WORKSHOP 5: Questions

Q1. If a random variable X has a beta distribution[1] with parameters α and β, derive the probability density function of $U = X/(1-X)$.

Q2. Suppose that X and Y are independent random variables with $X \sim \text{gamma}(n, \lambda)$ and $Y \sim \text{exponential}(\lambda)$. Use the transformation $U = X + Y$ and $V = Y$ to obtain the probability density function of U. What distribution does U have?

Q3. Suppose that X and Y are independent exponential($\lambda = 1$) random variables. Use the transformation $U = X - Y$ and $V = Y$ to obtain the probability density function of U.

Q4 - Will be Marked. Suppose a random variable X has a uniform distribution on the interval (a, b) so that the probability density function is $f_X(x) = 1/(b-a)$ for $a < x < b$ and is zero elsewhere. Obtain the moment generating function of X.

Q5 - Will be Marked. Suppose that a random variable X has a double exponential distribution[2] with parameter θ, $\theta > 0$, with probability density function

$$f_X(x) = \tfrac{1}{2}\theta e^{-\theta|x|}, \quad -\infty < x < +\infty.$$

Show that the moment generating function of X is

$$m_X(t) = \frac{\theta^2}{\theta^2 - t^2}.$$

For what values of t is this moment generating function defined?
Hint: Split the range of integration $x \in (-\infty, +\infty)$ into the two regions $x \in (-\infty, 0)$ and $x \in (0, +\infty)$. The value of t must be such that *both* integrals converge.

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[1] The English mathematician Karl Pearson (1857-1936) introduced the Pearson system of distributions as satisfying

$$f(x) \propto \exp\left(\int \frac{x-c}{a_2x^2 + a_1x + a_0}\,\mathrm{d}x\right).$$

Depending on the values of the constants a_0, a_1, a_2 and c, different families of distributions arise. The beta distribution is essentially a re-scaled form of the Pearson Type I distribution. The Pearson type III distribution became known as the gamma distribution in the first half of the 20th century. The Pearson types can be characterised by their skewness β_1 and their kurtosis β_2.

[2] Often called a Laplace distribution after the French mathematician Pierre-Simon Laplace (1749-1825). Laplace worked extensively in celestial mechanics and probability theory. The distribution was often used to model measurement error before being replaced with the normal distribution.