## Inverse and determinant of partitioned symmetric matrix

**Theorem 1**

$$(A + CBD)^{-1} = A^{-1} - A^{-1}C(B^{-1} + DA^{-1}C)^{-1}DA^{-1}$$

**Proof:**

$$(A + CBD)[A^{-1} - A^{-1}C(B^{-1} + DA^{-1}C)^{-1}DA^{-1}]$$

$$= (A + CBD)A^{-1} - (A + CBD)A^{-1}C(B^{-1} + DA^{-1}C)^{-1}DA^{-1}$$

$$= I + CBDA^{-1} - (C + CBDA^{-1}C)(B^{-1} + DA^{-1}C)^{-1}DA^{-1}$$

$$= I + CBDA^{-1} - CB(B^{-1} + DA^{-1}C)(B^{-1} + DA^{-1}C)^{-1}DA^{-1}$$

$$= I + CBDA^{-1} - CBDA^{-1} = I$$

**Theorem 2** (inverse of a partitioned symmetric matrix)

Divide an $n \times n$ symmetric matrix $A$ into four blocks

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix}$$

The inverse matrix $B = A^{-1}$ can also be divided into four blocks:

$$B = A^{-1} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix}$$

Here we assume the dimensionalities of these blocks are:

- $A_{11}$ and $B_{11}$ are $p \times p$,

- $A_{22}$ and $B_{22}$ are $q \times q$,

- $A_{12} = A_{21}^T$ and $B_{12} = B_{21}^T$ are $p \times q$

with $p + q = n$. Then we have

$$
\begin{aligned}
B_{11} &= (A_{11} - A_{12}A_{22}^{-1}A_{12}^T)^{-1} = A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{12}^T A_{11}^{-1} A_{12})^{-1} A_{12}^T A_{11}^{-1} \\
B_{22} &= (A_{22} - A_{12}^T A_{11}^{-1} A_{12})^{-1} = A_{22}^{-1} + A_{22}^{-1}A_{12}^T(A_{11} - A_{12}A_{22}^{-1}A_{12}^T)^{-1} A_{12} A_{22}^{-1} \\
B_{12}^T &= -A_{22}^{-1} A_{12}^T (A_{11} - A_{12}A_{22}^{-1}A_{12}^T)^{-1} \\
B_{12}^T &= -A_{11}^{-1} A_{12} (A_{22} - A_{12}^T A_{11}^{-1} A_{12})^{-1}
\end{aligned}
$$

**Proof:**

$$
I_n = AA^{-1} = AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{12}^T & B_{22} \end{bmatrix}
$$

$$
= \begin{bmatrix} A_{11}B_{11} + A_{12}B_{12}^T & A_{11}B_{12} + A_{12}B_{22} \\ A_{12}^T B_{11} + A_{22}B_{12}^T & A_{12}^T B_{12} + A_{22}B_{22} \end{bmatrix} = \begin{bmatrix} I_p & 0 \\ 0 & I_q \end{bmatrix}
$$

i.e.,

$$A_{11}B_{11} + A_{12}B_{12}^T = I_p \quad \text{or} \quad B_{11} = A_{11}^{-1} - A_{11}^{-1}A_{12}B_{12}^T$$

$$A_{11}B_{12} + A_{12}B_{22} = 0 \quad \text{or} \quad B_{12} = -A_{11}^{-1}A_{12}B_{22}$$

$$A_{12}^T B_{11} + A_{22}B_{12}^T = 0 \quad \text{or} \quad B_{12}^T = -A_{22}^{-1}A_{12}^T B_{11}$$

$$A_{12}^T B_{12} + A_{22}B_{22} = I_q \quad \text{or} \quad B_{22} = A_{22}^{-1} - A_{22}^{-1}A_{12}^T B_{12}$$

Plug $B_{12}^T$ into $B_{11}$ to get

$$B_{11} = A_{11}^{-1} + A_{11}^{-1}A_{12}A_{22}^{-1}A_{12}^T B_{11}$$

or

$$(I - A_{11}^{-1}A_{12}A_{22}^{-1}A_{12}^T)B_{11} = A_{11}^{-1} \quad \text{or} \quad (A_{11} - A_{12}A_{22}^{-1}A_{12}^T)B_{11} = I_p$$

or

$$B_{11} = (A_{11} - A_{12} A_{22}^{-1} A_{12}^T)^{-1}$$

Applying theorem 1 to this expression, we also get the other expression in the theorem. Similarly we can get

$$B_{22} = (A_{22} - A_{12}^T A_{11}^{-1} A_{12})^{-1}$$

and

$$B_{12}^T = -A_{22}^{-1} A_{12}^T (A_{11} - A_{12} A_{22}^{-1} A_{12}^T)^{-1}$$

$$B_{12} = -A_{11}^{-1} A_{12} (A_{22} - A_{12}^T A_{11}^{-1} A_{12}^{-1}$$

**Theorem 3** (Determinant of a partitioned symmetric matrix)

$$|A| = \left\| \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \right\| = |A_{22}||A_{11} - A_{12}A_{22}^{-1}A_{12}^T| = |A_{11}||A_{22} - A_{12}^T A_{11}^{-1}A_{12}|$$

**Proof:** Note that

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{12}^T & I \end{bmatrix} \begin{bmatrix} I & A_{11}^{-1} A_{12} \\ 0 & A_{22} - A_{12}^T A_{11}^{-1} A_{12} \end{bmatrix}$$

$$= \begin{bmatrix} I & A_{12} \\ 0 & A_{22} \end{bmatrix} \begin{bmatrix} A_{11} - A_{12} A_{22}^{-1} A_{12}^T & 0 \\ A_{22}^{-1} A_{21} & I \end{bmatrix}$$

The theorem is proved as we also know that

$$|AB| = |A| \, |B|$$

and

$$\begin{vmatrix} B & 0 \\ C & D \end{vmatrix} = \begin{vmatrix} B & C \\ 0 & D \end{vmatrix} = |B| \, |D|$$

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