1. Considere $\boldsymbol{x}_1,\dots,\boldsymbol{x}_n$ una muestra aleatoria desde $\mathsf{N}_p(\boldsymbol{\mu},\lambda\boldsymbol{\Sigma}_0)$, donde $\boldsymbol{\Sigma}_0$ es conocida. Tenemos,

$$\begin{aligned} L(\boldsymbol{\theta}) &= \prod_{i=1}^n |2\pi\lambda\boldsymbol{\Sigma}_0|^{-1/2} \exp\Big\{ -\frac{1}{2\lambda}(\boldsymbol{x}_i-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{x}_i-\boldsymbol{\mu})\Big\} \\ &= |2\pi\lambda\boldsymbol{\Sigma}_0|^{-n/2} \exp\Big\{ -\frac{1}{2\lambda}\sum_{i=1}^n(\boldsymbol{x}_i-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{x}_i-\boldsymbol{\mu})\Big\} \end{aligned}$$

con $\boldsymbol{\theta}=(\boldsymbol{\mu}^\top,\lambda)^\top$. De este modo,

$$\ell(\boldsymbol{\theta}) = -\frac{np}{2}\log\lambda - \frac{n}{2}\log|2\pi\boldsymbol{\Sigma}_0| - \frac{1}{2\lambda}\Big\{\operatorname{tr}(\boldsymbol{\Sigma}_0^{-1}\boldsymbol{Q}) + n(\overline{\boldsymbol{x}}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}_0^{-1}(\overline{\boldsymbol{x}}-\boldsymbol{\mu})\Big\},$$

donde $\boldsymbol{Q}=\sum_{i=1}^n(\boldsymbol{x}_i-\overline{\boldsymbol{x}})(\boldsymbol{x}_i-\overline{\boldsymbol{x}})^\top$. Evidentemente,

$$\begin{aligned} \frac{\partial\ell(\boldsymbol{\theta})}{\partial\boldsymbol{\mu}} &= \frac{1}{\lambda}\boldsymbol{\Sigma}_0^{-1}(\overline{\boldsymbol{x}}-\boldsymbol{\mu}) \\ \frac{\partial\ell(\boldsymbol{\theta})}{\partial\lambda} &= -\frac{np}{2\lambda} + \frac{1}{2\lambda^2}\Big\{\operatorname{tr}(\boldsymbol{\Sigma}_0^{-1}\boldsymbol{Q}) + n(\overline{\boldsymbol{x}}-\boldsymbol{\mu})^\top\boldsymbol{\Sigma}_0^{-1}(\overline{\boldsymbol{x}}-\boldsymbol{\mu})\Big\} \end{aligned}$$

Resolviendo las condiciones de primer orden, $\partial\ell(\boldsymbol{\theta})/\partial\boldsymbol{\mu}=\boldsymbol{0}$ y $\partial\ell(\boldsymbol{\theta})/\partial\lambda=0$. Obtenemos $\widehat{\boldsymbol{\mu}}=\overline{\boldsymbol{x}}$, y

$$-\frac{np}{2\lambda} + \frac{1}{2\lambda^2}\operatorname{tr}\boldsymbol{\Sigma}_0^{-1}\boldsymbol{Q} = 0.$$

Es decir,

$$\widehat{\lambda} = \frac{1}{np}\operatorname{tr}\boldsymbol{\Sigma}_0^{-1}\boldsymbol{Q} = \frac{1}{p}\operatorname{tr}\boldsymbol{\Sigma}_0^{-1}\boldsymbol{S}_*,$$

con $\boldsymbol{S}_*=\boldsymbol{Q}/n$.

2. Tenemos,

$$f(\boldsymbol{x}) = |2\pi\sigma^2\boldsymbol{I}|^{-1/2}\exp\Big(-\frac{1}{2\sigma^2}\|\boldsymbol{x}-\boldsymbol{\mu}\|^2\Big).$$

De este modo,

$$\ell(\boldsymbol{\mu}) = -\frac{p}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\|\boldsymbol{x}-\boldsymbol{\mu}\|^2.$$

Es decir, podemos escribir la función Lagrangiana,

$$F(\boldsymbol{\mu},\lambda) = -\frac{p}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\|\boldsymbol{x}-\boldsymbol{\mu}\|^2 - \frac{\lambda}{2\sigma^2}(\boldsymbol{\mu}^\top\boldsymbol{\mu}-1),$$

donde λ representa un multiplicador de Lagrange. Diferenciando $F(\boldsymbol{\mu},\lambda)$ con relación a $\boldsymbol{\mu}$ y λ, sigue que:

$$\begin{aligned} \frac{\partial F(\boldsymbol{\mu},\lambda)}{\partial\boldsymbol{\mu}} &= \frac{1}{\sigma^2}(\boldsymbol{x}-\boldsymbol{\mu}) - \frac{\lambda}{\sigma^2}\boldsymbol{\mu} \\ \frac{\partial F(\boldsymbol{\mu},\lambda)}{\partial\lambda} &= -\frac{1}{2\sigma^2}(\boldsymbol{\mu}^\top\boldsymbol{\mu}-1). \end{aligned}$$

Resolviendo la condición de primer orden, obtenemos

$$\boldsymbol{x} - \widehat{\boldsymbol{\mu}} - \widehat{\lambda}\widehat{\boldsymbol{\mu}} = \boldsymbol{0}, \tag{1}$$
$$\widehat{\boldsymbol{\mu}}^\top \widehat{\boldsymbol{\mu}} - 1 = 0. \tag{2}$$

Desde la Ecuación **(1)**, lleva

$$\boldsymbol{x} = (1 + \widehat{\lambda})\widehat{\boldsymbol{\mu}} \qquad \Rightarrow \qquad \widehat{\boldsymbol{\mu}} = \frac{\boldsymbol{x}}{1 + \widehat{\lambda}}. \tag{3}$$

Substituyendo en **(2)**, obtenemos

$$\widehat{\boldsymbol{\mu}}^\top \widehat{\boldsymbol{\mu}} = 1 \qquad \Rightarrow \qquad \Big(\frac{\boldsymbol{x}}{1 + \widehat{\lambda}}\Big)^\top \frac{\boldsymbol{x}}{1 + \widehat{\lambda}} = 1,$$

es decir, $\boldsymbol{x}^\top \boldsymbol{x} = (1 + \widehat{\lambda})^2$. O bien, $\sqrt{\boldsymbol{x}^\top \boldsymbol{x}} = 1 + \widehat{\lambda}$. Finalmente, por **(3)** tenemos:

$$\widehat{\boldsymbol{\mu}} = \frac{\boldsymbol{x}}{1 + \widehat{\lambda}} = \frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}.$$

3.a. Considere $\boldsymbol{Y} \sim \mathsf{N}_{n,k}(\boldsymbol{X}\boldsymbol{B}, \boldsymbol{I}_n \otimes \boldsymbol{\Sigma})$. En este caso tenemos $\boldsymbol{X}\boldsymbol{B} = \boldsymbol{1}\boldsymbol{\beta}_1^\top + \boldsymbol{X}_2\boldsymbol{B}_2$, con log-verosimilitud

$$\ell(\boldsymbol{\theta}) = -\frac{n}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}\operatorname{tr}\boldsymbol{\Sigma}^2(\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top - \boldsymbol{X}_2\boldsymbol{B}_2)^\top(\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top - \boldsymbol{X}_2\boldsymbol{B}_2),$$

con $\boldsymbol{\theta} = (\boldsymbol{\beta}_1, \boldsymbol{B}_2, \boldsymbol{\Sigma})$. Diferenciando con relación a $\boldsymbol{\beta}_1$, obtenemos

$$\mathsf{d}_{\beta_1}\ell(\boldsymbol{\theta}) = -\tfrac{1}{2}\operatorname{tr}\boldsymbol{\Sigma}^{-1}\,\mathsf{d}_{\beta_1} Q(\boldsymbol{B}), \qquad \mathsf{d}_{B_2}\ell(\boldsymbol{\theta}) = -\tfrac{1}{2}\operatorname{tr}\boldsymbol{\Sigma}^{-1}\,\mathsf{d}_{B_2} Q(\boldsymbol{B}).$$

Ahora,

$$\mathsf{d}_{\beta_1} Q(\boldsymbol{B}) = -(\mathsf{d}\,\boldsymbol{\beta}_1)\boldsymbol{1}^\top(\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top - \boldsymbol{X}_2\boldsymbol{B}_2) - (\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top - \boldsymbol{X}_2\boldsymbol{B}_2)^\top\boldsymbol{1}(\mathsf{d}\,\boldsymbol{\beta}_1)^\top,$$

luego,

$$\mathsf{d}_{\beta_1}\ell(\boldsymbol{\theta}) = \operatorname{tr}\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top - \boldsymbol{X}_2\boldsymbol{B}_2)^\top\boldsymbol{1}(\mathsf{d}\,\boldsymbol{\beta}_1)^\top.$$

El diferencial es cero si,

$$(\boldsymbol{Y} - \boldsymbol{1}\widehat{\boldsymbol{\beta}}_1^\top - \boldsymbol{X}_2\widehat{\boldsymbol{B}}_2)^\top\boldsymbol{1} = \boldsymbol{0}, \quad \Rightarrow \quad \boldsymbol{Y}^\top\boldsymbol{1} - \widehat{\boldsymbol{\beta}}_1\boldsymbol{1}^\top\boldsymbol{1} - \widehat{\boldsymbol{B}}_2{}^\top\boldsymbol{X}_2^\top\boldsymbol{1} = \boldsymbol{0},$$

como $\boldsymbol{X}_2^\top\boldsymbol{1} = \boldsymbol{0}$ sigue que

$$\widehat{\boldsymbol{\beta}}_1 = \frac{1}{n}\boldsymbol{Y}^\top\boldsymbol{1} = \overline{\boldsymbol{y}}.$$

Por otro lado,

$$\mathsf{d}_{B_2} Q(\boldsymbol{B}) = -(\mathsf{d}\,\boldsymbol{B}_2)\boldsymbol{X}_2^\top(\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top - \boldsymbol{X}_2\boldsymbol{B}_2) - (\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top - \boldsymbol{X}_2\boldsymbol{B}_2)^\top\boldsymbol{X}_2\,\mathsf{d}\,\boldsymbol{B}_2.$$

De este modo,

$$\mathsf{d}_{B_2}\ell(\boldsymbol{\theta}) = \operatorname{tr}\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top - \boldsymbol{X}_2\boldsymbol{B}_2)^\top\boldsymbol{X}_2\,\mathsf{d}\,\boldsymbol{B}_2,$$

lo que lleva a la ecuación de estimación

$$\boldsymbol{X}_2^\top(\boldsymbol{Y} - \boldsymbol{1}\widehat{\boldsymbol{\beta}}_1^\top - \boldsymbol{X}_2\widehat{\boldsymbol{B}}_2) = \boldsymbol{0},$$

tenemos que $\widehat{\boldsymbol{\beta}}_1 = \overline{\boldsymbol{y}}$, así podemos definir $\widetilde{\boldsymbol{Y}} = \boldsymbol{Y} - \boldsymbol{1}\overline{\boldsymbol{y}}^\top$ lo que permite escribir

$$\boldsymbol{X}_2^\top(\widetilde{\boldsymbol{Y}} - \boldsymbol{X}_2\widehat{\boldsymbol{B}}_2) = \boldsymbol{0},$$

es decir,

$$\boldsymbol{X}_2^\top\boldsymbol{X}_2\widehat{\boldsymbol{B}}_2 = \boldsymbol{X}_2^\top\widetilde{\boldsymbol{Y}} \qquad \Rightarrow \qquad \widehat{\boldsymbol{B}}_2 = (\boldsymbol{X}_2^\top\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2^\top\widetilde{\boldsymbol{Y}}.$$

3.b. Bajo H_0 tenemos el modelo $\boldsymbol{Y} \sim \mathsf{N}_{n,k}(\boldsymbol{1}\boldsymbol{\beta}_1^\top, \boldsymbol{I}_n \otimes \boldsymbol{\Sigma})$. De ahí que,

$$\ell(\boldsymbol{\beta}_1, \boldsymbol{\Sigma}) = -\frac{n}{2}\log|\boldsymbol{\Sigma}| - \frac{1}{2}\operatorname{tr}\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top)^\top(\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top).$$

De este modo,

$$\mathsf{d}_{\beta_1}\ell(\boldsymbol{\beta}_1, \boldsymbol{\Sigma}) = \operatorname{tr}\boldsymbol{\Sigma}^{-1}(\boldsymbol{Y} - \boldsymbol{1}\boldsymbol{\beta}_1^\top)^\top\boldsymbol{1}(\mathsf{d}\,\boldsymbol{\beta}_1)^\top,$$

y desde la condición de primer orden, obtenemos

$$(\boldsymbol{Y} - \boldsymbol{1}\widetilde{\boldsymbol{\beta}}_1^\top)^\top\boldsymbol{1} = \boldsymbol{0}, \qquad \Rightarrow \qquad \boldsymbol{Y}^\top\boldsymbol{1} - \widetilde{\boldsymbol{\beta}}_1\boldsymbol{1}^\top\boldsymbol{1} = \boldsymbol{0},$$

de ahí que $\widetilde{\boldsymbol{\beta}}_1 = \boldsymbol{Y}^\top\boldsymbol{1}/n = \overline{\boldsymbol{y}}$. Asimismo,

$$\widetilde{\boldsymbol{\Sigma}} = \frac{1}{n}Q(\widetilde{\boldsymbol{\beta}}_1), \qquad \widehat{\boldsymbol{\Sigma}} = \frac{1}{n}Q(\widehat{\boldsymbol{B}}),$$

con $\widehat{\boldsymbol{B}} = (\widehat{\boldsymbol{\beta}}_1, \widehat{\boldsymbol{B}}_2)$. De este modo,

$$\begin{aligned}
L(\widehat{\boldsymbol{\theta}}) &= (2\pi)^{-np/2}|\widehat{\boldsymbol{\Sigma}}|^{-n/2}\exp\Big\{-\frac{1}{2}\operatorname{tr}\widehat{\boldsymbol{\Sigma}}^{-1}Q(\widehat{\boldsymbol{B}})\Big\} \\
&= (2\pi)^{-np/2}|\widehat{\boldsymbol{\Sigma}}|^{-n/2}\exp\Big\{-\frac{n}{2}\operatorname{tr}\widehat{\boldsymbol{\Sigma}}^{-1}\widehat{\boldsymbol{\Sigma}}\Big\} \\
&= (2\pi)^{-np/2}|\widehat{\boldsymbol{\Sigma}}|^{-n/2}\exp(-np/2),
\end{aligned}$$

y análogamente,

$$L(\widetilde{\boldsymbol{\theta}}) = (2\pi)^{-np/2}|\widetilde{\boldsymbol{\Sigma}}|^{-n/2}\exp(-np/2).$$

Así, el estadístico de razón de verosimilitudes adopta la forma:

$$\Lambda = \frac{L(\widetilde{\theta})}{L(\widehat{\boldsymbol{\theta}})} = \frac{|\widetilde{\boldsymbol{\Sigma}}|^{-n/2}}{|\widehat{\boldsymbol{\Sigma}}|^{-n/2}} = \frac{|Q(\widetilde{\boldsymbol{\beta}}_1)/n|^{-n/2}}{|Q(\widehat{\boldsymbol{B}})/n|^{-n/2}}.$$

Tenemos que

$$\begin{aligned}
Q(\widehat{\boldsymbol{B}}) &= (\boldsymbol{Y} - \boldsymbol{1}\widehat{\boldsymbol{\beta}}_1^\top - \boldsymbol{X}_2\widehat{\boldsymbol{B}}_2)^\top(\boldsymbol{Y} - \boldsymbol{1}\widehat{\boldsymbol{\beta}}_1^\top - \boldsymbol{X}_2\widehat{\boldsymbol{B}}_2) \\
&= (\widetilde{\boldsymbol{Y}} - \boldsymbol{X}_2\widehat{\boldsymbol{B}}_2)^\top(\widetilde{\boldsymbol{Y}} - \boldsymbol{X}_2\widehat{\boldsymbol{B}}_2) \\
&= \widetilde{\boldsymbol{Y}}^\top(\boldsymbol{I} - \boldsymbol{H}_2)\widetilde{\boldsymbol{Y}},
\end{aligned}$$

con $\boldsymbol{H}_2 = \boldsymbol{X}_2(\boldsymbol{X}_2^\top\boldsymbol{X}_2)^{-1}\boldsymbol{X}_2^\top$, mientras que

$$Q(\widetilde{\boldsymbol{\beta}}_1) = (\boldsymbol{Y} - \boldsymbol{1}\widetilde{\boldsymbol{\beta}}_1^\top)^\top(\boldsymbol{Y} - \boldsymbol{1}\widetilde{\boldsymbol{\beta}}_1^\top) = \widetilde{\boldsymbol{Y}}^\top\widetilde{\boldsymbol{Y}}.$$

Finalmente podemos escribir:

$$T = \Lambda^{2/n} = \frac{|Q(\widehat{\boldsymbol{B}})|}{|Q(\widetilde{\boldsymbol{\beta}}_1)|} = \frac{|\widetilde{\boldsymbol{Y}}^\top(\boldsymbol{I} - \boldsymbol{H}_2)\widetilde{\boldsymbol{Y}}|}{|\widetilde{\boldsymbol{Y}}^\top\widetilde{\boldsymbol{Y}}|}.$$