Final report

Aleksandr Aksentev

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1 Detector counting rate model

We assume the following model for the detector counting rate:

$$N(t) = N_0(t) \cdot \left(1 + P \cdot e^{-t/\tau_d} \cdot \sin(\omega \cdot t + \phi)\right), \tag{1}$$

where τ_d is the decoherence lifetime, and $N_0(t)$ is the counting rate from the unpolarized cross-section.

Since the beam current can be expressed as a function of time as

$$I(t) \equiv N^b(t)\nu = I_0 \cdot e^{\lambda_b t},$$

λ_b the beam lifetime, the expected number of particles scattered in the direction of the detector during measurement time Δt_c is

$$\begin{aligned} N_0(t) &= p \cdot \int_{-\Delta t_c/2}^{+\Delta t_c/2} I(t+\tau)\mathrm{d}\tau \\ &= p \cdot \frac{\nu N_0^b}{\lambda_b} e^{\lambda_b t} \cdot \left(e^{\lambda_b \Delta t_c/2} - e^{-\lambda_b \Delta t_c/2}\right) \\ &\approx \underbrace{p \cdot \nu N_0^b e^{\lambda_b t}}_{\text{rate } r(t)} \cdot \Delta t_c, \end{aligned} \tag{2}$$

where p is the probability of "useful" scattering (approximately 1%).

The actual number of detected particles will be distributed as a Poisson distribution

$$P_{N_0(t)}(\tilde{N}_0) = \frac{(r(t)\Delta t_c)^{\tilde{N}_0}}{\tilde{N}_0!} \cdot e^{-r(t)\Delta t_c},$$

hence $\sigma^2_{\tilde{N}_0}(t) = N_0(t)$.

We are interested in the expectation value $N_0(t) = \mathrm{E}\left[\tilde{N}_0(t)\right]$, and its variance $\sigma_{N_0}(t)$. Those are estimated in the usual way, [1] as

$$\langle \tilde{N}_0(t) \rangle_{\Delta t_\epsilon} = \sum_{i=1}^{n_{c/\epsilon}} \tilde{N}_0(t_i),\ n_{c/\epsilon} = \Delta t_\epsilon / \Delta t_c,$$

and

$$\sigma_{\tilde{N}_0(t)|\Delta t_\epsilon} = \sum_{i=1}^{n_{c/\epsilon}} \left(\tilde{N}_0(t_i) - \langle \tilde{N}_0(t_i) \rangle_{\Delta t_\epsilon}\right)^2.$$

(Δt_ϵ is the event measurement time, Δt_c is the polarimetry measurement time.) A sum of random variables, $N_0(t)$ is normally distributed.

The standard error of the mean then is

$$\begin{aligned} \sigma_{N_0}(t) &= \sigma_{\tilde{N}_0}(t)/\sqrt{n_{c/\epsilon}} = \sqrt{N_0(t)\frac{\Delta t_c}{\Delta t_\epsilon}} \\ &\approx \sqrt{\frac{p \cdot \nu N_0^b}{\Delta t_\epsilon}} \cdot \Delta t_c \cdot \exp\left(\frac{\lambda_b}{2} \cdot t\right). \end{aligned}$$

Relative error grows:

$$\frac{\sigma_{N_0}(t)}{N_0(t)} \approx \frac{A}{\sqrt{\Delta t_\epsilon}} \cdot \exp\left(-\frac{\lambda_b}{2} t\right) = \frac{A}{\sqrt{\Delta t_\epsilon}} \cdot \exp\left(\frac{t}{2\tau_b}\right), \; A = \frac{1}{\sqrt{p \cdot \nu N_0^b}}. \tag{3}$$

2 Figure of merit

A measure of the beam's polarization is the relative asymmetry of detector counting rates: [2, p. 17]

$$\mathcal{A} = \frac{N(\frac{\pi}{2}) - N(-\frac{\pi}{2})}{N(\frac{\pi}{2}) + N(-\frac{\pi}{2})}. \tag{4}$$

In the simulation to follow, the function fitted to the asymmetry data is:

$$\mathcal{A}(t) = \mathcal{A}(0) \cdot e^{\lambda_d \cdot t} \cdot \sin\left(\omega \cdot t + \phi\right), \tag{5}$$

with three nuisance parameters $\mathcal{A}(0)$, λ_d, and ϕ.

Due to the decreasing beam size, the measurement of the figure of merit is heteroscedastic. From [2, p. 18], the heteroscedasticity model assumed is

$$\sigma^2_{\mathcal{A}}(t) \approx \frac{1}{2N_0(t)}. \tag{6}$$

3 Conditions for maximum precision

Assuming a Gaussian error distribution with mean zero and variance σ^2_ϵ, the maximum likelihood estimator for the variance of the frequency estimate of the cross-section asymmetry $\mathcal{A}$ can be expressed as

$$\mathrm{var}\left[\hat{\omega}\right] = \frac{\sigma^2_\epsilon}{X_{tot} \cdot \mathrm{var}_w\left[t\right]},$$

with

$$\begin{aligned}
X_{tot} &= \sum_{j=1}^{n_\epsilon} x_j = \sum_{s=1}^{n_{zc}} \sum_{j=1}^{n_{\epsilon/zc}} x_{js}, \\
\mathrm{var}_w\left[t\right] &= \sum_i w_i \left(t_i - \langle t \rangle_w\right)^2, \; \langle t \rangle_w = \sum_i w_i t_i, \\
w_i &= \frac{x_i}{\sum_j x_j}, \; x_i = \left(\mathcal{A}(0)\exp(\lambda_d t_i)\right)^2 \cos^2(\omega t_i + \phi) = \left(\mu'_\phi(t_i)\right)^2.
\end{aligned}$$

In the expression above, X_{tot} is the total Fisher information of the sample, and $\mathrm{var}_w\left[t\right]$ is a measure of its time-spread. It can be observed that by picking appropriate sampling times, one can raise the X_{tot} term, since it is proportional to a sum of the signal's time derivatives. If the oscillation frequency and phase are already known to a reasonable precision, further improvement can be achieved by the application of a sampling scheme in which measurements are taken only during rapid change in the signal (sampling modulation). Improvement here is limited by the polarimetry sampling rate.

Both the $\mathrm{var}_w\left[t\right]$ and X_{tot} terms are bounded as a result of spin tune decoherence. We can express $\sum_{j=1}^{n_{\epsilon/zc}} x_{js} = n_{\epsilon/zc} \cdot x_{0s}$, for some mean value x_{0s} at a given node s. $n_{\epsilon/zc}$ is the number of asymmetry measurements per node. The period of time during which measuring takes place, Δt_{zc}, is termed *compaction time*. The value of the sum $\sum_{j=1}^{n_{\epsilon/zc}} x_{js}$ falls exponentially due to decoherence, hence $x_{0s} = x_{01}\exp\left(\lambda_d \cdot \frac{(s-1)\cdot\pi}{\omega}\right)$. Therefore,

$$X_{tot} = n_{\epsilon/zc} \cdot x_{01} \cdot \frac{\exp\left(\frac{\lambda_d \pi}{\omega} n_{zc}\right) - 1}{\exp\left(\frac{\lambda_d \pi}{\omega}\right) - 1} \equiv n_{\epsilon/zc} \cdot x_{01} \cdot g(n_{zc}); \tag{7}$$

$$x_{01} = \frac{1}{\Delta t_{zc}} \int_{-\Delta t_{zc}/2}^{+\Delta t_{zc}/2} \cos^2(\omega \cdot t)\mathrm{d}t = \frac{1}{2} \cdot \left(1 + \frac{\sin \omega \Delta t_{zc}}{\omega \Delta t_{zc}}\right), \tag{8}$$

$$n_{\epsilon/zc} = \frac{\Delta t_{zc}}{\Delta t_\epsilon}. \tag{9}$$

Eq. (7) provides us with a means to estimating the limits on the duration of the experiment. In Table 1, the percentage of the total Fisher information limit, the time in decoherence lifetimes by which it is reached, and the signal-to-noise ratio by that time, are summarized. The signal-to-noise ratios are computed according to:

$$\mathrm{SNR} \triangleq \frac{\mathcal{A}(0)\cdot e^{-t/\tau_d}}{\sigma_{\mathcal{A}}(t)} \approx \sqrt{2\cdot p\cdot \nu N_0^b \cdot \Delta t_c}\cdot \mathcal{A}(0)\cdot \exp\left[-\frac{t}{\tau_d}\cdot\left(1+\frac{1}{2}\frac{\tau_d}{\tau_b}\right)\right], \tag{10}$$

in which, from $\sigma_{\mathcal{A}(0)}/\mathcal{A}(0) \approx 3\%$, the factor before the exponent is 33.

Table 1: Total Fisher information, by what time it is reached, and the corresponding signal-to-noise ratio.

FI limit (%)	Reached ($\times\tau_d$)	SNR
95	3.0	0.4
90	2.3	1.1
70	1.2	5.5
50	0.7	11.7

4 Simulation

We simulated data from two detectors with parameters gathered in Table 2 for $T_{tot} = 1000$ seconds, sampled uniformly at the rate $f_s = 375$ Hz. These figures are chosen for the following reason: the beam size in a fill is on the order of 10^{11} particles; if we want to keep the beam lifetime equal to the decoherence lifetime, we cannot exhaust more than 75% of it; only 1% of all scatterings are of the sort we need for polarimetry, so we're left with $7.5\cdot 10^8$ useful scatterings. A measurement of the counting rate $N_0(t)$ with a precision of approximately 3% requires somewhere in the neighborhood of 2000 detector counts, which further reduces the number of events to $3.75\cdot 10^5 = f_s \cdot T_{tot}$. One thousand seconds is the expected duration of a fill, hence $f_s = 375$ Hz.

Relative measurement error for the detector counting rates is depicted in Figure 1; the cross-section asymmetry, computed according to Eq. (4), is shown in Figure 2. To these data we fit via Maximum Likelihood a non-linear heteroscedastic model[1] given by Eq. (5), with the variance function for the weights given by Eq. (6). The fit results are summarized in Table 3.

Table 2: Detector counting rates' model parameters

	Left	Right	
ϕ	$-\pi/2$	$+\pi/2$	rad
ω	3		rad/sec
P	0.4		
τ_d	721		sec
τ_b	721		sec
$N_0(0)$	6730		

Table 3: Fit results

	Estimate	SE	Unit
$\mathcal{A}(0)$	0.400	$9.03\cdot 10^{-5}$	
λ_d	-0.001	$7.86\cdot 10^{-7}$	1/sec
ω	3.000	$7.55\cdot 10^{-7}$	rad/sec
ϕ	-1.571	$2.25\cdot 10^{-2}$	rad

4.1 Modulation gains

If our initial frequency estimate obtained from a time-uniform sample has a standard error on the order of $1\cdot 10^{-6}$ rad/sec, simulation shows the standard error of the estimate can be improved to $\approx 5.8\cdot 10^{-7}$ rad/sec.

[1] R package nlreg. [3]

References

[1] http://www.owlnet.rice.edu/~dodds/Files331/stat_notes.pdf.

[2] D. Eversmann et al. "Analysis of the Spin Coherence Time at the Cooler Synchrotron COSY," 2013. http://wwwo.physik.rwth-aachen.de/fileadmin/user_upload/www_physik/Institute/Inst_3B/Mitarbeiter/Joerg_Pretz/DEMasterarbeit.pdf.

[3] https://cran.r-project.org/web/packages/nlreg/index.html

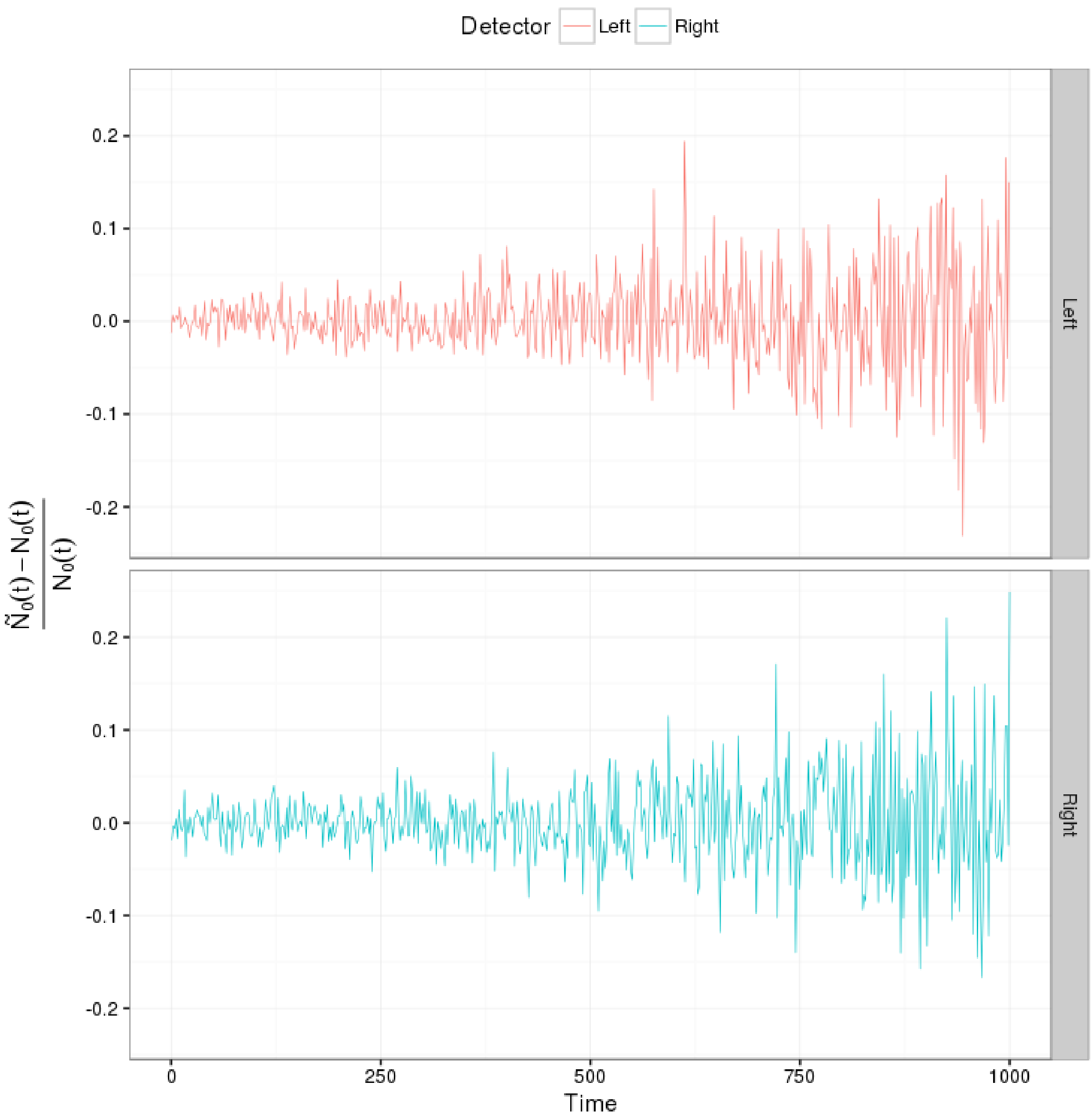

Figure 1: Relative counting rate measurement error for the left and right detectors as a function of time.

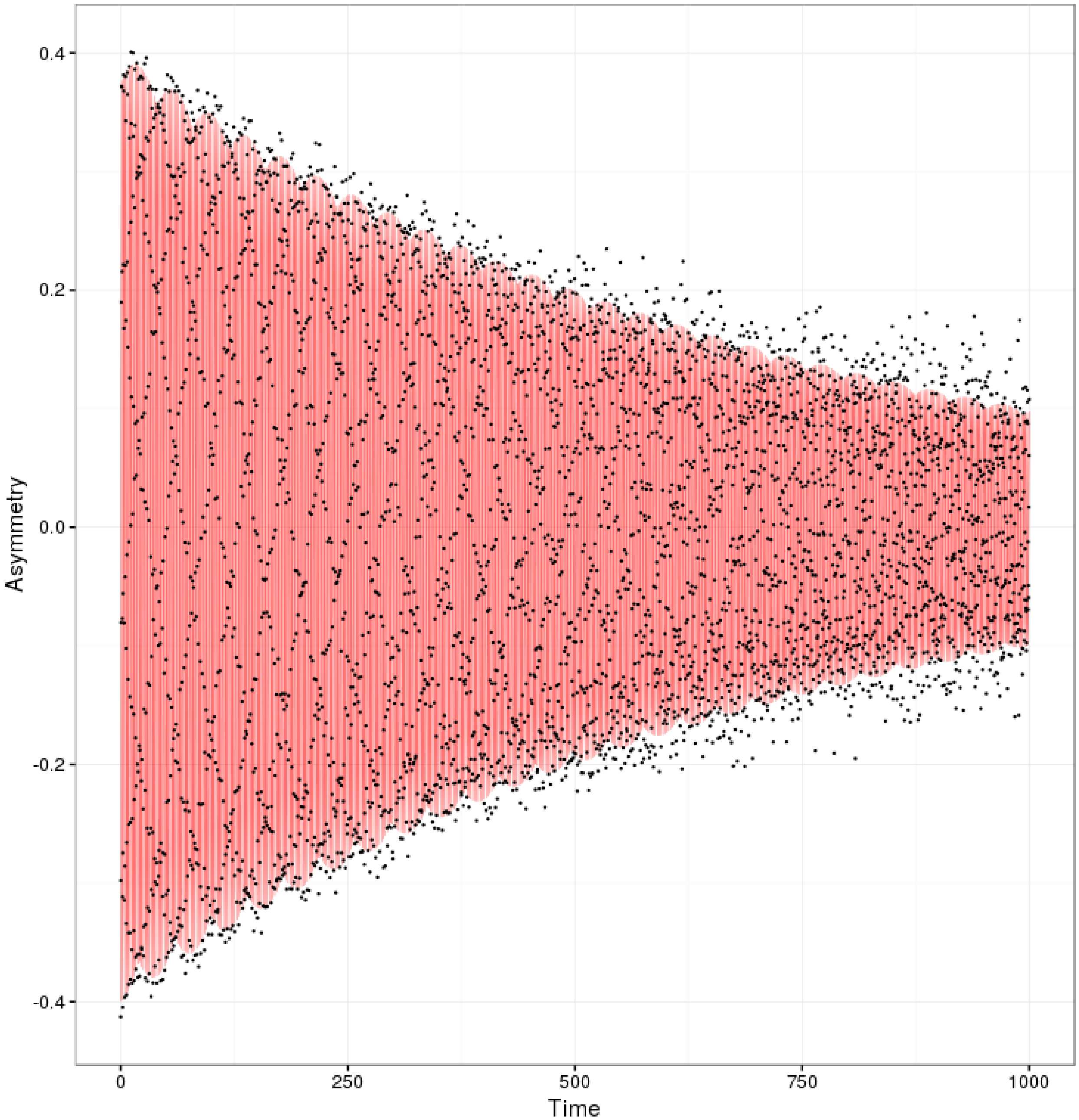

Figure 2: Expectation value (red line) and sample measurements (black dots) of the cross-section asymmetry.