

Topology Note

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Contents

1	Topology Spaces and Continuous Function	5
1.1	Basic Definition of Topology	5
1.2	Basis for a Topology	6
1.2.1	Exercise	7
1.3	The Order Topology	7
1.4	The Product Topology	8
1.5	The Subspace Topology	9
1.6	Closed Sets and Limit Points	11
1.6.1	Exercise	15

Definitions

B

basis, 6
boundary, 16

C

closed, 11
 closed in, 12
closure, 12
coarser, 5
 strictly coarser, 5
finer, 5
 strictly finer, 5
larger, 5
 strictly larger, 5
smaller, 5
 strictly smaller, 5
converge, 13
convex, 9

D

diagonal, 16
discrete topology, 5

F

finite complement topology, 5

H

Hausdorff space, 13

I

interior, 12
intersect, 12
interval, 7
 closed interval, 7
 half-open interval, 7
 open interval, 7

K

K-topology on \mathbb{R} , 6

L

limit, 14
cluster point, 13
limit point, 13
point of accumulation, 13
lower limit topology on \mathbb{R} , 6

N

neighbourhood, 13

O

open map, 10
open set, 5
open sets, 5
ordered square, 9
order topology, 7

P

product topology, 8
projection, 8

R

ray, 8
 closed ray, 8
 open ray, 8

S

standard topology on \mathbb{R} , 6
subbasis, 7
subspace, 9
subspace topology, 9

T

T_1 axiom, 14
topology, 5
topology generated by basis, 6
topology space, 5
trivial topology, 5

Chapter 1

Topology Spaces and Continuous Function

1.1 Basic Definition of Topology

Definition 1.1.1 (topology). A **topology** on a set \mathbb{X} is a collection \mathbb{T} of subsets of \mathbb{X} having the following properties:

- \emptyset and \mathbb{X} are in \mathbb{T}
- The union of the elements of any sub collection of \mathbb{T} is in \mathbb{T}
- The intersection of the elements of any **finite** sub collection of \mathbb{T} is in \mathbb{T}

Definition 1.1.2 (topology space). A **topological space** is a set \mathbb{X} for which a topology \mathbb{T} has been specified.

Definition 1.1.3 (open set). A **open set** \mathbb{U} is a subset of \mathbb{X} that belongs to a topology \mathbb{T} of \mathbb{X} .

Definition 1.1.4 (open sets). A topology can also be called a **open sets**

Definition 1.1.5 (discrete topology). The set of all subsets of a set \mathbb{X} formed a topology called **discrete topology**

Definition 1.1.6 (trivial topology). The set consisting the set \mathbb{X} and \emptyset only formed a topology of \mathbb{X} called **trivial topology**

Definition 1.1.7 (finite complement topology). Let \mathbb{X} be a set. Let \mathbb{T}_f be the collection of all subsets \mathbb{U} of \mathbb{X} such that $\mathbb{X} - \mathbb{U}$ either if a **finite**¹ or is all of \mathbb{X} . Then \mathbb{T}_f is a topology on \mathbb{X} , called the **finite complement topology**.

Definition 1.1.8 (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable). Let \mathbb{T} and \mathbb{T}' be two topology on a given set \mathbb{X} . If \mathbb{T} is a subset of \mathbb{T}' , we say that \mathbb{T}' is **finer** or **larger** than \mathbb{T} . If \mathbb{T} is a proper subset of \mathbb{T}' , we say that \mathbb{T}' is **strictly finer** or **strictly larger** than \mathbb{T} . We also say that \mathbb{T} is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than \mathbb{T}' . We say that \mathbb{T} and \mathbb{T}' is **comparable** if either \mathbb{T} is a subset of \mathbb{T}' or \mathbb{T}' is a subset of \mathbb{T} .

¹The set \mathbb{U} can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, \mathbb{U} will not be a topology.

1.2 Basis for a Topology

Definition 1.2.1 (basis). If \mathbb{X} is a set, a **basis** for a topology on \mathbb{X} is a collection \mathbb{B} of subsets of \mathbb{X} (called **basis elements**) such that:

- For each $x \in \mathbb{X}$, there is at least one basis element B containing x
- If x belongs to the intersection of two basis elements B_1 and B_2 , then there is another element $x \in B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$

Definition 1.2.2 (topology generated by basis). Let \mathbb{B} be a basis on \mathbb{X} . Let \mathbb{U} be a set containing all subsets U of \mathbb{X} such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such \mathbb{U} formed a topology on \mathbb{X} , called **topology \mathbb{T} generated by \mathbb{B}**

Lemma 1.2.1. Let \mathbb{X} be a set. Let \mathbb{B} be a basis for a topology \mathbb{T} on \mathbb{X} . Then \mathbb{T} equals to the set of all possible unions of elements of \mathbb{B} .

Proof. Let set \mathbb{U} be the set of all possible unions of elements of \mathbb{B} . For any $U \in \mathbb{U}$. $U = \cup B$ ² for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in \mathbb{B}$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, \mathbb{U} equals to \mathbb{T} . □

Lemma 1.2.2.³ Let \mathbb{X} be a topological space. Suppose that \mathbb{C} is a collection of open sets of \mathbb{X} such that for each open set U of \mathbb{X} and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq U$. Then \mathbb{C} is a basis for the topology of \mathbb{X} .

Lemma 1.2.3.⁴ Let \mathbb{B} and \mathbb{B}' be basis for the topologies \mathbb{T} and \mathbb{T}' , respectively, on \mathbb{X} . Then the following are equivalent:

- \mathbb{T}' is finer than \mathbb{T}
- For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing x , there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.

Definition 1.2.3 (standard topology on the real line). Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. \mathbb{B} formed a basis on real line. The topology generated by \mathbb{B} is called the **standard topology on the real line**⁵.

Definition 1.2.4 (lower limit topology on the real line). Let be $\mathbb{B} = \{B | B = \{x | a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. \mathbb{B} formed a basis on real line. The topology generated by \mathbb{B} is called the **lower limit topology on the real line**. When \mathbb{R} is given this topology, we denote it by \mathbb{R}_l .

Definition 1.2.5 (K-topology on the real line). Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$. $\mathbb{B} \cup \{B - K | B \in \mathbb{B}\}$ formed a basis on real line. The topology generated by \mathbb{B} is called the **K-topology on the real line**. When \mathbb{R} is given this topology, we denote it by \mathbb{R}_K .

Lemma 1.2.4.⁶ The topologies \mathbb{R}_l and \mathbb{R}_K is strictly finer than the standard topology on \mathbb{R} .

²Note that this expression may not be unique.

³We omit the proof of this lemma as it is obvious.

⁴We omit the proof of this lemma as it is obvious.

⁵Whenever we consider \mathbb{R} , we shall suppose it is given this topology unless we specifically state otherwise.

⁶We omit the proof of this lemma as it is obvious.

Lemma 1.2.5. *The topologies of \mathbb{R}_l and $\mathbb{R}_\mathbb{K}$ is not comparable.*

Proof. Let \mathbb{T}_l and $\mathbb{T}_\mathbb{K}$ be topologies of \mathbb{R}_l and $\mathbb{R}_\mathbb{K}$ respectively. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that \mathbb{T}_l is not finer than $\mathbb{T}_\mathbb{K}$. Let $U = \{x | -1 < x < 1\} - K, x = 0$. If there exist $B = \{x | a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus B is not a subset of U .

Then we proof that $\mathbb{T}_\mathbb{K}$ is not finer than \mathbb{T}_l . Let $U' = \{x | a' \leq x < b'\}$. If there exist $B' = \{x | a'' < x < b''\} \text{ or } \{x | a'' < x < b''\} - K$ such that $a' \in B \subseteq U$. Thus $a'' < a < b''$. Thus there exist c that $a'' < x < a, x \in B, x \notin U'$. Thus $B' \not\subseteq U'$.

Thus the topologies of \mathbb{R}_l and $\mathbb{R}_\mathbb{K}$ is not comparable. \square

Definition 1.2.6 (subbasis). *A **subbasis** \mathbb{S} for a topology on \mathbb{X} is a collection of subsets of \mathbb{X} whose union equals \mathbb{X} . The **topology generated by the subbasis** \mathbb{S} is defined to be the collection \mathbb{T}^7 of all unions of finite intersections of elements of \mathbb{S} .*

1.2.1 Exercise

1. Show that if \mathbb{A} is a basis for a topology on \mathbb{X} , then the topology generated by \mathbb{A} equals the intersection of all topologies on \mathbb{X} that contain \mathbb{A} . Prove the same if \mathbb{A} is a subbasis.

Proof. As a subbasis is also a basis, we will directly prove the case of subbasis here.

Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain \mathbb{A} . Let \mathbb{T} be the topology that \mathbb{A} generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.⁸

First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

Thus, $\mathbb{T} = \mathbb{T}'$ \square

1.3 The Order Topology

Definition 1.3.1 (interval). *Let \mathbb{X} is a set having a simple order relation $<$. Given elements a and b of \mathbb{X} such that $a < b$, there are four subsets of \mathbb{X} that are called **intervals** determined by a and b :*

- $(a, b) = \{x | a < x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $[a, b] = \{x | a \leq x \leq b\}$

(a, b) is called an **open interval** on \mathbb{X} . $[a, b]$ is called an **closed interval** on \mathbb{X} . $(a, b]$ and $[a, b)$ is called **half-open intervals**.

Definition 1.3.2 (order topology).⁹ *Let \mathbb{X} be a set with a simple order relation; assume \mathbb{X} has more than one element. Let \mathbb{B} be the collection of all sets of the following types:*

- All open intervals (a, b) in \mathbb{X} .

⁷It is obvious that \mathbb{T} is a topology, we just omit the proof here.

⁸It is obvious that \mathbb{T}' is also a topology, we just omit the proof here.

⁹The standard topology on \mathbb{R} is an order topology derived from the usual order on \mathbb{R} .

- All intervals of the form $[a_0, b)$, where a_0 is the smallest element(if exist) of \mathbb{X} .
- All intervals of the form $(a, b_0]$, where b_0 is the largest element(if exist) of \mathbb{X} .

The collection \mathbb{B} formed a basis for a topology on \mathbb{X} , which is called the order topology.

Definition 1.3.3 (ray).¹⁰¹¹ If \mathbb{X} is an ordered set, and a is an element of \mathbb{X} , there are four subsets of \mathbb{X} that are called **rays** determined by a :

- $(a, +\infty) = \{x | x > a\}$
- $(-\infty, a) = \{x | x < a\}$
- $[a, +\infty) = \{x | x \geq a\}$
- $(-\infty, a] = \{x | x \leq a\}$

$(a, +\infty)$ and $(-\infty, a)$ are called **open rays**. $[a, +\infty)$ and $(-\infty, a]$ are called **closed rays**.

1.4 The Product Topology

Definition 1.4.1 (product topology). Let \mathbb{X} and \mathbb{Y} be topological spaces. The **product topology** on $\mathbb{X} \times \mathbb{Y}$ having a basis \mathbb{B} containing all sets of the form $U \times V$, where U and V is open sets of \mathbb{X} and \mathbb{Y} respectively.

Theorem 1.4.1.¹² If \mathbb{B} and \mathbb{C} is basis for the topology of \mathbb{X} and \mathbb{Y} respectively, then the collection

$$\mathbb{D} = \{B \times C | B \in \mathbb{B} \text{ and } C \in \mathbb{C}\}$$

is a basis for the topology of $\mathbb{X} \times \mathbb{Y}$

Definition 1.4.2 (projection). Let $\pi_1 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$ be defined by the equation:

$$\pi_1(x, y) = x$$

Let $\pi_2 : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{Y}$ be defined by the equation:

$$\pi_2(x, y) = y$$

The maps π_1 and π_2 are called the **projections** of $\mathbb{X} \times \mathbb{Y}$ onto its first and second factors, respectively.

Theorem 1.4.2.¹³ The collection

$$\mathbb{S} = \{\pi_1^{-1}(U) | U \text{ open in } \mathbb{X}\} \cup \{\pi_2^{-1}(V) | V \text{ open in } \mathbb{Y}\}$$

is a subbasis for the product topology on $\mathbb{X} \times \mathbb{Y}$.

¹⁰open rays are always open sets in the order topology

¹¹the open rays also formed a subbasis of the order topology

¹²We omit the proof of this lemma as it is obvious.

¹³We omit the proof of this lemma as it is obvious.

1.5 The Subspace Topology

Definition 1.5.1 (subspace topology). *Let \mathbb{X} be a topological space with topology \mathbb{T} . If Y is a subset of \mathbb{X} , the collection $\mathbb{T}_Y = \{Y \cap U | U \in \mathbb{T}\}$ is a topology on Y , called the **subspace topology**.*

*Y is also called a **subspace** of \mathbb{X}*

Lemma 1.5.1. ¹⁴*If \mathbb{B} is basis for the topology of \mathbb{X} , Y is a subset of \mathbb{X} then the collection*

$$\mathbb{B}_Y = \{B \cap Y | B \in \mathbb{B}\}$$

is a basis for the subspace topology on Y

Lemma 1.5.2. ¹⁵*Let Y be a subspace of \mathbb{X} . If U is open in Y and Y is open in \mathbb{X} , then U is open in \mathbb{X} .*

Theorem 1.5.1. ¹⁶*If A is a subspace of \mathbb{X} and B is a subspace of \mathbb{Y} , then the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$*

Proof. Let $\mathbb{B}_\mathbb{X}$ and $\mathbb{B}_\mathbb{Y}$ and $\mathbb{B}_{\mathbb{X}\mathbb{Y}}$ be basis of topology of \mathbb{X} and \mathbb{Y} and $\mathbb{X} \times \mathbb{Y}$ respectively. Let $\mathbb{B}'_\mathbb{X}$ and $\mathbb{B}'_\mathbb{Y}$ and $\mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ be basis of topology of A and A and $A \times B$ respectively. We will show that $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} = \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$. Thus, the product topology on $A \times B$ is the same as the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$.

First, every element in $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$ can be represented by $B_A \cap A \times B_B \cap B = B_A \times B_B \cap A \times B \in \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$ where $B_A \in \mathbb{B}'_\mathbb{X}$, $B_B \in \mathbb{B}'_\mathbb{Y}$. Thus $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y} \subseteq \mathbb{B}'_{\mathbb{X}\mathbb{Y}}$.

Next, we show that $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$. For any open set U in $\mathbb{X} \times \mathbb{Y}$, and $\forall x \in U \cap A \times B$, $\exists B_\mathbb{X} \times B_\mathbb{Y} \in \mathbb{B}_{\mathbb{X}\mathbb{Y}}$, $x \in B_\mathbb{X} \times B_\mathbb{Y} \subseteq \mathbb{X} \times \mathbb{Y}$. Thus $x \in B_\mathbb{X} \times B_\mathbb{Y} \cap A \times B \subseteq A \times B$, $B_\mathbb{X} \times B_\mathbb{Y} \cap A \times B \in \mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$. Thus $\mathbb{B}'_\mathbb{X} \times \mathbb{B}'_\mathbb{Y}$ generate the topology $A \times B$ inherits as a subspace of $\mathbb{X} \times \mathbb{Y}$. \square

Definition 1.5.2 (ordered square). *Let $I = [0, 1]$. The set $I \times I$ in the dictionary order ¹⁷ topology will be called **ordered square**, and denoted by I_o^2*

Definition 1.5.3 (convex). *Given an ordered set \mathbb{X} , let us say that a subset \mathbb{Y} of \mathbb{X} is **convex** in \mathbb{X} if for each pair of points $a < b$ of \mathbb{Y} , the entire interval (a, b) of points of \mathbb{X} lies in \mathbb{Y}*

¹⁴We omit the proof of this lemma as it is obvious.

¹⁵We omit the proof of this lemma as it is obvious.

¹⁶If \mathbb{X} is an ordered set in the order topology, and \mathbb{Y} is a subset of \mathbb{X} . The order relation, when restricted to \mathbb{Y} , makes \mathbb{Y} into an ordered set. However, the resulting order topology on \mathbb{Y} need not be the same as the topology that \mathbb{Y} inherits as a subspace of \mathbb{X} .

¹⁷the dictionary means for $X_1, X_2 \in \mathbb{Y} = \mathbb{X}_1 \times \mathbb{X}_2 \times \mathbb{X}_3 \dots$ which:

$$\begin{aligned} X_1 &= (x_1, x_2, x_3 \dots) \\ X_2 &= (x'_1, x'_2, x'_3 \dots) \end{aligned}$$

$X_1 > X_2$ only when

$$\begin{aligned} \exists k \in \mathbb{Z}_+, \forall i \in \mathbb{Z}_+, 0 < i < k \\ x_i &= x'_i \\ x_k &> x'_k \end{aligned}$$

Theorem 1.5.2. ¹⁸ Let \mathbb{X} be an ordered set in the order topology. Let \mathbb{Y} be a subset of \mathbb{X} that is convex in \mathbb{X} . Then the order topology on \mathbb{Y} is the same as the topology \mathbb{Y} inherits as a subspace of \mathbb{X} .

Proof. Consider the ray $(a, +\infty)$ in \mathbb{X} . If $a \in \mathbb{Y}$, then

$$(a, +\infty) \cap \mathbb{Y} = \{x \mid x \in \mathbb{Y} \text{ and } x > a\}$$

This is an open ray of the ordered set of \mathbb{Y} . if $a \notin \mathbb{Y}$, then a is either a lower bound on \mathbb{Y} or an upper bound on \mathbb{Y} , since \mathbb{Y} is convex. In the former case, the set $(a, +\infty) \cap \mathbb{Y}$ equals all of \mathbb{Y} , in the latter case, it is empty.

A similar remark shows that the intersection of the ray $(-\infty, a)$ with \mathbb{Y} is either an open ray of \mathbb{Y} , or \mathbb{Y} itself, or empty. Since the sets $(a, +\infty) \cap \mathbb{Y}$ and $(-\infty, a) \cap \mathbb{Y}$ form a subbasis for the subspace topology on \mathbb{Y} , and since each is open in the order topology, the order topology contains the subspace topology.

To prove the reverse, note that any open ray of \mathbb{Y} equals the intersection of an open ray of \mathbb{X} with \mathbb{Y} , so it is open in the subspace topology on \mathbb{Y} . Since the open rays of \mathbb{Y} are a subbasis for the order topology on \mathbb{Y} , this topology is contained in the subspace topology. \square

Exercise

1. A map $f : \mathbb{X} \rightarrow \mathbb{Y}$ is said to be a **open map** if for every open set $U \subseteq \mathbb{X}$, the set $f(U)$ is open in \mathbb{Y} . Show that $\pi : \mathbb{X} \times \mathbb{Y} \rightarrow \mathbb{X}$ is open map.

Proof. An open set in $\mathbb{X} \times \mathbb{Y}$ can be represented by

$$\cup(U_i \times U'_i)$$

where U_i, U'_i are open sets in \mathbb{X} , \mathbb{Y} , respectively.

Also,

$$\cup(U_i \times U'_i) = \cup(U_i) \times \cup(U'_i)$$

Thus,

$$\pi(\cup(U_i \times U'_i)) = \cup(U_i)$$

Thus, $\pi(U)$ is open in \mathbb{X} . \square

2. Let \mathbb{X} and \mathbb{X}' denote a single set in the topologies \mathbb{T} and \mathbb{T}' , respectively; let \mathbb{Y} and \mathbb{Y}' denote a single set in the topologies \mathbb{U} and \mathbb{U}' , respectively. ¹⁹ Assume these sets are nonempty.

(a) Show that if $\mathbb{T}' \supseteq \mathbb{T}$ and $\mathbb{U}' \supseteq \mathbb{U}$, then the product topologies $\mathbb{X}' \times \mathbb{Y}'$ is finer than the product topology on $\mathbb{X} \times \mathbb{Y}$.

(b) Does the converse of the previous statement hold?

¹⁸Given \mathbb{X} is an ordered set in the order topology and \mathbb{Y} is a subset of \mathbb{X} , we shall assume that \mathbb{Y} is given the subspace topology unless we specifically state otherwise.

¹⁹what does $\mathbb{X}, \mathbb{X}', \mathbb{Y}, \mathbb{Y}'$ really mean here?? I do not know, so I just put the exercise here without a proof.

3. Show that the countable collection²⁰

$$\{(a, b) \times (c, d) | a < b, c < d, a \in \mathbb{Q}, b \in \mathbb{Q}, c \in \mathbb{Q}, d \in \mathbb{Q}\}$$

is a basis for \mathbb{R}^2

Proof. This is obvious if you prove that $(a, b) \times (c, d)$ is a rectangle in the \mathbb{R}^2 plane. \square

4. Let \mathbb{X} be an ordered set. If \mathbb{Y} is a proper subset of \mathbb{X} that is convex in \mathbb{X} prove that \mathbb{Y} may not be an interval or a ray in \mathbb{X} .

Proof. Let $\mathbb{X} = \mathbb{R}^2$ with dictionary order. Then $Y = \{(x, y) | -1 \leq x \leq 1\}$ is convex in \mathbb{X} , however it is not an interval or a ray. \square

There is a false prove given by myself.

Proof. Let \mathbb{S} be a set that contain all intervals and rays of \mathbb{Y} . We define a partial order on \mathbb{S} by inclusion. So if there is a chain in \mathbb{S} :

$$S_1 \subseteq S_2 \subseteq S_3 \dots$$

Let

$$S = S_1 \cup S_2 \cup S_3 \cup \dots$$

Thus, S is an upper bound of the chain.

Thus, by Zorn's Lemma, there is a maximal element of \mathbb{S} , say U , then we prove that $U = \mathbb{Y}$.

If $U \neq \mathbb{Y}$, then $\exists x, x \in \mathbb{Y} - U$.

If U is a ray say $(a, +\infty)$. Then $x < a$, thus $U \subseteq (x, +\infty) \subseteq \mathbb{B}$, then there is contradiction with the maximal element.

If U is an interval, the circumstance is similar with the proof of U is a ray.

Thus \mathbb{Y} is a ray or an interval. \square

However, there is issue with this proof, the set S does exists. However, it may not be an interval or ray, so it may not be contained in \mathbb{S}

1.6 Closed Sets and Limit Points

Definition 1.6.1 (closed).²¹ A subset A of a topological space is said to be closed if the set $\mathbb{X} - A$ is open.

Theorem 1.6.1.²² Let \mathbb{X} be a topological space. Then the following conditions hold

1. \emptyset and \mathbb{X} are closed.

²⁰The prove of this set is countable is typically similar to Cantor's enumeration of a countable collection of countable sets.

²¹A set can be open, or closed, or both, or neither

²²We omit the proof of this lemma as it is obvious.

2. Arbitrary intersections of closed sets are closed

3. Finite unions of closed sets are closed

Definition 1.6.2 (closed in). Let \mathbb{X} be a topological space; let \mathbb{Y} be a subspace of \mathbb{X} . We say that a set A is **closed in** \mathbb{Y} if A is a subset of \mathbb{Y} and A is closed in the subspace topology of \mathbb{Y} .

Theorem 1.6.2. Let \mathbb{Y} be a subspace of \mathbb{X} . Then a set A is closed in \mathbb{Y} if and only if it equals the intersection of a closed set of \mathbb{X} with \mathbb{Y} .

Proof. First we proof that if A is closed in \mathbb{Y} , then $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. As the origin topology form a surjective map to its subspace topology, there exists a B closed in \mathbb{X} that $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $B \cap \mathbb{Y} = A$.

Conversely, if $\exists B \subseteq \mathbb{X}, B \cap \mathbb{Y} = A$. Then, $\mathbb{Y} - A = (\mathbb{X} - B) \cap \mathbb{Y}$. Then $\mathbb{X} - B$ is open in \mathbb{Y} , $\mathbb{Y} - A$ is open in \mathbb{Y} . Then A is closed in \mathbb{Y} . \square

Theorem 1.6.3.²³ Let \mathbb{Y} be a subspace of \mathbb{X} . If A is closed in \mathbb{Y} and \mathbb{Y} is closed in \mathbb{X} , then A is closed in \mathbb{X} .

Definition 1.6.3 (interior). Given a subset A of a topological space \mathbb{X} , the **interior** of A is defined as the union of all open sets contained in A . Denoted by $\text{Int}(A)$.

Definition 1.6.4 (closure). Given a subset A of a topological space \mathbb{X} , the **closure** of A is defined as the intersection of all closed sets containing A . Denoted by $\text{Cl}(A)$ or \overline{A} .

Theorem 1.6.4.^{24,25} Let \mathbb{Y} be a subspace of a topological space \mathbb{X} ; let A be a subset of \mathbb{X} . Let \overline{A} denote the closure of A in \mathbb{X} . Then the closure of A in \mathbb{Y} equals $\overline{A} \cap \mathbb{Y}$.

Definition 1.6.5 (intersect). We say that a set A **intersects** B if $A \cap B$ is not empty.

Theorem 1.6.5. Let A be a subset of the topological space \mathbb{X} .

1. The $x \in \overline{A}$ if and only if every open set U containing x intersect A .
2. Supposing the topology of \mathbb{X} is given by a basis, then $x \in \overline{A}$ if and only if every basis element B containing x intersects A .

Proof. There are only two types of closed set U in \mathbb{X} :

1. $U \supseteq \overline{A}$
2. $U \cap A \neq A$

Thus, there are only two types of open set U in \mathbb{X} respectively.

1. U does not intersects A .
2. $U \cap \overline{A} \neq \emptyset$

²³As the proof is similar to the case in the open set, so we omit the proof here.

²⁴We omit the proof of this lemma as it is obvious.

²⁵As the closure of A in \mathbb{X} and the closure A in \mathbb{Y} will sometimes be different. We always use \overline{A} to denote the closure of A in \mathbb{X} .

1. If $x \in \bar{A}$, then every open set containing x is the open set of second type, thus every open set containing x intersects A .

If every open set containing x intersect A , suppose $x \notin \bar{A}$. Then $\mathbb{X} - \bar{A}$ is a open set containing x , however, it does not intersects A . Thus, $x \in \bar{A}$.

2. If $x \in \bar{A}$, as every basis element of \mathbb{X} is a open set, thus every basis element containing x intersects A .

If every open set containing x intersect A , suppose $x \notin \bar{A}$.

As every open sets can be represented by union of basis. Let

$$\mathbb{X} - \bar{A} = B_1 \cup B_2 \cup B_3 \cup \dots \cup B'_1 \cup B'_2 \cup B'_3 \cup \dots$$

where B are bases containing x , and B' are bases that does not contain x .

Thus,

$$x \in B_1 \cup B_2 \cup B_3 \cup \dots \subseteq \mathbb{X} - \bar{A}$$

Then $B_1 \cup B_2 \cup B_3 \cup \dots$ that is a open set can be generated by all the bases containing x , however, that does not intersects A . So, $x \in \bar{A}$.

□

Definition 1.6.6 (neighbourhood).²⁶ If we say U is a neighbourhood of x in \mathbb{X} , then U is an open set in \mathbb{X} containing x .

Definition 1.6.7 (limit point, point of accumulation, cluster point).²⁷ If A is a subset of topological space \mathbb{X} . We say that x is a limit point of A if and only if every open sets containing x intersects A with some points other than x .

This condition is also equivalent to the condition that if x is a limit point of A if and only if $x \in \overline{A - \{x\}}$.

Theorem 1.6.6.²⁸ Let A be a subset of topological space \mathbb{X} ; let A' be the set of all limit points of A . Then

$$\bar{A} = A \cup A'$$

Corollary 1.6.1.²⁹ A subset of a topological space is closed if and only if it contains all its limit point.

Definition 1.6.8 (converge).³⁰ We say that a sequence of $x_1, x_2, x_3 \dots$ converge to x . When for every neighbourhood U of x , there exists a positive integer N , such that for all $n > N$, $x_n \in U$.

Definition 1.6.9 (Hausdorff space). A topological space is called a **Hausdorff space**, if for every distinct x_1, x_2 in \mathbb{X} , there exists disjoint neighbourhood of U_1, U_2 of x_1, x_2 in \mathbb{X} .

²⁶Some other mathematicians use neighbourhood to say that U merely contains an open set containing x . The book does not give a formal definition for the word merely, and I am not sure either.

²⁷Note that, x may belong to A or not, this does not matter.

²⁸We omit the proof of this lemma as it is obvious.

²⁹We omit the proof of this lemma as it is obvious.

³⁰In real line, a sequence can not converge to multiple points, but for an arbitrary topological space, this is possible.

Theorem 1.6.7. ³¹³² Every finite point set in a Hausdorff space \mathbb{X} is closed.

Proof. Let A be a finite point set in a Hausdorff space \mathbb{X} .

Suppose A only have one element. Then for every $x \in \mathbb{X} - A$, there exists a neighbourhood of x that does not intersect with A . So A is closed.

Suppose A is a closed finite point set. We take $x_0 \in \mathbb{X} - A$. As finite union of closed set is closed, $A \cup \{x_0\}$ is closed.

Then, from induction, all finite point set in a Hausdorff space is closed. \square

Theorem 1.6.8. If \mathbb{X} is a Hausdorff space, then a sequence of points in \mathbb{X} converges to at most one point.

Proof. Suppose that the following sequence

$$x_1, x_2, x_3 \dots$$

Converge to more than one points say

$$y_1, y_2, y_3 \dots$$

Then there exists

$$n_1, n_2, n_3 \dots, U_1, U_2, U_3 \dots$$

Such that for $n > n_i$

$$x_n \in U_i, y_i \in U_i$$

If we take disjoint U_1, U_2 which is possible as this is a Hausdorff space.

Then the previous condition does not stand. So, every sequence of points in a Hausdorff space can only converge to at most one point. \square

Definition 1.6.10 (limit). If a sequence x_n of points in Hausdorff space converge to the point x , we denote this by $x_n \rightarrow x$ and we say the **limit** of x_n is x .

Definition 1.6.11 (T_1 axiom). The condition that all finite point set of a topological space is closed is called T_1 **axiom**.

Theorem 1.6.9. Let \mathbb{X} be a space satisfying the T_1 axiom; let A be a subset of \mathbb{X} . Then the point x is a limit point of A if and only if every neighbourhood of x contains infinitely many points of A .

Proof. If every neighbourhood of x contains infinitely many point of A . Then every neighbourhood of x intersect with A with infinite element other than x , then x is a limit point of A .

If x is a limit point of A . Suppose that there exists a open set U containing x and intersect with A for finite many points. Let

$$U' = U \cap (A - x)$$

Then, $x \notin U'$. Let

$$U'' = U - U'$$

³¹This implies that a sequence in a Hausdorff space cannot converge to multiple points. The following theorem prove this.

³²The condition every finite point set is closed is weaker than the Hausdorff space condition. For instance, the finite complement topology of \mathbb{R} met the condition of finite point set. However it is not a Hausdorff space.

Then U'' is open as U' is a finite point set and

$$U'' = U - U' = U \cap (\mathbb{X} - U')$$

Also, $x \in U''$. Thus, U'' is a open set containing x that only intersect A with x or do not intersect A . This is a contradiction of x is a limit point. Thus there does not exists a open set U containing x and intersect with A for finite many points. \square

Theorem 1.6.10. ³³Every simply ordered set is a Hausdorff space in order topology.

Theorem 1.6.11. ³⁴The product of two Hausdorff space is a Hausdorff space.

Theorem 1.6.12. ³⁵A subspace of a Hausdorff space is a Hausdorff space.

1.6.1 Exercise

1. Give an counter example why $\overline{\cup A_\alpha} = \cup \overline{A_\alpha}$ dose not hold.

Proof. Consider the X be the K -topology on the real line.

Let

$$\begin{aligned} A_n &= \left(\frac{1}{n+1}, \frac{1}{n}\right), n \in \mathbb{Z}_+ \\ A &= \cup A_n \end{aligned}$$

Then

$$\begin{aligned} \overline{A_n} &= \left[\frac{1}{n+1}, \frac{1}{n}\right] \\ \cup \overline{A_n} &= (0, 1] \end{aligned}$$

However, as every neighbourhood of 0 intersect $\cup A_\alpha$. $0 \in \overline{\cup A_\alpha}$.

Thus, $\overline{\cup A_\alpha} \neq \cup \overline{A_\alpha}$ \square

2. Prove that

$$\overline{A - B} \supseteq \overline{A} - \overline{B}$$

Proof. If $x \in \overline{A} - \overline{B}$. Then

$$x \in \overline{A}, x \notin \overline{B}$$

.

Thus for open set U containing x

$$\begin{aligned} \exists \quad U_1 \cap B &= \emptyset \\ \forall \quad U \cap A &\neq \emptyset \end{aligned}$$

³³We omit the proof of this lemma as it is obvious.

³⁴We omit the proof of this lemma as it is obvious.

³⁵We omit the proof of this lemma as it is obvious.

Suppose that $x \notin \overline{A - B}$. Then

$$\exists U_0 \cap (A - B) = \emptyset$$

Thus,

$$U_0 \cap A \subseteq B$$

Thus,

$$\begin{aligned} U_1 \cap B &= \emptyset \\ U_1 \cap U_0 \cap A &= \emptyset \end{aligned}$$

As $U_1 \cap U_0$ is an open set containing x , so there is contradiction with $x \in \overline{A}$. Thus $x \in \overline{A - B}$. \square

3. A **diagonal** is a subset $\Delta = \{x \times x | x \in \mathbb{X}\}$ of the product topology $\mathbb{X} \times \mathbb{X}$ where \mathbb{X} is a topological space. Show that the diagonal is closed in $\mathbb{X} \times \mathbb{X}$ if and only if \mathbb{X} is a Hausdorff space.

Proof. If \mathbb{X} is a Hausdorff space. For every element $x \times y$ of $\mathbb{X} \times \mathbb{X}$ that not in Δ . We take disjoint set U_x, U_y where $x \in U_x, y \in U_y$. Then $\mathbb{X} \times \mathbb{X} - \Delta = \cup_{x \neq y} U_x \times U_y$. Where $\cup_{x \neq y} U_x \times U_y$ is an open set. Thus Δ is a closed set.

Conversely, if Δ is a closed set, suppose that \mathbb{X} is not a Hausdorff space. Then there exists distinct x, y such that every neighbourhood of x and y intersect. Let \mathbb{B} be a basis of topology of \mathbb{X} . Then $x \times y \in \mathbb{X} \times \mathbb{X} - \Delta$. However we cannot find $B_1, B_2 \in \mathbb{B}, x \times y \in B_1 \times B_2 \subset \mathbb{X} \times \mathbb{X} - \Delta$. Then Δ is not a closed set. So there is a contradiction, then \mathbb{X} must be a Hausdorff space. \square

4. Prove that T_1 axiom is equivalent to the condition such that for every distinct pair x, y of \mathbb{X} , there exists neighbourhood of x does not contain y .

Proof. First if T_1 axiom hold, then for every pair x, y , the neighbourhood $\mathbb{X} - \{y\}$ of x does not contain y , so the second condition hold.

Conversely, if the second condition hold. Suppose that we can find a finite points set say $\{x_1, x_2, x_3 \dots\}$, then there must exists $x \in \{x_1, x_2, x_3 \dots\}$ such that the set $\{x\}$ is not closed. Then $\overline{\{x\}} - \{x\} \neq \emptyset$. Let $y \in \overline{\{x\}} - \{x\}$, then every neighbourhood of y must contain x , this is a contradiction to the second condition, so the T_1 axiom must hold. \square

5. If $A \subseteq \mathbb{X}$, we define the **boundary** of A by the equation

$$\text{Bd}A = \overline{A} \cap \overline{\mathbb{X} - A}$$

- (a) Show that $\text{Int}A$ and $\text{Bd}A$ are disjoint and $\overline{A} = \text{Int}A \cup \text{Bd}A$.

Proof. For every $x \in \text{Bd}A$, every open set contain x must intersect A and $\mathbb{X} - A$ so, there is no open set U contain x , $U \subseteq A$.

For every $x' \in \text{Int}A$, there exists $U' \subseteq A$, so $\text{Bd}A$ and $\text{Int}A$ are disjoint sets.

For every $x \in \overline{A}$, $x \in \text{Bd}A$ or $x \notin \text{Bd}A$. We discuss the condition that $x \notin \text{Bd}A$.

Then $x \notin \overline{\mathbb{X} - A}$, then there exists a open set U containing x , that does not intersect with $\mathbb{X} - A$. Thus $U \subseteq A$, thus $x \in \text{Int}A$. So $\overline{A} \subseteq \text{Int}A \cup \text{Bd}A$.

Then, $\text{Bd}A \subseteq \overline{A}$, $\text{Int}A \subseteq A \subseteq \overline{A}$. Thus, $\overline{A} \supseteq \text{Int}A \cup \text{Bd}A$

So, $\overline{A} = \text{Int}A \cup \text{Bd}A$ □

- (b) Show that $\text{Bd}A = \emptyset$ if and only if A is both open and closed.

Proof. So, $\text{Int}A = \overline{A}$, then $\text{Bd}A = \emptyset$ follows directly from $\overline{A} = \text{Int}A \cup \text{Bd}A$. □

- (c) Show that U is open if and only if $\text{Bd}U = \overline{U} - U$.

Proof. Suppose U is open. Then $\overline{\mathbb{X} - U} = \mathbb{X} - U$. Then for every $x \in U$, $x \notin \mathbb{X} - U$, $x \notin \overline{\mathbb{X} - U}$. Thus $\overline{U} \cap \overline{\mathbb{X} - U} = \overline{U} - U$.

Conversely, suppose $\text{Bd}U = \overline{U} - U$. Then for every $x \in U$, $x \notin \text{Bd}U$. Then as $\overline{U} = \text{Int}U \cup \text{Bd}U$, $x \in \text{Int}U$. So $\text{Int}U \supseteq U$. Thus $U = \text{Int}U$. Thus, U is open. □