1. Es fácil notar que

$$\overline{\boldsymbol{x}}_n = \frac{1}{W_n}\Big(\sum_{i=1}^{n-1}\omega_i\boldsymbol{x}_i + \omega_n\boldsymbol{x}_n\Big) = \frac{1}{W_n}\big(W_{n-1}\overline{\boldsymbol{x}}_{n-1} + \omega_n\boldsymbol{x}_n\big).$$

Ahora, $W_n = \sum_{i=1}^{n-1}\omega_i + \omega_n = W_{n-1} + \omega_n$. Luego,

$$\begin{aligned}\overline{\boldsymbol{x}}_n &= \frac{1}{W_n}\big\{(W_n - \omega_n)\overline{\boldsymbol{x}}_{n-1} + \omega_n\boldsymbol{x}_n\big\} = \overline{\boldsymbol{x}}_{n-1} + \frac{\omega_n}{W_n}(\boldsymbol{x}_n - \overline{\boldsymbol{x}}_{n-1}) \\ \overline{\boldsymbol{x}}_n &= \overline{\boldsymbol{x}}_{n-1} + \frac{\omega_n}{W_n}\boldsymbol{\delta}_n.\end{aligned}$$

Por otro lado,

$$\boldsymbol{Q}_n = \sum_{i=1}^{n-1}\omega_i(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_n)(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_n)^\top + \omega_n(\boldsymbol{x}_n - \overline{\boldsymbol{x}}_n)(\boldsymbol{x}_n - \overline{\boldsymbol{x}}_n)^\top$$

Tenemos que, $\boldsymbol{x}_i - \overline{\boldsymbol{x}}_n = \boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1} - (\omega_n/W_n)\boldsymbol{\delta}_n$, lo que lleva a,

$$\begin{aligned}\omega_i(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_n)(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_n)^\top &= \omega_i\Big\{\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1} - \frac{\omega_n}{W_n}\boldsymbol{\delta}_n\Big\}\Big\{\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1} - \frac{\omega_n}{W_n}\boldsymbol{\delta}_n\Big\}^\top \\ &= \omega_i(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1})(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1})^\top - \frac{\omega_i\omega_n}{W_n}(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1})\boldsymbol{\delta}_n^\top \\ &\quad - \frac{\omega_i\omega_n}{W_n}(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1})\boldsymbol{\delta}_n^\top + \omega_i\Big(\frac{\omega_n}{W_n}\Big)^2\boldsymbol{\delta}_n\boldsymbol{\delta}_n^\top.\end{aligned}$$

Como $\sum_{i=1}^{n-1}\omega_i(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1}) = \boldsymbol{0}$, sigue que

$$\sum_{i=1}^{n-1}\omega_i(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1})(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1})^\top + \sum_{i=1}^{n-1}\omega_i\Big(\frac{\omega_n}{W_n}\Big)^2\boldsymbol{\delta}_n\boldsymbol{\delta}_n^\top.$$

Además, $\boldsymbol{x}_n - \overline{\boldsymbol{x}}_n = \boldsymbol{x}_n - \overline{\boldsymbol{x}}_{n-1} - (\omega_n/W_n)\boldsymbol{\delta}_n = (1 - \omega_n/W_n)\boldsymbol{\delta}_n$. Esto permite escribir,

$$\begin{aligned}\boldsymbol{Q}_n &= \sum_{i=1}^{n-1}\omega_i(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1})(\boldsymbol{x}_i - \overline{\boldsymbol{x}}_{n-1})^\top + W_{n-1}\Big(\frac{\omega_n}{W_n}\Big)^2\boldsymbol{\delta}_n\boldsymbol{\delta}_n^\top + \omega_n\Big(1 - \frac{\omega_n}{W_n}\Big)^2\boldsymbol{\delta}_n\boldsymbol{\delta}_n^\top \\ &= \boldsymbol{Q}_{n-1} + \Big\{(W_n - \omega_n)\Big(\frac{\omega_n}{W_n}\Big)^2 + \omega_n\Big(1 - \frac{\omega_n}{W_n}\Big)^2\Big\}\boldsymbol{\delta}_n\boldsymbol{\delta}_n^\top.\end{aligned}$$

Como

$$(W_n - \omega_n)\Big(\frac{\omega_n}{W_n}\Big)^2 + \omega_n\Big(1 - \frac{\omega_n}{W_n}\Big)^2 = \frac{(W_n - \omega_n)\omega_n}{W_n} = U_n,$$

sigue la primera parte del resultado. La actualización para $\boldsymbol{Q}_n^{-1}$ sigue de la aplicación de la fórmula de Sherman-Morrison. En efecto,

$$\begin{aligned}\boldsymbol{Q}_n^{-1} = (\boldsymbol{Q}_{n-1} + U_n\boldsymbol{\delta}_n\boldsymbol{\delta}_n^\top)^{-1} &= \boldsymbol{Q}_{n-1}^{-1} - \frac{U_n\boldsymbol{Q}_{n-1}^{-1}\boldsymbol{\delta}_n\boldsymbol{\delta}_n^\top\boldsymbol{Q}_{n-1}^{-1}}{1 + U_n\boldsymbol{\delta}_n^\top\boldsymbol{Q}_{n-1}^{-1}\boldsymbol{\delta}_n} \\ &= \boldsymbol{Q}_{n-1}^{-1} - \frac{\boldsymbol{Q}_{n-1}^{-1}\boldsymbol{\delta}_n\boldsymbol{\delta}_n^\top\boldsymbol{Q}_{n-1}^{-1}}{U_n^{-1} + \boldsymbol{\delta}_n^\top\boldsymbol{Q}_{n-1}^{-1}\boldsymbol{\delta}_n}, \qquad U_n^{-1} + \boldsymbol{\delta}_n^\top\boldsymbol{Q}_{n-1}^{-1}\boldsymbol{\delta}_n \neq 0.\end{aligned}$$

2. Los coeficientes de sesgo y curtosis multivariados son dados por

$$b_{1p} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij}^3, \qquad b_{2p} = \frac{1}{n}\sum_{i=1}^{n} g_{ii}^2,$$

con $g_{ij} = (\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top \boldsymbol{S}^{-1}(\boldsymbol{x}_j - \overline{\boldsymbol{x}})$. Tenemos $\boldsymbol{y}_i = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{b}$, para $i = 1,\dots,n$, donde $\boldsymbol{A}$ es no singular. Sabemos que $\overline{\boldsymbol{y}} = \boldsymbol{A}\overline{\boldsymbol{x}} + \boldsymbol{b}$ y $\boldsymbol{S}_y = \boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^\top$. De este modo,

$$\boldsymbol{y}_i - \overline{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{b} - \boldsymbol{A}\overline{\boldsymbol{x}} - \boldsymbol{b} = \boldsymbol{A}(\boldsymbol{x}_i - \overline{\boldsymbol{x}}).$$

Luego,

$$\begin{aligned} g_{ij}^* &= (\boldsymbol{y}_i - \overline{\boldsymbol{y}})^\top \boldsymbol{S}_y^{-1}(\boldsymbol{y}_j - \overline{\boldsymbol{y}}) = (\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top \boldsymbol{A}^\top (\boldsymbol{A}\boldsymbol{S}\boldsymbol{A}^\top)^{-1}\boldsymbol{A}(\boldsymbol{x}_i - \overline{\boldsymbol{x}}) \\ &= (\boldsymbol{x}_i - \overline{\boldsymbol{x}})^\top \boldsymbol{A}^\top \boldsymbol{A}^{-\top}\boldsymbol{S}^{-1}\boldsymbol{A}^{-1}\boldsymbol{A}(\boldsymbol{x}_i - \overline{\boldsymbol{x}}) = g_{ij}, \end{aligned}$$

y el resultado sigue.

3. Considere,

$$\boldsymbol{X}^\top \boldsymbol{A}\boldsymbol{X} = \boldsymbol{X}^\top \boldsymbol{P}\boldsymbol{P}^\top \boldsymbol{X} = \boldsymbol{Y}^\top \boldsymbol{Y}, \qquad \boldsymbol{Y} = \boldsymbol{P}^\top \boldsymbol{X}.$$

Tenemos que

$$\operatorname{vec}(\boldsymbol{X}^\top) \sim \mathsf{N}_{np}\big(\operatorname{vec}(\boldsymbol{M}^\top), \boldsymbol{I}_n \otimes \boldsymbol{\Sigma}\big).$$

De ahí que,

$$\operatorname{vec}(\boldsymbol{Y}^\top) = \operatorname{vec}(\boldsymbol{X}^\top \boldsymbol{P}) = (\boldsymbol{P}^\top \otimes \boldsymbol{I}_p)\operatorname{vec}(\boldsymbol{X}^\top),$$

lo que lleva a

$$\operatorname{vec}(\boldsymbol{Y}^\top) \sim \mathsf{N}_{np}\big((\boldsymbol{P}^\top \otimes \boldsymbol{I}_p)\operatorname{vec}(\boldsymbol{M}^\top), (\boldsymbol{P}^\top \otimes \boldsymbol{I}_p)(\boldsymbol{I}_n \otimes \boldsymbol{\Sigma})(\boldsymbol{P} \otimes \boldsymbol{I}_p)\big).$$

Es decir,

$$\operatorname{vec}(\boldsymbol{Y}^\top) \sim \mathsf{N}_{np}\big((\boldsymbol{P}^\top \otimes \boldsymbol{I}_p)\operatorname{vec}(\boldsymbol{M}^\top), \boldsymbol{P}^\top \boldsymbol{P} \otimes \boldsymbol{\Sigma}\big).$$

El resultado sigue, luego de notar que $\boldsymbol{P}^\top \boldsymbol{P} = \boldsymbol{I}_r$, y

$$\boldsymbol{\Phi} = \tfrac{1}{2}\,\mathsf{E}(\boldsymbol{Y}^\top)\,\mathsf{E}(\boldsymbol{Y}) = \tfrac{1}{2}\,\mathsf{E}(\boldsymbol{X}^\top \boldsymbol{P})\,\mathsf{E}(\boldsymbol{P}^\top \boldsymbol{X}) = \tfrac{1}{2}\,\mathsf{E}^\top(\boldsymbol{X})\boldsymbol{P}\boldsymbol{P}^\top\,\mathsf{E}(\boldsymbol{X}) = \tfrac{1}{2}\boldsymbol{M}^\top \boldsymbol{A}\boldsymbol{M}.$$