Growth of function

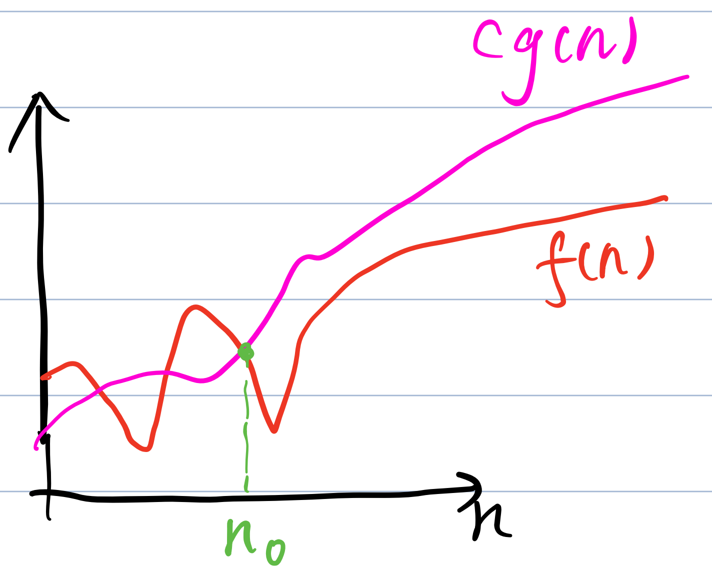
O , Ω , Θ , o , ω

O-notation

$O(g(n)) = \{ f(n)$: there exist positive constants c and n_0 such that $0 \le f(n) \le c\,g(n)$ for all $n \ge n_0 \}$

渐近性

$g(n)$ is an asymptotic upper bound for $f(n)$



Ex. $2n^2 \in O(n^3)$, $c=1$, $n_0=2$

$2n^2 \le 1 \cdot n^3$, $n \geq 2$

$2n^2 = O(n^3)$

$2n^2 = O(n^2)$, c, n_0

$0 \le 2n^2 \le cn^2$, $c = 3$, $n_0 = 1$

$\le 3n^2$

$n^2 = O(n^2)$, $n^2 + n = O(n^2)$

$0 \le n^2 + n \le cn^2$ c, n_0

$c = 2$, $n_0 = \cancel{4}\ 2$

$0 \le n^2 + n \le 2n^2$, if $n \geq 2$

$n^2 + 10000n = O(n^2)$, $1000n^2 + 1000n = O(n^2)$

$n = O(n^2)$, $n/10000 = O(n^2)$

$\frac{n^2}{\lg\lg n} = O(n^2)$, $n^{1.999999} = O(n^2)$

c, n_0.

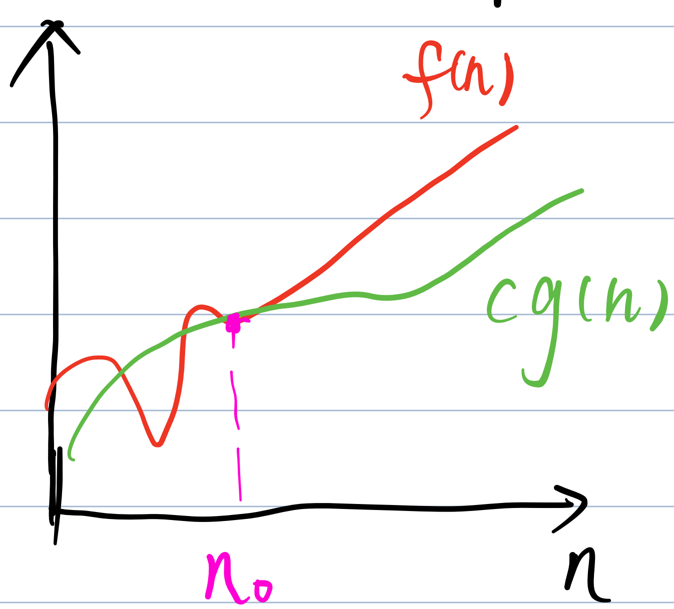
Ω - notation

$\Omega(g(n)) = \{ f(n)$: there exist positive constants c and n_0, such that

$$0 \le c\,g(n) \le f(n)$$

for all $n \geq n_0 \}$

$g(n)$ is an asymptotic lower bound for $f(n)$.



Ex. $\sqrt{n} = \Omega(\lg n)$, $c=1$, $n_0=16$

$n^2 = \Omega(n^2)$. $c = 0.5$, $n_0 = 1$

$0 \le 0.5n^2 \le n^2$ for $n \geq 1$

$n^2 + n = \Omega(n^2)$, $n^2 - 10n = \Omega(n^2)$

$n^2 + 10000n = \Omega(n^2)$, $n^2 - 1000n = \Omega(n^2)$

$n^3 = \Omega(n^2)$, $n^{2.00001} = \Omega(n^2)$

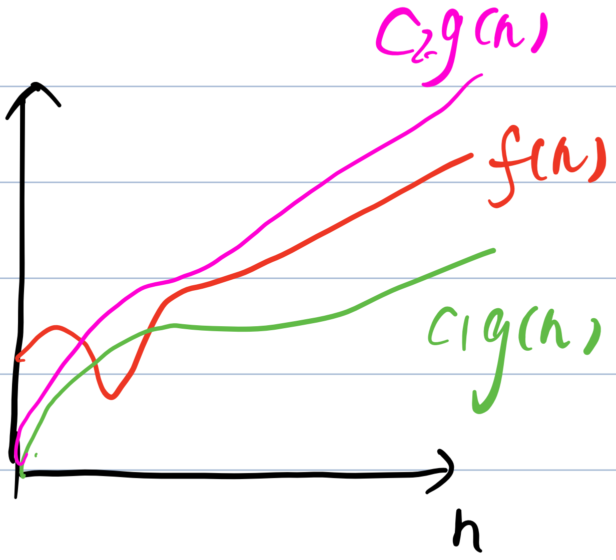
$n^2 \lg\lg n = \Omega(n^2)$, $2^{2n} = \Omega(n^2)$

Θ - notation

$\Theta(g(n)) = \{f(n)$: there exist positive constants c_1, c_2 and n_0 such that

$0 \le c_1 g(n) \le f(n) \le c_2 g(n)$

for all $n \geq n_0\}$



$g(n)$ is an asymptotic tight bound for $f(n)$.

Theorem.

$f(n) = \Theta(g(n))$ if and only if

$f(n) = O(n)$, and $f(n) = \Omega(n$

$O(1) = O(n)$

Step 1: $n=1$, $O(1) = O(1)$

Step 2: Suppose $k<n$

$O(1) = O(k)$

$O(n) = O(n-1) + O(1)$

$= O(1) + O(1)$

$= O(1)$

$O(n) = O(1)$

o-notation

$o(g(n)) = \{f(n):$ for all constants $c > 0$,

there exist a constant $n_0 > 0$

such that

$0 < f(n) < cg(n)$

for all $n \geq n_0\}$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$$

Ex. $\underbrace{n^{1.99999}}_{f} = o(\overset{g}{n^2})$ $\quad \lim_{n\to\infty} \frac{n^{1.99999}}{n^2} = 0$

$\frac{n^2}{\lg n} = o(n^2)$, $\frac{n^2}{10000} \neq o(n^2)$

ω-notation

$\omega(g(n)) = \{f(n):$ for all constants $c > 0$, there exist $n_0 > 0$ such that $0 < cg(n) < f(n)$ for all $n \geq n_0$.$\}$

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty$$

Ex. $n^{2.00001} = \omega(n^2)$, $\lim_{n\to\infty} \frac{n^{2.00001}}{n^2} = \infty$

$n^2 \lg \lg n = \omega(n^2)$

$n^2 \neq \omega(n^2)$

Relation properties:

Transitivity:

$f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n)) \Rightarrow$
$f(n) = \Theta(h(n))$

Same for O, Ω, o, ω

Reflexivity:

$f(n) = \Theta(f(n))$

Same for O, Ω

Symmetry:

$f(n) = \Theta(g(n))$ if and only if $g(n) = \Theta(f(n))$

Transpose Symmetry: iif = if and only if

$f(n) = O(g(n))$ iif $g(n) = \Omega(f(n))$

$f(n) = o(g(n))$ iif $g(n) = \omega(f(n))$