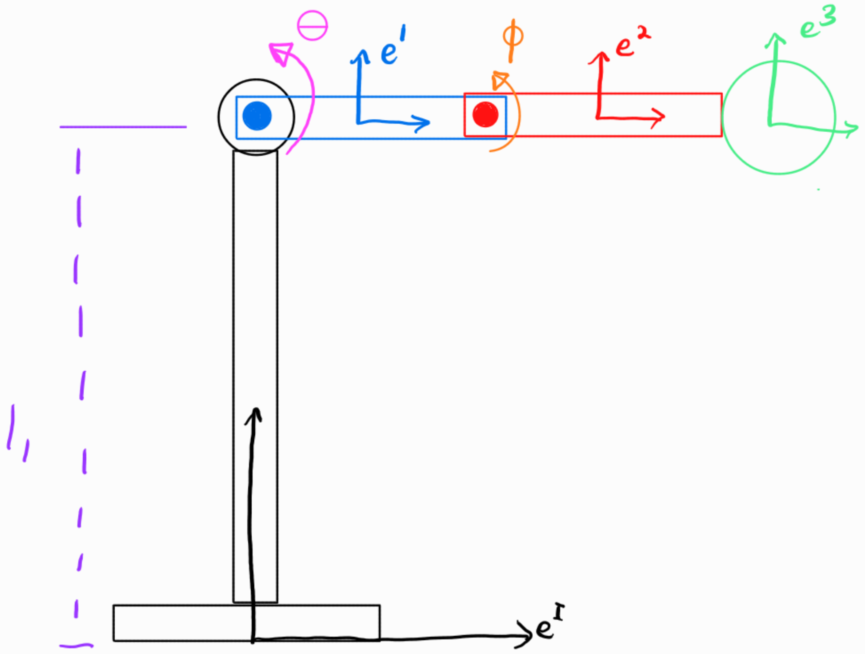
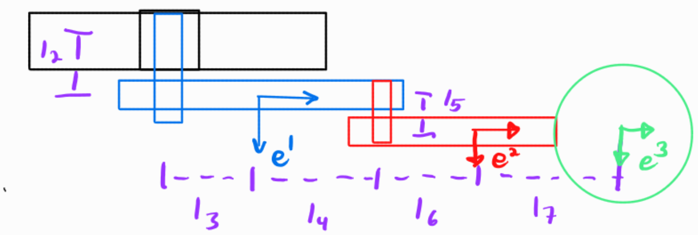
Structure and variables



Front-view



Top-view

Lengths : a, b, c, d, e, f, g (constants)

Angles : $\Theta(t)$, $\phi(t)$

Masses : m^I, m^1, m^2, m^3

Frames : e^I, e^1, e^2, e^3

Position-vectors : r^I, r^1, r^2, r^3

Definitions and notation of fram relationships

Definition of frame-/position-vector relations:

$$[e^{i} \ r^{i}] = [e^{i-1} \ r^{i-1}] \, E^{i/i-1}$$

$$[e^{i-1} \ r^{i-1}] = [e^{i} \ r^{i}] \left(E^{i/i-1}\right)^{-1}$$ ← inverse

Definition of relative frame-relation matrices $E^{i/i-1}$

$$E^{i/i-1} = \begin{bmatrix} R_j^{i/i-1} & s^{i/i-1} \\ 0_{3\times 1} & 1 \end{bmatrix}$$

R_j = rotation-matrix about j-axis

$j = \{1, 2, 3\}$

Definition of absolute frame-relation matrices E^{i}:

$$E^{i} = E^{i-1} \, E^{i/i-1} = \begin{bmatrix} R^{i} & x^{i} \\ 0_{3\times 1} & 1 \end{bmatrix}$$

Definition of the inverse frame-relatiom matrix

$$\left(E^{i}\right)^{-1} = \begin{bmatrix} (R^{i})^{T} & -(R^{i})^{T} x^{i} \\ 0_{3\times 1} & 1 \end{bmatrix}$$

Definition of angular velocity, omega-skew

$$\overset{\leftrightarrow}{\omega}^{i/i-1} = \left(R^{i/i-1}\right)^{T} \, \overset{\circ}{R}^{i/i-1}$$

$$\overset{\leftrightarrow}{\omega}^{i} = \left(R^{i/i-1}\right) \overset{\leftrightarrow}{\omega}^{i-1} R^{i/i-1} + \overset{\leftrightarrow}{\omega}^{i/i-1}$$

Definition of relative - and absolute omega matrices

$$\Omega^{i/i-1} = \left(E^{i/i-1}\right)^{-1} \dot{E}^{i/i-1} = \begin{bmatrix} \overleftrightarrow{\omega}^{i/i-1} & \left(R^{i/i-1}\right)^{T} \dot{s}^{i/i-1} \\ 0_{3\times 1} & 0 \end{bmatrix}$$

$$\Omega^{i} = \left(E^{i/i-1}\right)^{-1} \Omega^{i-1} E^{i/i-1} + \Omega^{i/i-1} = \begin{bmatrix} \left(R^{i/i-1}\right)^{T} \overleftrightarrow{\omega}^{i-1} R^{i/i-1} + \overleftrightarrow{\omega}^{i/i-1} & \left(R^{i/i-1}\right)^{T}\left(\overleftrightarrow{\omega}^{i-1} s^{i/i-1} + v^{i-1} + \dot{s}^{i/i-1}\right) \\ 0_{3\times -} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \overleftrightarrow{\omega}^{i} & \left(R^{i}\right)^{T} \dot{x}^{i} \\ 0_{3\times 1} & 0 \end{bmatrix}$$

Definition of rates of rotation and translation

$$\left[\dot{e}^{i} \;\; v^{i}\right] = \left[\dot{e}^{i-1} \;\; v^{i-1}\right] E^{i/i-1} + \left[e^{i-1} \;\; r^{i-1}\right] \dot{E}^{i/i-1}$$

$$= \left[\dot{e}^{i-1} \;\; v^{i-1}\right] E^{i/i-1} + \left[e^{i} \;\; r^{i}\right] \left(E^{i/i-1}\right)^{-1} \dot{E}^{i/i-1}$$

$$= \left[e^{i} \;\; r^{i}\right] \left(E^{i/i-1}\right)^{-1} \Omega^{i-1} E^{i/i-1} + \left[e^{i} \;\; r^{i}\right] \Omega^{i/i-1}$$

$$= \left[e^{i} \;\; r^{i}\right] \Omega^{i}$$

Setting up- and relating the body-frames on the pendulum-construction

from inertial to body-1

The inertial frame is considered as an approximate/relative fixed point at [0, 0, 0], and has no translation. We start here and move to the body-1 frame.

$$[e^1 \; r^1] = [e^I \; 0] \, E^1$$

The first frame-relation matrix from the inertial frame is always absolute.

To move to the body-1 frame, we go:

- translate up length l_1 along the e_3^I-axis: $R = I_{3\times3}$ $\quad s_1^1 = \begin{bmatrix} 0 \\ 0 \\ l_1 \end{bmatrix}$

$$E_1^1 = \begin{bmatrix} I_{3\times3} & s_1^1 \\ 0_{3\times1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & l_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Rotate about the e_1-axis: $R_1^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ $\quad s_2^1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$E_2^1 = \begin{bmatrix} R_1^1 & 0_{1\times3} \\ 0_{3\times1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Translate along the pendulum arm to the frame (at the center of mass)

$R = I$ $\quad s_3^1 = \begin{bmatrix} l_2 \\ l_3 \\ 0 \end{bmatrix}$

$$E_3^1 = \begin{bmatrix} I & 0_{1\times3} \\ 0_{3\times1} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & l_3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Combining the steps to our frame-relation matrix

$$E' = E_1' E_2' E_3' = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & \cos\theta & -\sin\theta & l_3 \cdot \cos\theta \\ 0 & \sin\theta & \cos\theta & l_1 + c \cdot \sin\theta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad R' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \qquad x' = \begin{bmatrix} l_2 \\ l_3 \cdot \cos\theta \\ l_1 + l_3 \cdot \sin\theta \end{bmatrix}$$

$$\begin{bmatrix} e' & r' \end{bmatrix} = \begin{bmatrix} e^I & o \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & \cos\theta & -\sin\theta & l_3 \cdot \cos\theta \\ 0 & \sin\theta & \cos\theta & l_1 + l_3 \cdot \sin\theta \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Finding the velocity and skew-symmetric angular velocity

$$(R')^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \qquad \dot{R}' = \dot{\theta} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & -\cos\theta \\ 0 & \cos\theta & -\sin\theta \end{bmatrix}$$

$$\overleftrightarrow{\omega}' = (R')^T \dot{R}' = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} \\ 0 & \dot{\theta} & 0 \end{bmatrix}$$ ← matrix multiply and trig. identities results in this.

$$\dot{x}' = \begin{bmatrix} 0 \\ -\dot{\theta} \cdot l_3 \cdot \sin\theta \\ \dot{\theta} \cdot l_3 \cdot \cos\theta \end{bmatrix}$$

$$\Omega' = \begin{bmatrix} \overleftrightarrow{\omega}' & (R')^T \dot{x}' \\ 0_{3\times 1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} & 0 \\ 0 & \dot{\theta} & 0 & \dot{\theta} \cdot l_3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

These will be necessary for the equation of motion later.

We continue this same procedure for the other bodies, but with less explanation along the way now.

Body 2-frame from body-1 frame and then from the inertial frame

$$[e^2 \; r^2] = [e^1 \; r^1] \, E^{2/1}$$

$$E^{2/1} = E^{2/1}_{transl.} \cdot E^{2/1}_{rot.} \cdot E^{2/1}_{transl.}$$

$$= \begin{bmatrix} I_{3\times3} & S_1^{2/1} \\ 0_{3\times1} & 1 \end{bmatrix} \begin{bmatrix} R_1 & 0_{1\times3} \\ 0_{3\times1} & 1 \end{bmatrix} \begin{bmatrix} I_{3\times3} & S_3^{2/1} \\ 0_{3\times1} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi & 0 \\ 0 & \sin\phi & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & l_5 \\ 0 & 1 & 0 & l_6 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & l_5 \\ 0 & \cos\phi & -\sin\phi & l_4 + l_6 \cdot \cos\phi \\ 0 & \sin\phi & \cos\phi & l_6 \cdot \sin\phi \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow R^{2/1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \qquad S^{2/1} = \begin{bmatrix} l_5 \\ l_4 + l_6 \cdot \cos\phi \\ l_6 \cdot \sin\phi \end{bmatrix}$$

$$[e^2 \; r^2] = [e^1 \; r^1] \begin{bmatrix} 1 & 0 & 0 & l_5 \\ 0 & \cos\phi & -\sin\phi & l_4 + l_6 \cdot \cos\phi \\ 0 & \sin\phi & \cos\phi & l_6 \cdot \sin\phi \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now finding the rates of the relative rotation matrix and translation vector

$$(R^{2/1})^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \qquad \dot{R}^{2/1} = \dot{\phi} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & \cos\phi \\ 0 & -\cos\phi & -\sin\phi \end{bmatrix}$$

$$\overleftrightarrow{\omega}^{2/1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\phi & \cos\phi \\ 0 & -\cos\phi & -\sin\phi \end{bmatrix} \cdot \dot{\phi}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi} \\ 0 & \dot{\phi} & 0 \end{bmatrix}$$

$$\dot{s}^{2/1} = \begin{bmatrix} 0 \\ -\dot{\phi}\, l_6 \sin\phi \\ \dot{\phi}\, l_6 \cos\phi \end{bmatrix}$$

Now we pull back to the inertial-frame.

There are 2 main ways of doing this.

1. $[e^2 \; r^2] = [e^1 r^1] \, E^{2/1} = [e^I \; 0] \, E^1 E^{2/1} = [e^I \; 0] \, E^2$

2. Using the terms found in the Ω-matrix.

We choose to use the latter, but both is fine.

$$\left[\dot{e}^2 \quad v^2\right] = \left[e^2 \quad r^2\right] \Omega^2$$

$$\Omega^2 = \left(E^{2/1}\right)^{-1} \Omega^1 E^{2/1} + \Omega^{2/1}$$

$$= \begin{bmatrix} 1 & 0 & 0 & -l_5 \\ 0 & \cos\phi & \sin\phi & -l_4\cdot\cos\phi - l_6 \\ 0 & -\sin\phi & \cos\phi & l_4 \sin\phi \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta} & 0 \\ 0 & \dot{\theta} & 0 & \dot{\theta}\cdot l_3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & l_5 \\ 0 & \cos\phi & -\sin\phi & l_4 + l_6\cdot\cos\phi \\ 0 & \sin\phi & \cos\phi & l_6\cdot\sin\phi \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{\phi} & 0 \\ 0 & \dot{\phi} & 0 & \dot{\phi}\cdot l_6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}-\dot{\phi} & \dot{\theta}\sin\phi\cdot(l_3+l_4) \\ 0 & \dot{\theta}+\dot{\phi} & 0 & \dot{\theta}\cos\phi\cdot(l_3+l_4) + \dot{\phi}\, l_6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This gives us the $\overset{\leftrightarrow}{\omega}^2$ and $\dot{x}^2$ terms needed for later, following the definition that:

$$\Omega^i = \begin{bmatrix} \overset{\leftrightarrow}{\omega}^i & (R^i)^T \dot{x}^i \\ 0_{3\times 1} & 0 \end{bmatrix}$$

finally relating the pendulum-ball at the end, the body-3 frame to the body-2 frame.

$$[e^3 \; r^3] = [e^2 \; r^2] \, E^{3/2}$$

$$E^{3/2} = E^{3/2}_{transl.} = \begin{bmatrix} I & s^{3/2} \\ \underset{3\times 1}{0} & 1 \end{bmatrix}$$

$$s^{3/2} = \begin{bmatrix} 0 \\ l_7 \\ 0 \end{bmatrix}$$

$$E^{3/2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find inverse, rate and relative capital-omega

$$(E^{3/2})^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -l_7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \dot{E}^{3/2} = \underset{4\times 4}{0}$$

$$\Omega^{3/2} = (E^{3/2})^{-1} \, \dot{E}^{3/2} = \underset{4\times 4}{0}$$

Finding the rates of the frame and the velocity

$$\left[\dot{e}^3 \;\; v^3\right] = \left[e^3 \;\; r^3\right] \Omega^3$$

$$\Omega^3 = \left(E^{3/2}\right)^{-1} \Omega^2 \, E^{3/2} + \cancel{\Omega^{3}_{2}} \quad 0_{4\times4}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -l_7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}-\dot{\phi} & \dot{\theta}\sin\phi \cdot (l_3+l_4) \\ 0 & \dot{\theta}+\dot{\phi} & 0 & \dot{\theta}\cos\phi \cdot (l_3+l_4) + \dot{\phi}\, l_6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}-\dot{\phi} & \dot{\theta}\sin\phi \cdot (l_3+l_4) \\ 0 & \dot{\theta}+\dot{\phi} & 0 & \dot{\theta}\cos\phi \cdot (l_3+l_4) + \dot{\phi}\, l_6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & l_7 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\dot{\theta}-\dot{\phi} & \dot{\theta}\sin\phi \cdot (l_3+l_4) \\ 0 & \dot{\theta}+\dot{\phi} & 0 & \dot{\theta}\cos\phi \cdot (l_3+l_4) + \dot{\phi}\, l_6 + l_7(\dot{\theta}+\dot{\phi}) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

with this we have what we need to setup the Equation of motion, following is a set of definitions for the EoM. in MFM-Dynamics.

We define a collection matrix for the rates of displacement and angular velocities. We also split it into columns corresponding to the number of rate variables. We call these variables the generalized coordinates.

$$\dot{q} = \begin{Bmatrix} \dot{\theta} \\ \dot{\phi} \end{Bmatrix}$$

The columns in the B-matrix contain the terms that depend on one of the generalized coordinates.

The B-matix multiplied with $\dot{q}$ should yield the $\dot{x}^i$- and ω^i- terms.

First unskew $\overleftrightarrow{\omega}$ terms.

$$\overleftrightarrow{\omega}^i = \begin{bmatrix} 0 & -\omega_3^i & \omega_2^i \\ \omega_3^i & 0 & -\omega_1^i \\ -\omega_2^i & \omega_1^i & 0 \end{bmatrix}, \quad \omega^i = \begin{bmatrix} \omega_1^i \\ \omega_2^i \\ \omega_3^i \end{bmatrix}$$

From this

$$\omega^1 = \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} \qquad \omega^2 = \begin{bmatrix} \dot{\theta} + \dot{\phi} \\ 0 \\ 0 \end{bmatrix} \qquad \omega^3 = \begin{bmatrix} \dot{\theta} + \dot{\phi} \\ 0 \\ 0 \end{bmatrix}$$

Now we find $\dot{x}^i$-vectors from the corresponding Ω^i-matrices by taking the column 4, rows 1 to 3 which are defined as $(R^i)^T \dot{x}^i$, and multiply this vector with $R^i \Rightarrow R^i (R^i)^T \dot{x}^i = \dot{x}^i$

for the absolute R^i-matrices that we dont have, we can simply find them from the following definition:

Absolute rotation matrix:

$$R^i = R^{i-1} \cdot R^i_{i-1}$$

what's nice here is that if we remember that for the first body from the inertial: The absolute displ. = relative disp. $\Rightarrow x = s \Rightarrow \dot{x} = \dot{s}$ And this gives us here a verification that this approach works.

From this we find

$$\dot{x}^1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta} \cdot l_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -\dot{\theta} \cdot l_3 \cdot \sin\theta \\ \dot{\theta} \cdot l_3 \cdot \cos\theta \end{bmatrix}$$

$$\dot{x}^2 = R^1 \cdot R^2_1 \cdot \begin{bmatrix} 0 \\ \dot{\theta} l_4 \sin\phi + \dot{\theta} l_3 \sin\phi \\ \dot{\theta} l_4 \cos\phi + \dot{\phi} l_6 + \dot{\theta} l_3 \cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \sin\phi \cdot (l_3 + l_4) \\ \dot{\theta} \cos\phi \cdot (l_3 + l_4) + \dot{\phi} l_6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta \cos\phi - \sin\theta \sin\phi & -\cos\theta \sin\phi - \sin\theta \cos\phi \\ 0 & \sin\theta \cos\phi + \cos\theta \sin\phi & \cos\theta \cos\phi - \sin\theta \sin\phi \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \sin\phi \cdot (l_3 + l_4) \\ \dot{\theta} \cos\phi \cdot (l_3 + l_4) + \dot{\phi} l_6 \end{bmatrix}$$

$$
= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta+\phi) & -\sin(\theta+\phi) \\ 0 & \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \sin\phi \cdot (l_3+l_4) \\ \dot{\theta} \cos\phi \cdot (l_3+l_4) + \dot{\phi}\, l_6 \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 \\ \dot{\theta} \cdot \sin\phi \cdot \cos(\theta+\phi)\left(l_3+l_4\right) - \dot{\theta} \cdot \cos\phi \cdot \sin(\theta+\phi)\left(l_3+l_4\right) - \dot{\phi} \cdot l_6 \cdot \sin(\theta+\phi) \\ \dot{\theta} \cdot \sin\phi \cdot \sin(\theta+\phi) \cdot (l_3+l_4) + \dot{\theta} \cos\phi \cdot \cos(\theta+\phi) \cdot (l_3+l_4) + \dot{\phi}\, l_6 \cdot \cos(\theta+\phi) \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 \\ \dot{\theta}(l_3+l_4) \cdot \sin(-\theta) - \dot{\phi} \cdot l_6 \cdot \sin(\theta+\phi) \\ \dot{\theta}(l_3+l_4) \cdot \cos(-\theta) + \dot{\phi} \cdot l_6 \cdot \cos(\theta+\phi) \end{bmatrix}
$$

$$
\dot{x}^3 = R^1 \cdot R^{2/1} \cdot \underbrace{R^{3/2}}_{=I} \cdot \begin{bmatrix} 0 \\ \dot{\theta} \sin\phi \cdot (l_3+l_4) \\ \dot{\theta} \cos\phi \cdot (l_3+l_4) + \dot{\phi}\, l_6 + l_7(\dot{\theta}+\dot{\phi}) \end{bmatrix}
$$

$$
= \begin{bmatrix} 0 \\ \dot{\theta}(l_3+l_4) \cdot \sin(-\theta) - (\dot{\phi} \cdot l_6 + l_7 \cdot (\dot{\theta}+\dot{\phi})) \cdot \sin(\theta+\phi) \\ \dot{\theta}(l_3+l_4) \cdot \cos(-\theta) + (\dot{\phi} \cdot l_6 + l_7 \cdot (\dot{\theta}+\dot{\phi})) \cdot \cos(\theta+\phi) \end{bmatrix}
$$

Now we have all the terms required to define the B-matrix

$$\dot{\mathbf{x}} = \begin{Bmatrix} \dot{x}^1 \\ \omega^1 \\ \dot{x}^2 \\ \omega^2 \\ \vdots \\ \dot{x}^i \\ \omega^i \end{Bmatrix} = B \cdot \dot{q}$$

$$B = \begin{bmatrix}
\dot{x}^1_1(\dot{\theta}) & \dot{x}^1_1(\dot{\phi}) \\
\dot{x}^1_2(\dot{\theta}) & \dot{x}^1_2(\dot{\phi}) \\
\dot{x}^1_3(\dot{\theta}) & \dot{x}^1_3(\dot{\phi}) \\
\omega^1_1(\dot{\theta}) & \omega^1_1(\dot{\phi}) \\
\omega^1_2(\dot{\theta}) & \omega^1_2(\dot{\phi}) \\
\omega^1_3(\dot{\theta}) & \omega^1_3(\dot{\phi}) \\
\dot{x}^2_1(\dot{\theta}) & \dot{x}^2_1(\dot{\phi}) \\
\dot{x}^2_2(\dot{\theta}) & \dot{x}^2_2(\dot{\phi}) \\
\dot{x}^2_3(\dot{\theta}) & \dot{x}^2_3(\dot{\phi}) \\
\omega^2_1(\dot{\theta}) & \omega^2_1(\dot{\phi}) \\
\omega^2_2(\dot{\theta}) & \omega^2_2(\dot{\phi}) \\
\omega^2_3(\dot{\theta}) & \omega^2_3(\dot{\phi}) \\
\dot{x}^3_1(\dot{\theta}) & \dot{x}^3_1(\dot{\phi}) \\
\dot{x}^3_2(\dot{\theta}) & \dot{x}^3_2(\dot{\phi}) \\
\dot{x}^3_3(\dot{\theta}) & \dot{x}^3_3(\dot{\phi}) \\
\omega^3_1(\dot{\theta}) & \omega^3_1(\dot{\phi}) \\
\omega^3_2(\dot{\theta}) & \omega^3_2(\dot{\phi}) \\
\omega^3_3(\dot{\theta}) & \omega^3_3(\dot{\phi})
\end{bmatrix}$$

Inserting every term into here would be too large, and even all the symbolic equations derived till now, all makes human error for too likely and so by using the symbolic matrix definition, a computer should be left to calculate the matrix values and perform all matrix multiplication using only the basic relative relations and the matrix operations we've shown here.

But first we'll finish defining the structure of the Equation of Motion.

The Equation of Motion is defined as the coupled, partial, second-order differential equation in matrix form as

$$M^{*}\ddot{q} + N^{*}\dot{q} = F^{*}$$

The following matrices are defined:

$$M^{*} = B^{T} \cdot M \cdot B$$

$$\rightarrow M = \begin{bmatrix} J_c^{1} & 0_{3\times3} & \cdots & & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & m^{1} \cdot I_3 & & & 0_{3\times3} & 0_{3\times3} \\ \vdots & & \ddots & & 0_{3\times3} & 0_{3\times3} \\ & & & \ddots & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & J_c^{i} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & 0_{3\times3} & m^{i} \cdot I_3 \end{bmatrix}$$

J_c^{i} = mass moment of inertia matrix for body-i

m^{i} = mass of body i.

$$N^{*} = B^{T}\left(M \cdot \dot{B} + D \cdot M \cdot B\right)$$

$$\rightarrow \dot{B} = \frac{d}{dt} \cdot B$$

$$\rightarrow D = \begin{bmatrix} \overleftrightarrow{\omega}^{1} & \underset{3\times3}{0} & \cdots & & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & & & \underset{3\times3}{0} & \underset{3\times3}{0} \\ & \vdots & \ddots & & \underset{3\times3}{0} & \underset{3\times3}{0} \\ & & & & \underset{3\times3}{0} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \overleftrightarrow{\omega}^{i} & \underset{3\times3}{0} \\ \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} & \underset{3\times3}{0} \end{bmatrix}$$

$$F^{*} = B^{T} \cdot f$$

$$\rightarrow f = \begin{bmatrix} f^{1} \\ m^{i} \\ \vdots \\ f^{i} \\ m^{i} \end{bmatrix}$$

With this we have what we need.

The dimensions, masses and mass-moments of inertia are determined in the design process of the construction.

When we wish to solve this numerically, we "turn" this into a first-order diff. eq. by defining $Q = \dot{q}$, $\dot{Q} = \ddot{q}$

$$\Rightarrow \quad M^* \mathring{Q} + N^* Q = f^*$$

And to solve for $\mathring{Q}$

$$\mathring{Q} = (M^*)^{-1} \cdot (f^* - N^* \cdot Q)$$

These coupled, partial diff.eq. gives us all the info we need on how the motion of the body develops.

The frame relation gives us info about positions of any point on any body for given angles.

Finally we are interested in the forces acting on each body as the pendulum swings.

First, for forces we may turn to Newton.

$$\sum F^i = m^i \cdot a^i.$$

The EOM, and more specifically the $\mathring{B}$-matrix within it gives us the acceleration terms, $\ddot{x}^i$.

$$\Rightarrow \quad \sum F^i = m^i \begin{bmatrix} \ddot{x}_1^i \\ \ddot{x}_2^i \\ \ddot{x}_3^i \end{bmatrix}$$

Secondly for the moments acting on each body, MFM-Dynamics expresses Euler's equation for moments as:

$$\sum M^i = \dot{H}^i = \tilde{\omega}^i J^i \omega^i + J^i \dot{\omega}^i$$

where $\dot{H}^i$ = rate of momentum of body-i.