

ETC3550/ETC5550

Applied forecasting

Ch10. Dynamic regression models

OTexts.org/fpp3/



Outline

- 1 Regression with ARIMA errors
- 2 Stochastic and deterministic trends
- 3 Dynamic harmonic regression
- 4 Lagged predictors

Outline

- 1 Regression with ARIMA errors
- 2 Stochastic and deterministic trends
- 3 Dynamic harmonic regression
- 4 Lagged predictors

Regression with ARIMA errors

Regression models

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t,$$

- y_t modeled as function of k explanatory variables $x_{1,t}, \dots, x_{k,t}$.
- In regression, we assume that ε_t was WN.
- Now we want to allow ε_t to be autocorrelated.

Regression with ARIMA errors

Regression models

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t,$$

- y_t modeled as function of k explanatory variables $x_{1,t}, \dots, x_{k,t}$.
- In regression, we assume that ε_t was WN.
- Now we want to allow ε_t to be autocorrelated.

Example: ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

where ε_t is white noise.

Residuals and errors

Example: $\eta_t = \text{ARIMA}(1,1,1)$

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

Residuals and errors

Example: $\eta_t = \text{ARIMA}(1,1,1)$

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

- Be careful in distinguishing η_t from ε_t .
- Only the errors ε_t are assumed to be white noise.
- In ordinary regression, η_t is assumed to be white noise and so $\eta_t = \varepsilon_t$.

Estimation

If we minimize $\sum \eta_t^2$ (by using ordinary regression):

- 1 Estimated coefficients $\hat{\beta}_0, \dots, \hat{\beta}_k$ are no longer optimal as some information ignored;
- 2 Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
- 3 p -values for coefficients usually too small ("spurious regression").
- 4 AIC of fitted models misleading.

Estimation

If we minimize $\sum \eta_t^2$ (by using ordinary regression):

- 1 Estimated coefficients $\hat{\beta}_0, \dots, \hat{\beta}_k$ are no longer optimal as some information ignored;
 - 2 Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
 - 3 p -values for coefficients usually too small (“spurious regression”).
 - 4 AIC of fitted models misleading.
- Minimizing $\sum \varepsilon_t^2$ avoids these problems.
 - Maximizing likelihood similar to minimizing $\sum \varepsilon_t^2$.

Regression with ARIMA errors

Model with ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

Regression with ARIMA errors

Model with ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

Equivalent to model with ARIMA(1,0,1) errors

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t,$$
$$(1 - \phi_1 B)\eta'_t = (1 + \theta_1 B)\varepsilon_t,$$

where $y'_t = y_t - y_{t-1}$, $x'_{t,i} = x_{t,i} - x_{t-1,i}$ and $\eta'_t = \eta_t - \eta_{t-1}$.

Regression with ARIMA errors

Any regression with an ARIMA error can be rewritten as a regression with an ARMA error by differencing all variables with the same differencing operator as in the ARIMA model.

Regression with ARIMA errors

Any regression with an ARIMA error can be rewritten as a regression with an ARMA error by differencing all variables with the same differencing operator as in the ARIMA model.

Original data

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t$$

where $\phi(B)(1 - B)^d \eta_t = \theta(B)\varepsilon_t$

Regression with ARIMA errors

Any regression with an ARIMA error can be rewritten as a regression with an ARMA error by differencing all variables with the same differencing operator as in the ARIMA model.

Original data

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t$$

where $\phi(B)(1-B)^d \eta_t = \theta(B)\varepsilon_t$

After differencing all variables

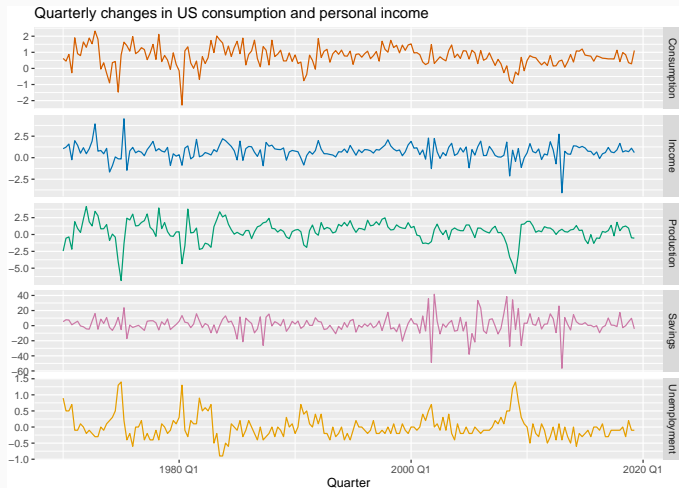
$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t$$

where $\phi(B)\eta'_t = \theta(B)\varepsilon_t$,
 $y'_t = (1-B)^d y_t$, $x'_{i,t} = (1-B)^d x_{i,t}$, and $\eta'_t = (1-B)^d \eta_t$

Regression with ARIMA errors

- In R, we can specify an $\text{ARIMA}(p, d, q)$ for the errors, and d levels of differencing will be applied to all variables $(y, x_{1,t}, \dots, x_{k,t})$.
- Check that ε_t series looks like white noise.
- AICc can be calculated for final model.
- Repeat procedure for all subsets of predictors to be considered, and select model with lowest AICc value.

US personal consumption and income



US personal consumption and income

```
fit <- us_change %>% model(ARIMA(Consumption ~ Income))  
report(fit)
```

```
## Series: Consumption
```

```
## Model: LM w/ ARIMA(1,0,2) errors
```

```
##
```

```
## Coefficients:
```

```
##          ar1          ma1          ma2  Income  intercept
```

```
##          0.707    -0.617    0.2066    0.1976          0.595
```

```
## s.e.    0.107     0.122    0.0741    0.0462          0.085
```

```
##
```

```
## sigma^2 estimated as 0.3113:  log likelihood=-163
```

```
## AIC=338    AICc=339    BIC=358
```

US personal consumption and income

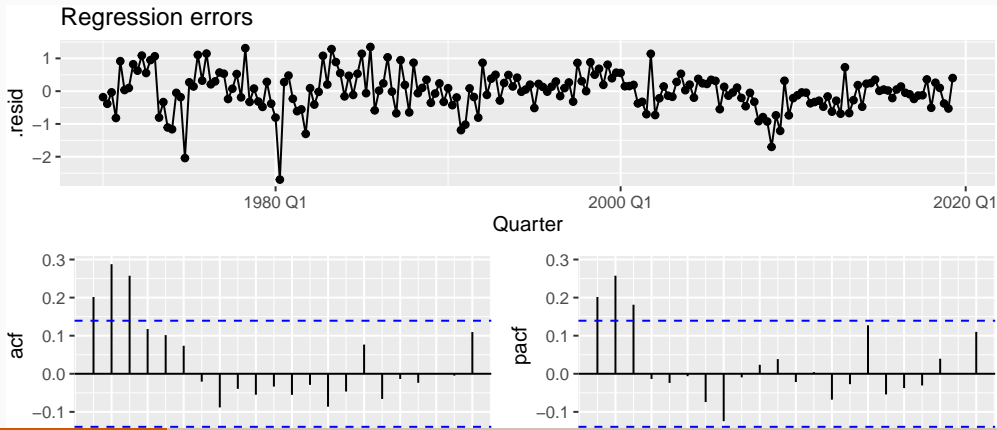
```
fit <- us_change %>% model(ARIMA(Consumption ~ Income))  
report(fit)
```

```
## Series: Consumption  
## Model: LM w/ ARIMA(1,0,2) errors  
##  
## Coefficients:  
##          ar1      ma1      ma2  Income  intercept  
##          0.707   -0.617   0.2066  0.1976         0.595  
## s.e.    0.107    0.122   0.0741  0.0462         0.085  
##  
## sigma^2 estimated as 0.3113:  log likelihood=-163  
## AIC=338   AICc=339   BIC=358
```

Write down the equations for the fitted model.

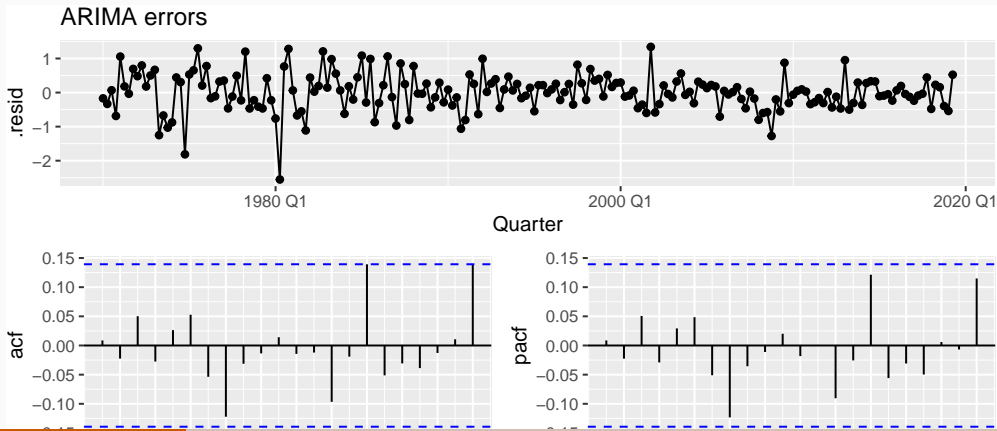
US personal consumption and income

```
residuals(fit, type='regression') %>%  
  gg_tsdisplay(.resid, plot_type = 'partial') +  
  labs(title = "Regression errors")
```



US personal consumption and income

```
residuals(fit, type='innovation') %>%  
  gg_tsdisplay(.resid, plot_type = 'partial') +  
  labs(title = "ARIMA errors")
```



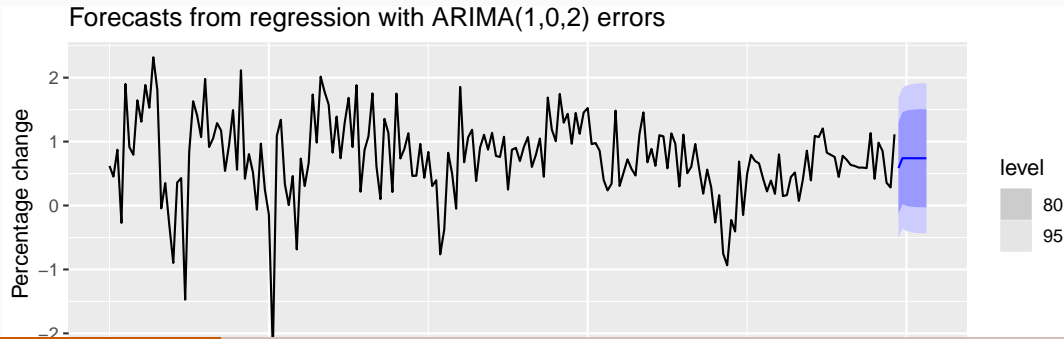
US personal consumption and income

```
augment(fit) %>%  
  features(.resid, ljung_box, dof = 5, lag = 12)
```

```
## # A tibble: 1 x 3  
##   .model                lb_stat lb_pvalue  
##   <chr>                <dbl>    <dbl>  
## 1 ARIMA(Consumption ~ Income)    5.54    0.595
```

US personal consumption and income

```
us_change_future <- new_data(us_change, 8) %>%  
  mutate(Income = mean(us_change$Income))  
forecast(fit, new_data = us_change_future) %>%  
  autoplot(us_change) +  
  labs(x = "Year", y = "Percentage change",  
        title = "Forecasts from regression with ARIMA(1,0,2) errors")
```



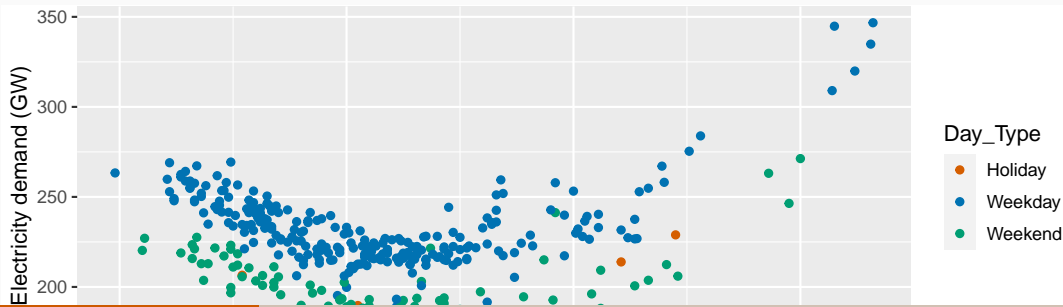
Forecasting

- To forecast a regression model with ARIMA errors, we need to forecast the regression part of the model and the ARIMA part of the model and combine the results.
- Some predictors are known into the future (e.g., time, dummies).
- Separate forecasting models may be needed for other predictors.
- Forecast intervals ignore the uncertainty in forecasting the predictors.

Daily electricity demand

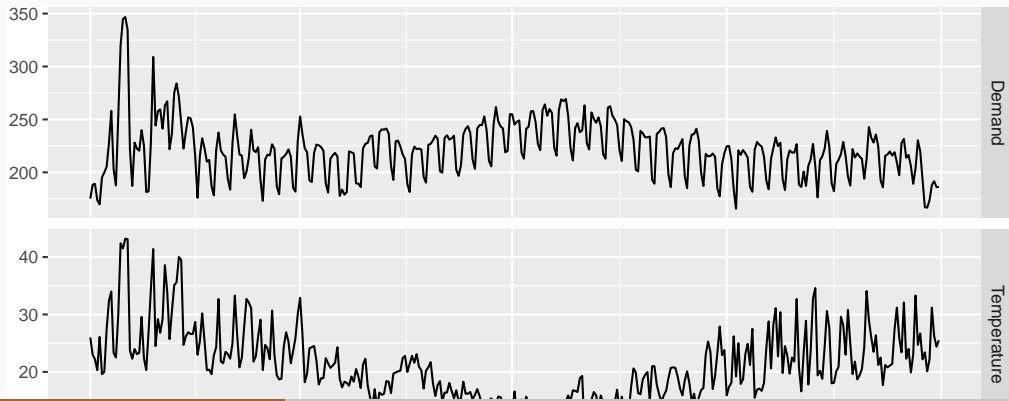
Model daily electricity demand as a function of temperature using quadratic regression with ARMA errors.

```
vic_elec_daily %>%  
  ggplot(aes(x = Temperature, y = Demand, colour = Day_Type)) +  
  geom_point() +  
  labs(x = "Maximum temperature", y = "Electricity demand (GW)")
```



Daily electricity demand

```
vic_elec_daily %>%  
  pivot_longer(c(Demand, Temperature)) %>%  
  ggplot(aes(x = Date, y = value)) + geom_line() +  
  facet_grid(name ~ ., scales = "free_y") + ylab("")
```



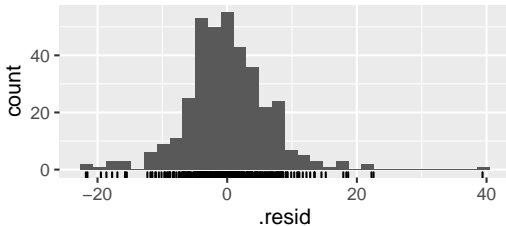
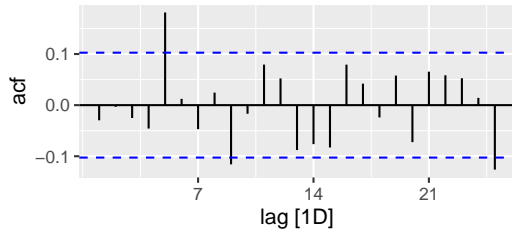
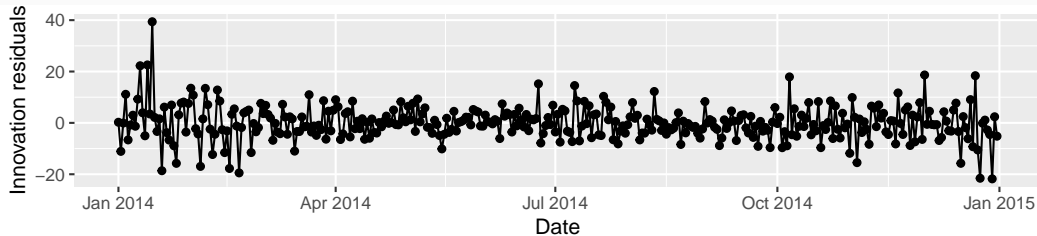
Daily electricity demand

```
fit <- vic_elec_daily %>%  
  model(ARIMA(Demand ~ Temperature + I(Temperature^2) +  
            (Day_Type=="Weekday")))  
report(fit)
```

```
## Series: Demand  
## Model: LM w/ ARIMA(2,1,2)(2,0,0)[7] errors  
##  
## Coefficients:  
##           ar1      ar2      ma1      ma2      sar1      sar2      Temperature  
##          -0.1093  0.7226 -0.0182 -0.9381  0.1958  0.417      -7.614  
## s.e.      0.0779  0.0739  0.0494  0.0493  0.0525  0.057      0.448  
##          I(Temperature^2) Day_Type == "Weekday"TRUE  
##                   0.1810                   30.40  
## s.e.                   0.0085                   1.33  
##  
## sigma^2 estimated as 44.91:  log likelihood=-1206  
## ATC=2432  ATCc=2433  BTC=2471
```

Daily electricity demand

```
gg_tsresiduals(fit)
```



Daily electricity demand

```
augment(fit) %>%  
  features(.resid, ljung_box, dof = 9, lag = 14)
```

```
## # A tibble: 1 x 3  
##   .model                                lb_stat lb_pvalue  
##   <chr>                                <dbl>     <dbl>  
## 1 "ARIMA(Demand ~ Temperature + I(Temperature^2) + (Day_Type ~ 28.4 0.0000304
```

Daily electricity demand

```
# Forecast one day ahead
```

```
vic_next_day <- new_data(vic_elec_daily, 1) %>%
```

```
  mutate(Temperature = 26, Day_Type = "Holiday")
```

```
forecast(fit, vic_next_day)
```

```
## # A tibble: 1 x 6 [1D]
```

```
## # Key:   .model [1]
```

```
##   .model          Date          Demand .mean Temperature Day_Type
```

```
##   <chr>          <date>          <dist> <dbl>          <dbl> <chr>
```

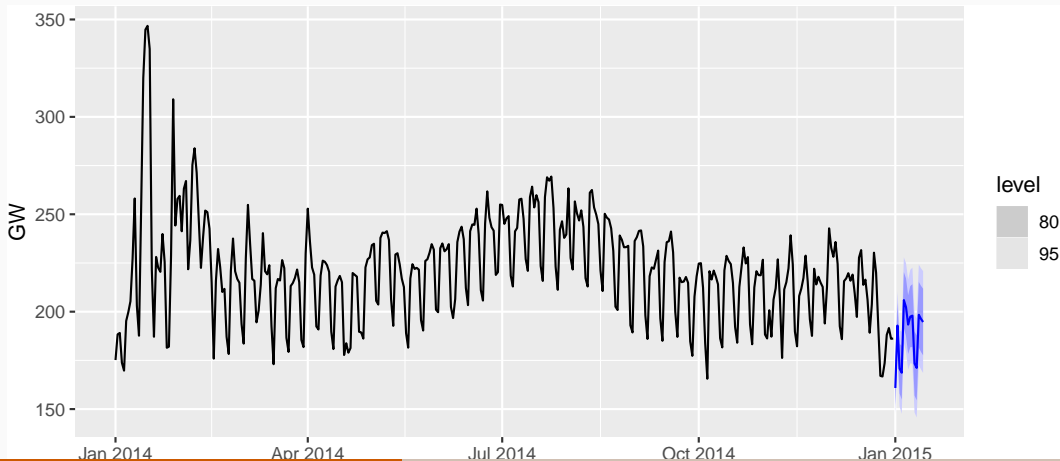
```
## 1 "ARIMA(Demand ~ Temperature ~ 2015-01-01 N(161, 45) 161.          26 Holiday
```

Daily electricity demand

```
vic_elec_future <- new_data(vic_elec_daily, 14) %>%  
  mutate(  
    Temperature = 26,  
    Holiday = c(TRUE, rep(FALSE, 13)),  
    Day_Type = case_when(  
      Holiday ~ "Holiday",  
      wday(Date) %in% 2:6 ~ "Weekday",  
      TRUE ~ "Weekend"  
    )  
  )
```

Daily electricity demand

```
forecast(fit, new_data = vic_elec_future) %>%  
  autoplot(vic_elec_daily) + labs(y="GW")
```



Outline

- 1 Regression with ARIMA errors
- 2 Stochastic and deterministic trends
- 3 Dynamic harmonic regression
- 4 Lagged predictors

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARMA process.

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARMA process.

Stochastic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARIMA process with $d \geq 1$.

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARMA process.

Stochastic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARIMA process with $d \geq 1$.

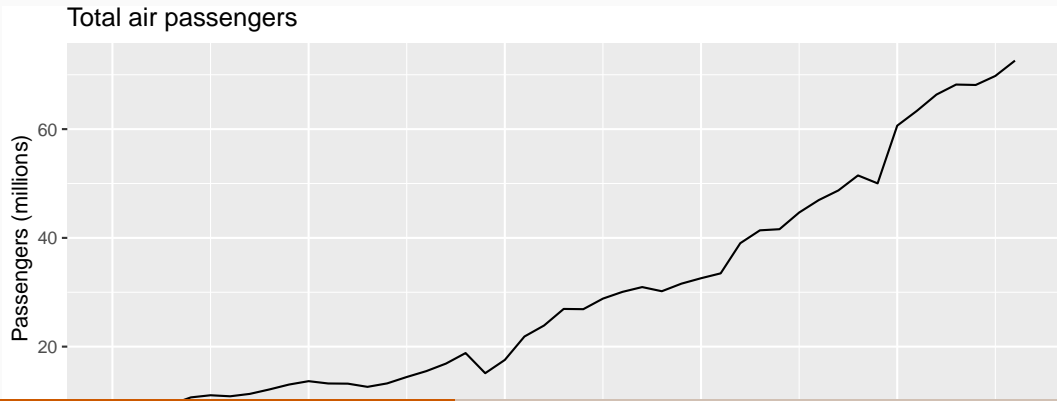
Difference both sides until η_t is stationary:

$$y'_t = \beta_1 + \eta'_t$$

where η'_t is ARMA process.

Air transport passengers Australia

```
aus_airpassengers %>%  
  autoplot(Passengers) +  
  labs(y = "Passengers (millions)",  
       title = "Total air passengers")
```



Air transport passengers Australia

Deterministic trend

```
fit_deterministic <- aus_airpassengers %>%  
  model(ARIMA(Passengers ~ 1 + trend() + pdq(d = 0)))  
report(fit_deterministic)
```

```
## Series: Passengers  
## Model: LM w/ ARIMA(1,0,0) errors  
##  
## Coefficients:  
##          ar1  trend()  intercept  
##      0.9564    1.415    0.901  
## s.e.  0.0362    0.197    7.075  
##  
## sigma^2 estimated as 4.343:  log likelihood=-101  
## AIC=210   AICc=211   BIC=217
```

Air transport passengers Australia

Deterministic trend

```
fit_deterministic <- aus_airpassengers %>%  
  model(ARIMA(Passengers ~ 1 + trend() + pdq(d = 0)))  
report(fit_deterministic)
```

```
## Series: Passengers  
## Model: LM w/ ARIMA(1,0,0) errors  
##  
## Coefficients:  
##          ar1  trend()  intercept  
##      0.9564    1.415    0.901  
## s.e.  0.0362    0.197    7.075  
##  
## sigma^2 estimated as 4.343:  log likelihood=-101  
## AIC=210   AICc=211   BIC=217
```

$$y_t = 0.901 + 1.415t + \eta_t$$

$$\eta_t = 0.956\eta_{t-1} + \varepsilon_t$$

Air transport passengers Australia

Stochastic trend

```
fit_stochastic <- aus_airpassengers %>%  
  model(ARIMA(Passengers ~ pdq(d = 1)))  
report(fit_stochastic)
```

```
## Series: Passengers  
## Model: ARIMA(0,1,0) w/ drift  
##  
## Coefficients:  
##      constant  
##      1.419  
## s.e.      0.301  
##  
## sigma^2 estimated as 4.271:  log likelihood=-98.2  
## AIC=200   AICc=201   BIC=204
```

Air transport passengers Australia

Stochastic trend

```
fit_stochastic <- aus_airpassengers %>%  
  model(ARIMA(Passengers ~ pdq(d = 1)))  
report(fit_stochastic)
```

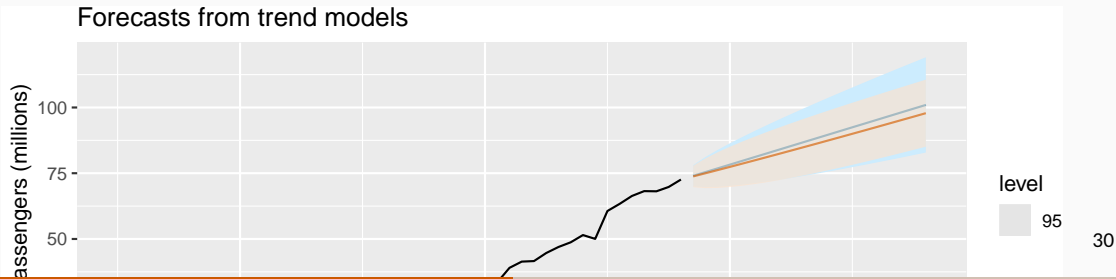
```
## Series: Passengers  
## Model: ARIMA(0,1,0) w/ drift  
##  
## Coefficients:  
##      constant  
##      1.419  
## s.e.      0.301  
##  
## sigma^2 estimated as 4.271:  log likelihood=-98.2  
## AIC=200   AICc=201   BIC=204
```

$$y_t - y_{t-1} = 1.419 + \varepsilon_t,$$

$$y_t = y_0 + 1.419t + \eta_t$$

Air transport passengers Australia

```
aus_airpassengers %>%  
  autoplot(Passengers) +  
  autolayer(fit_stochastic %>% forecast(h = 20),  
    colour = "#0072B2", level = 95) +  
  autolayer(fit_deterministic %>% forecast(h = 20),  
    colour = "#D55E00", alpha = 0.65, level = 95) +  
  labs(y = "Air passengers (millions)",  
    title = "Forecasts from trend models")
```



Forecasting with trend

- Point forecasts are almost identical, but prediction intervals differ.
- Stochastic trends have much wider prediction intervals because the errors are non-stationary.
- Be careful of forecasting with deterministic trends too far ahead.

Outline

- 1 Regression with ARIMA errors
- 2 Stochastic and deterministic trends
- 3 Dynamic harmonic regression
- 4 Lagged predictors

Dynamic harmonic regression

Combine Fourier terms with ARIMA errors

Advantages

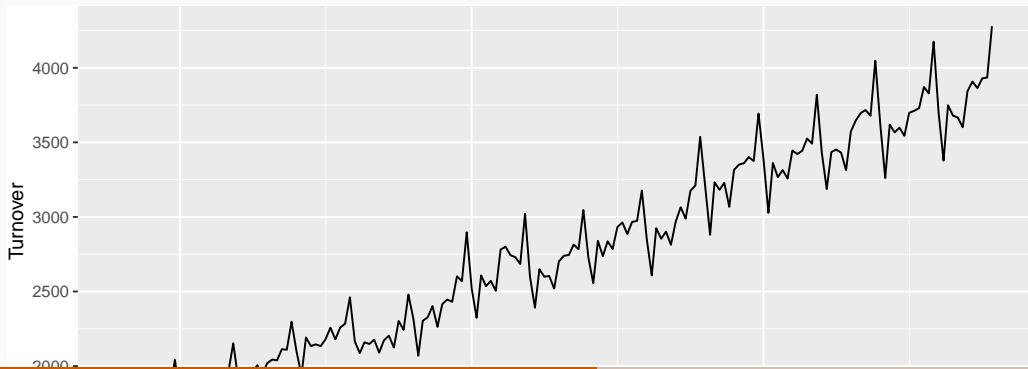
- it allows any length seasonality;
- for data with more than one seasonal period, you can include Fourier terms of different frequencies;
- the seasonal pattern is smooth for small values of K (but more wiggly seasonality can be handled by increasing K);
- the short-term dynamics are easily handled with a simple ARMA error.

Disadvantages

- seasonality is assumed to be fixed

Eating-out expenditure

```
aus_cafe <- aus_retail %>% filter(  
  Industry == "Cafes, restaurants and takeaway food services",  
  year(Month) %in% 2004:2018  
) %>% summarise(Turnover = sum(Turnover))  
aus_cafe %>% autoplot(Turnover)
```

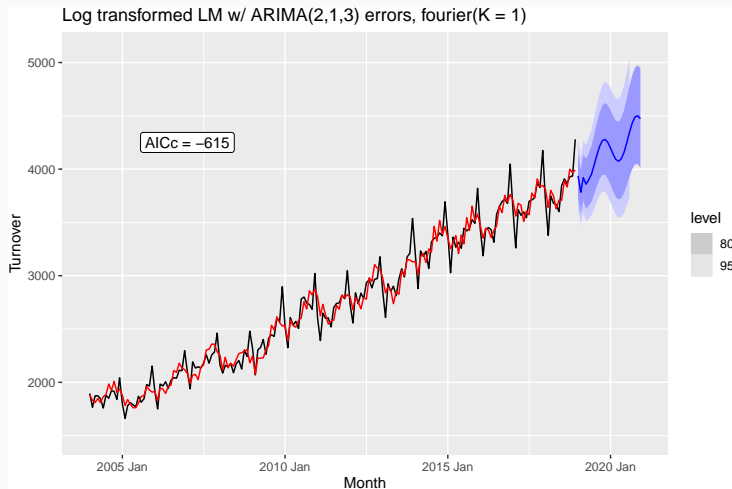


Eating-out expenditure

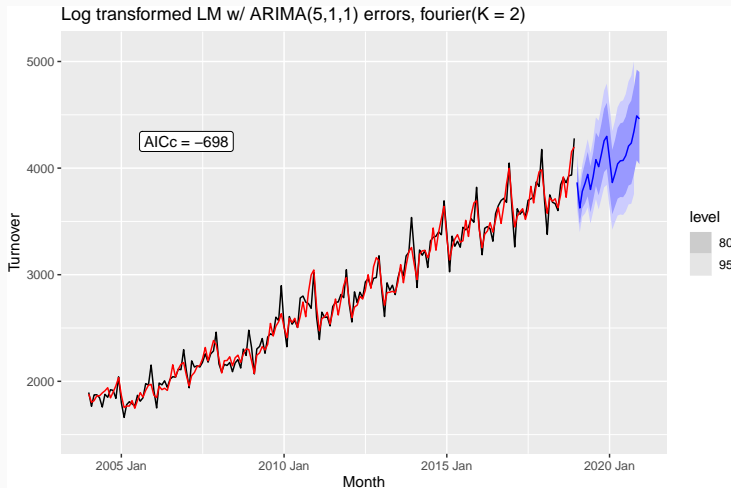
```
fit <- aus_cafe %>% model(  
  `K = 1` = ARIMA(log(Turnover) ~ fourier(K = 1) + PDQ(0,0,0)),  
  `K = 2` = ARIMA(log(Turnover) ~ fourier(K = 2) + PDQ(0,0,0)),  
  `K = 3` = ARIMA(log(Turnover) ~ fourier(K = 3) + PDQ(0,0,0)),  
  `K = 4` = ARIMA(log(Turnover) ~ fourier(K = 4) + PDQ(0,0,0)),  
  `K = 5` = ARIMA(log(Turnover) ~ fourier(K = 5) + PDQ(0,0,0)),  
  `K = 6` = ARIMA(log(Turnover) ~ fourier(K = 6) + PDQ(0,0,0))  
)  
glance(fit)
```

.model	sigma2	log_lik	AIC	AICc	BIC
K = 1	0.002	317	-616	-615	-588
K = 2	0.001	362	-700	-698	-661
K = 3	0.001	394	-763	-761	-725
K = 4	0.001	427	-822	-818	-771
K = 5	0.000	474	-919	-917	-875
K = 6	0.000	474	-920	-918	-875

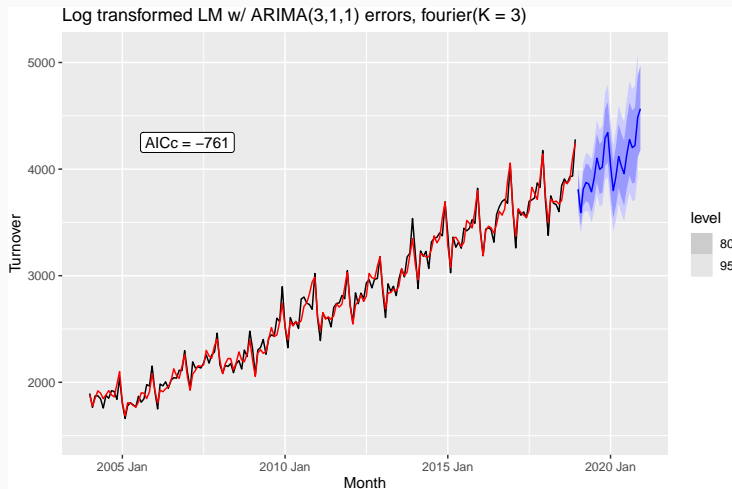
Eating-out expenditure



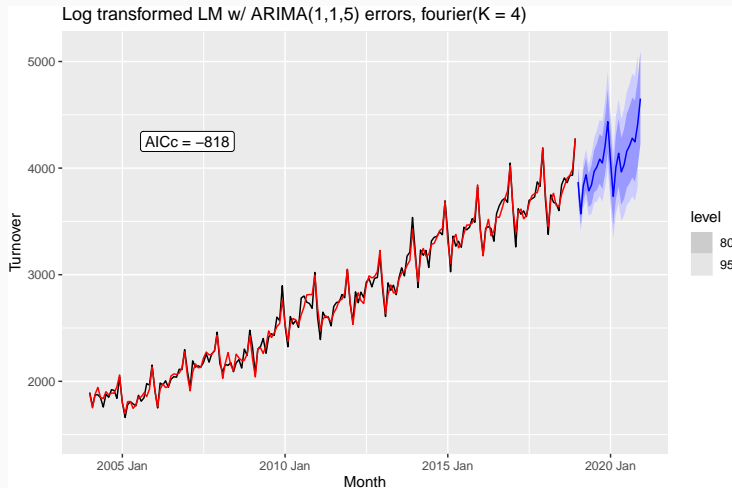
Eating-out expenditure



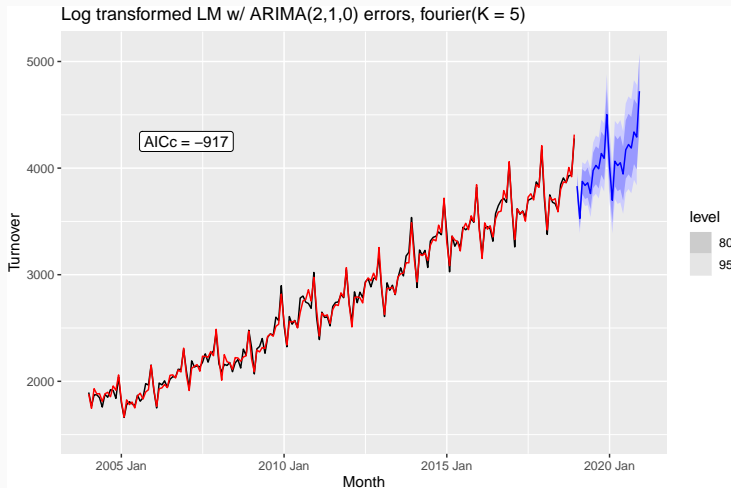
Eating-out expenditure



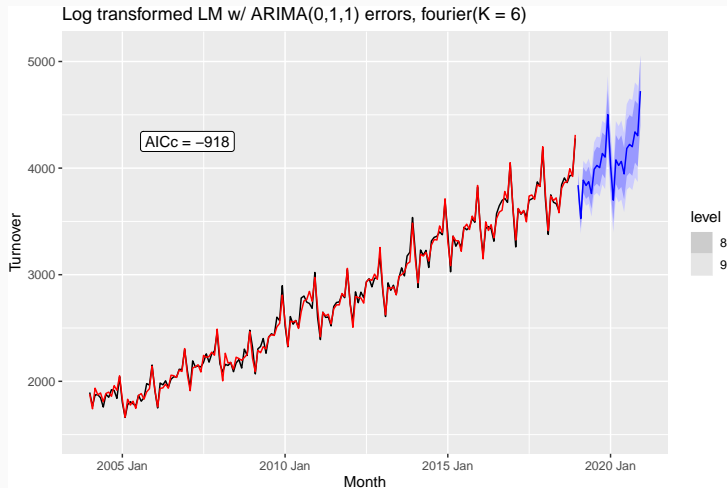
Eating-out expenditure



Eating-out expenditure



Eating-out expenditure



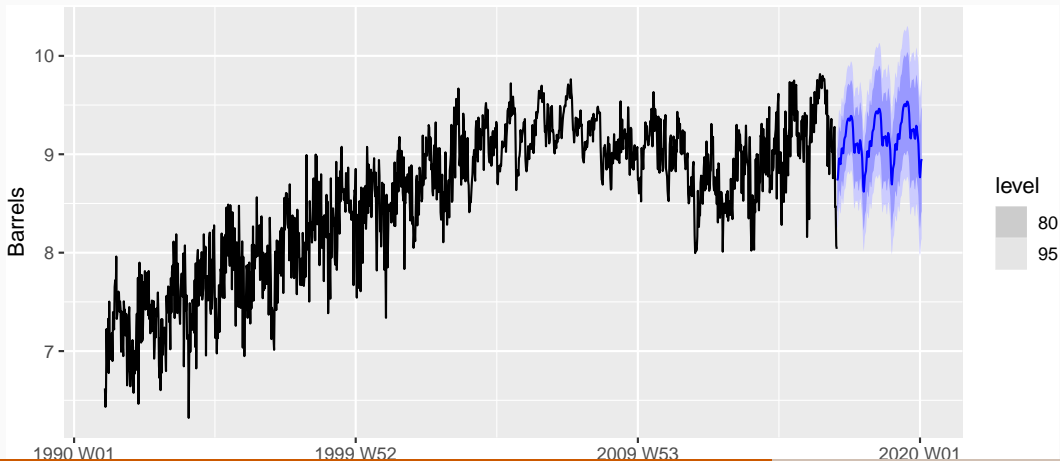
Example: weekly gasoline products

```
fit <- us_gasoline %>%  
  model(ARIMA(Barrels ~ fourier(K = 13) + PDQ(0,0,0)))  
report(fit)
```

```
## Series: Barrels  
## Model: LM w/ ARIMA(0,1,1) errors  
##  
## Coefficients:  
##          ma1  fourier(K = 13)C1_52  fourier(K = 13)S1_52  
##          -0.8934          -0.1121          -0.2300  
## s.e.       0.0132           0.0123           0.0122  
##          fourier(K = 13)C2_52  fourier(K = 13)S2_52  
##                   0.0420           0.0317  
## s.e.           0.0099           0.0099  
##          fourier(K = 13)C3_52  fourier(K = 13)S3_52  
##                   0.0832           0.0346  
## s.e.           0.0094           0.0094  
##          fourier(K = 13)C4_52  fourier(K = 13)S4_52  
##                   0.0185           0.0398  
## s.e.           0.0092           0.0092  
##          fourier(K = 13)C5_52  fourier(K = 13)S5_52
```

Example: weekly gasoline products

```
forecast(fit, h = "3 years") %>%  
  autoplot(us_gasoline)
```



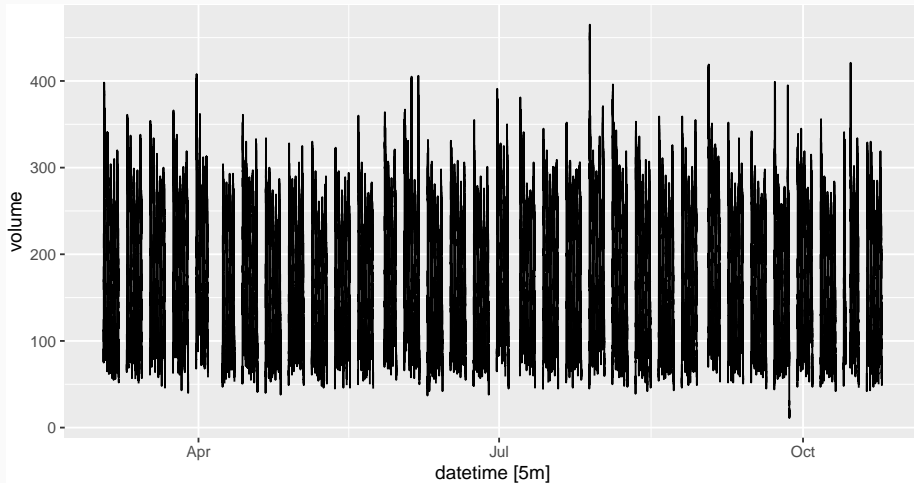
5-minute call centre volume

```
(calls <- readr::read_tsv("http://robjhyndman.com/data/callcenter.txt") %>%  
  rename(time = `...1`) %>%  
  pivot_longer(-time, names_to = "date", values_to = "volume") %>%  
  mutate(  
    date = as.Date(date, format = "%d/%m/%Y"),  
    datetime = as_datetime(date) + time  
  ) %>%  
  as_tsibble(index = datetime))
```

```
## # A tsibble: 27,716 x 4 [5m] <UTC>  
##   time    date      volume datetime  
##   <time> <date>      <dbl> <dtm>  
## 1 07:00  2003-03-03    111 2003-03-03 07:00:00  
## 2 07:05  2003-03-03    113 2003-03-03 07:05:00  
## 3 07:10  2003-03-03     76 2003-03-03 07:10:00  
## 4 07:15  2003-03-03     82 2003-03-03 07:15:00  
## 5 07:20  2003-03-03     91 2003-03-03 07:20:00  
## 6 07:25  2003-03-03     87 2003-03-03 07:25:00
```

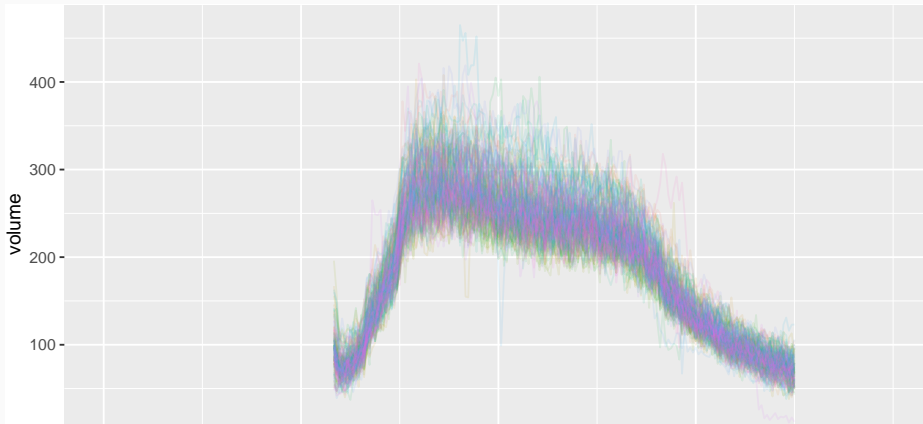
5-minute call centre volume

```
calls %>% fill_gaps() %>% autoplot(volume)
```



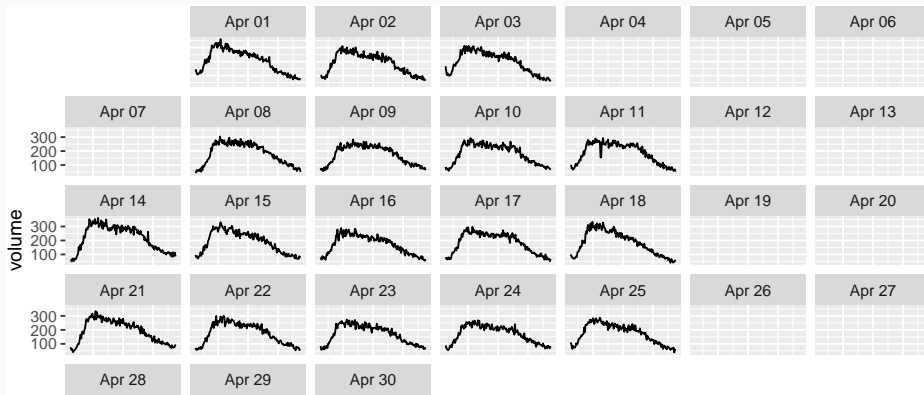
5-minute call centre volume

```
calls %>% fill_gaps() %>%  
  gg_season(volume, period = "day", alpha = 0.1) +  
  guides(colour = FALSE)
```



5-minute call centre volume

```
library(sugrants)
calls %>% filter(month(date, label = TRUE) == "Apr") %>%
  ggplot(aes(x = time, y = volume)) +
  geom_line() + facet_calendar(date)
```



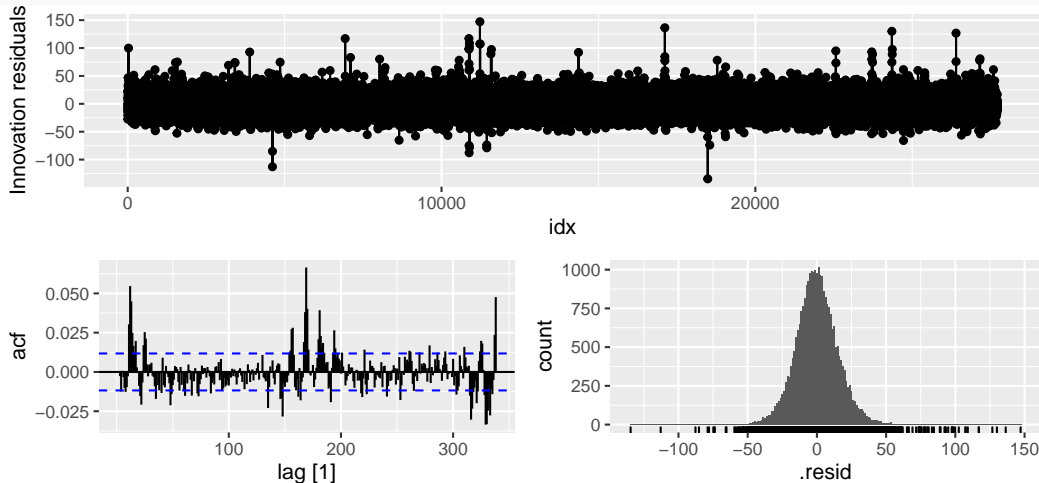
5-minute call centre volume

```
calls_mdl <- calls %>%  
  mutate(idx = row_number()) %>%  
  update_tsibble(index = idx)  
fit <- calls_mdl %>%  
  model(ARIMA(volume ~ fourier(169, K = 10) + pdq(d=0) + PDQ(0,0,0)))  
report(fit)
```

```
## Series: volume  
## Model: LM w/ ARIMA(1,0,3) errors  
##  
## Coefficients:  
##          ar1          ma1          ma2          ma3  fourier(169, K = 10)C1_169  
##          0.989   -0.7383   -0.0333   -0.0282                        -79.1  
## s.e.      0.001    0.0061    0.0075    0.0060                        0.7  
##          fourier(169, K = 10)S1_169  fourier(169, K = 10)C2_169  
##                                55.298                        -32.361  
## s.e.                                0.701                        0.378  
##          fourier(169, K = 10)S2_169  fourier(169, K = 10)C3_169  
##                                13.742                        -9.318  
## s.e.                                0.379                        0.273  
##          fourier(169, K = 10)S3_169  fourier(169, K = 10)C4_169
```

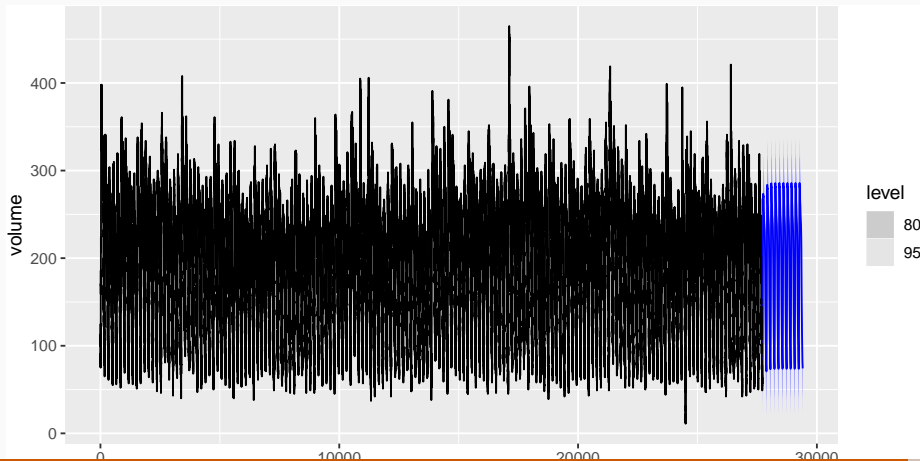
5-minute call centre volume

```
gg_tsresiduals(fit, lag = 338)
```



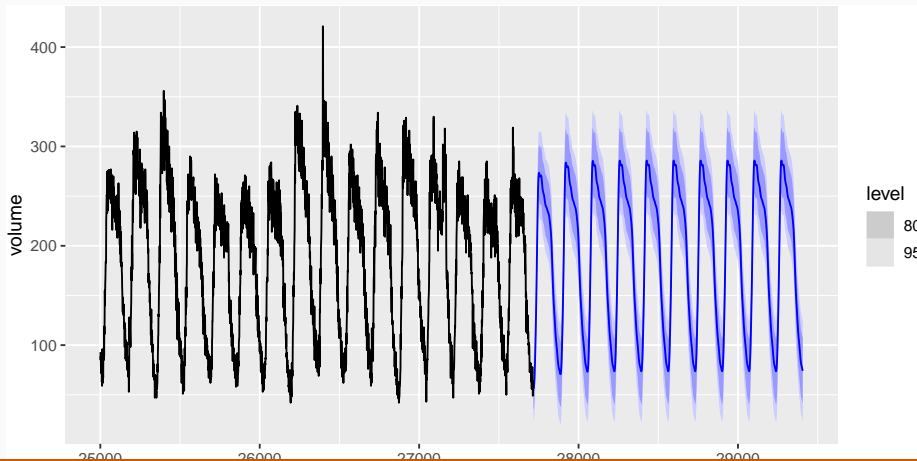
5-minute call centre volume

```
fit %>% forecast(h = 1690) %>%  
  autoplot(calls_md1)
```



5-minute call centre volume

```
fit %>% forecast(h = 1690) %>%  
  autoplot(filter(calls_mdl, idx > 25000))
```



Outline

- 1 Regression with ARIMA errors
- 2 Stochastic and deterministic trends
- 3 Dynamic harmonic regression
- 4 Lagged predictors

Lagged predictors

Sometimes a change in x_t does not affect y_t instantaneously

Lagged predictors

Sometimes a change in x_t does not affect y_t instantaneously

- y_t = sales, x_t = advertising.
- y_t = stream flow, x_t = rainfall.
- y_t = size of herd, x_t = breeding stock.

Lagged predictors

Sometimes a change in x_t does not affect y_t instantaneously

- y_t = sales, x_t = advertising.
 - y_t = stream flow, x_t = rainfall.
 - y_t = size of herd, x_t = breeding stock.
-
- These are dynamic systems with input (x_t) and output (y_t).
 - x_t is often a leading indicator.
 - There can be multiple predictors.

Lagged predictors

The model include present and past values of predictor:

$x_t, x_{t-1}, x_{t-2}, \dots$

$$y_t = a + \gamma_0 x_t + \gamma_1 x_{t-1} + \dots + \gamma_k x_{t-k} + \eta_t$$

where η_t is an ARIMA process.

Lagged predictors

The model include present and past values of predictor:

$x_t, x_{t-1}, x_{t-2}, \dots$

$$y_t = a + \gamma_0 x_t + \gamma_1 x_{t-1} + \dots + \gamma_k x_{t-k} + \eta_t$$

where η_t is an ARIMA process.

Rewrite model as

$$\begin{aligned} y_t &= a + (\gamma_0 + \gamma_1 B + \gamma_2 B^2 + \dots + \gamma_k B^k) x_t + \eta_t \\ &= a + \gamma(B) x_t + \eta_t. \end{aligned}$$

Lagged predictors

The model include present and past values of predictor:

$x_t, x_{t-1}, x_{t-2}, \dots$

$$y_t = a + \gamma_0 x_t + \gamma_1 x_{t-1} + \dots + \gamma_k x_{t-k} + \eta_t$$

where η_t is an ARIMA process.

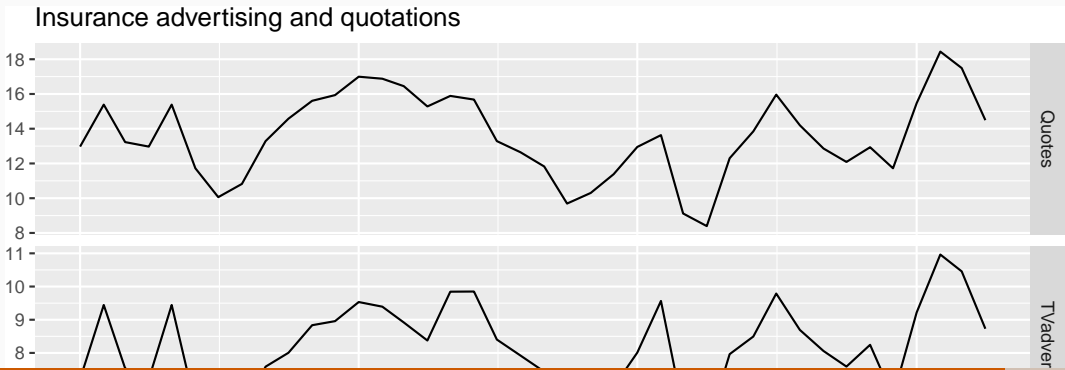
Rewrite model as

$$\begin{aligned} y_t &= a + (\gamma_0 + \gamma_1 B + \gamma_2 B^2 + \dots + \gamma_k B^k) x_t + \eta_t \\ &= a + \gamma(B) x_t + \eta_t. \end{aligned}$$

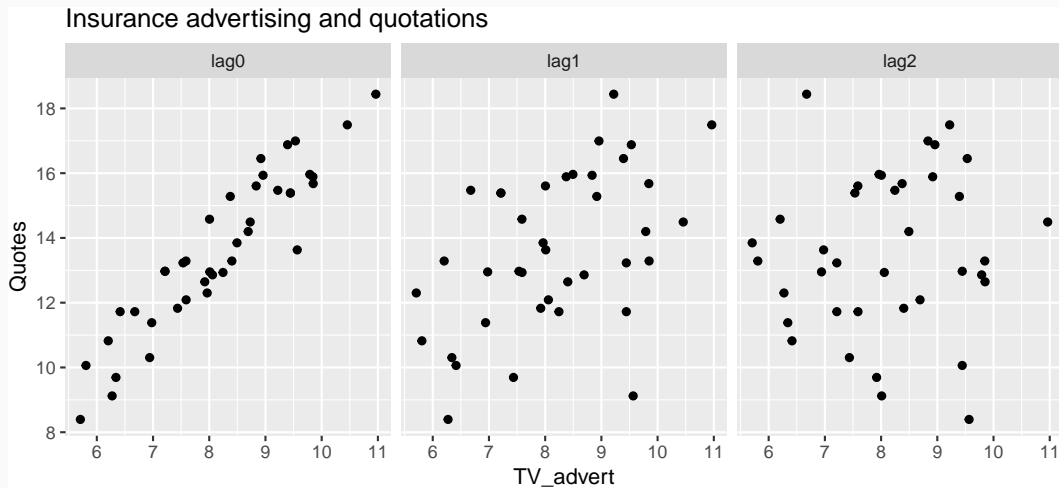
- $\gamma(B)$ is called a *transfer function* since it describes how change in x_t is transferred to y_t .

Example: Insurance quotes and TV adverts

```
insurance %>%  
  pivot_longer(Quotes:TVadverts) %>%  
  ggplot(aes(x = Month, y = value)) + geom_line() +  
  facet_grid(vars(name), scales = "free_y") +  
  labs(y = NULL, title = "Insurance advertising and quotations")
```



Example: Insurance quotes and TV adverts



Example: Insurance quotes and TV adverts

```
fit <- insurance %>%  
  # Restrict data so models use same fitting period  
  mutate(Quotes = c(NA,NA,NA,Quotes[4:40])) %>%  
  # Estimate models  
  model(  
    ARIMA(Quotes ~ pdq(d = 0) + TVadverts),  
    ARIMA(Quotes ~ pdq(d = 0) + TVadverts + lag(TVadverts)),  
    ARIMA(Quotes ~ pdq(d = 0) + TVadverts + lag(TVadverts) +  
      lag(TVadverts, 2)),  
    ARIMA(Quotes ~ pdq(d = 0) + TVadverts + lag(TVadverts) +  
      lag(TVadverts, 2) + lag(TVadverts, 3))  
  )
```


Example: Insurance quotes and TV adverts

```
glance(fit)
```

Lag order	sigma2	log_lik	AIC	AICc	BIC
0	0.265	-28.3	66.6	68.3	75.0
1	0.209	-24.0	58.1	59.9	66.5
2	0.215	-24.0	60.0	62.6	70.2
3	0.206	-22.2	60.3	65.0	73.8

Example: Insurance quotes and TV adverts

```
fit_best <- insurance %>%  
  model(ARIMA(Quotes ~ pdq(d=0) + TVadverts + lag(TVadverts)))  
report(fit_best)
```

```
## Series: Quotes  
## Model: LM w/ ARIMA(1,0,2) errors  
##  
## Coefficients:  
##          ar1      ma1      ma2 TVadverts lag(TVadverts) intercept  
##          0.512  0.917  0.459    1.2527         0.1464         2.16  
## s.e.    0.185  0.205  0.190    0.0588         0.0531         0.86  
##  
## sigma^2 estimated as 0.2166: log likelihood=-23.9  
## AIC=61.9   AICc=65.4   BIC=73.7
```

Example: Insurance quotes and TV adverts

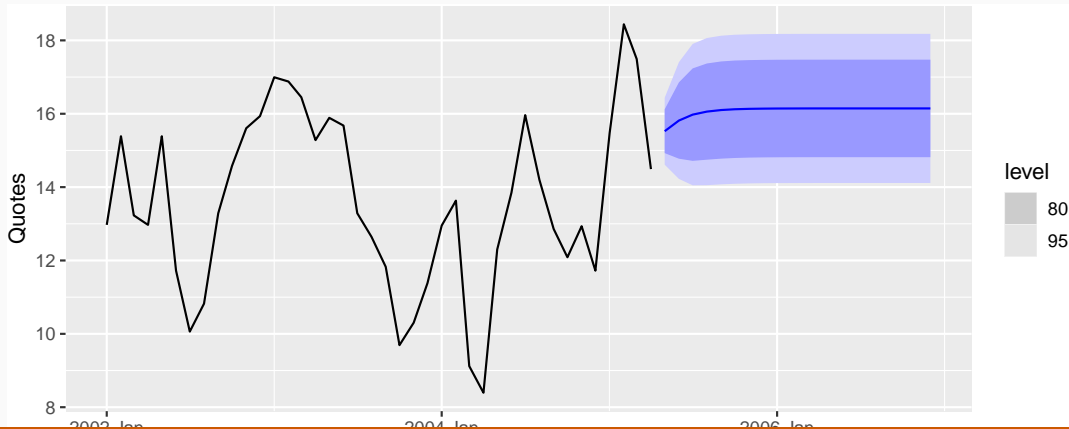
```
fit_best <- insurance %>%  
  model(ARIMA(Quotes ~ pdq(d=0) + TVadverts + lag(TVadverts)))  
report(fit_best)
```

```
## Series: Quotes  
## Model: LM w/ ARIMA(1,0,2) errors  
##  
## Coefficients:  
##          ar1      ma1      ma2 TVadverts lag(TVadverts) intercept  
##          0.512  0.917  0.459    1.2527          0.1464         2.16  
## s.e.    0.185  0.205  0.190    0.0588          0.0531         0.86  
##  
## sigma^2 estimated as 0.2166: log likelihood=-23.9  
## AIC=61.9   AICc=65.4   BIC=73.7
```

$$y_t = 2.155 + 1.253x_t + 0.146x_{t-1} + \eta_t,$$
$$\eta_t = 0.512\eta_{t-1} + \varepsilon_t + 0.917\varepsilon_{t-1} + 0.459\varepsilon_{t-2},$$

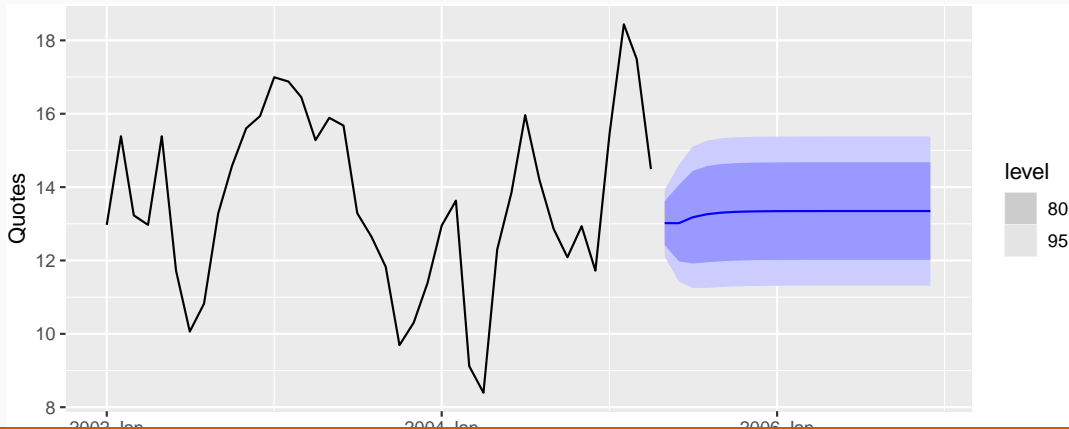
Example: Insurance quotes and TV adverts

```
advert_a <- new_data(insurance, 20) %>%  
  mutate(TVadverts = 10)  
forecast(fit_best, advert_a) %>% autoplot(insurance)
```



Example: Insurance quotes and TV adverts

```
advert_b <- new_data(insurance, 20) %>%  
  mutate(TVadverts = 8)  
forecast(fit_best, advert_b) %>% autoplot(insurance)
```



Example: Insurance quotes and TV adverts

```
advert_c <- new_data(insurance, 20) %>%  
  mutate(TVadverts = 6)  
forecast(fit_best, advert_c) %>% autoplot(insurance)
```

