Problem statement

Our goal is to estimate the frequency ω of the signal

$$N(t) = N_0 \cdot \left(1 + P \cdot e^{-t/\tau_d} \cdot \sin(\omega \cdot t + \phi)\right), \tag{1}$$

where τ_d is the spin tune lifetime. One measurement $N_i = N(t_i)$ takes anywhere between 1–10 milliseconds, and involves two to three thousand detector counts (Y. Senichev, personal communication Dec 9, 2016).

Assuming the Normal error distribution with mean zero and variance $\nu = \sigma_\epsilon^2$, the maximum likelihood estimator for the variance of the frequency estimate can be expressed as

$$\begin{cases} \mathrm{var}\,[\hat{\omega}] & = \nu \left(\sum_j x_j \cdot \mathrm{var}_w\,[t]\right)^{-1}, \\ \mathrm{var}_w\,[t] & = \sum_i w_i \left(t_i - \langle t\rangle_w\right)^2, \ \langle t\rangle_w = \sum_i t_i w_i, \\ w_i & = \frac{x_i}{\sum_j x_j}, \ x_i = (N_0 P \exp(\lambda t_i))^2 \cos^2(\omega t_i + \phi) = \left(\mu'_\phi(t_i)\right)^2. \end{cases} \tag{2}$$

The three factors, contributing to the standard error of the estimate are: *a*) the error variance $\nu \equiv \sigma_\epsilon^2$ (governed by the number $n_{c/\epsilon}$ of detector counts per measurement), *b*) the time spread $\sum_i w_i(t_i - \langle t\rangle_w)^2$ of the sample measurements, and *c*) their net informational content $X_{tot} = \sum_i \left(\mu'_\phi(t_i)\right)^2$.

The latter two quantities are related as a consequence of spin tune decoherence: in order to increase the measurement time spread, one has to raise the beam lifetime, which in turn increases the proportion of less informative measurements in the sample. The error variance is related to the time spread by virtue of

The variance is inversely proportional to the (weighted) spread of the predictor variable, and directly proportional to the variance of the error.

Regarding the former, the weighting by the derivative of the signal has a twofold effect: in the first place, measurements that are taken when the derivative is maximal contribute more to the spread than those made when the signal changes slowly. Considering the number of possible measurements during a fill is limited, a more cost-effective use of the beam is a concern; one that could be addressed by sampling only during the periods of rapid change in the signal. In the second place, due to spin tune decoherence, the measurements' contribution goes down with time. This aspect restricts our ability to maximize sampling efficiency. A possible trade-off would be to reduce the number of counts involved in, and thus the time of, taking a measurement. That way, more measurements could be squeezed in the periods when the sine changes sign (node), but simultaneously, the uncertainty of a measurement would be increased.

The above considerations prompt the following series of questions:

1. How long to measure the signal?
2. How many counts per measurement are optimal?
3. How congregated about the signal nodes the measurements should be?

We will try to answer them in what follows.

1 Spin tune decoherence time [Rewrite; cf. TradAccConf17 presentation for table]

A rough estimate of the maximum sensible experiment duration could be done by considering the time when the signal oscillation is indistinguishable from noise. If we denote by σ_ϵ the standard deviation of measurement error, sensibility would require

$$N_0 P \cdot e^{-t/\tau_d} \geq Z_\alpha \sigma_\epsilon.$$

Then

$$t_{max} = \tau_d \cdot \log\left(Z_\alpha^{-1} \frac{N_0 P}{\sigma_\epsilon}\right).$$

At a three percent error $\sigma_\epsilon = 3\% \cdot N_0 P$ the signal will be indistinguishable from noise at three standard deviations ($Z_\alpha = 3$), by $t_{max} = 2.4 \cdot \tau_d$.

2 Number of counts per measurement

Define the following variables: a) the number of counts per measurement: $n_{c/\epsilon}$; b) the number of measurements per node: $n_{\epsilon/zc}$; c) the number of nodes per experiment: n_{zc}.

The expected total number of scatterings in an experiment with a given number of nodes: $n_{tot} = \underbrace{n_{zc} \cdot n_{\epsilon/zc}}_{n_\epsilon} \cdot n_{c/\epsilon}$. ($n_\epsilon$ is the total number of measurements.)

$$\begin{cases} \mathrm{SE}\left[\hat{\omega}\right]^2 &= \frac{\sigma_\epsilon^2}{X_{tot} \cdot \sum_{j=1}^{n_\epsilon} w_j (t_j - \langle t \rangle_w)^2}, \\ \mathrm{SE}\left[\epsilon\right]^2 &= \frac{\sigma_\epsilon^2}{n_{c/\epsilon}}, \\ X_{tot} &= \sum_{j=1}^{n_\epsilon} x_j = \sum_{s=1}^{n_{zc}} \sum_{j=1}^{n_{\epsilon/zc}} x_{js}. \end{cases} \tag{3}$$

We can express $\sum_{j=1}^{n_{\epsilon/zc}} x_{js} = n_{\epsilon/zc} \cdot x_{0s}$, for some mean value x_{0s} in the given node s. The sum $\sum_{j=1}^{n_{\epsilon/zc}} x_{js}$ falls exponentially due to decoherence, hence $x_{0s} = x_{01} \exp\left(\lambda \cdot \frac{(s-1)\cdot\pi}{\omega}\right)$. Therefore,

$$X_{tot} = n_{\epsilon/zc} \cdot x_{01} \cdot \frac{\exp\left(\frac{\lambda\pi}{\omega} n_{zc}\right) - 1}{\exp\left(\frac{\lambda\pi}{\omega}\right) - 1} \equiv n_{\epsilon/zc} \cdot g(n_{zc}).$$

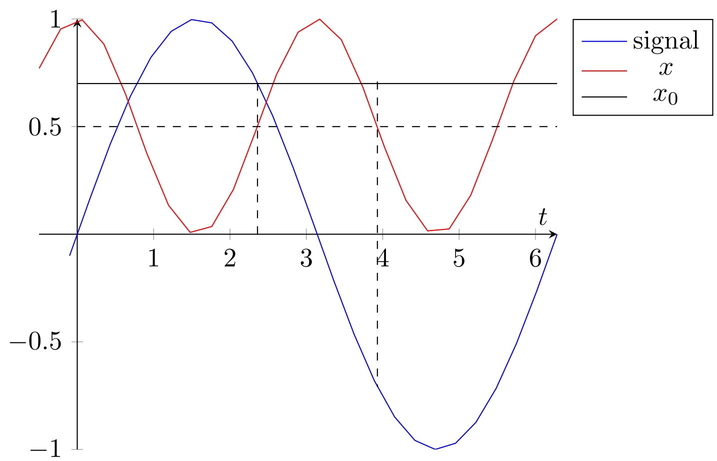

Figure 1: Explanation for x_0

The number of events per node

$$n_{\epsilon/zc} = \frac{\Delta t_{zc}}{n_{c/\epsilon} \cdot \Delta t_c},$$

hence

$$X_{tot} = g(n_{zc}) \cdot \frac{\Delta t_{zc}}{\Delta t_c} \cdot \frac{1}{n_{c/\epsilon}}.$$

The variance $\mathrm{var}_w\left[t\right]$ is also practically independent of how many measurements there are per one node (and by extension, the number $n_{c/\epsilon}$), and depends primarily on the n_{zc} and decoherence life time.

In sum, assuming one measurement is the mean of the counts ($\sigma_\epsilon = \sigma_\epsilon / \sqrt{n_{c/\epsilon}}$),

$$\begin{aligned} \mathrm{var}_w\left[\hat{\omega}\right] &= \frac{\sigma_\epsilon^2 \cdot {}^1/_{n_{c/\epsilon}}}{g(n_{zc}) \cdot \frac{\Delta t_{zc}}{\Delta t_\epsilon} \cdot {}^1/_{n_{c/\epsilon}} \cdot \mathrm{var}_w\left[t\right]} \\ &= \frac{\sigma_\epsilon^2}{g(n_{zc}) \cdot \frac{\Delta t_{zc}}{\Delta t_\epsilon} \cdot \mathrm{var}_w\left[t\right]}. \end{aligned}$$

There's no benefit to increasing the number of counts per measurement.

3 Modulation

$$
\begin{aligned}
X_{tot} &= n_{\epsilon/zc} \cdot \tilde{g}(n_{zc}) \cdot x_{01},\\
X_{tot} &= \frac{\sigma_\epsilon^2}{\mathrm{SE}\left[\hat{\omega}\right]^2 \cdot \mathrm{var}_w\left[t\right]},\\
x_{01} &= \frac{1}{\Delta t}\int_{-\Delta t/2}^{+\Delta t/2} \cos^2(\omega \cdot t)\mathrm{d}t = \frac{1}{2}\cdot\left(1+\frac{\sin\omega\Delta t}{\omega\Delta t}\right).
\end{aligned}
$$

$$
1+\frac{\sin\omega\Delta t}{\omega\Delta t} - 2\cdot X_{tot}\cdot\left(n_{\epsilon/zc}\cdot\tilde{g}(n_{zc})\right)^{-1} = 0, \tag{4}
$$

$$
n_{\epsilon/zc} = \frac{\Delta t}{\Delta t_\epsilon}.
$$

The factor X_{tot} reflects the amount of necessary Fisher information to reach the required statistical precision. The less the ratio $\mathrm{SE}\left[\hat{\omega}\right]/\sigma_\epsilon$, the more information is required. That information requirement can be fulfilled in two ways: via the increase in the number of nodes (total time), or the increase of the compaction time. The gain in information provided by the former is limited by the term $\tilde{g}(n_{zc})$ due to decoherence. Because of it, the compaction factor in my simulation is ≈ 1 (i.e. no sense in modulation), for the ratio ≈ 1, and total measurement time $\approx 2.4\tau_d$. For even less ratios, there doesn't even exist a solution for (4). That is, because of the decoherence, even if we collect the data uniformly in time (with frequency 5000 measurements per second), we won't have enough information to estimate ω with the required precision.

4 Polarization error

More pellets per second implies faster scattering rate, i.e.,

- less Δt_ϵ (faster collection of statistics),
- less τ_d.

Polarization is obtained as the ratio

$$
\begin{aligned}
P &= \frac{L-R}{L+R},\\
\mathrm{SE}\left[P\right] &= \sqrt{2}\cdot\frac{\sigma_N}{L+R}\cdot\sqrt{\left(L^2-R^2\right)^2+1},\\
\sigma_L &= \sigma_R = \sigma_N,
\end{aligned}
$$

where L, R are detector rates from the left, right detectors; N is from eq. (1), that is

$$
\begin{cases}
L(t) &= L_0\cdot\left(1+P\cdot e^{-t/\tau_d}\cdot\sin\left(\omega\cdot t+\phi_L\right)\right),\\
R(t) &= R_0\cdot\left(1+P\cdot e^{-t/\tau_d}\cdot\sin\left(\omega\cdot t+\phi_R\right)\right).
\end{cases} \tag{5}
$$

I'll assume $\phi_R = \phi_L + \pi/2$, so that when the signal is maximal at the L-detector, it is minimal at the R. Then

$$
\begin{cases}
L(t)-R(t) &= (L_0-R_0)+P\cdot e^{-t/\tau_d}\cdot\sqrt{L_0^2+R_0^2}\cdot\sin\left[\omega\cdot t+\phi_L-\arctan\left(R_0/L_0\right)\right],\\
L(t)+R(t) &= (L_0+R_0)+P\cdot e^{-t/\tau_d}\cdot\sqrt{L_0^2+R_0^2}\cdot\sin\left[\omega\cdot t+\phi_L+\arctan\left(R_0/L_0\right)\right].
\end{cases} \tag{6}
$$

5 Adding beam life-time into the picture

The beam decays as in

$$
J(t) = J_0\cdot\exp\left(\lambda_b(r)\cdot t\right).
$$

The detector registers

$$
N_0(t) = E\cdot r\cdot\int_0^{\Delta t_\epsilon} J(t+\tau)\mathrm{d}\tau \approx \Delta t_\epsilon\cdot ErJ_0\cdot\lambda_b\exp\left(\lambda_b\cdot t\right),
$$

with standard deviation

$$
\sigma_{N_0}(t)/N_0(t) = \sigma_r/r.
$$