Wavelet basics

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1. Introduction

For a given univariate function f, the Fourier transform of f and the inverse are given by

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt.$$

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega t}\, d\omega.$$

Parseval: $(f, g) = (\hat{f}, \hat{g})/2\pi, \quad (f, g) = \iint f(t)\,\overline{g(t)}\, dt.$

$e_\omega(t) = e^{-i\omega t}, \delta_{\omega_0}(\omega) = \delta(\omega - \omega_0)$

$\hat{f}(\omega_0) = (f, e_{\omega_0}) = (\hat{f}, \delta_{\omega_0})$

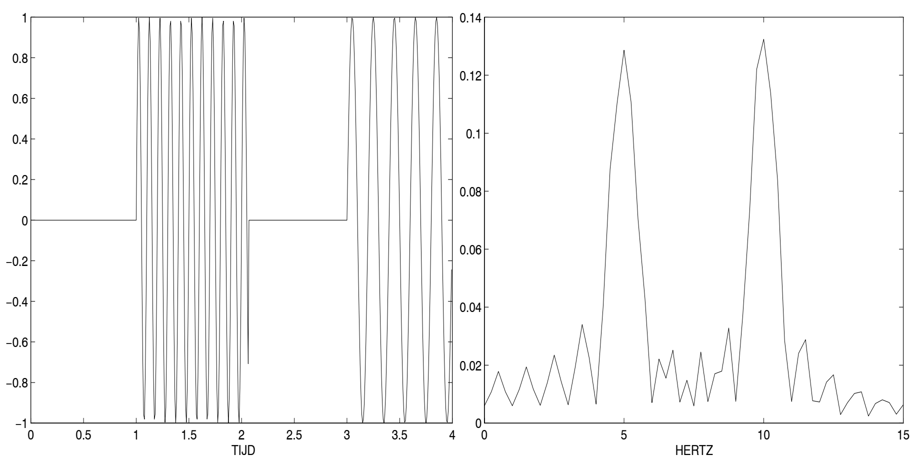


Figure 1: *The frequency break and its amplitude-spectrum*

The short time Fourier transform

Given a Window function g

$g \in L^2(\mathbb{R})$, $\|g\| = 1 \quad g$ is real-valued.

The short time Fourier transform $F(u, \tau)$ of a function f is defined by

$$F(u, \tau) = \int_{-\infty}^{\infty} f(t) e^{-iut} g(t - \tau)\, dt,$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(u, \tau) e^{iut} g(t - \tau)\, d\tau\, du,$$

$g_{u,\tau}(t) := e^{iut} g(t - \tau)$, $F(u, \tau) = (f, g_{u,\tau})$

$$(f, g_{u,\tau}) = \frac{1}{2\pi} (\hat{f}, \hat{g}_{u,\tau}) \text{ (Parseval).}$$

$$\hat{g}_{u,\tau}(\omega) = e^{-i(\omega - u)\tau} \hat{g}(\omega - u).$$

Fixed ” window width” in time and frequency.

2. The continous/discrete Wavelet transform

The continuous Wavelet transform

Given ψ in $L^2(\mathbb{R})$.
Introduce a family of functions $\psi_{a,b}$ $(a > 0,\ b \in \mathbb{R})$ as follows

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\, \psi((t-b)/a) \qquad (t \in \mathbb{R}),$$

$\|\psi_{a,b}\| = \|\psi\|$.

The continuous wavelet transform $F(a,b)$ of a function f is defined by

$$F(a,b) = (f, \psi_{a,b}) = \frac{1}{\sqrt{a}} \int_{-\infty}^{\infty} f(t)\, \psi((t-b)/a)\, dt.$$

$$(f, \psi_{a,b}) = \frac{1}{2\pi} (\hat{f}, \hat{\psi}_{a,b}) \text{ Parseval.}$$

where

$$\hat{\psi}_{a,b}(\omega) = \sqrt{a}\, e^{-i\omega b} \hat{\psi}(a\omega),$$

The inverse wavelet transform

$$f(t) = C_{\psi}^{-1} \int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{1}{a^2} F(a, b)\, \psi_{a,b}(t)\, da\, db.$$

$$C_{\psi} = \int_{0}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega.$$

Needed $\hat{\psi}(0) = 0$, i.e.,

$$\int_{-\infty}^{\infty} \psi(t)\, dt = 0.$$

This is the reason why the functions $\psi_{a,b}$ are called wavelets.

ψ is called the Motherwavelet.

Example: The Mexican hat (Morlet wavelet)

$$\psi(t) = \frac{2}{\sqrt{3}}\pi^{-\frac{1}{4}}(1 - t^2)e^{-t^2/2}.$$

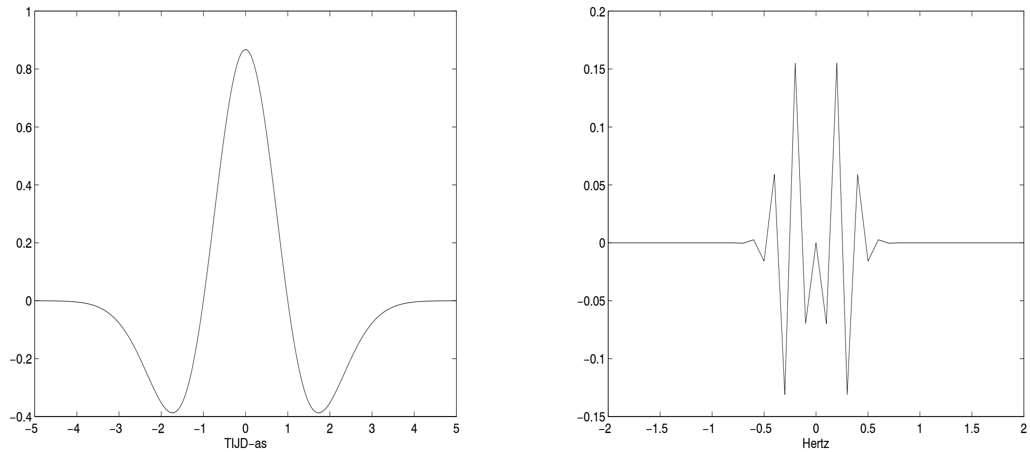


Figure 2: The Mexican hat

The wavelet transform of the frequency break using the Mexican hat

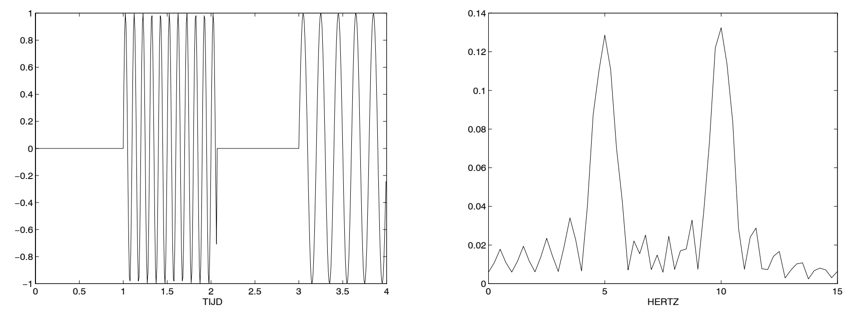


Figure 3: frequency break

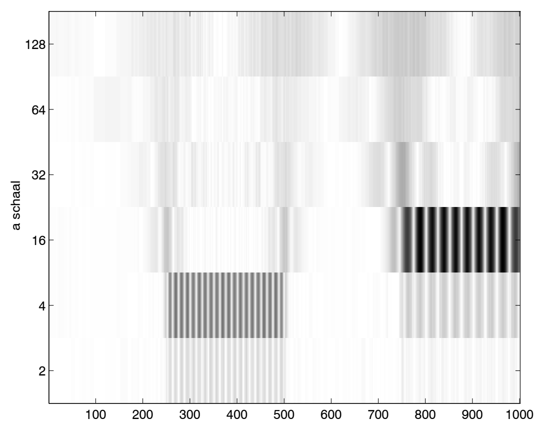


Figure 4: Grey value picture of the waveletcoëfficiënten

Horizontal b-axis contains 1000 samples on interval $[0, 1]$.

The vertical axis contains the a-values: $2, 4, \ldots, 128$.

The discrete wavelet transform

Sampling in the a-b plane.

$a_0 > 1, b_0 > 0$
$a = a_0^{-\ell}, \quad b = k\, a_0^{-\ell}\, b_0, \quad (k, \ell \in \mathbb{Z})$.
The translation step is adapted to the scale

$$\psi_{k,\ell}(t) = a_0^{\ell/2}\psi(a_0^{\ell}t - k\, b_0).$$

Dyadic wavelets: $a_0 = 2$, $b_0 = 1$.

$$\psi_{k,\ell}(t) = 2^{\ell/2}\, \psi(2^{\ell}t - k).$$

$(f, \psi_{k,\ell})$ are called *waveletcoefficients*.

Discrete Wavelet transform: $f \to (f, \psi_{k,\ell})$

a. Problem of reconstruction:

$f = \sum_{k,\ell}(f, \psi_{k,\ell})\tilde{\psi}_{k,\ell}$.

b. Problem of decomposition:

$f = \sum_{k,\ell} a_{k,\ell}\psi_{k,\ell}$

It would be nice if the functions $\psi_{k,\ell}$ constitute an orthonormal basis of $L^2(\mathbb{R})$. (orthogonal wavelets)

For orthogonal wavelets the reconstruction formula and the decomposition formula coincide.
A biorthogonal wavelets system consists of two sets of wavelets generated by a mother wavelet ψ and a dual wavelet $\tilde{\psi}$, for which

$$(\tilde{\psi}_{k,\ell}, \psi_{m,n}) = \delta_{k,m}\delta_{\ell,n},$$

for all integer values k, ℓ, m en n.
We assume that $(\psi_{k,\ell})$ constitute a so called Riesz basis (numerically stable) of $L^2(\mathbb{R})$, i.e.

$$A\,(f,\, f) \leq \| \sum_{k,\ell} \xi_{k,\ell} \|^2 \leq B\,(f,\, f)$$

for positive constants A en B, where $f = \sum_{k,\ell} \xi_{k,\ell}\psi_{k,\ell}$.

The reconstruction formula now reads

$$f = \sum_{k,\ell} (f,\, \psi_{k,\ell})\tilde{\psi}_{k,\ell}.$$

Examples of biorthogonal wavelets are the bior family implemented in the MATLAB Toolbox

3. Multi-resolution analysis

For a given function f, let

$$f_\ell = \sum_{k=-\infty}^{\infty} (f, \tilde{\psi}_{k,\ell}) \psi_{k,\ell},$$

Then

$$f = \sum_{\ell=-\infty}^{\infty} f_\ell.$$

f_ℓ can be interpreted as that part of f which belongs to the scale ℓ.
So, $f = \sum_{\ell=-\infty}^{\infty} f_\ell$ is a decomposition of f to different scale levels ℓ.
The function f_ℓ belongs to the scale space W_ℓ spanned by $(\psi_{k,\ell})$ with fixed ℓ.

The space W_0 is spanned by the integer translates of the mother wavelet ψ.

For integer n the function

$$g_n(t) = \sum_{\ell=-\infty}^{n-1} f_\ell(t)$$

contains all the information of f up to scale level $n-1$.
So $g_n \in V_n$, where

$$V_n = \sum_{\ell=-\infty}^{n-1} W_\ell.$$

It follows that $V_n = V_{n-1} \oplus W_{n-1} \quad (n \in \mathbb{Z})$ direct sum.

Properties of the sequence (V_n)

a) $V_{n-1} \subset V_n \quad (n \text{ geheel}),$

b) $\overline{\bigcup_{n\in\mathbb{Z}} V_n} = L^2(\mathbb{R}),$

c) $\bigcap_{n\in\mathbb{Z}} V_n = \{0\},$

d) $f(t) \in V_n \Leftrightarrow f(2t) \in V_{n+1},$

e) $f(t) \in V_0 \Rightarrow f(t+1) \in V_0.$

If a sequence of subspaces (V_n) satisfies the properties a) to e), then it is called a *Multi-Resolution-Analysis* (MRA) of $L^2(\mathbb{R})$.

If there exists a function ϕ such that V_0 is spanned by the integer translates of ϕ, then ϕ is called a scaling function for the MRA.
As a consequence one has that V_n is spanned by $\phi_{k,n}$, (n fixed),

$$\phi_{k,n} = 2^{n/2}\,\phi(2^n\,t-k)$$

4. Scaling functions

Sufficient conditions for a compactly supported function ϕ to be a scaling function for an MRA.

1. There exists a sequence of numbers (p_k), from which only a finite number differs from zero, such that

$$\phi(t) = \sum_{k=-\infty}^{\infty} p_k \phi(2t-k) \quad \text{2-scale relation.}$$

2. The so-called Riesz function has no zeros on the unit circle.

Autocorrelation function of ϕ: $\rho(\tau) := \int_{-\infty}^{\infty} \phi(t+\tau)\,\phi(t)\,dt$.

Riesz function

$$R(z) = \sum_{m=-\infty}^{\infty} \rho(m)\, z^m.$$

3. Partition of the unity

$$\sum_k \phi(t-k) \equiv 1.$$

The Laurent polynomial $P(z) = \frac{1}{2}\sum_k p_k\, z^k$ is called the two scale symbol of ϕ.

Examples

B-splines of order m:

$$P(z) = \left(\frac{z+1}{2}\right)^m$$

The Daubechies scaling function of order 2

$$P_2(z) = \frac{1}{2}\left\{\frac{1+\sqrt{3}}{4} + \frac{3+\sqrt{3}}{4}z + \frac{3-\sqrt{3}}{4}z^2 + \frac{1-\sqrt{3}}{4}z^3\right\}.$$

For an orthonormal system one has

$$R(z) \equiv 1,$$
$$|P(z)|^2 + |P(-z)|^2 \equiv 1 \quad (|z| = 1)$$

Based on a given MRA with scaling function ϕ one may construct wavelets by first completing the spaces V_ℓ to a space $V_{\ell+1}$ by means of a space W_ℓ, i.e.$V_{\ell+1} = V_\ell \oplus W_\ell$ in such a way that there exists a function ψ such that W_ℓ is spanned by $(\psi(2^\ell t - k))$.
To satisfy $V_1 = V_0 \oplus W_0$ the following conditions are necessary and sufficient:

1. $W_0 \subset V_1$,
2. $W_0 \bigcap V_0 = \{0\}$,
3. $\phi(2t) \in V_0 \oplus W_0$ and $\phi(2t-1) \in V_0 \oplus W_0$.

It follows that

$$\psi(t) = \sum_{k=-\infty}^{\infty} q_k \phi(2t-k),$$

$$\begin{aligned} \phi(2t) &= \sum_{k=-\infty}^{\infty} (a_k \phi(t-k) + b_k \psi(t-k)) \quad (t \in \mathbb{R}), \\ \phi(2t-1) &= \sum_{k=-\infty}^{\infty} (c_k \phi(t-k) + d_k \psi(t-k)) \quad (t \in \mathbb{R}). \end{aligned}$$

By introducing the Laurent series $A(z) = \sum_k a_k z^k$, $B(z) = \sum_k b_k z^k$, $C(z) = \sum_k c_k z^k$ and $D(z) = \sum_k d_k z^k$ and the symbol $Q(z) = \sum_k q_k z^k$ for the wavelet ψ, the application of the Fourier-transform to the previous equations and the 2-scale relation for the scaling function ϕ finally lead to the following set of equations, which must hold for complex z with $|z| = 1$.

$$
\begin{aligned}
&A(z^2)\, P(z) + B(z^2)\, Q(z) = 1/2,\\
&A(z^2)\, P(-z) + B(z^2)\, Q(-z) = 1/2,\\
&C(z^2)\, P(z) + D(z^2)\, Q(z) = z/2,\\
&C(z^2)\, P(-z) + D(z^2)\, Q(-z) = -z/2,
\end{aligned}
$$

Now let (assuming the inverse exists)

$$\begin{pmatrix} P(z) & Q(z) \\ P(-z) & Q(-z) \end{pmatrix}^{-1} = \begin{pmatrix} H(z) & H(-z) \\ G(z) & G(-z) \end{pmatrix},$$

where

$$H(z) = \sum_k h_k z^k,$$
$$G(z) = \sum_k g_k z^k.$$

Then

$$A(z^2) = (H(z) + H(-z))/2,$$
$$B(z^2) = (G(z) + G(-z))/2,$$
$$C(z^2) = z\,(H(z) - H(-z))/2,$$
$$D(z^2) = z\,(G(z) - G(-z))/2, .$$

We now have

$$\phi(2t-k) = \sum_{m=-\infty}^{\infty} \big(h_{2m-k}\phi(t-m)+g_{2m-k}\psi(t-m)\big) \quad (t \in \mathbb{R}).$$

It can be shown that the symbol $\tilde{P}(z)$ for the dual scaling $\tilde{\phi}$ and the symbol $\tilde{Q}(z)$ for the dual wavelet $\tilde{\psi}$ will satisfy

$$\tilde{P}(z) = H(z^{-1}),$$
$$\tilde{Q}(z) = Q(z^{-1}).$$

For orthogonal wavelets based on an orthogonal scaling function one may choose

$$q_k = (-1)^k \, p_{1-k}.$$

5. The Fast Wavelet Transform

To obtain a wavelet decomposition of a function f in practice, one first approximates f by a function from a space V_n, which is close to f. So let us assume that f itself belongs to V_n. So

$$f = \sum_{k=-\infty}^{\infty} a_{k,n} \phi_{k,n}$$

Since $V_n = \sum_{\ell=-\infty}^{n-1} W_\ell$, one has

$$f = \sum_{\ell=-\infty}^{n-1} \sum_{k=-\infty}^{\infty} d_{k,\ell} \psi_{k,\ell}$$

$V_n = V_{n-1} \oplus W_{n-1}$ implies

$$f = \sum_{k=-\infty}^{\infty} a_{k,n}\phi_{k,n} = \sum_{k=-\infty}^{\infty} a_{k,n-1}\phi_{k,n-1} + \sum_{k=-\infty}^{\infty} d_{k,n-1}\psi_{k,n-1}.$$

Due to

$$\phi_{k,n} = \sum_{m=-\infty}^{\infty} \sqrt{2}\, h_{2m-k}\phi_{m,n-1} + \sqrt{2}\, g_{2m-k}\psi_{m,n-1}.$$

we obtain

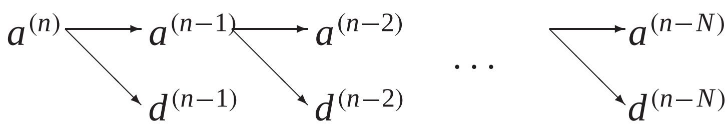
$$f = \sum_{k=-\infty}^{\infty} a_{k,n}\phi_{k,n} = \sum_{k=-\infty}^{\infty} a_{k,n}\sqrt{2}\,(\sum_{m=-\infty}^{\infty} (h_{2m-k}\phi_{m,n-1} + g_{2m-k}\psi_{m,n-1})).$$

Our conclusion is

$$a_{m,n-1} = \sum_{k=-\infty}^{\infty} \sqrt{2}\, h_{2m-k}a_{k,n}, \quad d_{m,n-1} = \sum_{k=-\infty}^{\infty} \sqrt{2}\, g_{2m-k}a_{k,n}.$$

convolution and subsequently downsampling ($m \to 2\,m$) yields the two sequences $a^{(n-1)} = (a_{m,n-1})$ en $d^{(n-1)} = (d_{m,n-1})$.

A repeated application of the previous operation leads to a decomposition of f to coarser levels, which can be expressed by the following scheme and filtering proces.



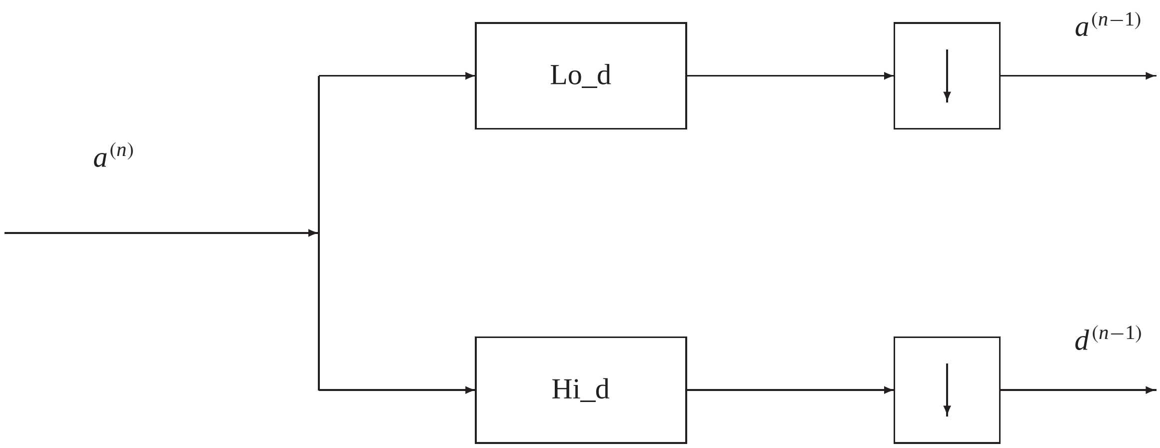


Figure 5: Decomposition

Filter coefficients are $\sqrt{2}\,h_k$ for the low pass filter and $\sqrt{2}\,g_k$ for the high pass filter.

Reconstruction

If $a^{\ell-1}$ and $d^{\ell-1}$ are given then we may reconstruct the approximation coefficients a^{ℓ}.

$f_\ell = f_{\ell-1} + w_{\ell-1}$

$$\begin{aligned}
f_\ell &= \sum_{k=-\infty}^{\infty} a_{k,\ell}\phi_{k,\ell} \\
&= \sum_{k=-\infty}^{\infty} a_{k,\ell-1}\phi_{k,\ell-1} + \sum_{k=-\infty}^{\infty} d_{k,\ell-1}\psi_{k,\ell-1} \\
&= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} a_{k,\ell-1}\frac{1}{\sqrt{2}}\, p_m\phi_{2k+m,\ell} \\
&+ \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} d_{k,\ell-1}\frac{1}{\sqrt{2}}\, q_m\phi_{2k+m,\ell}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty} a_{k,\ell}\phi_{k,\ell} \\
&= \sum_{k=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} \frac{1}{\sqrt{2}}\big(a_{k,\ell-1}p_{m-2k} + d_{k,\ell-1}q_{m-2k}\big)\phi_{m,\ell}.
\end{aligned}$$

Conclusion:

$$a_{k,\ell} = \frac{1}{\sqrt{2}}\sum_{m=-\infty}^{\infty}(a_{m,\ell-1}p_{k-2m} + d_{m,\ell-1}q_{k-2m}).$$

upsampling and subsequently convolution

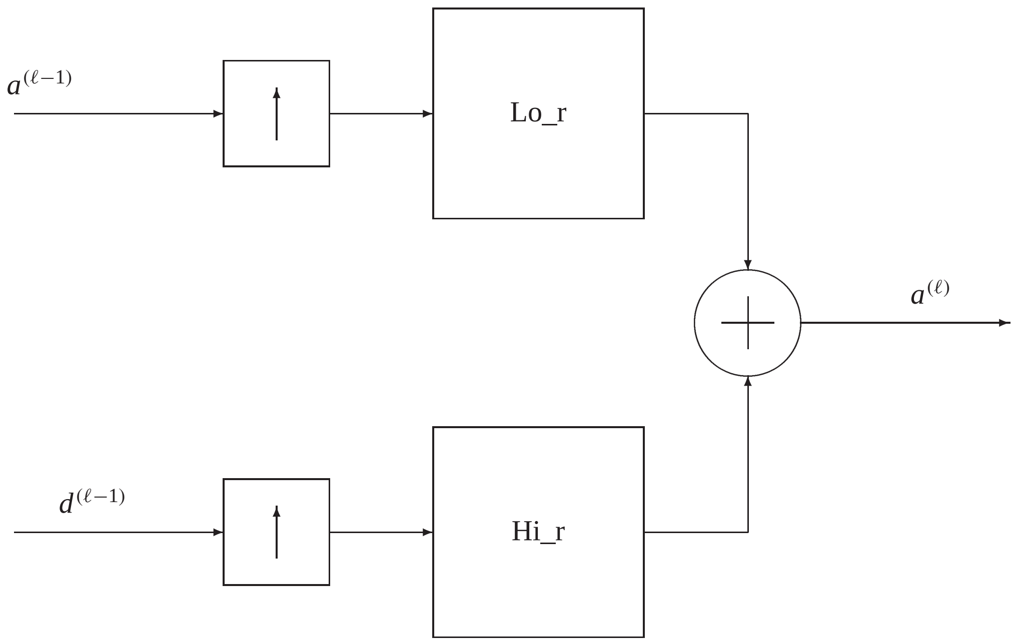


Figure 6: Reconstruction

6. Examples

1. Haar wavelet

General characteristics:

Orthogonal
Support width 1
Filters length 2
Number of vanishing moments for ψ: 1
Scaling function yes

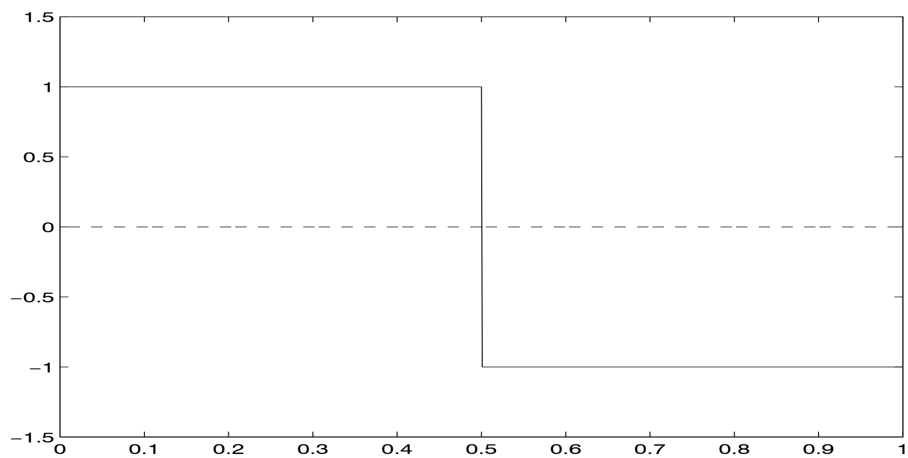


Figure 7: Haar wavelet

2. Daubechies family

General characteristics:

Order $N = 1, \ldots$
Orthogonal
Support width $2\,N - 1$
Filters length $2\,N$
Number of vanishing moments for ψ N
Scaling function yes

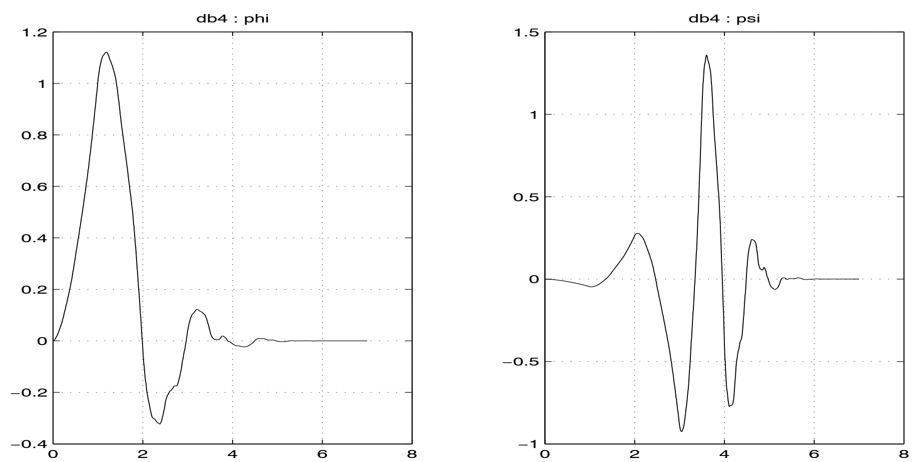


Figure 8: *Daubechies order 4*

3. Coiflet family

General characteristics:

Order $N = 1, \ldots, 5$
Orthogonal
Support width $6N - 1$
Filters length $6N$
Symmetry near from
Number of vanishing moments for ψ $2N$

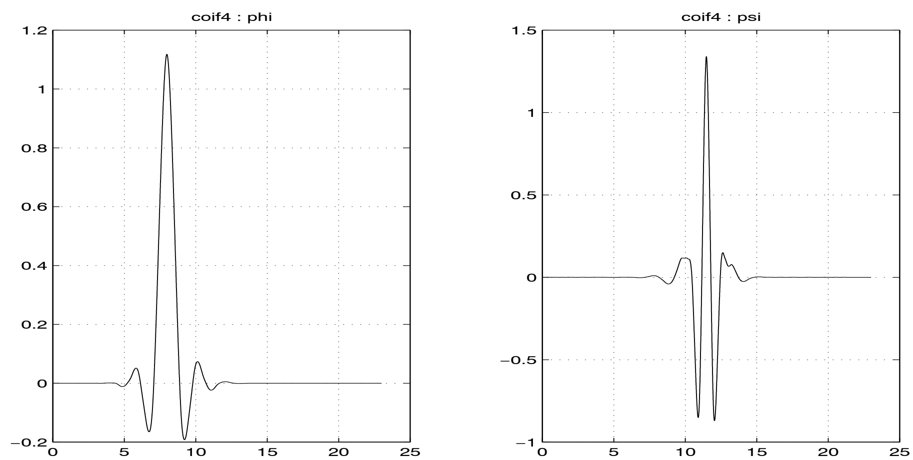


Figure 9: *Coiflet order 4*

Meyer wavelet

General characteristics:

Orthogonal
Compact support no
Effective support [-8, 8]
Symmetry yes
Scaling function yes

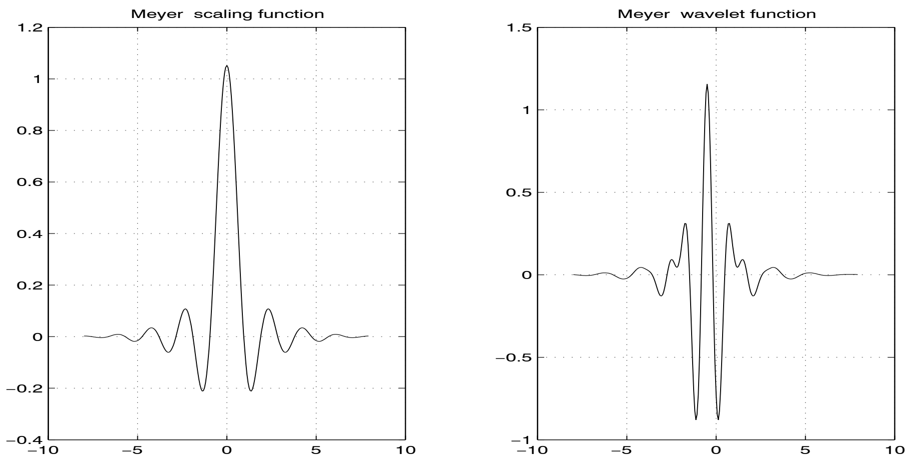


Figure 10: *Meyer*