

The probability of observing the value $y_i \equiv y(t_i)$ when the expectation value is $\mu(t_i)$ and the error is gaussian is

$$f(y_i|\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\nu}} \exp\left(-\frac{1}{2} \frac{(y_i - \mu(t_i))^2}{\nu}\right),$$

$$\boldsymbol{\theta} = (\nu, \omega, \phi),$$

$$\mu(t_i) = N_0 (1 + P \sin(\omega t_i + \phi)).$$

The likelihood of observing a set of observations $\mathbf{y} = (y_1, \dots, y_K)$, under the i.i.d. assumption, is the product of propabilities taken as a function of the parameters:

$$\mathcal{L}(\boldsymbol{\theta}|\mathbf{y}) = \prod_i f(y_i|\boldsymbol{\theta}),$$

and the log-likelihood

$$\ell(\boldsymbol{\theta}|\mathbf{y}) = -\frac{K}{2} \log 2\pi - \frac{K}{2} \log \nu - \frac{1}{2\nu} \sum_i \epsilon_i^2, \quad \epsilon_i = y_i - \mu(t_i).$$

The usual assumptions for the error term are zero expectation and strict exogeneity

$$\mathbb{E}[\epsilon_i | \boldsymbol{\theta}_0] = \mathbb{E}[t_i \epsilon_i | \boldsymbol{\theta}_0] = 0,$$

and the relations between the mean's derivatives are

$$\mu'_\phi = N_0 P \cos(\omega t + \phi),$$

$$\mu'_\omega = t \cdot \mu'_\phi, \quad \epsilon'_\xi = -\mu'_\xi.$$

The log-likelihood derivatives:

$$\begin{aligned} \ell'_\nu &= -\frac{K}{2\nu} + \frac{1}{2\nu^2} \sum_i \epsilon_i^2; & -\mathbb{E}[\ell''_{\nu^2} | \boldsymbol{\theta}_0] &= \frac{K}{2\nu^2} - \frac{1}{\nu^3} \sum_i \nu = \frac{K}{2\nu^2}; \\ \ell'_\omega &= \frac{1}{\nu} \sum_i \mu'_\phi(t_i) t_i \epsilon_i; & -\mathbb{E}[\ell''_{\nu\omega} | \boldsymbol{\theta}_0] &= \frac{1}{\nu^2} \sum_i \mu'_\phi(t_i) \mathbb{E}[t_i \epsilon_i | \boldsymbol{\theta}_0] = 0; \\ \ell'_\phi &= \frac{1}{\nu} \sum_i \mu'_\phi(t_i) \epsilon_i; & -\mathbb{E}[\ell''_{\nu\phi} | \boldsymbol{\theta}_0] &= \frac{1}{\nu^2} \sum_i \mu'_\phi(t_i) \mathbb{E}[\epsilon_i | \boldsymbol{\theta}_0] = 0; \\ \ell''_{\nu^2} &= \frac{K}{2\nu^2} - \frac{1}{\nu^3} \sum_i \epsilon_i^2, & -\mathbb{E}[\ell''_{\phi^2} | \boldsymbol{\theta}_0] &= \frac{1}{\nu} \sum_i \left((\mu'_\phi(t_i))^2 - \mu''_{\phi^2}(t_i) \mathbb{E}[\epsilon_i | \boldsymbol{\theta}_0] \right) = \frac{1}{\nu} \sum_i (\mu'_\phi(t_i))^2; \\ \ell''_{\nu\omega} &= -\frac{1}{\nu^2} \sum_i \mu'_\phi(t_i) t_i \epsilon_i, & -\mathbb{E}[\ell''_{\phi\omega} | \boldsymbol{\theta}_0] &= \frac{1}{\nu} \sum_i \left(t_i (\mu'_\phi(t_i))^2 - \mu''_{\phi^2}(t_i) \mathbb{E}[t_i \epsilon_i | \boldsymbol{\theta}_0] \right) = \frac{1}{\nu} \sum_i t_i (\mu'_\phi(t_i))^2; \\ \ell''_{\nu\phi} &= -\frac{1}{\nu^2} \sum_i \mu'_\phi(t_i) \epsilon_i, & -\mathbb{E}[\ell''_{\omega^2} | \boldsymbol{\theta}_0] &= \frac{1}{\nu} \sum_i \left((t_i \mu'_\phi(t_i))^2 - \mu''_{\phi^2}(t_i) \mathbb{E}[t_i^2 \epsilon_i | \boldsymbol{\theta}_0] \right) = \frac{1}{\nu} \sum_i (t_i \mu'_\phi(t_i))^2. \\ \ell''_{\phi^2} &= \frac{1}{\nu} \sum_i \left(\mu''_{\phi^2}(t_i) \epsilon_i - (\mu'_\phi(t_i))^2 \right), \\ \ell''_{\phi\omega} &= \frac{1}{\nu} \sum_i \left(\mu''_{\phi^2}(t_i) t_i \epsilon_i - (\mu'_\phi(t_i))^2 t_i \right), \\ \ell''_{\omega^2} &= \frac{1}{\nu} \sum_i \left(\mu''_{\phi^2}(t_i) t_i^2 \epsilon_i - (\mu'_\phi(t_i))^2 t_i \right), \end{aligned}$$

1 Variances

The Fisher matrix

$$I(\boldsymbol{\theta}_0) = \begin{pmatrix} K/2\nu & 0 & 0 \\ 0 & 1/\nu \sum_i (t_i \mu'_\phi(t_i))^2 & 1/\nu \sum_i t_i (\mu'_\phi(t_i))^2 \\ 0 & 1/\nu \sum_i t_i (\mu'_\phi(t_i))^2 & 1/\nu \sum_i (\mu'_\phi(t_i))^2 \end{pmatrix}.$$

The determinant

$$|I(\boldsymbol{\theta}_0)| = \frac{K}{2\nu^4} \underbrace{\left(\sum (t_i \mu'_\phi(t_i))^2 \sum (\mu'_\phi(t_i))^2 - \left(\sum t_i (\mu'_\phi(t_i))^2 \right)^2 \right)}_{\Omega}.$$

The variance-covariance matrix

$$vcov = \begin{pmatrix} 2\nu^2/K & 0 & 0 \\ 0 & \nu \frac{\sum (\mu'_\phi(t_i))^2}{\Omega} & \nu \frac{\sum t_i (\mu'_\phi(t_i))^2}{\Omega} \\ 0 & \nu \frac{\sum t_i (\mu'_\phi(t_i))^2}{\Omega} & \nu \frac{\sum (t_i \mu'_\phi(t_i))^2}{\Omega} \end{pmatrix}.$$

Variance of the frequency estimate

$$var(\hat{\omega}) = \nu \frac{\sum (\mu'_\phi(t_i))^2}{\sum (t_i \mu'_\phi(t_i))^2 \sum (\mu'_\phi(t_i))^2 - \left(\sum t_i (\mu'_\phi(t_i))^2 \right)^2}.$$

Cross-check. Let $\mu(t_i) = \phi + \omega t_i$. In that case $\mu'_\phi(t_i) = 1$, $\mu'_\omega(t_i) = t_i = t_i \cdot \mu'_\phi(t_i)$, the determinant of the Fisher matrix simplifies to

$$\begin{aligned} |I(\boldsymbol{\theta}_0)| &= \frac{K}{2\nu^4} \left(K \sum_i t_i^2 - \left(\sum t_i \right)^2 \right) \\ &= \frac{K^3}{2\nu^4} \left(\frac{1}{K} \sum t_i^2 - \langle t \rangle^2 \right) \\ &= \frac{K}{2\nu^4} \cdot \underbrace{K \sum (t_i - \langle t \rangle)^2}_{\Omega} \end{aligned}$$

and the variance-covariance matrix becomes

$$vcov = \begin{pmatrix} 2\nu^2/K & 0 & 0 \\ 0 & \frac{\nu}{\sum (t_i - \langle t \rangle)^2} & \nu \frac{\sum t_i}{K \sum (t_i - \langle t \rangle)^2} \\ 0 & \nu \frac{\sum t_i}{K \sum (t_i - \langle t \rangle)^2} & \nu \frac{\sum t_i^2}{K \sum (t_i - \langle t \rangle)^2} \end{pmatrix},$$

with the well-known expression for the slope variance

$$var(\hat{\omega}) = \frac{\nu}{\sum (t_i - \langle t \rangle)^2}.$$

In matrix form, the frequency variance is written as

$$var(\hat{\omega}) = \nu \frac{\underline{\mathbf{M}}' \underline{\mathbf{M}}}{(\underline{\mathbf{T}}' \mathcal{D}_\mu^2 \underline{\mathbf{T}}) (\underline{\mathbf{M}}' \underline{\mathbf{M}}) - (\underline{\mathbf{T}}' \mathcal{D}_\mu \underline{\mathbf{M}})^2},$$

with

$$\begin{aligned} \underline{\mathbf{T}} &= (t_0, \dots, t_{K-1})', \quad \underline{\mathbf{M}} = (\mu'_\phi(t_0), \dots, \mu'_\phi(t_{K-1}))', \\ \mathcal{D}_\mu &= \begin{pmatrix} \mu'_\phi(t_0) & 0 & \cdots & 0 \\ 0 & \mu'_\phi(t_1) & \cdots & 0 \\ \vdots & \cdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mu'_\phi(t_{K-1}) \end{pmatrix}. \end{aligned}$$

2 Estimates