# 工科数学分析下

李茂生

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# 7.4 多元复合函数的求导法则

- 复合函数的求导法则

- 全微分形式不变性

- 高阶偏导数与高阶微分

# 复合函数的求导法则(链导法则)

回忆(一元情形)： $y = f(u), u = \varphi(x)$, 则

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

**问题**

设而对于二元函数 $z = f(u, v), u = \varphi(x, y), v = \psi(x, y)$. 如何求

$$\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}?$$

# 复合函数的求导法则(链导法则)

**定理**

设二元函数$u=\varphi(x,y)$和$v=\psi(x,y)$都在点$P(x,y)$处有偏导数，且函数$z=f(u,v)$ 在对应点 $(u,v)$有连续的偏导数，则复合函数

$$z=f[\varphi(x,y),\psi(x,y)]$$

在对应点$(x,y)$的偏导数存在且有以下计算公式:

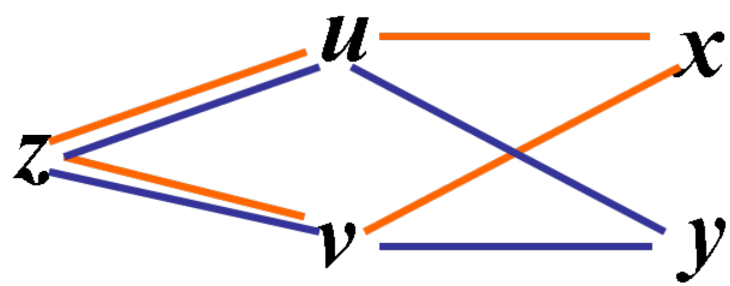
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x},\qquad \frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

# 网络图

$$z = f[\underbrace{\varphi(x,y)}_{u}, \underbrace{\psi(x,y)}_{v}]$$

网络图原则

网络图




$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}$$

# 练习题

$z=\ln(e^{\cdots}+x^2+y))$

**例**

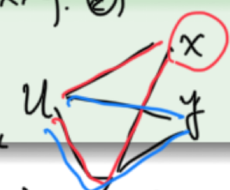
设$z=\ln(u^2+v)$, 而$u=e^{x+y^2}$, $v=x^2+y$. 求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.

解：
$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{2u}{u^2+v}\cdot e^{x+y^2}+\frac{1}{u^2+v}(2x)=$$
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{2u}{u^2+v}\cdot e^{x+y^2}(2y)+\frac{1}{u^2+v}\cdot(1)=\cdots$$

**例**

设$u=e^{x^2+y^2+z^2}$, 而$z=x^2\sin y$. 求 $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$.

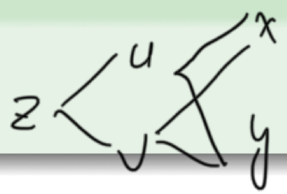
$u(x,y,z)$



解：
$$\frac{\partial u}{\partial x}=u_1'\cdot 1+\cancel{u_2'\cdot 0}+u_3'\cdot\frac{\partial z}{\partial x}=e^{x^2+y^2+z^2}(2x)+e^{x^2+y^2+z^2}\cdot(2z)\cdot 2x\sin y$$
$$\frac{\partial u}{\partial y}=u_2'\times 1+u_3'\cdot\frac{\partial z}{\partial y}=e^{x^2+y^2+z^2}\cdot(2y)+e^{x^2+y^2+z^2}\cdot(2z)\cdot x^2\cos y$$

**例**

设$z=f(xy,\frac{y}{x})$, $f$有连续偏导，求 $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$.



解：
$$\frac{\partial z}{\partial x}=f_1'\cdot\frac{\partial(xy)}{\partial x}+f_2'\cdot\frac{\partial(\frac{y}{x})}{\partial x}=f_1'\cdot y+f_2'\cdot(-\frac{y}{x^2})$$
$$\frac{\partial z}{\partial y}=f_1'\frac{\partial(xy)}{\partial y}+f_2'\cdot\frac{\partial(\frac{y}{x})}{\partial y}=f_1'\cdot x+f_2'\cdot(\frac{1}{x}).$$

# 练习题

## 例

设$z = f(x, y, z)$ 为$k$次齐次函数，即 $f(tx, ty, tz) = t^k f(x, y, z)$. 求证：

$$x\frac{\partial f}{\partial x}(x,y,z) + y\frac{\partial f}{\partial y}(x,y,z) + z\frac{\partial f}{\partial z}(x,y,z) = kf(x, y, z).$$

证明 由 $f(tx, ty, tz) = t^k f(x, y, z)$，两边同时对$t$求导有

$$x f_1'(tx, ty, tz) + y f_2'(tx, ty, tz) + z f_3'(tx, ty, tz) = k t^{k-1} f(x, y, z)$$

令 $t=1$，有

$$x f_1'(x, y, z) + y f_2'(x, y, z) + z f_3'(x, y, z) = k f(x, y, z)$$

## 例

已知$f(t)$可微, 证明 $z=\frac{y}{f(x^2-y^2)}$ 满足方程

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}.$$

证明: $\frac{\partial z}{\partial x}=\frac{-yf'\cdot 2x}{f^2}$ $\quad\frac{\partial z}{\partial y}=\frac{f-yf'(-2y)}{f^2}$

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=-\frac{2yf'}{f^2}+\frac{2yf'}{f^2}+\frac{1}{yf}$$

$$=\frac{1}{yf}=\frac{y}{y^2f}=\frac{z}{y^2}$$

# 全微分形式不变性

设$z=f(u,v)$具有连续偏导数, 两偏导数均存在，若有$u=\varphi(x,y)$, $v=\psi(x,y)$时，则复合后关于$x,y$的二元函数$z=f(u(x,y),v(x,y))$有全微分

$$\mathrm{d}z=\frac{\partial z}{\partial x}\mathrm{d}x+\frac{\partial z}{\partial y}\mathrm{d}y.$$

此外我们有以下式子成立

$$\mathrm{d}z=\frac{\partial z}{\partial u}\mathrm{d}u+\frac{\partial z}{\partial v}\mathrm{d}v.$$

$$\mathrm{d}(f(x(u,v)))=f_x(x(u,v))dx.$$

## 例

设$u = \ln\sqrt{x^2+y^2+z^2}$, 求$\mathrm{d}u$ 以及 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

$$du = \frac{1}{2}\frac{1}{x^2+y^2+z^2}\,d(x^2+y^2+z^2) = \frac{x\,dx+y\,dy+z\,dz}{x^2+y^2+z^2}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{x}{x^2+y^2+z^2},\quad \frac{\partial u}{\partial y} = \frac{y}{x^2+y^2+z^2},\quad \frac{\partial u}{\partial z} = \frac{z}{x^2+y^2+z^2}.$$

## 例

设$u = f(x^2-y^2, e^{xy}, z)$, 求 $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}$.

$$du = f_1'\cdot d(x^2-y^2) + f_2'\,de^{xy} + f_3'\,dz$$

$$= f_1'(2x\,dx-2y\,dy) + f_2'\cdot e^{xy}(y\,dx+x\,dy) + f_3'\,dz$$

$$= \underbrace{\boxed{(2xf_1'+ye^{xy}f_2')}}_{=\frac{\partial u}{\partial x}}dx + \underbrace{\boxed{(xe^{xy}f_2'-2yf_1')}}_{\frac{\partial u}{\partial y}}dy + \underbrace{\boxed{f_3'}}_{\frac{\partial u}{\partial z}}dz$$

通过全微分求所有一阶偏导数,比链导法则求偏导数有时会更灵活方便.

# 高阶偏导数和高阶全微分

二阶偏导数: 对偏导函数的偏导数. 函数$z = f(x, y)$的二阶偏导数为

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) := \frac{\partial^2 z}{\partial x^2} := f_{xx}(x, y), \qquad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) := \frac{\partial^2 z}{\partial y \partial x} := f_{xy}(x, y),$$

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) := \frac{\partial^2 z}{\partial x \partial y} := f_{yx}(x, y), \quad \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) := \frac{\partial^2 z}{\partial y^2} := f_{yy}(x, y).$$

通常我们把$f_{xy}(x, y)$和$f_{yx}(x, y)$称作混合偏导数.

二阶及二阶以上的偏导数统称为高阶偏导数.

**例**

求函数$f(x,y)=y\cos x+3x^2e^y$ 的所有二阶偏导数.

解：$\frac{\partial f}{\partial x}=-y\sin x+6xe^y$，$\frac{\partial f}{\partial y}=\cos x+3x^2e^y$

$\frac{\partial^2 f}{\partial x^2}=-y\cos x+6e^y$，$\frac{\partial^2 f}{\partial y^2}=3x^2e^y$，$\frac{\partial^2 f}{\partial x\partial y}=-\sin x+6xe^y$

$\frac{\partial^2 f}{\partial y\partial x}=-\sin x+6xe^y$

**例**

求函数$z=x^2y^3+xy^2$ 的所有二阶偏导数.

解：$\frac{\partial z}{\partial x}=2xy^3+y^2$，$\frac{\partial z}{\partial y}=3x^2y^2+2xy$

$\frac{\partial^2 z}{\partial x^2}=2y^3$，$\frac{\partial^2 z}{\partial y\partial x}=6xy^2+2y$，$\frac{\partial^2 z}{\partial y^2}=6x^2y+2x$

$\frac{\partial^2 z}{\partial x\partial y}=6xy^2+2y$

# 改变次序的混合偏导数

## 例 (混合偏导数的次序不一定能交换)

设函数

$$f(x,y)=\begin{cases}\frac{x^3y}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0),\end{cases}$$

求 $f_{xy}(0,0)$ 和 $f_{yx}(0,0)$.

解: $(x,y)\neq(0,0)$

$$\frac{\partial f}{\partial x}=\frac{3x^2y(x^2+y^2)-2x^4y}{(x^2+y^2)^2}=\frac{x^4y+3x^2y^3}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial y}=\frac{x^3(x^2+y^2)-2x^3y^2}{(x^2+y^2)^2}=\frac{x^5-x^3y^2}{(x^2+y^2)^2}$$

$$\frac{\partial f}{\partial x}(0,0)=\lim_{x\to0}\frac{f(x,0)-f(0,0)}{x}=0,\quad \frac{\partial f}{\partial y}(0,0)=\lim_{y\to0}\frac{f(0,y)-f(0,0)}{y}=0$$

$$f_{xy}(0,0)=\lim_{x\to0}\frac{f_y(x,0)-f_y(0,0)}{x}=\lim_{x\to0}\frac{x-0}{x}=1$$

$$f_{yx}(0,0)=\lim_{y\to0}\frac{f_x(0,y)-f_x(0,0)}{y}=\lim_{y\to0}\frac{0-0}{y}=0$$

# 改变次序的混合偏导数

**定理 (混合偏导数可改变次序的充分条件)**

如果函数$z=f(x,y)$的两个混合偏导数$f_{xy}(x,y)$和$f_{yx}(x,y)$在一个区域$D$上连续，则在该区域内该两混合偏导数都相等，即

$$f_{xy}(x,y)=f_{yx}(x,y),(x,y)\in D.$$

一般地，多元函数的高阶混合偏导数如果连续就与求导的次序无关.

## 例

设$z=f(x^2+y^2,\frac{y}{x})$，求$\frac{\partial^2 z}{\partial x^2},\frac{\partial^2 z}{\partial x\partial y},\frac{\partial^2 z}{\partial y^2}$.

解：$\frac{\partial z}{\partial x}=f_1'\cdot 2x+f_2'(-\frac{y}{x^2})$　　$\frac{\partial z}{\partial y}=f_1'\cdot 2y+f_2'\cdot\frac{1}{x}$

$$\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}(f_1')\cdot 2x+f_1'\cdot 2x+\frac{\partial f_2'}{\partial x}\cdot(-\frac{y}{x^2})+f_2'\frac{2y}{x^3}$$

$$=\left(f_{11}''2x+f_{21}''(-\frac{y}{x^2})\right)2x+f_1'\cdot 2x+\left(f_{12}''\cdot 2x+f_{22}''(\frac{-y}{x^2})\right)\cdot(\frac{-y}{x^2})+f_2'\cdot\frac{2y}{x^3}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}(f_1')\cdot 2y+\frac{\partial f_2'}{\partial x}\cdot\frac{1}{x}+f_2'\cdot\frac{-1}{x^2}$$

$$=\left(f_{11}''\cdot 2x+f_{21}''\cdot(-\frac{y}{x^2})\right)\cdot 2y+\left(f_{12}''2x+f_{22}''\cdot\frac{-y}{x^2}\right)\cdot\frac{1}{x}+f_2'\cdot\frac{-1}{x^2}$$

$$\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}f_1'\cdot 2y+f_1'\cdot 2+\frac{\partial}{\partial y}f_2'\cdot\frac{1}{x}$$

$$=\left(f_{11}''\cdot 2y+f_{21}''\cdot\frac{1}{x}\right)\cdot 2y+f_1'\cdot 2+\left(f_{12}''\cdot 2y+f_{22}''\cdot\frac{1}{x}\right)\cdot\frac{1}{x}$$

# 高阶全微分

## 定义 (二阶全微分)

设函数 $z = f(x, y)$ 在开区域$D$上每一点都存在全微分，则当自变量的改变量$\Delta x$ 和$\Delta y$任意固定时，全微分$\mathrm{d}z$是关于$x, y$的函数. 因此，可考虑$\mathrm{d}z$关于自变量的同一改变量的全微分. 即若

$$\mathrm{d}z = f_x \mathrm{d}x + f_y \mathrm{d}y,$$

则函数的二阶全微分 $\mathrm{d}(\mathrm{d}z) = \mathrm{d}^2 z$. 实际上，

$$\mathrm{d}^2 z = f_{xx} \mathrm{d}x^2 + 2f_{xy} \mathrm{d}x\mathrm{d}y + f_{yy} \mathrm{d}y^2.$$

## 例

求$z = x \sin y$ 的二阶全微分.

解:
$$\frac{\partial z}{\partial x} = \sin y \qquad \frac{\partial z}{\partial y} = x\cos y. \qquad \frac{\partial^2 z}{\partial x^2} = 0. \qquad \frac{\partial^2 z}{\partial x \partial y} = \cos y$$

$$\frac{\partial^2 z}{\partial y^2} = -x\sin x$$

$$\Rightarrow d^2 z = 2\cos y\, dx\, dy - x \sin x\, dy^2.$$

多元函数的偏导数常常用于建立某些偏微分方程.偏微分方程是描述自然现象、反映自然规律的一种重要手段.例如方程

$$\frac{\partial^2 z}{\partial x^2} = a\frac{\partial^2 z}{\partial y^2}$$

($a$是常数)称为波动方程, 它可用来描述各类波的运动.又如方程

$$\Delta z := \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

称为拉普拉斯(Laplace)方程,它在热传导、流体运动等问题中有着重要的作用.

## 例

设$f$满足Laplace 方程$\partial_{11}f+\partial_{22}f=0$. 证明:

$$u(x,y)=f(\frac{x}{x^2+y^2},\frac{y}{x^2+y^2})$$

也满足Laplace方程.

## 例

设$z=f(u,x,y), u=xe^y$, 其中$f$有二阶连续偏导数，求 $\frac{\partial^2 z}{\partial x\partial y}$.

解：$\frac{\partial z}{\partial y} = f_1'\cdot(xe^y) + f_2'\cdot 0 + f_3'\cdot 1$

$= f_1'(xe^y) + f_3'$

$\frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}f_1'\cdot(xe^y) + f_1'\cdot e^y + \frac{\partial f_3'}{\partial x}$

$= (f_{11}''\cdot e^y + f_{21}''\cdot 1 + f_{31}''\cdot 0)\cdot(xe^y) + f_1' e^y + (f_{13}''\cdot e^y + f_{23}''\cdot 1 + f_{33}''\cdot 0)$

例

设$z=f(2x-y,y\sin x)$，其中$f$有二阶连续偏导数，求 $\frac{\partial^2 z}{\partial x\partial y}$.

解：$\frac{\partial z}{\partial x}=f_1'\cdot 2+f_2'\cdot y\cos x$. $\frac{\partial z}{\partial y}=f_1'\cdot(-1)+f_2'\cdot\sin x$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\left(f_{11}''2+f_{21}''\cdot y\cos x\right)(-1)$$

$$+\left(f_{12}''\cdot 2+f_{22}''\cdot y\cos x\right)\sin x$$

$$+f_2'\cdot\cos x$$

$$=-2f_{11}''-y\cos x f_{21}''+2\sin x f_{12}''+y\cos x\sin x f_{22}''$$

$$+f_2'\cdot\cos x$$

# 作业

- 习题 7.4 (A)
  - 2. 奇数题
  - 3. 奇数题
  - 6. (3) (4)
  - 7. 偶数题
  - 10.
- 习题 7.4 (B)
  - 1. (2) (3) (5)
  - 2.

谢谢大家!