Statistical Methods MATH2715 info

Teaching material is all online!

- On Minerva `http://minerva.leeds.ac.uk`
- On GitHub `https://github.com/luisacutillo78/Statistical-Methods-Lecture-Notes`

R code submission

- No technichal issue - please submit your SURNAMEstudentid.R [or .Rmd as required] file in the assignment folder.

Resources

- Mathematical Statistics and Data Analysis - 3rd ed. (by J. A. Rice);
- Introduction to Statistics - Online Edition -D.M.Lane et al.
- http://www1.maths.leeds.ac.uk/statistics/R/Rintro.pdf;
- https://www.datacamp.com/courses/free-introduction-to-r.

Where We've Been, Where We're Going

In the previous Lecture

- Moment Generating Functions

Today

- More about the MGFs
- Multivariate Normal Distribution
- Exercises & Questions

Moment Generating Functions

Definition

Suppose X is a random variable with pdf f. Define,

$$E[X^n] = \int_{-\infty}^{\infty} x^n f(x) dx$$

We will call X^n the n^{th} moment of X

- By this definition $var(X) = \text{Second Moment} - \text{First Moment}^2$
- We are assuming that the integral converges

Moment Generating Functions

Proposition

Suppose X is a random variable with pdf $f(x)$. Call $M(t) = E[e^{tX}]$,

$$\begin{aligned} M(t) &= E[e^{tX}] \\ &= \int_{-\infty}^{\infty} e^{tx} f(x) dx \end{aligned}$$

We will call $M(t)$ the moment generating function, because:

$$\frac{\partial^n M(t)}{\partial^n t}\Big|_0 = E[X^n]$$

(Assuming that we can interchange derivative and integral)

Properties of moment-generating functions

- Moment-generating functions can be used to generate moments. To get $E(Y^k)$, differentiate $M_Y(t)$ with respect to t. Differentiate k times and set $t = 0$.

- Moment-generating functions correspond uniquely to probability distributions.

The function $M(t)$ is like a fingerprint of the probability distribution.

$Y \sim N(\mu, \sigma^2)$ if and only if $M_Y(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$

$Y \sim \chi^2(\nu)$ if and only if $M_Y(t) = (1 - 2t)^{-\nu/2}$ for $t < \frac{1}{2}$

Example: Using moment-generating functions to prove distribution facts

At the whiteboard

Let $X \sim N(\mu, \sigma^2)$. Show $Y = \frac{X-\mu}{\sigma} \sim N(0,1)$

Facts about moment-generating functions

Use these to find distributions of *functions* of random variables

- $M_{aY}(t) = M_Y(at)$
- $M_{Y+a}(t) = e^{at} M_Y(t)$
- If $Y_1, \ldots, Y_n$ are independent, $M_{\sum_{i=1}^n Y_i}(t) = \prod_{i=1}^n M_{Y_i}(t)$

Less well known

But very useful later

If $W = W_1 + W_2$ with W_1 and W_2 independent, $W \sim \chi^2(\nu_1 + \nu_2)$ and $W_2 \sim \chi^2(\nu_2)$ then $W_1 \sim \chi^2(\nu_1)$.

Multivariate Normal Distribution

Definition

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$ is a vector of random variables. If $\boldsymbol{X}$ has pdf

$$f(\boldsymbol{x}) \quad = \quad (2\pi)^{-N/2} \det(\boldsymbol{\Sigma})^{-1/2} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\mu})\right)$$

Then we will say $\boldsymbol{X}$ is a Multivariate Normal Distribution,

$$\boldsymbol{X} \quad \sim \quad \text{Multivariate Normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- Regularly used for likelihood, Bayesian, and other parametric inferences

Properties of the Multivariate Normal Distribution

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$
$X \sim$ Multivariate Normal$(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

$$\begin{aligned} E[\boldsymbol{X}] &= \boldsymbol{\mu} \\ \text{cov}(\boldsymbol{X}) &= \boldsymbol{\Sigma} \end{aligned}$$

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So that,

$$\boldsymbol{\Sigma} = \begin{pmatrix} \text{var}(X_1) & \text{cov}(X_1, X_2) & \ldots & \text{cov}(X_1, X_N) \\ \text{cov}(X_2, X_1) & \text{var}(X_2) & \ldots & \text{cov}(X_2, X_N) \\ \vdots & \vdots & \ddots & \vdots \\ \text{cov}(X_N, X_1) & \text{cov}(X_N, X_2) & \ldots & \text{var}(X_N) \end{pmatrix}$$

Multivariate Normal Distribution

Consider the (bivariate) special case where $\boldsymbol{\mu} = (0, 0)$ and

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$$f(x_1, x_2) = (2\pi)^{-2/2} 1^{-1/2} \exp\left(-\frac{1}{2}\left((\boldsymbol{x} - \mathbf{0})' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\boldsymbol{x} - \mathbf{0})\right)\right)$$

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$\rightsquigarrow$ product of univariate standard normally distributed random variables

Standard Multivariate Normal

definition

Suppose $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_N)$ is

$$\boldsymbol{Z} \quad \sim \quad \text{Multivariate Normal}(\mathbf{0}, \boldsymbol{I}_N).$$

Then we will call $\boldsymbol{Z}$ the standard multivariate normal.

Independence and Multivariate Normal

We have shown that:

Proposition

Suppose X and Y are independent. Then

$$\text{cov}(X, Y) = 0$$

- More generally if X and Y are independent, $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ for functions $g : \Re \to \Re$ and $h : \Re \to \Re$.

Zero covariance does not generally imply Independency

Except for the Normal case!

Proposition

Suppose $\boldsymbol{X} \sim \text{Multivariate Normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. where $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$.
If $\text{cov}(X_i, X_j) = 0$, then X_i and X_j are independent

Change of Coordinates

Proposition

Suppose X is a random variable and $Y = g(X)$, where $g : \Re \to \Re$ that is a *monotonic* function.
Define $g^{-1} : \Re \to \Re$ such that $g^{-1}(g(X)) = X$ and is differentiable. Then,

$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(y)}{\partial y} \right| \text{ if } y = g(x) \text{ for some } x \\ &= 0 \text{ otherwise} \end{aligned}$$

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$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{\partial g^{-1}(Y)}{\partial y} \right|$$

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$$\begin{aligned} f_Y(y) &= f_X(g^{-1}(y))\left|\frac{\partial g^{-1}(Y)}{\partial y}\right| \\ &= f_X(y^{1/n})\frac{y^{\frac{1}{n}-1}}{n} \end{aligned}$$

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$$
\begin{aligned}
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&= f_X(y^{1/n}) \frac{y^{\frac{1}{n}-1}}{n}
\end{aligned}
$$

We've used this to derive many of the pdfs

At the Whiteboard/ Home/ Next Lecture

- We are going to compute the MGF of $X \sim Expo(\lambda)$, the first two moments, expectation and variance.
- Solve the Problems 5,7,47 Chapter 4 of Mathem. Statistics and Data Analysis, 3rd edition, J.A. Rice.
- Solve the Problems 79, 81 Chapter 4 of Mathem. Statistics and Data Analysis, 3rd edition, J.A. Rice.

Workshop exercises: Sums of Random Variables

Exercise 1-Will be marked

Let X and Y be independent r.v. having Gamma distribution with parameters (n, λ) and $(1, \lambda)$. Given $Z = X + Y$. Compute the pdf of Z.

Exercise 2

Let X and Y be independent $N(0, 1)$ r.v. and $Z = X + Y$. Compute the pdf of Z.

Exercise 3

Let X and Y be independent *Poisson* r.v. with parameter, respectively, λ and μ. Compute the pmf of $Z = X + Y$.

Exercise 4 - Will be marked

Let $X \sim N(\mu, \sigma)$. Write a R Notebook containing:

- a function that, given the two parameters μ and σ, returns $P(a < X \leq b)$, $\forall a \leq b$. (make use of the base R function pnorm()).
- the output corresponding to $a = -2, b = 3, \mu = 1, \sigma = 2$
- a plot of the pdf and cdf of the same $N(\mu, \sigma)$, with the relative code
- a QQ plot showing the theoretical quantiles versus the empirical quantiles of the same $N(\mu, \sigma)$
- The notebook must be well documented.