2.30. Даны три вектора $\mathbf{a}(4,1,5)$, $\mathbf{b}(0,5,2)$ и $\mathbf{c}(-6,2,3)$. Найти вектор $\mathbf{x}$, удовлетворяющий системе уравнений $(\mathbf{x},\mathbf{a}) = 18$, $(\mathbf{x},\mathbf{b}) = 1$, $(\mathbf{x},\mathbf{c}) = 1$.

$$\vec{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \vec{a} = \begin{pmatrix} 4 \\ 1 \\ 5 \end{pmatrix} \quad \vec{b} = \begin{pmatrix} 0 \\ 5 \\ 2 \end{pmatrix} \quad \vec{c} = \begin{pmatrix} -6 \\ 2 \\ 3 \end{pmatrix}$$

$$(\vec{x}\cdot\vec{a}) = 4x + y + 5z = 18$$

$$(\vec{x}\cdot\vec{b}) = 5y + 2z = 1$$

$$(\vec{x}\cdot\vec{c}) = -6x + 2y + 3z = 1$$

$$\begin{cases} 4x + y + 5z = 18 \\ 5y + 2z = 1 \\ -6x + 2y + 3z = 1 \end{cases}$$

$$\begin{cases} z = \frac{1}{2} - \frac{5}{2}y \\ 4x + y + 5z = 18 \\ -6x + 2y + 3z = 1 \end{cases}$$

$$4x + y + 5\left(\frac{1}{2} - \frac{5}{2}y\right) = 18$$

$$4x + y + \frac{5}{2} - \frac{25}{2}y = 18$$

$$4x + \frac{2-25}{2}y = \frac{36-5}{2}$$

$$4x - \frac{23}{2}y = \frac{31}{2}$$

$$x = \frac{31}{8} + \frac{23}{8}y$$

$$\begin{cases} z = \frac{1}{2} - \frac{5}{2}y \\ x = \frac{31}{8} + \frac{23}{8}y \\ -6x + 2y + 3z = 1 \end{cases}$$

$$-6\left(\frac{31}{8} + \frac{23}{8}y\right) + 2y + 3\left(\frac{1}{2} - \frac{5}{2}y\right) = 1$$

$$-\frac{93}{4} - \frac{69}{4}y + 2y + \frac{3}{2} - \frac{15}{2}y = 1 \quad \Big|\cdot 4$$

$$-93 - 69y + 8y + 6 - 30y = 4$$

$$-61y - 30y = 4 + 87$$

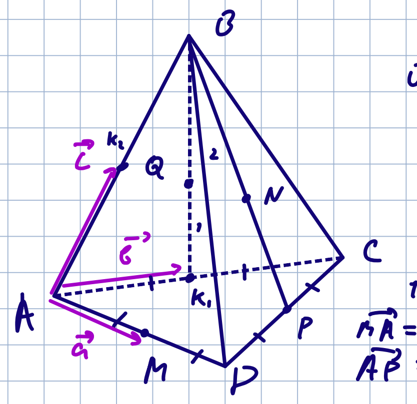
$$-91y = 91 \Rightarrow y = -1$$

$y = -1$
$x = 1$
$z = 3$

Ответ: $\vec{x} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$

2.47. В правильном тетраэдре $ABCD$ точки M и P — середины ребер AD и CD соответственно, точки N и Q — центры граней BCD и ABC соответственно. Найти угол между прямыми MN и PQ.

$|\vec{a}| = |\vec{b}| = |\vec{c}| = \frac{1}{2}AB = l$



Пусть K_1, K_2 - сер. AC, AB

Введём базис: $\vec{a} = \vec{AM}$, $\vec{b} = \vec{AK_1}$, $\vec{c} = \vec{AK_2}$

($\vec{a}, \vec{b}, \vec{c}$ - не компланарны)

1) $\vec{MN} = \vec{MA} + \vec{AP} + \vec{PN}$

$\vec{MA} = -\vec{a}$

$\vec{AP} = \vec{a} + \vec{b}$

$\vec{PN} = -\frac{1}{3}\vec{BP} = -\frac{1}{3}(\vec{BA} + \vec{AP}) = -\frac{1}{3}(-2\vec{c} + \vec{a} + \vec{b})$

$\vec{MN} = -\cancel{\vec{a}} + \cancel{\vec{a}} + \vec{b} - \frac{1}{3}(-2\vec{c} + \vec{a} + \vec{b}) =$

$= \vec{b} + \frac{2}{3}\vec{c} - \frac{1}{3}\vec{a} - \frac{1}{3}\vec{b} = \underline{\frac{2}{3}\vec{b} + \frac{2}{3}\vec{c} - \frac{1}{3}\vec{a}}$

2) $\vec{PQ} = \vec{PK_1} + \vec{K_1Q}$

$\cos 60° = \frac{1}{2}$

$\vec{PK_1} = -\vec{a}$

$\vec{K_1Q} = -\frac{1}{3}\vec{BK_1} = -\frac{1}{3}(-2\vec{c} + \vec{b}) = \frac{2}{3}\vec{c} - \frac{1}{3}\vec{b}$

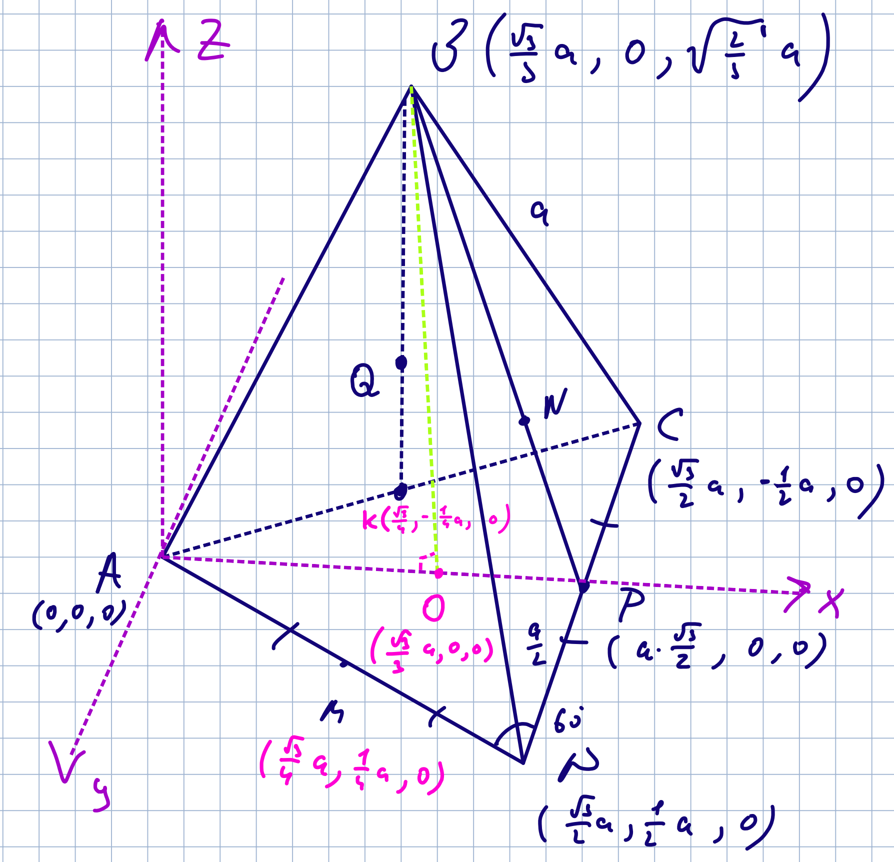
$\underline{\vec{PQ} = \frac{2}{3}\vec{c} - \frac{1}{3}\vec{b} - \vec{a}}$

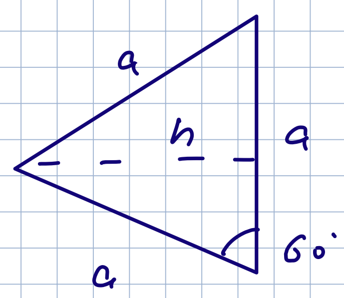
3) $PQ^2 = (\vec{PQ} \cdot \vec{PQ}) = \left(\frac{2}{3}\vec{c} - \frac{1}{3}\vec{b} - \vec{a}\right)\left(\frac{2}{3}\vec{c} - \frac{1}{3}\vec{b} - \vec{a}\right) =$

$= \frac{4}{9}c^2 - \frac{2}{9}(\vec{c}\cdot\vec{b})\cdot 2 - \frac{2}{3}(\vec{c}\cdot\vec{a})\cdot 2 + \frac{1}{3}(\vec{b}\cdot\vec{a})\cdot 2 + \frac{1}{9}b^2 + a^2$

$= \frac{4}{9}l^2 - \frac{4}{9}\cdot l^2 \cdot \frac{1}{2} - \frac{4}{3}\cdot\frac{1}{2}l^2 + \frac{1}{3}\cdot 2\cdot l^2\cdot\frac{1}{2} + \frac{1}{9}l^2 + l^2$

$= \frac{4}{9}l^2 - \frac{2}{9}l^2 - \frac{2}{3}l^2 + \frac{1}{3}l^2 + \frac{1}{9}l^2 + l^2 = l^2$





$h = a\cdot\sin 60°$

$\frac{\sqrt3}{2}a\cdot\frac{2}{3}$

$= \frac{2\sqrt3}{6}a$

$BO^2 = a^2 - \frac{1}{3}a^2 = \frac{2}{3}a^2$

$BO = \sqrt{\frac{2}{3}}a$

c) $\vec{BK} = \left(\frac{\sqrt3}{4}a - \frac{\sqrt3}{3}a, -\frac14 a, -\sqrt{\frac{2}{3}}a\right)$

$\vec{BK} = \left(\frac{3\sqrt3 - 4\sqrt3}{12}a; -\frac14 a; -\sqrt{\frac{2}{3}}a\right)$

$\vec{BK} = \left(-\frac{\sqrt3}{12}a; -\frac14 a; -\sqrt{\frac{2}{3}}a\right)$

$Q = B + \frac{2}{3}\vec{BK}$

$Q\left(\frac{\sqrt3}{3}a - \frac{1}{3}\cdot\frac{\sqrt3}{6}a; -\frac{\cancel{2}^{1}}{3}\cdot\frac{1}{\cancel{4}_{2}}a; \sqrt{\frac{2}{3}}a - \frac{2}{3}\sqrt{\frac{2}{3}}a\right)$

$$Q\left(\frac{5\sqrt{3}}{18}a\,;\,-\frac{1}{6}a\,;\,\frac{1}{3}\cdot\sqrt{\frac{2}{3}}\,a\right)$$

2) $$\vec{BP}=\left(\frac{\sqrt{3}}{6}a\,;\,0\,,\,-\sqrt{\frac{2}{3}}\,a\right)$$

$$B\left(\frac{\sqrt{3}}{3}a\,,\,0\,,\,\sqrt{\frac{2}{3}}\,a\right)$$

$$N\left(\frac{\sqrt{3}}{3}a+\frac{2}{3}\,\frac{\sqrt{3}}{6}a\,;\,0\,;\,\sqrt{\frac{2}{3}}\,a-\frac{2}{3}\cdot\sqrt{\frac{2}{3}}\,a\right)$$

$$N\left(\frac{4}{9}\sqrt{3}a\,;\,0\,;\,\frac{1}{3}\sqrt{\frac{2}{3}}\,a\right)$$

$$M\left(\frac{\sqrt{3}}{4}a\,,\,\frac{1}{4}a\,,\,0\right)$$

$$P\left(a\cdot\frac{\sqrt{3}}{2}\,,\,0\,,\,0\right)$$

$$\vec{MN}=\left(\frac{7}{36}\sqrt{3}a\,;\,-\frac{1}{4}a\,;\,\frac{1}{3}\sqrt{\frac{2}{3}}\,a\right)$$

$$\vec{PQ}=\left(-\frac{2}{9}\sqrt{3}a\,;\,-\frac{1}{6}a\,;\,\frac{1}{3}\sqrt{\frac{2}{3}}\,a\right)$$

$$\cos(\alpha)=\frac{47}{162}$$

$$MN^2 = \left(\frac{2}{3}\vec{b} + \frac{1}{3}\vec{c} - \frac{1}{3}\vec{a}\right)\left(\frac{2}{3}\vec{b} + \frac{1}{3}\vec{c} - \frac{1}{3}\vec{a}\right) =$$

$$= \frac{4}{9}\ell^2 + \frac{4}{9}\cdot\ell^2\cdot\frac{1}{\cancel{2}}\cdot\cancel{2} - \frac{2}{9}\ell^2\cdot\frac{1}{\cancel{2}}\cdot\cancel{2} - \frac{2}{9}\ell^2\cdot\frac{1}{\cancel{2}}\cdot\cancel{2} + \frac{4}{9}\ell^2 + \frac{1}{9}\ell^2 = \ell^2$$

$$\vec{PQ} = \frac{2}{3}\vec{c} - \frac{1}{3}\vec{b} - \vec{a}$$

$$(\vec{PQ}\cdot\vec{MN}) = \left(\frac{1}{3}\vec{c} + \frac{2}{3}\vec{b} - \frac{1}{3}\vec{a}\right)\left(\frac{2}{3}\vec{c} - \frac{1}{3}\vec{b} - \vec{a}\right) =$$

$$\frac{4}{9}c^2 - \frac{2}{9}(\vec{c}\cdot\vec{b}) - \frac{2}{3}(\vec{c}\cdot\vec{a}) + \frac{4}{9}(\vec{b}\cdot\vec{c}) - \frac{2}{9}b^2 - \frac{2}{3}(\vec{b}\cdot\vec{a}) - \frac{2}{9}(\vec{a}\cdot\vec{c}) + \frac{1}{9}(\vec{a}\cdot\vec{b}) + \frac{1}{3}a^2$$

$$\underbrace{\frac{4}{9}\ell^2 - \frac{1}{9}\ell^2}_{\frac{1}{3}} \cancel{- \frac{1}{3}\ell^2} + \cancel{\frac{2}{9}\ell^2 - \frac{2}{9}\ell^2} \cancel{- \frac{1}{3}\ell^2} - \frac{1}{9}\ell^2 + \frac{1}{18}\ell^2 + \cancel{\frac{1}{3}\ell^2} = -\frac{1}{18}\ell^2$$

$$\cos\alpha = \frac{(\vec{PQ}\cdot\vec{MN})}{|\vec{PQ}|\cdot|\vec{MN}|} = -\frac{1}{18}$$

15.2. Вычислить линейную комбинацию матриц:

1) $3\begin{Vmatrix} 1 & 2 \\ 1 & 2 \end{Vmatrix} - \begin{Vmatrix} 3 & 2 \\ 3 & 2 \end{Vmatrix} - 4\begin{Vmatrix} 0 & 1 \\ 0 & 1 \end{Vmatrix}$; 2) $2\begin{Vmatrix} 2 \\ 2 \\ 1 \end{Vmatrix} - 3\begin{Vmatrix} 0 \\ 5 \\ 6 \end{Vmatrix}$;

1) $3\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} - 4\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$

$$3\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 3 & 2 \\ 3 & 2 \end{pmatrix} - 4\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3-3-0 & 6-2-4 \\ 3-3-0 & 6-2-4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2) $2\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$

$$2\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 15 \\ 18 \end{pmatrix} = \begin{pmatrix} 4 \\ -11 \\ -16 \end{pmatrix}$$