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Quantum Abstract Interpretation

Seminar for the **Introduction to Quantum Computing** course

Università di Pisa
Dipartimento di Informatica

Pisa, 24 Luglio 2023

Roadmap

1. Introduction
   - Reasons
   - Abstract Interpretation
2. Preliminaries
   - Density Matrix
   - Reduced Density Matrix
3. Abstract Domain
   - Abstraction and Concretization Functions
   - Abstract Operations
   - Assertions

Introduction

As quantum computing advances, we would like to have some means to prove correctness properties on quantum programs, *especially* since quantum programming is counterintuitive.

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No: **exponential** space and time cost.

Example

$$n_{qubits} = 1$$

$$|0\rangle \langle 0|$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$2^2 = 4$ complex numbers

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$$n_{qubits} = 2$$

$$|00\rangle \langle 00|$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$2^4 = 16$ complex numbers

Example

$$n_{qubits} = 3$$

$$|000\rangle \langle 000|$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$2^6 = 64$ complex numbers

Example

$$n_{qubits} = 300$$

$$|0\rangle^{\otimes 300} \langle 0|^{\otimes 300}$$

?????

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$$2^{600} = 4149515568880992958512407863691161151012446232242436899995657329690652811412908146399707048947103794288197886611300789182395151075411775307886874834113963687061181803401509523685376$$

Bigger than the number of atoms in the universe.

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Solution: abstract interpretation

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: to capture the information we are interested in (e.g. the state of the quantum system belongs to the span of two vectors)

as coarse as possible
: to discard information we are not interested in, and thus be substantially more efficient to be analyzed

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- Assertions: properties we can prove with abstract interpretation

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Density Matrix

Instead of dealing with a state $|\phi\rangle$ in vector form, we use its *density matrix*:

$\rho_\phi = |\phi\rangle \langle\phi|$ (For a pure state)

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- $Tr(\rho) = 1$
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&= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) \\
&= \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}
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$$Tr_B[\rho] : AB \to A \qquad Tr_A[\rho] : AB \to B$$
$$Tr_B[\alpha \otimes \beta] = \alpha \cdot Tr(\beta) \qquad Tr_A[\alpha \otimes \beta] = Tr(\alpha) \cdot \beta$$

$$Tr_S[\rho + \sigma] = Tr_S[\rho] + Tr_S[\sigma] \text{ (Linearity)}$$

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Partial trace $Tr_B[\rho]$ **traces out** subsystem B.

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Loss of precision

Computing a reduced density matrix **discards information**!

$$\rho_{\beta_{00}} = \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2} \qquad \text{(Pure state)}$$

$$\rho_2 = \frac{|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|}{4} \qquad \text{(Mixed state)}$$

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$$Tr_B[\rho_{\beta_{00}}] = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = Tr_B[\rho_2]$$

The partial traces of two different initial states can be equal.

For a state $\rho \in A \otimes B$, even if we know $Tr_B[\rho]$ and $Tr_A[\rho]$, we cannot uniquely determine ρ.

Projection Subspaces

Each projection P corresponds to a linear subspace $\{v \mid Pv = v\}$.

The support of a matrix P is the subspace orthogonal to its kernel, i.e., the set $\{v \mid Pv \neq 0\}$.

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Abstract Domain

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Special case:

$$T = ([n]) \Rightarrow AbsDom(T) \simeq \mathcal{D}$$

Fineness Relation

Let $S = (s_1, ..., s_m)$ and $T = (t_1, ..., t_m)$ (with $1 \leq m \leq 2^n$), then:

$$\underbrace{S \trianglelefteq T}_{\text{"T is finer than S"}} \quad \triangleq \forall i \in [m].\ s_i \subseteq t_i$$

T is "more concrete" than S.

Least element: $\bot = (\emptyset, ..., \emptyset)$.
Greatest element: $\top = ([n], ...[n])$.

AbsDom($\bot$) corresponds to a state so abstract that it holds no information at all.
AbsDom($\top$) corresponds to tuples where every projection is a concrete state.

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4. Compute the intersection of the supports.

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$$\gamma_{S\to T} : AbsDom(S) \to AbsDom(T)$$

$$\gamma_{S\to T}(P_{s_1}, \dots, P_{s_m}) = (Q_{t_1}, \dots, Q_{t_m})$$
$$Q_{t_j} = \bigcap_{s_i.\ s_i \subseteq t_J} P_{s_i} \otimes I_{t_j \setminus s_i}$$

Given an abstract state $\overline{\sigma} \in AbsDom(S) = (P_{s_1}, \dots, P_{s_m})$, we want to compute $\overline{\tau} \in AbsDom(T) = (Q_{t_1}, \dots, Q_{t_m})$. For each $j \in [m]$:

1. Find all P_{s_i}s such that $s_i \subseteq t_j$. We know at least one exists (for $i = j$), since $S \trianglelefteq T$.
2. Extend the projection to the space of all qubits in t_j, by computing the tensor product with the identity matrix.

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3. Compute the intersection of the extended projections.

Order Relation on Abstract States

$$1 \leq m \leq 2^n, \quad S = (s_1, ...s_m), \quad \forall i \in [m].\ s_i \subseteq [n]$$

$$\overline{\sigma} \in AbsDom(S) = (P_{s_1}, ..., P_{s_m}), \quad \overline{\tau} \in AbsDom(S) = (Q_{s_1}, ..., Q_{s_m})$$

$$\overline{\sigma} \sqsubseteq \overline{\tau} \triangleq \forall i \in [m].\ \underbrace{P_{s_i} \subseteq Q_{s_i}}_{\substack{\text{Subspace interpretation} \\ \text{of projections}}}$$

Monotonicity

$$S \trianglelefteq T$$

$$\forall \overline{\sigma}, \overline{\tau} \in \mathit{AbsDom}(T). \quad \overline{\sigma} \sqsubseteq \overline{\tau} \Rightarrow \alpha_{T \to S}(\overline{\sigma}) \sqsubseteq \alpha_{T \to S}(\overline{\tau})$$

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Galois connection

$$S \trianglelefteq T$$

$$\forall \overline{\sigma} \in AbsDom(S).\ \forall \overline{\tau} \in AbsDom(T).$$

$$\left(\begin{array}{c} \overline{\tau} \sqsubseteq \gamma_{S \to T}(\overline{\sigma}) \Rightarrow \alpha_{T \to S}(\overline{\tau}) \sqsubseteq \overline{\sigma} \\ \wedge \\ \left(\exists \overline{\rho} \in AbsDom([n]^m).\ \overline{\tau} = \alpha_{[n]^m \to T}(\overline{\rho})\right) \ \Rightarrow \ \overline{\tau} \sqsubseteq \gamma_{S \to T}(\overline{\sigma}) \Leftrightarrow \alpha_{T \to S}(\overline{\tau}) \sqsubseteq \overline{\sigma} \end{array} \right)$$

$$\forall \overline{\sigma} \in AbsDom(S).\ \forall \overline{\tau} \in AbsDom([n]^m).$$

$$\overline{\tau} \sqsubseteq \gamma_{S \to [n]^m}(\overline{\sigma}) \quad \Leftrightarrow \quad \alpha_{[n]^m \to S}(\overline{\tau}) \sqsubseteq \overline{\sigma}$$

Abstract Operations

Operations on the concrete domain are unitary operators U.
To apply them to a concrete state ρ we compute $U\rho U^\dagger$.
We want to define $U^\sharp : AbsDom(S) \to AbsDom(S)$.

Let $T = (t_1, ..., t_m)$ s.t. $t_i = s_i \cup s_U$, where s_U denotes the set of qubits that matrix U acts on.
Let $U^{cg}(\overline{\sigma}) = (UT_{t_1}U^\dagger, ..., UT_{t_m}U^\dagger)$, then

$$U^\sharp = \alpha_{T\to S} \circ U^{cg} \circ \gamma_{S\to T}$$

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$$AbsDom(S) \xrightarrow[\text{Focus}]{\gamma_{S\to T}} AbsDom(T) \xrightarrow[\text{Apply}]{U^{cg}} AbsDom(T) \xrightarrow[\text{Unfocus}]{\alpha_{T\to S}} AbsDom(S)$$

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Focus - concretize to a new, finer domain with sufficient precision to accurately represent the operation;

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Unfocus - abstract back to the original abstract domain to keep the representation more compact

Assertions

We want to check that a state of a program lies in the span of two vectors.
We define an assertion as:

$$A = span\{v_1 = |a_1\rangle \dots |a_n\rangle\,, \quad v_2 = |b_1\rangle \dots |b_n\rangle\}$$

And a projection $proj(A)$ onto this subspace, such that:

$$proj(A)v_1 = v_1 \qquad proj(A)v_2 = v_2$$

Connectivity

$$S = (s_1, ..., s_m)$$

$$S \text{ is connected} \triangleq \forall k \in [n-1].\ \exists r \in [m].\ k \in s_r \wedge k+1 \in s_r$$

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Example:

$$n = 5, \quad m = 3$$
$$S = (\{0,2,3\}, \{0,1,2\}, \{3,4,5\})$$

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$$0, 1 \in \{0,1,2\}$$

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$$1, 2 \in \{0, 1, 2\}$$

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$$2, 3 \in \{0, 2, 3\}$$

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$$4, 5 \in \{3, 4, 5\}$$

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For an assertion A, if S is connected, then:

$$proj(A) = \gamma_{S \to [n]^m}(\alpha_{[n]^m \to S}(proj(A)))$$

Assertion Checking

Given an assertion A, if the final state of a computation is v and the final abstract state of the abstract interpretation is $\overline{v} \in AbsDom(S)$, with S connected, then:

$$\overline{v} \sqsubseteq \alpha_{[n]^m \to S}(proj(A)) \quad \Rightarrow \quad v \in A$$

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Fundamental choice for the shape of the abstract domain and, in turn, the success of the analysis.

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Less, smaller sets	Less computational cost and memory footprint	Less precision
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Examples:

- $S_0 =$ all 2^n combinations of up to n qubits
- $S_1 =$ all $\binom{n}{k}$ combinations of exactly k qubits
- $S_2 =$ sets that contain qubits used by at least two 3-qubit gates
- ...

Assertion Checking Example

We want to check (partial) correctness of Grover's algorithm.

Grover's algorithm steps:

1. Prepare the initial state $|\phi\rangle^{(0)}$
2. Repeat approx. $\frac{\pi}{4}\sqrt{N}$ times $|\phi\rangle^{(t+1)} = G\,|\phi\rangle^{(t)}$.
3. Measure in the computational basis.

With this abstract interpretation framework, we can prove the loop invariant:

$$|\phi\rangle^{(t)} \in A = span\{|\beta\rangle\,, |\phi\rangle\}$$

Assertion Checking Example (ctd.)

$$P(v) \triangleq v \in A = span\{|\beta\rangle, |\phi\rangle\}$$

$G|\phi\rangle^{(t)} \in A = G|\beta\rangle \in A \wedge G|\phi\rangle \in A$ By linearity of G and $|\phi\rangle^{(t)} \in A$

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$G\,|\phi\rangle^{(t)} \in A = G\,|\beta\rangle \in A \wedge G\,|\phi\rangle \in A$ By linearity of G and $|\phi\rangle^{(t)} \in A$

Computing the Projection of a Support

To compute a projection corresponding to $supp(A)$, we:

1. take the rows $\{r_1, ..., r_n\}$ of A;
2. extract an orthonormal set of vectors $\{b_1, ..., b_n\}$ that span the same subspace as the rows;
3. create the matrix $B = \begin{pmatrix} b_1 \\ ... \\ b_n \end{pmatrix}$;
4. return $BB^\dagger$.

Computing the Projection of an Intersection

$$\{P_1, ..., P_k\}, \quad \forall i \in [k].\, P_i = C^n \times C^n$$

$$\bigcap_{i \in [k]} P_i \triangleq I_n - supp(kI_n - \sum_{i \in [k]} P_i)$$

Computing the Assertion Projection

Given two vectors

$$v_1 = |a_1\rangle \dots |a_n\rangle$$
$$v_2 = |b_1\rangle \dots |b_n\rangle$$

1. Create a matrix $P = \begin{pmatrix} v_1^T \\ v_2^T \\ \mathbf{0} \\ \dots \\ \mathbf{0} \end{pmatrix}$
2. Return $supp(P)$

Partial trace

Alternatively:

$$Tr_B[\rho] = \sum_{v=0}^{2^m} (I_A \otimes \langle v|)\rho(I_A \otimes |v\rangle) \quad Tr_A[\rho] = \sum_{v=0}^{2^n} (\langle v| \otimes I_B)\rho(|v\rangle \otimes I_B)$$

Where v labels vectors of an orthonormal basis of the subspace we are tracing out.