Deriving models for ETS trend models

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ETS(A,A,N)

Component form:

$$\begin{aligned}
\hat{y}_{t+h|t} &= \ell_t + hb_t \\
\hat{y}_{t|t-1} &= \ell_{t-1} + b_{t-1} \\
\ell_t &= \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1}.
\end{aligned}$$

Model form:

$$\begin{aligned}
y_t &= \hat{y}_{t|t-1} + \varepsilon_t \\
&= \ell_{t-1} + b_{t-1} + \varepsilon_t \\
\ell_t &= \alpha y_t + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
&= \alpha(\ell_{t-1} + b_{t-1} + \varepsilon_t) + (1-\alpha)(\ell_{t-1} + b_{t-1}) \\
&= \ell_{t-1} + b_{t-1} + \alpha\varepsilon_t \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)b_{t-1} \\
&= \beta^*(b_{t-1} + \alpha\varepsilon_t) + (1-\beta^*)b_{t-1} \\
&= b_{t-1} + \alpha\beta^*\varepsilon_t \\
&= b_{t-1} + \beta\varepsilon_t \qquad \text{where } \beta = \alpha\beta^*.
\end{aligned}$$

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Component form:

$$\begin{aligned}
\hat{y}_{t+h|t} &= \ell_t + (\phi + \phi^2 + \cdots + \phi^h)b_t \\
\hat{y}_{t|t-1} &= \ell_{t-1} + \phi b_{t-1} \\
\ell_t &= \alpha y_t + (1-\alpha)(\ell_{t-1} + \phi b_{t-1}) \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1}.
\end{aligned}$$

Model form:

$$\begin{aligned}
y_t &= \hat{y}_{t|t-1} + \varepsilon_t \\
&= \ell_{t-1} + \phi b_{t-1} + \varepsilon_t \\
\ell_t &= \alpha y_t + (1-\alpha)(\ell_{t-1} + \phi b_{t-1}) \\
&= \alpha(\ell_{t-1} + \phi b_{t-1} + \varepsilon_t) + (1-\alpha)(\ell_{t-1} + \phi b_{t-1}) \\
&= \ell_{t-1} + \phi b_{t-1} + \alpha\varepsilon_t \\
b_t &= \beta^*(\ell_t - \ell_{t-1}) + (1-\beta^*)\phi b_{t-1} \\
&= \beta^*(\phi b_{t-1} + \alpha\varepsilon_t) + (1-\beta^*)\phi b_{t-1} \\
&= \phi b_{t-1} + \alpha\beta^*\varepsilon_t \\
&= \phi b_{t-1} + \beta\varepsilon_t \qquad \text{where } \beta = \alpha\beta^*.
\end{aligned}$$