Preliminary

The beam current falls according to the B-L law:

$$I(t) \equiv N^b(t)\nu = I_0 \cdot e^{\lambda_b t}.$$

A fraction p of the beam particles will be scattered in the direction of the detector; in time Δt_ϵ, the number of particles collected at the detector will be

$$\begin{aligned} N_0(t) &= p \cdot \int_{-\Delta t_c/2}^{+\Delta t_c/2} I(t+\tau)\mathrm{d}\tau \\ &= p \cdot \frac{\nu N_0^b}{\lambda_b} e^{\lambda_b t} \cdot \left(e^{\lambda_b \Delta t_c/2} - e^{-\lambda_b \Delta t_c/2}\right) \\ &\approx \underbrace{p \cdot \nu N_0^b e^{\lambda_b t}}_{\text{rate } r(t)} \cdot \Delta t_c. \, ^1 \end{aligned} \tag{1}$$

This is on average. To model the actual number, the Poisson distribution will be appropriate:

$$P_{N_0(t)}(\tilde{N}_0) = \frac{(r(t)\Delta t_c)^{\tilde{N}_0}}{\tilde{N}_0!} \cdot e^{-r(t)\Delta t_c}.$$

The variance is equal to the mean: $\sigma^2_{\tilde{N}_0}(t) = N_0(t)$. In the limit of large $N_0(t)$, the Poisson distribution becomes Gaussian.

$\tilde{N}_0$ is what we actually measure in Δt_c. To estimate the expectation $N_0(t)$ (and its variance), we have to take the mean (variance) of the Gaussian. For that we take several measurements, and estimate the parameters as usual, i.e. as sums of random variables, i.e. distributed normally. The number of measurements we can make during Δt_ϵ is $n_{c/\epsilon} = \Delta t_\epsilon / \Delta t_c$. The standard error of the mean then is

$$\begin{aligned} \sigma_{N_0}(t) &= \sigma_{\tilde{N}_0}(t)/\sqrt{n_{c/\epsilon}} = \sqrt{N_0(t)\frac{\Delta t_c}{\Delta t_\epsilon}} \\ &\approx \sqrt{\frac{p \cdot \nu N_0^b}{\Delta t_\epsilon}} \cdot \Delta t_c \cdot \exp\left(\frac{\lambda_b}{2} \cdot t\right). \end{aligned} \tag{2}$$

Relative error,

$$\frac{\sigma_{N_0}(t)}{N_0(t)} \approx \frac{A}{\sqrt{\Delta t_\epsilon}} \cdot \exp\left(-\frac{\lambda_b}{2} t\right) = \frac{A}{\sqrt{\Delta t_\epsilon}} \cdot \exp\left(\frac{t}{2\tau_b}\right), \; A = \frac{1}{\sqrt{p \cdot \nu N_0^b}},$$

grows.

1 Problem statement

Define the following variables: *a*) the number of measurements per node: $n_{\epsilon/zc}$, *b*) the number of nodes per experiment: n_{zc}.

We have to fit the function

$$N(t) = N_0(t) \cdot \left(1 + P \cdot e^{-t/\tau_d} \cdot \sin(\omega \cdot t + \phi)\right), \tag{3}$$

given $n_\epsilon = n_{zc} \cdot n_{\epsilon/zc}$ sample points.

Assuming the Gaussian error distribution with mean zero and variance $\nu \equiv \sigma^2_\epsilon = \sigma^2_{N_0}(t)$, the maximum likelihood estimator for the variance of the frequency estimate can be expressed as

$$\mathrm{var}\left[\hat{\omega}\right] = \frac{\nu}{X_{tot} \cdot \mathrm{var}_w\left[t\right]},$$

[1]Average number of events in a time interval is rate times interval: [1].

with

$$X_{tot} = \sum_{j=1}^{n_\epsilon} x_j = \sum_{s=1}^{n_{zc}} \sum_{j=1}^{n_{\epsilon/zc}} x_{js},$$

$$\mathrm{var}_w\,[t] = \sum_i w_i\,(t_i - \langle t\rangle_w)^2\,,\ \langle t\rangle_w = \sum_i w_i t_i,$$

$$w_i = \frac{x_i}{\sum_j x_j},\ x_i = (N_0 P \exp(\lambda t_i))^2 \cos^2(\omega t_i + \phi) = \left(\mu'_\phi(t_i)\right)^2.$$

The three factors, contributing to the standard error of the estimate are: *a*) the error variance $\nu = \sigma_{N_0}(t)^2$ (governed by the number $n_{c/\epsilon}$ of polarimetry measurements per signal measurement, eq. (2)), *b*) the time spread $\sum_i w_i(t_i - \langle t\rangle_w)^2$ of the sample measurements, and *c*) their net informational content X_{tot}.

2 Informational content

We can express $\sum_{j=1}^{n_{\epsilon/zc}} x_{js} = n_{\epsilon/zc} \cdot x_{0s}$, for some mean value x_{0s} in the given node s. The sum $\sum_{j=1}^{n_{\epsilon/zc}} x_{js}$ falls exponentially due to decoherence, hence $x_{0s} = x_{01} \exp\left(\lambda \cdot \frac{(s-1)\cdot\pi}{\omega}\right)$. Therefore,

$$X_{tot} = n_{\epsilon/zc} \cdot x_{01} \cdot \frac{\exp\left(\frac{\lambda\pi}{\omega} n_{zc}\right) - 1}{\exp\left(\frac{\lambda\pi}{\omega}\right) - 1} \equiv n_{\epsilon/zc} \cdot x_{01} \cdot g(n_{zc}); \tag{4}$$

$$x_{01} = \frac{1}{\Delta t_{zc}} \int_{-\Delta t_{zc}/2}^{+\Delta t/2} \cos^2(\omega \cdot t)\mathrm{d}t = \frac{1}{2} \cdot \left(1 + \frac{\sin \omega \Delta t_{zc}}{\omega \Delta t_{zc}}\right), \tag{5}$$

$$n_{\epsilon/zc} = \frac{\Delta t_{zc}}{\Delta t_\epsilon}. \tag{6}$$

Eq. (4) can be used to estimate the reasonable experiment duration. $g(n_{zc})$ is a limited function; in Figure 1 it is shown for different life-times. In Table 1 we have the time (in signal life-times) required to reach the different levels of information, $t(z)/\tau = -\ln(1-z)$.

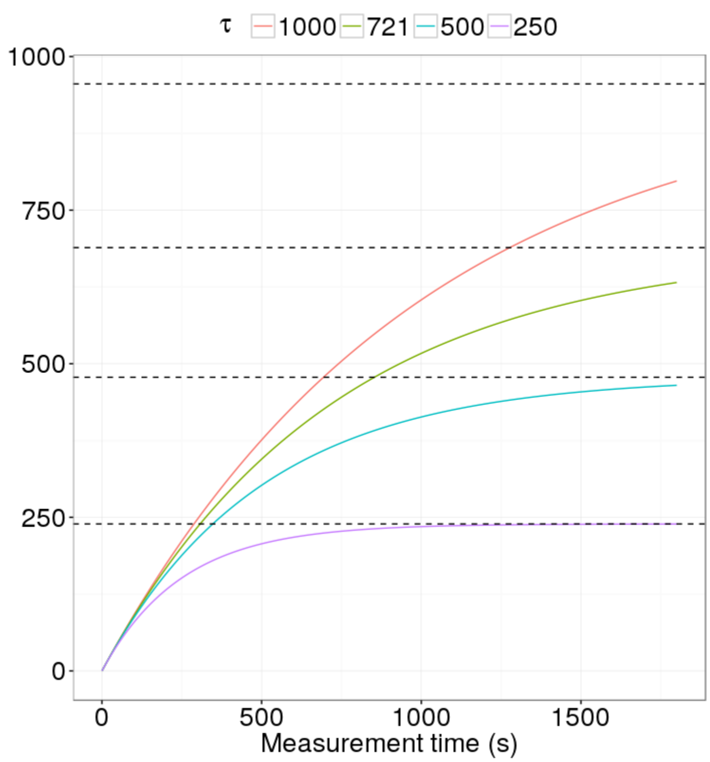

Figure 1: $[g \circ t](n_{zc})$ for different life-times $\tau = {}^1/\lambda$.

Table 1: Total Fisher information table.

FI limit (%)	Reached ($\times\tau$)	SNR
95	3.0	1.0
90	2.3	1.9
70	1.2	4.7
50	0.7	7.1

Eq. (5), together with eq. (6) and

$$X_{tot} = \frac{\sigma_\epsilon^2}{\mathrm{SE}\left[\hat{\omega}\right]^2 \cdot \mathrm{var}_w\left[t\right]},$$

produce an equation by which the required compaction time Δt_{zc} can be found:

$$\Delta t_{zc} + \frac{\sin \omega \Delta t_{zc}}{\omega} - 2 \cdot X_{tot} \cdot \Delta t_\epsilon \cdot g(n_{zc})^{-1} = 0. \tag{7}$$

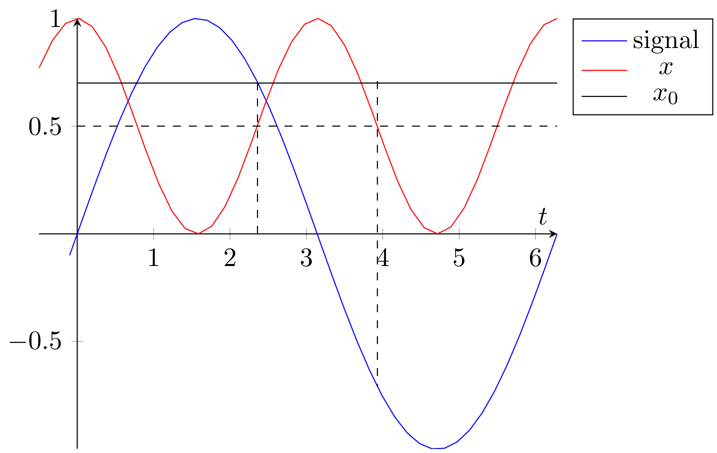

Figure 2: Explanation for x_0.

The signal-to-noise ratios in Table 2 are computed as

$$\mathrm{SNR} \triangleq \frac{N_0(t) \cdot P \cdot e^{-t/\tau_d}}{\sigma_{N_0}(t)} = \frac{P}{A}\sqrt{\Delta t_\epsilon} \cdot \exp\left(\frac{1-2x}{2x} \cdot \frac{t}{\tau_d}\right), \ x \equiv \frac{\tau_b}{\tau_d}. \tag{8}$$

3 Beam/decoherence life-times

Our want is to have the sample as spread out as possible; however, measurement time is limited by the signal life-time, which is

$$\tau = \left({}^1/_{\tau_b} + {}^1/_{\tau_d}\right)^{-1} = \frac{x}{1+x} \cdot \tau_d, \ x \equiv \frac{\tau_b}{\tau_d}.$$

By increasing x we raise the signal life-time, and hence the limit on $\mathrm{var}_w\left[t\right]$, but simultaneously the rate at which falls the SNR.

References

[1] http://www.owlnet.rice.edu/~dodds/Files331/stat_notes.pdf.