Recurrence (递归)

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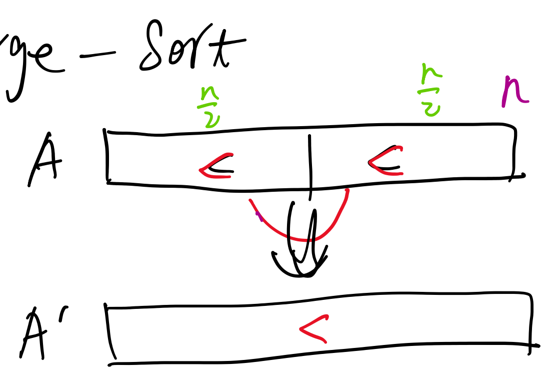
$$T(n)=\begin{cases}1, & n=1\\ T(n-1)+1, & n>1\end{cases}$$

$$F(n)=\begin{cases}1, & n=0,1\\ F(n-1)+F(n-2)\end{cases}$$

$$T(n)=\begin{cases}1, & n=1\\ 2T(\frac{n}{2})+n, & n>1\end{cases}$$

$$T(n)=\begin{cases}0, & n=2\\ T(\sqrt{n})+1, & n>2\end{cases}$$

Merge – Sort



Merge_Sort (A, p, q) $\quad$ if (p >= q) return; $\quad$ $T(n)$

$m=\lfloor \frac{p+q}{2} \rfloor$ $\quad$ $O(1)$

Merge_Sort (A, p, m) $\quad$ $T(\frac{n}{2})$

Merge_Sort (A, m+1, q) $\quad$ $T(\frac{n}{2})$

Merge (A, p, m, q) $\quad$ $\Theta(n)$: cn

Merge_Sort (A, 1, n)

$\cancel{T(n)} = 2T(\frac{n}{2}) + \underbrace{cn + O(1)}_{cn}$

$T(n) = 2T(\frac{n}{2}) + cn$

Substitution Method (递推法)

1. Guess the solution
2. Use induction to prove it.

Ex. $T(n)=\begin{cases}1, & n=1\\ 2T(\frac{n}{2})+n, & n>1\end{cases}$ $\quad T(n)=\Theta(n\lg n)$

1. Guess. $T(n)=O(n^3)$, c, n_0
   $0 \le T(n) \le cn^3$, for $n \ge n_0$

2. ① $k=1$ $\quad T(k) \le c\cdot 1^3$.
   $T(1)=1 \le c\cdot 1^3$

   ② $k<n$, $T(k)=O(k^3) \Longleftrightarrow T(k)\le ck^3$

$T(n) = 2T(\frac{n}{2}) + n$ $\qquad \frac{n}{2}<n$

$\le 2(c\cdot(\frac{n}{2})^3) + n$ $\qquad cn^3$

$= 2(\frac{cn^3}{8}) + n$

$= \frac{cn^3}{4} + n$

$= cn^3 - \frac{3cn^3}{4} + n$

$= cn^3 - (\frac{3c}{4}n^3 - n)$ $\quad \ge 0$

$\le cn^3$ $\quad \ge 0$

if $\frac{3c}{4}n^3 - n \ge 0$ 成立.

$n=1$, $c \ge \frac{4}{3}$. $\qquad n_0=1$, $c\ge\frac{4}{3}$

$$T(n)=\begin{cases}1, & n=1\\ 4T(\frac{n}{2})+100n, & n>1\end{cases} = O(n^2)$$

1. Guess. $T(n)=O(n^3)$
   $0 \le T(n) \le cn^3$

2. ① $k=1$, $T(k)=T(1)=1 \le c\cdot 1^3$.

   ② $k<n$, $T(k)\le ck^3$

   ③ $k=n$.

$T(n) = 4T(\frac{n}{2}) + 100n$ $\qquad \because \frac{n}{2}<n$

$\le 4(c(\frac{n}{2})^3) + 100n$

$= 4c\cdot\frac{n^3}{8} + 100n$

$= \frac{c}{2}n^3 + 100n$

$= cn^3 - \frac{c}{2}n^3 + 100n$

$T(n)\le cn^3$ $\qquad = cn^3 - (\frac{c}{2}n^3 - 100n)$ → Reminder

$\le cn^3$ $\quad \ge 0$

$\frac{c}{2}n^3 - 100n \ge 0$ $\quad$, $n_0=1$

$\frac{c}{2} - 100 \ge 0$, $\quad c \ge 200$

$$T(n)=\begin{cases}1, & n=1\\ 9T(\frac{n}{3})+n, & n>1\end{cases}$$

1. Guess $T(n)=O(n^2)$
   $0 \le T(n) \le cn^2$

2. $k<n$, $T(k)\le ck^2$ $\qquad \because \frac{n}{3}<n$

   $k=n$

$T(n) = 9T(\frac{n}{3}) + n$

$\le 9(c(\frac{n}{3})^2) + n$

$= \cancel{9}\cdot c\cdot\frac{n^2}{\cancel{9}} + n$

$= cn^2 + n = cn^2 - (-n)$ $\quad n\ge 0$ $\quad \ge 0$ $\quad \le 0$

$\not\le cn^2$

$T(n) \le cn^2$

$T(n) \le c_1 n^2 - c_2 n \le c_1 n^2$, $c_1, c_2 > 0$,

① $k<n$, $T(k)\le c_1k^2 - c_2k \le c_1k^2$

② $T(n) = 9T(\frac{n}{3}) + n$ $\qquad \frac{n}{3}<n$

$\le 9(c_1(\frac{n}{3})^2 - c_2(\frac{n}{3})) + n$

$= 9(\frac{c_1}{9}n^2 - \frac{c_2}{3}n) + n$

$= c_1n^2 - 3c_2n + n$

$= c_1n^2 - c_2n - (2c_2n - n)$ $\quad \ge 0$

$\le c_1n^2 - c_2n \le c_1n^2$

$2c_2\cancel{n} - \cancel{n} \ge 0$ $\qquad n_0=1$

$T(n)=O(n^2)$ $\qquad c_2 \ge \frac{1}{2}$