Problem

- Current formula [1]

$$\sigma_{\omega}^{2}=\frac{24}{K(PT)^{2}}$$

- Alternative formula

$$\sigma_{\hat{\omega}}^{2}=\frac{24}{KT^{2}}\cdot\left(\frac{\sigma_{\epsilon}}{N_{0}P}\right)^{2}$$

K –- number of measurements
P --- polarization
T --- cycle duration
N_0 --- unpolarized count rate
σ_ϵ--- measurement RMS

1. Joerg Pretz. Statistical uncertainties of frequency measurements [Internet]. 2014 [cited 2017 Jul 20]

Derivation: log-likelihood

- Current formula
- Alternative formula

$$N(t) = N_0(1 + P\sin(\omega t + \varphi))$$

Expectation value (structural model)

$$\mu(t_i) = N_0(1 + P\sin(\omega t_i + \varphi))$$

Probability distribution (probabilistic model)

$$f(y_i;\theta) = \frac{1}{\sqrt{2\pi\nu}}\exp\left(-\frac{1}{2}\frac{(y_i - \mu(t_i))^2}{\nu}\right)$$

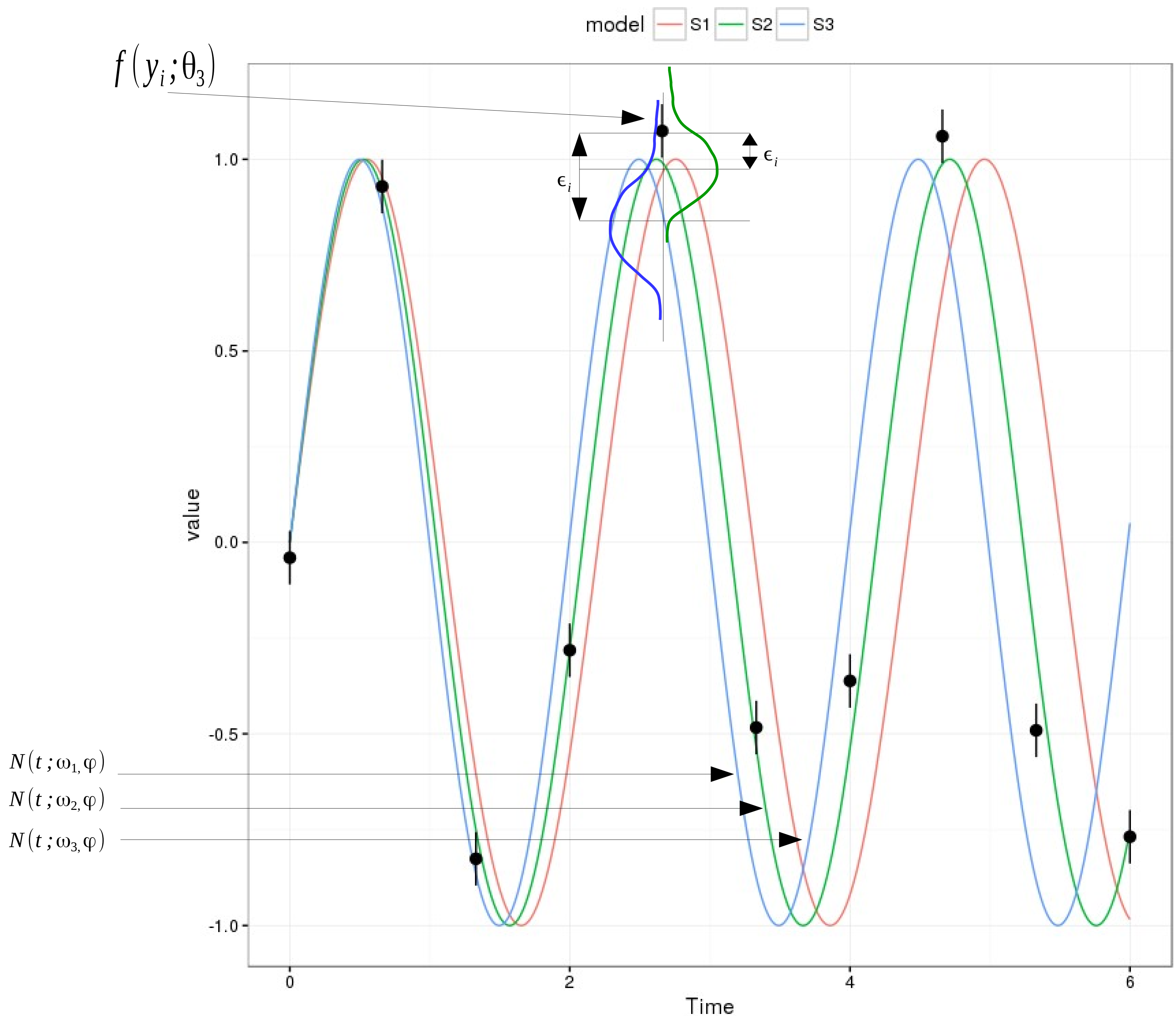
$$\theta \equiv (\nu, \omega, \varphi)$$

$$\log L = \sum_i \log(N_0(1 + P\sin(\omega t_i + \varphi)))$$

$$\log L(\theta;\vec{y}) = \prod_i f(y_i;\theta) = -\frac{K}{2}\log 2\pi - \frac{K}{2}\log\nu - \frac{1}{2\nu}\sum_i \epsilon_i^2$$

Error term

$$\epsilon_i \equiv y_i - \mu(t_i)$$



model
S1
S2
S3
$f(y_i;\theta_3)$
ϵ_i
ϵ_i
$N(t;\omega_1,\varphi)$
$N(t;\omega_2,\varphi)$
$N(t;\omega_3,\varphi)$
value
1.0
0.5
0.0
-0.5
-1.0
0
2
4
6
Time

MLE procedure

- First derivatives of log L
- Second derivatives
- Expectations of second derivatives
- The Fisher matrix
- The variance-covariance matrix

The variance-covariance matrix

- Current

$$cov(\omega,\varphi)=\begin{pmatrix}\frac{24}{K(PT)^2} & \frac{12}{KP^2T}\\ \frac{12}{KP^2T} & \frac{8}{KP^2}\end{pmatrix}$$

- Alternative

$$vcov(\hat{\nu},\hat{\omega},\hat{\varphi})=\begin{pmatrix}2\nu/K & 0 & 0\\ 0 & \frac{\nu}{\Omega}\sum_i(\mu'_\varphi(t_i))^2 & \frac{\nu}{\Omega}\sum_i t_i(\mu'_\varphi(t_i))^2\\ 0 & \frac{\nu}{\Omega}\sum_i t_i(\mu'_\varphi(t_i))^2 & \frac{\nu}{\Omega}\sum_i(t_i\mu'_\varphi(t_i))^2\end{pmatrix}$$

$$\Omega=\frac{2\nu^2}{K}|I(\theta_0)|$$

The alternative SE

- Sinusoidal model

$$\left(\mu'_{\varphi}(t_i)\right)^2 = \left(N_0 P\right)^2 \cos^2\left(\omega t_i + \varphi\right) \equiv x_i$$

$$\sigma^2_{\hat{\omega}} = \frac{\nu}{\sum_j x_j \sum_i w_i \left(t_i - \langle t \rangle_w\right)^2}$$

$$w_i = \frac{x_i}{\sum_j x_j}$$

- Linear regression cross-check

$$\mu(t_i) = \omega t_i + \varphi$$

$$\sigma^2_{\hat{\omega}} = \frac{\nu}{\sum_i \left(t_i - \langle t \rangle\right)^2}$$

Uniform sampling Δt

$$\sum_{j=1}^{K} x_j = \frac{1}{2}\left(N_0 P\right)^2 \cdot K$$

$$\sigma_{\hat{\omega}}^2 = \frac{\nu}{\sum_j x_j \sum_i w_i \left(t_i - \langle t \rangle_w\right)^2}$$

$$\sum_i \left(t_i - \langle t \rangle_w\right)^2 \underbrace{w_i}_{1/K} \approx \frac{\left(\Delta t\, K\right)^2}{12} = \frac{T^2}{12}$$

$$\sigma_{\hat{\omega}}^2 = \frac{24}{K T^2} \cdot \left(\frac{\sigma_\epsilon}{N_0 P}\right)^2$$