# Contents

# 1 Detector count rate model

We assume the following detector count rate model:

$$N(t) = N_0(t) \cdot \left(1 + P \cdot e^{-t/\tau_d} \cdot \sin(\omega \cdot t + \phi)\right), \tag{1}$$

where $N_0(t)$ is the count rate from the unpolarized cross-section, $\tau_d$ is the decoherence lifetime.

The current of a beam scattered on a target can be expressed as

$$I(t) = I_0 \cdot e^{t/\tau_b} = \nu N_0^b \cdot e^{t/\tau_b},$$

where $\tau_b$ is the beam lifetime, $N_0^b$ is the initial number of beam particles, and $\nu$ is the revolution frequency. Denoting $p$ the probability that a scattered particle flies in the direction of the detector, the expected number of particles detected during measurement time $\Delta t_c$ can is

$$\begin{aligned} N_0(t) &= p \cdot \int_{-\Delta t_c/2}^{+\Delta t_c/2} I(t+\tau)\mathrm{d}\tau \\ &= p \cdot \frac{\nu N_0^b}{\lambda_b} e^{\lambda_b t} \cdot \left(e^{\lambda_b \Delta t_c/2} - e^{-\lambda_b \Delta t_c/2}\right) \\ &\approx \underbrace{p \cdot \nu N_0^b e^{\lambda_b t}}_{\text{rate } r(t)} \cdot \Delta t_c. \end{aligned} \tag{2}$$

Its distribution is, therefore, a Poisson distribution

$$P_{N_0(t)}(\tilde{N}_0) = \frac{(r(t)\Delta t_c)^{\tilde{N}_0}}{\tilde{N}_0!} \cdot e^{-r(t)\Delta t_c},$$

with variance $\sigma\left[\tilde{N}_0\right]^2(t) = N_0(t)$.

We are interested in the expectation value $N_0(t) = \mathrm{E}\left[\tilde{N}_0(t)\right]$, and its standard deviation $\sigma\left[N_0\right](t)$. Denoting the count event measurement time $\Delta t_\epsilon$, polarimetry measurement time $\Delta t_c$, and the number of events per measurement $n_{c/\epsilon} = \Delta t_\epsilon / \Delta t_c$, the expectation is

$$\mathrm{E}\left[\tilde{N}_0(t)\right]_{\Delta t_\epsilon} = \frac{1}{n_{c/\epsilon}} \sum_{i=1}^{n_{c/\epsilon}} \tilde{N}_0(t_i).$$

Being a sum of random variables, $N_0(t)$ is normally distributed; hence the standard error of the mean is

$$\begin{aligned} \sigma\left[N_0\right](t) &= \sigma\left[\tilde{N}_0\right](t)/\sqrt{n_{c/\epsilon}} = \sqrt{N_0(t)\frac{\Delta t_c}{\Delta t_\epsilon}} \\ &\approx \sqrt{\frac{p \cdot \nu N_0^b}{\Delta t_\epsilon}} \cdot \Delta t_c \cdot \exp\left(\frac{\lambda_b}{2} \cdot t\right). \end{aligned}$$

Note that relative error grows with time:

$$\frac{\sigma\left[N_0\right](t)}{N_0(t)} \approx \frac{A}{\sqrt{\Delta t_\epsilon}} \cdot \exp\left(-\frac{\lambda_b}{2}t\right) = \frac{A}{\sqrt{\Delta t_\epsilon}} \cdot \exp\left(\frac{t}{2\tau_b}\right),\ A = \frac{1}{\sqrt{p \cdot \nu N_0^b}}. \quad (3)$$

# 2 Cross-section asymmetry

The relative asymmetry of detector count rates is used as a measure of the beam polarization. [**?** , p. 17] Cross-section asymmetry is defined as the normalized difference of the number of counts in the detectors placed on opposite sides of the vacuum tube:

$$\mathcal{A} = \frac{N(\frac{\pi}{2}) - N(-\frac{\pi}{2})}{N(\frac{\pi}{2}) + N(-\frac{\pi}{2})}. \quad (4)$$

Due to the decreasing beam size, the measurement of the figure of merit is heteroscedastic. From [**?** , p. 18], the assumed heteroscedasticity model is

$$\sigma\left[\mathcal{A}\right]^2(t) \approx \frac{1}{2N_0(t)}. \quad (5)$$

# 3 Measurement time frame

Assuming a Gaussian error distribution with zero mean and $\sigma\left[\epsilon\right]^2$ variance, the maximum likelihood estimator for the variance of the frequency estimate of the cross-section asymmetry $\mathcal{A}$ can be expressed as

$$\mathrm{var}\left[\hat{\omega}\right] = \frac{\sigma\left[\epsilon\right]^2}{X_{tot} \cdot \mathrm{var}_w\left[t\right]}, \quad (6)$$

with

$$\begin{aligned} X_{tot} &= \sum_{j=1}^{n_\epsilon} x_j = \sum_{s=1}^{n_{zc}} \sum_{j=1}^{n_{\epsilon/zc}} x_{js}, \\ \mathrm{var}_w\left[t\right] &= \sum_i w_i \left(t_i - \langle t\rangle_w\right)^2,\ \langle t\rangle_w = \sum_i w_i t_i, \\ w_i &= \frac{x_i}{\sum_j x_j},\ x_i = (\mathcal{A}(0)\exp(\lambda_d t_i))^2 \cos^2(\omega t_i + \phi) = \left(\mu'_\phi(t_i)\right)^2. \end{aligned}$$

In the expressions above, $X_{tot}$ is the total Fisher information of the sample, and $\mathrm{var}_w\left[t\right]$ is a measure of its time-spread. It can be observed that by picking appropriate sampling times, one can raise the $X_{tot}$ term, since it is proportional to a sum of the signal's time derivatives. If the oscillation frequency and phase are already known to a reasonable precision, further improvement can be achieved by the application of a sampling scheme in which measurements are taken only during rapid change in the signal (sampling modulation).

Both the $\mathrm{var}_w\left[t\right]$ and $X_{tot}$ terms are bounded as a result of spin tune decoherence. We can express $\sum_{j=1}^{n_{\epsilon/zc}} x_{js} = n_{\epsilon/zc} \cdot x_{0s}$, for some mean value $x_{0s}$ at a given node $s$, where $n_{\epsilon/zc}$ is the number of asymmetry measurements per node. We will call *compaction time* (denoted $\Delta t_{zc}$) the period of time during which polarimetry measurements are performed. The value of the sum $\sum_{j=1}^{n_{\epsilon/zc}} x_{js}$ falls exponentially due to decoherence, hence $x_{0s} = x_{01} \exp\left(\lambda_d \cdot \frac{(s-1)\cdot\pi}{\omega}\right)$. Therefore:

$$X_{tot} = n_{\epsilon/zc} \cdot x_{01} \cdot \frac{\exp\left(\frac{\lambda_d \pi}{\omega} n_{zc}\right) - 1}{\exp\left(\frac{\lambda_d \pi}{\omega}\right) - 1} \equiv n_{\epsilon/zc} \cdot x_{01} \cdot g(n_{zc}); \tag{7}$$

$$x_{01} = \frac{1}{\Delta t_{zc}} \int_{-\Delta t_{zc}/2}^{+\Delta t_{zc}/2} \cos^2(\omega \cdot t)\mathrm{d}t = \frac{1}{2} \cdot \left(1 + \frac{\sin \omega \Delta t_{zc}}{\omega \Delta t_{zc}}\right), \tag{8}$$

$$n_{\epsilon/zc} = \frac{\Delta t_{zc}}{\Delta t_\epsilon}. \tag{9}$$

Equation (7) can be used to estimate the limits on the duration of the experiment. In Table 1 we summarize the percentage of the total Fisher information limit, the time (in decoherence lifetimes) by which it is reached, and the corresponding signal-to-noise ratio. The signal-to-noise ratios are computed according to:

$$\mathrm{SNR} = \frac{\mathcal{A}(0) \cdot e^{-t/\tau_d}}{\sigma\left[\mathcal{A}\right](t)} \approx \sqrt{2 \cdot p \cdot \nu N_0^b \cdot \Delta t_c} \cdot \mathcal{A}(0) \cdot \exp\left[-\frac{t}{\tau_d} \cdot \left(1 + \frac{1}{2}\frac{\tau_d}{\tau_b}\right)\right], \tag{10}$$

in which, assuming $\sigma\left[\mathcal{A}(0)\right]/\mathcal{A}(0) \approx 3\%$ (polarimetry measurement precision), the factor before the exponent is 33.

Table 1: Total Fisher information, its time of reaching, and the corresponding signal-to-noise ratio.

| FI limit (%) | Reached ($\times\tau_d$) | SNR |
|---:|---:|---:|
| 95 | 3.0 | 0.4 |
| 90 | 2.3 | 1.1 |
| 70 | 1.2 | 5.5 |
| 50 | 0.7 | 11.7 |

Under the assumption of no decoherence ($\lambda_d = 0$) and uniform sampling rate $1/\Delta t$, eq (6) can be rewritten in physical terms as

$$\begin{aligned} X_{tot} &= \sum_{k=1}^{K} \mathcal{A}^2(0)\cos^2(\omega t_k + \phi) = \frac{1}{2}\mathcal{A}^2(0) \cdot K, \\ \mathrm{var}_w\left[t\right] &= \sum_{k=1}^{K} (k\Delta t - \langle t \rangle_w)^2 \underbrace{w_k}_{1/K} \\ &\approx \frac{\Delta t^2}{12} K^2 = \frac{T^2}{12}, \end{aligned}$$

and so

$$\mathrm{var}\left[\hat{\omega}\right] = \frac{24}{KT^2} \cdot \left(\frac{\sigma\left[\epsilon\right]}{\mathcal{A}(0)}\right)^2 .$$

# 4 Simulation

We simulated data from two detectors with parameters gathered in Table 2 for $T_{tot} = 1000$ seconds, sampled uniformly at the rate $f_s = 375$ Hz. These figures are chosen for the following reason: the beam size in a fill is on the order of $10^{11}$ particles; if we want to keep the beam lifetime equal to the decoherence lifetime, we cannot exhaust more than 75% of it; only 1% of all scatterings are of the sort we need for polarimetry, so we're left with $7.5 \cdot 10^8$ useful scatterings. A measurement of the count rate $N_0(t)$ with a precision of approximately 3% requires somewhere in the neighborhood of 2000 detector counts, which further reduces the number of events to $3.75 \cdot 10^5 = f_s \cdot T_{tot}$. One thousand seconds is the expected duration of a fill, hence $f_s = 375$ Hz.

Relative measurement error for the detector count rates is depicted in Figure 1; the cross-section asymmetry, computed according to eq (4), is shown in Figure 2. To these data we fit via Maximum Likelihood a non-linear heteroscedastic model given by

$$\mathcal{A}(t) = \mathcal{A}(0) \cdot e^{\lambda_d \cdot t} \cdot \sin\left(\omega \cdot t + \phi\right),$$

with the variance function for the weights given by eq (5). The fit results are summarized in Table 3.

Table 2: Count rate model parameters

| | Left | Right | |
|---|---|---|---|
| $\phi$ | $-\pi/2$ | $+\pi/2$ | rad |
| $\omega$ | 3 | | rad/sec |
| $P$ | 0.4 | | |
| $\tau_d$ | 721 | | sec |
| $\tau_b$ | 721 | | sec |
| $N_0(0)$ | 6730 | | |

Table 3: Fit results

| | Estimate | SE | Unit |
|---|---|---|---|
| $\mathcal{A}(0)$ | 0.400 | $9.03 \cdot 10^{-5}$ | |
| $\lambda_d$ | -0.001 | $7.86 \cdot 10^{-7}$ | 1/sec |
| $\omega$ | 3.000 | $7.55 \cdot 10^{-7}$ | rad/sec |
| $\phi$ | -1.571 | $2.25 \cdot 10^{-2}$ | rad |

If our initial frequency estimate obtained from a time-uniform sample has a standard error on the order of $1 \cdot 10^{-6}$ rad/sec, simulation shows the standard error of the estimate can be improved to $\approx 5.8 \cdot 10^{-7}$ rad/sec.

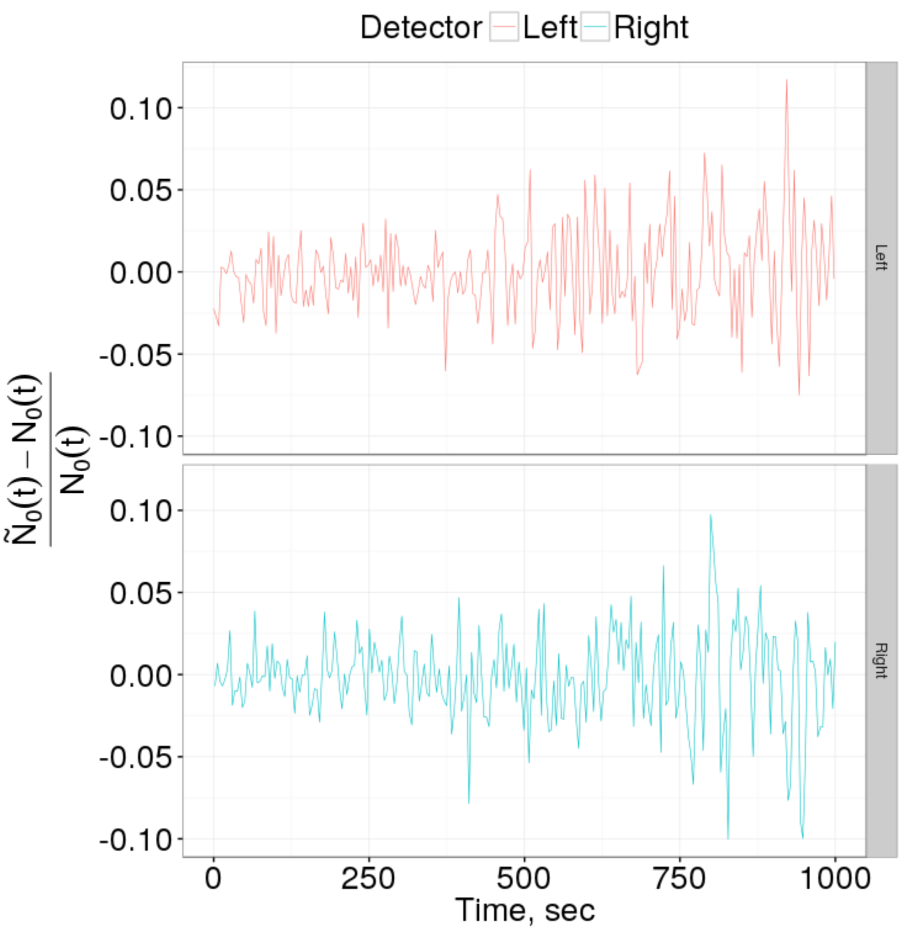


Figure 1: Relative count rate measurement error for the left and right detectors as a function of time.

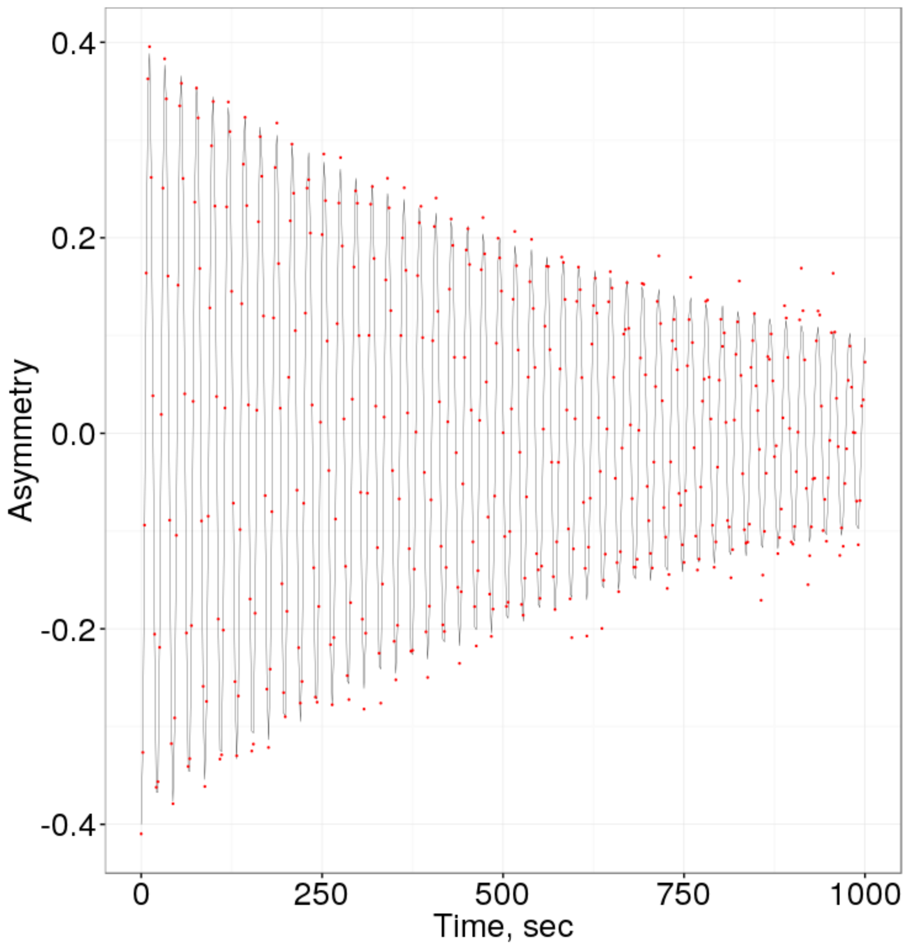


Figure 2: Expectation value (black line) and sample measurements (red dots) of the cross-section asymmetry.