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# ETC3550/ETC5550

## Applied forecasting

Ch9. ARIMA models

[OTexts.org/fpp3/](https://OTexts.org/fpp3/)

# Outline

- 1 Stationarity and differencing
- 2 Non-seasonal ARIMA models
- 3 Estimation and order selection
- 4 ARIMA modelling in R
- 5 Forecasting
- 6 Seasonal ARIMA models
- 7 ARIMA vs ETS

# ARIMA models

- AR:** autoregressive (lagged observations as inputs)
- I:** integrated (differencing to make series stationary)
- MA:** moving average (lagged errors as inputs)

# ARIMA models

**AR:** autoregressive (lagged observations as inputs)

**I:** integrated (differencing to make series stationary)

**MA:** moving average (lagged errors as inputs)

An ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns.

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# Stationarity

## Definition

If  $\{y_t\}$  is a stationary time series, then for all  $s$ , the distribution of  $(y_t, \dots, y_{t+s})$  does not depend on  $t$ .

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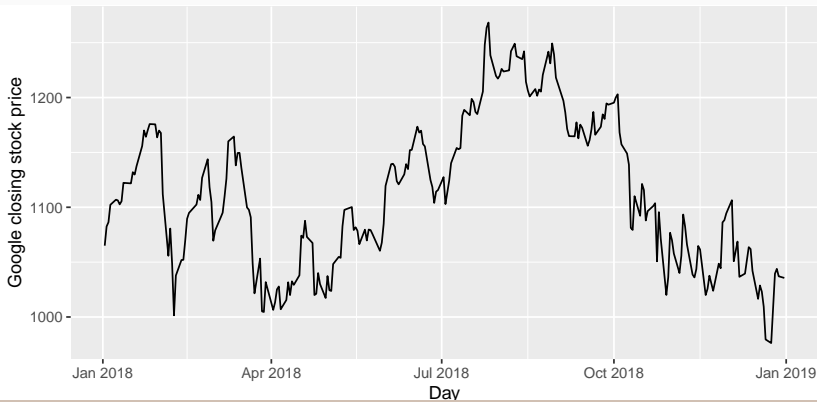
If  $\{y_t\}$  is a stationary time series, then for all  $s$ , the distribution of  $(y_t, \dots, y_{t+s})$  does not depend on  $t$ .

A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long-term

# Stationary?

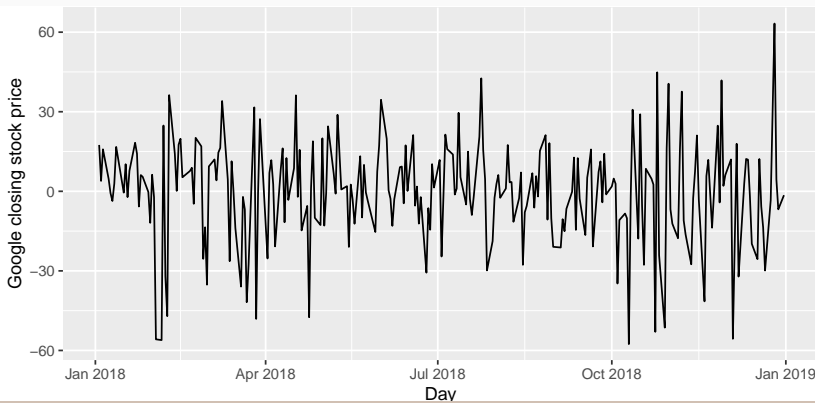
```
gafa_stock %>%  
  filter(Symbol == "GOOG", year(Date) == 2018)  
  autoplot(Close) + ylab("Google closing stock price")
```





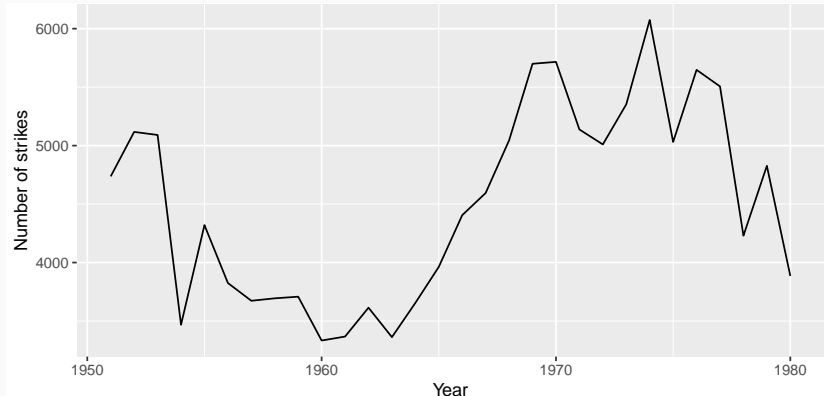
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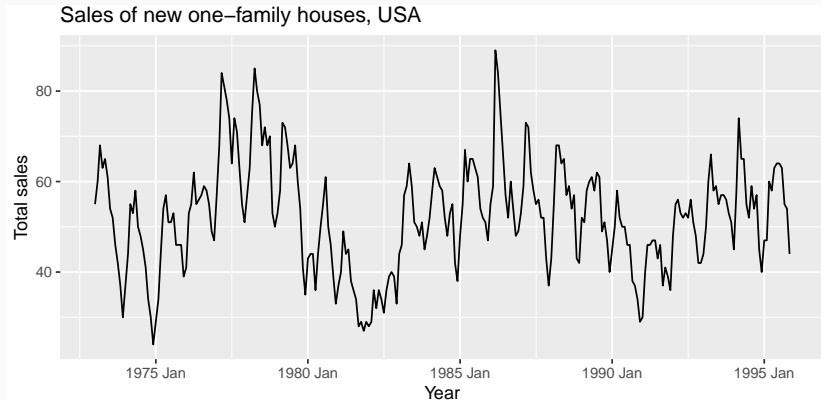
# Stationary?

```
as_tsibble(fma::strikes) %>% autoplot(value) +  
  ylab("Number of strikes") + xlab("Year")
```



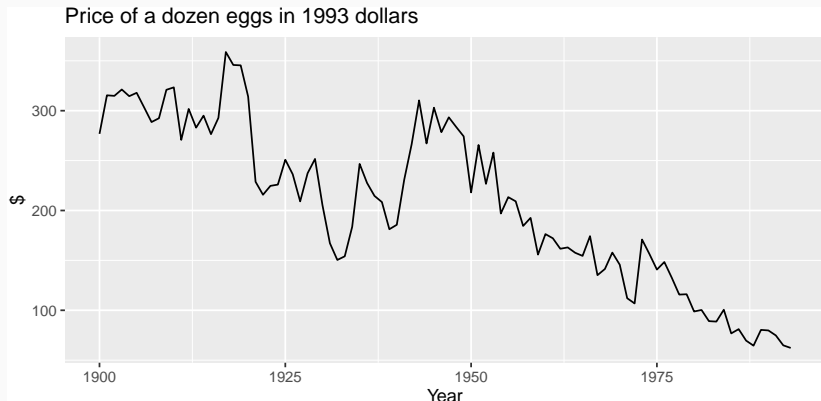
# Stationary?

```
as_tsibble(fma::hsales) %>% autoplot(value) +  
  ggtitle("Sales of new one-family houses, USA")
```



# Stationary?

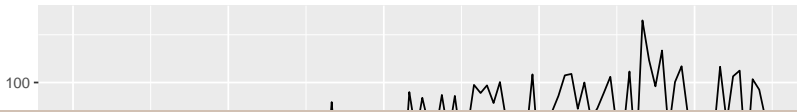
```
as_tsibble(fma::eggs) %>% autoplot(value) + xlab("Year") +  
ggtitle("Price of a dozen eggs in 1993 dollars")
```



# Stationary?

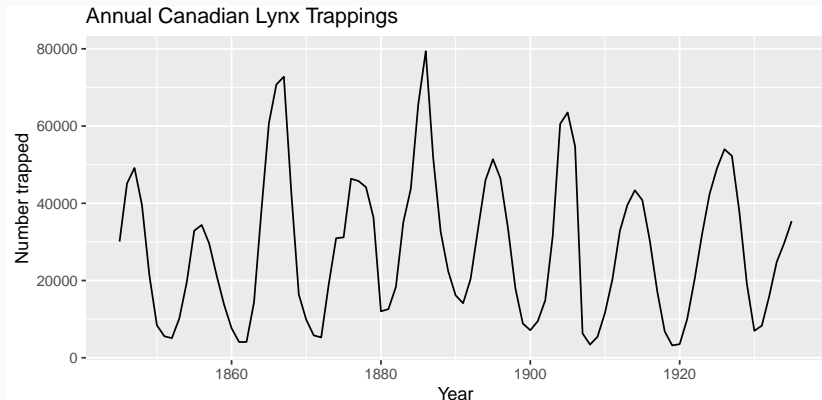
```
aus_livestock %>%  
  filter(  
    Animal == "Pigs",  
    State == "Victoria",  
    year(Month) >= 2010  
  ) %>%  
  autoplot(Count/1e3) + xlab("Year") + ylab("t  
  ggtitle("Number of pigs slaughtered in Victo
```

Number of pigs slaughtered in Victoria



# Stationary?

```
pelt %>% autoplot(Lynx) + xlab("Year") + ylab("Number trapped") + ggtitle("Annual Canadian Lynx Trappings")
```



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Transformations help to **stabilize the variance**.

For ARIMA modelling, we also need to **stabilize the mean**.



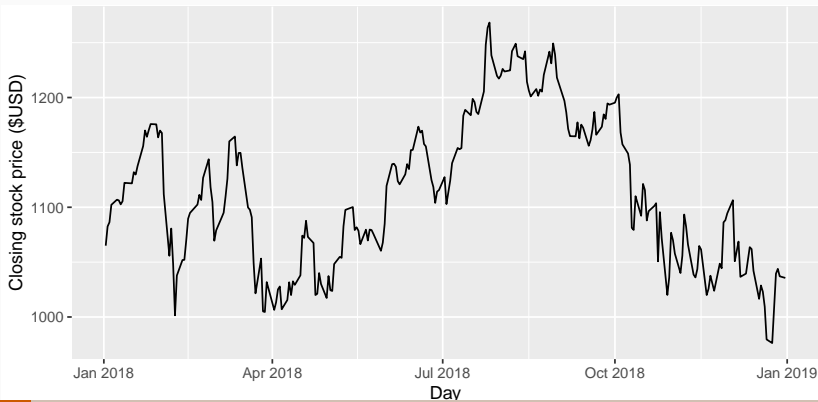
# Non-stationarity in the mean

## Identifying non-stationary series

- time plot.
- The ACF of stationary data drops to zero relatively quickly
- The ACF of non-stationary data decreases slowly.
- For non-stationary data, the value of  $r_1$  is often large and positive.

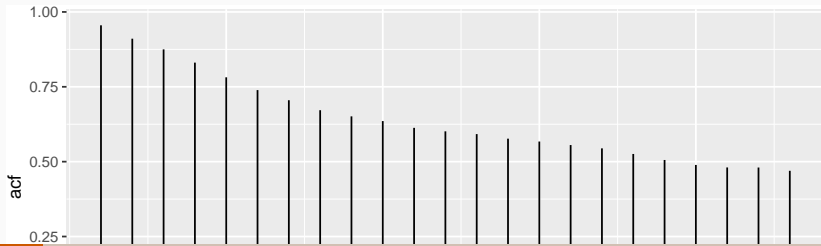
# Example: Google stock price

```
gafa_stock %>%  
  filter(Symbol == "GOOG", year(Date) == 2018)  
  autoplot(Close) + ylab("Closing stock price")
```



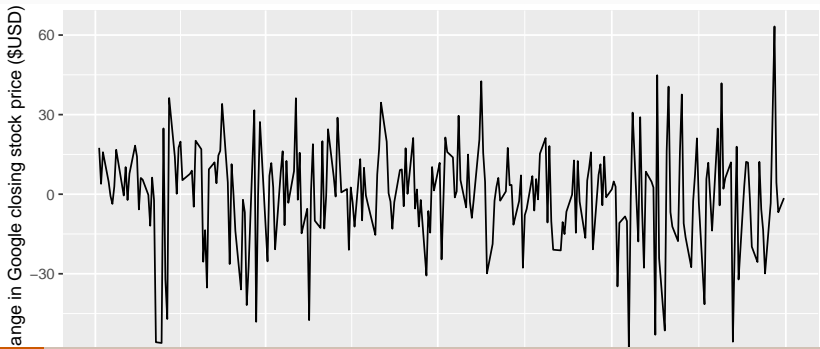
## Example: Google stock price

```
google_2018 <- gafa_stock %>%  
  filter(Symbol == "GOOG", year(Date) == 2018)  
  mutate(trading_day = row_number()) %>%  
  update_tsibble(index = trading_day, regular = TRUE)  
google_2018 %>%  
  ACF(Close) %>% autoplot()
```



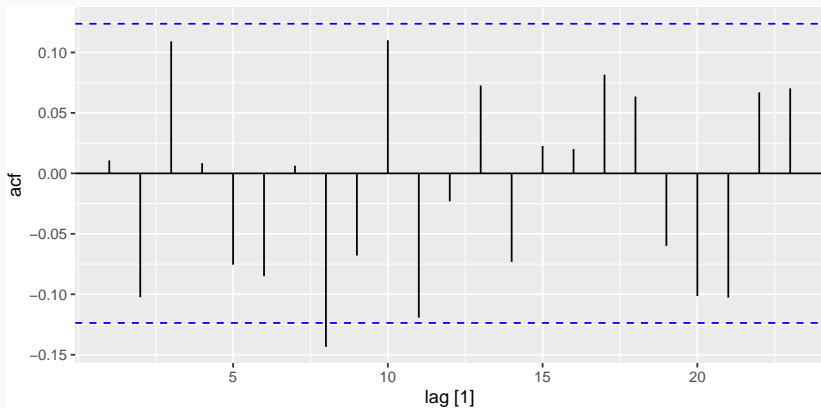
# Example: Google stock price

```
gafa_stock %>%  
  filter(Symbol == "GOOG", year(Date) == 2018)  
  autoplot(difference(Close)) +  
  ylab("Change in Google closing stock price ($USD)")
```



# Example: Google stock price

```
google_2018 %>% ACF(difference(Close)) %>% aut
```



# Differencing

- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series:

$$y'_t = y_t - y_{t-1}.$$

- The differenced series will have only  $T - 1$  values since it is not possible to calculate a difference  $y'_1$  for the first observation.

## Second-order differencing

Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time:

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$$\begin{aligned}y_t'' &= y_t' - y_{t-1}' \\&= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\&= y_t - 2y_{t-1} + y_{t-2}.\end{aligned}$$



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- $y_t''$  will have  $T - 2$  values.
- In practice, it is almost never necessary to go beyond second-order differences.

# Seasonal differencing

A seasonal difference is the difference between an observation and the corresponding observation from the previous year.

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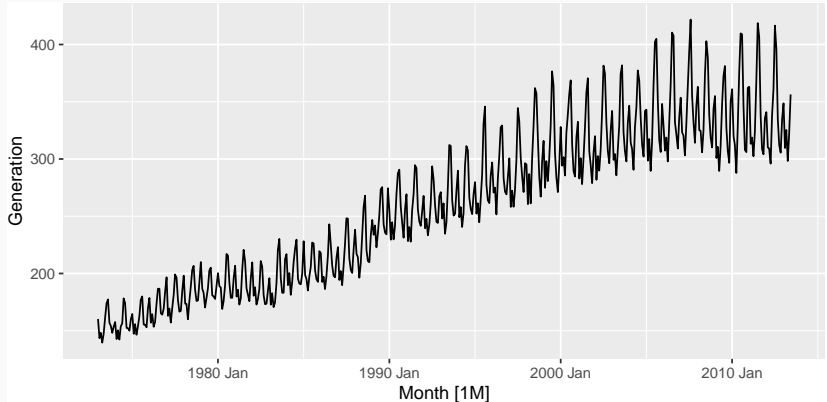
$$y'_t = y_t - y_{t-m}$$

where  $m$  = number of seasons.

- For monthly data  $m = 12$ .
- For quarterly data  $m = 4$ .

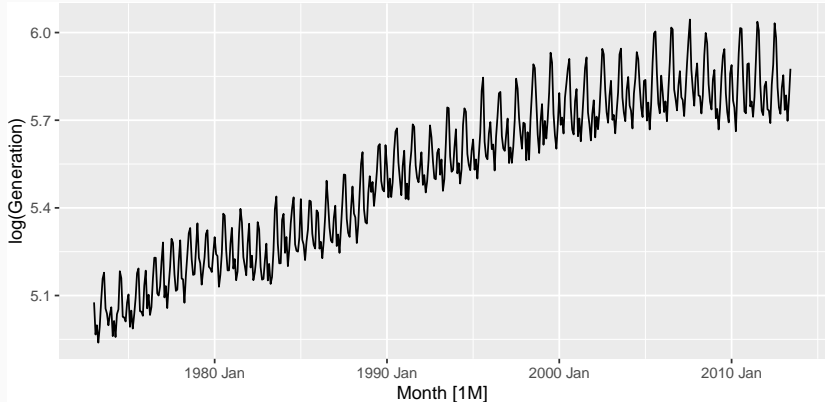
# Electricity production

```
usmelec %>% autoplot(  
  Generation  
)
```



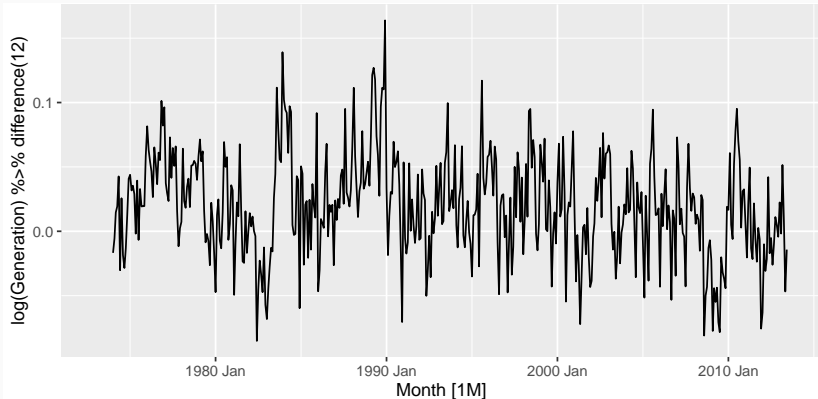
# Electricity production

```
usmelec %>% autoplot(  
  log(Generation)  
)
```



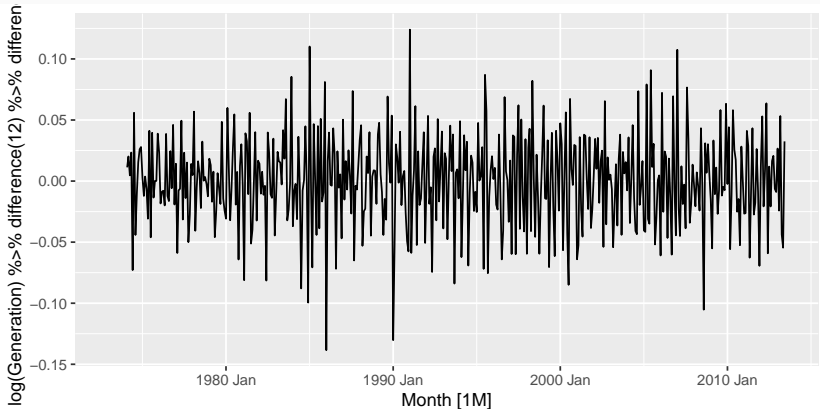
# Electricity production

```
usmelec %>% autoplot(  
  log(Generation) %>% difference(12)  
)
```



# Electricity production

```
usmelec %>% autoplot(  
  log(Generation) %>% difference(12) %>% difference()  
)
```





# Electricity production

- Seasonally differenced series is closer to being stationary.
- Remaining non-stationarity can be removed with further first difference.

If  $y'_t = y_t - y_{t-12}$  denotes seasonally differenced series, then twice-differenced series is

$$\begin{aligned}y_t^* &= y'_t - y'_{t-1} \\&= (y_t - y_{t-12}) - (y_{t-1} - y_{t-13}) \\&= y_t - y_{t-1} - y_{t-12} + y_{t-13} .\end{aligned}$$

# Seasonal differencing

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- it makes no difference which is done first—the result will be the same.
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It is important that if differencing is used, the differences are interpretable.

# Interpretation of differencing

- first differences are the change between **one observation and the next**;
- seasonal differences are the change between **one year to the next**.

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But taking lag 3 differences for yearly data, for example, results in a model which cannot be sensibly interpreted.

# Unit root tests

## Statistical tests to determine the required order of differencing.

- 1 Augmented Dickey Fuller test: null hypothesis is that the data are non-stationary and non-seasonal.
- 2 Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test: null hypothesis is that the data are stationary and non-seasonal.
- 3 Other tests available for seasonal data.

# KPSS test

```
google_2018 %>%  
  features(Close, unitroot_kpss)
```

```
## # A tibble: 1 x 3  
##   Symbol kpss_stat kpss_pvalue  
##   <chr>      <dbl>      <dbl>  
## 1 GOOG      0.573      0.0252
```



# KPSS test

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```

```
google_2018 %>%  
  features(Close, unitroot_ndiffs)
```

```
## # A tibble: 1 x 2  
##   Symbol ndiffs  
##   <chr>   <int>  
## 1 GOOG     1
```

# Automatically selecting differences

STL decomposition:  $y_t = T_t + S_t + R_t$

Seasonal strength  $F_s = \max\left(0, 1 - \frac{\text{Var}(R_t)}{\text{Var}(S_t + R_t)}\right)$

If  $F_s > 0.64$ , do one seasonal difference.

```
usmelec %>% mutate(log_gen = log(Generation)) %>%  
  features(log_gen, list(unitroot_nsdiffs, feat_stl))
```

```
## # A tibble: 1 x 10  
##   nsdiffs trend_strength seasonal_strength~ seasonal_peak_y~  
##   <int>          <dbl>          <dbl>          <dbl>  
## 1      1      0.994      0.941              7  
## # ... with 6 more variables: seasonal_trough_year <dbl>,  
## #   spikiness <dbl>, linearity <dbl>, curvature <dbl>,  
## #   stl_e_acf1 <dbl>, stl_e_acf10 <dbl>
```

# Automatically selecting differences

```
usmelec %>% mutate(log_gen = log(Generation)) %>%  
  features(log_gen, unitroot_nsdiffs)
```

```
## # A tibble: 1 x 1  
##   nsdiffs  
##   <int>  
## 1      1
```

```
usmelec %>% mutate(d_log_gen = difference(log(Generation), 12)) %>%  
  features(d_log_gen, unitroot_nsdiffs)
```

```
## # A tibble: 1 x 1  
##   nsdiffs  
##   <int>  
## 1      1
```

## Your turn

For the `tourism` dataset, compute the total number of trips and find an appropriate differencing (after transformation if necessary) to obtain stationary data.

# Backshift notation

A very useful notational device is the backward shift operator,  $B$ , which is used as follows:

$$By_t = y_{t-1}$$

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$$B(By_t) = B^2y_t = y_{t-2}$$

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$$B(By_t) = B^2y_t = y_{t-2}$$

For monthly data, if we wish to shift attention to “the same month last year”, then  $B^{12}$  is used, and the notation is  $B^{12}y_t = y_{t-12}$ .



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The backward shift operator is convenient for describing the process of *differencing*.

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Note that a first difference is represented by  $(1 - B)$ .

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Note that a first difference is represented by  $(1 - B)$ .

Similarly, if second-order differences (i.e., first differences of first differences) have to be computed, then:

$$y''_t = y_t - 2y_{t-1} + y_{t-2} = (1 - B)^2 y_t$$

# Backshift notation

- Second-order difference is denoted  $(1 - B)^2$ .
- *Second-order difference* is not the same as a *second difference*, which would be denoted  $1 - B^2$ ;
- In general, a  $d$ th-order difference can be written as

$$(1 - B)^d y_t$$

- A seasonal difference followed by a first difference can be written as

$$(1 - B)(1 - B^m)y_t$$

# Backshift notation

The “backshift” notation is convenient because the terms can be multiplied together to see the combined effect.

$$\begin{aligned}(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

# Backshift notation

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$$\begin{aligned}(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

For monthly data,  $m = 12$  and we obtain the same result as earlier.

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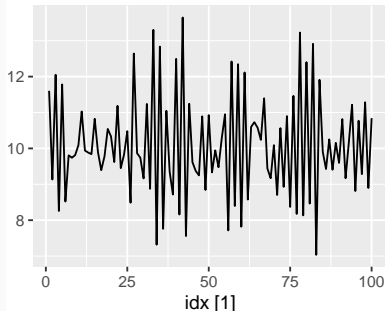
# Autoregressive models

## Autoregressive (AR) models:

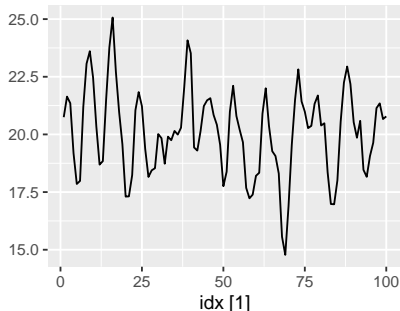
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where  $\varepsilon_t$  is white noise. This is a multiple regression with **lagged values** of  $y_t$  as predictors.

AR(1)



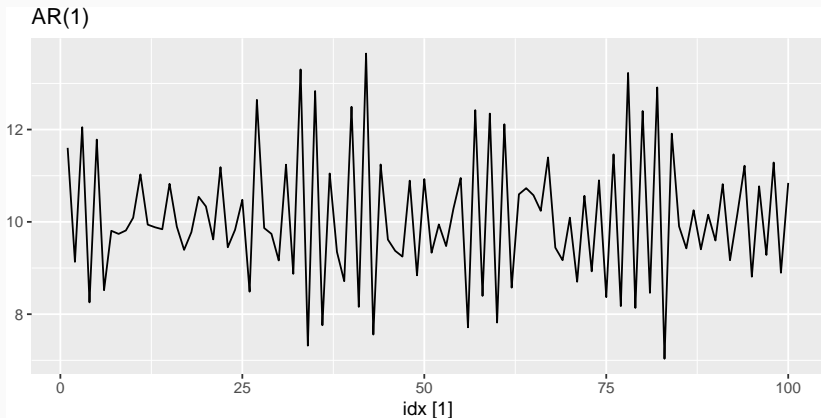
AR(2)



# AR(1) model

$$y_t = 18 - 0.8y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



# AR(1) model

$$y_t = c + \phi_1 y_{t-1} + \varepsilon_t$$

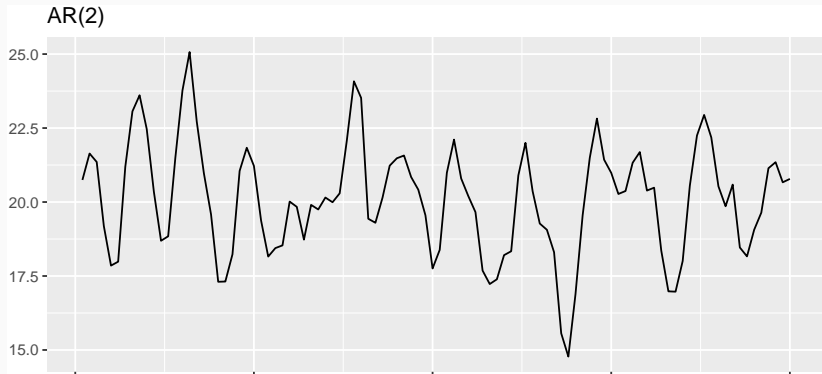
- When  $\phi_1 = 0$ ,  $y_t$  is **equivalent to WN**
- When  $\phi_1 = 1$  and  $c = 0$ ,  $y_t$  is **equivalent to a RW**
- When  $\phi_1 = 1$  and  $c \neq 0$ ,  $y_t$  is **equivalent to a RW with drift**
- When  $\phi_1 < 0$ ,  $y_t$  tends to **oscillate between positive and negative values.**

# AR(2) model

$$y_t = 8 + 1.3y_{t-1} - 0.7y_{t-2} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$

p2



# Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

## General condition for stationarity

Complex roots of  $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$  lie outside the unit circle on the complex plane.

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Complex roots of  $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$  lie outside the unit circle on the complex plane.

- For  $p = 1$ :  $-1 < \phi_1 < 1$ .
- For  $p = 2$ :  
 $-1 < \phi_2 < 1$        $\phi_2 + \phi_1 < 1$        $\phi_2 - \phi_1 < 1$ .
- More complicated conditions hold for  $p \geq 3$ .
- Estimation software takes care of this.

# Moving Average (MA) models

## Moving Average (MA) models:

$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q},$$

where  $\varepsilon_t$  is white noise. This is a multiple regression with **past errors** as predictors. *Don't confuse this with moving average smoothing!*

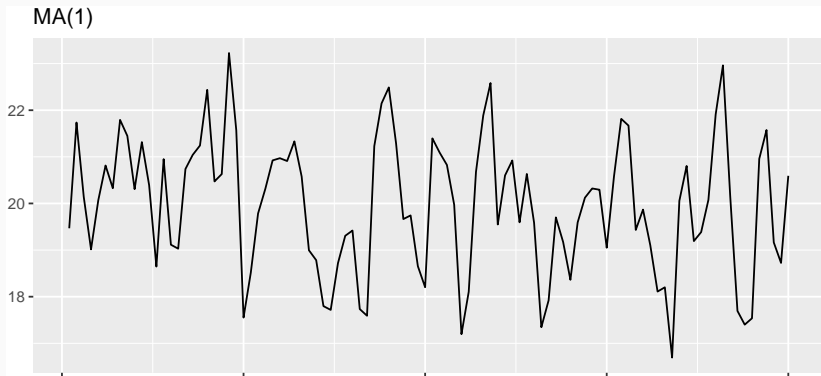
```
set.seed(2)
p1 <- tsibble(idx = seq_len(100), sim = 20 + a
  autoplot(sim) + ylab("") + ggtitle("MA(1)")
p2 <- tsibble(idx = seq_len(100), sim = arima.
  autoplot(sim) + ylab("") + ggtitle("MA(2)")
```

# MA(1) model

$$y_t = 20 + \varepsilon_t + 0.8\varepsilon_{t-1}$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$

p1



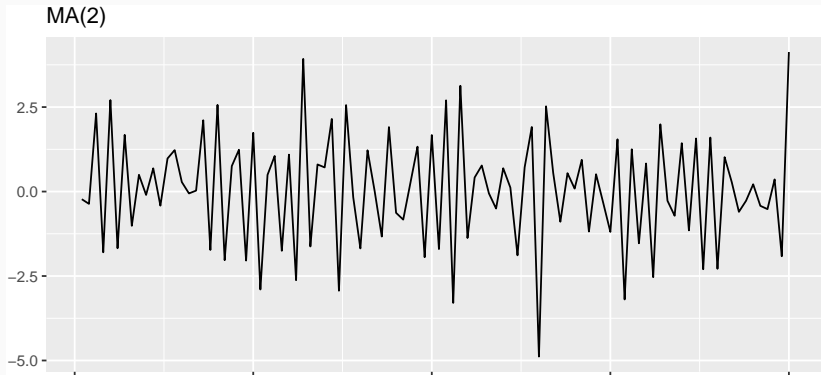


# MA(2) model

$$y_t = \varepsilon_t - \varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$

p2



# MA( $\infty$ ) models

It is possible to write any stationary AR( $p$ ) process as an MA( $\infty$ ) process.

## Example: AR(1)

$$\begin{aligned}y_t &= \phi_1 y_{t-1} + \varepsilon_t \\&= \phi_1(\phi_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\&= \phi_1^2 y_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\&= \phi_1^3 y_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\&\dots\end{aligned}$$

# MA( $\infty$ ) models

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$$\begin{aligned}y_t &= \phi_1 y_{t-1} + \varepsilon_t \\&= \phi_1(\phi_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\&= \phi_1^2 y_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\&= \phi_1^3 y_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\&\dots\end{aligned}$$

Provided  $-1 < \phi_1 < 1$ :

$$y_t = \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \phi_1^3 \varepsilon_{t-3} + \dots$$

# Invertibility

- Any  $MA(q)$  process can be written as an  $AR(\infty)$  process if we impose some constraints on the MA parameters.
- Then the MA model is called “invertible”.
- Invertible models have some mathematical properties that make them easier to use in practice.
- Invertibility of an ARIMA model is equivalent to forecastability of an ETS model.

# Invertibility

## General condition for invertibility

Complex roots of  $1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$  lie outside the unit circle on the complex plane.

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Complex roots of  $1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$  lie outside the unit circle on the complex plane.

- For  $q = 1$ :  $-1 < \theta_1 < 1$ .
- For  $q = 2$ :  
 $-1 < \theta_2 < 1 \quad \theta_2 + \theta_1 > -1 \quad \theta_1 - \theta_2 < 1$ .
- More complicated conditions hold for  $q \geq 3$ .
- Estimation software takes care of this.

# ARIMA models

## Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

# ARIMA models

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$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

- Predictors include both **lagged values of  $y_t$  and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.



# ARIMA models

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$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

- Predictors include both **lagged values of  $y_t$  and lagged errors.**
- Conditions on coefficients ensure stationarity.
- Conditions on coefficients ensure invertibility.

## Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing.**
- $(1 - B)^d y_t$  follows an ARMA model.

## Autoregressive Integrated Moving Average models

### ARIMA( $p, d, q$ ) model

AR:  $p$  = order of the autoregressive part

I:  $d$  = degree of first differencing involved

MA:  $q$  = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR( $p$ ): ARIMA( $p,0,0$ )
- MA( $q$ ): ARIMA(0,0, $q$ )

# Backshift notation for ARIMA

## ■ ARMA model:

$$y_t = c + \phi_1 B y_t + \dots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \dots + \theta_q B^q \varepsilon_t$$

or  $(1 - \phi_1 B - \dots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$

## ■ ARIMA(1,1,1) model:

$$\begin{array}{ccccc} (1 - \phi_1 B) & (1 - B) y_t & = & c + (1 + \theta_1 B) \varepsilon_t \\ \uparrow & \uparrow & & \uparrow \\ \text{AR}(1) & \text{First} & & \text{MA}(1) \\ & \text{difference} & & \end{array}$$

# Backshift notation for ARIMA

## ■ ARMA model:

$$y_t = c + \phi_1 B y_t + \dots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \dots + \theta_q B^q \varepsilon_t$$

or  $(1 - \phi_1 B - \dots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$

## ■ ARIMA(1,1,1) model:

$$\begin{array}{ccccc} (1 - \phi_1 B) & (1 - B) y_t & = & c + (1 + \theta_1 B) \varepsilon_t \\ \uparrow & \uparrow & & \uparrow \\ \text{AR}(1) & \text{First} & & \text{MA}(1) \\ & \text{difference} & & \end{array}$$

Written out:

$$y_t = c + y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$$

# R model

## Intercept form

$$(1 - \phi_1 B - \dots - \phi_p B^p) y'_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

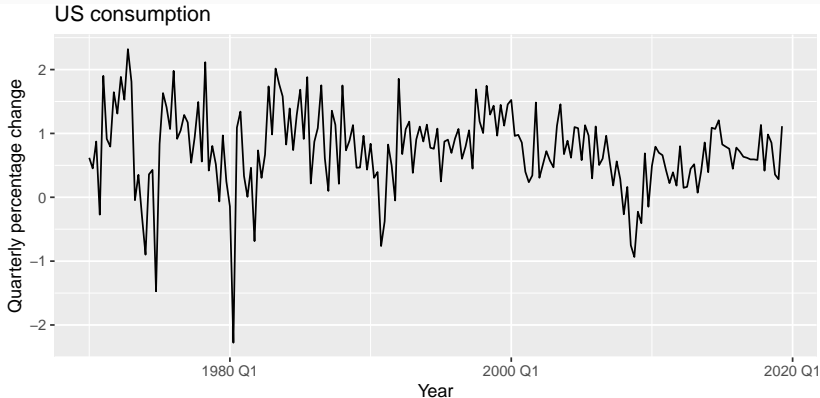
## Mean form

$$(1 - \phi_1 B - \dots - \phi_p B^p)(y'_t - \mu) = (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

- $y'_t = (1 - B)^d y_t$
- $\mu$  is the mean of  $y'_t$ .
- $c = \mu(1 - \phi_1 - \dots - \phi_p)$ .
- fable uses intercept form

# US consumption expenditure

```
us_change %>% autoplot(Consumption) +  
  xlab("Year") +  
  ylab("Quarterly percentage change") +  
  ggtitle("US consumption")
```



# US personal consumption

```
fit <- us_change %>% model(arima = ARIMA(Consumption ~ PDQ(0,0,0)))  
report(fit)
```

```
## Series: Consumption  
## Model: ARIMA(1,0,3) w/ mean  
##  
## Coefficients:  
##          ar1      ma1      ma2      ma3  constant  
##          0.573  -0.362  0.0925  0.1934    0.3160  
## s.e.    0.150    0.161  0.0787  0.0824    0.0371  
##  
## sigma^2 estimated as 0.3334:  log likelihood=-170  
## AIC=352    AICc=352    BIC=372
```

# US personal consumption

```
fit <- us_change %>% model(arima = ARIMA(Consumption ~ PDQ(0,0,0)))  
report(fit)
```

```
## Series: Consumption  
## Model: ARIMA(1,0,3) w/ mean  
##  
## Coefficients:  
##          ar1      ma1      ma2      ma3  constant  
##          0.573  -0.362  0.0925  0.1934   0.3160  
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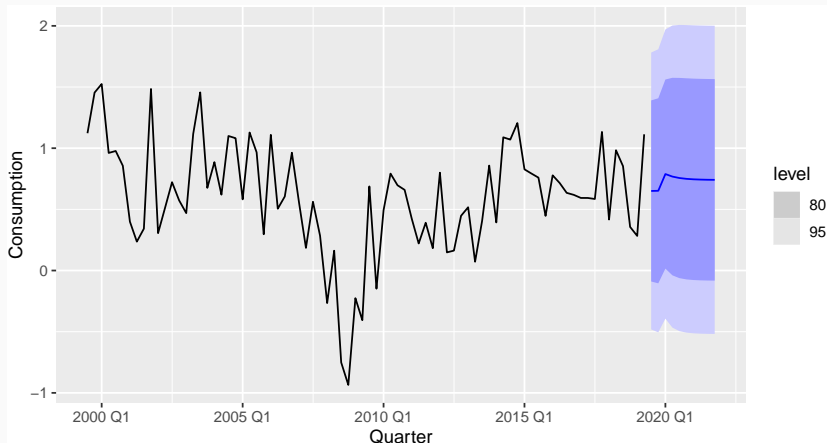
```
if(!grepl("ARIMA\\(1,0,3\\)", format(fit$arima)))  
  warning("Needs fixing")
```

**ARIMA(1,0,3) model:**



# US personal consumption

```
fit %>% forecast(h=10) %>%  
  autoplot(tail(us_change, 80))
```



# Understanding ARIMA models

- If  $c = 0$  and  $d = 0$ , the long-term forecasts will go to zero.
- If  $c = 0$  and  $d = 1$ , the long-term forecasts will go to a non-zero constant.
- If  $c = 0$  and  $d = 2$ , the long-term forecasts will follow a straight line.
- If  $c \neq 0$  and  $d = 0$ , the long-term forecasts will go to the mean of the data.
- If  $c \neq 0$  and  $d = 1$ , the long-term forecasts will follow a straight line.
- If  $c \neq 0$  and  $d = 2$ , the long-term forecasts will follow a quadratic trend.

# Understanding ARIMA models

## Forecast variance and $d$

- The higher the value of  $d$ , the more rapidly the prediction intervals increase in size.
- For  $d = 0$ , the long-term forecast standard deviation will go to the standard deviation of the historical data.

## Cyclic behaviour

- For cyclic forecasts,  $p \geq 2$  and some restrictions on coefficients are required.
- If  $p = 2$ , we need  $\phi_1^2 + 4\phi_2 < 0$ . Then average cycle of length

$$(2\pi) / \left[ \arccos(-\phi_1(1 - \phi_2)/(4\phi_2)) \right].$$

# Outline

- 1 Stationarity and differencing
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# Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters  $c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ .

# Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters  $c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ .

- MLE is very similar to least squares estimation obtained by minimizing

$$\sum_{t=1}^T e_t^2$$

- The `ARIMA()` function allows CLS or MLE estimation.
- Non-linear optimization must be used in either case.
- Different software will give different estimates.

# Partial autocorrelations

**Partial autocorrelations** measure relationship between  $y_t$  and  $y_{t-k}$ , when the effects of other time lags —  $1, 2, 3, \dots, k-1$  — are removed.

# Partial autocorrelations

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$\alpha_k$  =  $k$ th partial autocorrelation coefficient  
= equal to the estimate of  $\phi_k$  in regression:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k}.$$



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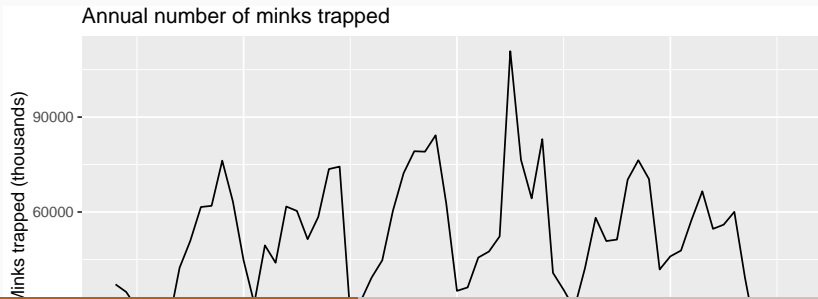
$\alpha_k$  =  $k$ th partial autocorrelation coefficient  
= equal to the estimate of  $\phi_k$  in regression:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k}.$$

- Varying number of terms on RHS gives  $\alpha_k$  for different values of  $k$ .
- $\alpha_1 = \rho_1$
- same critical values of  $\pm 1.96/\sqrt{T}$  as for ACF.
- Last significant  $\alpha_k$  indicates the order of an AR model.

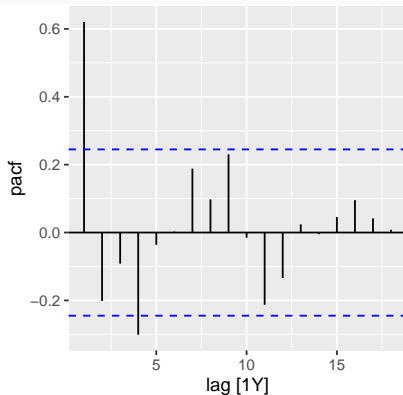
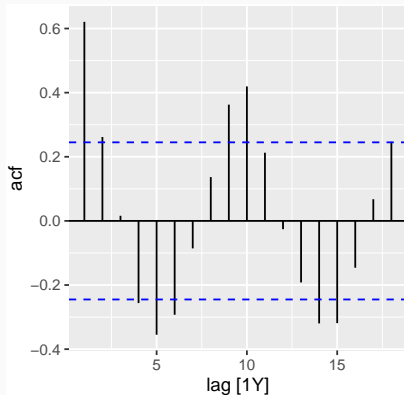
# Example: Mink trapping

```
mink <- as_tsibble(fma::mink)
mink %>% autoplot(value) +
  xlab("Year") +
  ylab("Minks trapped (thousands)") +
  ggtitle("Annual number of minks trapped")
```



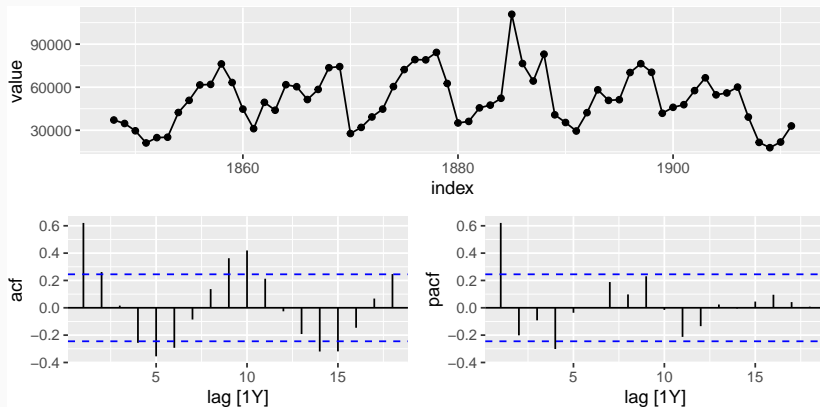
# Example: Mink trapping

```
p1 <- mink %>% ACF(value) %>% autoplot()  
p2 <- mink %>% PACF(value) %>% autoplot()  
p1 | p2
```



# Example: Mink trapping

```
mink %>% gg_tsdisplay(value, plot_type='partial')
```



# ACF and PACF interpretation

## AR(1)

$$\begin{aligned}\rho_k &= \phi_1^k && \text{for } k = 1, 2, \dots; \\ \alpha_1 &= \phi_1 && \alpha_k = 0 \quad \text{for } k = 2, 3, \dots\end{aligned}$$

So we have an AR(1) model when

- autocorrelations exponentially decay
- there is a single significant partial autocorrelation.

# ACF and PACF interpretation

## $AR(p)$

- ACF dies out in an exponential or damped sine-wave manner
- PACF has all zero spikes beyond the  $p$ th spike

So we have an  $AR(p)$  model when

- the ACF is exponentially decaying or sinusoidal
- there is a significant spike at lag  $p$  in PACF, but none beyond  $p$

# ACF and PACF interpretation

## MA(1)

$$\begin{aligned}\rho_1 &= \theta_1 & \rho_k &= 0 & \text{for } k = 2, 3, \dots; \\ \alpha_k &= -(-\theta_1)^k\end{aligned}$$

So we have an MA(1) model when

- the PACF is exponentially decaying and
- there is a single significant spike in ACF

# ACF and PACF interpretation

## MA( $q$ )

- PACF dies out in an exponential or damped sine-wave manner
- ACF has all zero spikes beyond the  $q$ th spike

So we have an MA( $q$ ) model when

- the PACF is exponentially decaying or sinusoidal
- there is a significant spike at lag  $q$  in ACF, but none beyond  $q$



# Information criteria

## Akaike's Information Criterion (AIC):

$$\text{AIC} = -2 \log(L) + 2(p + q + k + 1),$$

where  $L$  is the likelihood of the data,

$k = 1$  if  $c \neq 0$  and  $k = 0$  if  $c = 0$ .

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## Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p + q + k + 1)(p + q + k + 2)}{T - p - q - k - 2}.$$

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$$\text{BIC} = \text{AIC} + [\log(T) - 2](p + q + k + 1).$$

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## Bayesian Information Criterion:

$$\text{BIC} = \text{AIC} + [\log(T) - 2](p + q + k + 1).$$

Good models are obtained by minimizing either the AIC, AICc or BIC. Our preference is to use the AICc.

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# How does ARIMA() work?

## A non-seasonal ARIMA process

$$\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$$

Need to select appropriate orders:  $p, q, d$

## Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences  $d$  and  $D$  via KPSS test and seasonal strength measure.
- Select  $p, q$  by minimising AICc.
- Use stepwise search to traverse model space.

# How does ARIMA() work?

$$\text{AICc} = -2 \log(L) + 2(p + q + k + 1) \left[ 1 + \frac{(p+q+k+2)}{T-p-q-k-2} \right].$$

where  $L$  is the maximised likelihood fitted to the *differenced* data,  $k = 1$  if  $c \neq 0$  and  $k = 0$  otherwise.

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where  $L$  is the maximised likelihood fitted to the *differenced* data,  $k = 1$  if  $c \neq 0$  and  $k = 0$  otherwise.

**Step1:** Select current model (with smallest AICc) from:

ARIMA(2,  $d$ , 2)

ARIMA(0,  $d$ , 0)

ARIMA(1,  $d$ , 0)

ARIMA(0,  $d$ , 1)



# How does ARIMA() work?

$$\text{AICc} = -2 \log(L) + 2(p + q + k + 1) \left[ 1 + \frac{(p+q+k+2)}{T-p-q-k-2} \right].$$

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**Step1:** Select current model (with smallest AICc) from:

ARIMA(2,  $d$ , 2)

ARIMA(0,  $d$ , 0)

ARIMA(1,  $d$ , 0)

ARIMA(0,  $d$ , 1)

**Step 2:** Consider variations of current model:

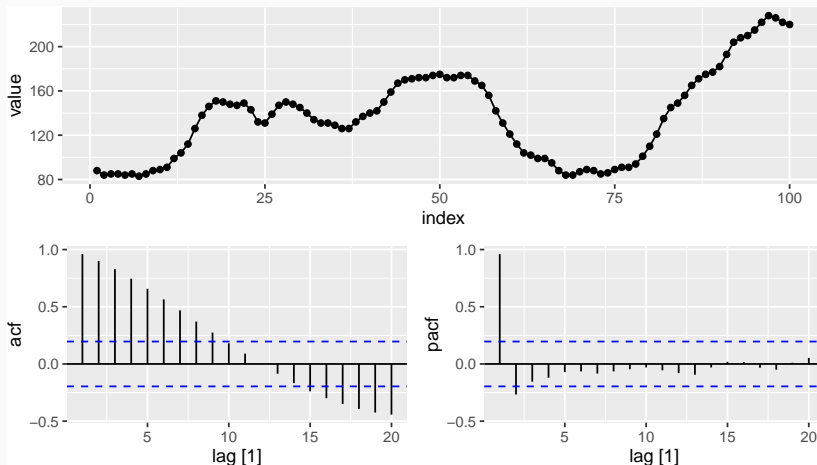
- vary one of  $p$ ,  $q$ , from current model by  $\pm 1$ ;
- $p$ ,  $q$  both vary from current model by  $\pm 1$ ;
- Include/exclude  $c$  from current model.

Model with lowest AICc becomes current model.

**Repeat Step 2 until no lower AICc can be found.**

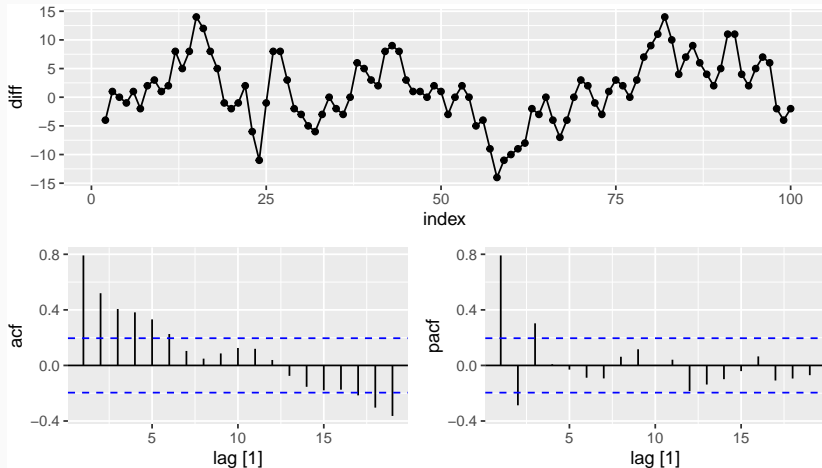
# Choosing your own model

```
web_usage <- as_tsibble(WWWusage)
web_usage %>% gg_tsdisplay(value, plot_type = 'partial')
```



# Choosing your own model

```
web_usage %>% mutate(diff = difference(value)) %>%  
  gg_tsdisplay(diff, plot_type = 'partial')
```



# Choosing your own model

```
fit <- web_usage %>%  
  model(arima = ARIMA(value ~ pdq(3, 1, 0)))  
report(fit)
```

```
## Series: value  
## Model: ARIMA(3,1,0)  
##  
## Coefficients:  
##          ar1      ar2      ar3  
##          1.151  -0.661  0.3407  
## s.e.    0.095    0.135  0.0941  
##  
## sigma^2 estimated as 9.656:  log likelihood=-252  
## AIC=512    AICc=512    BIC=522
```

# Choosing your own model

```
web_usage %>%  
  model(ARIMA(value ~ pdq(d=1))) %>%  
  report()
```

```
## Series: value  
## Model: ARIMA(1,1,1)  
##  
## Coefficients:  
##          ar1      ma1  
##      0.6504  0.5256  
## s.e.  0.0842  0.0896  
##  
## sigma^2 estimated as 9.995:  log likelihood=-254  
## AIC=514   AICc=515   BIC=522
```

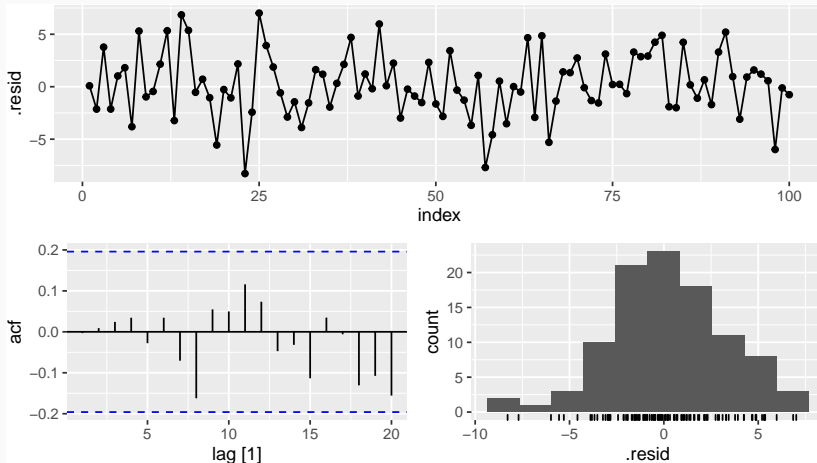
# Choosing your own model

```
web_usage %>%  
  model(ARIMA(value ~ pdq(d=1),  
    stepwise = FALSE, approximation = FALSE)) %>%  
  report()
```

```
## Series: value  
## Model: ARIMA(3,1,0)  
##  
## Coefficients:  
##          ar1      ar2      ar3  
##        1.151  -0.661   0.3407  
## s.e.   0.095   0.135   0.0941  
##  
## sigma^2 estimated as 9.656:  log likelihood=-252  
## AIC=512   AICc=512   BIC=522
```

# Choosing your own model

```
gg_tsresiduals(fit)
```



# Choosing your own model

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 10, dof = 3)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 arima      4.49      0.722
```



# Choosing your own model

```
fit %>% forecast(h = 10) %>%  
  autoplot(web_usage)
```



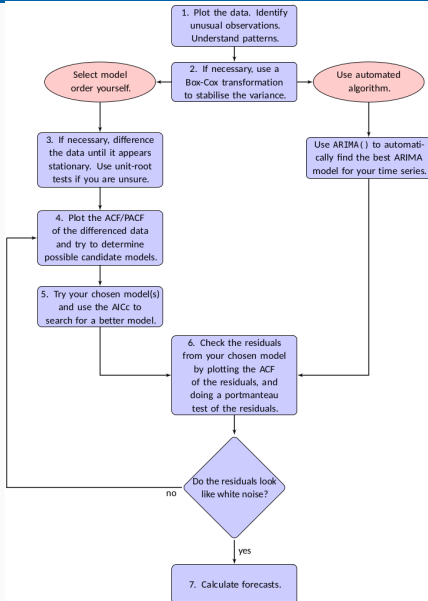
# Modelling procedure with ARIMA ( )

- 1 Plot the data. Identify any unusual observations.
- 2 If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
- 3 If the data are non-stationary: take first differences of the data until the data are stationary.
- 4 Examine the ACF/PACF: Is an  $AR(p)$  or  $MA(q)$  model appropriate?
- 5 Try your chosen model(s), and use the AICc to search for a better model.
- 6 Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
- 7 Once the residuals look like white noise, calculate forecasts.

# Automatic modelling procedure with ARIMA ( )

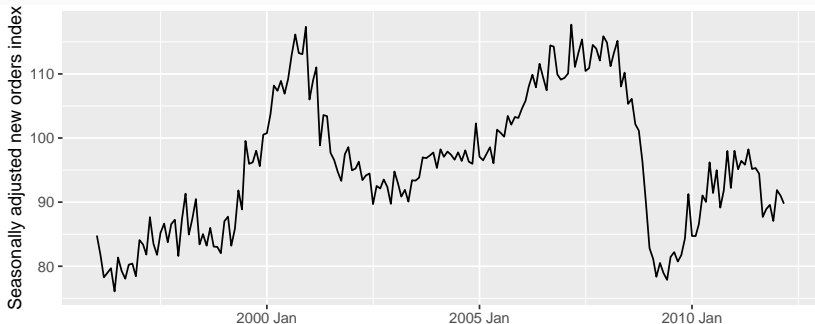
- 1 Plot the data. Identify any unusual observations.
- 2 If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
- 3 Use ARIMA to automatically select a model.
- 6 Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
- 7 Once the residuals look like white noise, calculate forecasts.

# Modelling procedure



# Seasonally adjusted electrical equipment

```
elecequip <- as_tsibble(fpp2::elecequip)
dcmp <- elecequip %>%
  model(STL(value ~ season(window = "periodic"))) %>%
  components() %>% select(-.model)
dcmp %>% as_tsibble %>%
  autoplot(season_adjust) + xlab("Year") +
  ylab("Seasonally adjusted new orders index")
```

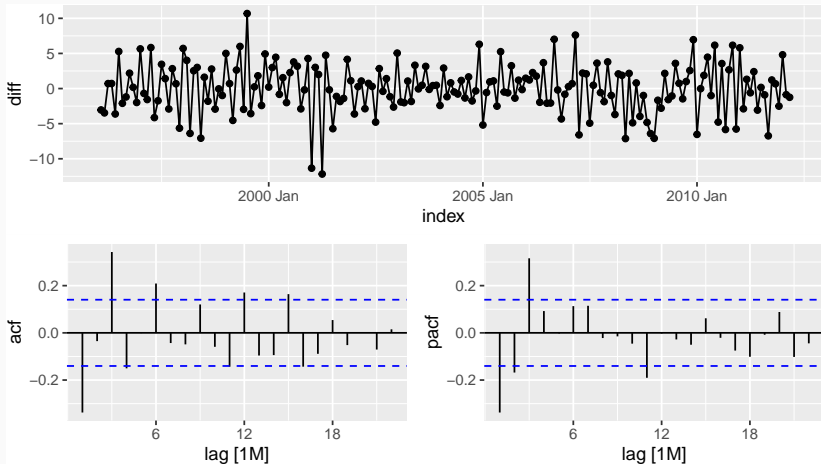


# Seasonally adjusted electrical equipment

- 1 Time plot shows sudden changes, particularly big drop in 2008/2009 due to global economic environment. Otherwise nothing unusual and no need for data adjustments.
- 2 No evidence of changing variance, so no Box-Cox transformation.
- 3 Data are clearly non-stationary, so we take first differences.

# Seasonally adjusted electrical equipment

```
dcmp %>% mutate(diff = difference(season_adjust)) %>%  
  gg_tsdisplay(diff, plot_type = 'partial')
```



# Seasonally adjusted electrical equipment

- 4 PACF is suggestive of AR(3). So initial candidate model is ARIMA(3,1,0). No other obvious candidates.
- 5 Fit ARIMA(3,1,0) model along with variations: ARIMA(4,1,0), ARIMA(2,1,0), ARIMA(3,1,1), etc. ARIMA(3,1,1) has smallest AICc value.



# Seasonally adjusted electrical equipment

```
fit <- dcmp %>%  
  model(arima = ARIMA(season_adjust))  
report(fit)
```

```
## Series: season_adjust  
## Model: ARIMA(3,1,0)  
##  
## Coefficients:  
##          ar1      ar2      ar3  
##      -0.3418  -0.0426   0.3185  
## s.e.   0.0681   0.0725   0.0682  
##  
## sigma^2 estimated as 9.639:  log likelihood=-494  
## AIC=996   AICc=996   BIC=1009
```

# Seasonally adjusted electrical equipment

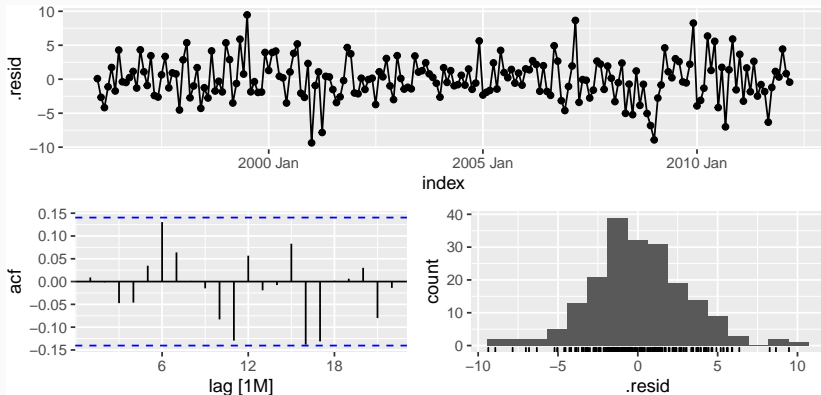
```
fit <- dcmp %>%  
  model(arima = ARIMA(season_adjust, approximation=FALSE))  
report(fit)
```

```
## Series: season_adjust  
## Model: ARIMA(3,1,1)  
##  
## Coefficients:  
##          ar1      ar2      ar3      ma1  
##      0.0044  0.0916  0.3698 -0.392  
## s.e.  0.2201  0.0984  0.0669  0.243  
##  
## sigma^2 estimated as 9.577:  log likelihood=-493  
## AIC=995   AICc=996   BIC=1012
```

# Seasonally adjusted electrical equipment

6

ACF plot of residuals from ARIMA(3,1,1) model look like white noise.



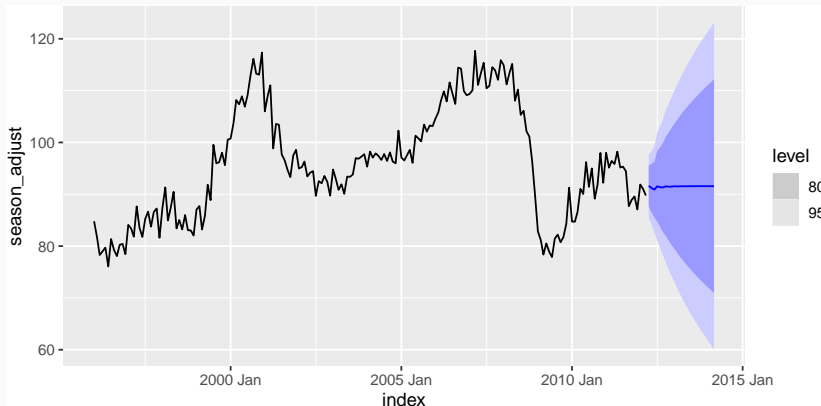
# Seasonally adjusted electrical equipment

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 24, dof = 4)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 arima      24.0      0.241
```

# Seasonally adjusted electrical equipment

```
fit %>% forecast %>% autoplot(dcmp)
```



# Outline

- 1 Stationarity and differencing
- 2 Non-seasonal ARIMA models
- 3 Estimation and order selection
- 4 ARIMA modelling in R
- 5 Forecasting
- 6 Seasonal ARIMA models
- 7 ARIMA vs ETS

# Point forecasts

- 1 Rearrange ARIMA equation so  $y_t$  is on LHS.
- 2 Rewrite equation by replacing  $t$  by  $T + h$ .
- 3 On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with  $h = 1$ . Repeat for  $h = 2, 3, \dots$

# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$



# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned} \left[ 1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4 \right] y_t \\ = (1 + \theta_1 B)\varepsilon_t, \end{aligned}$$

# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned} \left[ 1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4 \right] y_t \\ = (1 + \theta_1 B)\varepsilon_t, \end{aligned}$$

$$\begin{aligned} y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + \phi_3 y_{t-4} = \varepsilon_t + \theta_1 \varepsilon_{t-1}. \end{aligned}$$

# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned} \left[ 1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4 \right] y_t \\ = (1 + \theta_1 B)\varepsilon_t, \end{aligned}$$

$$\begin{aligned} y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + \phi_3 y_{t-4} = \varepsilon_t + \theta_1 \varepsilon_{t-1}. \end{aligned}$$

$$\begin{aligned} y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1 \varepsilon_{t-1}. \end{aligned}$$

# Point forecasts (h=1)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

# Point forecasts (h=1)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

## ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3y_{T-3} + \varepsilon_{T+1} + \theta_1\varepsilon_T.$$

# Point forecasts (h=1)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

## ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3y_{T-3} + \varepsilon_{T+1} + \theta_1\varepsilon_T.$$

## ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+1|T} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3y_{T-3} + \theta_1e_T.$$

## Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

# Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

## ARIMA(3,1,1) forecasts: Step 2

$$y_{T+2} = (1 + \phi_1)y_{T+1} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3y_{T-2} + \varepsilon_{T+2} + \theta_1\varepsilon_{T+1}.$$



# Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

## ARIMA(3,1,1) forecasts: Step 2

$$y_{T+2} = (1 + \phi_1)y_{T+1} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3y_{T-2} + \varepsilon_{T+2} + \theta_1\varepsilon_{T+1}.$$

## ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+2|T} = (1 + \phi_1)\hat{y}_{T+1|T} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3y_{T-2}.$$

# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where  $v_{T+h|T}$  is estimated forecast variance.

# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96 \sqrt{v_{T+h|T}}$$

where  $v_{T+h|T}$  is estimated forecast variance.

- $v_{T+1|T} = \hat{\sigma}^2$  for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[ 1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96 \sqrt{v_{T+h|T}}$$

where  $v_{T+h|T}$  is estimated forecast variance.

- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[ 1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where  $v_{T+h|T}$  is estimated forecast variance.

- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[ 1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

- AR(1): Rewrite as MA( $\infty$ ) and use above result.
- Other models beyond scope of this subject.

# Prediction intervals

- Prediction intervals **increase in size with forecast horizon.**
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed.**
- Prediction intervals tend to be too narrow.
  - ▶ the uncertainty in the parameter estimates has not been accounted for.
  - ▶ the ARIMA model assumes historical patterns will not change during the forecast period.
  - ▶ the ARIMA model assumes uncorrelated future errors

# Your turn

For the GDP data (from `global_economy`):

- fit a suitable ARIMA model to the logged data for all countries
- check the residual diagnostics for Australia;
- produce forecasts of your fitted model for Australia.

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# Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	↑	↑
	Non-seasonal part of the model	Seasonal part of of the model

where  $m$  = number of observations per year.

# Seasonal ARIMA models

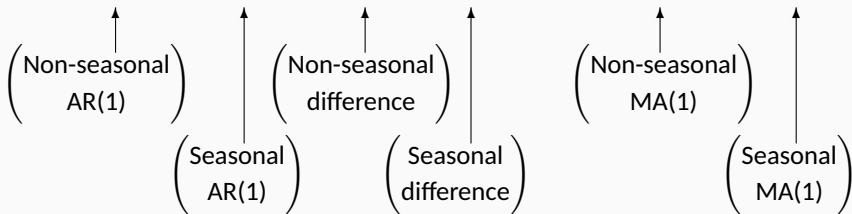
E.g.,  $\text{ARIMA}(1, 1, 1)(1, 1, 1)_4$  model (without constant)

# Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)<sub>4</sub> model (without constant)  
$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$

# Seasonal ARIMA models

E.g.,  $\text{ARIMA}(1, 1, 1)(1, 1, 1)_4$  model (without constant)  
 $(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$



# Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)<sub>4</sub> model (without constant)  
 $(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t = & (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + (1 + \Phi_1)y_{t-4} \\ & - (1 + \phi_1 + \Phi_1 + \phi_1 \Phi_1)y_{t-5} + (\phi_1 + \phi_1 \Phi_1)y_{t-6} \\ & - \Phi_1 y_{t-8} + (\Phi_1 + \phi_1 \Phi_1)y_{t-9} - \phi_1 \Phi_1 y_{t-10} \\ & + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \Theta_1 \varepsilon_{t-4} + \theta_1 \Theta_1 \varepsilon_{t-5}. \end{aligned}$$

# Common ARIMA models

The US Census Bureau uses the following models most often:

ARIMA(0,1,1)(0,1,1) <sub>m</sub>	with log transformation
ARIMA(0,1,2)(0,1,1) <sub>m</sub>	with log transformation
ARIMA(2,1,0)(0,1,1) <sub>m</sub>	with log transformation
ARIMA(0,2,2)(0,1,1) <sub>m</sub>	with log transformation
ARIMA(2,1,2)(0,1,1) <sub>m</sub>	with no transformation

# Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)<sub>12</sub> will show:**

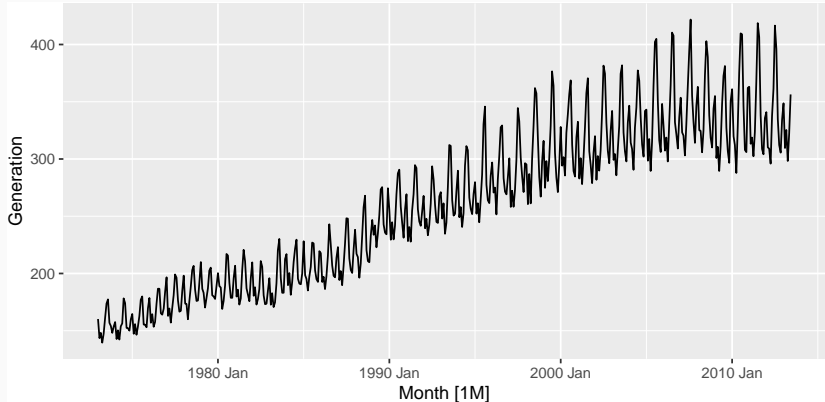
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ....

**ARIMA(0,0,0)(1,0,0)<sub>12</sub> will show:**

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

# Example: US electricity production

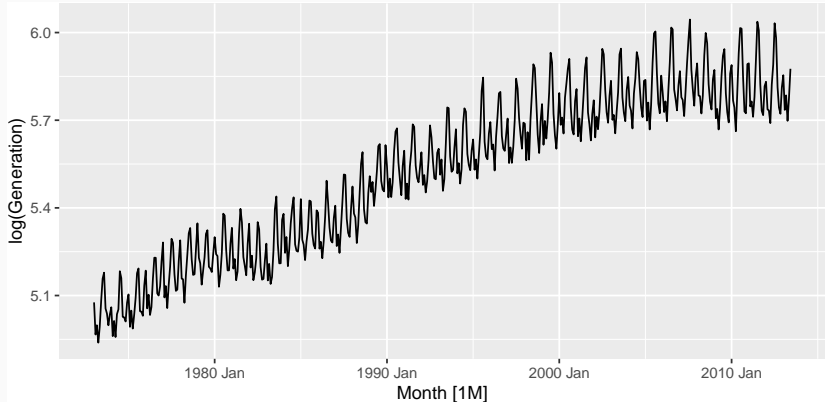
```
usmelec %>% autoplot(  
  Generation  
)
```





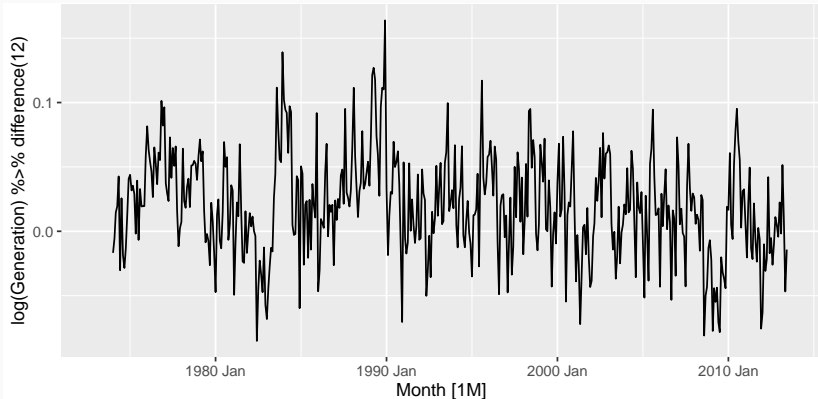
# Example: US electricity production

```
usmelec %>% autoplot(  
  log(Generation)  
)
```



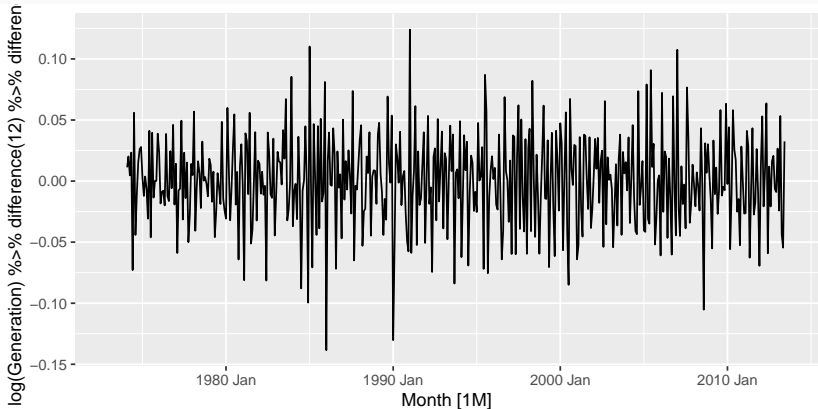
# Example: US electricity production

```
usmelec %>% autoplot(  
  log(Generation) %>% difference(12)  
)
```



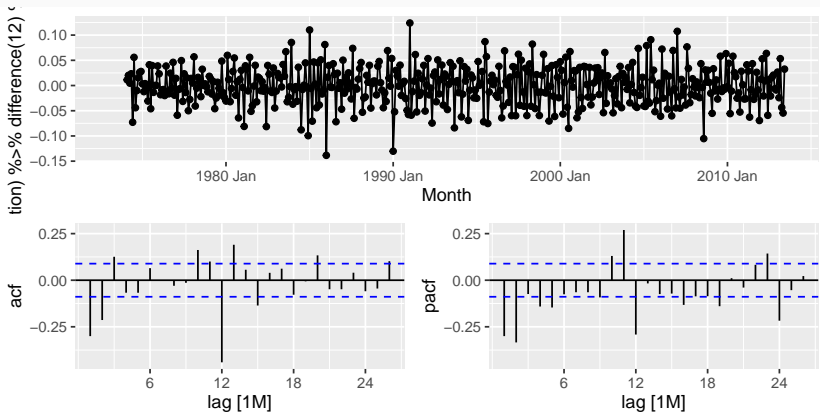
# Example: US electricity production

```
usmelec %>% autoplot(  
  log(Generation) %>% difference(12) %>% difference()  
)
```



# Example: US electricity production

```
usmelec %>% gg_tsdisplay(  
  log(Generation) %>% difference(12) %>% difference(),  
  plot_type = "partial")
```



## Example: US electricity production

- $d = 1$  and  $D = 1$  seems necessary
- $P = 0$  and  $Q = 1$  suggested by seasonal lags
- $p = 0$  and  $q = 3$  suggested by non-seasonal lags.

# Example: US electricity production

```
usmelec %>%  
  model(arima = ARIMA(log(Generation) ~ pdq(0,1,3) + PDQ(0,1,1))) %>%  
  report()
```

```
## Series: Generation  
## Model: ARIMA(0,1,3)(0,1,1)[12]  
## Transformation: log(Generation)  
##  
## Coefficients:  
##           ma1           ma2           ma3           sma1  
##      -0.4266   -0.2496   -0.0439   -0.8358  
## s.e.    0.0462    0.0516    0.0430    0.0262  
##  
## sigma^2 estimated as 0.0006904:  log likelihood=1045  
## AIC=-2080   AICc=-2080   BIC=-2059
```

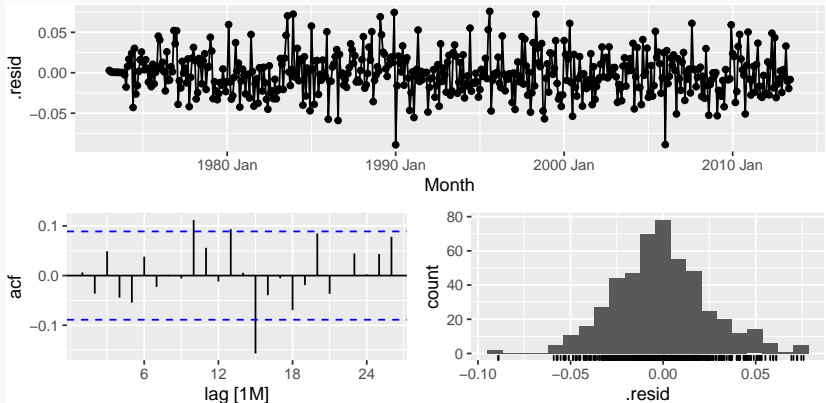
# Example: US electricity production

```
usmelec %>%  
  model(arima = ARIMA(log(Generation))) %>%  
  report()
```

```
## Series: Generation  
## Model: ARIMA(1,1,1)(2,1,1)[12]  
## Transformation: log(Generation)  
##  
## Coefficients:  
##          ar1          ma1          sar1          sar2          sma1  
##          0.4116   -0.8483   0.0100   -0.1017   -0.8204  
## s.e.    0.0617    0.0348   0.0561    0.0529    0.0357  
##  
## sigma^2 estimated as 0.0006841:  log likelihood=1047  
## AIC=-2082   AICc=-2082   BIC=-2057
```

# Example: US electricity production

```
fit <- usmelec %>%  
  model(arima = ARIMA(log(Generation)))  
gg_tsresiduals(fit)
```





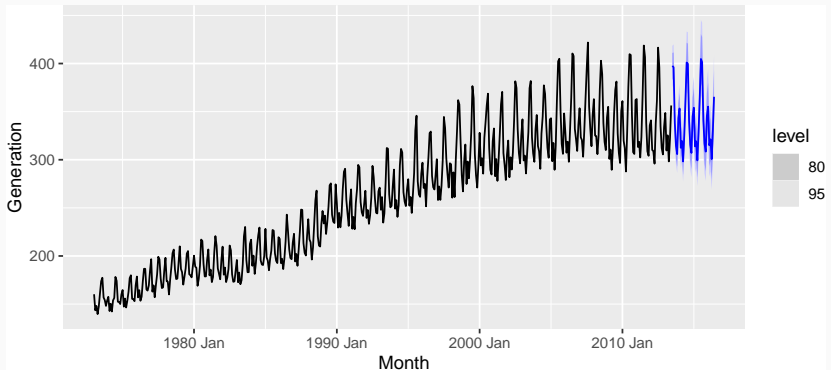
# Example: US electricity production

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 24, dof = 5)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 arima      44.5    0.000799
```

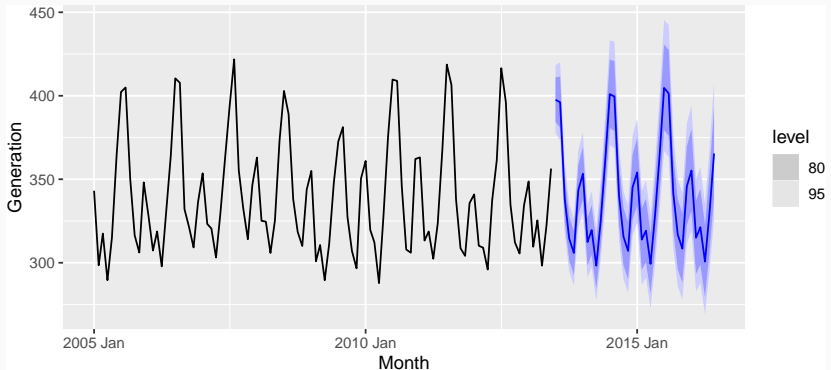
# Example: US electricity production

```
usmelec %>%  
  model(arima = ARIMA(log(Generation))) %>%  
  forecast(h = "3 years") %>%  
  autoplot(usmelec)
```



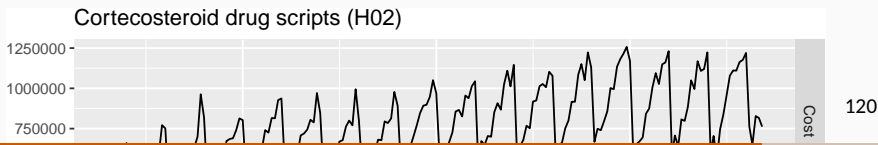
# Example: US electricity production

```
usmelec %>%  
  model(arima = ARIMA(log(Generation))) %>%  
  forecast(h = "3 years") %>%  
  autoplot(filter(usmelec, year(Month) >= 2005))
```



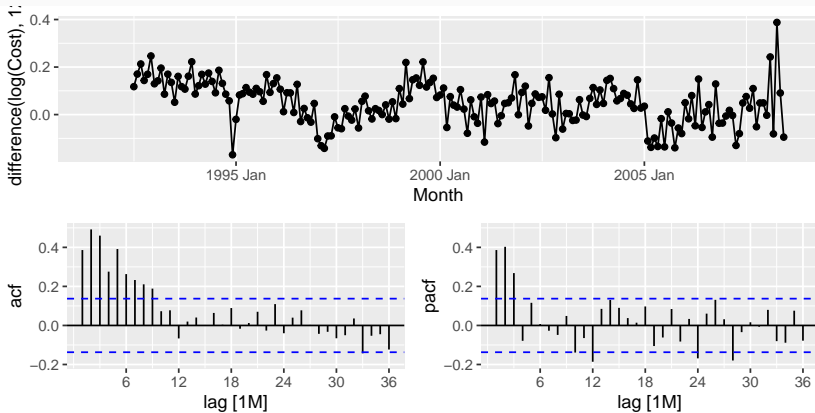
# Corticosteroid drug sales

```
h02 %>%  
  mutate(log(Cost)) %>%  
  gather() %>%  
  ggplot(aes(x = Month, y = value)) +  
  geom_line() +  
  facet_grid(key ~ ., scales = "free_y") +  
  xlab("Year") + ylab("") +  
  ggtitle("Corticosteroid drug scripts (H02)")
```



# Corticosteroid drug sales

```
h02 %>% gg_tsdisplay(difference(log(Cost),12),
```



# Corticosteroid drug sales

- Choose  $D = 1$  and  $d = 0$ .
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggests possible non-seasonal AR(3) term.
- Initial candidate model:  $\text{ARIMA}(3,0,0)(2,1,0)_{12}$ .

## Corticosteroid drug sales

.model	AICc
ARIMA(3,0,1)(0,1,2)[12]	-485
ARIMA(3,0,1)(1,1,1)[12]	-484
ARIMA(3,0,1)(0,1,1)[12]	-484
ARIMA(3,0,1)(2,1,0)[12]	-476
ARIMA(3,0,0)(2,1,0)[12]	-475
ARIMA(3,0,2)(2,1,0)[12]	-475
ARIMA(3,0,1)(1,1,0)[12]	-463

# Corticosteroid drug sales

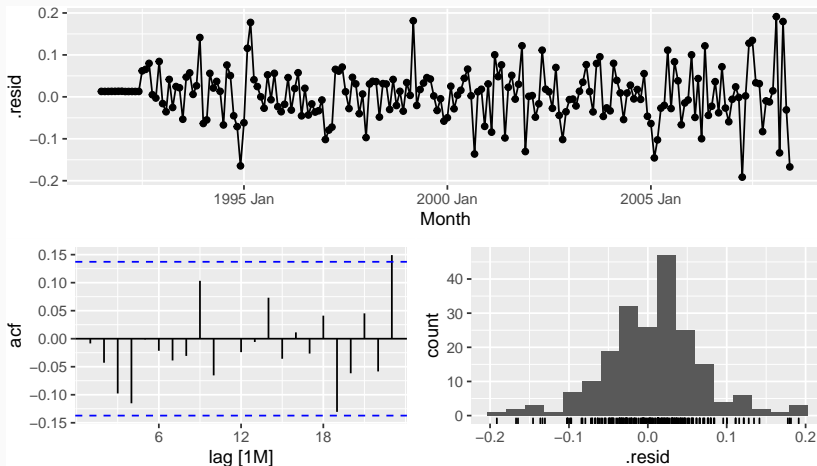
```
fit <- h02 %>%  
  model(best = ARIMA(log(Cost) ~ 0 + pdq(3,0,1) + PDQ(0,1,2)))  
report(fit)
```

```
## Series: Cost  
## Model: ARIMA(3,0,1)(0,1,2)[12]  
## Transformation: log(Cost)  
##  
## Coefficients:  
##          ar1      ar2      ar3      ma1      sma1      sma2  
##      -0.160  0.5481  0.5678  0.383  -0.5222  -0.1769  
## s.e.   0.164  0.0878  0.0942  0.190   0.0861   0.0872  
##  
## sigma^2 estimated as 0.004289:  log likelihood=250  
## AIC=-486   AICc=-485   BIC=-463
```



# Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



# Corticosteroid drug sales

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 36, dof = 6)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 best      58.0    0.00161
```

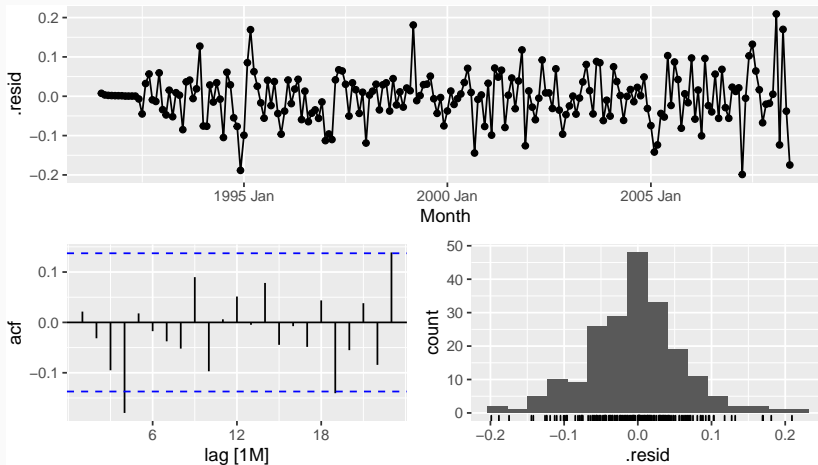
# Corticosteroid drug sales

```
fit <- h02 %>% model(auto = ARIMA(log(Cost)))  
report(fit)
```

```
## Series: Cost  
## Model: ARIMA(2,1,0)(0,1,1)[12]  
## Transformation: log(Cost)  
##  
## Coefficients:  
##           ar1      ar2      sma1  
##      -0.8491  -0.4207  -0.6401  
## s.e.   0.0712   0.0714   0.0694  
##  
## sigma^2 estimated as 0.004399:  log likelihood=245  
## AIC=-483   AICc=-483   BIC=-470
```

# Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



# Corticosteroid drug sales

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 36, dof = 3)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 auto      63.8    0.00102
```

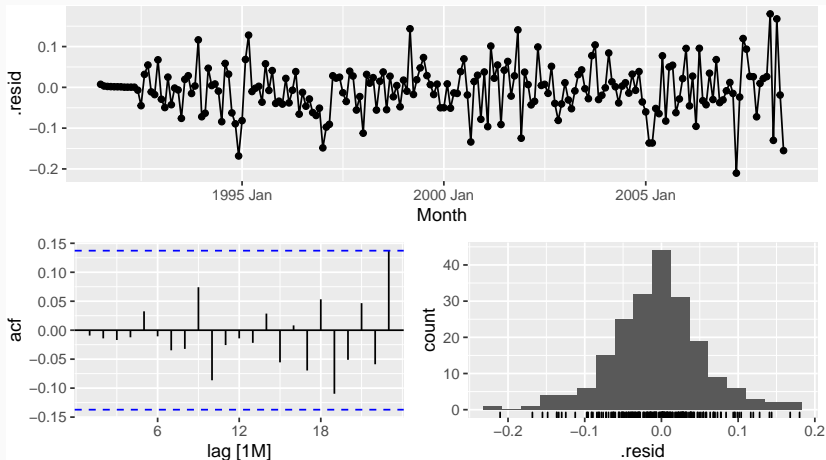
# Corticosteroid drug sales

```
fit <- h02 %>%  
  model(best = ARIMA(log(Cost), stepwise = FALSE,  
    approximation = FALSE,  
    order_constraint = p + q + P + Q <= 9))  
report(fit)
```

```
## Series: Cost  
## Model: ARIMA(4,1,1)(2,1,2)[12]  
## Transformation: log(Cost)  
##  
## Coefficients:  
##          ar1      ar2      ar3      ar4      ma1      sar1      sar2  
##      -0.0426  0.210  0.202  -0.227  -0.742  0.621  -0.383  
## s.e.   0.2167  0.181  0.114   0.081   0.207  0.242   0.118  
##          sma1     sma2  
##      -1.202  0.496  
## s.e.   0.249  0.214  
##  
## sigma^2 estimated as 0.004061:  log likelihood=254  
## AIC=-489   AICc=-487   BIC=-456
```

# Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



# Corticosteroid drug sales

```
augment(fit) %>%  
  features(.resid, ljung_box, lag = 36, dof = 9)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 best      44.1      0.0203
```



# Corticosteroid drug sales

Training data: July 1991 to June 2006

Test data: July 2006–June 2008

```
fit <- h02 %>%  
  filter_index(~ "2006 Jun") %>%  
  model(  
    ARIMA(log(Cost) ~ pdq(3, 0, 0) + PDQ(2, 1, 0)),  
    ARIMA(log(Cost) ~ pdq(3, 0, 1) + PDQ(2, 1, 0)),  
    ARIMA(log(Cost) ~ pdq(3, 0, 2) + PDQ(2, 1, 0)),  
    ARIMA(log(Cost) ~ pdq(3, 0, 1) + PDQ(1, 1, 0))  
    # ... #  
  )  
  
fit %>%  
  forecast(h = "2 years") %>%  
  accuracy(h02)
```

# Corticosteroid drug sales

```
models <- list(  
  c(3,0,1,0,1,2),  
  c(3,0,1,1,1,1),  
  c(3,0,1,0,1,1),  
  c(3,0,1,2,1,0),  
  c(3,0,0,2,1,0),  
  c(3,0,2,2,1,0),  
  c(3,0,1,1,1,0),  
  c(2,1,0,0,1,1),  
  c(4,1,1,2,1,2))
```

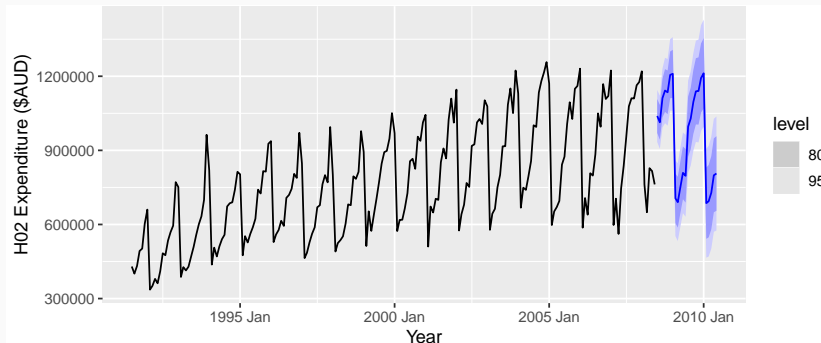
```
model_defs <- map(models, ~ ARIMA(log(Cost) ~ 0 + pdq(!  
model_defs <- set_names(model_defs, map_chr(models, 134  
  ~ sprintf("ARIMA(%i.%i.%i)(%i.%i.%i)[12]" . . [1] . . [2]
```

# Corticosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing. But RMSE test set comparisons can involve any models.
- Use the best model available, even if it does not pass all tests.

# Corticosteroid drug sales

```
fit <- h02 %>%  
  model(ARIMA(Cost ~ 0 + pdq(3,0,1) + PDQ(0,1,2)))  
fit %>% forecast %>% autoplot(h02) +  
  ylab("H02 Expenditure ($AUD)") + xlab("Year")
```



# Outline

- 1 Stationarity and differencing
- 2 Non-seasonal ARIMA models
- 3 Estimation and order selection
- 4 ARIMA modelling in R
- 5 Forecasting
- 6 Seasonal ARIMA models
- 7 ARIMA vs ETS

# ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

# ARIMA vs ETS

```
library(latex2exp)
cols = c(ets = "#e41a1c", arima = "#377eb8")
tibble(
  x = c(-0.866, 0.866),
  y = c(-0.5, -0.5),
  labels = c("ets", "arima"),
) %>%
  ggplot(aes(color = labels, fill=labels)) +
  ggforce::geom_circle(aes(x0 = x, y0 = y, r =
  scale_color_manual(values=cols) + scale_fill
  coord_fixed() + guides(fill = FALSE) +
```

# Equivalences

ETS model	ARIMA model	Parameters
ETS(A,N,N)	ARIMA(0,1,1)	$\theta_1 = \alpha - 1$
ETS(A,A,N)	ARIMA(0,2,2)	$\theta_1 = \alpha + \beta - 2$ $\theta_2 = 1 - \alpha$
ETS(A,A <sub>d</sub> ,N)	ARIMA(1,1,2)	$\phi_1 = \phi$ $\theta_1 = \alpha + \phi\beta - 1 - \phi$ $\theta_2 = (1 - \alpha)\phi$
ETS(A,N,A)	ARIMA(0,0,m)(0,1,0) <sub>m</sub>	
ETS(A,A,A)	ARIMA(0,1,m + 1)(0,1,0) <sub>m</sub>	
ETS(A,A <sub>d</sub> ,A)	ARIMA(1,0,m + 1)(0,1,0) <sub>m</sub>	