Communicate Without Errors

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Definition 0.1 (graph)**.** *A graph G is a set of vertices with a set of edges connecting pairs of vertices.*

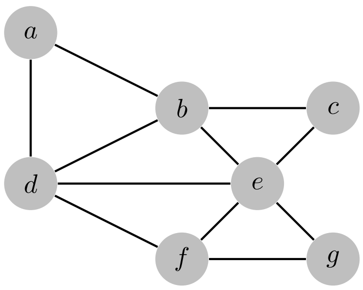


Figure 1: An example of a graph.

Definition 0.2 ($\alpha(G)$)**.** *Given a graph G. Given a subgraph of G, H, such that every vertex of H is not connected in G. Then $\alpha(G)$ is the maximum number of vertices of H.*

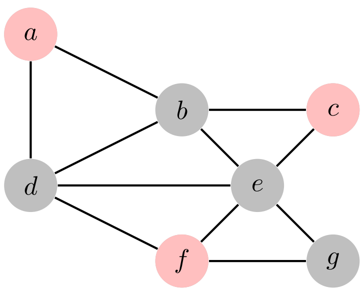


Figure 2: Example of $\alpha(G)$. Here $\alpha(G) = 3$.

Definition 0.3 (graph product)**.** *Given two graph G and H. The graph product $G \times H$ is the graph with vertices $V(G \times H) = V(G) \times V(H)$ in which (x, y) is adjacent to (x', y') in $G \times H$ if and only if x is adjacent to x' in G and y is adjacent to y' in H.*

A graph G product itself for n times will always be denoted by G^n.

Lemma 0.1. *Given graph G and H.*

$$\alpha(G \times H) \geq \alpha(G)\alpha(H) \tag{1}$$

Proof. Given graph G and H. Let G' and H' be subgraph of G and H such that no vertex of G' or H' is adjacent in G or H, respectively. Then $G' \times H'$ is a subgraph of $G \times H$ such that no vertex of $G' \times H'$ is adjacent in $G \times H$. $\square$

Definition 0.4 (Shannon capacity). *Given a graph G, the Shannon capacity $\Theta(G)$ is defined by*

$$\Theta(G) = \lim_{n \to \infty} \sqrt[n]{\alpha(G^n)} \tag{2}$$

We will see that this sequence is monotonically increasing and has an upper bound later. So, this sequence will always converge.

Definition 0.5 (adjacency matrix). *Given a graph G, the adjacency matrix A is defined by*

$$A_{ij} = \begin{cases} 1 & \textit{if } i \textit{ and } j \textit{ are adjacent in } G \\ 0 & \textit{otherwise} \end{cases} \tag{3}$$

Definition 0.6 (orthonormal representation). *Given a graph G with vertices $1, 2, \ldots, n$, the orthonormal representation of G is a set of unit vectors $\{v_1, v_2, \ldots, v_n\}$ such that if i and j are not adjacent in G, then v_i and v_j are orthogonal.*

Lemma 0.2 (existence of orthonormal representation). *Given a graph G with vertices $1, 2, \ldots, n$, there exists an orthonormal representation of G.*

Proof. Any orthonormal basis of $\mathbb{R}^n$ can be used as an orthonormal representation of G. $\square$

Definition 0.7 (tensor product). *Given two vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, the tensor product $v \circ w$ is defined by*

$$v \circ w = (v_1 w_1, \ldots, v_1 w_n, v_2 w_1, \ldots, v_2 w_n, \ldots, v_n w_1, \ldots, v_n w_n) \tag{4}$$

Lemma 0.3 (inner product of tensor product). *Given two vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$, the inner product of $v \circ w$ is defined by*

$$\langle v \circ w, v' \circ w' \rangle = \langle v, v' \rangle \langle w, w' \rangle \tag{5}$$

Lemma 0.4 (product of orthonormal representation). *Given a graph G with vertices $1, 2, \ldots, n$, and a graph H with vertices $1, 2, \ldots, m$. Then vectors $\{v_i \circ w_j\}$ is an orthonormal representation of $G \times H$.*

Definition 0.8 (value of a orthonormal representation). *Given an orthonormal representation $\{v_1, v_2, \ldots, v_n\}$ of a graph G the value of the representation is defined by*

$$\textit{value}(\{v_1, v_2, \ldots, v_n\}) = \inf_c \max_i \frac{1}{\langle c, v_i \rangle^2} \tag{6}$$

which $i = 1, 2, \ldots, n$ and c is any vector does not orthogonal to v_i.

Definition 0.9 ($\theta(G)$). *Given a graph G, the $\theta(G)$ is defined by*

$$\theta(G) = \inf_{\{v_1, v_2, \ldots, v_n\}} \textit{value}(\{v_1, v_2, \ldots, v_n\}) \tag{7}$$

where $\{v_1, v_2, \ldots, v_n\}$ is an orthonormal representation of G.

Lemma 0.5. *Given graph G and H, then*

$$\theta(G \times H) \leq \theta(G)\theta(H) \tag{8}$$

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_m\}$ be orthonormal representation of G and $H$and c_v and c_w such that

$$\max\left\{\frac{1}{\langle c_v, v_i\rangle^2} : i = 1, 2, \ldots, n\right\} = \theta(G) \tag{9}$$

and

$$\max\left\{\frac{1}{\langle c_w, w_i\rangle^2} : i = 1, 2, \ldots, m\right\} = \theta(H) \tag{10}$$

Then

$$\begin{aligned}
\theta(G \times H) &\leq \max\left\{\frac{1}{\langle c_v \circ c_w, v_i \circ w_j\rangle^2}\right\} && (11)\\
&= \max\left\{\frac{1}{\langle c_v, v_i\rangle^2 \langle c_w, w_j\rangle^2}\right\} && (12)\\
&= \max\left\{\frac{1}{\langle c_v, v_i\rangle^2}\right\} \max\left\{\frac{1}{\langle c_w, w_j\rangle^2}\right\} && (13)\\
&= \theta(G)\theta(H) && (14)\\
& && (15)
\end{aligned}$$

□

Lemma 0.6.

$$\theta(G) \geq \alpha(G) \tag{16}$$

Proof. Let $\{1, 2, \ldots, k\}$ be the set of vertices of G such that every point is not adjacency in G. And $k = \alpha(G)$

Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal representation of G and c such that

$$\max\left\{\frac{1}{\langle c, v_i\rangle^2}\right\} = \theta(G) \tag{17}$$

Then

$$\begin{aligned}
1 &= c^2 && (18)\\
&\geq \sum_{i=1}^{k} \langle c, v_i\rangle^2 && (19)\\
&\geq \frac{k}{\theta(G)} && (20)
\end{aligned}$$

□

Theorem 0.1. *Given a graph G, then*

$$\theta(G) \geq \Theta(G) \tag{21}$$

Proof.

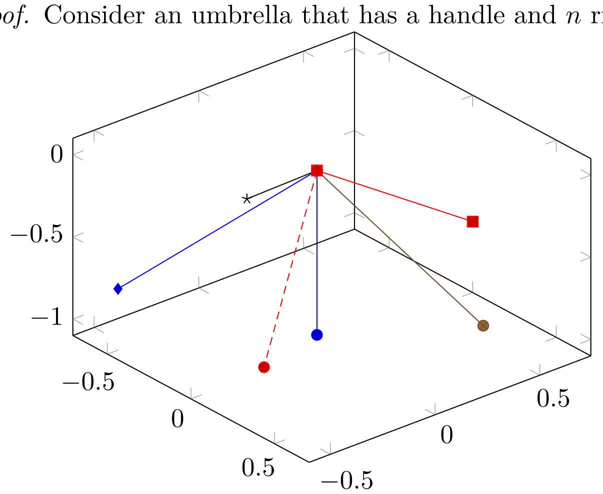
$$\begin{aligned}
\Theta(G) &= \lim_n \sqrt[n]{\alpha(G^n)} && (22)\\
&\leq \lim_n \sqrt[n]{\theta(G^n)} && (23)\\
&\leq \lim_n \sqrt[n]{\theta(G)^n} && (24)\\
&= \theta(G) && (25)
\end{aligned}$$

□

Theorem 0.2. *For odd n,*

$$\theta(C_n) \geq \frac{n\cos(\pi/n)}{\cos(\pi/n)+1} \tag{26}$$

Proof. Consider an umbrella that has a handle and n ribs that all have unit length.



Hear, angles between two consecutive ribs are same. Let v be a rib. And let w be one of the rib that have the largest angle with v. Then, we let the angle between v and w be $\pi/2$.

Then, the n ribs of such an umbrella form an orthonormal representation of $\bar{C}_n$.

Let d be the vector represent the handle, and $v_1, v_2, \ldots, v_n$ be the n ribs. And let γ be the angle between the handle and any rib.

So, by some calculation, we get

$$\begin{aligned}
\theta(C_n) &= \max \sum_{i=1}^{n} (d^T v_i)^2 && (27)\\
&\geq n\,(\cos(\gamma))^2 && (28)\\
&= n\left(\sqrt{1-\frac{1}{1+\cos(\pi/n)}}\right)^2 && (29)\\
&= \frac{n\cos(\pi/n)}{\cos(\pi/n)+1} && (30)
\end{aligned}$$

□