Statistical Methods MATH2715 info

Teaching material is all online!

- On Minerva `http://minerva.leeds.ac.uk`
- On GitHub `https://github.com/luisacutillo78/Statistical-Methods-Lecture-Notes`

R code submission

- No technichal issue - please submit your SURNAMEstudentid.R file in the assignment folder.

Resources

- Mathematical Statistics and Data Analysis - 3rd ed. (by J. A. Rice);
- Introduction to Statistics - Online Edition -D.M.Lane et al.
- http://www1.maths.leeds.ac.uk/statistics/R/Rintro.pdf;
- https://www.datacamp.com/courses/free-introduction-to-r.

Expected Values

The expectation of a random variable is connected to the concept of weighted average.

Discrete Case

$$E(X) = \sum_i x_i p(x_i)$$

Limitation: If it's an infinite sum and the x_i are both positive and negative, the sum can fail to converge! $=>$ We restrict to cases where the sum converges absolutely:

$$\sum_i |x_i| p(x_i) < \infty$$

Otherwise, we say that the expectation is undefined.

Expected Values

Continuous Case

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Corresponding limitation: If

$$\int_{-\infty}^{\infty} |x| f(x)\, dx = \infty$$

we say that the expectation is undefined.

Table of Common Distributions

taken from *Statistical Inference* by Casella and Berger

Discrete Distrbutions

distribution	pmf	mean	variance	mgf/moment
Bernoulli(p)	$p^x(1-p)^{1-x};\ x=0,1;\ p\in(0,1)$	p	$p(1-p)$	$(1-p)+pe^t$
Beta-binomial(n,α,β)	$\binom{n}{x}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(\alpha+\beta+n)}$	$\frac{n\alpha}{\alpha+\beta}$	$\frac{n\alpha\beta}{(\alpha+\beta)^2}$	
Notes: If $X\|P$ is binomial (n,P) and P is beta(α,β), then X is beta-binomial(n,α,β).				
Binomial(n,p)	$\binom{n}{x}p^x(1-p)^{n-x};\ x=1,\dots,n$	np	$np(1-p)$	$[(1-p)+pe^t]^n$
Discrete Uniform(N)	$\frac{1}{N};\ x=1,\dots,N$	$\frac{N+1}{2}$	$\frac{(N+1)(N-1)}{12}$	$\frac{1}{N}\sum_{i=1}^N e^{it}$
Geometric(p)	$p(1-p)^{x-1};\ p\in(0,1)$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Note: $Y=X-1$ is negative binomial$(1,p)$. The distribution is *memoryless*: $P(X>s\|X>t)=P(X>s-t)$.				
Hypergeometric(N,M,K)	$\frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}};\ x=1,\dots,K$	$\frac{KM}{N}$	$\frac{KM}{N}\frac{(N-M)(N-k)}{N(N-1)}$	?
	$M-(N-K)\le x\le M;\ N,M,K>0$			
Negative Binomial(r,p)	$\binom{r+x-1}{x}p^r(1-p)^x;\ p\in(0,1)$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-(1-p)e^t}\right)^r$
	$\binom{y-1}{r-1}p^r(1-p)^{y-r};\ Y=X+r$			
Poisson(λ)	$\frac{e^{-\lambda}\lambda^x}{x!};\ \lambda\ge0$	λ	λ	$e^{\lambda(e^t-1)}$
Notes: If Y is gamma(α,β), X is Poisson$(\frac{x}{\beta})$, and α is an integer, then $P(X\ge\alpha)=P(Y\le y)$.				

Continuous Distributions

distribution	pdf	mean	variance	mgf/moment
Beta(α,β)	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1};\ x\in(0,1),\ \alpha,\beta>0$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$1+\sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!}$
Cauchy(θ,σ)	$\frac{1}{\pi\sigma}\frac{1}{1+(\frac{x-\theta}{\sigma})^2};\ \sigma>0$	does not exist	does not exist	does not exist
Notes: Special case of Students's t with 1 degree of freedom. Also, if X,Y are iid $N(0,1)$, $\frac{X}{Y}$ is Cauchy				
χ^2_p	$\frac{1}{\Gamma(\frac{p}{2})2^{\frac{p}{2}}}x^{\frac{p}{2}-1}e^{-\frac{x}{2}};\ x>0,\ p\in N$	p	$2p$	$\left(\frac{1}{1-2t}\right)^{\frac{p}{2}},\ t<\frac{1}{2}$
Notes: Gamma$(\frac{p}{2},2)$.				
Double Exponential(μ,σ)	$\frac{1}{2\sigma}e^{-\frac{\lvert x-\mu\rvert}{\sigma}};\ \sigma>0$	μ	$2\sigma^2$	$\frac{e^{\mu t}}{1-(\sigma t)^2}$
Exponential(θ)	$\frac{1}{\theta}e^{-\frac{x}{\theta}};\ x\geq 0,\ \theta>0$	θ	θ^2	$\frac{1}{1-\theta t},\ t<\frac{1}{\theta}$
Notes: Gamma$(1,\theta)$. Memoryless. $Y=X^{\frac{1}{\gamma}}$ is *Weibull*. $Y=\sqrt{\frac{2X}{\beta}}$ is *Rayleigh*. $Y=\alpha-\gamma\log\frac{X}{\beta}$ is *Gumbel*.				
F_{ν_1,ν_2}	$\frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}}\frac{x^{\frac{\nu_1-2}{2}}}{\left(1+(\frac{\nu_1}{\nu_2})x\right)^{\frac{\nu_1+\nu_2}{2}}};\ x>0$	$\frac{\nu_2}{\nu_2-2},\ \nu_2>2$	$2(\frac{\nu_2}{\nu_2-2})^2\frac{\nu_1+\nu_2-2}{\nu_1(\nu_2-4)},\ \nu_2>4$	$EX^n=\frac{\Gamma(\frac{\nu_1+2n}{2})\Gamma(\frac{\nu_2-2n}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\left(\frac{\nu_2}{\nu_1}\right)^n,\ n<\frac{\nu_2}{2}$
Notes: $F_{\nu_1,\nu_2}=\frac{\chi^2_{\nu_1}/\nu_1}{\chi^2_{\nu_2}/\nu_2}$, where the χ^2s are independent. $F_{1,\nu}=t^2_\nu$.				
Gamma(α,β)	$\frac{1}{\Gamma(\alpha)\beta^\alpha}x^{\alpha-1}e^{-\frac{x}{\beta}};\ x>0,\ \alpha,\beta>0$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta t}\right)^{\alpha},\ t<\frac{1}{\beta}$
Notes: Some special cases are exponential ($\alpha=1$) and χ^2 ($\alpha=\frac{p}{2},\beta=2$). If $\alpha=\frac{3}{2}$, $Y=\sqrt{\frac{X}{\beta}}$ is *Maxwell*. $Y=\frac{1}{X}$ is *inverted gamma*.				
Logistic(μ,β)	$\frac{1}{\beta}\frac{e^{-\frac{x-\mu}{\beta}}}{\left[1+e^{-\frac{x-\mu}{\beta}}\right]^2};\ \beta>0$	μ	$\frac{\pi^2\beta^2}{3}$	$e^{\mu t}\Gamma(1+\beta t),\ \lvert t\rvert<\frac{1}{\beta}$
Notes: The cdf is $F(x\mid\mu,\beta)=\frac{1}{1+e^{-\frac{x-\mu}{\beta}}}$.				
Lognormal(μ,σ^2)	$\frac{1}{\sqrt{2\pi}\sigma}\frac{1}{x}e^{-\frac{(\log x-\mu)^2}{2\sigma^2}};\ x>0,\sigma>0$	$e^{\mu+\frac{\sigma^2}{2}}$	$e^{2(\mu+\sigma^2)}-e^{2\mu+\sigma^2}$	$EX^n=e^{n\mu+\frac{n^2\sigma^2}{2}}$
Normal(μ,σ^2)	$\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}};\ \sigma>0$	μ	σ^2	$e^{\mu t+\frac{\sigma^2t^2}{2}}$
Pareto(α,β)	$\frac{\beta\alpha^\beta}{x^{\beta+1}};\ x>\alpha,\ \alpha,\beta>0$	$\frac{\beta\alpha}{\beta-1},\ \beta>1$	$\frac{\beta\alpha^2}{(\beta-1)^2(\beta-2)},\ \beta>2$	does not exist
t_ν	$\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})}\frac{1}{\sqrt{\nu\pi}}\frac{1}{(1+\frac{x^2}{\nu})^{\frac{\nu+1}{2}}}$	$0,\ \nu>1$	$\frac{\nu}{\nu-2},\nu>2$	$EX^n=\frac{\Gamma(\frac{\nu+1}{2})\Gamma(\nu-\frac{n}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})}\nu^{\frac{n}{2}},\ n$ even
Notes: $t^2_\nu=F_{1,\nu}$.				
Uniform(a,b)	$\frac{1}{b-a},\ a\leq x\leq b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Notes: If $a=0,\ b=1$, this is special case of beta ($\alpha=\beta=1$).				
Weibull(γ,β)	$\frac{\gamma}{\beta}x^{\gamma-1}e^{-\frac{x^\gamma}{\beta}};\ x>0,\ \gamma,\beta>0$	$\beta^{\frac{1}{\gamma}}\Gamma(1+\frac{1}{\gamma})$	$\beta^{\frac{2}{\gamma}}\left[\Gamma(1+\frac{2}{\gamma})-\Gamma^2(1+\frac{1}{\gamma})\right]$	$EX^n=\beta^{\frac{n}{\gamma}}\Gamma(1+\frac{n}{\gamma})$
Notes: The mgf only exists for $\gamma\geq 1$.				

Example: Gamma and Exponential Expectation

Gamma

$$g(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \ , \ t \geq 0$$

where $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$ Note $g(x) = \lambda e^{-\lambda x}$ if $\alpha = 1$ exponential.

Gamma Expectation $E(X) = \frac{\alpha}{\lambda}$

$$E(X) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x} dx$$

Note: $\int_0^\infty \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\lambda x} dx = 1$ hence $\int_0^\infty x^\alpha e^{-\lambda x} = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}$ It follows that:

$$E(X) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} = \frac{\alpha}{\lambda}$$

(we used $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$) Note. $E[X] = 1/\lambda$ for the exponential.

R Notebook

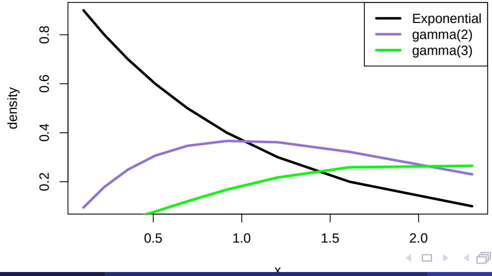
This is an R Markdown Notebook. When you execute code within the notebook, the results appear beneath the code.

Try executing this chunk by clicking the *Run* button within the chunk or by placing your cursor inside it and pressing *Cmd+Shift+Enter*.

```r
a=1
####generating quantiles for gamma
p <- (1:9)/10
x=qgamma(p, shape = a)

f1<-dgamma(x, shape = a)
f2=dgamma(x, shape = a+1)
f3=dgamma(x, shape = a+2)
#pdf('gamma_density.pdf')
plot(x, f1, xlab = "x", ylab = "density",lwd = 3,type = "l")
lines(x, f2, xlab ="x",ylab = "density",lwd = 3,col = 'mediumpurple')
lines(x, f3, xlab ="x",ylab = "density",lwd = 3,col = 'green')

legend('topright', legend = c('Exponential',
      'gamma(2)','gamma(3)'), col = c('black',  'mediumpurple','green'),
      lwd = 3)
```



Example: Normal Distribution $X \sim N(\mu, \sigma^2)$.

Given the pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

Can we compute the Expectation?

$$E(x) = \int_{-\infty}^{\infty} xf(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} xe^{-(x-\mu)^2/2\sigma^2}.$$

Whiteboard Solution.

Expectations of Functions of Random Variables

Theorem A

Let $g(x)$ be a fixed function.

- Discrete case

$$E(g(X)) = \sum_{x_i} g(x_i)p(x_i)$$

  with $\sum_{x_i} |g(x_i)|p(x_i) < \infty$

- Continuous case

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

  with $\int_{-\infty}^{\infty} |g(x)|f(x)dx < \infty$

We will prove this result for the discrete case (Whiteboard Solution)

Expectations of Functions of Random Variables

Theorem B

Suppose $X_1, \ldots, X_n$ are jointly distributed r.v. Let $Y = g(X_1, \ldots, X_n)$.

- Discrete case

$$E(Y) = \sum_{x_1, \ldots, x_n} g(x_1, \ldots, x_n) p(x_1, \ldots, x_n)$$

with $\sum_{x_1, \ldots, x_n} |g(x_1, \ldots, x_n)| p(x_1, \ldots, x_n) < \infty$

- Continuous case

$$E(Y) = \int \ldots \int g(x_1, \ldots, x_n) f(x_1, \ldots, x_n) dx_1, \ldots, x_n$$

provided that the integral with $|g|$ in place of g converges.

The proof is similar to that of Theorem A.

Expectations of Functions of Random Variables

Theorem C

Suppose $X_1, \ldots, X_n$ are jointly distributed r.v. with expectations $E(X_i)$ and $Y = a + \sum_{i=1}^{n} b_i X_i$, then

$$E(Y) = a + \sum_{i=1}^{n} b_i E(X_i)$$

Variance and Standard Deviation

Definition

The variance of a random variable X is defined as:

$$Var(X) = E[X - E(X)]^2$$

The standard deviation , denoted by σ, is given by the square root of the variance.

Theorem

If $Y = a + bX$ then $Var(Y) = b^2 Var(X)$
Proof. Since $E(Y) = a + bE(X)$,

$$\begin{aligned} E[(Y - E(Y))^2] &= E[a + bX - a - bE(X)]^2 \\ &= b^2 E[[X - E(X)]^2 \\ &= b^2 Var(X) \end{aligned}$$

Alternative way of calculating the variance

Theorem

The variance of X, if it exists, might also be computed as:

$$Var(X) = E(X^2) - [E(X)]^2$$

Proof

Let $E(X) = \mu$, then

$$\begin{aligned} Var(X) &= E[(X-\mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E(X^2) - 2\mu E(X) + \mu^2] \\ &= E(X^2) - \mu^2 \end{aligned}$$

Covariance

Definition

For jointly continous random variables X and Y define, the covariance of X and Y as,

Covariance

Definition

For jointly continous random variables X and Y define, the covariance of X and Y as,

$$\text{cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

Covariance

Definition

For jointly continous random variables X and Y define, the covariance of X and Y as,

$$\begin{aligned} \mathrm{cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E\left[XY - E[X]Y - E[Y]X + E[X]E[Y]\right] \end{aligned}$$

Covariance

Definition

For jointly continous random variables X and Y define, the covariance of X and Y as,

$$
\begin{aligned}
\mathrm{cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\
&= E\left[XY - E[X]Y - E[Y]X + E[X]E[Y]\right] \\
&= E[XY] - 2E[X]E[Y] + E[E[X]E[Y]]
\end{aligned}
$$

Covariance

Definition

For jointly continous random variables X and Y define, the covariance of X and Y as,

$$\begin{aligned}
\mathrm{cov}(X,Y) &= E[(X-E[X])(Y-E[Y])] \\
&= E\left[XY - E[X]Y - E[Y]X + E[X]E[Y]\right] \\
&= E[XY] - 2E[X]E[Y] + E[E[X]E[Y]] \\
&= E[XY] - E[X]E[Y]
\end{aligned}$$

Observation

Variance is the covariance of a random variable with itself!

Covariance

Definition

For jointly continous random variables X and Y define, the covariance of X and Y as,

$$\begin{aligned} \text{cov}(X, Y) &= E[(X - E[X])(Y - E[Y])] \\ &= E\left[XY - E[X]Y - E[Y]X + E[X]E[Y]\right] \\ &= E[XY] - 2E[X]E[Y] + E[E[X]E[Y]] \\ &= E[XY] - E[X]E[Y] \end{aligned}$$

Observation

Variance is the covariance of a random variable with itself!

$$\begin{aligned} cov(X, X) &= E[XX] - E[X]E[X] \\ &= E[X^2] - E[X]^2 \end{aligned}$$

Correlation Coefficient

Definition

Define the correlation coefficient of X and Y as,

$$\rho = \text{cor}(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \tag{1}$$

Correlation measures the linear relationship between two random variables!
It is possible to show that

$$|\rho| <= 1$$

Observations about the Correlation

Correlation is between -1 and 1

Suppose $X = Y$

Observations about the Correlation

Correlation is between -1 and 1

Suppose $X = Y$

$$cor(X, Y) = \frac{cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

Observations about the Correlation

Correlation is between -1 and 1

Suppose $X = Y$

$$
\begin{aligned}
cor(X, Y) &= \frac{cov(X, Y)}{\sqrt{Var(X)Var(Y)}} \\
&= \frac{Var(X)}{Var(X)}
\end{aligned}
$$

Observations about the Correlation

Correlation is Between -1 and 1

Suppose $X = -Y$

Observations about the Correlation

Correlation is Between -1 and 1

Suppose $X = -Y$

$$cor(X, Y) = \frac{cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

Observations about the Correlation

Correlation is Between -1 and 1

Suppose $X = -Y$

$$
\begin{aligned}
cor(X, Y) &= \frac{cov(X, Y)}{\sqrt{Var(X)Var(Y)}} \\
&= \frac{-Var(X)}{Var(X)}
\end{aligned}
$$

Example: $X + Y$

Suppose X and Y have joint pdf $f(x, y) = x + y$ for $x, y \in [0, 1]$.

Example: $X + Y$

Suppose X and Y have joint pdf $f(x, y) = x + y$ for $x, y \in [0, 1]$.
$\text{Cov}(X, Y)$

Example: $X + Y$

Suppose X and Y have joint pdf $f(x, y) = x + y$ for $x, y \in [0, 1]$.
$\text{Cov}(X, Y)$

$$E[XY] = \int_0^1 \int_0^1 xy(x + y)dxdy$$

Example: $X + Y$

Suppose X and Y have joint pdf $f(x, y) = x + y$ for $x, y \in [0, 1]$.
$\mathrm{Cov}(X, Y)$

$$
\begin{aligned}
E[XY] &= \int_0^1 \int_0^1 xy(x + y)dxdy \\
&= \int_0^1 \int_0^1 (x^2y + y^2x)dxdy
\end{aligned}
$$

Example: $X+Y$

Suppose X and Y have joint pdf $f(x,y) = x + y$ for $x, y \in [0,1]$.
Cov(X,Y)

$$
\begin{aligned}
E[XY] &= \int_0^1 \int_0^1 xy(x+y)dxdy \\
&= \int_0^1 \int_0^1 (x^2y + y^2x)dxdy \\
&= \int_0^1 (\frac{y}{3} + \frac{y^2}{2})dy
\end{aligned}
$$

Example: $X + Y$

Suppose X and Y have joint pdf $f(x, y) = x + y$ for $x, y \in [0, 1]$.
Cov(X, Y)

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^1 xy(x+y)dxdy \\ &= \int_0^1 \int_0^1 (x^2y + y^2x)dxdy \\ &= \int_0^1 (\frac{y}{3} + \frac{y^2}{2})dy \\ &= \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \end{aligned}$$

Example: $X + Y$

Suppose X and Y have joint pdf $f(x, y) = x + y$ for $x, y \in [0, 1]$.
Cov(X, Y)

$$
\begin{aligned}
E[XY] &= \int_0^1 \int_0^1 xy(x+y)dxdy \\
&= \int_0^1 \int_0^1 (x^2y + y^2x)dxdy \\
&= \int_0^1 (\frac{y}{3} + \frac{y^2}{2})dy \\
&= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
\end{aligned}
$$

$$
E[X] = \int_0^1 \int_0^1 x(x+y)dxdy
$$

Example: $X + Y$

Suppose X and Y have joint pdf $f(x,y) = x + y$ for $x, y \in [0,1]$.
Cov(X, Y)

$$
\begin{aligned}
E[XY] &= \int_0^1 \int_0^1 xy(x+y)dxdy \\
&= \int_0^1 \int_0^1 (x^2y + y^2x)dxdy \\
&= \int_0^1 (\frac{y}{3} + \frac{y^2}{2})dy \\
&= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
E[X] &= \int_0^1 \int_0^1 x(x+y)dxdy \\
&= \frac{7}{12}
\end{aligned}
$$

Example: $X + Y$

Suppose X and Y have joint pdf $f(x, y) = x + y$ for $x, y \in [0, 1]$.
$\text{Cov}(X, Y)$

$$
\begin{aligned}
E[XY] &= \int_0^1 \int_0^1 xy(x+y)dxdy \\
&= \int_0^1 \int_0^1 (x^2y + y^2x)dxdy \\
&= \int_0^1 (\frac{y}{3} + \frac{y^2}{2})dy \\
&= \frac{1}{6} + \frac{1}{6} = \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
E[Y] &= \int_0^1 \int_0^1 y(x+y)dxdy \\
&=
\end{aligned}
$$

Example: $X + Y$

Suppose X and Y have joint pdf $f(x, y) = x + y$ for $x, y \in [0, 1]$.
$\text{Cov}(X, Y)$

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^1 xy(x+y)dxdy \\ &= \int_0^1 \int_0^1 (x^2y + y^2x)dxdy \\ &= \int_0^1 (\frac{y}{3} + \frac{y^2}{2})dy \\ &= \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \end{aligned}$$

$$\begin{aligned} E[Y] &= \int_0^1 \int_0^1 y(x+y)dxdy \\ &= \frac{7}{12} \end{aligned}$$

Example: $X + Y$

$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

Example: $X + Y$

$$\begin{aligned}
\mathrm{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\
&= \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}
\end{aligned}$$

Example: $X + Y$

$$
\begin{aligned}
\text{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\
&= \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}
\end{aligned}
$$

$$
\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}
$$

Example: $X + Y$

$$
\begin{aligned}
\mathrm{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\
&= \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{Cor}(X, Y) &= \frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \\
&= \frac{-\frac{1}{144}}{\frac{11}{144}}
\end{aligned}
$$

Example: $X + Y$

$$\begin{aligned}\mathrm{Cov}(X,Y) &= E[XY] - E[X]E[Y] \\ &= \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}\end{aligned}$$

$$\begin{aligned}\mathrm{Cor}(X,Y) &= \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \\ &= \frac{-\frac{1}{144}}{\frac{11}{144}} \\ &= \frac{-1}{11}\end{aligned}$$

Sums of Random Variable

Variance of the sum

Suppose X_i is a sequence of random variables with joint pdf,

$$f(\boldsymbol{x}) = f(x_1, x_2, \ldots, x_n)$$

We have:

$$\mathrm{var}(\sum_{i=1}^{N} X_i) = \sum_{i=1}^{N} \mathrm{var}(X_i) + 2\sum_{i<j} \mathrm{cov}(X_i, X_j)$$

Sums of Random Variable

Proof.

Consider two random variables, X_1 and X_2. Then,

Proof.

Consider two random variables, X_1 and X_2. Then,

$$\mathrm{var}(X_1 + X_2) = E[(X_1 + X_2)^2] - (E[X_1] + E[X_2])^2$$

Sums of Random Variable

Proof.

Consider two random variables, X_1 and X_2. Then,

$$\begin{aligned} \text{var}(X_1 + X_2) &= E[(X_1 + X_2)^2] - (E[X_1] + E[X_2])^2 \\ &= E[X_1^2] + 2E[X_1 X_2] + E[X_2^2] \\ &\quad -(E[X_1])^2 - 2E[X_1]E[X_2] - 2E[X_2]^2 \end{aligned}$$

Sums of Random Variable

Proof.

Consider two random variables, X_1 and X_2. Then,

$$
\begin{aligned}
\text{var}(X_1 + X_2) &= E[(X_1 + X_2)^2] - (E[X_1] + E[X_2])^2 \\
&= E[X_1^2] + 2E[X_1X_2] + E[X_2^2] \\
&\quad -(E[X_1])^2 - 2E[X_1]E[X_2] - 2E[X_2]^2 \\
&= \underbrace{E[X_1^2] - (E[X_1])^2}_{\text{var}(X_1)} + \underbrace{E[X_2^2] - E[X_2]^2}_{\text{var}(X_2)} \\
&\quad +2\underbrace{(E[X_1X_2] - E[X_1]E[X_2])}_{\text{cov}(X_1,X_2)}
\end{aligned}
$$

Sums of Random Variable

Proof.

Consider two random variables, X_1 and X_2. Then,

$$\begin{aligned}
\text{var}(X_1 + X_2) &= E[(X_1 + X_2)^2] - (E[X_1] + E[X_2])^2 \\
&= E[X_1^2] + 2E[X_1 X_2] + E[X_2^2] \\
&\quad -(E[X_1])^2 - 2E[X_1]E[X_2] - 2E[X_2]^2 \\
&= \underbrace{E[X_1^2] - (E[X_1])^2}_{\text{var}(X_1)} + \underbrace{E[X_2^2] - E[X_2]^2}_{\text{var}(X_2)} \\
&\quad +2\underbrace{(E[X_1 X_2] - E[X_1]E[X_2])}_{\text{cov}(X_1, X_2)} \\
&= \text{var}(X_1) + \text{var}(X_2) + 2\text{cov}(X_1, X_2)
\end{aligned}$$

$\square$

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$ is a vector of random variables. If $\boldsymbol{X}$ has pdf

$$f(\boldsymbol{x}) = (2\pi)^{-N/2} \det(\boldsymbol{\Sigma})^{-1/2} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})' \boldsymbol{\Sigma} (\boldsymbol{x}-\boldsymbol{\mu})\right)$$

Then we will say $\boldsymbol{X}$ is a Multivariate Normal Distribution,

$$\boldsymbol{X} \sim \text{Multivariate Normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

- Regularly used for likelihood, Bayesian, and other parametric inferences

Independence and Covariance

Theorem

Suppose X and Y are independent rv. Then

$$\operatorname{cov}(X, Y) = 0$$

Proof.

Suppose X and Y are independent.

Proof.

Suppose X and Y are independent.

$$\text{cov}(X, Y) \quad = \quad E[XY] - E[X]E[Y]$$

Proof.

Suppose X and Y are independent.

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y]$$

Calculating $E[XY]$

Proof.

Suppose X and Y are independent.

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y]$$

Calculating $E[XY]$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)dxdy$$

Proof.

Suppose X and Y are independent.

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y]$$

Calculating $E[XY]$

$$\begin{aligned} E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy \end{aligned}$$

Proof.

Suppose X and Y are independent.

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y]$$

Calculating $E[XY]$

$$\begin{aligned} E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} xf_X(x)dx \int_{-\infty}^{\infty} yf_Y(y)dy \end{aligned}$$

Proof.

Suppose X and Y are independent.

$$\text{cov}(X, Y) = E[XY] - E[X]E[Y]$$

Calculating $E[XY]$

$$\begin{aligned}
E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x, y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy \\
&= \int_{-\infty}^{\infty} xf_X(x)dx \int_{-\infty}^{\infty} yf_Y(y)dy \\
&= E[X]E[Y]
\end{aligned}$$

Proof.

Suppose X and Y are independent.

$$\text{cov}(X,Y) = E[XY] - E[X]E[Y]$$

Calculating $E[XY]$

$$\begin{aligned} E[XY] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy \\ &= \int_{-\infty}^{\infty} xf_X(x)dx \int_{-\infty}^{\infty} yf_Y(y)dy \\ &= E[X]E[Y] \end{aligned}$$

Then $\text{cov}(X,Y) = 0$.

□

Observation on null correlation

Zero covariance does not generally imply Independence!

Workshop exercises: Sums of Random Variables

Exercise 1-Will be marked

Let X and Y be independent r.v. having Gamma distribution with parameters (n, λ) and $(1, \lambda)$. Given $Z = X + Y$. Compute the pdf of Z.

Exercise 2

Let X and Y be independent $N(0, 1)$ r.v. and $Z = X + Y$. Compute the pdf of Z.

Exercise 3

Let X and Y be independent *Poisson* r.v. with parameter, respectively, λ and μ. Compute the pmf of $Z = X + Y$.

Exercise 4 - Will be marked

Let $X \sim N(\mu, \sigma)$. Write a R Notebook containing:

- a function that, given the two parameters μ and σ, returns $P(a < X \leq b)$, $\forall a \leq b$. (make use of the base R function pnorm()).
- the output corresponding to $a = -2, b = 3, \mu = 1, \sigma = 2$
- a plot of the pdf and cdf of the same $N(\mu, \sigma)$, with the relative code
- a QQ plot showing the theoretical quantiles versus the empirical quantiles of the same $N(\mu, \sigma)$
- The notebook must be well documented.