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1 Detector count rate model

We assume the following detector count rate model:

$$N(t) = N_0(t) \cdot \left(1 + P \cdot e^{-t/\tau_d} \cdot \sin(\omega \cdot t + \phi)\right), \quad (1)$$

where $N_0(t)$ is the count rate from the unpolarized cross-section, τ_d is the decoherence lifetime.

The current of a beam scattered on a target can be expressed as

$$I(t) = I_0 \cdot e^{t/\tau_b} = \nu N_0^b \cdot e^{t/\tau_b},$$

where τ_b is the beam lifetime, N_0^b is the initial number of beam particles, and ν is the revolution frequency. Denoting p the probability that a scattered particle flies in the direction of the detector, the expected number of particles detected during measurement time Δt_c can be

$$\begin{aligned} N_0(t) &= p \cdot \int_{-\Delta t_c/2}^{+\Delta t_c/2} I(t + \tau) d\tau \\ &= p \cdot \frac{\nu N_0^b}{\lambda_b} e^{\lambda_b t} \cdot \left(e^{\lambda_b \Delta t_c/2} - e^{-\lambda_b \Delta t_c/2}\right) \\ &\approx \underbrace{p \cdot \nu N_0^b e^{\lambda_b t}}_{\text{rate } r(t)} \cdot \Delta t_c. \end{aligned} \quad (2)$$

Its distribution is, therefore, a Poisson distribution

$$P_{N_0(t)}(\tilde{N}_0) = \frac{(r(t)\Delta t_c)^{\tilde{N}_0}}{\tilde{N}_0!} \cdot e^{-r(t)\Delta t_c},$$

with variance $\sigma \left[\tilde{N}_0 \right]^2(t) = N_0(t)$.

We are interested in the expectation value $N_0(t) = \mathbb{E} \left[\tilde{N}_0(t) \right]$, and its standard deviation $\sigma \left[N_0 \right](t)$. Denoting the count event measurement time Δt_ϵ , polarimetry measurement time Δt_c , and the number of events per measurement $n_{c/\epsilon} = \Delta t_\epsilon / \Delta t_c$, the expectation is

$$\mathbb{E} \left[\tilde{N}_0(t) \right]_{\Delta t_\epsilon} = \frac{1}{n_{c/\epsilon}} \sum_{i=1}^{n_{c/\epsilon}} \tilde{N}_0(t_i).$$

Being a sum of random variables, $N_0(t)$ is normally distributed; hence the standard error of the mean is

$$\begin{aligned} \sigma \left[N_0 \right](t) &= \sigma \left[\tilde{N}_0 \right](t) / \sqrt{n_{c/\epsilon}} = \sqrt{N_0(t) \frac{\Delta t_c}{\Delta t_\epsilon}} \\ &\approx \sqrt{\frac{p \cdot \nu N_0^b}{\Delta t_\epsilon}} \cdot \Delta t_c \cdot \exp \left(\frac{\lambda_b}{2} \cdot t \right). \end{aligned}$$

Note that relative error grows with time:

$$\frac{\sigma[N_0](t)}{N_0(t)} \approx \frac{A}{\sqrt{\Delta t_\epsilon}} \cdot \exp\left(-\frac{\lambda_b}{2}t\right) = \frac{A}{\sqrt{\Delta t_\epsilon}} \cdot \exp\left(\frac{t}{2\tau_b}\right), \quad A = \frac{1}{\sqrt{p \cdot \nu N_0^b}}. \quad (3)$$

2 Cross-section asymmetry

The relative asymmetry of detector count rates is used as a measure of the beam polarization. [?, p. 17] Cross-section asymmetry is defined as the normalized difference of the number of counts in the detectors placed on opposite sides of the vacuum tube:

$$\mathcal{A} = \frac{N(\frac{\pi}{2}) - N(-\frac{\pi}{2})}{N(\frac{\pi}{2}) + N(-\frac{\pi}{2})}. \quad (4)$$

Due to the decreasing beam size, the measurement of the figure of merit is heteroscedastic. From [?, p. 18], the assumed heteroscedasticity model is

$$\sigma[\mathcal{A}]^2(t) \approx \frac{1}{2N_0(t)}. \quad (5)$$

3 Measurement time frame

Assuming a Gaussian error distribution with zero mean and $\sigma[\epsilon]^2$ variance, the maximum likelihood estimator for the variance of the frequency estimate of the cross-section asymmetry \mathcal{A} can be expressed as

$$\text{var}[\hat{\omega}] = \frac{\sigma[\epsilon]^2}{X_{tot} \cdot \text{var}_w[t]}, \quad (6)$$

with

$$\begin{aligned} X_{tot} &= \sum_{j=1}^{n_\epsilon} x_j = \sum_{s=1}^{n_{zc}} \sum_{j=1}^{n_{\epsilon/zc}} x_{js}, \\ \text{var}_w[t] &= \sum_i w_i (t_i - \langle t \rangle_w)^2, \quad \langle t \rangle_w = \sum_i w_i t_i, \\ w_i &= \frac{x_i}{\sum_j x_j}, \quad x_i = (\mathcal{A}(0) \exp(\lambda_d t_i))^2 \cos^2(\omega t_i + \phi) = (\mu'_\phi(t_i))^2. \end{aligned}$$

In the expressions above, X_{tot} is the total Fisher information of the sample, and $\text{var}_w[t]$ is a measure of its time-spread. It can be observed that by picking appropriate sampling times, one can raise the X_{tot} term, since it is proportional to a sum of the signal's time derivatives. If the oscillation frequency and phase are already known to a reasonable precision, further improvement can be achieved by the application of a sampling scheme in which measurements are taken only during rapid change in the signal (sampling modulation).

Both the $\text{var}_w[t]$ and X_{tot} terms are bounded as a result of spin tune decoherence. We can express $\sum_{j=1}^{n_{\epsilon/zc}} x_{js} = n_{\epsilon/zc} \cdot x_{0s}$, for some mean value x_{0s} at a

given node s , where $n_{\epsilon/zc}$ is the number of asymmetry measurements per node. We will call *compaction time* (denoted Δt_{zc}) the period of time during which polarimetry measurements are performed. The value of the sum $\sum_{j=1}^{n_{\epsilon/zc}} x_{js}$ falls exponentially due to decoherence, hence $x_{0s} = x_{01} \exp(\lambda_d \cdot \frac{(s-1) \cdot \pi}{\omega})$. Therefore:

$$X_{tot} = n_{\epsilon/zc} \cdot x_{01} \cdot \frac{\exp(\frac{\lambda_d \pi}{\omega} n_{zc}) - 1}{\exp(\frac{\lambda_d \pi}{\omega}) - 1} \equiv n_{\epsilon/zc} \cdot x_{01} \cdot g(n_{zc}); \quad (7)$$

$$x_{01} = \frac{1}{\Delta t_{zc}} \int_{-\Delta t_{zc}/2}^{+\Delta t_{zc}/2} \cos^2(\omega \cdot t) dt = \frac{1}{2} \cdot \left(1 + \frac{\sin \omega \Delta t_{zc}}{\omega \Delta t_{zc}} \right), \quad (8)$$

$$n_{\epsilon/zc} = \frac{\Delta t_{zc}}{\Delta t_{\epsilon}}. \quad (9)$$

Equation (7) can be used to estimate the limits on the duration of the experiment. In Table 1 we summarize the percentage of the total Fisher information limit, the time (in decoherence lifetimes) by which it is reached, and the corresponding signal-to-noise ratio. The signal-to-noise ratios are computed according to:

$$\text{SNR} = \frac{\mathcal{A}(0) \cdot e^{-t/\tau_d}}{\sigma[\mathcal{A}](t)} \approx \sqrt{2 \cdot p \cdot \nu N_0^b \cdot \Delta t_c} \cdot \mathcal{A}(0) \cdot \exp \left[-\frac{t}{\tau_d} \cdot \left(1 + \frac{1}{2} \frac{\tau_d}{\tau_b} \right) \right], \quad (10)$$

in which, assuming $\sigma[\mathcal{A}(0)]/\mathcal{A}(0) \approx 3\%$ (polarimetry measurement precision), the factor before the exponent is 33.

Table 1: Total Fisher information, its time of reaching, and the corresponding signal-to-noise ratio.

FI limit (%)	Reached ($\times \tau_d$)	SNR
95	3.0	0.4
90	2.3	1.1
70	1.2	5.5
50	0.7	11.7

Under the assumption of no decoherence ($\lambda_d = 0$) and uniform sampling rate $1/\Delta t$, eq (6) can be rewritten in physical terms as

$$\begin{aligned} X_{tot} &= \sum_{k=1}^K \mathcal{A}^2(0) \cos^2(\omega t_k + \phi) = \frac{1}{2} \mathcal{A}^2(0) \cdot K, \\ \text{var}_w[t] &= \sum_{k=1}^K (k\Delta t - \langle t \rangle_w)^2 \underbrace{w_k}_{1/K} \\ &\approx \frac{\Delta t^2}{12} K^2 = \frac{T^2}{12}, \end{aligned}$$

and so

$$\text{var} [\hat{\omega}] = \frac{24}{KT^2} \cdot \left(\frac{\sigma[\epsilon]}{\mathcal{A}(0)} \right)^2.$$

4 Simulation

We simulated data from two detectors with parameters gathered in Table 2 for $T_{tot} = 1000$ seconds, sampled uniformly at the rate $f_s = 375$ Hz.

These figures are chosen for the following reason: the beam size in a fill is on the order of 10^{11} particles; if we want to keep the beam lifetime equal to the decoherence lifetime, we cannot exhaust more than 75% of it; only 1% of all scatterings are of the sort we need for polarimetry, so we're left with $7.5 \cdot 10^8$ useful scatterings. A measurement of the count rate $N_0(t)$ with a precision of approximately 3% requires somewhere in the neighborhood of 2000 detector counts, which further reduces the number of events to $3.75 \cdot 10^5 = f_s \cdot T_{tot}$. One thousand seconds is the expected duration of a fill, hence $f_s = 375$ Hz.

Relative measurement error for the detector count rates is depicted in Figure 1; the cross-section asymmetry, computed according to eq (4), is shown in Figure 2. The asymmetry data are fitted with a non-linear heteroscedastic model given by

$$\mathcal{A}(t) = \mathcal{A}(0) \cdot e^{\lambda_d \cdot t} \cdot \sin(\omega \cdot t + \phi),$$

with the variance function for the weights given by eq (5). The fit results are summarized in Table 3.

Table 2: Count rate model parameters

	Left	Right	
ϕ	$-\pi/2$	$+\pi/2$	rad
ω	3		rad/sec
P	0.4		
τ_d	721		sec
τ_b	721		sec
$N_0(0)$	6730		

Table 3: Fit results

	Estimate	SE	Unit
$\mathcal{A}(0)$	0.400	$9.03 \cdot 10^{-5}$	
λ_d	-0.001	$7.86 \cdot 10^{-7}$	1/sec
ω	3.000	$7.55 \cdot 10^{-7}$	rad/sec
ϕ	-1.571	$2.25 \cdot 10^{-2}$	rad

If the initial frequency estimate obtained from a time-uniform sample has a standard error on the order of 10^{-6} rad/sec, simulation shows that the standard error of the estimate can be improved to approximately $5.8 \cdot 10^{-7}$ rad/sec.

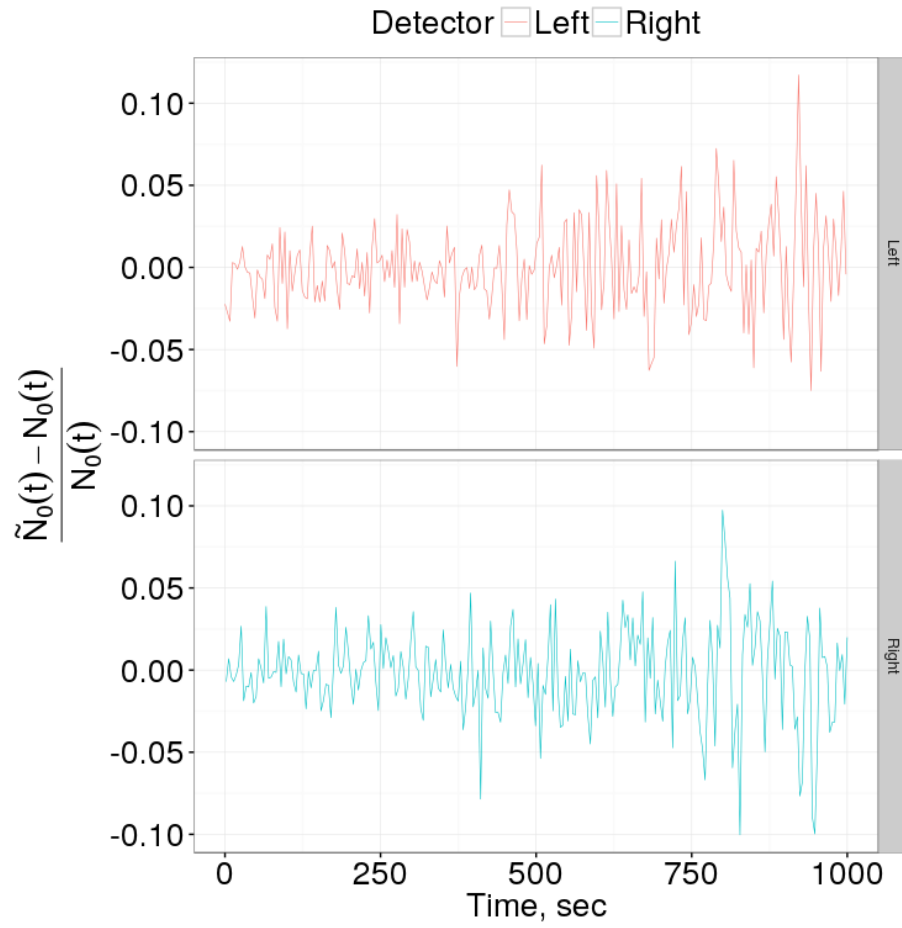


Figure 1: Relative count rate measurement error for the left and right detectors as a function of time.

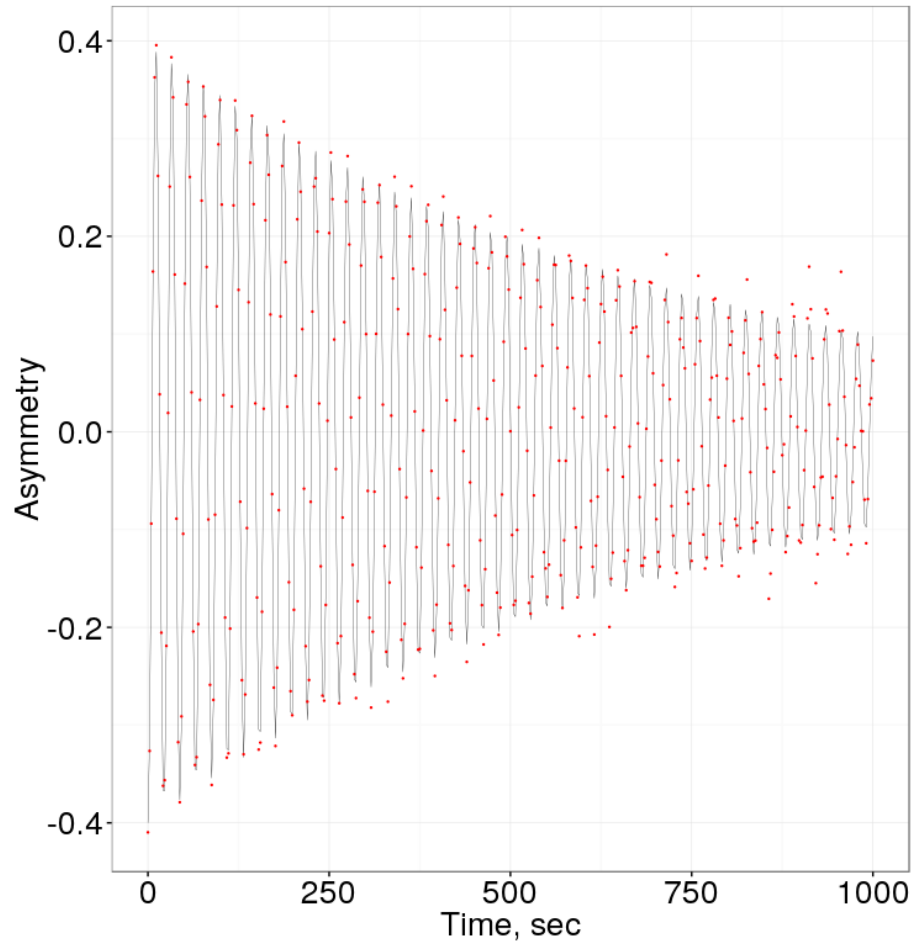


Figure 2: Expectation value (black line) and sample measurements (red dots) of the cross-section asymmetry.