Numerical solution of EBM-SCM model

Jake Aylmer

February 10, 2018

Reference: T. Wagner and I. Eisenman, 2015: *How Climate Model Complexity Influences Sea Ice Stability*

1 Physical model

The surface enthalpy per unit area, $E(x,t)$ in W yr m^{-2}, where $x = \sin\phi$, evolves according to

$$\frac{\partial E(x,t)}{\partial t} = a(x)S(x,t) - A - B\left[T(x,t) - T_{\mathrm{m}}\right] + F_{\mathrm{b}}(x) + F(x) + D\nabla^2 T(x,t) \tag{1}$$

where $T(x,t)$ is the surface temperature, $a(x,t)$ is the co-albedo, $S(x,t)$ is insolation, $A+B\left[T(x,t) - T_{\mathrm{m}}\right]$ is the outgoing longwave radiation (OLR), $F_{\mathrm{b}}(x)$ is the ocean basal heat-flux (set to 4 Wm^{-2} by default), $F(x)$ is a prescribed forcing (set to 0 by default) and $D\nabla^2 T(x,t)$ represents atmospheric meridional transport of heat.

$E(x,t)$ can be converted to sea-ice thickness $h(x,t)$ when sea-ice is present ($E(x,t) < 0$) or sea surface temperature (SST) $T(x,t)$ when sea-ice is not present ($E(x,t) \geq 0$):

$$E(x,t) = \begin{cases} -L_{\mathrm{f}} h(x,t) & E(x,t) < 0 \\ c_{\mathrm{w}}\left[T(x,t) - T_{\mathrm{m}}\right] & E(x,t) \geq 0, \end{cases} \tag{2}$$

where L_{f} is the latent heat of fusion of sea-water and c_{w} is the heat capacity of a unit area of ocean mixed-layer (the product of the the specific heat capacity and density of sea-water, and the ocean mixed-layer depth H_{ml}).

2 Numerical model

See reference appendix A. The equations to solve numerically are:

$$\left[\frac{\partial}{\partial t} - \frac{D}{c_{\mathrm{g}}}\frac{\partial^2}{\partial x^2}\right] T_{\mathrm{g}}(x,t) = \frac{T(x,t) - T_{\mathrm{g}}(x,t)}{\tau_{\mathrm{g}}} \tag{3}$$

and

$$\frac{\partial E(x,t)}{\partial t} = a(x)S(x,t) - A - B\left[T(x,t) - T_{\mathrm{m}}\right] - \frac{c_{\mathrm{g}}}{\tau_{\mathrm{g}}}\left[T(x,t) - T_{\mathrm{g}}(x,t)\right] + F_{\mathrm{b}}(x) + F(x). \tag{4}$$

Equation (3) is integrated forward one time-step Δt to get $T_{\mathrm{g}}(x, t+\Delta t)$. The Wagner and Eisenman code provided does this using the implicit backwards in time Euler scheme. Equation (4) is solved for $E(x, t+\Delta t)$. The Wagner and Eisenman code provided does this using the explicit forwards in time Euler scheme.

With each time-step, $T(x, t+\Delta t)$ must also be found since $T(x,t)$ appears in both (3) and (4). Firstly, if $E(x,t) > 0$, then sea-ice is not present and $T(x,t) = T_{\mathrm{m}} + E(x,t)/c_{\mathrm{w}}$. If $E(x,t) < 0$, then sea-ice is present and the surface temperature depends on the atmosphere-ice boundary heat-flux balance. If it is balanced by a temperature $T_0 < T_\mathrm{m}$ then the surface temperature is T_0. If it is balanced by a temperature $T_0 > T_\mathrm{m}$ then surface-melting is occuring (i.e. in this case fluxes are not balanced) and the top-of-ice surface temperature is T_m. This leads to the condition (8) in Wagner and Eisenman 2015 which is solved for T_0 in order to determine the regime. In the numerical model the condition takes the form

$$k\frac{T_\mathrm{m} - T_0(x,t)}{h(x,t)} = -a(x)S(x,t) + A + B\left[T_0(x,t) - T_\mathrm{m}\right] - F(x) + \frac{c_\mathrm{g}}{\tau_\mathrm{g}}\left[T_0(x,t) - T_\mathrm{g}\right] \tag{5}$$

which is re-arranged to give

$$T_0(x,t) = \frac{a(x)S(x,t) - A + F(x) + \frac{c_\mathrm{g}}{\tau_\mathrm{g}}T_\mathrm{g}(x,t) + \left(\frac{k}{h(x,t)} + B\right)T_\mathrm{m}}{\frac{c_\mathrm{g}}{\tau_\mathrm{g}} + B + \frac{k}{h(x,t)}}. \tag{6}$$

Note that (6) is only solved if $E(x,t) < 0$ in which case $h(x,t) = -E(x,t)/L_\mathrm{f}$.

So after calculating $T_\mathrm{g}(x, t+\Delta t)$ and $E(x, t+\Delta t)$, the surface temperature profile is given by

$$T(x, t+\Delta t) = \begin{cases} T_\mathrm{m} + E(x,t+\Delta t)/c_\mathrm{w} & E(x,t+\Delta t) > 0 \\ T_\mathrm{m} & E(x,t+\Delta t) < 0 \quad T_0 > T_\mathrm{m} \\ T_0 & E(x,t+\Delta t) < 0 \quad T_0 < T_\mathrm{m}. \end{cases} \tag{7}$$