Properties of eigenvalues and eigenvectors

A square matrix $\mathbf{A}$ and its transpose have the same eigenvalues.

Proof. We have that

$$\begin{aligned} \boldsymbol{det}(\mathbf{A}^{T} - \lambda\mathbf{I}) &= \boldsymbol{det}(\mathbf{A}^{T} - \lambda\mathbf{I}^{T}) \\ &= \boldsymbol{det}(\mathbf{A} - \lambda\mathbf{I})^{T} \\ &= \boldsymbol{det}(\mathbf{A} - \lambda\mathbf{I}) \end{aligned}$$

so any solution of $\boldsymbol{det}(\mathbf{A} - \lambda\mathbf{I}) = 0$ is a solution of $\boldsymbol{det}(\mathbf{A} - \lambda\mathbf{I})^{T} = 0$ and vice versa. Thus $\mathbf{A}$ and $\mathbf{A}^{T}$ have the same eigenvalues.

The matrices $\mathbf{A}$ and $\mathbf{A}^{T}$ will usually have different eigen***vectors***.

The eigenvalues of a diagonal or triangular matrix are its diagonal elements.

Proof. Suppose the matrix $\mathbf{A}$ is diagonal or triangular. If you subtract λ's from its diagonal elements, the result $\mathbf{A} - \lambda\mathbf{I}$ is still diagonal or triangular. Its determinant is the product of its diagonal elements, so it is just the product of factors of the form (*diagonal element* – λ). The roots of the characteristic equation must then be the diagonal elements.

Another addition to the square matrix theorem.

An n x n matrix is invertible if and only if it doesn't have 0 as an eigenvalue.

Proof. An n x n matrix $\mathbf{A}$ has an eigenvalue 0 if and only if $\boldsymbol{det}(\mathbf{A} - 0\mathbf{I}) = 0$, i.e. if and only if $\boldsymbol{det}(\mathbf{A}) = 0$. Since A is invertible if and only if $\boldsymbol{det}\mathbf{A} \neq 0$, $\mathbf{A}$ is invertible if and only if 0 is not an eigenvalue of $\mathbf{A}$.

If a matrix **A** has eigenvalue λ with corresponding eigenvector $\mathbf{x}$, then for any k = 1, 2, ... , $\mathbf{A}^k$ has eigenvalue λ^k corresponding to the same eigenvector $\mathbf{x}$.

Proof. Suppose the matrix **A** has eigenvalue λ with eigenvector $\mathbf{x}$, i.e. suppose that $\mathbf{Ax} = \lambda\mathbf{x}$. Then $\mathbf{A}^2\mathbf{x} = \mathbf{A}(\mathbf{Ax}) = \mathbf{A}(\lambda\mathbf{x}) = \lambda(\mathbf{Ax}) = \lambda(\lambda\mathbf{x}) = \lambda^2\mathbf{x}$. Multiply by more **A**'s to get $\mathbf{A}^3\mathbf{x} = \lambda^3\mathbf{x}$, $\mathbf{A}^4\mathbf{x} = \lambda^4\mathbf{x}$ and so on.

If **A** is an invertible matrix with eigenvalue λ corresponding to eigenvector $\mathbf{x}$, then $\mathbf{A}^{-1}$ has eigenvalue λ^{-1} corresponding to the same eigenvector $\mathbf{x}$.

Proof. Multiply the equation $\mathbf{Ax} = \lambda\mathbf{x}$ by $\lambda^{-1}\mathbf{A}^{-1}$:

$$\lambda^{-1}\mathbf{A}^{-1}(\mathbf{Ax}) = \lambda^{-1}\mathbf{A}^{-1}\lambda\mathbf{x},$$

i.e.

$$\lambda^{-1}\mathbf{x} = \mathbf{A}^{-1}\mathbf{x}.$$

Thus $\mathbf{A}^{-1}$ has eigenvalue λ^{-1} corresponding to the same eigenvector $\mathbf{x}$.

Eigenvectors of a matrix **A** with distinct eigenvalues are linearly independent.

Proof. Suppose the statement is not true, i.e. suppose that **A** has a linearly dependent set of eigenvectors each with a different eigenvalue. "Thin out" this set of vectors to get a linearly independent subset $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_k$, with distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_k$.

Suppose $\mathbf{u}$ is one of the eigenvectors you thinned out because it was linearly dependent on the others:

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k \qquad *$$

for some scalars $c_1, c_2, ..., c_k$.

First multiply * by **A**:

$\mathbf{Au}$

$$= \mathbf{A}(c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + ... + c_k\mathbf{v}_k)$$
$$= c_1\mathbf{A}\mathbf{v}_1 + c_2\mathbf{A}\mathbf{v}_2 + ... + c_k\mathbf{A}\mathbf{v}_k$$
$$= c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + ... + c_k\lambda_k\mathbf{v}_k.$$

Since $\mathbf{u}$ is also an eigenvector, $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$ for some eigenvalue λ, so this equation gives

$$\lambda\mathbf{u} = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 + ... + c_k\lambda_k\mathbf{v}_k. \qquad **$$

Now multiply * by λ:

$$\lambda\mathbf{u} = \lambda c_1\mathbf{v}_1 + \lambda c_2\mathbf{v}_2 + ... + \lambda c_k\mathbf{v}_k\ . \qquad ***$$

Subtract *** from ** to get

$$\mathbf{0} = (c_1\lambda_1 - \lambda c_1)\mathbf{v}_1 + c_2\lambda_2 - \lambda c_2)\mathbf{v}_2 + ... + (c_k\lambda_k - \lambda c_k)\mathbf{v}_k$$
$$= c_1(\lambda_1 - \lambda)\mathbf{v}_1 + c_2(\lambda_2 - \lambda)\mathbf{v}_2 + ... + c_k(\lambda_k - \lambda)\mathbf{v}_k.$$

Since the $\mathbf{v}_i$'s are linearly independent, $c_i(\lambda_i - \lambda) = 0$ for all $i = 1, 2, ..., k$. Since the eigenvalues (including λ) are all different, $c_i = 0$ for all i. But this implies (from equation *) that $\mathbf{u} = \mathbf{0}$, which is impossible since $\mathbf{u}$ is an eigenvector.

The original assumption must be false, i.e. it is not possible to have a linearly dependent set of eigenvectors with distinct eigenvalues; any eigenvectors with distinct eigenvalues must be linearly independent.