Introduction

In this note we try to derive the form of the measured signal, as well as investigate its properties under the assumption of a finite particle phase and oscillation frequency distributions.

1 Projection of polarization

The polarization of the beam is the sum of the spins of the particles in it. We measure the projection of the polarization on the y-axis:

$$\begin{aligned}
\vec{P} &= \sum_{i=1}^{n_b} \vec{s}_i, \\
\pi_{\hat{\boldsymbol{y}}}\vec{s} &\equiv \hat{\boldsymbol{y}} \cdot \vec{s} = |\vec{s}| \cos\Theta, \\
\pi_{\hat{\boldsymbol{y}}}\vec{P}(t) &= \sum_i \pi_{\hat{\boldsymbol{y}}}\vec{s_i} = |\vec{s}| \sum_i \cos\Theta_i(t), \\
\Theta_i(t) &= \omega_i \cdot t + \phi_i.
\end{aligned} \tag{1}$$

The signal at time t is a sum of random variables, and hence has expectation

$$\begin{aligned}
\mathrm{E}\left[\pi_{\hat{\boldsymbol{y}}}\vec{P}(t)\right] &= |\vec{s}| \sum_i \mathrm{E}\left[\cos\Theta_i(t)\right] = |\vec{s}| \sum_i \int_\infty \cos x \cdot f_{\Theta_i(t)}(x)\mathrm{d}x \\
&= |\vec{s}| \sum_i \int_{-1}^{+1} f_{\Theta_i(t)}(\arcsin y)\mathrm{d}y.
\end{aligned}$$

1.1 Distribution of $\Theta_i(t)$

If the distribution of ω_i is $f_\omega(x)$, then that of $\omega_i t$ is $f_{\omega t}(x) = \frac{1}{t} f_\omega(\frac{x}{t})$. The distribution of $\Theta_i(t)$ is the convolution

$$(f_{\omega t} * f_\phi)(\theta) \triangleq \int_\infty f_{\omega t}(\theta - y) f_\phi(y)\mathrm{d}y = \frac{1}{t}\int_\infty f_\omega\left(\frac{\theta - y}{t}\right) f_\phi(y)\mathrm{d}y.$$

Assuming $\omega_i = \omega_0 + G_6\Delta\gamma_i^2$ and $f_{\Delta\gamma}(y) = \mathcal{N}(0, \sigma_{\Delta\gamma})(y)$,

$$f_{\Delta\gamma^2}(y) = \begin{cases} \frac{1}{\sigma^2_{\Delta\gamma}} \cdot \chi^2_1(\frac{y}{\sigma^2_{\Delta\gamma}}), & y \geq 0, \\ 0, & y < 0; \end{cases}$$

and

$$f_\omega(x) = \begin{cases} \frac{1}{G_6\sigma^2_{\Delta\gamma}} \chi^2_1\left(\frac{x - \omega_0}{G_6\sigma^2_{\Delta\gamma}}\right), & x \geq \omega_0, \\ 0 & x < \omega_0. \end{cases} \tag{2}$$

$$f_\omega(x) = \begin{cases} A(a, \omega_0) \cdot \frac{\exp(-\frac{1}{2}ax)}{\sqrt{x - \omega_0}}, & x \geq \omega_0, \\ 0, & x < \omega_0, \end{cases}$$

with

$$A(a, \omega_0) \equiv \frac{a \exp(\frac{1}{2}a\omega_0)}{\sqrt{2\pi a}}, \quad a \equiv \frac{1}{G\sigma^2_{\Delta\gamma}}. \tag{3}$$

$$f_\omega\left(\frac{\theta - y}{t}\right) = \begin{cases} A(a, \omega_0) \cdot B(\theta, t) \cdot \frac{\exp\left(\frac{ay}{2t}\right)}{\sqrt{\Delta\theta(t) - y}}, & y \leq \Delta\theta(t), \\ 0, & y > \Delta\theta(t), \end{cases}$$

where

$$B(\theta, t) \equiv \sqrt{t} \exp\left(-\frac{a\theta}{2t}\right), \quad \Delta\theta(t) \equiv \theta - \omega_0 t. \tag{4}$$

We'll assume a normally distributed $\phi_i \sim \mathcal{N}(\phi_0, \sigma_\phi)$.

Hence,

$$f_{\Theta(t)}(\theta) = (f_{\omega t} * f_\phi)(\theta) = \frac{1}{t} A(a, \omega_0) C(\phi_0, \sigma_\phi) \cdot B(\theta, t) D(\theta, \omega_0 t),$$

where

$$D(\theta, \omega_0 t) \equiv \int_{-\infty}^{\Delta\theta(t)} \frac{\exp(\xi y - y^2/2\sigma_\phi^2)}{\sqrt{\Delta\theta(t) - y}} \mathrm{d}y, \tag{5}$$

$$C(\phi_0, \sigma_\phi) \equiv \exp\left(\frac{\phi_0^2}{2\sigma_\phi^2}\right), \quad \xi \equiv \frac{a\sigma_\phi^2 + 2t\phi_0}{2t\sigma_\phi^2}. \tag{6}$$

2 Simulation

In the simulation, the following model was assumed:

$$\begin{aligned} \Delta\gamma &\sim \mathcal{N}(0, \sigma_{\Delta\gamma}), \\ \sigma_{\Delta\gamma}(t) &= \sigma_{\Delta\gamma}(0) + h \cdot t, \\ \omega_i &= \omega_0 + G_6 \cdot \Delta\gamma_i^2, \\ \phi_i &\sim \mathcal{N}(\phi_0, \sigma_\phi). \end{aligned}$$

Parameter	Value	Dimension
$\sigma_{\Delta\gamma}(0)$	$1 \cdot 10^{-3}$	
h	$5 \cdot 10^{-6}$	$1/sec$
ω_0	3	rad/sec
G_6	$7 \cdot 10^2$	rad/sec
ϕ_0	$\pi/2$	
σ_ϕ	$2 \cdot 10^{-2}$	rad

2.1 Parameter G_6

From eq. (2), $\mathrm{var}\left[\frac{\omega}{G_6\sigma_{\Delta\gamma}^2}\right] = \left(G_6\sigma_{\Delta\gamma}^2\right)^{-2} \cdot \mathrm{var}\left[\omega\right] = 2$, hence $\mathrm{var}\left[\omega\right] = 2\left(G_6\sigma_{\Delta\gamma}^2\right)^2$.

$\mathrm{var}\left[\omega T\right] = T^2 \cdot \mathrm{var}\left[\omega\right] = 2T^2\left(G_6\sigma_{\Delta\gamma}^2\right)^2$; $\sigma_{\omega T} = \sqrt{2} \cdot T \cdot G_6\sigma_{\Delta\gamma}^2$.

Therefore, for G_6 we have

$$G_6 = \frac{\sigma_{\omega T}}{\sqrt{2} \cdot T \cdot \sigma_{\Delta\gamma}^2}. \tag{7}$$

If in $T = 1000$ secs $\sigma_{\omega T} = 1$ rad, assuming $\sigma_{\Delta\gamma} = 1 \cdot 10^{-3}$, $G_6 \approx 7 \cdot 10^2$.

2.2 Simulation results

The distributions of the beam particles' phase and oscillation frequency are presented in Figure 1a. The resulting polarization signal has the form shown in Figure 2a.

The signal's power spectrum (Figure 1b) mimics the particles' ω density distribution.

2.3 Normally distributed ω

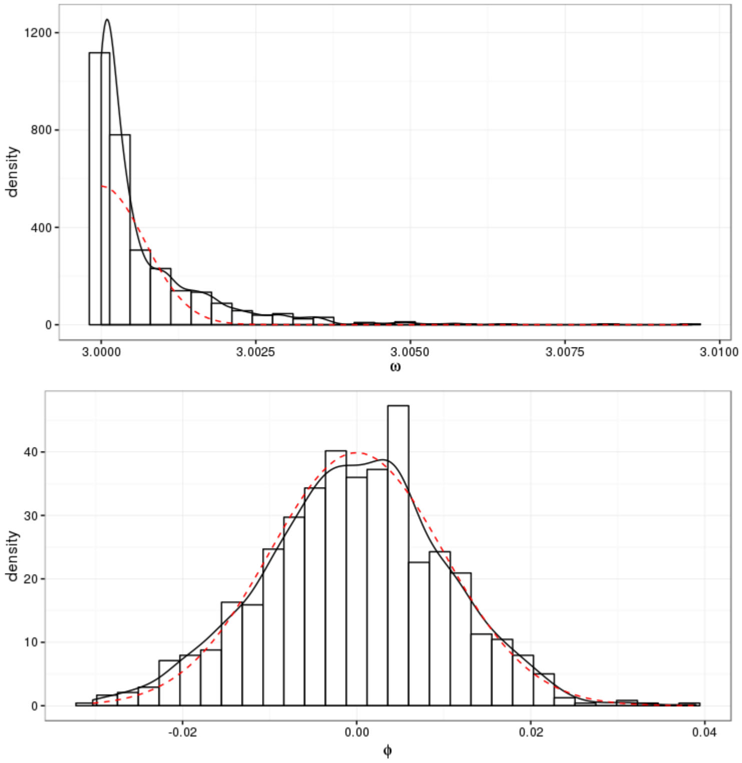
For comparison, we simulated the signal in the case of $\omega_i \sim \mathcal{N}(\omega_0, G_6 \cdot \sigma_{\Delta\gamma}^2)$ (Figure 2b).

3 Frequency creep

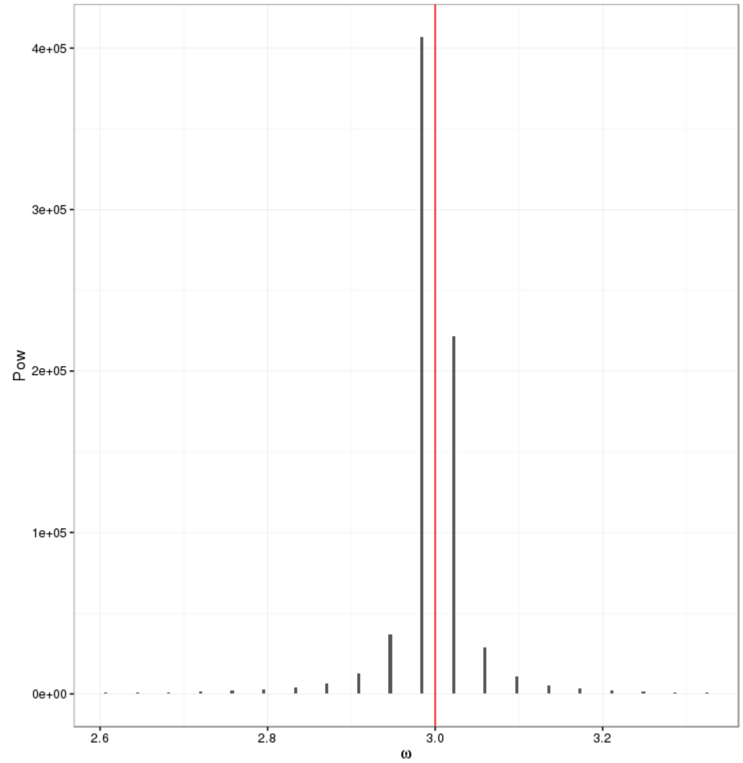
The analysis of the frequency creep is based on the single-harmonic representation of the signal:

$$f(t) = E(t) \cdot sin(\omega \cdot t + \phi). \tag{8}$$

The methodology is described in what follows.



(a) Phase space distributions.



(b) Power spectral density plot.

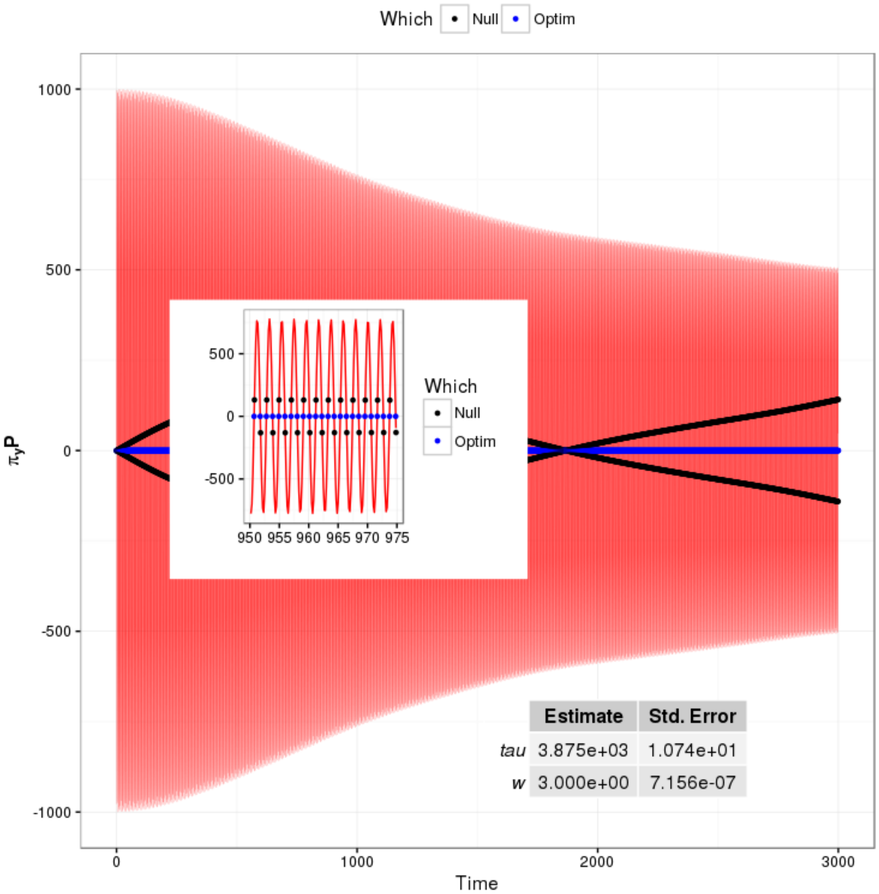
Figure 1: Distributions and power spectra.

First, we find the positions of the special points (either the nodes or the extrema) on the signal using a combination of the golden section search and parabolic interpolation,[1] optimizing the square of eq. (1) (See Figure 4).

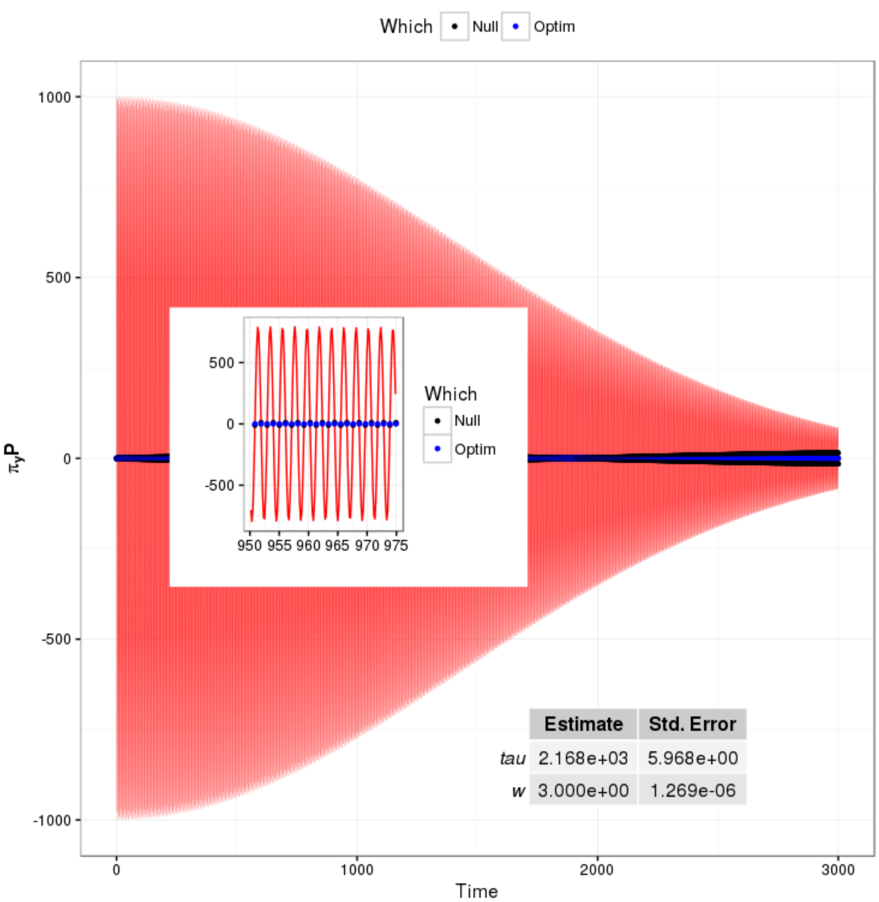
After that we take the difference between consecutive points; by observing changes in the time difference over time, we can infer the behavior of $\omega(t)$. Under the assumption of eq. (8), the time difference Δt between two consecutive nodes yields us the value of $\omega(t) \triangleq \pi/\Delta t$. The result of this estimation is shown in Figure 5a. The corresponding signal plot is in Figure

The single-harmonic model, however, does not seem to be valid.

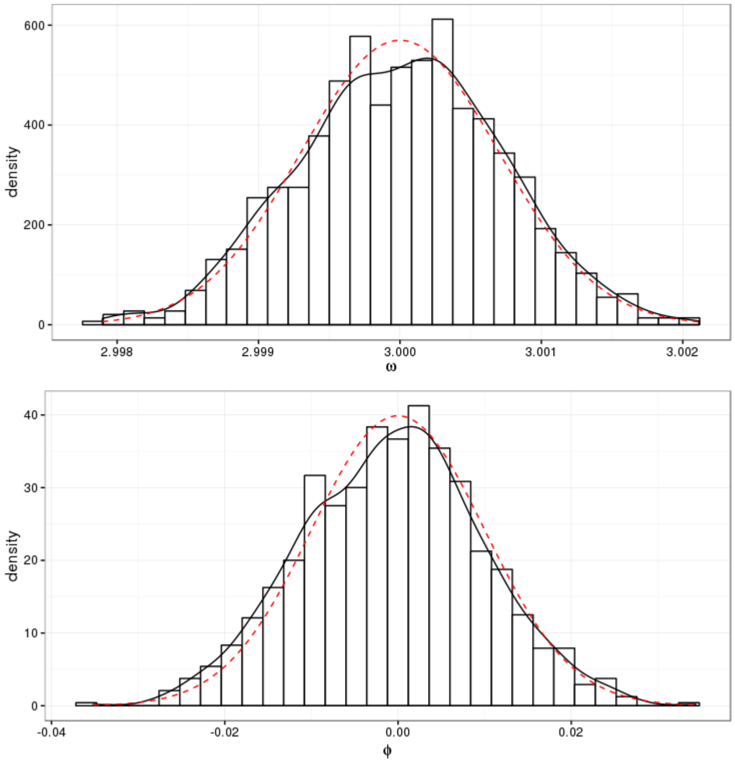
[1]The Stats::optimize function in R.



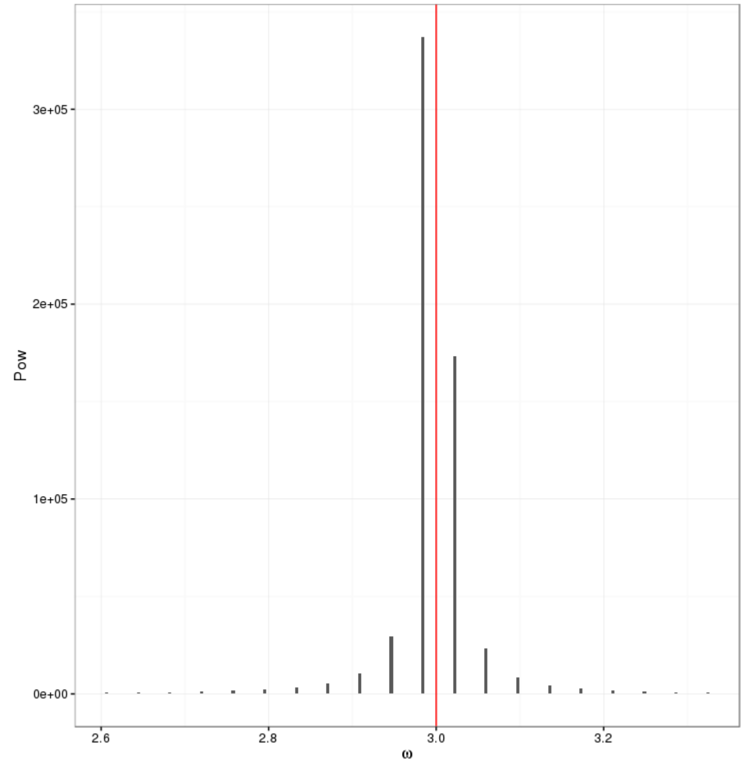
(a) Polarization signal in the case of constant phase space distributions. The black lines mark the measurements taken at the points $\sin(\omega_0 \cdot t_n + \phi_0) = \pm 1$.



(b) Signal in the case of a normally distributed oscillation frequency.



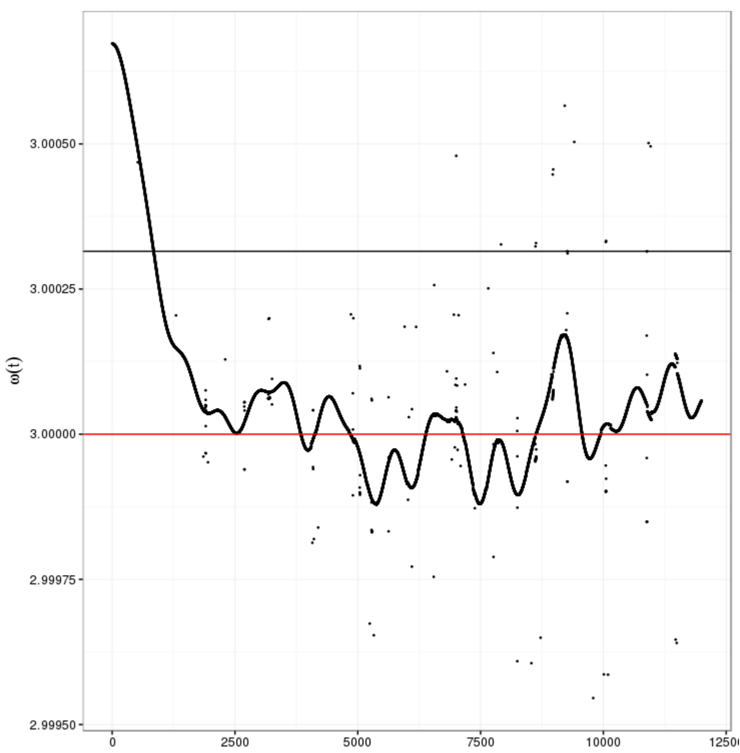
(a) Phase space distributions.



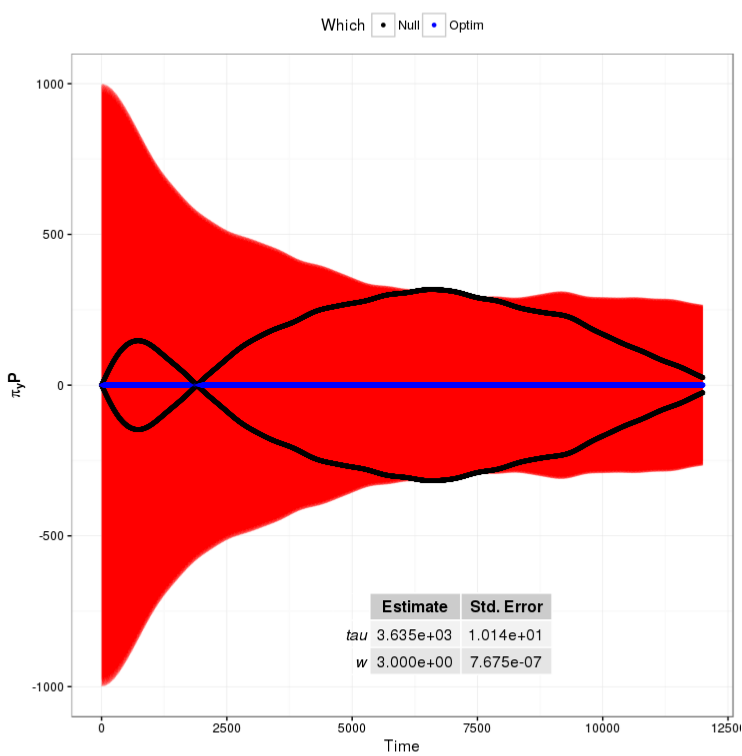
(b) Power spectral density plot.

Figure 3: Frequency power spectrum.

Figure 4: The signal and optimization target function. The target function was scaled to fit on the plot.



(a) The red horizontal line corresponds to ω_0, the black one to the $\hat{\omega}$ from the fit by eq. (8) with $E(t) = a \cdot \exp(\lambda \cdot t)$.



(b) Signal.

Figure 5: Simulation results.