MATH2715 − Workshop 4

Q1. If X has a uniform distribution over the interval $(0, 1)$, prove that $U = -\log X$ has an exponential distribution with mean one.
Hint: X has probability density function $f_X(x) = 1$ for $0 < x < 1$. Find the probability density function of U.

Q2.

(a) If X has a uniform distribution over the interval $(0, 1)$, obtain the probability density function of $U = X^2$.

(b) If X has a uniform distribution over the interval $(-\frac{1}{2}, +\frac{1}{2})$, obtain the probability density function of $U = X^2$.
Hint: Is this a 1–1 mapping?

Q3. A non-negative random variable X has probability density function $f_X(x) = e^{-x}$, $x > 0$. Derive the probability density function $f_U(u)$ of $U = \log X$. Sketch this function $f_U(u)$.
Hint: Don't forget that you should always specify the range of a probability density function. To sketch $f_U(u)$ you could look at what happens for large u and small u and check for turning points. You could also try substituting different numerical values for u.

Q4. Suppose that X has an exponential distribution with parameter $\lambda = \frac{1}{2}$.

(a) Find the probability density function of $U = +\sqrt{X}$.

(b) Obtain the mean of U.

Hint: Recall that if $Z \sim \mathrm{N}(0, 1)$ with probability density function $f_Z(z) = \dfrac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$, defined for $-\infty < z < \infty$, then $\mathrm{E}[Z^2] = \mathrm{Var}[Z] + \{\mathrm{E}[Z]\}^2 = 1 + 0^2 = 1$.

(c) Simulate 1000 values of X and hence generate 1000 values of U. What is the mean of your simulated U values?
Hint: R commands:

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x=rexp(1000,0.5)                # Puts 1000 exponential(lambda=0.5) values into x.
u=sqrt(x)
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Q5. Suppose that random variables X and Y are independent standard normal random variables.

(a) If $U = X^2 + Y^2$ and $V = X/Y$, obtain the joint probability density function $f_{UV}(u, v)$.

Hint: To find the Jacobian notice that $J = \left|\dfrac{\partial(x, y)}{\partial(u, v)}\right| = \left|\dfrac{\partial(u, v)}{\partial(x, y)}\right|^{-1}$. Is the mapping $(x, y) \to (u, v)$ a 1–1 mapping?

(b) Prove that U and V are independent.
Hint: Show that $f_{UV}(u, v)$ factorises as $f_U(u) f_V(v)$ where $f_U(u)$ and $f_V(v)$ are recognisable probability density functions.