Statistical Methods MATH2715 info

Teaching material is all online!

- On Minerva `http://minerva.leeds.ac.uk`
- On GitHub `https://github.com/luisacutillo78/Statistical-Methods-Lecture-Notes`

R code submission

- No technichal issue - please submit your SURNAMEstudentid.R [or .Rmd as required] file in the assignment folder.

Resources

- Mathematical Statistics and Data Analysis - 3rd ed. (by J. A. Rice);
- Introduction to Statistics - Online Edition -D.M.Lane et al.
- http://www1.maths.leeds.ac.uk/statistics/R/Rintro.pdf;
- https://www.datacamp.com/courses/free-introduction-to-r.

Where We've Been, Where We're Going

In the previous Lecture

- More about Moment Generating Functions
- Multivariate Normal Distribution

Today

- Univariate and Multivariate Change of Variables
- Markov's Inequality
- Chebyshev's Inequality
- Weak Law of Large Numbers
- Exercises & Questions

Change of Coordinates

Proposition

Suppose X is a random variable and $Y = g(X)$, where $g : \Re \to \Re$ that is a *monotonic* function.
Define $g^{-1} : \Re \to \Re$ such that $g^{-1}(g(X)) = X$ and is differentiable. Then,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right| \text{ if } y = g(x) \text{ for some } x \\ &= 0 \text{ otherwise} \end{aligned}$$

Change of Coordinates

Suppose X is a random variable with pdf $f_X(x)$. Suppose $Y = X^n$. Find $f_Y(y)$.
Then $g^{-1}(x) = x^{1/n}$.

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(Y)}{\partial y} \right| \\
&= f_X(y^{1/n}) \frac{y^{\frac{1}{n}-1}}{n}
\end{aligned}
$$

Change of Coordinates: Bivariate case

Suppose X_1 and X_2 has a joint density $f(x_1, x_2)$ and support S_X.
Let $Y_1 = u_1(X_1, X_2)$ and $Y_1 = u_1(X_1, X_2)$ with the single-valued inverse $X_1 = v_1(Y_1, Y_2)$ and $X_2 = v_2(Y_1, Y_2)$.

Joint distribution

The joint pdf of Y_1 and $Y2$ is:

$$g(y_1, y_2) = |J| f\left[v_1(y_1, y_2), v_2(y_1, y_2)\right]$$

where $|J|$ is the determinant of the Jacobian Matrix:

$$\begin{pmatrix} \frac{\partial v_1(y_1,y_2)}{\partial y_1} & \frac{\partial v_1(y_1,y_2)}{\partial y_2} \\ \frac{\partial v_2(y_1,y_2)}{\partial y_1} & \frac{\partial v_2(y_1,y_2)}{\partial y_2} \end{pmatrix}$$

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Note that S_Y, the support of (Y_1, Y_2), is usually found by considering the image of S_X under the transformation $Y_1, Y2$. Meaning that $\forall (x_1, x_2) \in S_X$ we find $(y_1, y_2) \in S_Y$

$$x_1 = v_1(y_1, y_2), \qquad x_2 = v_2(y_1, y_2)$$

Markov's Inequality

Proposition

Suppose X is a random variable that takes on non-negative values. Then, for all $a > 0$,

$$P(X \geq a) \quad \leq \quad \frac{E[X]}{a}$$

Chebyshev's Inequality

Proposition

If X is a random variable with mean μ and variance σ^2, then, for any value $k > 0$,

$$P(|X - \mu| \geq k) \quad \leq \quad \frac{\sigma^2}{k^2}$$

Chebyshev's Inequality

Proof.

Define the random variable

$$Y = (X - \mu)^2$$

Where $\mu = E[X]$.

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$$P((X - \mu)^2 \geq k^2) \leq \frac{E[(X - \mu)^2]}{k^2}$$

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Sequence of Random Variables

Sequence of Independent and Identically, Distributed Random variables.

- Sequence: $X_1, X_2, \ldots, X_n, \ldots$
- Think of a sequence as sampled data:
  - Suppose we are drawing a sample of N observations
  - Each observation will be a random variable, say X_i
  - With realization x_i

Weak Law of Large Numbers

Proposition

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from a distribution with mean μ and $Var(X_i) = \sigma^2$. Then, for all $\epsilon > 0$,

$$P\left\{\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq \epsilon\right\} \to 0 \text{ as } n \to \infty$$

Weak Law of Large Numbers

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$$\frac{\mathrm{E}[X_1 + X_2 + \cdots + X_n]}{n} = \frac{\sum_{i=1}^{n} E[X_i]}{n} = \mu$$

Further,

$$\mathrm{E}[(\frac{\sum_{i=1}^{n} X_i - \mu}{n})^2] = \frac{\mathrm{Var}(X_1 + X_2 + \cdots + X_n)}{n^2} = \frac{\sum_{i=1}^{n} \mathrm{Var}(X_i)}{n^2} = \frac{\sigma^2}{n}$$

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Apply Chebyshev's Inequality:

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Apply Chebyshev's Inequality:

$$P\left\{\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq \epsilon\right\} \leq \frac{\sigma^2}{n\epsilon^2}$$

$\square$

Weak Law of Large Numbers: EXAMPLE

Suppose $X_1, X_2, \ldots$ are iid normal distributions,

$$X_i \quad \sim \quad \text{Normal}(0, 10)$$

$$P\left\{\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq 0.1\right\} \text{ as } n \to \infty$$

Suppose we want to guarantee that we have at most a 0.01 probability of being more than 0.1 away from the true μ. How big do we need n?

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$$\begin{aligned} 0.01 &= \frac{10}{n(0.1^2)} \\ n &= \frac{1000}{0.01} \\ n &= 100,000 \end{aligned}$$

WHITE BOARD EXERCISES

Today

- Univariate and Multivariate Change of Variables Exercises
- Questions