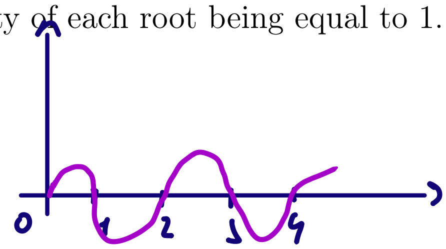
Mathematical Analysis. Assignment 4.
Mean Value Theorems & l'Hôpital's rule

1. Prove that the derivative of a function $f(x) = x(x-1)(x-2)(x-3)(x-4)$ has four distinct roots that belong to intervals $(0;1)$, $(1;2)$, $(2;3)$, $(3;4)$, the multiplicity of each root being equal to 1.



2. Using mean value theorems prove the inequalities[1]

   (a) $\frac{x}{x+1} < \ln(1+x) < x$ for $x > 0$;

   (b) $e^x \geqslant 1 + x$, $x \in \mathbb{R}$.

3. Rolle's theorem states that if function $f(x)$

   (a) is continuous on $[a;b]$,

   (b) is differentiable on $(a;b)$,

   (c) has equal values at the endpoints of the interval, that is $f(a) = f(b)$ then there exists $c \in (a;b)$ such that $f'(c) = 0$.

   Show that all conditions of the theorem are substantial, i.e. that the theorem does not hold if you omit at least one of them[2].

4. Find the following limits (use l'Hôpital's rule[3]):

   (a) $\lim\limits_{x\to 1} \frac{x^{100}-100x+99}{x^{50}-50x+49}$;

   (b) $\lim\limits_{x\to 0} \frac{e^{\sin x}-e^x}{\sin x - x}$; $= \frac{\cos x \cdot e^{\sin x} - e^x}{\cos x - 1} \underset{\lim\limits_{x\to 0}}{=} \frac{-\sin x \cdot e^{\sin x} + \cos^2 x\, e^{\sin x} - e^x}{-\sin x} =$

   $= \frac{1}{-\cos x}\Big(-\sin x \cdot e^{\sin x} \cdot \sin x + -\cos x \cdot e^{\sin x} - 2\cos x \cdot (-\sin x)\, e^{\sin x} + \cos^2 x \cdot e^{\sin x} - e^x\Big)$

   $= -(0 - 1 - 2 + 1 \cdot 1) =$

   $= +1$

   (c) $\lim\limits_{x\to 0+} \frac{3+\ln x}{2-3\ln(\sin x)}$;

   (d) $\lim\limits_{x\to 0} \sin x \ln(\cot x)$;

   (e) $\lim\limits_{x\to +\infty} (\pi - 2\arctan\sqrt{x})\sqrt{x}$;

   (f) $\lim\limits_{x\to 0} \left(\frac{1}{x} - \frac{1}{e^x-1}\right)$;

   (g) $\lim\limits_{x\to 1} x^{\frac{1}{x-1}}$;

   (h) $\lim\limits_{x\to 0} \left(\frac{(1+x)^{\frac{1}{x}}}{e}\right)^{\frac{1}{x}}$;

   (i) $\lim\limits_{x\to \frac{\pi}{2}-} (\tan x)^{\cos x}$;

   (j) $\lim\limits_{x\to 0} \frac{\ln\frac{1+x}{1-x} - 2x}{x - \sin x}$;

   (k) $\lim\limits_{x\to 0} \frac{(a+x)^x - a^x}{x^2}$, $a > 0$.

   (l) $\lim\limits_{x\to 0} \frac{2\tan 3x - 6\tan x}{3\arctan x - \arctan 3x}$;

   (m) $\lim\limits_{x\to 0+} x^\alpha \ln^\beta\left(\frac{1}{x}\right)$;

   (n) $\lim\limits_{x\to +\infty} x^\alpha a^x$, $a > 0$, $a \neq 1$;

   (o) $\lim\limits_{x\to +\infty} \left(x^{\frac{7}{8}} - x^{\frac{6}{7}} \ln^2 x\right)$;

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[1](a) Apply Lagrange mean value theorem to function $f(t) = \ln(1+t)$, $t \in [0;x]$. (b) If $x \in (-1;0)$ consider $g(t) = e^t - t$, $t \in [x;0]$; if $x > 0$ consider $g(t) = e^t$, $t \in [0;x]$. Otherwise this inequality is obvious.

[2]It implies that you have to provide 3 counterexamples.

[3]The dreams come true...

(a) $\frac{x}{x+1} < \ln(1+x) < x$ for $x > 0$;

1) Т. Лагранжа $\quad f(x) = \ln(x+1)$

$$\frac{f(b)-f(a)}{b-a} = f'(\xi)$$

$$\frac{\ln(x+1)}{x} = f'(\xi) = \frac{1}{\xi+1} \qquad 0 < \xi < x$$

$\xi+1 < x+1$

$(\xi+1)^{-1} > (x+1)^{-1}$

$$\frac{\ln(x+1)}{x} = \frac{1}{\xi+1}$$

$$\frac{\ln(x+1)}{x} = \frac{1}{\xi+1} < 1$$

$$\ln(x+1) < x$$

2) $\frac{\ln(x+1)}{x} = \frac{1}{\xi+1} > \frac{1}{x+1} \Rightarrow \ln(x+1) > \frac{x}{x+1}$

(b) (b) $e^x \geqslant 1+x$, $x \in \mathbb{R}$.

Т. Лагранжа

$f(x) = e^x \qquad \xi \in (0, x)$

$$\frac{f(b)-f(a)}{b-a} = f'(\xi) \qquad 0 < \xi < x$$

$1 < e^\xi < e^x$

$$\frac{e^x}{x} = e^\xi > 1$$

$$e^x > x$$

$$\frac{e^x - e^{-1}}{x+1} = e^\xi \qquad -1 < \xi < x \qquad \frac{1}{e} < e^\xi < e^x$$

$$e^x - e^{-1} > -(x+1)$$

Пусть $f(x) = e^x$ По т. Коши
$g(x) = x$

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

$$\frac{e^x - e^0}{x-0} = \frac{e^\xi}{1} > 1 \qquad 0 < \xi < x$$

$$1 < e^\xi < e^x$$

$$e^x - 1 > x$$

$$\boxed{e^x > x+1}$$

(p) $\lim\limits_{x\to 1}\left(\frac{\alpha}{1-x^\alpha}-\frac{\beta}{1-x^\beta}\right)$;

(q) $\lim\limits_{x\to +\infty}(3x^2+3^x)^{\frac{1}{x}}$.

Answer. (a) $\frac{198}{49}$; (b) 1; (c) $-\frac{1}{3}$; (d) 0; (e) 2; (f) $\frac{1}{2}$; (g) e; (h) $e^{-\frac{1}{2}}$; (i) 1; (j) 4; (k) $\frac{1}{a}$; (l) 2; (m) 0; (n) 0 if $0<a<1$; $+\infty$ if $a>1$; (o) $+\infty$; (p) $\frac{\alpha-\beta}{2}$; (q) 3.

5. Show that l'Hôpital's rule is not applicable for the limits below and calculate them using some other methods:

   (a) $\lim\limits_{x\to\infty}\frac{x+\cos x}{x-\cos x}$;

   (b) $\lim\limits_{x\to 0}\frac{x^3\sin\frac{1}{x}}{\sin^2 x}$.

   Answer. (a) 1; (b) 0.

6. Let us suppose that $f(x)$ has at least three derivatives in the neighborhood of point a. Calculate the limits

   (a) $\lim\limits_{h\to 0}\frac{f(a+h)+f(a-h)-2f(a)}{h^2}$;

   (b) $\lim\limits_{h\to 0}\frac{f(a+3h)-3f(a+2h)+3f(a+h)-f(a)}{h^3}$.

   Answer. (a) $f''(a)$; (b) $f'''(a)$.

7. Let us consider $f(x)=\begin{cases} e^{-\frac{1}{x^2}}, & x\neq 0,\\ 0, & x=0.\end{cases}$ Prove that this function is infinitely differentiable for all $x\in\mathbb{R}$. Find $f^{(k)}(0)$ as well.

   Answer. $f^{(k)}(0)=0$, $k\in\mathbb{N}$.

8. Find the following limits:

   (a) $\lim\limits_{x\to 1^-}\ln x\cdot\ln(1-x)$;

   (b) $\lim\limits_{x\to 0+}\frac{\ln x\cdot\ln(1+x)}{\sqrt{x}}$;

   (c) $\lim\limits_{x\to 0}\frac{\cos x-\cos 3x+x^3\cos\frac{\pi}{x}}{x^2}$.

   Answer. (a) 0; (b) 0; (c) 4.

(q) $\lim\limits_{x\to+\infty}(3x^2+3^x)^{\frac{1}{x}}$.

$$\lim_{x\to+\infty}(3x^2+3^x)^{\frac{1}{x}} = \lim_{x\to+\infty} e^{\frac{1}{x}\ln(3x^2+3^x)} =$$

$$= \lim_{x\to+\infty} e^{\frac{\ln(3x^2+3^x)}{x}} = e^{\ln 3} = 3$$

$$\lim_{x\to+\infty}\frac{\ln(3x^2+3^x)}{x} = \lim_{x\to+\infty}\frac{\frac{1}{3x^2+3^x}\cdot(6x+\ln 3\cdot 3^x)}{1} =$$

$$= \lim_{x\to+\infty}\frac{(6x+\ln 3\cdot 3^x)''_x}{(3x^2+3^x)''_x} = \lim_{x\to+\infty}\frac{(\ln 3)^4\cdot 3^x}{(\ln 3)^3\cdot 3^x} = \ln 3$$

(p) $\lim\limits_{x\to 1}\left(\frac{\alpha}{1-x^\alpha}-\frac{\beta}{1-x^\beta}\right)$;

$$\lim_{x\to 1}\left(\frac{\alpha}{1-x^\alpha}-\frac{\beta}{1-x^\beta}\right) =$$

$$= \lim_{x\to 1}\frac{\alpha(1-x^\beta)-\beta(1-x^\alpha)}{(1-x^\alpha)(1-x^\beta)} =$$

$$\lim_{x\to 1}\frac{(\alpha(1-x^\beta)-\beta(1-x^\alpha))(1+x^\alpha)(1+x^\beta)}{(1+x^{2\alpha})(1+x^{2\beta})} = \lim_{x\to 1}\frac{\alpha(1+x^{\beta})(1+x^\alpha)-\beta(1+x^{\alpha})(1+x^\beta)}{(1+x^{2\alpha})(1+x^{2\beta})}$$

$$= \frac{4\alpha-4\beta}{2\cdot 2} = \alpha-\beta$$

$$(\alpha(1-x^{\beta})-\beta(1-x^{\alpha}))(1+x^{\alpha})(1+x^{\beta}) =$$

$$= \alpha(1+x^{\beta})(1+x^{\alpha}) - \beta(1+x^{\alpha})(1+x^{\beta})$$

$$= \lim_{x\to 1} \frac{\alpha(1-x^{\beta})-\beta(1-x^{\alpha})}{(1-x^{\alpha})(1-x^{\beta})} = \lim_{x\to 1} \frac{\alpha-\alpha x^{\beta}-\beta+\beta x^{\alpha}}{1-x^{\beta}-x^{\alpha}+x^{\beta\alpha}} =$$

$$= \lim_{x\to 1} \frac{-\alpha\beta x^{\beta-1}+\beta\alpha x^{\alpha-1}}{-\beta x^{\beta-1}-\alpha x^{\alpha-1}+\alpha\beta x^{\alpha\beta-1}} =$$

$$= \lim_{x\to 1} \frac{-\alpha\beta(\beta-1)x^{\beta-2}+\beta\alpha(\alpha-1)x^{\alpha-2}}{-\beta(\beta-1)x^{\beta-2}-\alpha(\alpha-1)x^{\alpha-2}+\alpha\beta(\alpha\beta-1)x^{\alpha\beta-2}}$$

$$= \frac{-\alpha\beta^{2}+\alpha\beta+\beta\alpha^{2}-\beta\alpha}{-\beta^{2}+\beta-\alpha^{2}+\alpha+\alpha^{2}\beta^{2}-\alpha\beta} =$$

$$= \frac{\beta\alpha^{2}-\alpha\beta^{2}}{-\beta^{2}+\beta-\alpha^{2}+\alpha+\alpha^{2}\beta^{2}-\alpha\beta}$$