№1

(a) $\lim\limits_{x\to x_0^-} f(x) = a \iff$

$$\forall \varepsilon>0\ \exists \delta(\varepsilon)>0\ \forall x : \Big(0<(x_0-x)<\delta(\varepsilon) \Rightarrow |f(x)-a|<\varepsilon\Big)$$

(d) $\lim\limits_{x\to x_0^+} f(x) = +\infty \iff$

$$\forall \varepsilon>0\ \exists \delta(\varepsilon)>0\ \forall x : \Big(0<|x-x_0|<\delta(\varepsilon) \Rightarrow f(x)>\frac{1}{\varepsilon}\Big)$$

(b) $\lim\limits_{x\to -\infty} f(x) = \infty \iff$

$$\forall \varepsilon>0\ \exists \delta(\varepsilon)>0\ \forall x : \Big(x<\frac{-1}{\delta(\varepsilon)} \Rightarrow |f(x)|>\frac{1}{\varepsilon}\Big)$$

№3

$\lim\limits_{x\to 4} x^3 = 64$

$$\forall \varepsilon>0\ \exists \delta(\varepsilon)>0\ \forall x : \Big(0<|x-4|<\delta(\varepsilon) \Rightarrow |x^3-64|<\varepsilon\Big)$$

$4-\delta(\varepsilon) < x < 4+\delta(\varepsilon)$ $\qquad$ // $64-\varepsilon < x^3 < 64+\varepsilon$ //

$\iff$

$(4-\delta(\varepsilon))^3 < x^3 < (4+\delta(\varepsilon))^3$

$$64 - 3\cdot 4^2\cdot\delta(\varepsilon) + 3\cdot 4\,\delta^2(\varepsilon) - (\delta(\varepsilon))^3 < x^3 < 64 + 3\cdot 4^2\cdot\delta(\varepsilon) + 3\cdot 4\,\delta^2(\varepsilon) + (\delta(\varepsilon))^3$$

$\sqrt[3]{64-\varepsilon} < x < \sqrt[3]{64+\varepsilon}$

$4-\delta \le \sqrt[3]{64-\varepsilon}$

$\sqrt[3]{64+\varepsilon} \le 4+\delta$

$-4-\delta \le -\sqrt[3]{64+\varepsilon}$

$\cancel{-6\delta < -12\delta^2}$

$\cancel{\varepsilon = 48\delta - 6\delta + \delta^3}$ $\cancel{-6\delta = 12\delta^2}$

$\cancel{\varepsilon = 48\delta + 12\delta^2 + \delta^3}$ $\cancel{0 = 6\delta(2\delta+1)}$

$-2\delta \le \sqrt[3]{64-\varepsilon} - \sqrt[3]{64+\varepsilon}$

$2\delta \ge \sqrt[3]{64+\varepsilon} - \sqrt[3]{64+\varepsilon}$

$$\delta \ge \underbrace{\frac{1}{2}\left(\sqrt[3]{64+\varepsilon} - \sqrt[3]{64+\varepsilon}\right)}_{} = \delta(\varepsilon)$$

№4

$\{x_n\}$ - расход. $\forall p \geq 1$ верно: $\lim\limits_{k\to\infty} |x_{k+p} - x_k| = 0$

$$\sqrt{n+p} - \sqrt{n} = \frac{n+p-n}{\sqrt{n+p}+\sqrt{n}} = \frac{p}{\sqrt{n+p}+\sqrt{n}}$$

№5

$b>1$: $\lim\limits_{k\to\infty} \frac{k^\alpha}{b^k} = 0$ где $\forall \alpha > 0$

(k)

$$\lim_{x\to-\infty}\left(\sqrt{x^2+8x+3} - \sqrt{x^2+4x+3}\right) = 2$$

$$\frac{x^2+8x+3-x^2-4x-3}{\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}} = \frac{4x}{\sqrt{x^2+8x+3}+\sqrt{x^2+4x+3}} = \frac{4}{\sqrt{1+\frac{8}{x}+\frac{3}{x^2}}+\sqrt{1+4\frac{1}{x}+\frac{1}{x^2}}}$$

$$\lim_{x\to 0}\frac{(x+8)^{\frac{1}{3}}-2}{(1+2x)^{\frac{1}{2}}-1} = \lim_{x\to 0}\frac{\frac{1}{3}(x+8)^{-\frac{2}{3}}}{\frac{1}{2}(1+2x)^{-\frac{1}{2}}\cdot 2} = \frac{1}{3}\lim_{x\to 0}\frac{\sqrt{1+2x}}{(\sqrt[3]{x+8})^2} = \frac{1}{3}\cdot\frac{1}{4} = \frac{1}{12}$$

$$\lim_{x\to\infty} x\cdot\sin\frac{\pi}{x} = \lim_{t\to 0}\frac{\pi}{t}\cdot\sin(t) = \pi\cdot 1 = \pi$$

$t = \frac{\pi}{x}$

$x = \frac{\pi}{t}$