Conversely, if $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. Then, $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $\mathbb{X} - B$ is open in \mathbb{Y} , $\mathbb{Y} - A$ is open in \mathbb{Y} . Then A is closed in \mathbb{Y} \square

Theorem 1.6.3. ³⁰ Let \mathbb{Y} be a subspace of \mathbb{X} . If A is closed in \mathbb{Y} and \mathbb{Y} is closed in \mathbb{X} , then A is closed in \mathbb{X} .

Definition 1.6.3 (interior). Given a subset A of a topological space \mathbb{X} , the **interior** of A is defined as the union of all open sets contained in A . Denoted by $\text{Int}(A)$.

Definition 1.6.4 (closure). Given a subset A of a topological space \mathbb{X} , the **closure** of A is defined as the intersection of all closed sets containing A . Denoted by $\text{Cl}(A)$ or \overline{A}

Theorem 1.6.4. ³¹³² Let \mathbb{Y} be a subspace of a topological space \mathbb{X} ; let A be a subset of \mathbb{X} . Let \overline{A} denote the closure of A in \mathbb{X} . Then the closure of A in \mathbb{Y} equals $\overline{A} \cap \mathbb{Y}$

Definition 1.6.5 (intersect). We say that a set A **intersects** B if $A \cap B$ is not empty.

Theorem 1.6.5. Let A be a subset of the topological space \mathbb{X}

1. The $x \in \overline{A}$ if and only if every open set U containing x intersect A .
2. Supposing the topology of \mathbb{X} is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A

Proof. There are only two types of closed set U in \mathbb{X} :

1. $U \supseteq \overline{A}$
2. $U \cap A \neq A$

Thus, there are only two types of open set U in \mathbb{X} respectively.

1. U does not intersects A .
 2. $U \cap \overline{A} \neq \emptyset$
1. If $x \in \overline{A}$, then every open set containing x is the open set of second type, thus every open set containing x intersects A

If every open set containing x intersect \mathbb{A} , suppose $x \notin \overline{A}$. Then $\mathbb{X} - \overline{A}$ is a open set containing x , however, it does not intersects A . Thus, $x \in \overline{A}$.

³⁰As the proof is similar to the case in the open set, so we omit the proof here.

³¹We omit the proof of this theorem as it is obvious.

³²As the closure of A in \mathbb{X} and the closure A in \mathbb{Y} will sometimes be different. We always use \overline{A} to denote the closure of A in \mathbb{X}

2. If $x \in \overline{A}$, as every basis element of \mathbb{X} is a open set, thus every basis element containing x intersects A

If every open set containing x intersect A , suppose $x \notin \overline{A}$.

As every open sets can be represented by union of basis. Let

$$\mathbb{X} - \overline{A} = B_1 \cup B_2 \cup B_3 \cup \dots \cup B'_1 \cup B'_2 \cup B'_3 \cup \dots$$

where B are bases containing x , and B' are bases that does not contain x .

Thus,

$$x \in B_1 \cup B_2 \cup B_3 \cup \dots \subseteq \mathbb{X} - \overline{A}$$

Then $B_1 \cup B_2 \cup B_3 \cup \dots$ that is a open set can be generated by all the bases containing x , however, that does not intersects A . So, $x \in \overline{A}$.

□

Definition 1.6.6 (neighbourhood).³³ If we say U is a neighbourhood of x in \mathbb{X} , then U is an open set in \mathbb{X} containing x

Definition 1.6.7 (limit point, point of accumulation, cluster point).³⁴ If A is a subset of topological space \mathbb{X} . We say that x is a limit point of A if and only if every open sets containing x intersects A with some points other than x .

This condition is also equivalent to the condition that if x is a limit point of A if and only if $x \in A - \{x\}$

Theorem 1.6.6.³⁵ Let A be a subset of topological space \mathbb{X} ; let A' be the set of all limit points of A . Then

$$\overline{A} = A \cup A'$$

Corollary 1.6.1.³⁶ A subset of a topological space is closed if and only if it contains all its limit point.

Definition 1.6.8 (converge).³⁷ We say that a sequence of $x_1, x_2, x_3 \dots$ converge to x . When for every neighbourhood U of x , there exists a positive integer N , such that for all $n > N$, $x_n \in U$.

Definition 1.6.9 (Hausdorff space). A topological space is called a **Hausdorff space**, if for every distinct x_1, x_2 in \mathbb{X} , there exists disjoint neighbourhood of U_1, U_2 of x_1, x_2 in \mathbb{X} .

³³Some other mathematicians use neighbourhood to say that U merely contains an open set containing x . The book does not give a formal definition for the word merely, and I am not sure either.

³⁴Note that, x may belong to A or not, this does not matter.

³⁵We omit the proof of this theorem as it is obvious.

³⁶We omit the proof of this corollary as it is obvious.

³⁷In real line, a sequence can not converge to multiple points, but for an arbitrary topological space, this is possible.

Theorem 1.6.7. ³⁸³⁹ *Every finite point set in a Hausdorff space \mathbb{X} is closed.*

Proof. Let A be a finite point set in a Hausdorff space \mathbb{X} .

Suppose A only have one element. Then for every $x \in \mathbb{X} - A$, there exists a neighbourhood of x that does not intersect with A . So A is closed.

Suppose A is a closed finite point set. We take $x_0 \in \mathbb{X} - A$. As finite union of closed set is closed, $A \cup \{x_0\}$ is closed.

Then, from induction, all finite point set in a Hausdorff space is closed. \square

Theorem 1.6.8. *If \mathbb{X} is a Hausdorff space, then a sequence of points in \mathbb{X} converges to at most one point.*

Proof. Suppose that the following sequence

$$x_1, x_2, x_3 \dots$$

Converge to more than one points say

$$y_1, y_2, y_3 \dots$$

Then there exists

$$n_1, n_2, n_3 \dots, U_1, U_2, U_3 \dots$$

Such that for $n > n_i$

$$x_n \in U_i, y_i \in U_i$$

If we take disjoint U_1, U_2 which is possible as this is a Hausdorff space.

Then the previous condition does not stand. So, every sequence of points in a Hausdorff space can only converge to at most one point. \square

Definition 1.6.10 (limit). *If a sequence x_n of points in Hausdorff space converge to the point x , we denote this by $x_n \rightarrow x$ and we say the **limit** of x_n is x .*

Definition 1.6.11 (T_1 axiom). *The condition that all finite point set of a topological space is closed is called T_1 **axiom**.*

Theorem 1.6.9. *Let \mathbb{X} be a space satisfying the T_1 axiom; let A be a subset of \mathbb{X} . Then the point x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A .*

Proof. If every neighbourhood of x contains infinitely many point of A . Then every neighbourhood of x intersect with A with infinite element other than x , then x is a limit point of A .

³⁸This implies that a sequence in a Hausdorff space cannot converge to multiple points. The following theorem prove this.

³⁹The condition every finite point set is closed is weaker than the Hausdorff space condition. For instance, the finite complement topology of \mathbb{R} met the condition of finite point set. However it is not a Hausdorff space.

If x is a limit point of A . Suppose that there exists a open set U containing x and intersect with A for finite many points. Let

$$U' = U \cap (A - x)$$

Then, $x \notin U'$. Let

$$U'' = U - U'$$

Then U'' is open as U' is a finite point set and

$$U'' = U - U' = U \cap (\mathbb{X} - U')$$

Also, $x \in U''$. Thus, U'' is a open set containing x that only intersect A with x or do not intersect A . This is a contradiction of x is a limit point. Thus there does not exists a open set U containing x and intersect with A for finite many points. \square

Theorem 1.6.10. ⁴⁰Every simply ordered set is a Hausdorff space in order topology.

Theorem 1.6.11. ⁴¹The product of two Hausdorff space is a Hausdorff space.

Theorem 1.6.12. ⁴²A subspace of a Hausdorff space is a Hausdorff space.

1.6.1 Exercise

1. Give an counter example why $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$ dose not hold.

Proof. Consider the X be the K-topology on the real line.

Let

$$\begin{aligned} A_n &= \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{Z}_+ \\ A &= \cup A_n \end{aligned}$$

Then

$$\begin{aligned} \overline{A_n} &= \left[\frac{1}{n+1}, \frac{1}{n}\right] \\ \cup \overline{A_n} &= (0, 1] \end{aligned}$$

However, as every neighbourhood of 0 intersect $\cup A_\alpha$. $0 \in \overline{\cup A_\alpha}$.

Thus, $\overline{\cup A_\alpha} \neq \cup \overline{A_\alpha}$ \square

⁴⁰We omit the proof of this theorem as it is obvious.

⁴¹We omit the proof of this theorem as it is obvious.

⁴²We omit the proof of this theorem as it is obvious.

2. Prove that

$$\overline{A - B} \supseteq \overline{A} - \overline{B}$$

Proof. If $x \in \overline{A} - \overline{B}$. Then

$$x \in \overline{A}, x \notin \overline{B}$$

.

Thus for open set U containing x

$$\exists U_1 \cap B = \emptyset$$

$$\forall U \cap A \neq \emptyset$$

Suppose that $x \notin \overline{A - B}$. Then

$$\exists U_0 \cap (A - B) = \emptyset$$

Thus,

$$U_0 \cap A \subseteq B$$

Thus,

$$U_1 \cap B = \emptyset$$

$$U_1 \cap U_0 \cap A = \emptyset$$

As $U_1 \cap U_0$ is an open set containing x , so there is contradiction with $x \in \overline{A}$. Thus $x \in \overline{A - B}$. \square

3. A **diagonal** is a subset $\Delta = \{x \times x | x \in \mathbb{X}\}$ of the product topology $\mathbb{X} \times \mathbb{X}$ where \mathbb{X} is a topological space. Show that the diagonal is closed in $\mathbb{X} \times \mathbb{X}$ if and only if \mathbb{X} is a Hausdorff space.

Proof. If \mathbb{X} is a Hausdorff space. For every element $x \times y$ of $\mathbb{X} \times \mathbb{X}$ that not in Δ . We take disjoint set U_x, U_y where $x \in U_x, y \in U_y$. Then $\mathbb{X} \times \mathbb{X} - \Delta = \cup_{x \neq y} U_x \times U_y$. Where $\cup_{x \neq y} U_x \times U_y$ is an open set. Thus Δ is a closed set.

Conversely, if Δ is a closed set, suppose that \mathbb{X} is not a Hausdorff space. Then there exists distinct x, y such that every neighbourhood of x and y intersect. Let \mathbb{B} be a basis of topology of \mathbb{X} . Then $x \times y \in \mathbb{X} \times \mathbb{X} - \Delta$. However we cannot find $B_1, B_2 \in \mathbb{B}, x \times y \in B_1 \times B_2 \subset \mathbb{X} \times \mathbb{X} - \Delta$. Then Δ is not a closed set. So there is a contradiction, then \mathbb{X} must be a Hausdorff space. \square

4. Prove that T_1 axiom is equivalent to the condition such that for every distinct pair x, y of \mathbb{X} , there exists neighbourhood of x does not contain y .

Proof. First if T_1 axiom hold, then for every pair x, y , the neighbourhood $\mathbb{X} - \{y\}$ of x does not contain y , so the second condition hold.

Conversely, if the second condition hold. Suppose that we can find a finite points set say $\{x_1, x_2, x_3 \dots\}$, then there must exists $x \in \{x_1, x_2, x_3 \dots\}$ such that the set $\{x\}$ is not closed. Then $\overline{\{x\}} - \{x\} \neq \emptyset$. Let $y \in \overline{\{x\}} - \{x\}$, then every neighbourhood of y must contain x , this is a contradiction to the second condition, so the T_1 axiom must hold. \square

5. If $A \subseteq \mathbb{X}$, we define the **boundary** of A by the equation

$$\text{Bd}A = \overline{A} \cap \overline{\mathbb{X} - A}$$

- (a) Show that $\text{Int}A$ and $\text{Bd}A$ are disjoint and $\overline{A} = \text{Int}A \cup \text{Bd}A$.

Proof. For every $x \in \text{Bd}A$, every open set contain x must intersect A and $\mathbb{X} - A$ so, there is no open set U contain x , $U \subseteq A$.

For every $x' \in \text{Int}A$, there exists $U' \subseteq A$, so $\text{Bd}A$ and $\text{Int}A$ are disjoint sets.

For every $x \in \overline{A}$, $x \in \text{Bd}A$ or $x \notin \text{Bd}A$. We discuss the condition that $x \notin \text{Bd}A$.

Then $x \notin \overline{\mathbb{X} - A}$, then there exists a open set U containing x , that does not intersect with $\mathbb{X} - A$. Thus $U \subseteq A$, thus $x \in \text{Int}A$. So $\overline{A} \subseteq \text{Int}A \cup \text{Bd}A$.

Then, $\text{Bd}A \subseteq \overline{A}$, $\text{Int}A \subseteq A \subseteq \overline{A}$. Thus, $\overline{A} \supseteq \text{Int}A \cup \text{Bd}A$

So, $\overline{A} = \text{Int}A \cup \text{Bd}A$ \square

- (b) Show that $\text{Bd}A = \emptyset$ if and only if A is both open and closed.

Proof. So, $\text{Int}A = \overline{A}$, then $\text{Bd}A = \emptyset$ follows directly from $\overline{A} = \text{Int}A \cup \text{Bd}A$. \square

- (c) Show that U is open if and only if $\text{Bd}U = \overline{U} - U$.

Proof. Suppose U is open. Then $\overline{\mathbb{X} - U} = \mathbb{X} - U$. Then for every $x \in U$, $x \notin \mathbb{X} - U$, $x \notin \overline{\mathbb{X} - U}$. Thus $\overline{U} \cap \overline{\mathbb{X} - U} = \overline{U} - U$.

Conversely, suppose $\text{Bd}U = \overline{U} - U$. Then for every $x \in U$, $x \notin \text{Bd}U$. Then as $\overline{U} = \text{Int}U \cup \text{Bd}U$, $x \in \text{Int}U$. So $\text{Int}U \supseteq U$. Thus $U = \text{Int}U$. Thus, U is open. \square

1.7 Continuous Function

Definition 1.7.1 (continuous).⁴³ Let \mathbb{X} and \mathbb{Y} be topological spaces. A function $f : \mathbb{X} \rightarrow \mathbb{Y}$ is said to be **continuous** if for each open subset V of \mathbb{Y} , the set $f^{-1}(V)$ is an open subset of \mathbb{X} .

Theorem 1.7.1. Let \mathbb{X} and \mathbb{Y} be topological spaces; let $f : \mathbb{X} \rightarrow \mathbb{Y}$. Then the following are equivalent.

1. f is continuous.
2. For every subset A of \mathbb{X} , one has $f(\overline{A}) \subseteq \overline{f(A)}$.
3. For every closed set B of \mathbb{Y} , the set $f^{-1}(B)$ is closed in \mathbb{X} .
4. For each $x \in \mathbb{X}$ and each neighbourhood V of $f(x)$, there is a neighbourhood U of x such that $f(U) \subseteq V$.

Proof.

1 \Rightarrow 3:

Let A be a open set in \mathbb{Y} . $f^{-1}(\mathbb{Y} - A) = \mathbb{X} - f^{-1}(A)$.

3 \Rightarrow 1:

Let A be a closed set in \mathbb{Y} . $f^{-1}(\mathbb{Y} - A) = \mathbb{X} - f^{-1}(A)$.

1 \Rightarrow 2:

For $x \in \overline{A}$, we take a open set $f(x) \in U \subseteq \mathbb{Y}$. Thus $x \in f^{-1}(U) \cap A \neq \emptyset$. Thus $U \cap f(A) \neq \emptyset$. So $f(x) \in \overline{f(A)}$. Thus $f(\overline{A}) \subseteq \overline{f(A)}$.

2 \Rightarrow 3:

Suppose f is not continuous. Then there must exists V , such that $f^{-1}(V) = U$ is not closed. Thus $\overline{U} \supset B = f^{-1}(A)$. Thus $f\overline{B} \supset A$. However $f(\overline{B}) \subseteq \overline{f(B)} = A$. There is a contradiction. So f must be continuous.

1 \Rightarrow 4:

For every neighbourhood V of $f(x)$, $f^{-1}(V)$ is a neighbourhood of x that $f(f^{-1}(V)) \subseteq V$.

4 \Rightarrow 1:

We take a open set V of \mathbb{Y} . Let S be the collection of all open set U in \mathbb{X} such that $f(U) \subseteq V$. The set cannot be empty unless $f^{-1}(V) = \emptyset$. Let U_0 denote the union of all the element in S . We prove that $U_0 = f^{-1}(V)$.

For all element $x \in U_0$, $f(x) \in V$. Thus $U_0 \subseteq f^{-1}(V)$.

⁴³As the continuity of a function is different as the topological spaces are different. So if we want to emphasis this fact, we say that f is continuous **relative** to specific topologies on \mathbb{X} and \mathbb{Y} .

For all element $x \in f^{-1}(V)$. There is a U' such that $x \in U'$, $f(U') \subseteq V$. This follows from the condition 4. Thus $U' \in S$. Thus $x \in U_0$. Thus $U_0 \subseteq f^{-1}(V)$. As U_0 is union of open set, U_0 is also open. Thus, $f^{-1}(V)$ is also open.

Thus f is continuous. \square

Definition 1.7.2 (homeomorphism).⁴⁴ Let \mathbb{X} and \mathbb{Y} be topological space; let $f : \mathbb{X} \rightarrow \mathbb{Y}$ be a bijection. If both the function f and the inverse function

$$f^{-1} : \mathbb{Y} \rightarrow \mathbb{X}$$

are continuous, then f is called a **homeomorphism**

Definition 1.7.3 (topological imbedding). Suppose that $f : \mathbb{X} \rightarrow \mathbb{Y}$ is an injective continuous map, where \mathbb{X} and \mathbb{Y} are topological spaces. Let \mathbb{Z} be the image set $f(\mathbb{X})$, considered as a subspace of \mathbb{Y} ; then the function $f' : \mathbb{X} \rightarrow \mathbb{Z}$ obtained by restricting the range of f is bijective. If f' happens to be a homeomorphism of \mathbb{X} with \mathbb{Z} , we say that the map $f : \mathbb{X} \rightarrow \mathbb{Y}$ is a **topological imbedding**, or simply an **imbedding**, of \mathbb{X} in \mathbb{Y} .

Theorem 1.7.2 (Rules for constructing continuous functions). Let \mathbb{X} , \mathbb{Y} , and \mathbb{Z} be topological spaces.

1. (Constant function) If $f : \mathbb{X} \rightarrow \mathbb{Y}$ maps all of \mathbb{X} into the single point y_0 of \mathbb{Y} , then f is continuous.
2. (Inclusion) If A is a subspace of \mathbb{X} , the inclusion function $j : A \rightarrow \mathbb{X}$ is continuous.
3. (Composites) If $f : \mathbb{X} \rightarrow \mathbb{Y}$ and $g : \mathbb{Y} \rightarrow \mathbb{Z}$ are continuous, then the map $g \circ f : \mathbb{X} \rightarrow \mathbb{Z}$ is continuous.
4. (Restricting the domain) If $f : \mathbb{X} \rightarrow \mathbb{Y}$ is continuous, and if A is a subspace of \mathbb{X} , then the restriction function $f|_A : A \rightarrow \mathbb{Y}$ is continuous.
5. (Restricting or expanding the range) Let $f : \mathbb{X} \rightarrow \mathbb{Y}$ is continuous. Let \mathbb{Z} be a subspace of \mathbb{Y} containing the image $f(\mathbb{X})$, the function $h : \mathbb{X} \rightarrow \mathbb{Z}$ obtained by restricting the range of f is continuous. If \mathbb{Z} is a space having \mathbb{Y} as a subspace, then the function $h : \mathbb{X} \rightarrow \mathbb{Y}$ obtained by expanding the range of f is continuous.
6. (Local formulation of continuity) The map $f : \mathbb{X} \rightarrow \mathbb{Y}$ is continuous if \mathbb{X} can be written as the union of open sets U_α such set $f|_{U_\alpha}$ is continuous for each α .

Proof.

⁴⁴A equivalent way to define homeomorphism, is that for any open subset U of \mathbb{X} , $f(U)$ is open if and only if U is open.

1. $f^{-1}(U)$ of any open set U is \mathbb{X} , thus f is continuous.
2. For every open subset U of \mathbb{X} , $j^{-1}(U) = U \cap A$ is continuous in A . Thus j is a continuous function.
3. For every open subset U of \mathbb{Z} , $f^{-1}(U)$ is open in \mathbb{Y} , and $g^{-1}(f^{-1}(U))$ is open in \mathbb{X} . Thus, $g \circ f$ is continuous.
4. For every open subset U of \mathbb{Y} , $f^{-1}(U)$ is open in \mathbb{X} , thus $f^{-1}(U) \cap A$ is open in A . Thus the function $f|_A$ is continuous.
5. If \mathbb{Z} is a subspace of \mathbb{Y} , then every open subset of \mathbb{Z} can be represented as $U \cap \mathbb{Z}$, where U is a open subset of \mathbb{Y} . Thus $h^{-1}(U \cap \mathbb{Z}) = g^{-1}(\mathbb{Z}) \cap g^{-1}(U) = \mathbb{X} \cap g^{-1}(U)$ which is a open subset of X , thus h is continuous.
If \mathbb{Y} is a subspace of \mathbb{Z} . Then we take a open subset U of \mathbb{Z} . $h^{-1}(U) = g^{-1}(U) \cap \mathbb{X}$ which is open in \mathbb{X} , thus h is continuous.
6. if $f|_{U_\alpha}$ is continuous for each α . For every open subset U of \mathbb{Y} .

$$U = \cup_\alpha (U_\alpha \cap U)$$

where $U_\alpha \cap U$ is open both in U_α and in \mathbb{Y} .

Thus,

$$\begin{aligned} f^{-1}(U) &= f^{-1}(\cup_\alpha (U_\alpha \cap U)) \\ &= \cup_\alpha ((f|_{U_\alpha})^{-1}(U_\alpha \cap U)) \end{aligned}$$

and each $(f|_{U_\alpha})^{-1}(U_\alpha \cap U)$ is open, thus $f^{-1}(U)$ is open.

□

Theorem 1.7.3 (The pasting lemma).⁴⁵ Let $\mathbb{X} = A \cup B$, where A, B are closed in \mathbb{X} . Let $f : A \rightarrow \mathbb{Y}$ and $g : B \rightarrow \mathbb{Y}$ be continuous. If $f(x) = g(x)$ for every $x \in A \cap B$, then f, g combine to give a continuous function $h : \mathbb{X} \rightarrow \mathbb{Y}$, defined by setting $h(x) = f(x), x \in A$ and $h(x) = g(x), x \in B$.

Theorem 1.7.4 (Maps into products).⁴⁶ Let $f : A \rightarrow \mathbb{X} \times \mathbb{Y}$ be given by the equation

$$f(a) = (f_1(a), f_2(a))$$

Then, the function f is continuous if and only if the functions

$$f_1 : A \rightarrow \mathbb{X}, f_2 : A \rightarrow \mathbb{Y}$$

are continuous.

⁴⁵The proof of this theorem is similar to the "Local formulation of continuity" condition of "Rules for constructing continuous functions", so we omit the proof here.

⁴⁶The map f_1, f_2 are called the **coordinate functions** of f

Proof. Let π_1, π_2 be the projection function

$$\begin{aligned}\pi_1 &: \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X} \\ \pi_2 &: \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}\end{aligned}$$

We first proof that if U is an open subset of $\mathbb{X} \times \mathbb{Y}$,

$$f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$$

Let $x \times y \in U$, $f^{-1}(x \times y)$ contains all a such that $f(a) = x \times y$.
Then for any $a \in f^{-1}(x \times y)$, $a \in f_1^{-1}(\pi_1(x \times y)), a \in f_2^{-1}(\pi_2(x \times y))$.
Thus, $f^{-1}(x \times y) \subseteq f_1^{-1}(\pi_1(x \times y)) \cap f_2^{-1}(\pi_2(x \times y))$.
Thus $f^{-1}(U) \subseteq f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$.

Also, if $a \in f_1^{-1}(\pi_1(x \times y)), a \in f_2^{-1}(\pi_2(x \times y))$, $f_1(a) = x, f_2(a) = y$.
Thus $f(a) = x \times y$. Thus $a \in f^{-1}(x \times y)$.
Thus $f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$

Let U be any open subset of $\mathbb{X} \times \mathbb{Y}$

$$f^{-1}(U) = f_1^{-1}(\pi_1(U)) \cap f_2^{-1}(\pi_2(U))$$

Where $f_1^{-1}(\pi_1(U))$ and $f_2^{-1}(\pi_2(U))$ are both open set. Thus $f^{-1}(U)$ is open. \square

1.7.1 Exercise

1. Let \mathbb{Y} be an ordered set in the order topology. Let $f, g : \mathbb{X} \rightarrow \mathbb{Y}$ be continuous, show that the set $A = \{x | f(x) \leq g(x)\}$ is closed in \mathbb{X} .

Proof. We only need to proof $\mathbb{X} - A$ is open in \mathbb{X} . We take $x \in \mathbb{X} - A$.
Thus $f(x) > g(x)$.

Let U_1, U_2 be the open set in \mathbb{Y} that met the following demand

$$\begin{aligned}\forall y_1 \in U_1, y_2 \in U_2, y_1 > y_2 \\ f(x) \in U_1, g(x) \in U_2\end{aligned}$$

As \mathbb{Y} is an ordered set, U_1, U_2 must exist.

Let $U = f^{-1}(U_1) \cap g^{-1}(U_2)$. It is obvious that U is a open set, and $x \in U$.

Also, for any $y \in U$. $f(y) > g(y)$. Thus $U \subseteq A$. Thus A is an open set. \square

2. Let $\{A_\alpha\}$ be a collection of subsets of \mathbb{X} ; let $\mathbb{X} = \cup_\alpha A_\alpha$. Let $f : \mathbb{X} \rightarrow \mathbb{Y}$; suppose that $f|_{A_\alpha}$ is continuous for each α . An indexed family of sets $\{A_\alpha\}$ is said to be **locally finite** if each point x of \mathbb{X} has a neighbourhood that intersect A_α for only finitely many values of α . Show that if the family $\{A_\alpha\}$ is locally finite and each A_α is closed, then f is continuous.

Proof. For any closed subset U of \mathbb{Y} . Let

$$V = \cup f|_{A_\alpha}(U)$$

We prove that V is closed, so, f is continuous.

To prove that V is closed, we prove that $\bar{V} = V$. That is for any $x \in \bar{V}$, we prove $x \in V$. For any neighbourhood B of x , let C_B denote the set that contain all α , such that $f|_{A_\alpha(U)}$ intersect with B . As B intersect with V , C_B can not be empty.

Let

$$\mathbb{C} = \{C_B | B \text{ be a neighbourhood of } x\}$$

As $\{A_\alpha\}$ is locally finite, \mathbb{C} contain at least one element with finite elements.

Also

$$C_{B_1 \cap B_2} \subseteq C_{B_1} \cap C_{B_2}$$

Let \leq be a partial order on the \mathbb{C} . If $C_{B_1} \subseteq C_{B_2}$, we say that $C_{B_1} \geq C_{B_2}$.

If there is chain in \mathbb{C}

$$C_{B_1} \leq C_{B_2} \dots$$

Let C_{B_0} be a element of \mathbb{C} with finite element. If $C_{B_0} \subseteq C_{B_1}, C_{B_0} \subseteq C_{B_2} \dots$. Then C_{B_0} is a upper bound of the chain.

If C is not a subset of all element of the chain. Then we construct a new set say

$$D = \{C_{B_0 \cap B_1}, C_{B_0 \cap B_2} \dots\}$$

Let

$$\mathbb{D} = \{C_{D_1 \cap D_2 \cap \dots} | C_{D_1}, C_{D_2} \dots \in D\}$$

As C_{B_0} is a finite set, D is a finite set, \mathbb{D} is also a finite set. Thus there must be a maximal element $E \in \mathbb{D}$ that is the subset of all element of \mathbb{D} . Then E is a subset of all element of the chain. Thus E is a upper bound of the chain.

Thus, there must be a maximal element C_F of \mathbb{C} , that is a subset of all element of \mathbb{C} .

Let G be the set be the union of all element of C_F .

As C_F is finite, G is closed. And all neighbourhood of x intersect with G . Thus $x \in G$.

As G is a subset of V , $x \in V$. So V is closed. And f is a continuous function on \mathbb{X} .

□

3. Let A be a subset of topological space \mathbb{X} , let \mathbb{Y} be a Hausdorff space. Let $f : A \rightarrow \mathbb{Y}$ be a continuous function. Let $g : \overline{A} \rightarrow \mathbb{Y}$ also be a continuous function where $g(x) = f(x), x \in A$. Prove that g is uniquely determined by f .⁴⁷

Proof. Say g and h are two distinct function that met the demand.

So there exist x_0 such that $g(x_0) \neq h(x_0)$.

As \mathbb{Y} is a Hausdorff space, so there exist disjoint open subset $g(x_0) \in U$ and $h(x_0) \in V$.

Then $g^{-1}(U)$ and $h^{-1}(V)$ are both open subset of \mathbb{X} that contain x_0 .

If $g^{-1}(U) \cap h^{-1}(V) \cap A \neq \emptyset$. Then there exist $x_1 \in g^{-1}(U) \cap h^{-1}(V) \cap A$ such that $g(x_1) \in U$ and $h(x_1) \in V$ and $g(x_1) = h(x_1)$. However U and V are disjoint. So there is a contradiction.

As $g^{-1}(U) \cap h^{-1}(V)$ is a open subset contain x_0 . So $g^{-1}(U) \cap h^{-1}(V)$ must intersect with A . So it is impossible that $g^{-1}(U) \cap h^{-1}(V) \cap A = \emptyset$.

So $g = h$.

□

⁴⁷It is possible that g does not exist.

Let \mathbb{X} be the real line with order topology. Let \mathbb{Y} be $\{0, 1\}$.

Let $A = \mathbb{X} - \{0\}$.

Let,

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

So, it is obvious that f is a continuous function on \mathbb{X} . However g does not exist in this case.