Ejercicio 3

Sea $x_1, \dots, x_n$ TIAS de $X \sim (\mu; \Sigma)$, $\bar{x}_n = \frac{1}{n}\sum_{i=1}^{n} x_i$

a $E(\bar{x}_n) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n} E\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) =$

$= \frac{1}{n}\sum_{i=1}^{n} \mu = \frac{1}{n}(n\mu) = \mu$

$V(\bar{x}_n) = V\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2} V\left(\sum_{i=1}^{n} x_i\right) =$

$= \frac{1}{n^2}\left[\sum_{i=1}^{n} V(x_i) + 2\sum_{j=1}^{n}\sum_{i<j} \mathrm{cov}(x_i; x_j)\right] = \boxed{\text{por TIAS}} =$

$= \frac{1}{n^2}\left(n\Sigma + 2\,0\right) = \frac{1}{n}\Sigma$

b Si $S = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x}_n)(x_i - \bar{x}_n)' \Rightarrow E(S) = \frac{n-1}{n}\Sigma$

si $E(S) = \left(\frac{n-1}{n}\right)\Sigma \Rightarrow \frac{n}{n-1}E(S) = \Sigma \Rightarrow$ por

linealidad de la esperanza:

$E\left[\left(\frac{n}{n-1}\right)S\right] = \Sigma \Rightarrow \left(\frac{n}{n-1}\right)S$ es un estimador

insesgado de Σ

Ejercicio 4

$Y_{n\times m} = X_{n\times p} A_{p\times m}$

Del ejercicio 2: $V(AX) = AV(X)A' \Rightarrow$ de forma análoga se demuestra que $V(XA) = AV(X)A' \Rightarrow$

$\Rightarrow V(Y) = V(XA) = A'V(X)A \Rightarrow S_Y = A'S_X A$

Ejercicio 5

$$\frac{1}{np}\sum_{i=1}^{n}(x_i-\bar{x})'S^{-1}(x_i-\bar{x}) = tr\left(\frac{1}{np}\sum_{i=1}^{n}(x_i-\bar{x})'S^{-1}(x_i-\bar{x})\right) =$$

$$= \frac{1}{np}\sum_{i=1}^{n} tr\left((x_i-\bar{x})'S^{-1}(x_i-\bar{x})\right) =$$

$$= \frac{1}{np}\sum_{i=1}^{n} tr\left(S^{-1}(x_i-\bar{x})'(x_i-\bar{x})\right) =$$

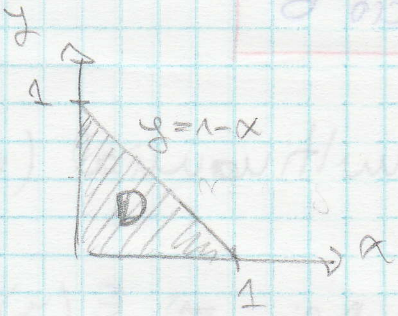
$$= \frac{1}{np} tr\left(\sum_{i=1}^{n} S^{-1}(x_i-\bar{x})'(x_i-\bar{x})\right) =$$

$$= \frac{1}{np} tr\left(S^{-1}\sum_{i=1}^{n}(x_i-\bar{x})'(x_i-\bar{x})\right) = \frac{1}{p} tr\left(S^{-1}\underbrace{\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})'(x_i-\bar{x})}_{S}\right)$$

$$= \frac{1}{p} tr(S^{-1}S) = \frac{1}{p} tr(I_p) = \frac{1}{p}\cdot p = 1$$

Ejercicio 7

$$f_{XY}(x,y) = \begin{cases} kx & \text{si } \begin{cases} 0 < x < 1 \\ 0 < y < 1-x \end{cases} \\ 0 & \text{en } \%c \end{cases}$$



a) $f(x,y) = \int_{x=0}^{x=1} \int_{y=0}^{y=1-x} kx \, dx\,dy = \int_0^1 kx \left(y \Big|_0^{1-x} \right) dx =$

$= \int_0^1 kx(1-x)\, dx = k\left(\int_0^1 x\, dx - \int_0^1 x^2 dx \right) =$

$= k\left(\frac{x^2}{2}\Big|_0^1 - \frac{x^3}{3}\Big|_0^1 \right) = k \; \frac{1}{6} = 1 \Rightarrow \boxed{k = 6}$

b) $f_X(x) = \int_0^{1-x} 6x\, dy = 6x\left(y\Big|_0^{1-x} \right) = 6x(1-x)$

$$f_X(x) = \begin{cases} 6x(1-x) & \text{si } 0 \leq x \leq 1 \\ 0 & \text{en } \%c \end{cases}$$

Verificación

$f(x) = \int_0^1 6x(1-x)\, dx = 6\int_0^1 x\,dx - 6\int_0^1 x^2 dx =$

$= 6\left(\frac{x^2}{2}\Big|_0^1 \right) - 6\left(\frac{x^3}{3}\Big|_0^1 \right) = 3 - 2 = 1$ ✓

$$f_Y(y) = \int_0^{1-y} 6x\,dx = 6\left(\frac{x^2}{2}\Big|_0^{1-y}\right) = 3(1-y)^2$$

Verificación

$$f_Y(y) = \int_0^1 3(1-y)^2\,dy = 3\int_0^1 (1-2y+y^2)\,dy =$$

$$= 3\int_0^1 1\,dy - 6\int_0^1 y\,dy + 3\int_0^1 y^2\,dy =$$

$$= 3\left(y\Big|_0^1\right) - 6\left(\frac{y^2}{2}\Big|_0^1\right) + 3\left(\frac{y^3}{3}\Big|_0^1\right) =$$

$$= 3 - 3 + 1 = 1 \checkmark$$

Por lo tanto: $f_Y(y) = \begin{cases} 3(1-y)^2 & \text{si } 0 \le y \le 1 \\ 0 & \text{en c/c} \end{cases}$

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{6x}{3(1-y)^2} = \frac{2x}{(1-y)^2} \quad \text{si } x,y \in [0,1]$$

$$f_{Y|X}(y|x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\cancel{6x}}{\cancel{6x}(1-x)} = \frac{1}{1-x} \quad \text{si } x,y \in [0,1]$$

$$E(X,Y) = \begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}$$

$$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x\,6x(1-x)\,dx =$$

$$= 6\int_0^1 x^2\,dx - 6\int_0^1 x^3\,dx = 6\left(\frac{x^3}{3}\Big|_0^1\right) - 6\left(\frac{x^4}{4}\Big|_0^1\right) = \frac{1}{2}$$

$$E(Y) = \int_0^1 y f_Y(y)\, dy = \int_0^1 y\, 3(1-y)^2\, dy =$$

$$= 3\int_0^1 y - 2y^2 + y^3\, dy = 3\left[\left(\frac{y^2}{2}\Big|_0^1\right) - 2\left(\frac{y^3}{3}\Big|_0^1\right) + \frac{y^4}{4}\Big|_0^1\right] =$$

$$= \frac{3}{2} - 2 + \frac{3}{4} = 0{,}25 \Rightarrow E(X,Y) = \begin{pmatrix} 0{,}5 \\ 0{,}25 \end{pmatrix}$$

$$\circledast \quad \Sigma_{XY} = \begin{pmatrix} \sigma_X^2 & \sigma_{XY} \\ \sigma_{XY} & \sigma_Y^2 \end{pmatrix}$$

$$* V(X) = E(X^2) - E^2(X) = 0{,}3 - 0{,}5^2 = 0{,}05$$

$$E(X^2) = \int_0^1 x^2\, 6x(1-x)\, dx = 6\int_0^1 x^3\, dx - 6\int_0^1 x^4\, dx =$$

$$= 6\left(\frac{x^4}{4}\Big|_0^1\right) - 6\left(\frac{x^5}{5}\Big|_0^1\right) = 1{,}5 - 1{,}2 = 0{,}3$$

$$* V(Y) = E(Y^2) - E^2(Y) = 0{,}1 - 0{,}25^2 = 0{,}0375$$

$$E(Y^2) = \int_0^1 y^2\, 3(1-y)^2\, dy = 3\int_0^1 y^2 - 2y^3 + y^4\, dy =$$

$$= 3\left(\frac{y^3}{3}\Big|_0^1\right) - 6\left(\frac{y^4}{4}\Big|_0^1\right) + 3\left(\frac{y^5}{5}\Big|_0^1\right) =$$

$$= 1 - 1{,}5 + 0{,}6 = 0{,}1$$

* $\mathrm{COV}(X,Y) = E(XY) - E(X)\,E(Y) = -0{,}025$

$$E(XY) = \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} x\,y\,6x\,dx\,dy = 6\int_0^1 x^2\left[\int_0^{1-y} y\,dy\right]dx =$$

$$= 6\int_0^1 x^2 \left(\frac{y^2}{2}\Big|_0^{1-x}\right)dx = 6\int_0^1 x^2\,\frac{(1-x)^2}{2}\,dx =$$

$$= 3\int_0^1 x^2 - 2x^3 + x^4\,dx =$$

$$= 3\int_0^1 x^2 dx - 6\int_0^1 x^3 dx + 3\int_0^1 x^4 dx =$$

$$= 3\left(\frac{x^3}{3}\Big|_0^1\right) - 6\left(\frac{x^4}{4}\Big|_0^1\right) + 3\left(\frac{x^5}{5}\Big|_0^1\right) =$$

$$= 1 - 1{,}5 + 0{,}6 = 0{,}1$$

Por lo tanto: $\Sigma_{XY} = \begin{pmatrix} 0{,}05 & -0{,}025 \\ -0{,}025 & 0{,}0375 \end{pmatrix}$