Statistical modeling

November 15, 2016

Introduction

...Since the estimate $\hat{A}_{y,y}$ is obtained as a difference statistic of the slopes in the up- and down-polarized trials, its variance $\sigma\left[\hat{A}_{y,y}\right] = C\sqrt{2} \cdot \sigma\left[\hat{\beta}\right]$, with the proportionality coefficient (see Table 1) $C = (\nu\sigma_0\Theta P^t\Delta P)^{-1} \approx 1.3 \cdot 10^5$.

For the mean,

$$\sigma\left[\langle\hat{A}_{y,y}\rangle\right] = \frac{\sigma\left[\hat{A}_{y,y}\right]}{\sqrt{N}} = \sqrt{2}\sqrt{\frac{h}{H}} \cdot \sigma\left[\hat{A}_{y,y}\right], \tag{1}$$

where H is the beam time, h the cycle length, and so the number of estimate pairs $N = {H}/{2h}$.

1 Required beam time

Under the Gauss-Markov conditions, the slope estimate's standard error is

$$\sigma\left[\hat{\beta}\right] = \frac{\sigma\left[\epsilon\right]}{\sqrt{\sum_k(t_k - \langle t\rangle)^2}}, \tag{2}$$

where $\sigma\left[\epsilon\right]$ is the measurement resolution.

Table 1: Parameter values (June 2016)

Parameter	Value	Dimension
ν	0.79	MHz
Θ	$1.1 \cdot 10^{14}$	$\text{at} \cdot \text{cm}^{-2}$
P^t	0.88	–
ΔP	1.48	–
σ_0 [a]	70	mb

[a] From Particle Data Group http://pdg.lbl.gov/2016/hadronic-xsections/rpp2014-pd_pn_plots.pdf

Since the measurements are taken uniformly in time with the step Δt, rewriting eq. (2) in physical terms gets:

$$\sum_{k=1}^{K}(t_k - \langle t \rangle)^2 = \sum_k \left(k\Delta t - \frac{1}{K}\sum_{k=1}^{K}k\Delta t\right)^2;$$

$$\frac{1}{K}\sum_{k=1}^{K}k\Delta t = \frac{\Delta t}{2}(K+1) \equiv \Delta t\mu,$$

$$\sum_k (k\Delta t - \mu\Delta t)^2 = \Delta t^2 \sum_k \left(k^2 - 2k\mu + \mu^2\right)$$

$$= \Delta t^2 \left(\sum_k k^2 - 2\mu\sum_k k + \mu^2 K\right)$$

$$= \Delta t^2 \left(\frac{2K+1}{3}K\mu - 2\mu^2 K + \mu^2 K\right) = \Delta t^2 \mu K\left(\frac{2K+1}{3} - \mu\right)$$

$$= \Delta t^2 \mu K\frac{K-1}{6}$$

$$= \frac{\Delta t^2}{12}K(K^2 - 1),$$

and hence

$$\sqrt{\sum_{k=1}^{K}(t_k - \langle t \rangle)^2} = \frac{\Delta t}{2\sqrt{3}}\sqrt{K}\sqrt{K^2 - 1}.$$

As the number of measurements grows, $K \gg 1$, $K^2 - 1 \approx K^2$, and so

$$\sqrt{\sum_k (t_k - \langle T \rangle)^2} \approx \frac{\Delta t}{2\sqrt{3}}K\sqrt{K}.$$

The number K of measurements that go into evaluating a slope is related to the state time h as $K\Delta t = h$, hence

$$\sqrt{\sum_{k=1}^{K}(t_k - \langle t \rangle)^2} \approx \frac{h\sqrt{h}}{2\sqrt{3}\sqrt{\Delta t}}.$$

Finally, for the slope variance,

$$\sigma\left[\hat{\beta}\right] = 2\sqrt{3}\sqrt{\frac{\Delta t}{h}}\frac{\sigma\left[\epsilon\right]}{h}. \tag{3}$$

By plugging it in eq. (1) we get

$$\sigma\left[\langle \hat{A}_{y,y}\rangle\right] = 4\sqrt{3}C \cdot \frac{\sqrt{\Delta t}}{h\sqrt{H}} \cdot \sigma\left[\epsilon\right]. \tag{4}$$

The estimated beam time for different values of measurement resolution and precision, computed for the case of 15-minute cycles, is shown in Figure 1.

2 Appropriate cycle length

Suppose the slope varies; in that case the law of total variance says that

$$\sigma^2\left[\hat{\beta}\right] = \mathrm{E}_{\beta}\left[\sigma^2\left[\hat{\beta}\middle|\ \beta\right]\right] + \sigma^2_{\beta}\left[\mathrm{E}\left[\hat{\beta}\middle|\ \beta\right]\right]. \tag{5}$$

$$\mathrm{E}\left[\hat{\beta}\middle|\ \beta\right] = \beta,$$

$$\sigma^2\left[\hat{\beta}\middle|\ \beta\right] = 12\frac{\Delta t}{h}\frac{\sigma^2\left[\epsilon\right]}{h^2}, \tag{eq. (3)}$$

and hence

$$\sigma^2\left[\langle \hat{A}_{y,y}\rangle\right] = \frac{4C^2}{H}\left(12\Delta t \cdot \frac{\sigma^2\left[\epsilon\right]}{h^2} + h\cdot\sigma^2\left[\beta\right]\right).$$

The first term describes the estimate's statistical precision, the second its accuracy. By decreasing the cycle length we reduce precision, but improve accuracy. The accuracy is improved simply by virtue of better averaging of $\sigma^2[\beta]$.

The slope variation determines the best achievable precision (see Table 2). From the last expression, the best variance is achieved at

$$h_{best} = \sqrt[3]{24 \cdot \Delta t} \left(\frac{\sigma^2[\epsilon]}{\sigma^2[\beta]} \right)^{1/3}. \tag{6}$$

Table 2: For Figure 2

$\sigma[\epsilon]$	h_{best} (min)	$\sigma\left[\langle \hat{A}_{y,y} \rangle\right]$
$5 \cdot 10^{-3}$	141	$8 \cdot 10^{-4}$
$1 \cdot 10^{-3}$	48	$3 \cdot 10^{-4}$
$5 \cdot 10^{-4}$	30	$3 \cdot 10^{-4}$
$1 \cdot 10^{-4}$	10	$2 \cdot 10^{-4}$
$5 \cdot 10^{-5}$	7	$1 \cdot 10^{-4}$

2.1 Inherent slope variation

By definition, the slope at time t is

$$\beta_t = \frac{\Delta \ln I_t}{\Delta t} \approx \frac{1}{\Delta t} \frac{\Delta I_t}{I_t} \propto \frac{\Delta N_t}{N_t}, \quad \Delta N_t \sim B(N_t, p).$$

Since this is a sample proportion,

$$\begin{cases} \mathrm{E}[\beta_t] & = p, \\ \sigma[\beta_t] & = \frac{1}{\Delta t} \frac{\sqrt{p(1-p)}}{\sqrt{N_t}}, \\ p & = \nu \sigma_0 \Theta \Delta t. \end{cases}$$

Again, from the defition

$$N_t = N_{t-1}\left(1 + \beta_{t-1}\Delta t\right).$$

Therefore,

$$\sigma^2[\beta_t] = \frac{\alpha}{N_t} = \sigma^2[\beta_{t-1}]\left(1 + \beta_{t-1}\Delta t\right)^{-1}, \tag{7}$$

where, for the parameters of Table 1, $\sigma^2[\beta_0] \approx 8 \cdot 10^{-16}$.
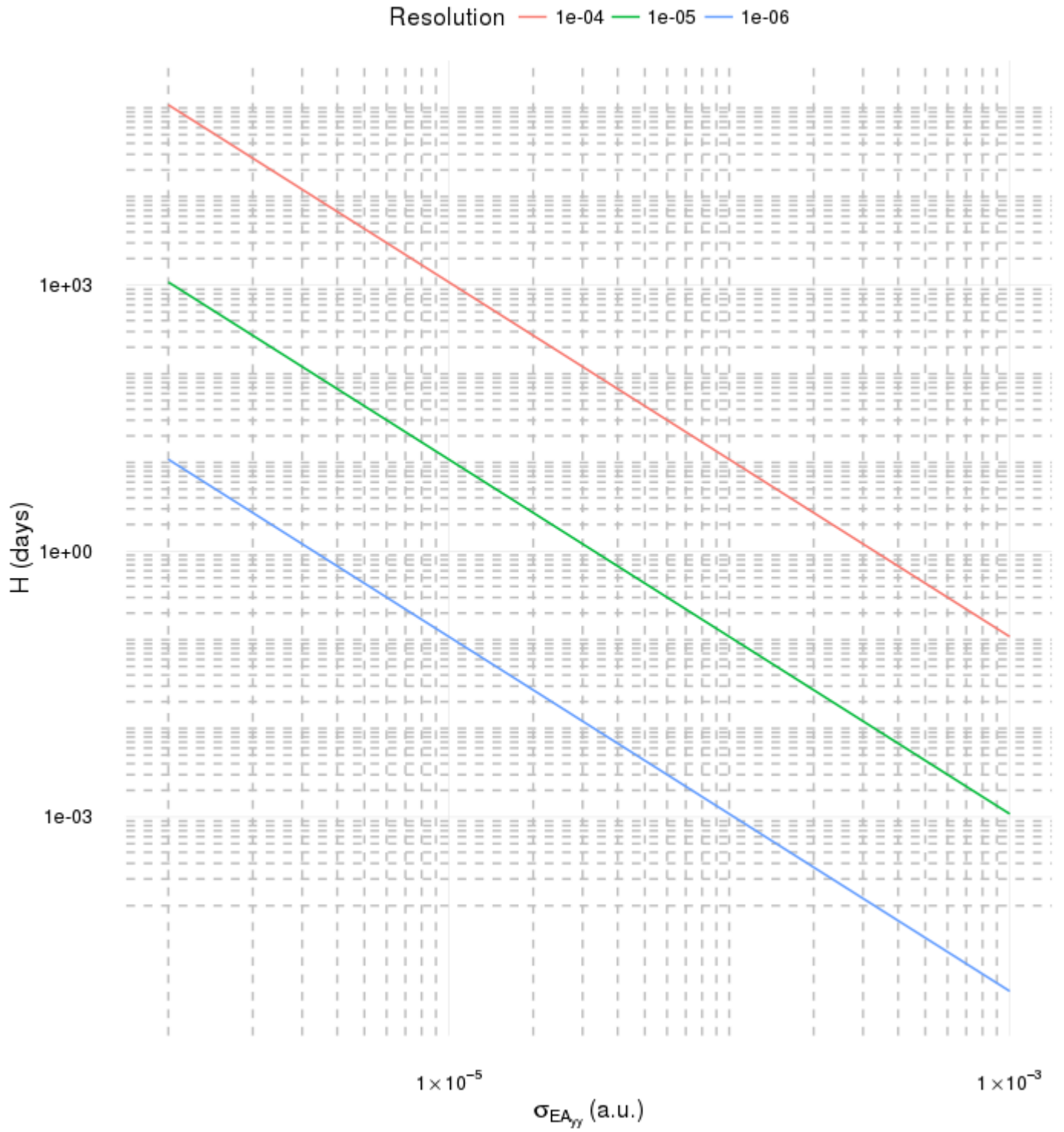
Figure 1: Beam time (in full days) as a function of the standard error of the mean $A_{y,y}$ estimate, required in the case of 15-minute cycles.
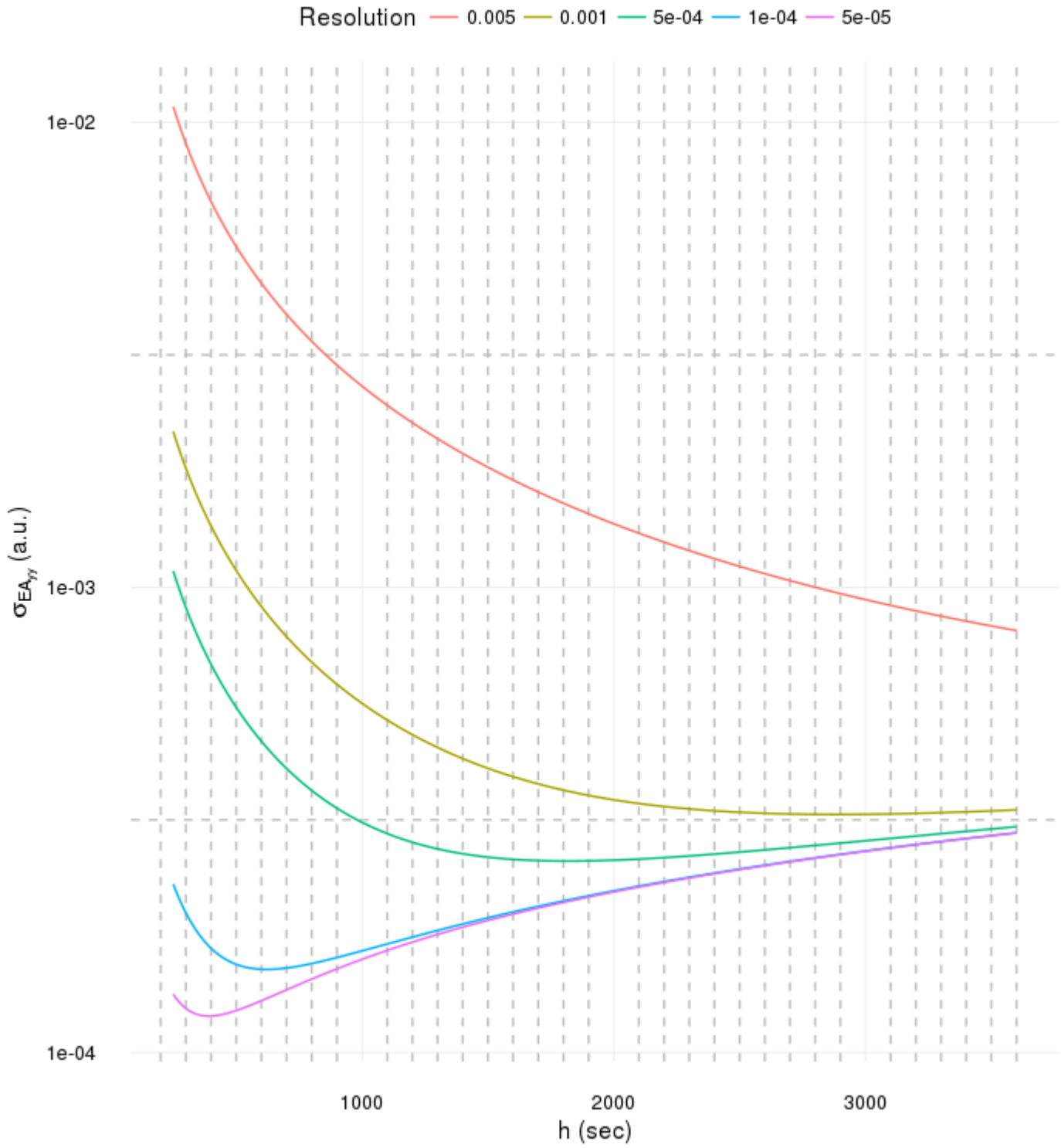
Figure 2: The standard error of the mean $A_{y,y}$ estimate as a function of cycle length when the inherent slope variation $\sigma^2[\beta] = 1 \cdot 10^{-15}$.