Statistical modeling

November 15, 2016

Introduction

...Since the estimate $\hat{A}_{y,y}$ is obtained as a difference statistic of the slopes in the up- and down-polarized trials, its variance $\sigma\left[\hat{A}_{y,y}\right] = C\sqrt{2} \cdot \sigma\left[\hat{\beta}\right]$, with the proportionality coefficient $C = (\nu\sigma_0\Theta P^t \Delta P)^{-1} \approx 1.3 \cdot 10^5$.

For the mean,

$$\sigma\left[\langle\hat{A}_{y,y}\rangle\right] = \frac{\sigma\left[\hat{A}_{y,y}\right]}{\sqrt{N}} = \sqrt{2}\sqrt{\frac{h}{H}} \cdot \sigma\left[\hat{A}_{y,y}\right], \tag{1}$$

where H is the beam time, h the cycle length, and so the number of estimate pairs $N = {}^{H}/_{2h}$.

1 Necessary beam time

Under the Gauss-Markov conditions, the slope estimate's standard error is

$$\sigma\left[\hat{\beta}\right] = \frac{\sigma\left[\epsilon\right]}{\sqrt{\sum_k (t_k - \langle t\rangle)^2}}. \tag{2}$$

Table 1: Parameter values (June 2016)

Parameter	Value	Dimension
ν	0.79	MHz
Θ	$1.1 \cdot 10^{14}$	$\text{at} \cdot \text{cm}^{-2}$
P^t	0.88	–
ΔP	1.48	–
σ_0[a]	70	mb

[a] From Particle Data Grpoup `http://pdg.lbl.gov/2016/hadronic-xsections/rpp2014-pd_pn_plots.pdf`

Since the measurements are taken uniformly in time with the step Δt, rewriting eq. (2) in physical terms gets:

$$\begin{aligned}
\sum_{k=1}^{K}(t_k - \langle t\rangle)^2 &= \sum_k (k\Delta t - \frac{1}{K}\sum_{k=1}^{K} k\Delta t)^2; \\
\frac{1}{K}\sum_{k=1}^{K} k\Delta t &= \frac{\Delta t}{2}(K+1) \equiv \Delta t\mu, \\
\sum_k (k\Delta t - \mu\Delta t)^2 &= \Delta t^2 \sum_k \left(k^2 - 2k\mu + \mu^2\right) \\
&= \Delta t^2 \left(\sum_k k^2 - 2\mu\sum_k k + \mu^2 K\right) \\
&= \Delta t^2 \left(\frac{2K+1}{3}K\mu - 2\mu^2 K + \mu^2 K\right) = \Delta t^2 \mu K \left(\frac{2K+1}{3} - \mu\right) \\
&= \Delta t^2 \mu K \frac{K-1}{6} \\
&= \frac{\Delta t^2}{12} K(K^2-1),
\end{aligned}$$

and hence

$$\sqrt{\sum_{k=1}^{K}(t_k - \langle t\rangle)^2} = \frac{\Delta t}{2\sqrt{3}}\sqrt{K}\sqrt{K^2-1}.$$

As the number of measurements grows, $K \gg 1$, $K^2 - 1 \approx K^2$, and so

$$\sqrt{\sum_k (t_k - \langle T\rangle)^2} \approx \frac{\Delta t}{2\sqrt{3}} K\sqrt{K}.$$

The number K of measurements that go into evaluating a slope is related to the state time h as $K\Delta t = h$, hence

$$\sqrt{\sum_{k=1}^{K}(t_k - \langle t\rangle)^2} \approx \frac{h\sqrt{h}}{2\sqrt{3}\sqrt{\Delta t}}.$$

Finally, for the slope variance,

$$\sigma\left[\hat{\beta}\right] = 2\sqrt{3}\sqrt{\frac{\Delta t}{h}}\frac{\sigma\left[\epsilon\right]}{h}.$$

By plugging it in eq. (1) we get

$$\sigma\left[\langle \hat{A}_{y,y}\rangle\right] = 4\sqrt{3}C \cdot \frac{\sqrt{\Delta t}}{h\sqrt{H}} \cdot \sigma\left[\epsilon\right]. \tag{3}$$

2 Necessary cycle length

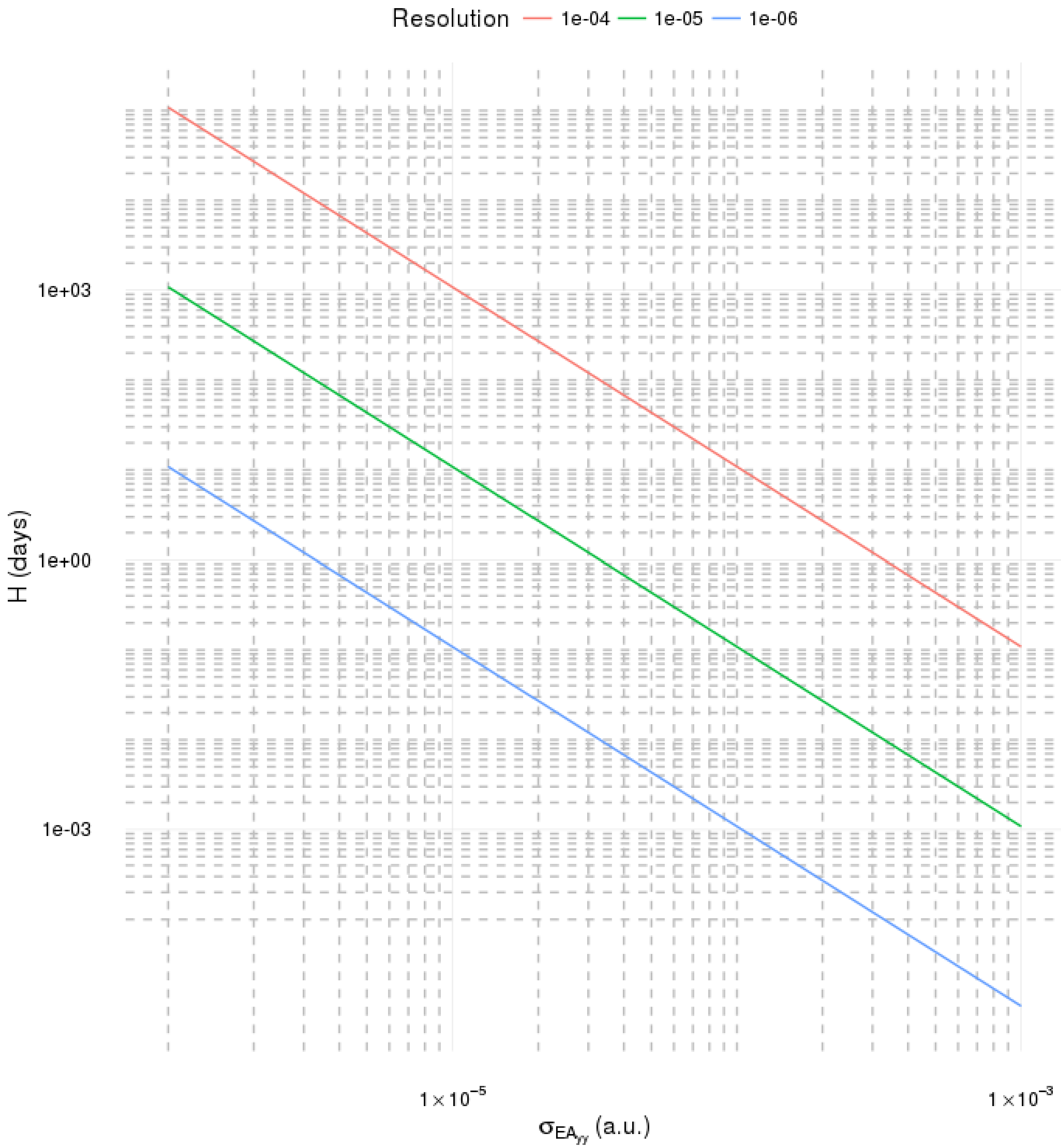

Figure 1: Beam time (in full days) as a function of the standard error of the mean $A_{y,y}$ estimate, required in the case of 15-minute cycles.