Workshop 3

Solutions

Sum of Independent Poisson

Let X and Y be independent Poisson random variables with parameter λ and μ, respectively. Let $Z = X + Y$. Let's compute the probability mass function:

$$
\begin{aligned}
P(Z = n) &= P(X + Y = n) = \sum_{k=0}^{n} P(X = k,\ Y = n - k) \\
&= \sum_{k=0}^{n} P(X = k)\, P(Y = n - k) \\
&= \sum_{k=0}^{n} \frac{\lambda^k}{k!} e^{-\lambda} \frac{\mu^{n-k}}{(n-k)!} e^{-\mu} = e^{-(\lambda+\mu)} \sum_{k=0}^{n} \frac{\lambda^k}{k!} \frac{\mu^{n-k}}{(n-k)!} \\
&= e^{-(\lambda+\mu)} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} \lambda^k \mu^{n-k} = e^{-(\lambda+\mu)} \frac{(\lambda+\mu)^n}{n!}
\end{aligned}
$$

Conclusion: The sum is Poisson with parameter $\lambda + \mu$. The result can be extended to a sum of any number of *independent* Poisson random variables:

$$
X_k \sim \text{Poisson}(\lambda_k) \quad \Longrightarrow \quad \sum_k X_k \sim \text{Poisson}\left(\sum_k \lambda_k\right)
$$

Sum of Independent Standard Normals

Let X and Y be independent Normal(0,1) r.v.'s and $Z = X + Y$. Compute Z's cdf:

$$P(Z \leq z) \;=\; P(X+Y \leq z) \;=\; \int_{-\infty}^{\infty} f(x) P(Y \leq z - x) dx$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2/2} \int_{-\infty}^{z-x} e^{-y^2/2} dy dx$$

Differentiating, we compute the density function for Z:

$$\begin{aligned} f_Z(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2/2} e^{-(z-x)^2/2} dx \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-x^2+xz-z^2/2} dx \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-(x-z/2)^2+z^2/4-z^2/2} dx \;=\; \frac{1}{2\pi} e^{-z^2/4} \int_{-\infty}^{\infty} e^{-(x-z/2)^2} dx \\ &= \frac{1}{2\pi} e^{-z^2/4} \int_{-\infty}^{\infty} e^{-x^2} dx \;=\; \frac{1}{\sqrt{2\pi}\sqrt{2}} e^{-z^2/4} \end{aligned}$$

Conclusion: The sum is Normal with mean 0 and variance 2. The result can be extended to a sum of any number of *independent* Normal random variables:

$$X_k \sim \text{Normal}(\mu_k,\ \sigma_k^2) \quad \Longrightarrow \quad \sum_k X_k \sim \text{Normal}\left(\sum_k \mu_k,\ \sum_k \sigma_k^2\right)$$

THE FUNDAMENTAL THEOREM OF CALCULUS

Let:

- ***f*** be a function that is continuous on an open interval **I,**
- a is any point in the interval I.

Let *f* be a continuous real-valued function defined on a closed interval [*a*, *b*]. Let *F* be the function defined, for all *x* in [*a*, *b*], by

$$F(x) = \int_a^x f(t)\,dt.$$

Then, *F* is uniformly continuous on [*a*, *b*], differentiable on the open interval (*a*, *b*), and

$$F'(x) = f(x)$$

for all *x* in (*a*, *b*).

then the derivative of F(x) is F'(x) = f(x) for every x in the interval I.

(Sometimes this theorem is called the second fundamental theorem of calculus.)