Mathematical Analysis. Assignment 6.
Using Derivatives for Exploring Functions

1. Find the asymptotes of the following functions:

   (a) $y = \sqrt{x^4 + x^3} - \sqrt{x^4 - x^3}$;

   (b) $y = x \tanh x$;

   (c) $y = \ln(1 + e^x)$;

   (d) $y = \frac{1}{\operatorname{arccot} \frac{1}{x}}$;

   (e) $y = x + \sqrt{4x^2 + 1}$;

   (f) $y = \frac{(x-3)^2}{4x-4}$;

   (g) $x = \frac{t^2}{t-1}$, $y = \frac{t}{t^2-1}$.

   Answer: (a) $y = x$; (b) $y = x$, $y = -x$; (c) $y = 0$, $y = x$; (d) $x = 0$, $y = \frac{2}{\pi}$; (e) $y = -x$, $y = 3x$; (f) $x = 1$, $y = \frac{x-5}{4}$; (g) $x = \frac{1}{2}$, $y = 0$, $y = \frac{2x-3}{4}$.

2. Find the equations of tangent lines and normal lines to the following curves at the indicated points:

   (a) $x^3 + y^2 + 2x - 6 = 0$, $M(-1; 3)$;

   (b) $x = \frac{2t-1}{t^2}$, $y = \frac{3t^2-1}{t^3}$, $M(1; 2)$.

   Answer: (a) $5x + 6y - 13 = 0$ & $6x - 5y + 21 = 0$; (b) $3x - y - 1 = 0$ & $x + 3y - 7 = 0$.

3. Find the angles between the graphs of the functions at their intersection points:

   (a) $f(x) = \ln x$ and $g(x) = \frac{x^2}{2e}$;

   (b) $f(x) = 4x^2 + 2x - 8$ and $g(x) = x^3 - x + 10$.

   Answer: (a) the intersection point is $\left(\sqrt{e}; \frac{1}{2}\right)$, the angle is 0; (b) two intersection points $(3; 34)$ and $(-2; 4)$, the angles are equal to 0 and $\arctan \frac{25}{153}$ respectively.

4. Let us consider a function $f(x)$ continuous on an open interval $(a; b)$ such that its derivative is positive on this interval but for a finite number of points. Prove that $f(x)$ is strictly increasing on $(a; b)$.

5. Let us consider $f(x) = \begin{cases} x^2 \left(2 + \cos \frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$ Prove that

   (a) $f(x)$ has a strict minimum at $x = 0$;

   (b) for any $\delta > 0$ function is not decreasing on $(-\delta; 0)$ and is not increasing on $(0; \delta)$.

6. Two functions are given:

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} xe^{-\frac{1}{x^2}}, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

   Prove that

   (a) $f^{(n)}(0) = g^{(n)}(0) = 0$, $n \in \mathbb{N}$;

   (b) $f(x)$ has a strict minimum at $x = 0$ whereas $x = 0$ yields neither minimum nor maximum for $g(x)$.

(a) $y = \sqrt{x^4 + x^3} - \sqrt{x^4 - x^3}$;

$$y'_x = \frac{1}{2} \cdot \frac{1}{(x^4 + x^3)^{3/2}} \cdot (4x^3 + 3x^2) - \frac{1}{2} \cdot \frac{1}{(x^4 - x^3)^{3/2}} \cdot (4x^3 - 3x^2)$$

7. Find the inflection points of the following functions:

   (a) $f(x) = x^4 - 12x^3 + 48x^2$;
   (b) $f(x) = x^3 e^{-4x}$.

   Answer: (a) $x = 2$, $x = 4$; (b) $x = e^{-1.5}$.

8. Let $f(x) = \begin{cases} x^3\left(2 + \cos\frac{1}{x^2}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$ Prove that

   (a) there exists a tangent line to the graph of $f(x)$ at the origin;
   (b) when x passes through 0, the graph passes from one side of the tangent line to the other side of the tangent line;
   (c) $x = 0$ is not an inflection point of $f(x)$.

9. Draw the graphs of the following functions[1]:

   (a) $y = (x-1)^2(x+2)$;
   (b) $y = \frac{(x-1)^2}{(x+1)^3}$;
   (c) $y = \frac{x^3+2x^2}{(x-1)^2}$;
   (d) $y = \frac{(x-5)^3}{(x-7)^2}$;
   (e) $y = x - \sqrt{x^2 - 2x}$;
   (f) $y = \frac{3x-2}{\sqrt{x^2-1}}$;
   (g) $y = \sqrt[3]{x^2(3-x)}$;
   (h) $y = \frac{x}{\sqrt[3]{(x-2)^2}}$;
   (i) $y = \sqrt[3]{x^2|2-x|}$;
   (j) $y = \left(x^2 - 2\right)e^{-2x}$;
   (k) $y = \frac{\ln x}{\sqrt{x}}$.

   Answer:

   (a) $(1;0)$ is a local minimum, $(-1;4)$ is a local maximum, $(0;2)$ is an inflection point.
   (b) The asymptotes are $x = -1$ and $y = 0$; $(1;0)$ is a local minimum, $\left(5;\frac{2}{27}\right)$ is a local maximum; there are two inflection points at $x = 5 \pm 2\sqrt{3}$.
   (c) The asymptotes are $y = x + 4$ and $x = 1$; there are two local minima $(0;0)$, $\left(4;\frac{32}{3}\right)$ and one local maximum $\left(-1;\frac{1}{4}\right)$. $\left(-\frac{2}{7};\frac{16}{189}\right)$ is an inflection point.
   (d) The asymptotes are $x = 7$ and $y = x - 1$. $(11;13.5)$ is a local minimum and $(5;0)$ is an inflection point.
   (e) The asymptotes are $y = 0$ and $y = 2x$. There are no local maxima, local minima or inflection points.
   (f) The asymptotes are $y = 3$, $y = -3$, $x = 1$, $x = -1$. $\left(\frac{3}{2};\sqrt{5}\right)$ is a local minimum, $\left(2;\frac{4}{\sqrt{3}}\right)$ is an inflection point.
   (g) The asymptote is $y = 1 - x$; $(0;0)$ is a local minimum, $\left(2;\sqrt[3]{4}\right)$ is a local maximum, $(3;0)$ is an inflection point.

[1] You have to find asymptotes, minima, maxima and inflection points.

(e) $y = x - \sqrt{x^2 - 2x}$;

1) Област опр. $\qquad D(y): (-\infty;0] \cup [2;+\infty)$

$x^2 - 2x \geq 0$

$x(x-2) \geq 0$

$+ \quad 0 \quad - \quad 2 \quad +$ $\qquad x \in (-\infty;0] \cup [2;+\infty)$

2) Верт. асимп. $\qquad$ Граница обл. опр.

$$\lim_{x \to +\infty} \left(x - \sqrt{x^2 - 2x}\right) = 1$$

$$\lim_{x \to -\infty} \left(x - \sqrt{x^2 - 2x}\right) = -\infty$$

$\lim\limits_{x \to 0} y = 0 \qquad \lim\limits_{x \to 2} y \nexists$

$\lim\limits_{x \to 0^+} y \nexists \qquad \lim\limits_{x \to 2} y = 2$

3) Наклонная асимптота

$$a_1 = \lim_{x \to +\infty} \frac{x - \sqrt{x^2 - 2x}}{x} = \lim_{x \to +\infty} \left(1 - \sqrt{1 - \frac{2}{x}}\right) = 0$$

$$b_1 = \lim_{x \to +\infty} \left(x - \sqrt{x^2 - 2x} - 0\right) = 0$$

$$a_2 = \lim_{x \to -\infty} \frac{x - \sqrt{x^2 - 2x}}{x} = \lim_{x \to -\infty} \frac{x^2 - x^2 + 2x}{x\left(x + \sqrt{x^2 - 2x}\right)} = \lim_{x \to -\infty} \frac{2}{x + \sqrt{x^2 - 2x}} =$$

$$= \text{“} t = -x \text{”} = \lim_{t \to +\infty} \frac{-t - \sqrt{t^2 + 2t}}{-t} = \lim_{t \to +\infty} \frac{t + \sqrt{t^2 + 2t}}{t} = \lim_{t \to +\infty} \frac{t^2 - t^2 - 2t}{t\left(t - \sqrt{t^2 + 2t}\right)} =$$

$$= \lim_{t \to +\infty} \frac{-2}{\left(t - \sqrt{t^2 + 2t}\right)} = \lim_{t \to +\infty} \frac{2}{\sqrt{t^2 + 2t} - t} = \frac{2}{1} = 2$$

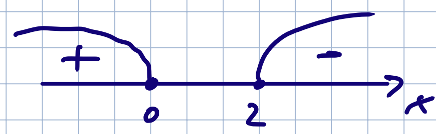
$$\lim_{t \to +\infty} \sqrt{t^2+2t} - t = \lim_{t \to +\infty} \frac{\cancel{t^2} + 2t - \cancel{t^2}}{\sqrt{t^2+2t}+t} = \underline{1}$$

$$b_2 = \lim_{x \to -\infty} \left(x - \sqrt{x^2-2x} - 2x \right) = \underline{-1}$$

$$\lim_{t \to +\infty} \left(-\sqrt{x^2+2t} + t \right)$$

$$y'_x = 0 - \frac{1}{2\left(x^2-2x\right)^{3/2}}\left(2x-2\right)$$

$$y'_x = +\frac{1-x}{\sqrt{\left(x^2-2x\right)^3}}$$



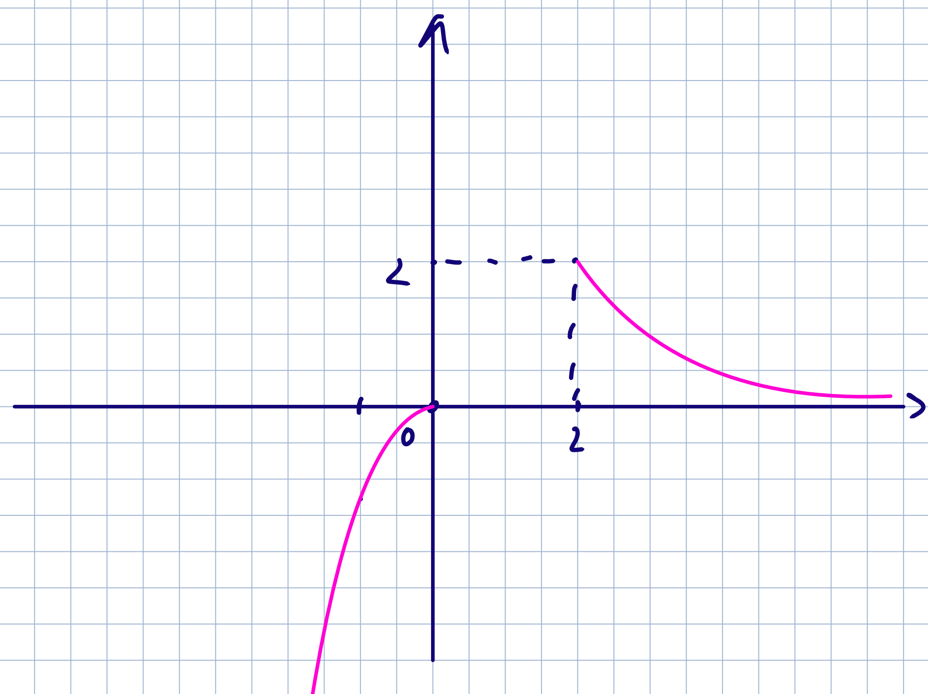
$$y = x - \sqrt{x(x-2)}$$

$$\lim_{x\to+\infty}\left(x-\sqrt{x^2-2x}\right)$$

$$x-\sqrt{x^2-2x} = \left(x-\sqrt{x^2-2x}\right)\cdot\frac{x+\sqrt{x^2-2x}}{x+\sqrt{x^2-2x}} =$$

$$= \frac{x-x^2+2x}{x+\sqrt{x^2-2x}} =$$

$$= \frac{2}{1+\sqrt{1-\frac{2}{x}}} \xrightarrow[x\to+\infty]{} 1$$



$$\lim_{x \to -\infty} \left(x - \sqrt{x^2 - 2x} \right)$$

(h) $x=2$ is an asymptote, $\left(6; \frac{3}{\sqrt[3]{2}}\right)$ is a local minimum, $\left(12; \frac{12}{\sqrt[3]{100}}\right)$ is an inflection point.

(i) The asymptotes are $y=-x+\frac{2}{3}$ and $y=x-\frac{2}{3}$; $(0;0)$ and $(2;0)$ are local minima, $\left(\frac{4}{3}; \frac{2\sqrt[3]{4}}{3}\right)$ is a local maximum.

(j) $y=0$ is an asymptote; $(-1;-e^2)$ is a local minimum, $(2;e^{-4})$ is a local maximum; there are two inflection points at $x=1\pm\frac{\sqrt{10}}{2}$.

(k) The asymptotes are $y=0$ and $x=0$; $\left(e^2; \frac{2}{e}\right)$ is a local maximum, $\left(e^{\frac{8}{3}}; \frac{8}{3e^{\frac{4}{3}}}\right)$ is an inflection point.

10. Draw the curves given by parametric equations[2]:

(a) $x=(t-1)^2(t-2)$, $y=(t-1)^2(t-3)$;

(b) $x=\frac{1}{t^3-t^2}$, $y=\frac{1}{t^2-t}$;

(c) $x=t^3-3t$, $y=\left(\frac{t-1}{t}\right)^2$;

(d) $x=\frac{(t+1)^2}{t}$, $y=\frac{t+1}{t+2}$.

[2]You have to find asymptotes, minima and maxima, inflection points and cusps.