

Topology Note

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Chapter 1

Topology Spaces and Continuous Function

1.1 Basic Definition of Topology

Definition 1.1.1 (topology). A **topology** on a set \mathbb{X} is a collection \mathbb{T} of subsets of \mathbb{X} having the following properties:

- \emptyset and \mathbb{X} are in \mathbb{T}
- The union of the elements of any sub collection of \mathbb{T} is in \mathbb{T}
- The intersection of the elements of any **finite** sub collection of \mathbb{T} is in \mathbb{T}

Definition 1.1.2 (topology space). A **topological space** is a set \mathbb{X} for which a topology \mathbb{T} has been specified.

Definition 1.1.3 (open set). A **open set** \mathbb{U} is a subset of \mathbb{X} that belongs to a topology \mathbb{T} of \mathbb{X} .

Definition 1.1.4 (open sets). A topology can also be called a **open sets**

Definition 1.1.5 (discrete topology). The set of all subsets of a set \mathbb{X} formed a topology called **discrete topology**

Definition 1.1.6 (trivial topology). The set consisting the set \mathbb{X} and \emptyset only formed a topology of \mathbb{X} called **trivial topology**

Definition 1.1.7 (finite complement topology). Let \mathbb{X} be a set. Let \mathbb{T}_f be the collection of all subsets \mathbb{U} of \mathbb{X} such that $\mathbb{X} - \mathbb{U}$ either if a **finite**¹ of is all of \mathbb{X} . Then \mathbb{T}_f is a topology on \mathbb{X} , called the .

Definition 1.1.8 (finer, larger, strictly finer, strictly larger, coarser, smaller, strictly coarser, strictly smaller, comparable). Let \mathbb{T} and \mathbb{T}' be two topology on a given set \mathbb{X} . If \mathbb{T} is a subset of \mathbb{T}' , we say that \mathbb{T}' is **finer** or **larger** than \mathbb{T} . If \mathbb{T} is a proper subset of \mathbb{T}' , we say that \mathbb{T}' is **strictly finer** or **strictly larger** than \mathbb{T} . We also say that \mathbb{T} is **coarser** or **smaller** or **strictly coarser** or **strictly smaller** than \mathbb{T}' . We say that \mathbb{T} and \mathbb{T}' is **comparable** if either \mathbb{T} is a subset of \mathbb{T}' or \mathbb{T}' is a subset of \mathbb{T} .

¹The set \mathbb{U} can form a topology because of the definition of topology is intersection of finite sub collection. If this can be intersection of infinite sub collection, \mathbb{U} will not be a topology.

1.2 Basis for a Topology

Definition 1.2.1 (basis). If \mathbb{X} is a set, a **basis** for a topology on \mathbb{X} is a collection \mathbb{B} of subsets of \mathbb{X} (called **basis elements**) such that:

- For each $x \in \mathbb{X}$, there is at least one basis element B containing x
- If x belongs to the intersection of two basis elements B_1 and B_2 , then there is another element $x \in B_3 \in \mathbb{B}$ such that $B_3 \subseteq B_1 \cap B_2$

Definition 1.2.2 (topology generated by basis). Let \mathbb{B} be a basis on \mathbb{X} . Let \mathbb{U} be a set containing all subsets U of \mathbb{X} such that for each element $x \in U$, there is $B \in \mathbb{B}$ that $x \in B \subseteq U$. Such \mathbb{U} formed a topology on \mathbb{X} , called **topology \mathbb{T} generated by \mathbb{B}**

Lemma 1.2.1. Let \mathbb{X} be a set. Let \mathbb{B} be a basis for a topology \mathbb{T} on \mathbb{X} . Then \mathbb{T} equals to the set of all possible unions of elements of \mathbb{B} .

Proof. Let set \mathbb{U} be the set of all possible unions of elements of \mathbb{B} . For any $U \in \mathbb{U}$. $U = \cup B$ ² for some $B \in \mathbb{B}$. Thus, for every $x \in U$, there exist a $B' \in \mathbb{B}$ that $x \in B' \subseteq U$. Thus, $U \in \mathbb{T}$.

Conversely, for any $U \in \mathbb{T}$. For any $x \in U$, let $x \in B_x \in \mathbb{B}$. Then, $U = \cup_{x \in U} B_x$. Thus, $U \in \mathbb{U}$.

Therefore, \mathbb{U} equals to \mathbb{T} . □

Lemma 1.2.2. ³ Let \mathbb{X} be a topological space. Suppose that \mathbb{C} is a collection of open sets of \mathbb{X} such that for each open set U of \mathbb{X} and each $x \in U$, there is an element $C \in \mathbb{C}$ such that $x \in C \subseteq U$. Then \mathbb{C} is a basis for the topology of \mathbb{X} .

Lemma 1.2.3. ⁴ Let \mathbb{B} and \mathbb{B}' be basis for the topologies \mathbb{T} and \mathbb{T}' , respectively, on \mathbb{X} . Then the following are equivalent:

- \mathbb{T}' is finer than \mathbb{T}
- For each $x \in \mathbb{X}$ and each basis element $B \in \mathbb{B}$ containing x , there is a basis element $B' \in \mathbb{B}'$ such that $x \in B' \subseteq B$.

Definition 1.2.3 (standard topology on the real line). Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. \mathbb{B} formed a basis on real line. The topology generated by \mathbb{B} is called the **standard topology on the real line** ⁵.

Definition 1.2.4 (lower limit topology on the real line). Let be $\mathbb{B} = \{B | B = \{x | a \leq x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. \mathbb{B} formed a basis on real line. The topology generated by \mathbb{B} is called the **lower limit topology on the real line**. When \mathbb{R} is given this topology, we denote it by \mathbb{R}_l .

Definition 1.2.5 (K-topology on the real line). Let be $\mathbb{B} = \{B | B = \{x | a < x < b\}, a < b, a \in \mathbb{R}, b \in \mathbb{R}\}$. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$. $\mathbb{B} \cup \{B - K | B \in \mathbb{B}\}$ formed a basis on real line. The topology generated by \mathbb{B} is called the **K-topology on the real line**. When \mathbb{R} is given this topology, we denote it by \mathbb{R}_K .

Lemma 1.2.4. ⁶ The topologies \mathbb{R}_l and \mathbb{R}_K is strictly finer than the standard topology on \mathbb{R} .

²Note that this expression may not be unique.

³We omit the proof of this lemma as it is obvious.

⁴We omit the proof of this lemma as it is obvious.

⁵Whenever we consider \mathbb{R} , we shall suppose it is given this topology unless we specifically state otherwise.

⁶We omit the proof of this lemma as it is obvious.

Lemma 1.2.5. *The topologies of \mathbb{R}_l and $\mathbb{R}_\mathbb{K}$ is not comparable.*

Proof. Let \mathbb{T}_l and $\mathbb{T}_\mathbb{K}$ be topologies of \mathbb{R}_l and $\mathbb{R}_\mathbb{K}$ respectively. Let $K = \{x | x = \frac{1}{n}, n \in \mathbb{Z}_+\}$.

We first proof that \mathbb{T}_l is not finer than $\mathbb{T}_\mathbb{K}$. Let $U = \{x | -1 < x < 1\} - K, x = 0$. If there exist $B = \{x | a \leq x < b\} \in \mathbb{T}_l$ such that $x \in B \subseteq U$, then $0 < b < 1$. Thus, there exist $n \in \mathbb{Z}_+$ that $0 < \frac{1}{n} < b$. Thus B is not a subset of U .

Then we proof that $\mathbb{T}_\mathbb{K}$ is not finer than \mathbb{T}_l . Let $U' = \{x | a' \leq x < b'\}$. If there exist $B' = \{x | a'' < x < b''\} \text{ or } \{x | a'' < x < b''\} - K$ such that $a' \in B \subseteq U$. Thus $a'' < a < b''$. Thus there exist c that $a'' < x < a, x \in B, x \notin U'$. Thus $B' \not\subseteq U'$.

Thus the topologies of \mathbb{R}_l and $\mathbb{R}_\mathbb{K}$ is not comparable. \square

Definition 1.2.6 (subbasis). *A **subbasis** \mathbb{S} for a topology on \mathbb{X} is a collection of subsets of \mathbb{X} whose union equals \mathbb{X} . The **topology generated by the subbasis** \mathbb{S} is defined to be the collection \mathbb{T} ⁷ of all unions of finite intersections of elements of \mathbb{S} .*

1.2.1 Exercise

1. Show that if \mathbb{A} is a basis for a topology on \mathbb{X} , then the topology generated by \mathbb{A} equals the intersection of all topologies on \mathbb{X} that contain \mathbb{A} . Prove the same if \mathbb{A} is a subbasis.

Proof. As a subbasis is also a basis, we will directly prove the case of subbasis here.

Let $\mathbb{S} = \{\mathbb{T}_\alpha\}$ be set contain all the topologies that contain \mathbb{A} . Let \mathbb{T} be the topology that \mathbb{A} generated. Let $\mathbb{T}' = \cap \mathbb{T}_\alpha$.⁸

First, $\mathbb{A} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}_\alpha$. Thus, $\mathbb{T} \subseteq \mathbb{T}'$.

Also, $\mathbb{A} \subseteq \mathbb{T}$. Thus, $\mathbb{T} \in \mathbb{S}$. Thus, $\mathbb{T}' \subseteq \mathbb{T}$.

Thus, $\mathbb{T} = \mathbb{T}'$ \square

1.3 The Order Topology

Definition 1.3.1 (interval). *Let \mathbb{X} is a set having a simple order relation $<$. Given elements a and b of \mathbb{X} such that $a < b$, there are four subsets of \mathbb{X} that are called **intervals** determined by a and b :*

- $(a, b) = \{x | a < x < b\}$
- $(a, b] = \{x | a < x \leq b\}$
- $[a, b) = \{x | a \leq x < b\}$
- $[a, b] = \{x | a \leq x \leq b\}$

(a, b) is called an **open interval** on \mathbb{X} . $[a, b]$ is called an **closed interval** on \mathbb{X} . $(a, b]$ and $[a, b)$ is called **half-open intervals**.

Definition 1.3.2 (order topology).⁹ *Let \mathbb{X} be a set with a simple order relation; assume \mathbb{X} has more than one element. Let \mathbb{B} be the collection of all sets of the following types:*

- All open intervals (a, b) in \mathbb{X} .

⁷It is obvious that \mathbb{T} is a topology, we just omit the proof here.

⁸It is obvious that \mathbb{T}' is also a topology, we just omit the proof here.

⁹The standard topology on \mathbb{R} is an order topology derived from the usual order on \mathbb{R} .

- All intervals of the form $[a_0, b)$, where a_0 is the smallest element(if exist) of \mathbb{X} .
- All intervals of the form $(a, b_0]$, where b_0 is the largest element(if exist) of \mathbb{X} .

The collection \mathbb{B} formed a basis for a topology on \mathbb{X} , which is called the order topology.

Definition 1.3.3 (ray).¹⁰¹¹ If \mathbb{X} is an ordered set, and a is an element of \mathbb{X} , there are four subsets of \mathbb{X} that are called **rays** determined by a :

- $(a, +\infty) = \{x | x > a\}$
- $(-\infty, a) = \{x | x < a\}$
- $[a, +\infty) = \{x | x \geq a\}$
- $(-\infty, a] = \{x | x \leq a\}$

$(a, +\infty)$ and $(-\infty, a)$ are called **open rays**. $[a, +\infty)$ and $(-\infty, a]$ are called **closed rays**.

¹⁰open rays are always open sets in the order topology

¹¹the open rays also formed a subbasis of the order topology