

Teaching material is all online!

- On Minerva <http://minerva.leeds.ac.uk>
- On GitHub <https://github.com/luisacutillo78/Statistical-Methods-Lecture-Notes>

R code submission

- No technical issue - please submit your SURNAMEstudentid.R file in the assignment folder.

Resources

- Mathematical Statistics and Data Analysis - 3rd ed. (by J. A. Rice);
- Introduction to Statistics - Online Edition -D.M.Lane et al.
- <http://www1.maths.leeds.ac.uk/statistics/R/Rintro.pdf>;
- <https://www.datacamp.com/courses/free-introduction-to-r>.

Expected Values

The expectation of a random variable is connected to the concept of weighted average.

Discrete Case

$$E(X) = \sum_i x_i p(x_i)$$

Limitation: If it's an infinite sum and the x_i are both positive and negative, the sum can fail to converge! \Rightarrow We restrict to cases where the sum converges **absolutely**:

$$\sum_i |x_i| p(x_i) < \infty$$

Otherwise, we say that the expectation is **undefined**.

Continuous Case

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x) dx$$

Corresponding **limitation**: If

$$\int_{-\infty}^{\infty} |x|f(x) dx = \infty$$

we say that the expectation is **undefined**.

Table of Common Distributions

taken from *Statistical Inference* by Casella and Berger

Discrete Distributions

distribution	pmf	mean	variance	mgf/moment
Bernoulli(p)	$p^x(1-p)^{1-x}; x = 0, 1; p \in (0, 1)$	p	$p(1-p)$	$(1-p) + pe^t$
Beta-binomial(n, α, β)	$\binom{n}{x} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{\Gamma(x+\alpha)\Gamma(n-x+\beta)}{\Gamma(\alpha+\beta+n)}$	$\frac{n\alpha}{\alpha+\beta}$	$\frac{n\alpha\beta}{(\alpha+\beta)^2}$	
Notes: If $X P$ is binomial (n, P) and P is beta(α, β), then X is beta-binomial(n, α, β).				
Binomial(n, p)	$\binom{n}{x} p^x(1-p)^{n-x}; x = 1, \dots, n$	np	$np(1-p)$	$[(1-p) + pe^t]^n$
Discrete Uniform(N)	$\frac{1}{N}; x = 1, \dots, N$	$\frac{N+1}{2}$	$\frac{(N+1)(N-1)}{12}$	$\frac{1}{N} \sum_{i=1}^N e^{it}$
Geometric(p)	$p(1-p)^{x-1}; p \in (0, 1)$	$\frac{1}{p}$	$\frac{1-p}{p^2}$	$\frac{pe^t}{1-(1-p)e^t}$
Note: $Y = X - 1$ is negative binomial($1, p$). The distribution is <i>memoryless</i> : $P(X > s X > t) = P(X > s - t)$.				
Hypergeometric(N, M, K)	$\frac{\binom{M}{x}\binom{N-M}{K-x}}{\binom{N}{K}}; x = 1, \dots, K$ $M - (N - K) \leq x \leq M; N, M, K > 0$	$\frac{KM}{N}$	$\frac{KM}{N} \frac{(N-M)(N-K)}{N(N-1)}$?
Negative Binomial(r, p)	$\binom{r+x-1}{x} p^r(1-p)^x; p \in (0, 1)$ $\binom{y-1}{y-r-1} p^r(1-p)^{y-r}; Y = X + r$	$\frac{r(1-p)}{p}$	$\frac{r(1-p)}{p^2}$	$\left(\frac{p}{1-(1-p)e^t}\right)^r$
Poisson(λ)	$\frac{e^{-\lambda}\lambda^x}{x!}; \lambda \geq 0$	λ	λ	$e^{\lambda(e^t-1)}$
Notes: If Y is gamma(α, β), X is Poisson($\frac{\lambda}{\beta}$), and α is an integer, then $P(X \geq \alpha) = P(Y \leq y)$.				

Continuous Distributions

distribution	pdf	mean	variance	mgf/moment
Beta(α, β)	$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1}; x \in (0, 1), \alpha, \beta > 0$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	$1 + \sum_{k=1}^{\infty} \left(\prod_{r=0}^{k-1} \frac{\alpha+r}{\alpha+\beta+r} \right) \frac{t^k}{k!}$
Cauchy(θ, σ)	$\frac{1}{\pi\sigma} \frac{1}{1+(\frac{x-\theta}{\sigma})^2}; \sigma > 0$	does not exist	does not exist	does not exist
Notes: Special case of Student's t with 1 degree of freedom. Also, if X, Y are iid $N(0, 1)$, $\frac{X}{Y}$ is Cauchy				
χ_p^2	$\frac{1}{\Gamma(\frac{p}{2})2^{\frac{p}{2}}} x^{\frac{p}{2}-1} e^{-\frac{x}{2}}; x > 0, p \in \mathbb{N}$	p	$2p$	$\left(\frac{1}{1-2t} \right)^{\frac{p}{2}}, t < \frac{1}{2}$
Notes: Gamma($\frac{p}{2}, 2$).				
Double Exponential(μ, σ)	$\frac{1}{2\sigma} e^{-\frac{ x-\mu }{\sigma}}; \sigma > 0$	μ	$2\sigma^2$	$\frac{e^{st}}{1-(\sigma t)^2}$
Exponential(θ)	$\frac{1}{\theta} e^{-\frac{x}{\theta}}; x \geq 0, \theta > 0$	θ	θ^2	$\frac{1}{1-\theta t}, t < \frac{1}{\theta}$
Notes: Gamma(1, θ). Memoryless. $Y = X^{\frac{1}{\beta}}$ is Weibull. $Y = \sqrt{\frac{2X}{\beta}}$ is Rayleigh. $Y = \alpha - \gamma \log \frac{X}{\beta}$ is Gumbel.				
F_{ν_1, ν_2}	$\frac{\Gamma(\frac{\nu_1+\nu_2}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_1}{\nu_2} \right)^{\frac{\nu_1}{2}} \frac{x^{\frac{\nu_1-2}{2}}}{(1+(\frac{\nu_1}{\nu_2})x)^{\frac{\nu_1+\nu_2}{2}}}; x > 0$	$\frac{\nu_2}{\nu_2-2}, \nu_2 > 2$	$2\left(\frac{\nu_2}{\nu_2-2}\right)^2 \frac{\nu_1+\nu_2-2}{\nu_1(\nu_2-4)}, \nu_2 > 4$	$EX^n = \frac{\Gamma(\frac{\nu_1+2n}{2})\Gamma(\frac{\nu_2-2n}{2})}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})} \left(\frac{\nu_2}{\nu_1} \right)^n, n < \frac{\nu_2}{2}$
Notes: $F_{\nu_1, \nu_2} = \frac{\chi_{\nu_1}^2/\nu_1}{\chi_{\nu_2}^2/\nu_2}$, where the χ^2 s are independent. $F_{1, \nu} = t_{\nu}^2$.				
Gamma(α, β)	$\frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}}; x > 0, \alpha, \beta > 0$	$\alpha\beta$	$\alpha\beta^2$	$\left(\frac{1}{1-\beta t} \right)^{\alpha}, t < \frac{1}{\beta}$
Notes: Some special cases are exponential ($\alpha = 1$) and χ^2 ($\alpha = \frac{p}{2}, \beta = 2$). If $\alpha = \frac{3}{2}$, $Y = \sqrt{\frac{X}{\beta}}$ is Maxwell. $Y = \frac{1}{X}$ is inverted gamma.				
Logistic(μ, β)	$\frac{1}{\beta} \frac{e^{-\frac{x-\mu}{\beta}}}{\left[1 + e^{-\frac{x-\mu}{\beta}} \right]^2}; \beta > 0$	μ	$\frac{\pi^2\beta^2}{3}$	$e^{\mu t} \Gamma(1 + \beta t), t < \frac{1}{\beta}$
Notes: The cdf is $F(x \mu, \beta) = \frac{1}{1 + e^{-\frac{x-\mu}{\beta}}}$.				
Lognormal(μ, σ^2)	$\frac{1}{\sqrt{2\pi}\sigma} \frac{1}{x} e^{-\frac{(\log x - \mu)^2}{2\sigma^2}}; x > 0, \sigma > 0$	$e^{\mu + \frac{\sigma^2}{2}}$	$e^{2(\mu + \sigma^2)} - e^{2\mu + \sigma^2}$	$EX^n = e^{n\mu + \frac{n^2\sigma^2}{2}}$
Normal(μ, σ^2)	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}; \sigma > 0$	μ	σ^2	$e^{\mu t + \frac{\sigma^2 t^2}{2}}$
Pareto(α, β)	$\frac{\beta\alpha^{\beta}}{x^{\beta+1}}; x > \alpha, \alpha, \beta > 0$	$\frac{\beta\alpha}{\beta-1}, \beta > 1$	$\frac{\beta\alpha^2}{(\beta-1)(\beta-2)}, \beta > 2$	does not exist
t_{ν}	$\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})} \frac{1}{\sqrt{\nu\pi}} \frac{1}{(1+\frac{x^2}{\nu})^{\frac{\nu+1}{2}}}$	$0, \nu > 1$	$\frac{\nu}{\nu-2}, \nu > 2$	$EX^n = \frac{\Gamma(\frac{\nu+1}{2})\Gamma(\frac{\nu-n}{2})}{\sqrt{\pi}\Gamma(\frac{\nu}{2})} \nu^{\frac{n}{2}}, n \text{ even}$
Notes: $t_{\nu}^2 = F_{1, \nu}$.				
Uniform(a, b)	$\frac{1}{b-a}, a \leq x \leq b$	$\frac{b+a}{2}$	$\frac{(b-a)^2}{12}$	$\frac{e^{bt}-e^{at}}{(b-a)t}$
Notes: If $a = 0, b = 1$, this is special case of beta ($\alpha = \beta = 1$).				
Weibull(γ, β)	$\frac{\gamma}{\beta} x^{\gamma-1} e^{-\frac{x^{\gamma}}{\beta}}; x > 0, \gamma, \beta > 0$	$\beta^{\frac{1}{\gamma}} \Gamma(1 + \frac{1}{\gamma})$	$\beta^{\frac{2}{\gamma}} \left[\Gamma(1 + \frac{2}{\gamma}) - \Gamma^2(1 + \frac{1}{\gamma}) \right]$	$EX^n = \beta^{\frac{n}{\gamma}} \Gamma(1 + \frac{n}{\gamma})$
Notes: The mgf only exists for $\gamma \geq 1$.				

Example: Gamma and Exponential Expectation

Gamma

$$g(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}, \quad t \geq 0$$

where $\Gamma(\alpha) = \int_0^\infty u^{\alpha-1} e^{-u} du$ **Note** $g(x) = \lambda e^{-\lambda x}$ if $\alpha = 1$ **exponential**.

Gamma Expectation $E(X) = \frac{\alpha}{\lambda}$

$$E(X) = \int_0^\infty \frac{\lambda^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\lambda x} dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty x^\alpha e^{-\lambda x} dx$$

Note: $\int_0^\infty \frac{\lambda^{\alpha+1}}{\Gamma(\alpha+1)} x^\alpha e^{-\lambda x} dx = 1$ hence $\int_0^\infty x^\alpha e^{-\lambda x} = \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}}$ It follows that:

$$E(X) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} = \frac{\alpha}{\lambda}$$

(we used $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$) **Note.** $E[X] = 1/\lambda$ for the **exponential**.

R Notebook

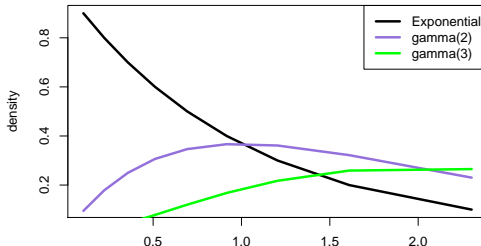
This is an R Markdown Notebook. When you execute code within the notebook, the results appear beneath the code.

Try executing this chunk by clicking the *Run* button within the chunk or by placing your cursor inside it and pressing *Cmd+Shift+Enter*.

```
a=1
####generating quantiles for gamma
p <- (1:9)/10
x=qgamma(p, shape = a)

f1<-dgamma(x, shape = a)
f2=dgamma(x, shape = a+1)
f3=dgamma(x, shape = a+2)
#pdf('gamma_density.pdf')
plot(x, f1, xlab = "x", ylab = "density",lwd = 3,type = "l")
lines(x, f2, xlab = "x",ylab = "density",lwd = 3,col = 'mediumpurple')
lines(x, f3, xlab = "x",ylab = "density",lwd = 3,col = 'green')

legend('topright', legend = c('Exponential',
                              'gamma(2)', 'gamma(3)'), col = c('black', 'mediumpurple','green'),
      lwd = 3)
```



Example: Normal Distribution $X \sim N(\mu, \sigma^2)$.

Given the pdf:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}$$

Can we compute the Expectation?

$$E(x) = \int_{-\infty}^{\infty} xf(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} xe^{-(x-\mu)^2/2\sigma^2}.$$

Whiteboard Solution.

Theorem A

Let $g(x)$ be a fixed function.

- Discrete case

$$E(g(X)) = \sum_{x_i} g(x_i)p(x_i)$$

with $\sum_{x_i} |g(x_i)|p(x_i) < \infty$

- Continuous case

$$E(g(X)) = \int_{-\infty}^{\infty} g(x)f(x)dx$$

with $\int_{-\infty}^{\infty} |g(x)|f(x)dx < \infty$

We will prove this result for the discrete case (Whiteboard Solution)

Expectations of Functions of Random Variables

Theorem B

Suppose X_1, \dots, X_n are jointly distributed r.v. Let $Y = g(X_1, \dots, X_n)$.

- Discrete case

$$E(Y) = \sum_{x_1, \dots, x_n} g(x_1, \dots, x_n) p(x_1, \dots, x_n)$$

with $\sum_{x_1, \dots, x_n} |g(x_1, \dots, x_n)| p(x_1, \dots, x_n) < \infty$

- Continuous case

$$E(Y) = \int \dots \int g(x_1, \dots, x_n) f(x_1, \dots, x_n) dx_1, \dots, x_n$$

provided that the integral with $|g|$ in place of g converges.

The proof is similar to that of Theorem A.

Theorem C

Suppose X_1, \dots, X_n are jointly distributed r.v. with expectations $E(X_i)$ and $Y = a + \sum_{i=1}^n b_i X_i$, then

$$E(Y) = a + \sum_{i=1}^n b_i E(X_i)$$

Variance and Standard Deviation

Definition

The variance of a random variable X is defined as:

$$\text{Var}(X) = E[X - E(X)]^2$$

The standard deviation, denoted by σ , is given by the square root of the variance.

Theorem

If $Y = a + bX$ then $\text{Var}(Y) = b^2 \text{Var}(X)$

Proof. Since $E(Y) = a + bE(X)$,

$$\begin{aligned} E[(Y - E(Y))^2] &= E[a + bX - a - bE(X)]^2 \\ &= b^2 E[(X - E(X))^2] \\ &= b^2 \text{Var}(X) \end{aligned}$$

Alternative way of calculating the variance

Theorem

The variance of X , if it exists, might also be computed as:

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

Proof

Let $E(X) = \mu$, then

$$\begin{aligned}\text{Var}(X) &= E[(X - \mu)^2] \\ &= E[X^2 - 2\mu X + \mu^2] \\ &= E(X^2) - 2\mu E(X) + \mu^2 \\ &= E(X^2) - \mu^2\end{aligned}$$

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Observation

Variance is the covariance of a random variable with itself!

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Observation

Variance is the covariance of a random variable with itself!

$$\begin{aligned}\text{cov}(X, X) &= E[XX] - E[X]E[X] \\ &= E[X^2] - E[X]^2\end{aligned}$$

Correlation Coefficient

Definition

Define the correlation coefficient of X and Y as,

$$\rho = \text{cor}(X, Y) = \frac{\text{cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \quad (1)$$

Correlation measures the linear relationship between two random variables!
It is possible to show that

$$|\rho| \leq 1$$

Observations about the Correlation

Correlation is between -1 and 1

Suppose $X = Y$

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Observations about the Correlation

Correlation is Between -1 and 1

Suppose $X = -Y$

$$\begin{aligned} \text{cor}(X, Y) &= \frac{\text{cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \\ &= \frac{-\text{Var}(X)}{\text{Var}(X)} \end{aligned}$$

Example: $X + Y$

Suppose X and Y have joint pdf $f(x, y) = x + y$ for $x, y \in [0, 1]$.

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$$\begin{aligned} E[XY] &= \int_0^1 \int_0^1 xy(x + y) dx dy \\ &= \int_0^1 \int_0^1 (x^2 y + y^2 x) dx dy \end{aligned}$$

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$$\begin{aligned}E[X] &= \int_0^1 \int_0^1 x(x + y) dx dy \\&= \frac{7}{12}\end{aligned}$$

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$$\text{Cov}(X, Y) = E[XY] - E[X]E[Y]$$

Example: $X + Y$

$$\begin{aligned}\text{Cov}(X, Y) &= E[XY] - E[X]E[Y] \\ &= \frac{1}{3} - \frac{49}{144} = -\frac{1}{144}\end{aligned}$$

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$$\text{Cor}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

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$$\begin{aligned}\text{Cor}(X, Y) &= \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \\ &= \frac{-\frac{1}{144}}{\frac{11}{144}}\end{aligned}$$

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$$\begin{aligned}\text{Cor}(X, Y) &= \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} \\ &= \frac{-\frac{1}{144}}{\frac{11}{144}} \\ &= \frac{-1}{11}\end{aligned}$$

Variance of the sum

Suppose X_i is a sequence of random variables with joint pdf,

$$f(\mathbf{x}) = f(x_1, x_2, \dots, x_n)$$

We have:

$$\text{var}\left(\sum_{i=1}^N X_i\right) = \sum_{i=1}^N \text{var}(X_i) + 2 \sum_{i < j} \text{cov}(X_i, X_j)$$

Sums of Random Variable

Proof.

Consider two random variables, X_1 and X_2 . Then,

Sums of Random Variable

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Consider two random variables, X_1 and X_2 . Then,

$$\text{var}(X_1 + X_2) = E[(X_1 + X_2)^2] - (E[X_1] + E[X_2])^2$$

Sums of Random Variable

Proof.

Consider two random variables, X_1 and X_2 . Then,

$$\begin{aligned}\text{var}(X_1 + X_2) &= E[(X_1 + X_2)^2] - (E[X_1] + E[X_2])^2 \\ &= E[X_1^2] + 2E[X_1X_2] + E[X_2^2] \\ &\quad - (E[X_1])^2 - 2E[X_1]E[X_2] - 2E[X_2]^2\end{aligned}$$

Sums of Random Variable

Proof.

Consider two random variables, X_1 and X_2 . Then,

$$\begin{aligned}\text{var}(X_1 + X_2) &= E[(X_1 + X_2)^2] - (E[X_1] + E[X_2])^2 \\&= E[X_1^2] + 2E[X_1X_2] + E[X_2^2] \\&\quad - (E[X_1])^2 - 2E[X_1]E[X_2] - 2E[X_2]^2 \\&= \underbrace{E[X_1^2] - (E[X_1])^2}_{\text{var}(X_1)} + \underbrace{E[X_2^2] - E[X_2]^2}_{\text{var}(X_2)} \\&\quad + 2 \underbrace{(E[X_1X_2] - E[X_1]E[X_2])}_{\text{cov}(X_1, X_2)}\end{aligned}$$

Sums of Random Variable

Proof.

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$$\begin{aligned}\text{var}(X_1 + X_2) &= E[(X_1 + X_2)^2] - (E[X_1] + E[X_2])^2 \\&= E[X_1^2] + 2E[X_1X_2] + E[X_2^2] \\&\quad - (E[X_1])^2 - 2E[X_1]E[X_2] - 2E[X_2]^2 \\&= \underbrace{E[X_1^2] - (E[X_1])^2}_{\text{var}(X_1)} + \underbrace{E[X_2^2] - E[X_2]^2}_{\text{var}(X_2)} \\&\quad + 2 \underbrace{(E[X_1X_2] - E[X_1]E[X_2])}_{\text{cov}(X_1, X_2)} \\&= \text{var}(X_1) + \text{var}(X_2) + 2\text{cov}(X_1, X_2)\end{aligned}$$



Theorem

Suppose X and Y are independent rv. Then

$$\text{cov}(X, Y) = 0$$

Proof.

Suppose X and Y are independent.

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$$\text{cov}(X, Y) = E[XY] - E[X]E[Y]$$

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Suppose X and Y are independent.

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Calculating $E[XY]$

$$\begin{aligned} E[XY] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf(x, y) dx dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xyf_X(x)f_Y(y) dx dy \end{aligned}$$

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Then $\text{cov}(X, Y) = 0$.



Observation on null correlation

Zero covariance does not **generally** imply Independence!

Workshop exercises: Sums of Random Variables

Exercise 1-Will be marked

Let X and Y be independent r.v. having Gamma distribution with parameters (n, λ) and $(1, \lambda)$. Given $Z = X + Y$. Compute the pdf of Z .

Exercise 2

Let X and Y be independent $N(0, 1)$ r.v. and $Z = X + Y$. Compute the pdf of Z .

Exercise 3

Let X and Y be independent *Poisson* r.v. with parameter, respectively, λ and μ . Compute the pmf of $Z = X + Y$.

Exercise 4 - Will be marked

Let $X \sim N(\mu, \sigma)$. Write a R Notebook containing:

- a function that, given the two parameters μ and σ , returns $P(a < X \leq b)$, $\forall a \leq b$. (make use of the base R function `pnorm()`).
- the output corresponding to $a = -2, b = 3, \mu = 1, \sigma = 2$
- a plot of the pdf and cdf of the same $N(\mu, \sigma)$, with the relative code
- a QQ plot showing the theoretical quantiles versus the empirical quantiles of the same $N(\mu, \sigma)$
- The notebook must be well documented.