



Design and Analysis of Algorithms

Linear Programming

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Topics

- **An Example**
- **Standard Form**
- **Geometry**
- **Linear Algebra**
- **Simplex Algorithm**



Linear Programming

Linear programming. Optimize a linear function subject to linear inequalities.

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s. t.} \quad & \sum_{j=1}^n a_{ij} x_j \geq b_i \quad 1 \leq i \leq m \\ & x_j \geq 0 \quad 1 \leq j \leq n \end{aligned}$$

Ranked among most important scientific advances of 20th century.



Linear Programming

Linear programming. Optimize a linear function subject to linear inequalities.

Generalizes: $AX=B$, 2-person zero-sum games, shortest path, max flow, assignment problem, ...



Brewery Problem

Small brewery produces ale and beer.

- Production limited by scarce resources: corn, hops, barley malt.
- Recipes for ale and beer require different proportions of resources.

Beverage	Corn (pounds)	Hops (ounces)	Malt (pounds)	Profit (\$)
Ale (barrel)	5	4	35	13
Beer (barrel)	15	4	20	23
constraint	480	160	1190	

How can brewer maximize profit?

- Devote all resources to ale: 34 barrels of ale -> \$442
- Devote all resources to beer: 32 barrels of beer -> \$736
- 7.5 barrels of ale, 29.5 barrels of beer -> \$ 776
- 12 barrels of ale, 28 barrels of beer -> \$800



Brewery Problem

objective function

Ale

Beer

$$\begin{array}{ll} \max & 13A + 23B \\ \text{s. t.} & 5A + 15B \leq 480 \\ & 4A + 4B \leq 160 \\ & 35A + 20B \leq 1190 \\ & A, B \geq 0 \end{array}$$

Profit

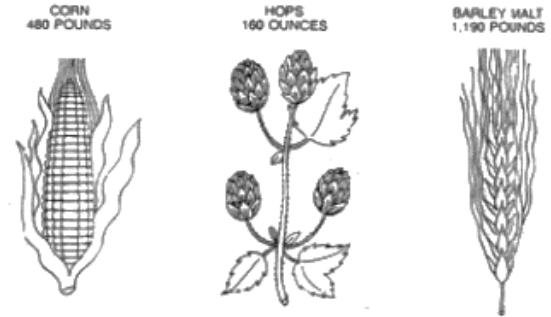
Corn

Hops

Malt

constraint

decision variable



\$13 PROFIT



\$23 PROFIT

MOST PROFITABLE PRODUCT MIX



Standard Form

“Standard form” of a linear program.

- Input: real numbers a_{ij}, c_j, b_i .
- Output: real numbers x_j .
- $n = \#$ decision variables, $m = \#$ constraints.
- Maximize linear objective function subject to linear equalities.

$$\begin{aligned} \max \quad & \sum_{j=1}^n c_j x_j \\ \text{s. t.} \quad & \sum_{j=1}^n a_{ij} x_j = b_i \quad 1 \leq i \leq m \\ & x_j \geq 0 \quad 1 \leq j \leq n \end{aligned}$$



Brewery Problem: Converting to Standard Form

Original input.

$$\begin{array}{ll}\max & 13A + 23B \\ \text{s. t.} & 5A + 15B \leq 480 \\ & 4A + 4B \leq 160 \\ & 35A + 20B \leq 1190 \\ & A, B \geq 0\end{array}$$

Standard form?

- Add **slack** variable for each inequality.
- Now a 5-dimensional problem.

$$\begin{array}{llllll}\max & 13A + 23B & & & & \\ \text{s. t.} & 5A + 15B + S_C & & & & = 480 \\ & 4A + 4B & & + S_H & & = 160 \\ & 35A + 20B & & & + S_M & = 1190 \\ & A, B, S_C, S_H, S_M & \geq & 0 & & \end{array}$$



Basic and Non-basic Variables

Basic variables are selected arbitrarily with the restriction that there will be as many basic variables as the equations. The remaining variables are **non-basic variables**.

$$x_1 + 2x_2 + s_1 = 32$$

$$3x_1 + 4x_2 + s_2 = 84$$

This system has two equations, we can select any two of the four variables as basic variables. The remaining two variables are then non-basic variables. A solution found by setting the two non-basic variables equal to 0 and solving for the two basic variables is a basic solution. If a basic solution has no negative values, it is a basic feasible solution.



Equivalent Forms

Easy to convert variants to standard form.

$$\begin{array}{ll}\max & c^T x \\ \text{s. t.} & Ax = b \\ & x \geq 0\end{array}$$

- Less than to equality.

$$x + 2y - 3z \leq 17$$

- Greater than to equality.

$$x + 2y - 3z \geq 17$$

- Min to max.

$$\min x + 2y - 3z$$

- Unrestricted to nonnegative.

$$x \text{ unrestricted}$$



Equivalent Forms

Easy to convert variants to standard form.

$$\begin{aligned} \max & c^T x \\ \text{s.t.} & Ax = b \\ & x \geq 0 \end{aligned}$$

- Less than to equality.

$$x + 2y - 3z \leq 17 \rightarrow x + 2y - 3z + s = 17, s \geq 0$$

- Greater than to equality.

$$x + 2y - 3z \geq 17 \rightarrow x + 2y - 3z - s = 17, s \geq 0$$

- Min to max.

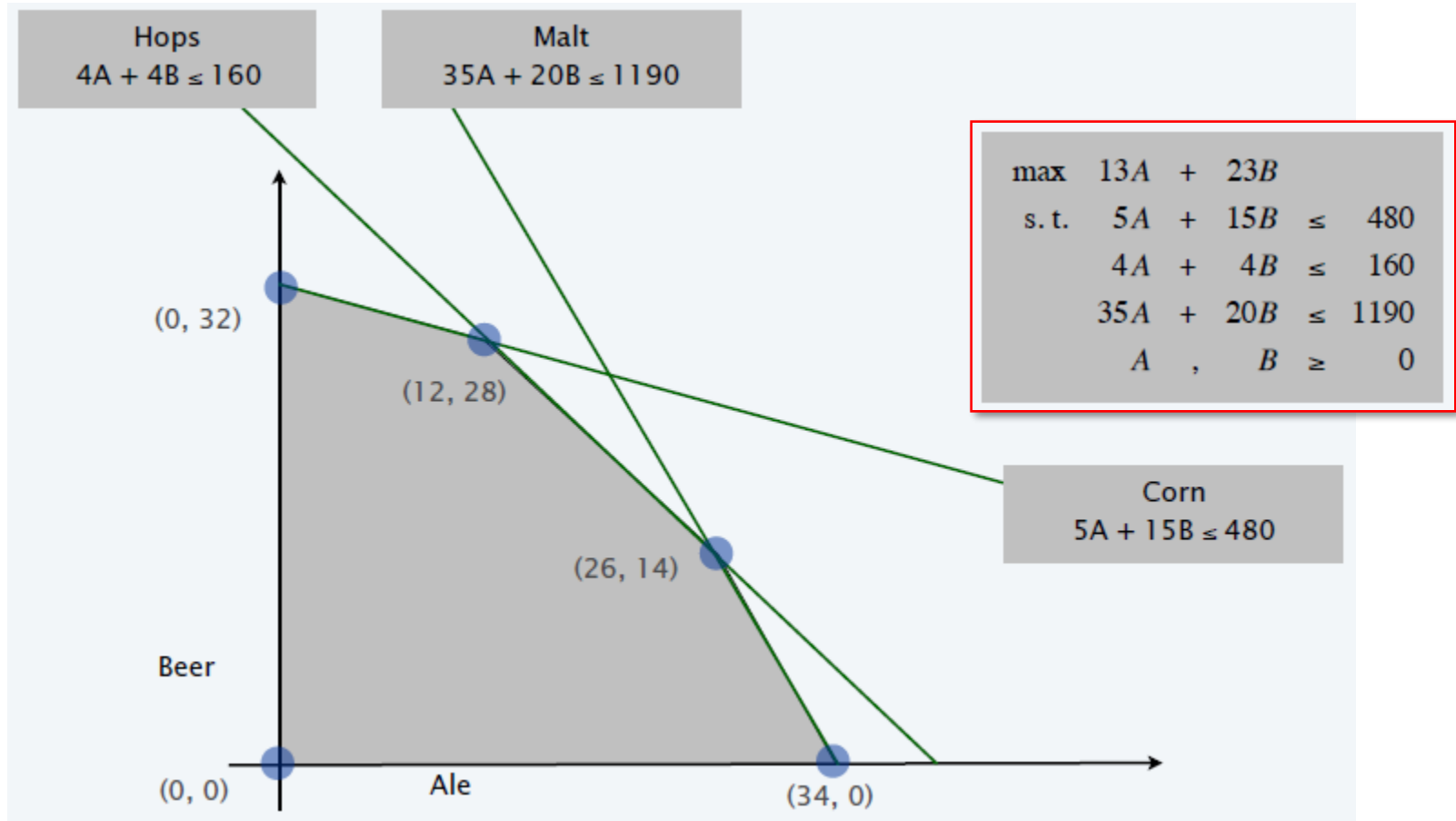
$$\min x + 2y - 3z \rightarrow \max -x - 2y + 3z$$

- Unrestricted to nonnegative.

$$x \text{ unrestricted} \rightarrow x = x^+ - x^-, x^+ \geq 0, x^- \geq 0$$



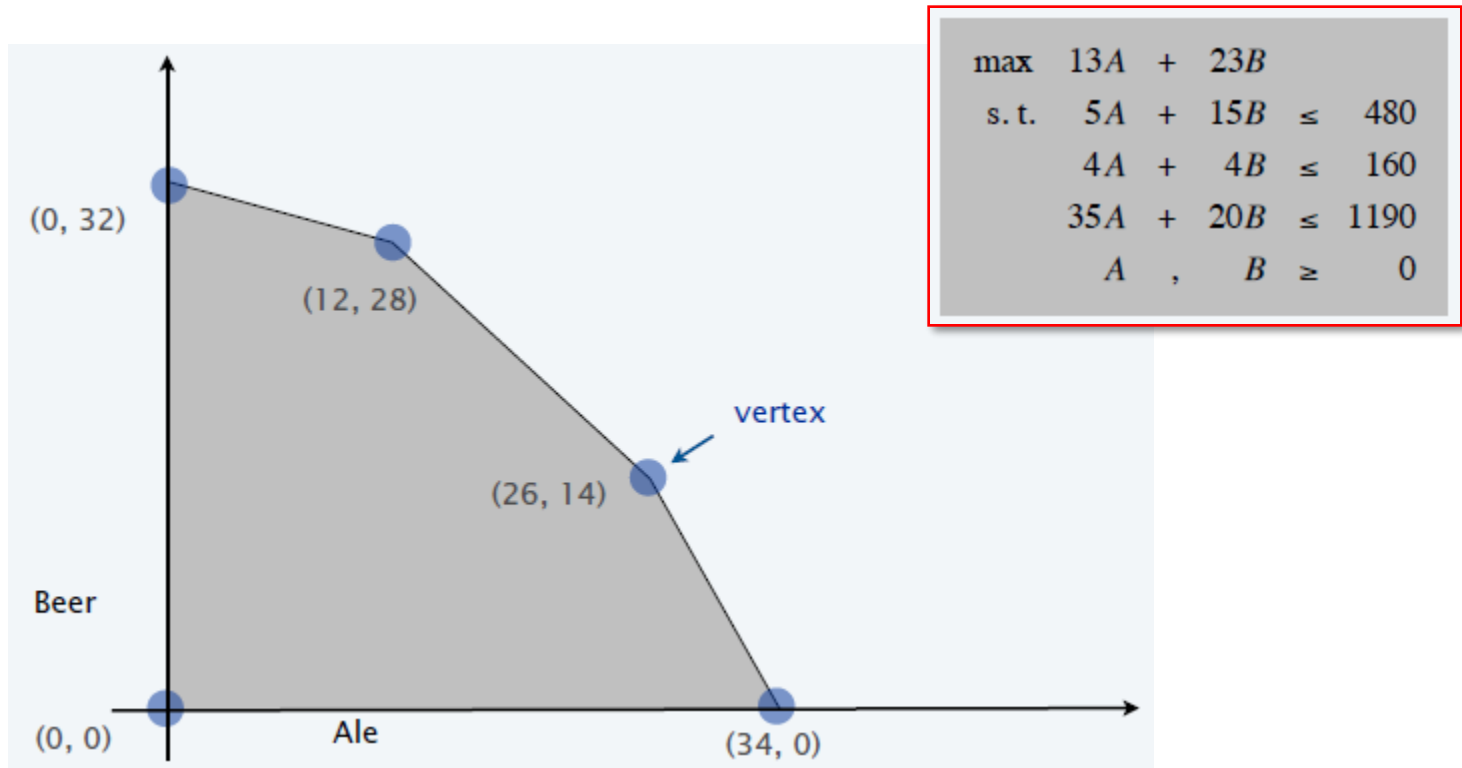
Brewery Problem: Feasible Region





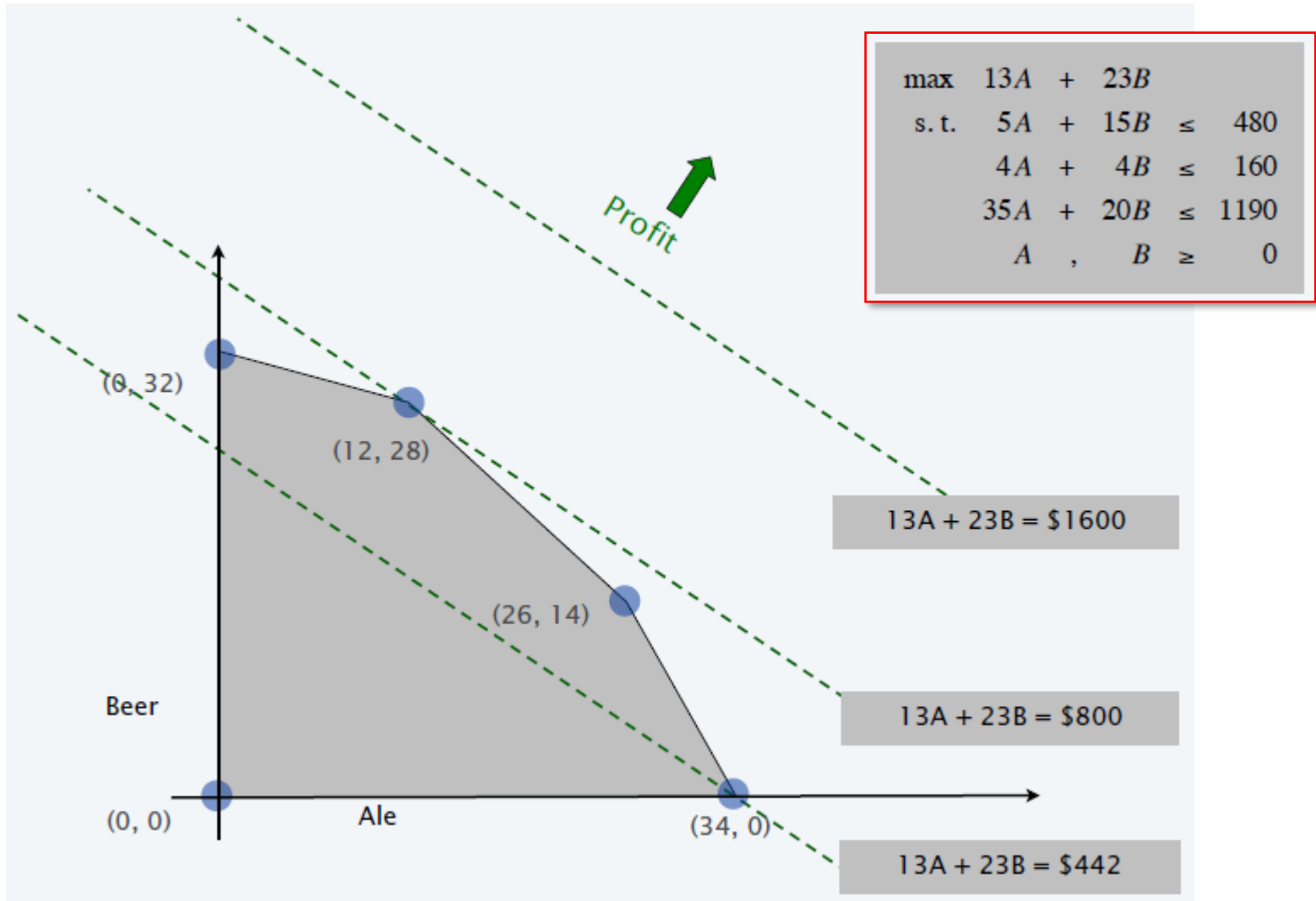
Brewery Problem: Geometry

Brewery problem observation. Regardless of objective function coefficients, an optimal solution occurs at a **vertex**.





Brewery Problem: Objective Function

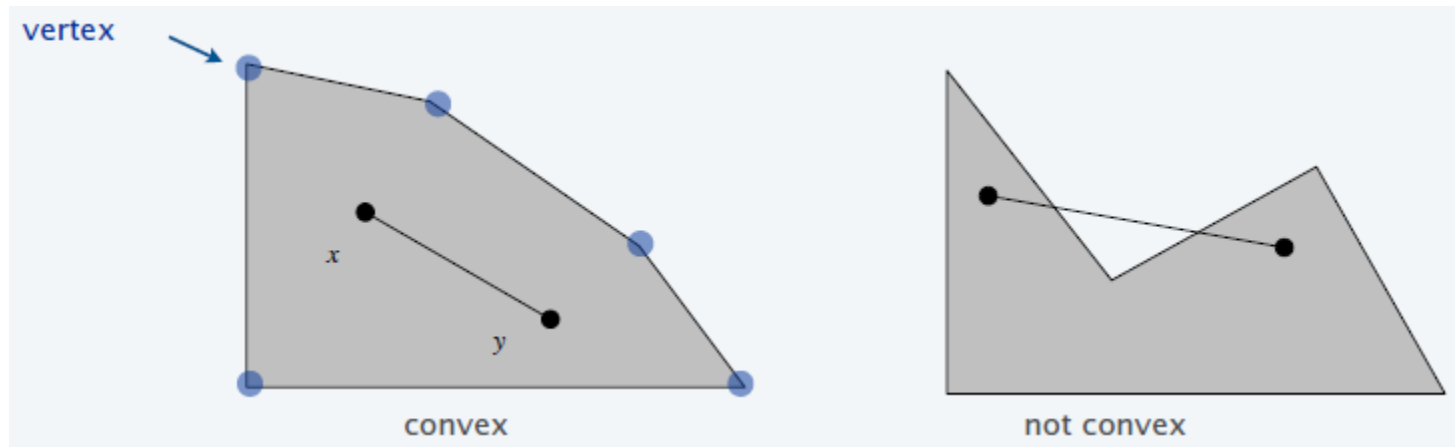




Convexity

Convex set. If two points x and y are in the set, then so is $\lambda x + (1 - \lambda)y$ for $0 \leq \lambda \leq 1$.

Vertex. A point x in the set that can't be written as a strict convex combination of two distinct points in the set.

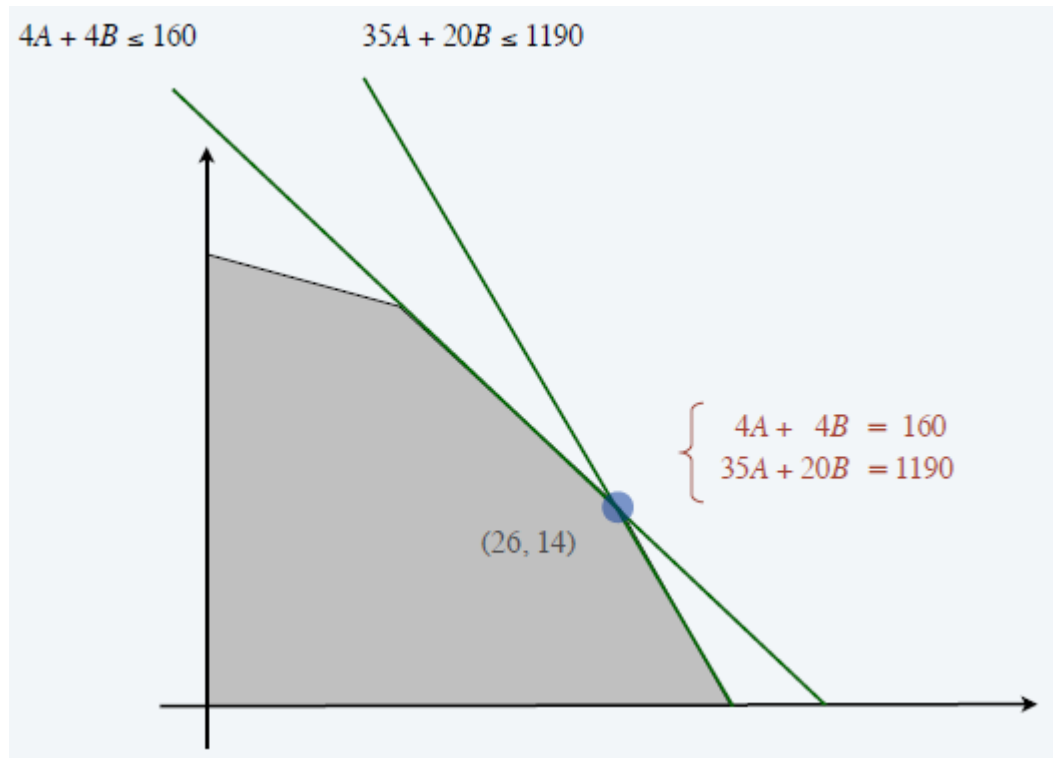


Observation. LP feasible region is a convex set.



Vertex

Intuition. A vertex in R^m is uniquely specified by m linearly independent equations.



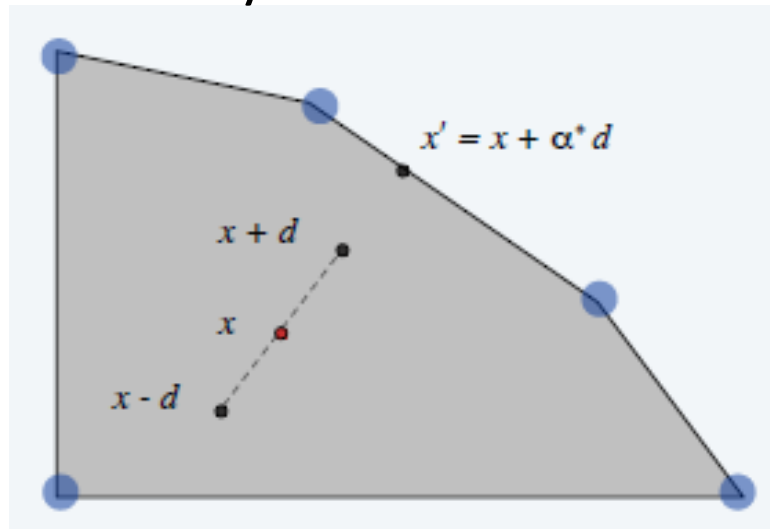


Vertex

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Intuition. If the optimum is not a vertex, move in a non-decreasing direction until you reach a boundary.





Vertex

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

Pf.

Since there exists an optimal solution, there exists an optimal solution x with *a minimal number of non-zero components*.

Suppose x is not a vertex, so that

$$x = \lambda u + (1 - \lambda)v,$$

for some $u \neq v, \lambda \in (0,1)$.



Vertex

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

Since x is optimal, $c^T u \leq c^T x$ and $c^T v \leq c^T x$.

But also $c^T x = \lambda c^T u + (1 - \lambda)c^T v$ so in fact $c^T u = c^T v = c^T x$.

Now consider the line defined by

$$x(\epsilon) = x + \epsilon(u - v)$$

Then

- $Ax = Au = Av = b$ so $Ax(\epsilon) = b$ for all ϵ ,
- $c^T x(\epsilon) = c^T x$ for all ϵ ,
- If $x_i = 0$ then $u_i = v_i = 0$, which implies $x(\epsilon)_i = 0$ for all ϵ ,
- If $x_i > 0$ then $x(0)_i > 0$, and $x(\epsilon)_i$ is continuous in ϵ .



Vertex

Theorem. If there exists an optimal solution to (P), then there exists one that is a vertex.

So we can increase ϵ from zero, in a positive or a negative direction as appropriate, until at least one extra component of $x(\epsilon)$ becomes zero.

This gives an optimal solution $x(\epsilon)$ with fewer non-zero components than x .

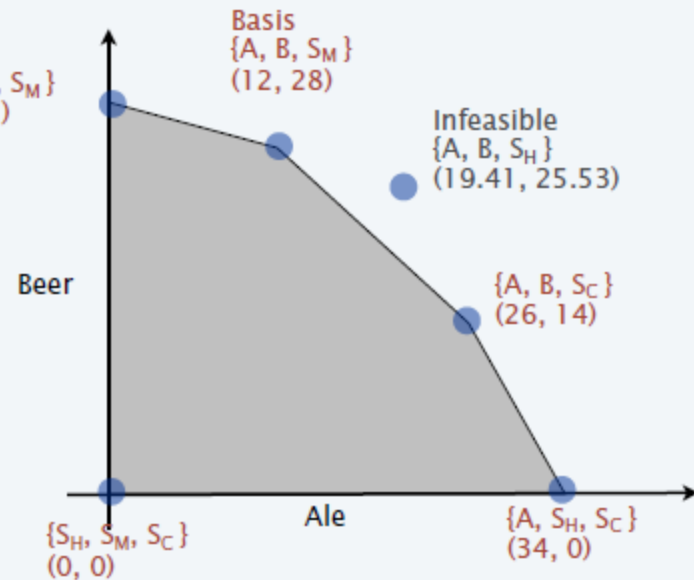
So x must be a vertex.



Basic Feasible Solution: Example

Basic feasible solutions.

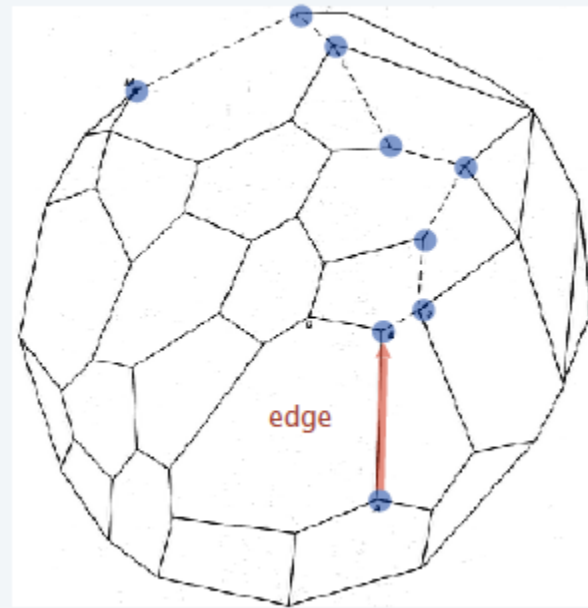
$$\begin{array}{llll}
 \max & 13A & + & 23B \\
 \text{s. t.} & 5A & + & 15B + S_C = 480 \\
 & 4A & + & 4B + S_H = 160 \\
 & 35A & + & 20B + S_M = 1190 \\
 & A, B, S_C, S_H, S_M & \geq & 0
 \end{array}$$





Simplex Algorithm: Intuition

Simplex algorithm. Move from BFS (Basic Feasible Solution) to adjacent BFS, without decreasing objective function (replace one basic variable with another).



Greedy property. BFS optimal iff no adjacent BFS is better.



Simplex Algorithm: Initialization

max Z subject to									
13A	+	23B					-	Z	= 0
5A	+	15B	+	S_C					= 480
4A	+	4B			+	S_H			= 160
35A	+	20B					+	S_M	= 1190
A	,	B	,	S_C	,	S_H	,	S_M	≥ 0

Basis = $\{S_C, S_H, S_M\}$

$A = B = 0$

$Z = 0$

$S_C = 480$

$S_H = 160$

$S_M = 1190$



Simplex Algorithm: Pivot 1

$$\begin{array}{rcll}
 \text{max } Z \text{ subject to} & & & \\
 13A + 23B & & - Z & = 0 \\
 5A + 15B + S_C & & & = 480 \\
 4A + 4B & + S_H & & = 160 \\
 35A + 20B & & + S_M & = 1190 \\
 A, B, S_C, S_H, S_M & & & \geq 0
 \end{array}$$

$$\text{Basis} = \{S_C, S_H, S_M\}$$

$$A = B = 0$$

$$Z = 0$$

$$S_C = 480$$

$$S_H = 160$$

$$S_M = 1190$$

Substitute: $B = 1/15 (480 - 5A - S_C)$

$$\begin{array}{rcll}
 \text{max } Z \text{ subject to} & & & \\
 \frac{16}{3}A & - \frac{23}{15}S_C & - Z & = -736 \\
 \frac{1}{3}A + B + \frac{1}{15}S_C & & & = 32 \\
 \frac{8}{3}A & - \frac{4}{15}S_C + S_H & & = 32 \\
 \frac{85}{3}A & - \frac{4}{3}S_C & + S_M & = 550 \\
 A, B, S_C, S_H, S_M & & & \geq 0
 \end{array}$$

$$\text{Basis} = \{B, S_H, S_M\}$$

$$A = S_C = 0$$

$$Z = 736$$

$$B = 32$$

$$S_H = 32$$

$$S_M = 550$$



Simplex Algorithm: Pivot 1

max Z subject to									
13A	+	23B				-	Z	=	0
5A	+	15B	+	S _C				=	480
4A	+	4B			+	S _H		=	160
35A	+	20B				+	S _M	=	1190
A	,	B	,	S _C	,	S _H	,	S _M	≥ 0

Basis = {S_C, S_H, S_M}

A = B = 0

Z = 0

S_C = 480

S_H = 160

S_M = 1190

Q. Why pivot on column 2 (or 1)?

A. Each unit increase in B increases objective value by \$23.

Q. Why pivot on row 2.

A. Preserves feasibility by ensuring *RHS (Right Hand Side)* ≥ 0 .
 (min ratio rule: $\min\{480/15, 160/4, 1190/20\}$)



Simplex Algorithm: Pivot 2

$$\begin{array}{rcll}
 \text{max } Z \text{ subject to} & & & \\
 \frac{16}{3} A & - & \frac{23}{15} S_C & - Z = -736 \\
 \frac{1}{3} A + B + \frac{1}{15} S_C & & & = 32 \\
 \frac{8}{3} A & - & \frac{4}{15} S_C + S_H & = 32 \\
 \frac{85}{3} A & - & \frac{4}{3} S_C + S_M & = 550 \\
 A, B, S_C, S_H, S_M & \geq & & 0
 \end{array}$$

$$\begin{aligned}
 \text{Basis} &= \{B, S_H, S_M\} \\
 A = S_C &= 0 \\
 Z &= 736 \\
 B &= 32 \\
 S_H &= 32 \\
 S_M &= 550
 \end{aligned}$$

Substitute: $A = 3/8 (32 + 4/15 S_C - S_H)$

$$\begin{array}{rcll}
 \text{max } Z \text{ subject to} & & & \\
 & - & S_C - 2 S_H & - Z = -800 \\
 B + \frac{1}{10} S_C + \frac{1}{8} S_H & & & = 28 \\
 A - \frac{1}{10} S_C + \frac{3}{8} S_H & & & = 12 \\
 & - & \frac{25}{6} S_C - \frac{85}{8} S_H + S_M & = 110 \\
 A, B, S_C, S_H, S_M & \geq & & 0
 \end{array}$$

$$\begin{aligned}
 \text{Basis} &= \{A, B, S_M\} \\
 S_C = S_H &= 0 \\
 Z &= 800 \\
 B &= 28 \\
 A &= 12 \\
 S_M &= 110
 \end{aligned}$$



Simplex Algorithm: Optimality

Q. When to stop pivoting?

A. When all coefficients in top row are non-positive.

Q. Why is the resulting solution optimal?

A. Any feasible solution satisfies systems of equations in tableau.

- In particular: $Z = 800 - S_C - 2S_H$, $S_C \geq 0$, $S_H \geq 0$.
- Thus, optimal objective value $Z^* \leq 800$.
- Current BFS has value 800 \rightarrow optimal.

max Z subject to						
		$-$	S_C	$-$	$2 S_H$	$- Z = -800$
	B	$+$	$\frac{1}{10} S_C$	$+$	$\frac{1}{8} S_H$	$= 28$
A		$-$	$\frac{1}{10} S_C$	$+$	$\frac{3}{8} S_H$	$= 12$
		$-$	$\frac{25}{6} S_C$	$-$	$\frac{85}{8} S_H$	$+ S_M = 110$
A	$,$	B	$,$	S_C	$,$	S_H
					$,$	S_M
						≥ 0

Basis = $\{A, B, S_M\}$

$S_C = S_H = 0$

$Z = 800$

$B = 28$

$A = 12$

$S_M = 110$



Variant Tableau

The constraints are a linear system including m equations and n variables. m of the variables can be evaluated in terms of the other $n - m$ variables

$$x_1 = b_1 - a_{1,m+1}x_{m+1} - \cdots - a_{1,n}x_n$$

$$x_2 = b_2 - a_{2,m+1}x_{m+1} - \cdots - a_{2,n}x_n$$

.....

$$x_m = b_m - a_{m,m+1}x_{m+1} - \cdots - a_{m,n}x_n$$

Objective function $z = \sum_{j=1}^n c_j x_j$
 $= \sum_{i=1}^m c_i b_i + \sum_{j=m+1}^n (c_j - \sum_{i=1}^m c_i a_{ij}) x_j.$

Let $z^0 = \sum_{i=1}^m c_i b_i$, $\sigma_j = c_j - \sum_{i=1}^m c_i a_{ij}$, and we have

$$z = z^0 + \sum_{j=m+1}^n \boxed{\sigma_j x_j}$$

indicator



Variant Tableau

C_j		C_1	C_2	\dots	C_m	C_{m+1}	\dots	C_n	\mathbf{b}	θ
$\mathbf{C_B}$	$\mathbf{X_B}$	x_1	x_2	\dots	x_m	x_{m+1}	\dots	x_n		
c_1	x_1	1	0	\dots	0	$a'_{1,m+1}$	\dots	a'_{1n}	b'_1	
c_2	x_2	0	1	\dots	0	$a'_{2,m+1}$	\dots	a'_{2n}	b'_2	
\dots	\dots								\dots	
c_m	x_m	0	0	\dots	1	$a'_{m,m+1}$	\dots	a'_{mn}	b'_m	
σ_j		0	0	\dots	0	$c_{m+1} - \sum_{i=1}^m c_i a'_{i,m+1}$				



Variant Tableau

To solve a linear programming problem, use the following steps:

1. Convert each inequality in the set of constraints to an equation by adding slack variables.
2. Create the initial simplex tableau.
3. Select the pivot column (*The column with the “most positive value” element in the last row*).
4. Select the pivot row (*The row with the smallest non-negative result when the last element in the row is divided by the corresponding in the pivot column*).
5. Use elementary row operations calculate new values for the pivot row so that the pivot is 1.
6. Use elementary row operations to make all numbers in the pivot column equal to 0 except for the pivot.
7. If all entries in the bottom row are non-positive, this the final tableau. If not, go back to Step 3.



Variant Tableau: An Example

$$\max \quad z = 2x_1 + 3x_2$$

$$s.t. \begin{cases} 2x_1 + x_2 \leq 4 \\ x_1 + 2x_2 \leq 5 \\ x_1, x_2 \geq 0 \end{cases}$$



$$\max \quad z = 2x_1 + 3x_2$$

$$s.t. \begin{cases} 2x_1 + x_2 + x_3 = 4 \\ x_1 + 2x_2 + x_4 = 5 \\ x_1, x_2, x_3, x_4 \geq 0 \end{cases}$$



Variant Tableau: An Example

Pivot column. The column of the tableau representing the variable to be entered into the solution mix.

Pivot row. The row of the tableau representing the variable to be replaced in the solution mix.

Basic variable. Variables in the solution mix.

Initial tableau

Pivot column

C_j		2	3	0	0	b	θ
C_B	X_B	x_1	x_2	x_3	x_4		
0	x_3	2	1	1	0	4	4/1
0	x_4	1	2	0	1	5	5/2
σ_j		2	3	0	0		

Min ratio rule

Pivot row



Variant Tableau: An Example

c_j		2	3	0	0	b	θ
C_B	X_B	x_1	x_2	x_3	x_4		
0	x_3	2	1	1	0	4	4/1
0	x_4	1	2	0	1	5	5/2
σ_j		2	3	0	0		

- Since the entry 3 is the most positive entry in the last row of the tableau, the second column in the tableau is the pivot column.
- Divide each positive number of the pivot column into the corresponding entry in the column of constants. The ratio 5/2 is less than the ratio 4/1, so row 2 is the pivot row.



Variant Tableau: An Example

c_j		2	3	0	0	b	θ
C_B	X_B	x_1	x_2	x_3	x_4		
0	x_3	3/2	0	1	-1/2	3/2	1
3	x_2	1/2	1	0	1/2	5/2	5
σ_j		1/2	0	0	-3/2		

- Since the entry 1/2 is the most positive entry in the last row of the tableau, the first column in the tableau is the pivot column.
- Divide each positive number of the pivot column into the corresponding entry in the column of constants. The ratio 3/2 is less than the ratio 5/2, so row 1 is the pivot row.



Variant Tableau: An Example

c_j		2	3	0	0	b	θ
C_B	X_B	x_1	x_2	x_3	x_4		
2	x_1	1	0	$2/3$	$-1/3$	1	
3	x_2	0	1	$-1/3$	$2/3$	2	
σ_j		0	0	$-1/3$	$-4/3$		

- The last row of the tableau contains no positive numbers, so an optimal solution has been reached.



Matrix Form

Initial simplex tableau

$$\begin{aligned} c_B^T x_B + c_N^T x_N &= Z \\ A_B x_B + A_N x_N &= b \\ x_B, x_N &\geq 0 \end{aligned}$$

Simplex tableau corresponding to basis B .

$$\begin{aligned} (c_N^T - c_B^T A_B^{-1} A_N) x_N &= Z - c_B^T A_B^{-1} b \quad \leftarrow \text{subtract } c_B^T A_B^{-1} \text{ times constraints} \\ I x_B + A_B^{-1} A_N x_N &= A_B^{-1} b \quad \leftarrow \text{multiply by } A_B^{-1} \\ x_B, x_N &\geq 0 \end{aligned}$$

$$\begin{aligned} x_B &= A_B^{-1} b \geq 0 \\ x_N &= 0 \end{aligned}$$

basic feasible solution

$$c_N^T - c_B^T A_B^{-1} A_N \leq 0$$

optimal basis



Matrix Form

Standard form:

$$\begin{aligned} \max Z &= C^T X \\ \text{s.t. } AX &= b \\ X &\geq 0 \end{aligned}$$

Let $A = [A_B, A_N]$, $X = \begin{bmatrix} X_B \\ X_N \end{bmatrix}$, $C = \begin{bmatrix} C_B \\ C_N \end{bmatrix}$, we have

$$\begin{aligned} A_B X_B + A_N X_N &= b \\ \rightarrow X_B &= A_B^{-1} b - A_B^{-1} A_N X_N \end{aligned}$$

For the basis B ,

$$\begin{aligned} Z &= C^T X = [C_B^T, C_N^T] \begin{bmatrix} X_B \\ X_N \end{bmatrix} = C_B^T X_B + C_N^T X_N \\ &= C_B^T (A_B^{-1} b - A_B^{-1} A_N X_N) + C_N^T X_N \\ &= C_B^T A_B^{-1} b + (C_N^T - C_B^T A_B^{-1} A_N) X_N \end{aligned}$$



Matrix Form: Variant Tableau

	C_B^T	C_N^T	
	X_B^T	X_N^T	
$C_B \ X_B$	I	$A_B^{-1} A_N$	$A_B^{-1} b$
Indicator	0	$C_N^T - C_B^T A_B^{-1} A_N$	