Final report

A. Aksentev

1. Standard error of the fit;
2. Improvement by sampling modulation?

1 Detector counting rate model

We assume the following model for the detector counting rate:

$$N(t) = N_0(t) \cdot \left(1 + P \cdot e^{-t/\tau_d} \cdot \sin(\omega \cdot t + \phi)\right), \tag{1}$$

where τ_d is the decoherence lifetime, and $N_0(t)$ is the counting rate from the unpolarized cross-section.

Since the beam current can be expressed as a function of time as

$$I(t) \equiv N^b(t)\nu = I_0 \cdot e^{\lambda_b t},$$

λ_b the beam lifetime, the expected number of particles scattered in the direction of the detector during measurement time Δt_c is

$$\begin{aligned} N_0(t) &= p \cdot \int_{-\Delta t_c/2}^{+\Delta t_c/2} I(t+\tau)\mathrm{d}\tau \\ &= p \cdot \frac{\nu N_0^b}{\lambda_b} e^{\lambda_b t} \cdot \left(e^{\lambda_b \Delta t_c/2} - e^{-\lambda_b \Delta t_c/2}\right) \\ &\approx \underbrace{p \cdot \nu N_0^b e^{\lambda_b t}}_{\text{rate } r(t)} \cdot \Delta t_c, \end{aligned} \tag{2}$$

where p is the probability of "useful" scattering (approximately 1%).

The actual number of detected particles will be distributed as a Poisson distribution

$$P_{N_0(t)}(\tilde{N}_0) = \frac{(r(t)\Delta t_c)^{\tilde{N}_0}}{\tilde{N}_0!} \cdot e^{-r(t)\Delta t_c},$$

hence $\sigma^2_{\tilde{N}_0}(t) = N_0(t)$.

We are interested in the expectation value $N_0(t) = \mathrm{E}\left[\tilde{N}_0(t)\right]$, and its variance $\sigma_{N_0}(t)$. Those are estimated in the usual way, [1] as

$$\langle \tilde{N}_0(t) \rangle_{\Delta t_\epsilon} = \sum_{i=1}^{n_{c/\epsilon}} \tilde{N}_0(t_i), \;\; n_{c/\epsilon} = \Delta t_\epsilon / \Delta t_c,$$

and

$$\sigma_{\tilde{N}_0(t)|\Delta t_\epsilon} = \sum_{i=1}^{n_{c/\epsilon}} \left(\tilde{N}_0(t_i) - \langle \tilde{N}_0(t_i) \rangle_{\Delta t_\epsilon}\right)^2.$$

(Δt_ϵ is the event measurement time, Δt_c is the polarimetry measurement time.) A sum of random variables, $N_0(t)$ is normally distributed.

The standard error of the mean then is

$$\begin{aligned} \sigma_{N_0}(t) &= \sigma_{\tilde{N}_0}(t)/\sqrt{n_{c/\epsilon}} = \sqrt{N_0(t)\frac{\Delta t_c}{\Delta t_\epsilon}} \\ &\approx \sqrt{\frac{p \cdot \nu N_0^b}{\Delta t_\epsilon}} \cdot \Delta t_c \cdot \exp\left(\frac{\lambda_b}{2} \cdot t\right). \end{aligned}$$

Relative error grows:

$$\frac{\sigma_{N_0}(t)}{N_0(t)} \approx \frac{A}{\sqrt{\Delta t_\epsilon}} \cdot \exp\left(-\frac{\lambda_b}{2}t\right) = \frac{A}{\sqrt{\Delta t_\epsilon}} \cdot \exp\left(\frac{t}{2\tau_b}\right), \; A = \frac{1}{\sqrt{p \cdot \nu N_0^b}}. \tag{3}$$

2 Figure of merit

A measure of the beam's polarization is the relative asymmetry of detector counting rates: [2, p. 17]

$$\mathcal{A} = \frac{N(\frac{\pi}{2}) - N(-\frac{\pi}{2})}{N(\frac{\pi}{2}) + N(-\frac{\pi}{2})}. \tag{4}$$

In the simulation to follow, the following function is fitted to the asymmetry data:

$$\mathcal{A}(t) = \mathcal{A}(0) \cdot e^{\lambda_d \cdot t} \cdot \sin(\omega \cdot t + \phi), \tag{5}$$

with three nuisance parameters $\mathcal{A}(0)$, λ_d, and ϕ.

Due to the decreasing beam size, the measurement of the figure of merit is heteroscedastic. From [2, p. 18], the heteroscedasticity model assumed is

$$\sigma^2_{\mathcal{A}}(t) \approx \frac{1}{2N_0(t)}. \tag{6}$$

3 Simulation

We simulated data from two detectors with parameters gathered in Table 1 for $T_{tot} = 1000$ seconds, sampled uniformly at the rate $f_s = 500$ Hz. These figures are chosen for the following reason: if the beam size in a fill is on the order of 10^{11} particles, and only 1% of them will be registered, we're left with 10^9 useful scatterings. A measurement of the counting rate $N_0(t)$ with a precision of approximately 3% requires somewhere in the neighborhood of 2000 detector counts, which further reduces the number of events to $5 \cdot 10^5 = f_s \cdot T_{tot}$. One thousand seconds is the expected duration of a fill, hence $f_s = 500$ Hz.

Relative measurement error for the detector counting rates is depicted in Figure 1; the cross-section asymmetry, computed according to Eq. (4), is shown in Figure 2. To these data we fit via Maximum Likelihood a non-linear heteroscedastic model[1] given by Eq. (5), with the variance function for the weights given by Eq. (6). The fit results are summarized in Table 2.

Table 1: Detector counting rates' model parameters

	Left	Right	
ϕ	$-\pi/2$	$+\pi/2$	rad
ω	3		rad/sec
P	0.4		
τ_d	721		sec
τ_b	721		sec
$N_0(0)$	6730		

Table 2: Fit results

	Estimate	SE
$\mathcal{A}(0)$	0.4065	$8.4 \cdot 10^{-5}$
λ_d	-0.0016	$8.7 \cdot 10^{-7}$
ω	3.0000	$7.0 \cdot 10^{-7}$
ϕ	-1.5707	$2.0 \cdot 10^{-2}$

[1] R package nlreg. [3]

References

[1] http://www.owlnet.rice.edu/~dodds/Files331/stat_notes.pdf.

[2] D. Eversmann et al. Analysis of the Spin Coherence Time at the Cooler Synchrotron COSY, 2013. http://wwwo.physik.rwth-aachen.de/fileadmin/user_upload/www_physik/Institute/Inst_3B/Mitarbeiter/Joerg_Pretz/DEMasterarbeit.pdf.

[3] https://cran.r-project.org/web/packages/nlreg/index.html

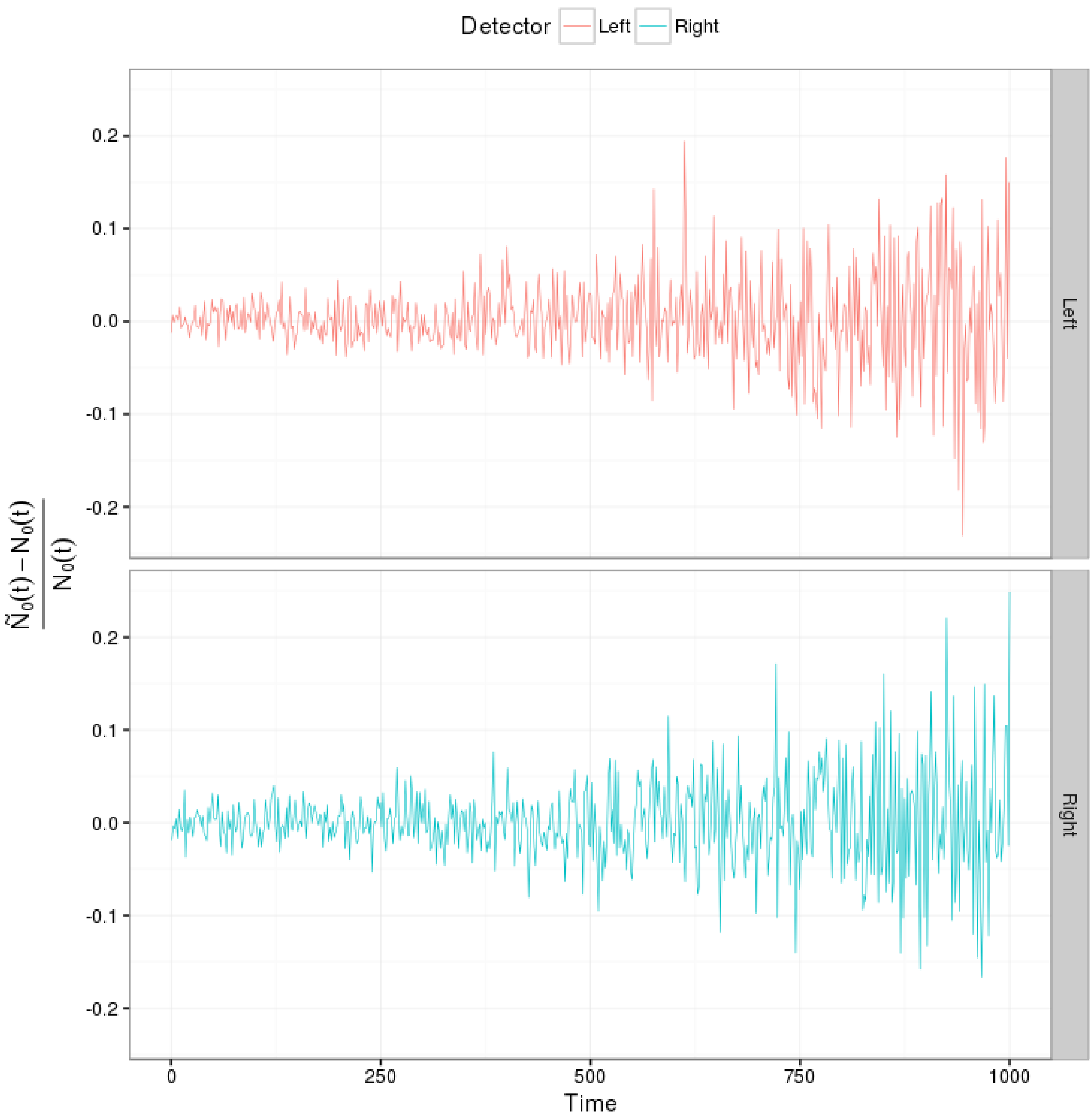

Figure 1: Relative counting rate measurement error for the left and right detectors as a function of time.

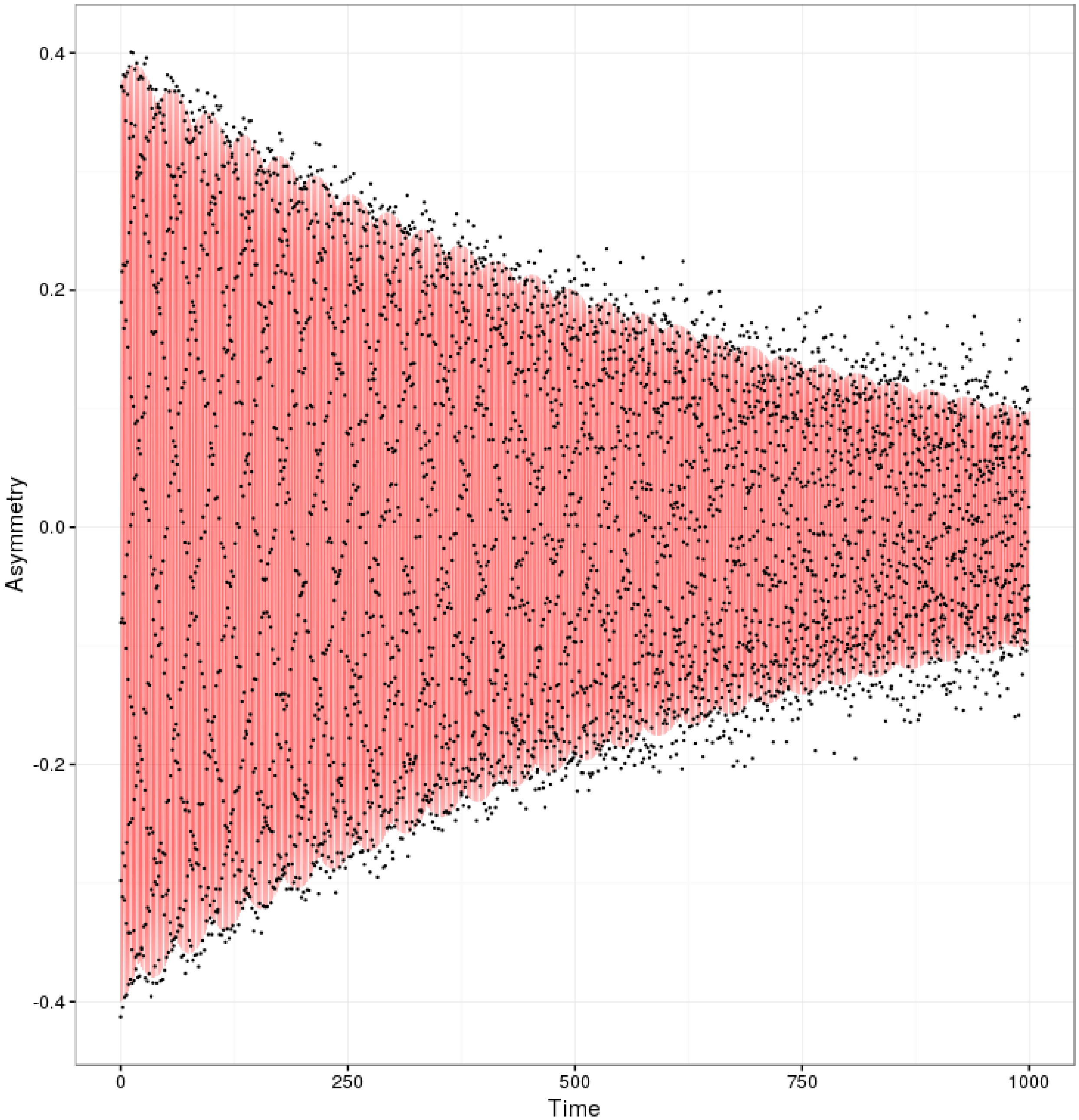

Figure 2: Expectation value (red line) and sample measurements (black dots) of the cross-section asymmetry.