Introduction to Number Theory

Spring Semester 2024

Homework 1

Homework 1 counts as 5% of your final grade for this course. Homework 1 is worth 10 points. Please submitt your answers to all Homework 1 questions before Monday, 29th January at 15.00.

1. Given $n > 1$, write the polynomial $x^n - 1$ as a product of two polynomials each of degree less than n. [1 mark]
2. Deduce that if for integer $d > 1$ the number $d^n - 1$ is prime, then $d = 2$ (where $n > 1$). [1 mark]
3. Show that if n is composite, say $n = ab$ with $a > 1$, $b > 1$ then $2^n - 1$ is composite. [1 mark]
4. Deduce that if $2^n - 1$ is prime then n is prime. [1 mark]
5. Given odd $n > 1$, write the polynomial $x^n + 1$ as a product of two polynomials each of degree less than n. [1 mark]
6. Show that if $2^n + 1$ is prime then n cannot be odd unless $n = 1$. [2 marks]
7. Show that if $2^n + 1$ is prime then n cannot be divisible by an odd number $q > 1$. [2 marks]
8. Conclude that if $2^n + 1$ is prime then $n = 2^m$ for some m. [1 mark]

For your interest. A prime of the form $2^{2^m} + 1$ is called a Fermat number, and prime Fermat numbers are called Fermat primes. Notice that $F_5 = 2^{2^5} + 1$ is a Fermat number which is not prime. As of 2019, the only known Fermat primes are F_0, F_1, F_2, F_3, and F_4. A prime of the form $2^p - 1$ is called a Mersenne prime after the French mathematician and theologian Marin Mersenne who studied them in 1644. Notice that $2^{11} - 1 = 2047 = 23 \cdot 8$ is not a prime number. As in the case of Fermat primes it is not known if there are infinitely many Mersenne primes.