

The probability of observing the value $y_i \equiv y(t_i)$ when the expectation value is $\mu(t_i)$ and the error is gaussian is

$$f(y_i|\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\nu}} \exp\left(-\frac{1}{2} \frac{(y_i - \mu(t_i))^2}{\nu}\right),$$

$$\boldsymbol{\theta} = (\nu, \omega, \phi),$$

$$\mu(t_i) = N_0 (1 + P \sin(\omega t_i + \phi)).$$

The likelihood of observing a set of observations $\mathbf{y} = (y_1, \dots, y_K)$, under the i.i.d. assumption, is the product of propabilities taken as a function of the parameters:

$$\mathcal{L}(\boldsymbol{\theta}|\mathbf{y}) = \prod_i f(y_i|\boldsymbol{\theta}),$$

and the log-likelihood

$$\ell(\boldsymbol{\theta}|\mathbf{y}) = -\frac{K}{2} \log 2\pi - \frac{K}{2} \log \nu - \frac{1}{2\nu} \sum_i \epsilon_i^2, \quad \epsilon_i = y_i - \mu(t_i).$$

The usual assumptions for the error term are zero expectation and strict exogeneity

$$\mathbb{E}[\epsilon_i | \boldsymbol{\theta}_0] = \mathbb{E}[t_i \epsilon_i | \boldsymbol{\theta}_0] = 0,$$

and the relations between the mean's derivatives are

$$\mu'_\phi = N_0 P \cos(\omega t + \phi),$$

$$\mu'_\omega = t \cdot \mu'_\phi, \quad \epsilon'_\xi = -\mu'_\xi.$$

The log-likelihood derivatives:

$$\begin{aligned} \ell'_\nu &= -\frac{K}{2\nu} + \frac{1}{2\nu^2} \sum_i \epsilon_i^2; & -\mathbb{E}[\ell''_{\nu^2} | \boldsymbol{\theta}_0] &= \frac{K}{2\nu^2} - \frac{1}{\nu^3} \sum_i \nu = \frac{K}{2\nu^2}; \\ \ell'_\omega &= \frac{1}{\nu} \sum_i \mu'_\phi(t_i) t_i \epsilon_i; & -\mathbb{E}[\ell''_{\nu\omega} | \boldsymbol{\theta}_0] &= \frac{1}{\nu^2} \sum_i \mu'_\phi(t_i) \mathbb{E}[t_i \epsilon_i | \boldsymbol{\theta}_0] = 0; \\ \ell'_\phi &= \frac{1}{\nu} \sum_i \mu'_\phi(t_i) \epsilon_i; & -\mathbb{E}[\ell''_{\nu\phi} | \boldsymbol{\theta}_0] &= \frac{1}{\nu^2} \sum_i \mu'_\phi(t_i) \mathbb{E}[\epsilon_i | \boldsymbol{\theta}_0] = 0; \\ \ell''_{\nu^2} &= \frac{K}{2\nu^2} - \frac{1}{\nu^3} \sum_i \epsilon_i^2, & -\mathbb{E}[\ell''_{\phi^2} | \boldsymbol{\theta}_0] &= \frac{1}{\nu} \sum_i \left((\mu'_\phi(t_i))^2 - \mu''_{\phi^2}(t_i) \mathbb{E}[\epsilon_i | \boldsymbol{\theta}_0] \right) = \frac{1}{\nu} \sum_i (\mu'_\phi(t_i))^2; \\ \ell''_{\nu\omega} &= -\frac{1}{\nu^2} \sum_i \mu'_\phi(t_i) t_i \epsilon_i, & -\mathbb{E}[\ell''_{\phi\omega} | \boldsymbol{\theta}_0] &= \frac{1}{\nu} \sum_i \left(t_i (\mu'_\phi(t_i))^2 - \mu''_{\phi^2}(t_i) \mathbb{E}[t_i \epsilon_i | \boldsymbol{\theta}_0] \right) = \frac{1}{\nu} \sum_i t_i (\mu'_\phi(t_i))^2; \\ \ell''_{\nu\phi} &= -\frac{1}{\nu^2} \sum_i \mu'_\phi(t_i) \epsilon_i, & -\mathbb{E}[\ell''_{\omega^2} | \boldsymbol{\theta}_0] &= \frac{1}{\nu} \sum_i \left((t_i \mu'_\phi(t_i))^2 - \mu''_{\phi^2}(t_i) \mathbb{E}[t_i^2 \epsilon_i | \boldsymbol{\theta}_0] \right) = \frac{1}{\nu} \sum_i (t_i \mu'_\phi(t_i))^2. \\ \ell''_{\phi^2} &= \frac{1}{\nu} \sum_i \left(\mu''_{\phi^2}(t_i) \epsilon_i - (\mu'_\phi(t_i))^2 \right), \\ \ell''_{\phi\omega} &= \frac{1}{\nu} \sum_i \left(\mu''_{\phi^2}(t_i) t_i \epsilon_i - (\mu'_\phi(t_i))^2 t_i \right), \\ \ell''_{\omega^2} &= \frac{1}{\nu} \sum_i \left(\mu''_{\phi^2}(t_i) t_i^2 \epsilon_i - (\mu'_\phi(t_i))^2 t_i \right), \end{aligned}$$

1 Variances

The Fisher matrix

$$I(\boldsymbol{\theta}_0) = \begin{pmatrix} K/2\nu & 0 & 0 \\ 0 & 1/\nu \sum_i (t_i \mu'_\phi(t_i))^2 & 1/\nu \sum_i t_i (\mu'_\phi(t_i))^2 \\ 0 & 1/\nu \sum_i t_i (\mu'_\phi(t_i))^2 & 1/\nu \sum_i (\mu'_\phi(t_i))^2 \end{pmatrix}.$$

The determinant

$$|I(\boldsymbol{\theta}_0)| = \frac{K}{2\nu^4} \underbrace{\left(\sum (t_i \mu'_\phi(t_i))^2 \sum (\mu'_\phi(t_i))^2 - \left(\sum t_i (\mu'_\phi(t_i))^2 \right)^2 \right)}_{\Omega}.$$

The variance-covariance matrix

$$vcov = \begin{pmatrix} 2\nu^2/K & 0 & 0 \\ 0 & \nu \frac{\sum (\mu'_\phi(t_i))^2}{\Omega} & \nu \frac{\sum t_i (\mu'_\phi(t_i))^2}{\Omega} \\ 0 & \nu \frac{\sum t_i (\mu'_\phi(t_i))^2}{\Omega} & \nu \frac{\sum (t_i \mu'_\phi(t_i))^2}{\Omega} \end{pmatrix}.$$

The variance of the frequency estimate

$$\text{var} [\hat{\omega}] = \nu \frac{\sum (\mu'_\phi(t_i))^2}{\sum (t_i \mu'_\phi(t_i))^2 \sum (\mu'_\phi(t_i))^2 - \left(\sum t_i (\mu'_\phi(t_i))^2 \right)^2}. \quad (1)$$

Cross-check. Let $\mu(t_i) = \phi + \omega t_i$. In that case $\mu'_\phi(t_i) = 1$, $\mu'_\omega(t_i) = t_i = t_i \cdot \mu'_\phi(t_i)$, the determinant of the Fisher matrix simplifies to

$$\begin{aligned} |I(\boldsymbol{\theta}_0)| &= \frac{K}{2\nu^4} \left(K \sum_i t_i^2 - \left(\sum t_i \right)^2 \right) \\ &= \frac{K^3}{2\nu^4} \left(\frac{1}{K} \sum t_i^2 - \langle t \rangle^2 \right) \\ &= \frac{K}{2\nu^4} \cdot \underbrace{K \sum (t_i - \langle t \rangle)^2}_{\Omega} \end{aligned}$$

and the variance-covariance matrix becomes

$$vcov = \begin{pmatrix} 2\nu^2/K & 0 & 0 \\ 0 & \nu \frac{\sum (t_i - \langle t \rangle)^2}{K} & \nu \frac{\sum t_i (t_i - \langle t \rangle)^2}{K} \\ 0 & \nu \frac{\sum t_i (t_i - \langle t \rangle)^2}{K} & \nu \frac{\sum t_i^2 (t_i - \langle t \rangle)^2}{K} \end{pmatrix},$$

with the well-known expression for the slope variance

$$\text{var} [\hat{\omega}] = \frac{\nu}{\sum (t_i - \langle t \rangle)^2}.$$

Let us denote $\left(\mu'_\phi(t_i) \right)^2 = (N_0 P)^2 \cos^2(\omega t_i + \phi) \equiv x_i$. Eq. (1) can be rewritten in the following form:

$$\begin{aligned} \text{var} [\hat{\omega}] &= \frac{\nu}{\sum_j x_j \left(\sum_i t_i^2 \frac{x_i}{\sum_j x_j} - \left(\sum_i t_i \frac{x_i}{\sum_j x_j} \right)^2 \right)} \\ &= \frac{\nu}{\sum_j x_j \sum_i w_i (t_i - \langle t \rangle_w)^2} \\ &= \frac{\nu}{\sum_j x_j \cdot \text{var}_w [t]}. \end{aligned} \quad (2)$$

In matrix form, the frequency variance is written as

$$\text{var} [\hat{\omega}] = \frac{\nu}{(\mathbf{T}' \mathcal{D}_\mu^2 \mathbf{T}) - (\mathbf{T}' \mathcal{D}_\mu \mathbf{M})^2 / (\mathbf{M}' \mathbf{M})},$$

with

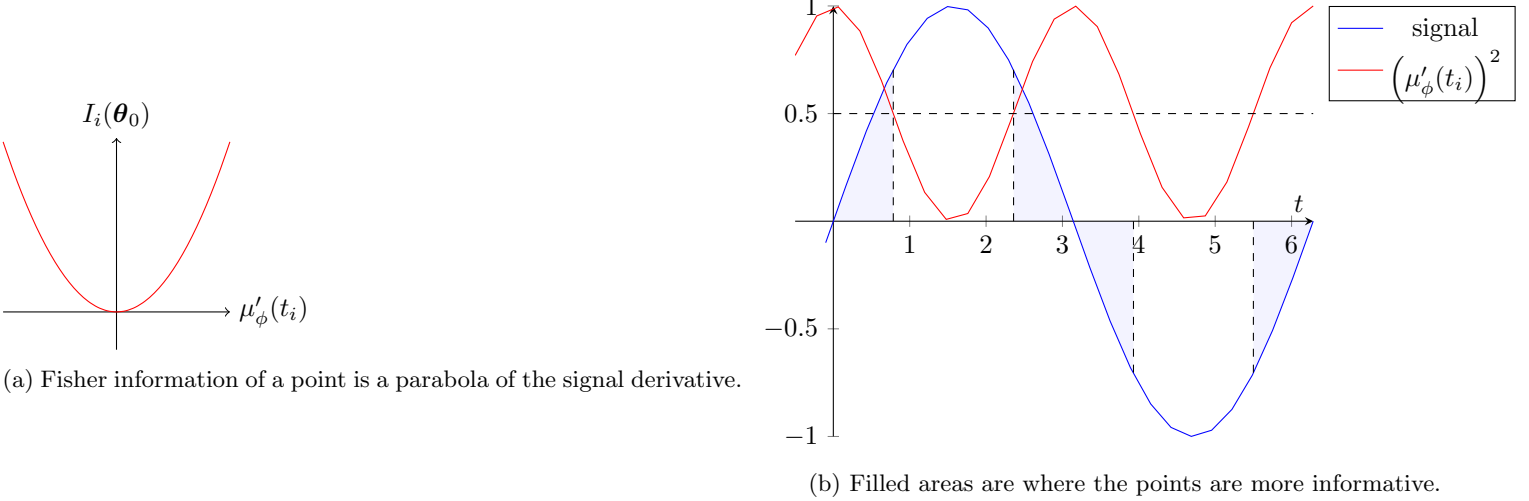
$$\begin{aligned} \mathbf{T} &= (t_0, \dots, t_{K-1})', \quad \mathbf{M} = (\mu'_\phi(t_0), \dots, \mu'_\phi(t_{K-1}))', \\ \mathcal{D}_\mu &= \begin{pmatrix} \mu'_\phi(t_0) & 0 & \cdots & 0 \\ 0 & \mu'_\phi(t_1) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu'_\phi(t_{K-1}) \end{pmatrix}. \end{aligned}$$

2 Sampling modulation

Suppose we write the Fisher matrix as a sum:

$$I(\theta_0) = \sum_i I_i(\theta_0); \quad I_i(\theta_0) = \frac{1}{\nu} \begin{pmatrix} (\sqrt{2} \cdot \mu'_\phi(t_i))^{-2} & 0 & 0 \\ 0 & t_i^2 & t_i \\ 0 & t_i & 1 \end{pmatrix} \cdot (\mu'_\phi(t_i))^2. \quad (3)$$

$I_i(\theta_0) = -\mathbb{E} \left[\frac{\partial^2}{\partial \theta^2} \log f(y_i | \theta) |_{\theta=\theta_0} | \theta_0 \right]$ could be¹ interpreted as the information about the parameter that's carried in y_i .



If we attribute each point a weight proportional to its Fisher information, i.e. $w_i = \cos^2(\omega t_i + \phi)$,² the weight of a region where $(\mu'_\phi(t_i))^2 \geq 1/2$ is greater than that of an equivalent region with $(\mu'_\phi(t_i))^2 < 1/2$ by the factor:

$$\int_{t_0}^{t_1} \cos^2(\omega t + \phi) dt = \frac{1}{\omega} \int_{\omega t_0}^{\omega t_1} \cos^2 \theta d\theta = \frac{\Delta t}{2} + \frac{1}{2\omega} \sin \omega \Delta t \cos \omega \Sigma t \approx 1.9.$$

The implication is that increasing the number of points measured during the signal's rise and fall is roughly twice as beneficial as doing so during the peaks and troughs.

3 The dependence of $\text{var}[\hat{\omega}]$ on ω

The variance in eq. (2) is conditional on the frequency ω and phase ϕ of the signal. The conditioning happens because the parameters determine the sampling range of the distribution of the x_i weights.

That distribution is

$$f(x_i) = \frac{1}{\pi} \cdot \left[\frac{x_i}{N_0 P} \left(1 - \frac{x_i}{N_0 P} \right) \right]^{-1/2}.$$

$$\mathbb{E} \left[\sum_i x_i | (\omega, \phi) \right] = \sum_i \int_{\cos^2 \phi}^{\cos^2(\omega t_i + \phi)} x_i f(x_i) dx_i;$$

¹The t_i in the structural matrix in eq. (3) worries me, because it appears that a point carries more information simply by virtue of it being measured later in time; but as far as I can tell the reason for it is that it is assumed that the point labeled as i is the i -th point in a series, and so a later point is more informative than a point closer to the origin, all other things being equal. And it's nothing new; in linear regression we also want our predictors to be as spread out as possible.

²The variance of ω is proportional to the (2,2)-minor, in which time doesn't figure, only the squared cosine.