

Alessandro Scala



Quantum Abstract Interpretation

Seminar for the **Introduction to Quantum Computing** course

Università di Pisa
Dipartimento di Informatica

Pisa, 24 Luglio 2023

Roadmap

- ① Introduction
 - Reasons
 - Abstract Interpretation
- ② Preliminaries
 - Density Matrix
 - Reduced Density Matrix
- ③ Abstract Domain
 - Abstraction and Concretization Functions
 - Abstract Operations
 - Assertions
- ④ Conclusions



Introduction

As quantum computing advances, we would like to have some means to prove correctness properties on quantum programs, *especially* since quantum programming is unintuitive.



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No: **exponential** space and time cost.



Example

$$n_{qubits} = 1$$

$$|0\rangle \langle 0|$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

$$2^2 = 4 \text{ complex numbers}$$



Example

$$n_{\text{qubits}} = 2$$

$$|00\rangle \langle 00|$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$2^4 = 16 \text{ complex numbers}$$



Example

$$n_{\text{qubits}} = 3$$

$$|000\rangle \langle 000|$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$2^6 = 64 \text{ complex numbers}$$



Example

$$n_{\text{qubits}} = 300$$

$$|0\rangle^{\otimes 300} \langle 0|^{\otimes 300}$$

?????



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$2^{600} = 41495155688809929585124078636911611510124462322424368$
999956573296906528114129081463997070489471037942881978866113
007891823951510754117753078868748341139636870611818034015095
23685376

Bigger than the number of atoms in the universe.



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Yes, but time consuming. Needs to be adapted to the specific program.



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Solution: abstract interpretation



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Instead of considering the **concrete domain**, we restrict our analysis to a **more coarse domain**, an **abstract domain**.



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Abstract interpretation is usually sound, but not complete:

- AI returns **true** \Rightarrow Property is **true**
- AI returns **false** \Rightarrow Property can be either **true** or **false**



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- Assertions: properties we can prove with abstract interpretation



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Density Matrix

Instead of dealing with a state $|\phi\rangle$ in vector form, we use its *density matrix*:

$$\rho_\phi = |\phi\rangle \langle\phi| \quad (\text{For a pure state})$$

- positive semi-definite
- $\text{Tr}(\rho) = 1$
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$$\text{Tr}_B[\rho] : AB \rightarrow A$$

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$$\text{Tr}_B[\alpha \otimes \beta] = \alpha \cdot \text{Tr}(\beta)$$

$$\text{Tr}_A[\alpha \otimes \beta] = \text{Tr}(\alpha) \cdot \beta$$

$$\left. \begin{aligned} \text{Tr}_S[\rho + \sigma] &= \text{Tr}_S[\rho] + \text{Tr}_S[\sigma] \\ \text{Tr}_S[a \cdot \rho] &= a \cdot \text{Tr}_S[\rho] \end{aligned} \right\} \text{ (Linearity)}$$



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Partial trace $Tr_B[\rho]$ **traces out** subsystem B .



Example

$$A = \mathbb{C}^2 \times \mathbb{C}^2 \quad B = \mathbb{C}^2 \times \mathbb{C}^2 \quad AB = A \otimes B$$

$$\rho_{\beta_{00}} = |\beta_{00}\rangle \langle \beta_{00}| = \frac{|00\rangle \langle 00| + |00\rangle \langle 11| + |11\rangle \langle 00| + |11\rangle \langle 11|}{2}$$

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Loss of precision

Computing a reduced density matrix **discards information!**

$$\rho_{\beta_{00}} = \frac{|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|}{2} \quad (\text{Pure state})$$

$$\rho_2 = \frac{|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|}{4} \quad (\text{Mixed state})$$



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$$\text{Tr}_B[\rho_{\beta_{00}}] = \frac{|0\rangle\langle 0| + |1\rangle\langle 1|}{2} = \text{Tr}_B[\rho_2]$$

The partial traces of two different initial states can be equal.

For a state $\rho \in A \otimes B$, even if we know $\text{Tr}_B[\rho]$ and $\text{Tr}_A[\rho]$, we cannot uniquely determine ρ .



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This duality between projections and subspaces will be employed multiple times and will often be implied.



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Abstract Domain

$$\mathcal{D} = \mathbb{C}^{2^n} \otimes \mathbb{C}^{2^n}, \quad S = (s_1, \dots, s_m), \quad 1 \leq m \leq 2^n, \quad s_i \subseteq [n]$$



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Special case:

$$T = ([n]) \Rightarrow AbsDom(T) \simeq \mathcal{D}$$



Fineness Relation

Let $S = (s_1, \dots, s_m)$ and $T = (t_1, \dots, t_m)$ (with $1 \leq m \leq 2^n$), then:

$$\underbrace{S \sqsubseteq T} \triangleq \forall i \in [m]. s_i \subseteq t_i$$

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$AbsDom(\perp)$ corresponds to a state so abstract that it holds no information at all.

$AbsDom(\top)$ corresponds to tuples where every projection is a concrete state.



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- ③ Compute the support of the traced matrices (to preserve the structure of projections).



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- ② For each Q_{t_j} found, trace out the qubits in t_j that are not in s_i .
- ③ Compute the support of the traced matrices (to preserve the structure of projections).
- ④ Compute the intersection of the supports.



Concretization Function

$$S \sqsubseteq T \triangleq \forall i \in [m]. s_i \subseteq t_i$$

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- ③ Compute the intersection of the extended projections.



Abstract Operations

Operations on the concrete domain are unitary operators U .
To apply them to a concrete state ρ we compute $U\rho U^\dagger$.
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$$U^\# = \alpha_{T \rightarrow S} \circ U^{cg} \circ \gamma_{S \rightarrow T}$$



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Apply - apply the unitary operator to all the elements in the tuple

Unfocus - abstract back to the original abstract domain to keep the representation more compact



Assertions

We want to check that a state of a program lies in the span of two vectors.



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$$A = \text{span}\{v_1 = |a_1\rangle \dots |a_n\rangle, v_2 = |b_1\rangle \dots |b_n\rangle\}$$

And a projection $\text{proj}(A)$ onto this subspace, such that:

$$\text{proj}(A)v_1 = v_1 \quad \text{proj}(A)v_2 = v_2$$



Order Relation on Abstract States

$$1 \leq m \leq 2^n, \quad S = (s_1, \dots, s_m), \quad \forall i \in [m]. s_i \subseteq [n]$$

$$\bar{\sigma} \in \text{AbsDom}(S) = (P_{s_1}, \dots, P_{s_m}), \quad \bar{\tau} \in \text{AbsDom}(S) = (Q_{s_1}, \dots, Q_{s_m})$$

$$\bar{\sigma} \sqsubseteq \bar{\tau} \triangleq \forall i \in [m]. \underbrace{P_{s_i} \subseteq Q_{s_i}}$$

Subspace interpretation
of projections



Monotonicity and Galois Connection

Monotonicity of *abstraction*, *concretization*, and *abstract operations*:

$$\begin{aligned}
 &S \sqsubseteq T \\
 &\forall \bar{\sigma}, \bar{\tau} \in \text{AbsDom}(T). \\
 &\left(\begin{array}{c} \bar{\sigma} \sqsubseteq \bar{\tau} \Rightarrow \alpha_{T \rightarrow S}(\bar{\sigma}) \sqsubseteq \alpha_{T \rightarrow S}(\bar{\tau}) \\ \wedge \\ \bar{\sigma} \sqsubseteq \bar{\tau} \Rightarrow \gamma_{T \rightarrow S}(\bar{\sigma}) \sqsubseteq \gamma_{T \rightarrow S}(\bar{\tau}) \\ \wedge \\ \bar{\sigma} \sqsubseteq \bar{\tau} \Rightarrow U^\#(\bar{\sigma}) \sqsubseteq U^\#(\bar{\tau}) \end{array} \right)
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Galois connection:

$$\begin{aligned}
 & \forall \bar{\sigma} \in \text{AbsDom}(S). \forall \bar{\tau} \in \text{AbsDom}([n]^m). \\
 & \bar{\tau} \sqsubseteq \gamma_{S \rightarrow [n]^m}(\bar{\sigma}) \quad \Leftrightarrow \quad \alpha_{[n]^m \rightarrow S}(\bar{\tau}) \sqsubseteq \bar{\sigma}
 \end{aligned}$$



Connectivity

$$S = (s_1, \dots, s_m)$$

S is connected $\triangleq \forall k \in [n-1]. \exists r \in [m]. k \in s_r \wedge k+1 \in s_r$



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Example:

$$n = 5, \quad m = 3$$

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$$4, 5 \in \{3, 4, 5\}$$

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For an assertion A , if S is connected, then:

$$proj(A) = \gamma_{S \rightarrow [n]^m}(\alpha_{[n]^m \rightarrow S}(proj(A)))$$



Assertion Checking

Given an assertion A , if the final state of a computation is v and the final abstract state of the abstract interpretation is $\bar{v} \in \text{AbsDom}(S)$, with S connected, then:

$$\bar{v} \sqsubseteq \alpha_{[n]^m \rightarrow S}(\text{proj}(A)) \quad \Rightarrow \quad v \in A$$



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Essentially, if the abstract state satisfies the assertion, then the concrete one does as well.



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Fundamental choice for the shape of the abstract domain and, in turn, the success of the analysis.



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Less, smaller sets	Less computational cost and memory footprint	Less precision
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Examples:

- $S_0 =$ all 2^n combinations of up to n qubits
- $S_1 =$ all $\binom{n}{k}$ combinations of exactly k qubits
- $S_2 =$ sets that contain qubits used by at least two 3-qubit gates
- ...



Choice of Tuple S (ctd.)

Another important factor in the choice for the abstract state is which qubits are put together in projections.

A good choice would make the abstract interpretation *flow-sensitive*.



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- U_1 acts on qubits $\{1, 2, 3\}$ with $\{1, 2\}$ as input and $\{3\}$ as output.
- U_2 acts on qubits $\{3, 4, 5\}$ with $\{3, 4\}$ as input and $\{5\}$ as output.



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A good choice which would improve the precision of the abstract interpretation is to have the set $\{1, 2, 3, 4, 5\}$ in the abstract state.



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With this abstract interpretation framework, we can prove the loop invariant:

$$|\phi\rangle^{(t)} \in A = \text{span}\{|\beta\rangle, |\phi\rangle\}$$



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- ② $P(|\phi\rangle^{(t)}) \Rightarrow P(|\phi\rangle^{(t+1)})$



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$$G|\phi\rangle^{(t)} \in A = G|\beta\rangle \in A \wedge G|\phi\rangle \in A \quad \text{By linearity of } G \text{ and } |\phi\rangle^{(t)} \in A$$



Roadmap

- ① Introduction
 - Reasons
 - Abstract Interpretation
- ② Preliminaries
 - Density Matrix
 - Reduced Density Matrix
- ③ Abstract Domain
 - Abstraction and Concretization Functions
 - Abstract Operations
 - Assertions
- ④ Conclusions



Future developments

- Programs with measurements
- Conditionals
- Loops
- Mix of classical and quantum computation
- Choosing the optimal abstract space
- Other kinds of assertions



Computing the Projection of a Support

[noframenumbering] To compute a projection corresponding to $\text{supp}(A)$, we:

- 1 take the rows $\{r_1, \dots, r_n\}$ of A ;
- 2 extract an orthonormal set of vectors $\{b_1, \dots, b_n\}$ that span the same subspace as the rows;
- 3 create the matrix $B = \begin{pmatrix} b_1 \\ \dots \\ b_n \end{pmatrix}$;
- 4 return BB^\dagger .



Computing the Projection of an Intersection

[noframenumbering]

$$\{P_1, \dots, P_k\}, \quad \forall i \in [k]. \ P_i = C^n \times C^n$$
$$\bigcap_{i \in [k]} P_i \triangleq I_n - \text{supp}(kl_n - \sum_{i \in [k]} P_i)$$



Computing the Assertion Projection

[noframenumbering] Given two vectors

$$v_1 = |a_1\rangle \dots |a_n\rangle$$

$$v_2 = |b_1\rangle \dots |b_n\rangle$$

① Create a matrix $P = \begin{pmatrix} v_1^T \\ v_2^T \\ \mathbf{0} \\ \dots \\ \mathbf{0} \end{pmatrix}$

② Return $\text{supp}(P)$



Partial trace

$$Tr_B[\rho] = \sum_{v=0}^{2^m} (I_A \otimes \langle v|) \rho (I_A \otimes |v\rangle) \quad Tr_A[\rho] = \sum_{v=0}^{2^n} (\langle v| \otimes I_B) \rho (|v\rangle \otimes I_B)$$

Where v labels vectors of an orthonormal basis of the subspace we are tracing out.



Weak Galois Connection

$$\begin{aligned}
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 & \forall \bar{\sigma} \in \text{AbsDom}(S). \forall \bar{\tau} \in \text{AbsDom}(T). \\
 & \left(\begin{array}{c} \bar{\tau} \sqsubseteq \gamma_{S \rightarrow T}(\bar{\sigma}) \Rightarrow \alpha_{T \rightarrow S}(\bar{\tau}) \sqsubseteq \bar{\sigma} \\ \wedge \\ (\exists \bar{\rho} \in \text{AbsDom}([n]^m). \bar{\tau} = \alpha_{[n]^m \rightarrow T}(\bar{\rho})) \Rightarrow \bar{\tau} \sqsubseteq \gamma_{S \rightarrow T}(\bar{\sigma}) \Leftrightarrow \alpha_{T \rightarrow S}(\bar{\tau}) \sqsubseteq \bar{\sigma} \end{array} \right)
 \end{aligned}$$



