

# Problem

- Current formula
- Alternative formula

$$\sigma_{\omega}^2 = \frac{24}{K(P T)^2}$$

$$\sigma_{\hat{\omega}}^2 = \frac{24}{K T^2} \cdot \left( \frac{\sigma_{\epsilon}}{N_0 P} \right)^2$$

# Derivation: log-likelihood

- Current formula

$$N(t) = N_0(1 + P \sin(\omega t + \varphi))$$

**Expectation value  
(structural model)**

- Alternative formula

$$\mu(t_i) = N_0(1 + P \sin(\omega t_i + \varphi))$$

**Probability distribution  
(probabilistic model)**

$$f(y_i; \theta) = \frac{1}{\sqrt{2\pi v}} \exp\left(-\frac{1}{2} \frac{(y_i - \mu(t_i))^2}{v}\right)$$

$$\theta \equiv (v, \omega, \varphi)$$

$$\log L = \sum_i \log(N_0(1 + P \sin(\omega t_i + \varphi))) \quad \log L(\theta; \vec{y}) = \prod_i f(y_i; \theta) = -\frac{K}{2} \log 2\pi - \frac{K}{2} \log v - \frac{1}{2v} \sum_i \epsilon_i^2$$

**Error term**

$$\epsilon_i \equiv y_i - \mu(t_i)$$

# MLE procedure

- First derivatives of  $\log L$
- Second derivatives
- Expectations of second derivatives
- The Fisher matrix
- The variance-covariance matrix

# The variance-covariance matrix

- Current

$$\text{cov}(\omega, \varphi) = \begin{pmatrix} \frac{24}{K(P T)^2} & \frac{12}{K P^2 T} \\ \frac{12}{K P^2 T} & \frac{8}{K P^2} \end{pmatrix}$$

- Alternative

$$v\text{cov}(\hat{v}, \hat{\omega}, \hat{\varphi}) = \begin{pmatrix} 2v/K & 0 & 0 \\ 0 & \frac{v}{\Omega} \sum_i (\mu'_{\varphi}(t_i))^2 & \frac{v}{\Omega} \sum_i t_i (\mu'_{\varphi}(t_i))^2 \\ 0 & \frac{v}{\Omega} \sum_i t_i (\mu'_{\varphi}(t_i))^2 & \frac{v}{\Omega} \sum_i (t_i \mu'_{\varphi}(t_i))^2 \end{pmatrix}$$

$$\Omega = \frac{2v^2}{K} |I(\theta_0)|$$

# The alternative SE

- Sinusoidal model
- Linear regression cross-check

$$\left(\mu'_{\varphi}(t_i)\right)^2 = (N_0 P)^2 \cos^2(\omega t_i + \varphi) \equiv x_i \quad \mu(t_i) = \omega t_i + \varphi$$

$$\sigma_{\hat{\omega}}^2 = \frac{v}{\sum_j x_j \sum_i w_i (t_i - \langle t \rangle_w)^2}$$

$$\sigma_{\hat{\omega}}^2 = \frac{v}{\sum_i (t_i - \langle t \rangle)^2}$$

$$w_i = \frac{x_i}{\sum_j x_j}$$

# Uniform sampling $\Delta t$

$$\sum_{j=1}^K x_j = \frac{1}{2} (N_0 P)^2 \cdot K$$

$$\sigma_{\hat{\omega}}^2 = \frac{v}{\sum_j x_j \sum_i w_i (t_i - \langle t \rangle_w)^2}$$

$$\sum_i (t_i - \langle t \rangle_w)^2 \underbrace{w_j}_{1/K} \approx \frac{(\Delta t K)^2}{12} = \frac{T^2}{12}$$

$$\sigma_{\hat{\omega}}^2 = \frac{24}{KT^2} \cdot \left( \frac{\sigma_{\epsilon}}{N_0 P} \right)^2$$