Notes for Alternate Theories of Gravity Project

August 28, 2020

1 Overview

The goal of this project is to use GW170817 to test alternate theories of gravity. This has several steps

1. Determine/Calculate modified TOV equations for alternate theories to be considered
2. Develop code to solve modified TOV system of eq
3. Calculate mass-radius curves for chrial-eft in alternate theories of gravity
4. Generate EOS priors for theories of gravity at different critical densities
5. Use PyCBC to do parameter estimation in alternate theories
6. Use dynesty and TI to do model selection across different theories.

2 Literature Review

2.1 GR

GR, proposed in 1915 by Einstein, is the current accepted theory of gravity. In GR, gravity is a result of space-time geometry rather than a force. Mass/energy bends space time and objects move along the curves. GR is described mathematically in tensors, through Einstein's equations.

The action in GR is known as the Einstein-Hilbert action and is as follows:

$$S = \frac{1}{2\kappa} \int R\sqrt{-g}\; d^4x \tag{2.1}$$

By varying this equation with respect to the metric, one can get the field equations.

Above R is the Ricci scalar. This scalar is a contraction of the Ricci Curvature Tensor;

$$R_{\alpha\beta} \equiv R^{\rho}_{\alpha\rho\beta} = -R^{\rho}_{\alpha\beta\rho} = \partial_\rho \Gamma^{\rho}_{\beta\alpha} - \partial_\beta \Gamma^{\rho}_{\rho\alpha} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\beta\alpha} - \Gamma^{\rho}_{\beta\lambda}\Gamma^{\lambda}_{\rho\alpha} \tag{2.2}$$

The Ricci tensor is a contraction of the Riemann Tensor. [2]

$$R^{\delta}_{\alpha\beta\gamma} = \Gamma^{\delta}_{\alpha\gamma,\beta} - \Gamma^{\delta}_{\beta\gamma,\alpha} + \Gamma^{\sigma}_{\alpha\gamma}\Gamma^{\delta}_{\sigma\beta} - \Gamma^{\sigma}_{\beta\gamma}\Gamma^{\delta}_{\sigma\alpha} \tag{2.3}$$

Here we see that the Ricci tensor/scalar depend on the christoffel symbol.

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu}(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}) \tag{2.4}$$

2.2 Scalar-Tensor Theories

2.2.1 Jordan or Metric frame

The action in scalar tensor theories includes a term thats a function of the scalar.

$$S = \frac{1}{2\kappa^2} \int \sqrt{-g} \bigg(\phi R + \frac{\omega(\phi)}{\phi} g^{\mu\nu} \phi_{,\mu} \phi_{,\nu} + 2\phi\lambda(\phi) \bigg) \, d^4x + S_{matter} \tag{2.5}$$

The field equations can be calculated by varying the action with respect to the metric and the scalar field. I will write out the entire process at the end of this document. But there are a handful of tricks I should note [3].

The field equations by variation with respect to the metric and the scalar field respectively.

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R - \lambda(\phi) g_{\mu\nu} = \frac{\kappa^2}{\phi} T_{\mu\nu} + \frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2} g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}) + \frac{1}{\phi}(\phi_{;\mu\nu} - g_{\mu\nu}\Box\phi) \tag{2.6}$$

$$\Box\phi + \frac{1}{2}\phi_{,\lambda}\phi^{,\lambda} \frac{d}{d\phi} \ln\Big(\frac{\omega(\phi)}{\phi}\Big) + \frac{1}{2}\frac{\phi}{\omega(\phi)}\Big[R + 2\frac{d}{d\phi}(\phi\lambda(\phi))\Big] = 0 \tag{2.7}$$

In this case $T_{\mu\nu} = 2(-g)^{-1/2}\delta S_M / \delta g^{\mu\nu}$

2.2.2 Einstein Frame

There are two ways to express the equations for scalar-tensor theory. One is in terms of the scalar-tensor metric (This is called the Jordan Frame). The other is called the 'Einstein Frame' because it causes the equations to look more similar to those of GR. The thesis notes that the Einstein frame is better for solving numerically as well.

The metric is transformed and so the 'new metric' is not the physically measured metric:

$$g_{\mu\nu} \to \tilde{g_{\mu\nu}} = \Omega^2(\phi) g_{\mu\nu} \tag{2.8}$$

Ω is a non-vanishing regular function. It is known as a Weyl or conformal transformation. It affects the lengths of spice/time-like intervals and vectors, but not the light cone. The spacetimes $(M, g_{\mu\nu})$ and $(M, \tilde{g}_{\mu\nu})$ have the same causality (a space-like vector remains a space-like vector and a time-like vector remains a time-like vector. [2]

In the Mandolidis he chooses [3]

$$\Omega^2(\phi) = \phi \tag{2.9}$$

The curvature (Ricci) scalar also changes. This can be derived by transforming the Ricci tensor with the rescaling of the christoffel symbols. In the case of four dimensional spacetime R can be defined as follows

$$R = \Omega^2 \tilde{R} + 6\frac{\Box\Omega}{\Omega} \tag{2.10}$$

The rescaling of the christoffel symbol is:

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \Omega^{-1}(\delta^{\alpha}_{\beta}\nabla_{\gamma}\Omega + \delta^{\alpha}_{\gamma}\nabla_{\beta}\Omega - g_{\beta\gamma}\nabla^{\alpha}\Omega) \tag{2.11}$$

The determinant of the new metric will also be different and it has the form:

$$\sqrt{-\tilde{g}} = \Omega^4\sqrt{-g} \tag{2.12}$$

A new definition of the scalar field is also needed. It is defined in terms of $d\phi$

$$d\varphi = \frac{1}{2}\sqrt{2\omega(\phi)+3}\frac{d\phi}{\phi} \tag{2.13}$$

The Scalar-Tensor action can be rewritten in terms of the new scalar potential. This equation is simpler than the action in terms of ϕ. The thesis chooses $\lambda(\phi) = 0$. However, the field equations are much simpler when written in the Einstein Frame.

$$S = \frac{1}{2\kappa^2}\int d^4x\sqrt{\tilde{g}}\big[\tilde{R} - 2\tilde{g}^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu}\big] \tag{2.14}$$

This gives the field equations in a new/simpler form:

$$\tilde{R}_{\mu\nu} = 2\tilde{g}^{\mu\nu}\partial_{\mu}\varphi\partial_{nu}\varphi + \kappa^2(\tilde{T}_{\mu\nu} - \frac{1}{2}\tilde{T}\tilde{g}_{\mu\nu}) \tag{2.15a}$$

$$\Box_{\tilde{g}}\varphi = -\frac{\kappa^2}{2}\tilde{\alpha}\tilde{T} \tag{2.15b}$$

The term $\tilde{\alpha}$ appears in the scalar field equation. This is a term defined from, Damour and Esposito-Farese. They define three new parameters that make ity simpler to express the field equations and the post newtonian parameters.

$$A(\varphi) = \Omega^{-1} \tag{2.16a}$$

$$\tilde{\alpha}(\varphi) \equiv \frac{d\ln A(\varphi)}{d\varphi} = \frac{1}{\sqrt{3+2\omega(\phi)}} \tag{2.16b}$$

$$\tilde{\beta}(\varphi) = \frac{d\tilde{\alpha}(\varphi)}{d\varphi} \tag{2.16c}$$

3 Calculating TOV equations

This mainly follows the Manolidis thesis chapter titled "Theoretical Framework"

3.1 Scalar Tensor Theories

As mentioned above there are two separate frames in which to solve for the TOV eqs. First, start with the Jordan or metric frame. He first shows the process using the FJBD theory ($\omega' = \lambda = 0$).

We first need the field equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{\kappa^2}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}) + \frac{1}{\phi}(\phi_{;\mu\nu} - g_{\mu\nu}\Box\phi) \tag{3.1a}$$

$$= \phi_{;\mu}^{;\mu} = \frac{1}{\sqrt{-g}}(\sqrt{-g}\phi^{,\mu})_{,\mu} = \frac{8\pi T}{3+2\omega} \tag{3.1b}$$

Next we need to define the spacetime metric. He uses a spherically symmetric spacetime metric:

$$ds^2 = -e^{2\Phi(r)}dt^2 + e^{2\Lambda(r)}dr^2 + r^2 d\Omega^2 \tag{3.2}$$

The second term is then recast in terms of the mass:

$$e^{2\Lambda(r)} = \Big(1 - \frac{2m(r)}{r}\Big)^{-1} \tag{3.3}$$

Finally we need the stress-energy tensor. In this case, we the stress energy tensor of a perfect fluid and it has the following form:

$$T^{\mu\nu} = (p+\rho)u^\mu u^\nu + pg^{\mu\nu} \tag{3.4}$$

When you combine these three, you get the following
For the 00 or tt term

$$\frac{dm}{dr} = 4\pi r^2\Big[\frac{\rho}{\phi} + \frac{\omega\phi_{,r}^2}{16\pi\phi^2}(1 - \frac{2m}{r}) - \frac{3p-\rho}{\phi(2\omega+3))}\Big] \tag{3.5}$$

and for the 11 or rr term:

$$\frac{d\Phi}{dr} = \frac{1}{r(r-2m)}\Big[m + \frac{4\pi r^3 p}{\phi} - \frac{\omega\phi_{,r}^2 r^3}{4\phi^2}\Big(1 - \frac{2m}{r}\Big)\Big] \tag{3.6}$$

For the scalar field, you have [Where is this coming from? I'm not sure how this is derived]

$$\phi_{,rr} + \phi_{,r}\Big(\frac{2}{r} + \Phi' - \Lambda'\Big) = \frac{8\pi r(3p-\rho)}{(r-2m)(2\omega+3)} \tag{3.7}$$

Using energy-momentum conservation ($\nabla_{nu}T^{\mu\nu} = 0$) you get the additional constraint:

$$\frac{dp}{dr} = -(p+\rho)\frac{d\Phi}{dr} \tag{3.8}$$

Substituting (3.6) inti (3.8) you get the following:

$$\frac{dp}{dr} = -\frac{p+\rho}{r(r-2m)}\Big[m + \frac{4\pi r^3 p}{\phi} - \frac{\omega\phi_{,r}^2 r^3}{4\phi^2}\Big(1 - \frac{2m}{r}\Big)\Big] \tag{3.9}$$

Next, you can substitute 3.5 and 3.6 into 3.7. Note: You can get the relationship between Λ' and dm/dr using the fact that $e^{2\Lambda} = (1 - 2m(r)/r)^{-1}$

This gives you:

$$\phi_{,rr} = \left(\frac{r\omega\phi_{,r}^2}{2\phi^2} - \frac{2(2\pi r^3(\rho - p) - \phi r + m\phi}{(2m - r)r\phi}\right)\phi_{,r} + \frac{4\pi r(3p - \rho)(r\phi_{,r} - 2\phi)}{(2m - r)\phi(2\omega + 3)} \tag{3.10}$$

In addition to these we need an equation of state is needed to related pressure and density. The Jordan frame has shown itself to be ill suited to numerical integration, so the Einstein Frame is used for calculations.

3.2 Sensitivities

The scalar field has an additional (non-geometric?) affect on the structure of the compact object. The scalar field affects the value of G as follows:

$$G \equiv \left[\frac{4 + \omega(\phi_0)}{3 + 2\omega(\phi_0)}\right]\phi_0^{-1} = 1 \tag{3.11}$$

Here ϕ_0 is the asymptotic value of the scalar field.
Since mass is a function of ϕ, it is also a function of G. They refer to the mass as a function of $\ln G$, **I am not sure why they chose $\ln G$ instead of G. I guess it has something to do with the fact that the derivative of $\ln G$ is $1/G$ and $1/G \propto \phi_0$.**

To define the sensitivies, we start with the weak field approximation:

$$\phi = \phi_0 + \Delta\phi \tag{3.12}$$

where ϕ_0 is the asymptotic value of the scalar field. Writing the mass with this expansion is:

$$\begin{aligned} m_A(\phi) &= m_A(\ln G) \\ &= m_{A0} - \left(\frac{\partial m_A}{\partial \ln G}\right)_0 \left[\frac{\Delta\phi}{\phi_0} - \frac{1}{2}\left(\frac{\Delta\phi}{\phi_0}\right)^2\right] \\ &\quad \frac{1}{2}\left(\frac{\partial^2 m_A}{\partial \ln G^2}\right)_0 \left(\frac{\Delta\phi}{\phi_0}\right)^2 + O\left(\frac{\Delta\phi}{\phi_0}\right)^3 \end{aligned}$$

The two sensitivities are then:

$$s_A = -\left(\frac{\partial \ln m_A}{\partial \ln G}\right)_0 \tag{3.13}$$

$$s'_A = -\left(\frac{\partial^2 \ln m_A}{\partial \ln G^2}\right)_0 \tag{3.14}$$

They are called sensitivities because they indicate how 'sensitive' the mass of the object is to the scalar field/gravitational constant. Since the mass is part of the post-Newtonian equations of motion and gravitational wave forms, the sensitivities are needed for those calculations. A model is uniquely defined by an EOS ($p = p(\rho)$), ϕ_0, and ρ_c. Holding the baryon number fixed, it can be shown that:

$$s = -\left(\frac{\partial m}{\partial \ln G}\right)_N = \left(\frac{\partial m}{\partial \ln G}\right)_{\rho_c} + \left(\frac{\partial m}{\partial \ln N}\right)_G \left(\frac{\partial N}{\partial \ln G}\right)_{\rho_c} \tag{3.15}$$

Sensitivities can alternatively be defined directly in terms of the scalar field:

$$\hat{s}_A = \left(\frac{\partial m}{\partial \phi}\right)_0 \tag{3.16}$$

$$\hat{s}'_A = \left(\frac{\partial \hat{s}_A}{\partial \phi}\right)_0 \tag{3.17}$$

In the Einstein Frame there is a direct equivalent to these sensitivities:

$$\alpha_A = \frac{\partial \tilde{m}_A}{\partial \varphi_0} \tag{3.18}$$

$$\beta_A = \frac{\partial \alpha_A}{\partial \varphi_0} \tag{3.19}$$

$\tilde{m}_A$ is the mass in the Einstein and it's not the same as the mass in the Jordan Frame. Thet are easily related:

$$\tilde{m}_A = A(\varphi) m_A \tag{3.20}$$

The sensitivities in the Einstein Frame and the Jordan frame are related by

$$\hat{s}_a = \frac{\tilde{\alpha} - \alpha_A}{2\tilde{\alpha}} \tag{3.21}$$

$$\hat{s}'_a = \frac{1}{4\tilde{\alpha}^2}\left(\beta_A - \frac{\tilde{\beta}}{\tilde{\alpha}}\alpha_A\right) \tag{3.22}$$

4 Solving Field Equations

The goal of this section is to create a c++ code that can solve modified TOV equations. The final goal will be to choose a theory and pass it a table from Ingo's EOS ad return a mass radius curve. This is a complex process, so will break it into several steps.

1. make a code that solves GR TOV from a tabulated file containing polytropic equation of state values
2. modify the GR TOV to the simplest Scalar-Tensor case ($\omega = const$)
3. generalize code further for $\omega = \omega(\phi)$
4. attempt to use code in combination with eos tables from Ingo

4.1 Newtonian star

Looking at [4] equations for a newtonian star are:

$$\frac{dp}{dr} = -\frac{G\rho M(r)}{r^2} \tag{4.1}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \tag{4.2}$$

equation of state used for initial:

$$P = K\rho^{\gamma} \tag{4.3}$$

In this case, $\gamma = 1 + 1/n$. Exact solutions are known for $n = 0, 1, 5$. However, $n = 0$ is a singularity and so, I wont test it with the code. First, I use $n = 1$ case which gives constant radius.
K is the polytropic constant. It can be calculated from the fermi gas model [5]. This gives the following result:

$$K_{rel} = \frac{\hbar c}{12\pi^2}\left(\frac{3\pi^2 Z}{A m_N c^2}\right)^{4/3} \tag{4.4}$$

$$K_{non-rel} = \frac{\hbar^2}{15\pi^2 m_e}\left(\frac{3\pi^2 Z}{A m_N c^2}\right)^{5/3} \tag{4.5}$$

- m_N: nucleon mass
- m_e: electron mass
- A/Z: number of nucleons per electron

We want to integrate in dimensionless variables $\bar{\rho}$ an d $\bar{p}$. So, these dimensionless quantities calculated using the following $p = p_c\bar{p}$ and $\rho = \rho_c\bar{\rho}$ where ρ_c and p_c are the central density and central pressure respectively.

To use the EOS, I need to solve for $\rho(p)$ and substitute in the dimensionless quantities. This gives:

$$\bar{\rho} = \left(\frac{p_c\bar{p}}{K}\right)^{\frac{n}{n+1}} \frac{1}{\rho_c} \tag{4.6}$$

The Newtonian TOV equations:

```
def TOV(r,y):
    M = y[0]
    p = y[1]

    rho = EOS(p)

    dMdr = 4 * np.pi * r**2. * rho * rho_c
    dpdr = - ( 1. * rho * M * rho_c) / (r**2. * p_c)
```

```
    if p<= 0:
        dMdr = 0.
        dpdr = 0.

    return [dMdr,dpdr]
```

This calls the EOS function to calculate rho. The EOS code is:

```
def EOS(p):
    rho = ((p_c * p)/(K))**(1./gamma) / rho_c
    #print p,rho
    return rho
```

Note that constants G (gravitational constant), c (speed of light), M_sun (mass of the sun), and K are defined globally.

To solve the system, I use from scipy.integrate.solve_ivp.

```
soln =
    solve_ivp(TOV,t_span,y0,method='RK45',events=star_boundary,dense_output=True)
```

- t_span: tuple of 2 floats that define interval of integration. For this its $r = 0.1, r = 20,000$ m.
- y0: the initial conditions. $m(r = 0) = 0$ and pressure is varied from $\bar{p}(r = 0) = (1, 100)$.
- method='RK45': chooses a Runge Kutta integrator. "The error is controlled assuming 4th order accuracy, but steps are taken using a 5th oder accurate formula "
- events=star_boundary: causes the integration to terminate when pressure reaches zero.
- dense_output=True: gives all the integration steps and is useful for plotting the solutions.

In order to stop the code at $p = 0$, I use events.

```
def star_boundary(r,y):
    return y[1]

star_boundary.terminal = True
```

The function star_boundary returns the value of the pressure, and star_boundary.terminal = True tells the code to terminate when star_boundary outputs 0.

By solving the system of equations with various initial pressures, I can create a mass-radius plot. The mass and radius for each initial pressure and saved.

I start this with $n = 1$. The exact mass radius solution is known for this polytropic index. According to [6] t is given by:

$$R = \left(\frac{\pi K}{2G}\right)^{1/2} \tag{4.7}$$

Thus, the radius is independent of mass and also of central pressure. The calculated radius for all $\bar{p}$ values is 4.8 km and the integrator calculates a radius of 4.8 km.
As a secondary check, I looked at the pressure radius curves for the polytropes. For n=01,5 this can be calculated analytically. The first step to this is to define dimensionless quantities for the equation of state.

$$\theta = \theta(r)$$
$$\rho = \rho_c \theta^n$$
$$P = P_c \theta^{n+1}$$
$$P_c = K \rho_c^{1+\frac{1}{n}}$$

For the numerical integration, I make slightly different substitutions, namely $p = p_c\bar{p}$ and $\rho = \rho_c\bar{\rho}$. I do this because θ is not used for GR TOV integration and scalar tensor tov. The analytical solution of the Newtonian stellar structure equations uses a dimensionless radius variable ξ where $r = r_n\xi$. I do not make this substitution in the integrator. So, to get $p(r)$, I need to transform the $\theta(\xi)$ function.
For $n = 1$

$$\theta(\xi) = \frac{\sin\xi}{\xi} \quad \xi_1 = \pi \tag{4.8}$$

then

$$p = p_c\theta^{n+1} = p_c\left(\frac{\sin\xi}{\xi}\right)^{n+1} \tag{4.9}$$

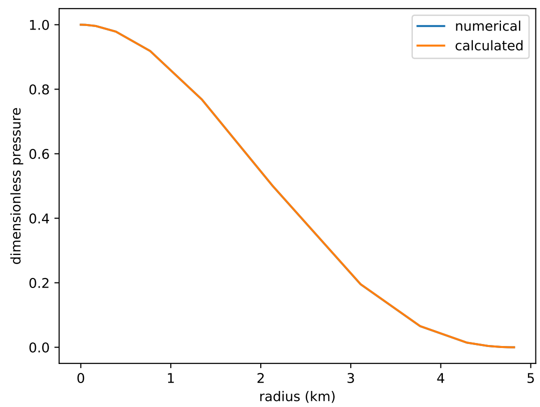
In order to get this in terms of the radius, we need

$$r_n^2 = \frac{(n+1)K}{4\pi G}\rho_c^{\frac{1}{n}-1} \tag{4.10}$$

The code for this transformation is:

```
ps = []
r_n = (((n+1.) * K )/(1.* 4.*np.pi)) * rho_c**(1./n-1.) #G= 1
r_n = r_n**(1./2.)
xi = r / r_n
for j in xi:
   ps.append((math.sin(j)/j)**(n+1.))
```

This gave the following results:



The next test is to do $n = 3/2$ case, which is the nonrelativistic limit for fermi gas.

4.2 GR TOV Solver

Polytropic equations of state are defined by their polytropic index Γ in the following way

$$\rho = m_b n + \frac{K m_b n_0}{\Gamma - 1}\left(\frac{n}{n_0}\right)^{\Gamma} \tag{4.11}$$

$$p = K m_b n_0 \left(\frac{n}{n_0}\right)^{\Gamma} \tag{4.12}$$

K and Γ are free parameters that uniquely define the EOS. The values are chosen by fitting the polytrope to a more physical eos. The constants in the equation have values ($m_b = 1.66 \times 10^{-27}$ kg and $n_0 = 0.1$ fm$^-3$) which comes from the PhD Thesis.

4.3 Solving Scalar-Tensor Theory TOV

4.3.1 Mathematics

THe plan here is to go along with the method outlined in chapter 3 of the thesis.

Starting with equations in Jordan Frame.

$$\boxed{\begin{aligned}
\frac{dm}{dr} &= 4\pi r^2\Big[\frac{\rho}{\phi} + \frac{\omega\phi_{,r}^2}{16\pi\phi^2}\big(1 - \frac{2m}{r}\big) - \frac{3p-\rho}{\phi(2\omega+3))}\Big] && (4.13)\\
\frac{dp}{dr} &= -\frac{p+\rho}{r(r-2m)}\left[m + \frac{4\pi r^3 p}{\phi} - \frac{\omega\phi_{,r}^2 r^3}{4\phi^2}\left(1 - \frac{2m}{r}\right)\right] && (4.14)\\
\phi_{,rr} &= \left(\frac{r\omega\phi_{,r}^2}{2\phi^2} - \frac{2(2\pi r^3(\rho - p) - \phi r + m\phi}{(2m-r)r\phi}\right)\phi_{,r} + \frac{4\pi r(3p-\rho)(r\phi_{,r} - 2\phi)}{(2m-r)\phi(2\omega+3)} && (4.15)
\end{aligned}}$$

These equations can be transformed into the Einstein Frame using equations (2.16a-c)

$$
\boxed{\begin{aligned}
M' &= \frac{dm}{dr} = \frac{\kappa^2}{2} r^2 A^4(\varphi)\rho + \frac{1}{2} r(r-2M)\psi^2 && (4.16)\\
\nu' &= 2\frac{d\Phi}{dr} = \kappa^2 \frac{r^2 A^4(\varphi) p}{r-2M} + r\psi^2 + \frac{2M}{r(r-2M)} && (4.17)\\
\varphi' &= \frac{d\varphi}{dr} = \psi && (4.18)\\
\psi' &= \frac{d\psi}{dr} = \frac{d^2\varphi}{dr^2} = \frac{\kappa^2}{2}\frac{rA^4(\varphi)}{r-2M}\big[\alpha(\varphi)(\rho-3p) + r\psi(\rho-p)\big] - \frac{2(r-M)}{r(r-2M)}\psi && (4.19)\\
p' &= \frac{dp}{dr} = -(\rho+p)\Big[\frac{\kappa^2}{2}\frac{r^2 A^4(\varphi)p}{r-2M} + \frac{1}{2} r\psi^2 + \frac{M}{r(r-2M)} + \alpha(\varphi)\psi\Big] && (4.20)
\end{aligned}}
$$

We also need a set of initial conditions for the integration. As $1/r$ appears in several expressions, the equations are tricky at r=0. For this reason, the initial conditions are actuall set at some small radius $r = \Delta r$, which is typically one integration step from the center.

$$M(\Delta r) = 0 \tag{4.21}$$

$$\nu(\Delta r) = 0 \tag{4.22}$$

$$\varphi(\Delta r) = \varphi_c \tag{4.23}$$

$$p(\Delta r) = p_c \tag{4.24}$$

$$\psi(\Delta r) = \frac{4\pi}{3}\Delta r A^4(\varphi_c)\alpha(\varphi_c)(\rho_c - 3p_c) \tag{4.25}$$

The baryonic mass, $\bar{M}$, is also of interest. We need a separate equation for this.

$$\bar{M}' = \frac{d\bar{M}}{dr} = 4\pi m_b n A^3(\varphi)\frac{r^2}{\sqrt{1-2M/r}} \tag{4.26}$$

The initial condition for the baryonic mass is:

$$\bar{M}(\Delta r) = 0 \tag{4.27}$$

These equations apply to the interior of the star. Beyond the surface of the star (where the pressure drops to zero) the spacetime equations are different. The internal and external solution must match smoothly at the boundary. This can be done by choosing the correct boundary conditions. The metric of the space outside the star is defined as follows

$$ds^2 = -e^{\nu}dt^2 + e^{-\nu}[d\xi^2 + e^{\lambda}(d\theta^2 + \sin^2\theta d\phi^2)] \tag{4.28}$$

$$e^{\lambda} = \xi^2 - a\xi \tag{4.29}$$

$$e^{\nu} = \big(1 - \frac{a}{\xi}\big)^{b/a} \tag{4.30}$$

$$\varphi(r) = \varphi_0 + \frac{d}{a}\ln\big(1 - \frac{a}{\xi}\big) \tag{4.31}$$

a and b are constants of integration. They are related to each other and another constant d by the equation $a^2 - b^2 = 4d^2$. a, b, and d can a;sp be expressed in terms of the total mass and the coupling constant.

$$b = 2\frac{G}{c^2}m_A \tag{4.32}$$

$$\frac{b}{a} = (1 + \alpha_a^2)^{1/2} \tag{4.33}$$

$$\frac{d}{b} = \frac{1}{2}\alpha_A \tag{4.34}$$

According to the thesis, if you want to relate these back to Schwarzschild coordinates you can use the following: (However, e^μ appears nowhere in the metric above, so I suspect that he means e^ν)

$$r = \xi\Big(1 - \frac{a}{\xi}\Big)^{(a-b)/2a} \tag{4.35}$$

$$e^\mu = \Big(1 - \frac{a}{\xi}\Big)\Big(1 + \frac{a+b}{2\xi}\Big)^{-2} \tag{4.36}$$

This gives us the surface boundary conditions:

$$\nu_s' = R\psi_s + \frac{2M_s}{R(R - 2M_s)} \tag{4.37}$$

$$\alpha_A = \frac{2\psi_s}{\nu_s'} \tag{4.38}$$

$$\varphi_0 = \varphi_s - \frac{1}{2}\alpha_A\hat{\nu}_s \tag{4.39}$$

$$m_A = \frac{1}{2}\nu_s' R^2 Q_2 e^{\frac{1}{2}\hat{\nu}_S} \tag{4.40}$$

$$\bar{m}_A = \bar{M}_s \tag{4.41}$$

For ease of writing the above, these values have been defined:

$$R \equiv r_s \tag{4.42}$$

$$\alpha_A \equiv \frac{2\psi_s}{\nu_s'} \tag{4.43}$$

$$Q_1 \equiv (1 + \alpha_A^2)^{1/2} \tag{4.44}$$

$$Q_2 \equiv \Big(1 - \frac{2M_s}{R}\Big)^{1/2} \tag{4.45}$$

$$\hat{\nu}_s \equiv -\frac{2}{Q_1}\tanh^{-1}\Big(\frac{Q_1}{1 + 2(R\nu_s')^{-1}}\Big) \tag{4.46}$$

Because the variables of interest span several orders of magnitude and because unitless variables are preferred for numerical analysis, the variables are rescaled as follows:

$$r = r_0\hat{r} \tag{4.47}$$

$$M = r_0\hat{M} \tag{4.48}$$

$$\rho = \rho_0\hat{\rho} \tag{4.49}$$

$$p = \rho_0 p \tag{4.50}$$

$\rho_0 = \rho_c$ is generally chosen, which makes $\hat{\rho}_c = 1$

Need to rescale the equation of state as well. To do this, the values if the pressure and density are divided by the central density. The structure equations are the same, except some terms are multiplied by $r_0^2\rho_0$. They choose $r_0^2\rho_0 = 1$ so that the structure equations retain their original form. The initial conditions are similarly rescaled. The value for the scalar field at the center of the star is set at the same order of magnutude as φ_∞. Theyt use $\varphi_0 < 4.3 \times 10^{-3}$ for "spontaneous scalarization case" following Damour and Esposito-Farese [1].

4.3.2 First Solution with FJBD

As with the thesis, I want to use a Runge-Kutta method. I am planning on using scipy.integrate.RK45.

Starting with G=c=1 units.

The FJBD theory is the simplest example of Scalar-Tensor theories and so I will start here. $\omega'(\phi) = 0$ and so $\omega(\phi) = const$. According to Chapter1 ω is constrained by solar system experiments to $\omega > 400,000$. This gives $\alpha < 1.1 \times 10^{-3}$. I'll start with $\alpha = 10^{-3}$ For the equation of state, I use the rescaled polytrope from Section 3.1.3. Starting with EOS1

	Γ	K
EOS 1	2.00	0.1
EOS 2	2.34	0.0195
EOS 3	2.46	0.00936

The rescaled EOS has the functional form:

$$\hat{\rho} = \frac{\hat{p}}{\Gamma - 1} + A\hat{p}^{1/\Gamma} \quad where \quad A = \frac{m_b n_0}{(KLm_b n_0)^{1/\Gamma}\rho_c^{1-1/\Gamma}} \tag{4.51}$$

m_b and n_0 are constants.

Initial conditions. $\rho_0 = \rho_c$.

	rescaled	original
M	0	0
ν	0	0
φ	φ	φ
p	1-A	$p_c = (1 - A) * 10^{20}$
ψ		
$\bar{M}$	0	0
ρ	1	10^{20}kg/m^3

The step size will be $\Delta r = 10^{-5}$. They start with a pressure $\approx 10^{20}$ kg. To get the initial density, solve EQ 4.41 for $\hat{\rho}_c = 1$

A General Relativity: Variation of the Einstein-Hilbert action

$$\delta S = \delta \frac{1}{2\kappa} \int R\sqrt{-g}\, d^4x = 0 \tag{A.1}$$

$$\delta \int R\sqrt{-g}\, d^4x = 0 \tag{A.2}$$

$$\delta \int g^{\mu\nu} R_{\mu\nu}\sqrt{-g}\, d^4x = 0 \tag{A.3}$$

$$\int (\delta g^{\mu\nu}) R_{\mu\nu}\sqrt{-g} + g^{\mu\nu}(\delta R_{\mu\nu})\sqrt{-g} + g^{\mu\nu} R_{\mu\nu}\delta(\sqrt{-g})\, d^4x = 0 \tag{A.4}$$

Use the following:

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} \tag{A.5}$$

$$\int (\delta g^{\mu\nu}) R_{\mu\nu}\sqrt{-g} - \frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} R d^4x + \int g^{\mu\nu}(\delta R_{\mu\nu})\sqrt{-g}\, d^4x \tag{A.6}$$

We can prove that the second integral is equal to zero:
Recall the definition of R

$$R_{\alpha\beta} \equiv R^{\rho}_{\alpha\rho\beta} = -R^{\rho}_{\alpha\beta\rho} = \partial_\rho \Gamma^{\rho}_{\beta\alpha} - \partial_\beta \Gamma^{\rho}_{\rho\alpha} + \Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\beta\alpha} - \Gamma^{\rho}_{\beta\lambda}\Gamma^{\lambda}_{\rho\alpha} \tag{A.7}$$

$$\delta R_{\alpha\beta} = \partial_\rho \delta\Gamma^{\rho}_{\beta\alpha} - \partial_\beta \delta\Gamma^{\rho}_{\rho\alpha} + \delta\Gamma^{\rho}_{\rho\lambda}\Gamma^{\lambda}_{\beta\alpha} + \Gamma^{\rho}_{\rho\lambda}\delta\Gamma^{\lambda}_{\beta\alpha} - \delta\Gamma^{\rho}_{\beta\lambda}\Gamma^{\lambda}_{\rho\alpha} - \Gamma^{\rho}_{\beta\lambda}\delta\Gamma^{\lambda}_{\rho\alpha} \tag{A.8}$$

$$\delta R_{\alpha\beta} = \nabla_\rho(\delta\Gamma^{\rho}_{\beta\alpha}) - \nabla_\beta(\delta\Gamma^{\rho}_{\rho\alpha}) \tag{A.9}$$

Back to the integral

$$\int g^{\mu\nu}\sqrt{-g}\big(\nabla_\alpha(\delta\Gamma^{\alpha}_{\mu\nu}) - \nabla_\nu(\delta\Gamma^{\alpha}_{\mu\alpha})\big)\, d^4x = 0 \tag{A.10}$$

$$\int \sqrt{-g}\big(\nabla_\alpha(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu}) - \nabla_\nu(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\alpha})\big)\, d^4x = 0 \tag{A.11}$$

$$\int \sqrt{-g}\big(\nabla_\alpha(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu}) - \nabla_\alpha(g^{\mu\alpha}\delta\Gamma^{\nu}_{\mu\nu})\big) d^4x = 0 \tag{A.12}$$

$$\int \sqrt{-g}\big(\nabla_\alpha(g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\nu}_{\mu\nu})\big) d^4x = 0 \tag{A.13}$$

Note that $g^{\mu\nu}\delta\Gamma^{\alpha}_{\mu\nu} - g^{\mu\alpha}\delta\Gamma^{\nu}_{\mu\nu} = A^\alpha$ is a rank one tensor. Then the divergence theorem we know

$$\int_V \nabla_\alpha A^\alpha \sqrt{-g}\, d^4x = 0 \tag{A.14}$$

That gives us:

$$\int (\delta g^{\mu\nu}) R_{\mu\nu}\sqrt{-g} - \frac{1}{2}\sqrt{-g} g_{\mu\nu}\delta g^{\mu\nu} R d^4x = 0 \tag{A.15}$$

$\delta g^{\mu\nu}$ is arbitrary. (divide by $\sqrt{-g}$)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \tag{A.16}$$

B Scalar Tensor Theories: Variation of the action with respect to the scalar field

$$\delta_\phi \left(\frac{1}{2\kappa^2} \int \sqrt{-g} \Big(\phi R + \frac{\omega(\phi)}{\phi} g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} + 2\phi\lambda(\phi) \Big)\ d^4x + S_{matter} \right) = 0 \tag{B.1}$$

$$\delta_\phi \left(\sqrt{-g} \Big(\phi R + \frac{\omega(\phi)}{\phi} g^{\mu\nu} \phi_{,\mu}\phi_{,\nu} + 2\phi\lambda(\phi) \Big) \right) = 0 \tag{B.2}$$

$$\sqrt{-g} \Big[R\delta\phi + \delta\Big(\frac{\omega(\phi)}{\phi}\Big) g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \frac{\omega(\phi)}{\phi} g^{\mu\nu} \delta(\phi_{,\mu}\phi_{,\nu}) + 2\delta(\phi\lambda(\phi)) \Big] = 0 \tag{B.3}$$

$$R\delta\phi + \delta\Big(\frac{\omega(\phi)}{\phi}\Big) g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \frac{\omega(\phi)}{\phi} \delta(g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + g^{\mu\nu}\phi_{,\mu}\delta\phi_{,\nu}) + 2\delta(\phi\lambda(\phi)) = 0 \tag{B.4}$$

$$R\delta\phi + g^{\mu\nu}\Big(\frac{\omega'(\phi)}{\phi} - \frac{\omega(\phi)}{\phi^2}\Big)\phi_{,\mu}\phi_{,\nu} + \frac{\omega(\phi)}{\phi}(2\delta\phi^{,\lambda}\phi_{,\lambda}) + 2\frac{d}{d\phi}(\phi\lambda(\phi)) = 0 \tag{B.5}$$

$$\tag{B.6}$$

Now, we need to use the following:

$$\frac{d}{d\phi} \ln\Big(\frac{\omega(\phi)}{\phi}\Big) = \frac{\phi}{\omega(\phi)}\Big(\frac{\omega'(\phi)}{\phi} - \frac{\omega(\phi)}{\phi}\Big) \tag{B.7a}$$

so

$$\Big(\frac{\omega'(\phi)}{\phi} - \frac{\omega(\phi)}{\phi}\Big) = \frac{\phi}{\omega(\phi)}\frac{d}{d\phi} \ln\Big(\frac{\omega(\phi)}{\phi}\Big) \tag{B.7b}$$

Using this we get the following:

$$R\delta\phi + g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}\frac{\phi}{\omega(\phi)}\frac{d}{d\phi}\ln\Big(\frac{\omega(\phi)}{\phi}\Big) + 2\Big(\frac{\omega(\phi)}{\phi}\Big)\Box\phi + 2\frac{d}{d\phi}(\phi\lambda(\phi)) = 0 \tag{B.8}$$

$$\Box\phi + \frac{1}{2}\frac{\phi}{\omega(\phi)}\Big[R\delta + 2\frac{d}{d\phi}(\phi\lambda(\phi))\Big] + \frac{1}{2}\phi_{,\lambda}\phi^{,\lambda}\frac{d}{d\phi}\ln\Big(\frac{\omega(\phi)}{\phi}\Big) = 0 \tag{B.9}$$

C Scalar Tensor Theories: Variation with respect to the metric

This derivation is not correct, I am missing the following term: $\frac{1}{\phi}(\phi_{;\mu\nu} - g_{\mu\nu}\Box\phi)$

$$\delta_g\left(\frac{1}{2\kappa^2}\int\sqrt{-g}\left(\phi R + \frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + 2\phi\lambda(\phi)\right)\,d^4x + S_{matter}\right) = 0 \tag{C.1}$$

$$\delta_g\left(\frac{1}{2\kappa^2}\int\sqrt{-g}\left(\phi R + \frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + 2\phi\lambda(\phi)\right)\,d^4x + \frac{1}{2\kappa^2}\frac{2\kappa^2}{1}\int\delta S_{matter}\right) = 0 \tag{C.2}$$

$$\delta_g\left(\sqrt{-g}\left(\phi R + \frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + 2\phi\lambda(\phi)\right)\right) + 2\kappa^2\delta S_M = 0 \tag{C.3}$$

I find it useful to rearrange the terms here:

$$\delta_g\Big(\sqrt{-g}(\phi R) + \sqrt{-g}\Big(\frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + 2\phi\lambda(\phi)\Big)\Big) + 2\kappa^2\delta S_M = 0 \tag{C.4}$$

Note that the first term is the the same as the variation of the action in GR multiplied by ϕ. So you can substitute in the GR field eq.

$$\delta_g\Big(\sqrt{-g}(\phi R)\Big) + (\delta_g\sqrt{-g})\Big(\frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + 2\phi\lambda(\phi)\Big) +$$
$$\sqrt{-g}\Big(\delta_g\frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \delta_g 2\phi\lambda(\phi)\Big) + 2\kappa^2\delta S_M = 0$$

Now we need to use this:

$$\delta_g\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g_{\mu\nu} \tag{C.5}$$

This gives

$$\phi\Big(\delta_g\sqrt{-g}R\Big) - \frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g_{\mu\nu}\Big(\frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + 2\phi\lambda(\phi)\Big) + \sqrt{-g}\big(\delta_g g^{\mu\nu}\big)\Big(\frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi_{,\nu}\Big) + 2\kappa^2\delta S_M = 0$$

$$\delta g_{\mu\nu}\phi\Big(-R_{\mu\nu}+\frac{1}{2}g_{\mu\nu}R\Big)-\frac{1}{2}g_{\mu\nu}\delta g_{\mu\nu}\Big(\frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu}+2\phi\lambda(\phi)\Big)+\delta_g g^{\mu\nu}\Big(\frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi_{,\nu}\Big)+2\frac{\kappa^2}{\sqrt{-g}}\delta S_M = 0 \tag{C.6}$$

I realize here that something is funny with the signs (I calculated a negative for the λ term)

$$\phi\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\lambda(\phi)\Big) = -\frac{1}{2}g_{\mu\nu}\frac{\omega(\phi)}{\phi}g^{\mu\nu}\phi_{,\mu}\phi_{,\nu} + \frac{\omega(\phi)}{\phi}\phi_{,\mu}\phi_{,\nu} + 2\frac{\kappa^2}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}} \tag{C.7}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\lambda(\phi) = \frac{\omega(\phi)}{\phi^2}\Big(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\Big) + 2\frac{\kappa^2}{\phi\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}} \tag{C.8}$$

Now define $T_{\mu\nu} = 2\frac{1}{\sqrt{-g}}\frac{\delta S_M}{\delta g_{\mu\nu}}$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - g_{\mu\nu}\lambda(\phi) = \frac{\omega(\phi)}{\phi^2}\Big(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\Big) + \frac{\kappa^2}{\phi}T_{\mu\nu} \tag{C.9}$$

The solution from the thesis is:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu R} - g_{\mu\nu}\lambda(\phi) = \frac{\kappa^2}{\phi}T_{\mu\nu} + \frac{\omega(\phi)}{\phi^2}\Big(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda}\Big) + \frac{1}{\phi}(\phi_{;\mu\nu} - g_{\mu\nu}\Box\phi) \tag{C.10}$$

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