Problem statement

Our goal is to estimate the frequency ω of the signal

$$N(t) = N_0 \cdot \left(1 + P \cdot e^{-t/\tau_d} \cdot \sin(\omega \cdot t + \phi)\right), \tag{1}$$

where τ_d is the spin tune lifetime. One observation $N_i = N(t_i)$ takes anywhere between 1–10 milliseconds, and involves two to three thousands of polarimetry measurements.

Assuming the Normal error distribution with mean zero and variance $\nu = \sigma_\epsilon^2$, the maximum likelihood estimator for the variance of the frequency estimate can be expressed as[1]

$$\begin{cases} \mathrm{var}\left[\hat{\omega}\right] & = \nu \left(\sum_j x_j \cdot \mathrm{var}_w\left[t\right]\right)^{-1}, \\ \mathrm{var}_w\left[t\right] & = \sum_i w_i \left(t_i - \langle t\rangle_w\right)^2, \ \langle t\rangle_w = \sum_i t_i w_i, \\ w_i & = \frac{x_i}{\sum_j x_j}, \ x_i = (N_0 P \exp(\lambda t_i))^2 \cos^2(\omega t_i + \phi) = \left(\mu_\phi'(t_i)\right)^2. \end{cases} \tag{2}$$

As expected, the variance is inversely proportional to the (weighted) spread of the predictor variable, and directly proportional to the variance of the error.

Regarding the former, the weighting by the derivative of the signal has a twofold effect: in the first place, observations that are made when the derivative is maximal contribute more to the spread than those made when the signal changes slowly. Considering the number of possible observations during a fill is limited, a more cost-effective use of the beam is a concern; one that could be addressed by sampling only during the periods of rapid change in the signal. In the second place, due to spin tune decoherence, the observations' contribution goes down with time. This aspect restricts our ability to maximize sampling efficiency. A possible trade-off would be to reduce the number of polarimetry measurements involved in, and thus the time of, making an observation. That way, more observations could be squeezed in the periods when the sine changes sign (zero-crossings), but simultaneously, the uncertainty of an observation would be increased.

The above considerations prompt the following series of questions:

1. How long to measure the signal?
2. How many measurements per observations are optimal?
3. How congregated about the zero-crossings the measurements should be?

We will try to answer them in what follows.

1 Number of measurements per observation (event)

Define the following variables: *a*) the number of measurements per observation: $n_{m/\epsilon}$; *b*) the number of observations per zero-crossing: $n_{\epsilon/zc}$; *c*) the number of zero-crossings per experiment: n_{zc}.

The expected total number of scatterings in an experiment with a given number of zero-crossings: $n_m = \underbrace{n_{zc} \cdot n_{\epsilon/zc}}_{n_\epsilon} \cdot n_{m/\epsilon}$. ($n_\epsilon$ is the total number of observations.)

$$\begin{cases} \mathrm{SE}\left[\hat{\omega}\right]^2 & = \frac{\sigma_\epsilon^2}{X_{tot} \cdot \sum_{j=1}^{n_\epsilon} w_j (t_j - \langle t\rangle_w)^2}, \\ \mathrm{SE}\left[\epsilon\right]^2 & = \frac{\sigma_m^2}{n_{m/\epsilon}}, \\ X_{tot} & = \sum_{j=1}^{n_\epsilon} x_j = \sum_{s=1}^{n_{zc}} \sum_{j=1}^{n_{\epsilon/zc}} x_{js}. \end{cases} \tag{3}$$

We can express $\sum_{j=1}^{n_{\epsilon/zc}} x_{js} = n_{\epsilon/zc} \cdot x_{0s}$, for some mean value x_{0s} in the given zero-crossing s. The sum $\sum_{j=1}^{n_{\epsilon/zc}} x_{js}$ falls exponentially due to decoherence, hence $x_{0s} = x_{01} \exp\left(\lambda \cdot \frac{(s-1)\cdot\pi}{\omega}\right)$. Therefore,

$$X_{tot} = n_{\epsilon/zc} \cdot x_{01} \cdot \frac{\exp\left(\frac{\lambda\pi}{\omega} n_{zc}\right) - 1}{\exp\left(\frac{\lambda\pi}{\omega}\right) - 1} \equiv n_{\epsilon/zc} \cdot g(n_{zc}).$$

[1]This expression consistently overestimates the standard error by a factor of 2; however, when used on a function with $x_i = (N_0 P)^2$ (linear regression), it yields the same expression as the fitter. The fitter also computes the standard error according to some formula (which I haven't seen yet).

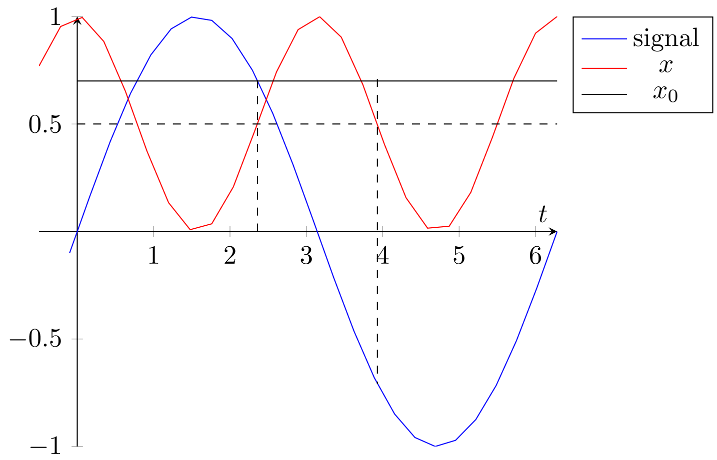


Figure 1: Explanation for x_0

The number of events per zero-crossing

$$n_{\epsilon/zc} = \frac{\Delta t_{zc}}{n_{m/\epsilon} \cdot \Delta t_m},$$

hence

$$X_{tot} = g(n_{zc}) \cdot \frac{\Delta t_{zc}}{\Delta t_m} \cdot \frac{1}{n_{m/\epsilon}}.$$

The variance $\mathrm{var}_w[t]$ is also practically independent of how many measurements there are per one zero-crossing (and by extension, the number $n_{m/\epsilon}$), and depends primarily on the n_{zc} and decoherence life time.

In sum, assuming one observation is the mean of the measurements ($\sigma_\epsilon = \sigma_m/\sqrt{n_{m/\epsilon}}$),

$$\begin{aligned}\mathrm{var}_w[\hat{\omega}] &= \frac{\sigma_m^2 \cdot {}^1/_{n_{m/\epsilon}}}{g(n_{zc}) \cdot \frac{\Delta t_{zc}}{\Delta t_m} \cdot {}^1/_{n_{m/\epsilon}} \cdot \mathrm{var}_w[t]} \\ &= \frac{\sigma_m^2}{g(n_{zc}) \cdot \frac{\Delta t_{zc}}{\Delta t_m} \cdot \mathrm{var}_w[t]}.\end{aligned}$$

2 Spin tune decoherence time

A rough estimate of the maximum sensible experiment duration could be done by considering the time when the signal oscillation is indistinguishable from noise. If we denote by σ_ϵ the standard deviation of the observation error, the sensibility condition would require

$$N_0 P \cdot e^{-t/\tau_d} \geq Z_\alpha \sigma_\epsilon.$$

Then

$$t_{max} = \tau_d \cdot \log\left(Z_\alpha^{-1} \frac{N_0 P}{\sigma_\epsilon}\right).$$

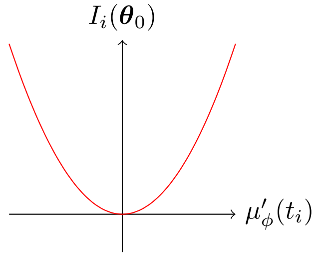
At a three percent error $\sigma_\epsilon = 3\% \cdot N_0 P$, the signal will be indistinguishable from noise at three standard deviations ($Z_\alpha = 3$), by $t_{max} = 2.4 \cdot \tau_d$. However, sampling will become impractical long before that.

We have the Cramér-Rao inequality to tell us what's the minimum variance of an estimator is possible:

$$\mathrm{var}[\hat{\omega}] \geq \frac{1}{I(\omega)}.$$

Fisher information is additive, and what I call *point Fisher information* can be expressed as:

$$I_i(\boldsymbol{\theta}_0) = \frac{1}{\nu} \begin{pmatrix} \left(\sqrt{2} \cdot \mu'_\phi(t_i)\right)^{-2} & 0 & 0 \\ 0 & t_i^2 & t_i \\ 0 & t_i & 1 \end{pmatrix} \cdot \left(\mu'_\phi(t_i)\right)^2.$$



$I_i(\boldsymbol{\theta}_0)$
$\mu'_\phi(t_i)$