1.a. Considere

$$\begin{aligned}\boldsymbol{Q}(\boldsymbol{a}) &= \sum_{i=1}^n \{(\boldsymbol{x}_i - \overline{\boldsymbol{x}} + \overline{\boldsymbol{x}} - \boldsymbol{a})(\boldsymbol{x}_i - \overline{\boldsymbol{x}} + \overline{\boldsymbol{x}} - \boldsymbol{a})^\top\} \\ &= \sum_{i=1}^n \{(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top + (\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\overline{\boldsymbol{x}} - \boldsymbol{a})^\top + (\overline{\boldsymbol{x}} - \boldsymbol{a})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top + (\overline{\boldsymbol{x}} - \boldsymbol{a})(\overline{\boldsymbol{x}} - \boldsymbol{a})^\top\} \\ &= \sum_{i=1}^n (\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top + n(\overline{\boldsymbol{x}} - \boldsymbol{a})(\overline{\boldsymbol{x}} - \boldsymbol{a})^\top,\end{aligned}$$

pues $\sum_{i=1}^n (\boldsymbol{x}_i - \overline{\boldsymbol{x}}) = \boldsymbol{0}$.

1.b. Note que,

$$\begin{aligned}|\boldsymbol{Q}(\boldsymbol{a})| &= |\boldsymbol{Q} + n(\overline{\boldsymbol{x}} - \boldsymbol{a})(\overline{\boldsymbol{x}} - \boldsymbol{a})^\top| = |\boldsymbol{Q}(\boldsymbol{I}_p + n\boldsymbol{Q}^{-1}(\overline{\boldsymbol{x}} - \boldsymbol{a})(\overline{\boldsymbol{x}} - \boldsymbol{a})^\top)| \\ &= |\boldsymbol{Q}||\boldsymbol{I}_p + n\boldsymbol{Q}^{-1}(\overline{\boldsymbol{x}} - \boldsymbol{a})(\overline{\boldsymbol{x}} - \boldsymbol{a})^\top| = |\boldsymbol{Q}|\{1 + n(\overline{\boldsymbol{x}} - \boldsymbol{a})^\top \boldsymbol{Q}^{-1}(\overline{\boldsymbol{x}} - \boldsymbol{a})\},\end{aligned}$$

como $n(\overline{\boldsymbol{x}} - \boldsymbol{a})^\top \boldsymbol{Q}^{-1}(\overline{\boldsymbol{x}} - \boldsymbol{a}) \geq 0$, sigue que

$$|\boldsymbol{Q}(\boldsymbol{a})| \geq |\boldsymbol{Q}|,$$

lo que permite verificar el resultado.

1.c. Evidentemente,

$$\sum_{i=1}^n g_{ij} = \sum_{i=1}^n (\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top \boldsymbol{S}_*^{-1}(\boldsymbol{x}_j - \overline{\boldsymbol{x}}) = 0,$$

pues $\sum_{i=1}^n (\boldsymbol{x}_i - \overline{\boldsymbol{x}}) = \boldsymbol{0}$. Asimismo,

$$\begin{aligned}\sum_{i=1}^n g_{ii} &= \sum_{i=1}^n \operatorname{tr} \boldsymbol{S}_*^{-1}(\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top = \operatorname{tr} \boldsymbol{S}_*^{-1} \sum_{i=1}^n (\boldsymbol{x}_i - \overline{\boldsymbol{x}})(\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top \\ &= \operatorname{tr}((\boldsymbol{Q}/n)^{-1}\boldsymbol{Q}) = n \operatorname{tr} \boldsymbol{Q}^{-1}\boldsymbol{Q} = n \operatorname{tr} \boldsymbol{I}_p = np.\end{aligned}$$

2. Considere la función Lagrangiana

$$\psi(\boldsymbol{G}) = \frac{1}{2}\operatorname{tr}(\boldsymbol{G}\boldsymbol{G}^\top) - \operatorname{tr} \boldsymbol{L}^\top(\boldsymbol{G}\boldsymbol{Z} - \boldsymbol{I}_k),$$

donde $\boldsymbol{L}$ es matriz de multiplicadores de Lagrange de orden $k \times k$. Así,

$$\begin{aligned}\mathsf{d}\,\psi(\boldsymbol{G}) &= \frac{1}{2}\operatorname{tr}(\mathsf{d}\,\boldsymbol{G})\boldsymbol{G}^\top + \frac{1}{2}\operatorname{tr} \boldsymbol{G}(\mathsf{d}\,\boldsymbol{G})^\top - \operatorname{tr} \boldsymbol{L}^\top(\mathsf{d}\,\boldsymbol{G})\boldsymbol{Z} \\ &= \operatorname{tr} \boldsymbol{G}^\top \mathsf{d}\,\boldsymbol{G} - \operatorname{tr} \boldsymbol{Z}\boldsymbol{L}^\top \mathsf{d}\,\boldsymbol{G} \\ &= \operatorname{tr}(\boldsymbol{G}^\top - \boldsymbol{Z}\boldsymbol{L}^\top)\,\mathsf{d}\,\boldsymbol{G}.\end{aligned}$$

Esto lleva a las condiciones de primer orden:

$$\boldsymbol{G}^\top = \boldsymbol{Z}\boldsymbol{L}^\top, \qquad \text{y} \qquad \boldsymbol{G}\boldsymbol{Z} = \boldsymbol{I}_k.$$

Resolviendo para $\boldsymbol{L}$ desde,

$$\boldsymbol{I}_k = \boldsymbol{G}\boldsymbol{Z} = \boldsymbol{L}\boldsymbol{Z}^\top\boldsymbol{Z} \quad \Longrightarrow \quad \widehat{\boldsymbol{L}} = (\boldsymbol{Z}^\top\boldsymbol{Z})^{-1}.$$

De ahí que

$$\widehat{\boldsymbol{G}} = \widehat{\boldsymbol{L}}\boldsymbol{Z}^\top = (\boldsymbol{Z}^\top\boldsymbol{Z})^{-1}\boldsymbol{Z}^\top.$$

Notando que ψ es estrictamente convexa, sigue que tiene un mínimo global en $\widehat{\boldsymbol{G}} = (\boldsymbol{Z}^\top\boldsymbol{Z})^{-1}\boldsymbol{Z}^\top$.

3. Como $\boldsymbol{P}$ es matriz de proyección de rango k, podemos escribir:

$$\boldsymbol{X}^\top\boldsymbol{X} = \boldsymbol{X}^\top\boldsymbol{P}\boldsymbol{X} + \boldsymbol{X}^\top(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{X}.$$

Ahora $(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{X} \sim \mathsf{N}_{n,p}(\boldsymbol{0}, \boldsymbol{0} \otimes \boldsymbol{\Sigma})$. De modo que, $\boldsymbol{X}^\top(\boldsymbol{I} - \boldsymbol{P})\boldsymbol{X}$ es cero con probabilidad uno. Además, existe una matriz $\boldsymbol{M}$ tal que $\boldsymbol{P} = \boldsymbol{M}\boldsymbol{M}^\top$ y $\boldsymbol{M}^\top\boldsymbol{M} = \boldsymbol{I}_k$. Lo que permite escribir $\boldsymbol{X}^\top\boldsymbol{P}\boldsymbol{X} = (\boldsymbol{M}^\top\boldsymbol{X})^\top\boldsymbol{M}^\top\boldsymbol{X}$. Tenemos que

$$\boldsymbol{M}^\top\boldsymbol{X} \sim \mathsf{N}_{n,p}(\boldsymbol{0}, \boldsymbol{M}^\top\boldsymbol{P}\boldsymbol{M}, \boldsymbol{\Sigma}) \stackrel{\mathsf{d}}{=} \mathsf{N}_{n,p}(\boldsymbol{0}, \boldsymbol{I}_k \otimes \boldsymbol{\Sigma}),$$

de donde sigue que $\boldsymbol{X}^\top\boldsymbol{X} \sim \mathsf{W}_p(k, \boldsymbol{\Sigma})$.