Exercise 1.13

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Prove that $Fib(n)$ **is the closest integer to** $\phi^n/\sqrt{5}$**, where** $\phi = (1+\sqrt{5})/2$**.**

We begin by making the conjecture $S(n) : Fib(n) = (\phi^n - \psi^n)/\sqrt{5}$, where $\psi = (1-\sqrt{5}/2)$, and proving that $S(n)$ is true for all positive integers n. The proof proceeds by induction on n.

Base case ($n = 0$):

$$\begin{aligned} \frac{(\psi^0 - \phi^0)}{\sqrt{5}} &= \frac{(1-1)}{\sqrt{5}} \\ &= \frac{0}{\sqrt{5}} \\ &= 0 \end{aligned}$$

and $Fib(0) = 0$, therefore $S(0)$ is true.

Base case ($n = 1$):

$$\begin{aligned} \frac{(\psi^1 - \phi^1)}{\sqrt{5}} &= \frac{\frac{(1+\sqrt{5})}{2} - \frac{(1-\sqrt{5})}{2}}{\sqrt{5}} \\ &= \frac{\frac{2\sqrt{5}}{2}}{\sqrt{5}} \\ &= 1 \end{aligned}$$

and $Fib(1) = 1$, therefore $S(1)$ is true.

Induction step: Assume the truth of the statements $S(0)$, $S(1)$, $\cdots$, $S(k-2)$, $S(k-1)$, and $S(k)$ for some positive integer $k \geq 1$. Then

$$S(k+1) : Fib(k+1) = \frac{\left(\phi^{k+1} - \psi^{k+1}\right)}{\sqrt{5}}$$

which is equivalent to

$$Fib(k) + Fib(k-1) \quad = \quad \frac{\left(\phi^k - \psi^k\right)}{\sqrt{5}} + \frac{\left(\phi^{k-1} - \psi^{k-1}\right)}{\sqrt{5}}$$

by the definition of Fibonacci numbers. We know that this is true from the truth of $S(k)$ and $S(k-1)$. Therefore $S(n)$ is true for all positive integers n by induction. ■