工科数学分析下

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7.6 隐函数微分法

- 一个方程的情形
- 方程组的情形
- 隐函数存在定理

隐函数在实际问题中是常见的. 如

- 平面曲线方程 $F(x,y)=0$;
- 空间曲面方程 $F(x,y,z)=0$;
- 空间曲线方程 $\begin{cases} F(x,y,z)=0, \\ G(x,y,z)=0. \end{cases}$

下面讨论如何由隐函数方程求偏导数.

隐函数存在定理

定理 (隐函数存在定理)

设二元函数 $F(x, y)$满足以下条件：

1. 在矩形区域$D = \{(x, y) \mid |x - x_0| < a, |y - y_0| < b\}$内有关于$x, y$的连续偏导数；
2. $F(x_0, y_0) = 0$;
3. $F_y(x_0, y_0) \neq 0$.

则有

1. 在点(x_0, y_0)的某邻域内，由方程$F(x, y) = 0$可以确定唯一的函数$y = f(x)$. 即存在$\eta > 0$ 当$x \in U(x_0, \eta)$时有 $F(x, f(x)) \equiv 0$, 且$y_0 = f(x_0)$;
2. f在 $U(x_0, \eta)$内连续；
3. f 在$U(x_0, \eta)$内有连续的导数，且有

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y}.$$

注:

1. 该定理只说明了隐函数的存在性,并不一定能解出.
2. 定理的结论是局部的.
3. 隐函数的导数仍含有x与y, 理解为
$$\frac{\mathrm{dy}}{\mathrm{d}x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y}\Big|_{y=f(x)}.$$
4. 定理的条件只是充分条件. 如: $F(x, y) = (x - y)^2 = 0$.
5. 注意哪个是隐函数,哪个是自变量.

定理

假定函数$y = f(x)$满足方程$F(x, y) = 0$ 即 $F(x, f(x)) \equiv 0$. 假设$F(x, y)$与函数$f(x)$都可微且 $\frac{\partial F}{\partial y} \neq 0$. 则有

$$\frac{\mathrm{dy}}{\mathrm{d}x} = -\frac{\partial F}{\partial x} \Big/ \frac{\partial F}{\partial y}.$$

例

设y是由下列方程确定为x的隐函数：

$$F(x,y) = xy^5 - x^5y - 2 = 0.$$

求$\frac{\mathrm{dy}}{\mathrm{dx}}$.

解：把 y 看作 x 的函数，对上述方程两边对 x 求导得

$$y^5 + x\cdot 5y^4\cdot y'(x) - 5x^4\cdot y + x^5\cdot y'(x) = 0$$

$$(5xy^4 + x^5)\, y'(x) = 5x^4y - y^5$$

故 $y'(x) = \dfrac{5x^4y - y^5}{5xy^4 + x^5}$

例

已知 $\ln\sqrt{x^2+y^2}=\arctan\frac{y}{x}$, 求 $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

解: 把 y 看作 x 的函数 对上述方程两边关于 x 求导 可得

$$\frac{1}{\sqrt{x^2+y^2}}\cdot\frac{2x+2y\cdot y'(x)}{2\sqrt{x^2+y^2}}=\frac{\frac{y'(x)\cdot x-y}{x^2}}{1+\left(\frac{y}{x}\right)^2}=\frac{y'(x)\,x-y}{x^2+y^2}$$

$$\Rightarrow\quad x+y\cdot y'(x)=y'(x)\,x-y\quad\Rightarrow\quad y'(x)=\frac{x+y}{x-y}$$

$$\frac{d^2y}{dx^2}=y''(x)=\frac{(1+y'(x))(x-y)-(x+y)\cdot(1-y'(x))}{(x-y)^2}$$

注意 y 是关于 x 的函数

$$=\frac{-2y+2x\cdot y'(x)}{(x-y)^2}=\frac{-2y+2x\cdot\frac{x+y}{x-y}}{(x-y)^2}=\frac{2x^2+2y^2}{(x-y)^3}$$

定理

设函数$z = z(x,y)$是由方程$F(x,y,z) = 0$确定的隐函数，若$\frac{\partial F}{\partial z} \neq 0$, 则

$$\frac{\partial z}{\partial x} = -\frac{\partial F}{\partial x}\Big/\frac{\partial F}{\partial z}, \quad \frac{\partial z}{\partial y} = -\frac{\partial F}{\partial y}\Big/\frac{\partial F}{\partial z}.$$

例

求由 $\frac{x}{z} = \ln\frac{z}{y}$确定的隐函数 $z = z(x, y)$的一阶偏导数.

解: 把 z 看作 $x.y$ 的函数 对 $\frac{x}{z} = \ln\frac{z}{y}$ 两边同时关于 x 求偏导

$$\frac{z - x\frac{\partial z}{\partial x}}{z^2} = \frac{y}{z}\cdot\frac{1}{y}\cdot\frac{\partial z}{\partial x} = \frac{\partial z}{\partial x}\Big/z \Rightarrow z = (x+z)\frac{\partial z}{\partial x} \Rightarrow \frac{\partial z}{\partial x} = \frac{z}{x+z}$$

对 $\frac{x}{z} = \ln\frac{z}{y}$ 两边关于 y 求偏导有

$$-\frac{x}{z^2}\frac{\partial z}{\partial y} = \frac{y}{z}\cdot\frac{\frac{\partial z}{\partial y}\cdot y - z}{y^2} = \frac{1}{z}\frac{\partial z}{\partial y} - \frac{1}{y} \qquad \frac{z^2}{y} = (x+z)\cdot\frac{\partial z}{\partial y}. \Rightarrow \frac{\partial z}{\partial y} = \frac{z^2}{y(x+z)}$$

例

设$z = z(x, y)$是由$F(xy, y+z, xz) = 0$确定的隐函数，求 $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{\partial^2 z}{\partial x \partial y}$.

解：把 z 看作为 x, y 的函数，对上述方程两边关于 x 求偏导得

$$F_1' \cdot y + F_2' \cdot \frac{\partial z}{\partial x} + F_3' \cdot \left(z + x \cdot \frac{\partial z}{\partial x}\right) = 0 \Rightarrow \frac{\partial z}{\partial x} = -\frac{yF_1' + zF_3'}{F_2' + xF_3'}$$

对上述方程两边关于 y 求偏导得

$$F_1' \cdot x + F_2'\left(1 + \frac{\partial z}{\partial y}\right) + F_3' \cdot \left(x \cdot \frac{\partial z}{\partial y}\right) = 0 \Rightarrow \frac{\partial z}{\partial y} = -\frac{xF_1' + F_2'}{F_2' + xF_3'}$$

注意：F_1', F_2', F_3', z 为复合函数，都含 x, y.

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left(-\frac{yF_1' + zF_3'}{F_2' + xF_3'}\right)$$

$$= -\frac{(F_2' + xF_3')\boxed{\frac{\partial}{\partial y}(yF_1' + zF_3')} - (yF_1' + zF_3')\boxed{\frac{\partial}{\partial y}(F_2' + xF_3')}}{(F_2' + xF_3')^2}$$

最后还需要求 $\frac{\partial z}{\partial y}$ 代入化简

$$= -\frac{\left\{(F_2' + xF_3')\left[F_1' + y\left(F_{11}''x + F_{12}''\left(1 + \frac{\partial z}{\partial y}\right) + F_{13}'' \cdot x\frac{\partial z}{\partial y}\right) + \frac{\partial z}{\partial y} \cdot F_3' + z\left(F_{31}''x + F_{32}''\left(1 + \frac{\partial z}{\partial y}\right) + F_{33}'' \cdot x\frac{\partial z}{\partial y}\right)\right] - (yF_1' + zF_3')\left[F_{21}''x + F_{22}''\left(1 + \frac{\partial z}{\partial y}\right) + F_{23}'' \cdot x\frac{\partial z}{\partial y} + x\left(F_{31}''x + F_{32}''\left(1 + \frac{\partial z}{\partial y}\right) + F_{33}'' \cdot x\frac{\partial z}{\partial y}\right)\right]\right\}}{(F_2' + xF_3')^2}$$

例

设$z = z(x, y)$是由方程

$$z - y - x + xe^{z-y-x} = 0$$

所确定的隐函数，求$\mathrm{d}z$.

方法1. 先求 $\frac{\partial z}{\partial x}$ $\frac{\partial z}{\partial y}$. 把 z 看作 x, y 的函数 对上式两端分别关于 x, y 求偏导.

$$\begin{cases} \frac{\partial z}{\partial x} - 1 + e^{z-y-x} + xe^{z-y-x}\cdot\left(\frac{\partial z}{\partial x} - 1\right) = 0 \\ \frac{\partial z}{\partial y} - 1 + xe^{z-y-x}\cdot\left(\frac{\partial z}{\partial y} - 1\right) = 0 \end{cases}$$

解得

$$\begin{cases} \frac{\partial z}{\partial x} = \frac{e^{z-y-x}(x-1)+1}{xe^{z-y-x}+1} \\ \frac{\partial z}{\partial y} = \frac{xe^{z-y-x}+1}{xe^{z-y-x}+1} = 1 \end{cases} \Rightarrow dz = \frac{e^{z-y-x}(x-1)+1}{xe^{z-y-x}+1}dx + dy$$

方法二. 直接求微分

$$dz - dy - dx + dx\cdot e^{z-y-x} + x\cdot e^{z-y-x}(dz - dy - dx) = 0$$

$$dz = \frac{e^{z-y-x}(x-1)+1}{xe^{z-y-x}+1}dx + dy$$

方程组的情形

下面讨论由联立方程组所确定的隐函数的求导方法.假设由方程组

$$\begin{cases} F(x,y,z)=0, \\ G(x,y,z)=0 \end{cases}$$

确定两个一元函数 $y=y(x), z=z(x)$. 求 $\frac{\mathrm{d}y}{\mathrm{d}x}, \frac{\mathrm{d}z}{\mathrm{d}x}$? 将恒等式

$$\begin{cases} F(x,y(x),z(x))\equiv 0, \\ G(x,y(x),z(x))\equiv 0 \end{cases}$$

两边关于x求偏导，由链式法则得:

$$\begin{cases} \frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\partial F}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}=0, \\ \frac{\partial G}{\partial x}+\frac{\partial G}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}+\frac{\partial G}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}x}=0. \end{cases}$$

$$\begin{cases} \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{\partial F}{\partial z}\dfrac{\mathrm{d}z}{\mathrm{d}x} = 0, \\ \dfrac{\partial G}{\partial x} + \dfrac{\partial G}{\partial y}\dfrac{\mathrm{d}y}{\mathrm{d}x} + \dfrac{\partial G}{\partial z}\dfrac{\mathrm{d}z}{\mathrm{d}x} = 0. \end{cases}$$

当系数行列式 (称为雅可比 (Jacobi) 行列式)不为零时，即

$$J = \frac{\partial(F, G)}{\partial(y, z)} = \begin{vmatrix} \dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial z} \end{vmatrix} \neq 0.$$

解得

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial z} \end{vmatrix} \Bigg/ \begin{vmatrix} \dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial z} \end{vmatrix} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(x, z)},$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = -\begin{vmatrix} \dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial x} \\ \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial x} \end{vmatrix} \Bigg/ \begin{vmatrix} \dfrac{\partial F}{\partial y} & \dfrac{\partial F}{\partial z} \\ \dfrac{\partial G}{\partial y} & \dfrac{\partial G}{\partial z} \end{vmatrix} = -\frac{1}{J}\frac{\partial(F, G)}{\partial(y, x)}.$$

方程组的情形

下面讨论由联立方程组所确定的隐函数的求导方法.假设由方程组

$$\begin{cases} F(x,y,u,v)=0, \\ G(x,y,u,v)=0 \end{cases}$$

确定两个二元函数 $u=u(x,y), v=v(x,y)$. 求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$. 将恒等式

$$\begin{cases} F(x,y,u(x,y),v(x,y))\equiv 0, \\ G(x,y,u(x,y),v(x,y))\equiv 0 \end{cases}$$

两边关于x求偏导，由链式法则得：

$$\begin{cases} \frac{\partial F}{\partial x}+\frac{\partial F}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\frac{\partial v}{\partial x}=0, \\ \frac{\partial G}{\partial x}+\frac{\partial G}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial G}{\partial v}\frac{\partial v}{\partial x}=0 \end{cases}$$

$$\begin{cases} \dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial F}{\partial v}\dfrac{\partial v}{\partial x} = 0, \\ \dfrac{\partial G}{\partial x} + \dfrac{\partial G}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial G}{\partial v}\dfrac{\partial v}{\partial x} = 0 \end{cases}$$

当雅可比行列式不为零时，即

$$J = \frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} \neq 0.$$

解得

$$\frac{\partial u}{\partial x} = -\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial x} & \frac{\partial G}{\partial v} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(x,v)},$$

$$\frac{\partial v}{\partial x} = -\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial x} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial x} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,x)}.$$

同理，两边关于y求偏导，由链式法则得：

$$\begin{cases} \frac{\partial F}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y} = 0, \\ \frac{\partial G}{\partial y} + \frac{\partial G}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial G}{\partial v}\frac{\partial v}{\partial y} = 0 \end{cases}$$

解得，

$$\frac{\partial u}{\partial y} = -\begin{vmatrix} \frac{\partial F}{\partial y} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial y} & \frac{\partial G}{\partial v} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(y,v)},$$

$$\frac{\partial v}{\partial y} = -\begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial y} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial y} \end{vmatrix} \Bigg/ \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = -\frac{1}{J}\frac{\partial(F,G)}{\partial(u,y)}.$$

例

设$x=x(z), y=y(z)$由$\begin{cases} x^2+y^2+z^2-1=0, \\ x^2+2y^2-z^2-1=0 \end{cases}$确定,求$\frac{\mathrm{d}x}{\mathrm{d}z}, \frac{\mathrm{d}y}{\mathrm{d}z}$.

解：把x、y看作z的函数，对上述方程组两边对z求导得

$$\begin{cases} 2x\cdot x'(z)+2y\cdot y'(z)+2z=0 \\ 2x\,x'(z)+4y\cdot y'(z)-2z=0 \end{cases} \Rightarrow \begin{cases} x\,x'(z)+y\,y'(z)+z=0 \\ x\cdot x'(z)+2y\,y'(z)-z=0 \end{cases}$$

解得
$$\begin{cases} x'(z)=-\dfrac{3z}{x} \\ y'(z)=\dfrac{2z}{y} \end{cases}$$

例

设方程组 $\begin{cases} x^2+y^2-uv=0, \\ xy^2-u^2+v^2=0 \end{cases}$ 确定函数 $u=u(x,y), v=v(x,y)$,求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$.

解：把 u、v 看作 $x.y$ 的函数 对上述方程组两边关于 x 求偏导得

$$\begin{cases} 2x - \frac{\partial u}{\partial x}\cdot v - u\cdot\frac{\partial v}{\partial x} = 0 & ① \\ y^2 - 2u\cdot\frac{\partial u}{\partial x} + 2v\cdot\frac{\partial v}{\partial x} = 0 & ② \end{cases}$$

①$\times 2u$ − ②$\times v$

$$4xu - vy^2 = (2u^2+2v^2)\frac{\partial v}{\partial x} \Rightarrow \frac{\partial v}{\partial x} = \frac{4xu - vy^2}{2(u^2+v^2)}$$

①$\times 2v$ + ②$\times u$ 得 $4xv + uy^2 = (2u^2+2v^2)\frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = \frac{4xv+uy^2}{2(u^2+v^2)}$

同理关于 y 求偏导得
$$\begin{cases} 2y - \frac{\partial u}{\partial y}\cdot v - u\cdot\frac{\partial v}{\partial y} = 0 \\ 2xy - 2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y} = 0 \end{cases}$$

解得
$$\begin{cases} \frac{\partial u}{\partial y} = \frac{2vy + uxy}{u^2+v^2} \\ \frac{\partial v}{\partial y} = \frac{2uy - vxy}{u^2+v^2} \end{cases}$$

隐函数存在定理

定理 (隐函数存在定理)

设二元函数 $F(x,y)$满足以下条件：

1. 在矩形区域$D=\{(x,y)||x-x_0|<a,|y-y_0|<b\}$内有关于$x,y$的连续偏导数；
2. $F(x_0,y_0)=0$;
3. $F_y(x_0,y_0)\neq 0$.

则有

1. 在点(x_0,y_0)的某邻域内，由方程$F(x,y)=0$可以确定唯一的函数 $y=f(x)$. 即存在$\eta>0$ 当$x\in U(x_0,\eta)$时有 $F(x,f(x))\equiv 0$, 且 $y_0=f(x_0)$;
2. f在 $U(x_0,\eta)$内连续；
3. f 在$U(x_0,\eta)$内有连续的导数.

隐函数存在定理的证明

隐函数存在定理的证明

隐函数存在定理的证明

练习题

例

求证: 方程 $xyz+\sqrt{x^2+y^2+z^2}=\sqrt{2}$在点$(1,0,-1)$的某个邻域内可以确定一个隐函数 $z=z(x,y)$, 并在该点处求微分$\mathrm{d}z$.

解: 两边求微分得

$$dx\cdot yz+dy\cdot xz+dz\cdot xy+\frac{2xdx+2ydy+2zdz}{2\sqrt{x^2+y^2+z^2}}=0$$

令 $(x,y,z)=(1,0,-1)$ 有.

$$dx\cdot 0+dy(1\times(-1))+dz\cdot 0+\frac{dx+0dy-dz}{\sqrt{2}}=0$$

$$-dy+\frac{dx-dz}{\sqrt{2}}=0 \Rightarrow dz=dx-\sqrt{2}dy$$

7.7 泰勒多项式

1. 一元函数的局部线性化: $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$.
2. 设二元函数$f(x, y)$在点(x_0, y_0)处可微，则$f(x, y)$在点(x_0, y_0)附近可局部线性化:

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

如何找一个二次的二元多项式$P(x, y)$在点(x_0, y_0)处充分地接近$f(x, y)$呢？

他们在点(x_0, y_0)处具有相同的一阶、二阶偏导数!

泰勒公式

定理

设二元函数 $z=f(x,y)$在(a,b)处的某一邻域内连续，且有直到$n+1$阶的连续偏导数，$(a+h,b+k)$为此邻域内一点，则称下式为带 *Lagrange* 余项的n阶泰勒展式：

$$\begin{aligned} f(a+h,b+k) &= f(a,b)+\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)f(a,b) \\ &+\frac{1}{2!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^2 f(a,b)+\cdots+\frac{1}{n!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^n f(a,b) \\ &+\frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^{n+1} f(a+\theta h,b+\theta k)\quad (0<\theta<1). \end{aligned}$$

其中 $\left(h\frac{\partial}{\partial x}+k\frac{\partial}{\partial y}\right)^m f(a,b):=\sum_{p=0}^{m} C_m^p h^p k^{m-p}\frac{\partial^m f(a,b)}{\partial x^p\partial y^{m-p}}.$

例

求函数$f(x,y,z)=\sqrt{x+2y+1}$在点 $(0,0)$处的二阶泰勒多项式.

例

求函数$f(x,y)=e^{x^2-y^2}$在点 $(0,0)$处的二阶泰勒多项式.

例

求函数$f(x,y)=\frac{\cos x}{1+y}$在点 $(0,0)$处带Lagrange余项的一阶泰勒展式.

作业

- 习题 7.6 (A)
  - 2. (1) (2)
  - 3.
  - 5.
- 习题 7.6 (B)
  - 1.
  - 3.
- 习题 7.7 (A)
  - 2.

7.8 向量值函数的导数

- 向量值函数的概念
- 向量值函数的极限与连续性
- 向量值函数的导数

向量值函数的概念

定义 (向量值函数)

设$D\subseteq\mathbb{R}^n$是一个点集，称映射 $\boldsymbol{f}:D\to\mathbb{R}^m\quad(m\geq 2)$ 为定义于D上、在$\mathbb{R}^m$中取值的向量值函数．记为，

$$\boldsymbol{y}=\boldsymbol{f}(\boldsymbol{x})$$

其中$\boldsymbol{x}=(x_1,\cdots,x_n)^T\in\mathbb{R}^n,\boldsymbol{y}=(y_1,\cdots,y_m)^T\in\mathbb{R}^m.$

若将它们的坐标分量一个一个写出来，就是一个多元函数组

$$\begin{cases}y_1=f_1(x_1,\cdots,x_n),\\ y_2=f_2(x_1,\cdots,x_n),\\ \vdots\\ y_m=f_m(x_1,\cdots,x_n).\end{cases}$$

向量值函数的极限

定义 (向量值函数的极限ϵ-δ)

设$D \subseteq \mathbb{R}^n$, $\boldsymbol{f} : D \to \mathbb{R}^m \quad (m \geq 2)$为定义在$D$上的向量值函数，$\boldsymbol{x}_0$ 为D的极限点. 若存在$\boldsymbol{A} = (A_1, \cdots, A_m)^T \in \mathbb{R}^m$有 $\forall \epsilon > 0, \exists \delta > 0$ 使得任意满足$0 < ||\boldsymbol{x} - \boldsymbol{x}_0|| < \delta, \boldsymbol{x} \in D$ 的$\boldsymbol{x}$均有 $||\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{A}|| < \epsilon$. 则称当$\boldsymbol{x}$在$D$内趋于$\boldsymbol{x}_0$时，$\boldsymbol{f}$的极限为$\boldsymbol{A}$. 记为

$$\lim_{D \ni \boldsymbol{x} \to \boldsymbol{x}_0} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}.$$

简记为 $\lim\limits_{\boldsymbol{x} \to \boldsymbol{x}_0} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A}$.

若 $\boldsymbol{f} = (f_1, \cdots, f_m)^T$, 其中 $f_i : D \to \mathbb{R}$的 n元函数，则

$$\lim_{\boldsymbol{x} \to \boldsymbol{x}_0} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{A} \Leftrightarrow \lim_{\boldsymbol{x} \to \boldsymbol{x}_0} f_i(\boldsymbol{x}) = A_i (i = 1, \cdots, m).$$

向量值函数的极限归结为多元函数的极限.

向量值函数的连续性

定义 (向量值函数的连续性)

设$D\subseteq\mathbb{R}^n$, $\boldsymbol{f}:D\to\mathbb{R}^m\quad(m\geq 2)$为定义在$D$上的向量值函数，$\boldsymbol{x}_0$ 为D的极限点. 若有

$$\lim_{D\ni\boldsymbol{x}\to\boldsymbol{x}_0}\boldsymbol{f}(\boldsymbol{x})=\boldsymbol{f}(\boldsymbol{x}_0),$$

则称$\boldsymbol{f}$在点$\boldsymbol{x}_0$处连续.

若 $\boldsymbol{f}=(f_1,\cdots,f_m)^T$, 其中 $f_i:D\to\mathbb{R}$的 n元函数，则

$\boldsymbol{f}$ 在点 $\boldsymbol{x}_0\in D$ 连续 $\Leftrightarrow$ n 元函数 f_i 在点 $\boldsymbol{x}_0\in D$ 连续 $(i=1,\cdots,m)$.

向量值函数的连续性归结为多元函数的连续性.

向量值函数的导数

定义 (一元向量值函数的导数)

设$D \subseteq \mathbb{R}$, $\boldsymbol{f} : D \to \mathbb{R}^m \quad (m \geq 2)$为定义在$D$上的向量值函数，若$\boldsymbol{f} = (f_1, \cdots, f_m)$, 其中 $f_i : D \to \mathbb{R}$的函数，若$f_i(x)$在点x_0处可导($i = 1, \cdots, m$)，则定义$\boldsymbol{f}$在点x_0处的导数为

$$\begin{aligned}
\frac{\mathrm{d}\boldsymbol{f}}{\mathrm{d}x} &= \lim_{\Delta x \to 0} \frac{\boldsymbol{f}(x_0 + \Delta x) - \boldsymbol{f}(x_0)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \begin{bmatrix} \frac{f_1(x_0+\Delta x)-f_1(x_0)}{\Delta x} \\ \frac{f_2(x_0+\Delta x)-f_2(x_0)}{\Delta x} \\ \vdots \\ \frac{f_m(x_0+\Delta x)-f_m(x_0)}{\Delta x} \end{bmatrix} = \begin{bmatrix} f_1'(x_0) \\ f_2'(x_0) \\ \vdots \\ f_m'(x_0) \end{bmatrix} = \boldsymbol{f}'(x_0).
\end{aligned}$$

向量值函数的导数

定义 (一般向量值函数的导数)

设$D \subseteq \mathbb{R}^m$, $\boldsymbol{f} : D \to \mathbb{R}^m \quad (m \geq 2)$为定义在$D$上的向量值函数，若$\boldsymbol{f} = (f_1, \cdots, f_m)$, 其中 $f_i : D \to \mathbb{R}$的n元函数，若$f_i(x)$在点$\boldsymbol{x}_0$处关于每个分量$x_j(j = 1, \cdots, n)$的偏导数都存在$(i = 1, \cdots, m)$，则定义$\boldsymbol{f}$在点$\boldsymbol{x}_0$处的导数为下列雅可比矩阵

$$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}.$$

简记为 $D\boldsymbol{f}(\boldsymbol{x}_0) = \left(\frac{\partial f_i}{\partial x_j}\right)_{i,j}$.

例

求向量值函数 $\boldsymbol{f}(x,y,z)=\begin{bmatrix} 3x+e^y z \\ x^3+y^2\sin z \end{bmatrix}$在点$(x_0,y_0,z_0)$处的导数.

谢谢大家!