

The questions are:

1. How long to measure the signal?
2. How many measurements per event are optimal?
3. How congregated about the zero-crossings the measurements should be?

The analytical expression of the variance of the estimate is¹

$$\begin{cases} \text{var} [\hat{\omega}] &= \nu / \sum_j x_j \text{var}_w [t], \\ \text{var}_w [t] &= \sum_i w_i (t_i - \langle t \rangle_w)^2, \quad \langle t \rangle_w = \sum_i t_i w_i, \\ w_i &= \frac{x_i}{\sum_j x_j}, \quad x_i = (N_0 P \exp(\lambda t_i))^2 \cos^2(\omega t_i + \phi) = \left(\mu'_\phi(t_i) \right)^2. \end{cases} \quad (1)$$

As one can see, the variance inversely depends on the spread of the predictor variable. This spread is determined mainly by the total duration of measurement, and hence the spin tune decoherence time. Variation in the number of polarimetry measurements per event has only a second-order effect on $\text{var}_w [t]$.

The two questions that follow both condition the measurement distribution across time. The number of measurements per event determine, with regard to this statistical aspect, the number of events per second, and hence the number of events that can fit within a compaction region. If that number is large, there will be fewer (but more precise) measurements within the high-information regions, and we will have to increase the compaction factor in order to collect enough events, at which point our use of the beam is sub-optimal.

1 Number of measurements per observation (event)

Define the following variables: a) the number of measurements per observation: $n_{m/\epsilon}$; b) the number of observations per zero-crossing: $n_{\epsilon/zc}$; c) the number of zero-crossings per experiment: n_{zc} .

The expected total number of scatterings in an experiment with a given number of zero-crossings: $n_m = \underbrace{n_{zc} \cdot n_{\epsilon/zc} \cdot n_{m/\epsilon}}_{n_\epsilon}$. (n_ϵ is the total number of observations.)

$$\begin{cases} \text{SE} [\hat{\omega}]^2 &= \frac{\text{SE}[\epsilon]^2}{X_{tot} \cdot \sum_{j=1}^{n_\epsilon} w_j (t_j - \langle t \rangle_w)^2}, \\ \text{SE} [\epsilon]^2 &= \frac{\text{SE}[m]^2}{n_{m/\epsilon}}, \\ X_{tot} &= \sum_{j=1}^{n_\epsilon} x_j = \sum_{s=1}^{n_{zc}} \sum_{j=1}^{n_{\epsilon/zc}} x_{js}. \end{cases} \quad (2)$$

We can express $\sum_{j=1}^{n_{\epsilon/zc}} x_{js} = n_{\epsilon/zc} \cdot x_{0s}$, for some mean value x_{0s} in the given zero-crossing s . The sum $\sum_{j=1}^{n_{\epsilon/zc}} x_{js}$ falls exponentially due to decoherence, hence $x_{0s} = x_{01} \exp(\lambda \cdot \frac{(s-1) \cdot \pi - \phi}{\omega})$. Therefore,

$$X_{tot} = n_{\epsilon/zc} \cdot x_{01} \cdot \frac{\exp\left(\frac{\lambda \phi}{\omega}\right) \cdot [\exp\left(\frac{\lambda \pi}{\omega} n_{zc}\right) - 1]}{\exp\left(\frac{\lambda \pi}{\omega}\right) - 1} \equiv n_{\epsilon/zc} \cdot g(n_{zc}).$$

The number of events per zero-crossing

$$n_{\epsilon/zc} = \frac{\Delta t_{zc}}{n_{m/\epsilon} \cdot \Delta t_m},$$

hence

$$X_{tot} = g(n_{zc}) \cdot \frac{\Delta t_{zc}}{\Delta t_m} \cdot \frac{1}{n_{m/\epsilon}}.$$

The variance $\text{var}_w [t]$ is also practically independent of how many measurements there are per one zero-crossing (and by extension, the number $n_{m/\epsilon}$), and depends primarily on the n_{zc} and decoherence life time.

¹This expression consistently overestimates the standard error by a factor of 2; however, when used on a function with $x_i = (N_0 P)^2$ (linear regression), it yields the same expression as the fitter. The fitter also computes the standard error according to some formula. As far as I can tell, the difference between this one and the one the fitter uses (which I haven't seen yet), is the $\sum_j x_j$ factor in the denominator.

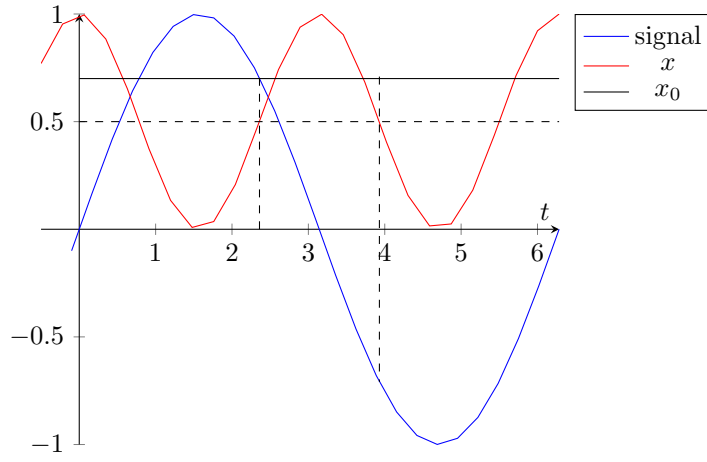


Figure 1: Explanation for x_0

In sum, assuming one observation is the mean of the measurements ($\text{SE}[\epsilon] = \text{SE}[m] / \sqrt{n_{m/\epsilon}}$),

$$\begin{aligned} \text{var}_w[\hat{\omega}] &= \frac{\text{SE}[m]^2 \cdot 1/n_{m/\epsilon}}{g(n_{zc}) \cdot \frac{\Delta t_{zc}}{\Delta t_m} \cdot 1/n_{m/\epsilon} \cdot \text{var}_w[t]} \\ &= \frac{\text{SE}[m]^2}{g(n_{zc}) \cdot \frac{\Delta t_{zc}}{\Delta t_m} \cdot \text{var}_w[t]} \end{aligned}$$

2 Spin tune decoherence time

We have the Cramér-Rao inequality to tell us what's the minimum variance of an estimator is possible:

$$\text{var}[\hat{\omega}] \geq \frac{1}{I(\omega)}.$$

Fisher information is additive, and what I call *point Fisher information* can be expressed as:

$$I_i(\theta_0) = \frac{1}{\nu} \begin{pmatrix} (\sqrt{2} \cdot \mu'_\phi(t_i))^{-2} & 0 & 0 \\ 0 & t_i^2 & t_i \\ 0 & t_i & 1 \end{pmatrix} \cdot (\mu'_\phi(t_i))^2.$$

