

# ETC3550/ETC5550

## Applied forecasting

Ch5. The forecasters' toolbox

[OTexts.org/fpp3/](https://OTexts.org/fpp3/)



# Outline

- 1 A tidy forecasting workflow
- 2 Some simple forecasting methods
- 3 Residual diagnostics
- 4 Distributional forecasts and prediction intervals
- 5 Forecasting with transformations
- 6 Forecasting and decomposition
- 7 Evaluating forecast accuracy
- 8 Time series cross validation

# Outline

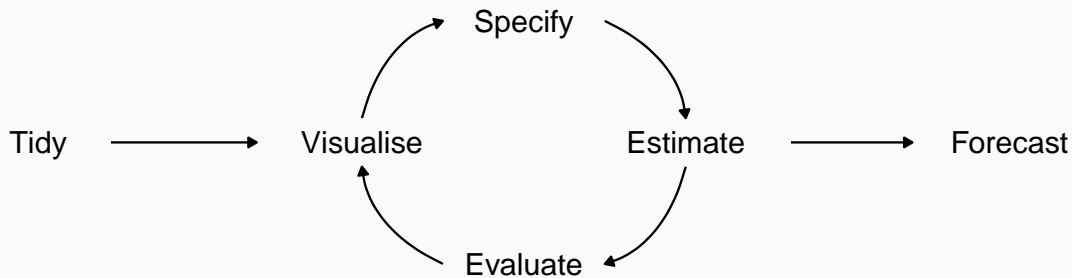
- 1 A tidy forecasting workflow
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# A tidy forecasting workflow

The process of producing forecasts can be split up into a few fundamental steps.

- 1 Preparing data
- 2 Data visualisation
- 3 Specifying a model
- 4 Model estimation
- 5 Accuracy & performance evaluation
- 6 Producing forecasts

# A tidy forecasting workflow



# Data preparation (tidy)

```
gdppc <- global_economy %>%  
  mutate(GDP_per_capita = GDP/Population) %>%  
  select(Year, Country, GDP, Population, GDP_per_capita)  
gdppc
```

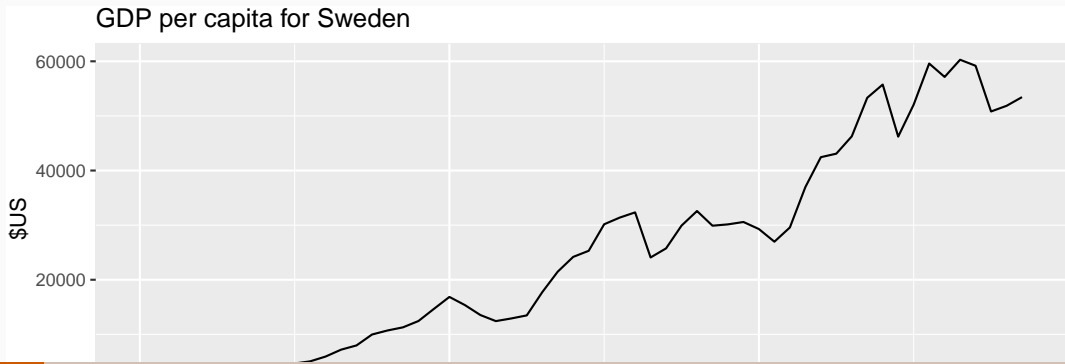
```
## # A tsibble: 15,150 x 5 [1Y]
```

```
## # Key:      Country [263]
```

##	Year	Country	GDP	Population	GDP_per_capita	
##	<dbl>	<fct>	<dbl>	<dbl>	<dbl>	
##	1	1960	Afghanistan	537777811.	8996351	59.8
##	2	1961	Afghanistan	548888896.	9166764	59.9
##	3	1962	Afghanistan	546666678.	9345868	58.5
##	4	1963	Afghanistan	751111191.	9533954	78.8
##	5	1964	Afghanistan	800000044.	9731361	82.2

# Data visualisation

```
gdppc %>%  
  filter(Country=="Sweden") %>%  
  autoplot(GDP_per_capita) +  
    labs(title = "GDP per capita for Sweden", y = "$US")
```



# Model estimation

The `model()` function trains models to data.

```
fit <- gdppc %>%  
  model(trend_model = TSLM(GDP_per_capita ~ trend()))  
fit
```

```
## # A mable: 263 x 2  
## # Key:      Country [263]  
##   Country      trend_model  
##   <fct>         <model>  
## 1 Afghanistan <TSLM>  
## 2 Albania      <TSLM>  
## 3 Algeria      <TSLM>  
## 4 American Samoa <TSLM>
```



# Model estimation

The `model()` function trains models to data.

```
fit <- gdppc %>%  
  model(trend_model = TSLM(GDP_per_capita ~ trend()))  
fit
```

```
## # A mable: 263 x 2  
## # Key:      Country [263]  
##   Country      trend_model  
##   <fct>         <model>  
## 1 Afghanistan <TSLM>  
## 2 Albania      <TSLM>  
## 3 Algeria      <TSLM>  
## 4 American Samoa <TSLM>
```

# Producing forecasts

```
fit %>% forecast(h = "3 years")
```

```
## # A tibble: 789 x 5 [1Y]
```

```
## # Key:      Country, .model [263]
```

##	Country	.model	Year	GDP_per_capita	.mean
##	<fct>	<chr>	<dbl>	<dist>	<dbl>
##	1 Afghanistan	trend_model	2018	N(526, 9653)	526.
##	2 Afghanistan	trend_model	2019	N(534, 9689)	534.
##	3 Afghanistan	trend_model	2020	N(542, 9727)	542.
##	4 Albania	trend_model	2018	N(4716, 476419)	4716.
##	5 Albania	trend_model	2019	N(4867, 481086)	4867.
##	6 Albania	trend_model	2020	N(5018, 486012)	5018.
##	7 Algeria	trend_model	2018	N(4410, 643094)	4410.
##	8 Algeria	trend_model	2019	N(4489, 645311)	4489.

# Producing forecasts

```
fit %>% forecast(h = "3 years")
```

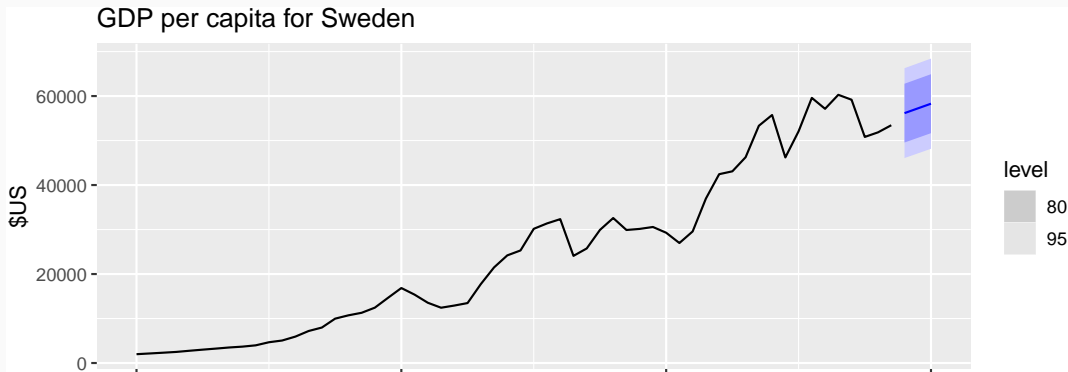
```
## # A tibble: 789 x 5 [1Y]
```

```
## # Key:      Country, .model [263]
```

##	Country	.model	Year	GDP_per_capita	.mean
##	<fct>	<chr>	<dbl>	<dist>	<dbl>
##	1 Afghanistan	trend_model	2018	N(526, 9653)	526.
##	2 Afghanistan	trend_model	2019	N(534, 9689)	534.
##	3 Afghanistan	trend_model	2020	N(542, 9727)	542.
##	4 Albania	trend_model	2018	N(4716, 476419)	4716.
##	5 Albania	trend_model	2019	N(4867, 481086)	4867.
##	6 Albania	trend_model	2020	N(5018, 486012)	5018.
##	7 Algeria	trend_model	2018	N(4410, 643094)	4410.
##	8 Algeria	trend_model	2019	N(4489, 645211)	4489.

# Visualising forecasts

```
fit %>% forecast(h = "3 years") %>%  
  filter(Country=="Sweden") %>%  
  autoplot(gdppc) +  
    labs(title = "GDP per capita for Sweden", y = "$US")
```



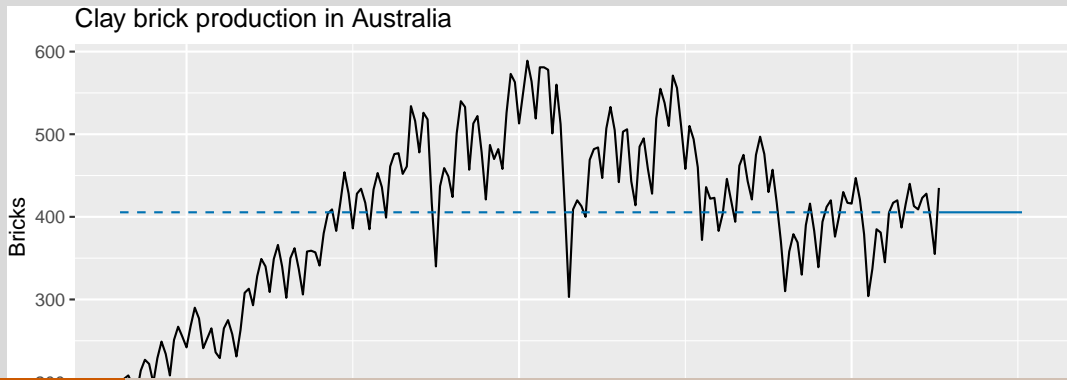
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# Some simple forecasting methods

## MEAN( $y$ ): Average method

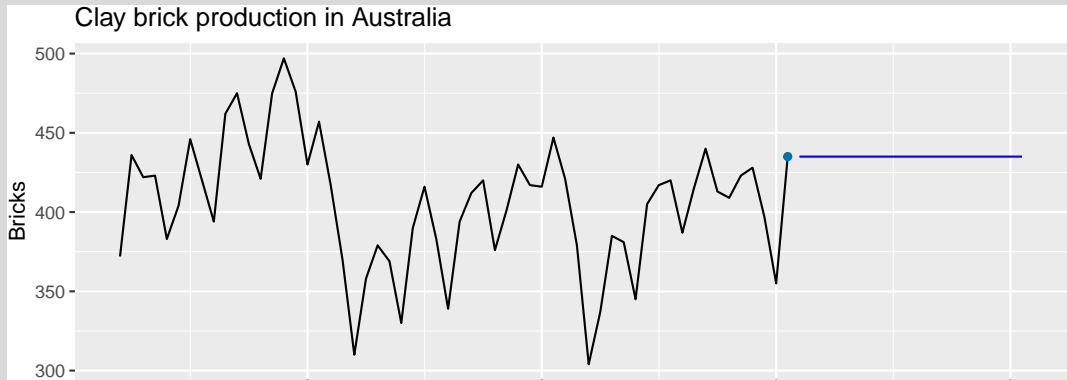
- Forecast of all future values is equal to mean of historical data  $\{y_1, \dots, y_T\}$ .
- Forecasts:  $\hat{y}_{T+h|T} = \bar{y} = (y_1 + \dots + y_T)/T$



# Some simple forecasting methods

## NAIVE( $y$ ): Naïve method

- Forecasts equal to last observed value.
- Forecasts:  $\hat{y}_{T+h|T} = y_T$ .
- Consequence of efficient market hypothesis.

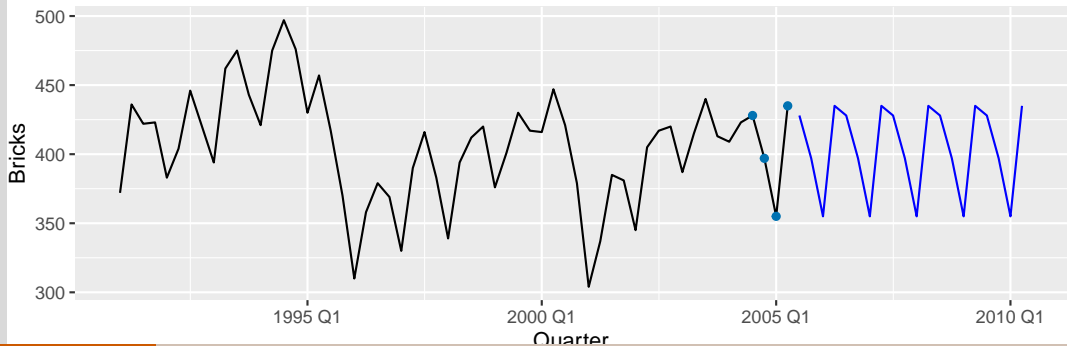


# Some simple forecasting methods

## SNAIVE( $y \sim \text{lag}(m)$ ): Seasonal naïve method

- Forecasts equal to last value from same season.
- Forecasts:  $\hat{y}_{T+h|T} = y_{T+h-m(k+1)}$ , where  $m$  = seasonal period and  $k$  is the integer part of  $(h-1)/m$ .

Clay brick production in Australia





# Some simple forecasting methods

## `RW(y ~ drift())`: Drift method

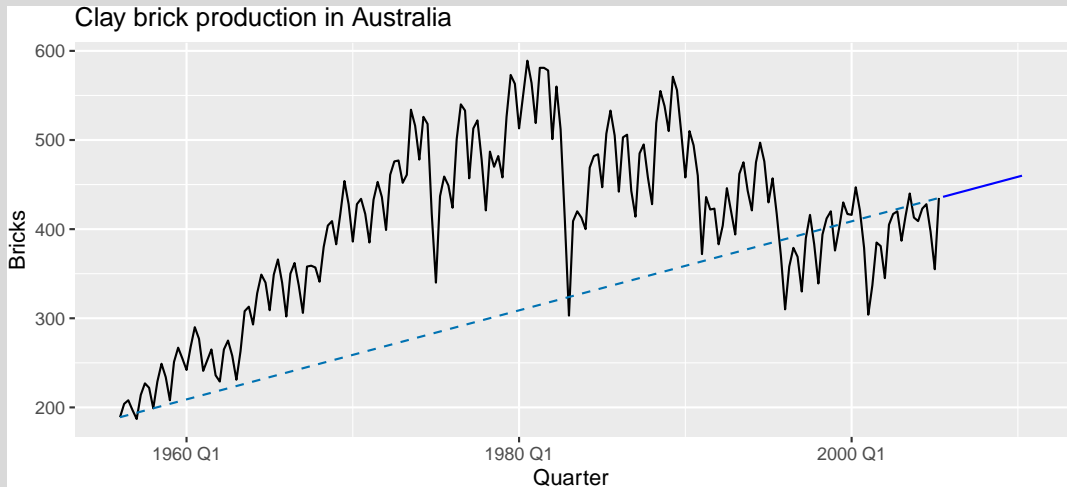
- Forecasts equal to last value plus average change.
- Forecasts:

$$\begin{aligned}\hat{y}_{T+h|T} &= y_T + \frac{h}{T-1} \sum_{t=2}^T (y_t - y_{t-1}) \\ &= y_T + \frac{h}{T-1} (y_T - y_1).\end{aligned}$$

- Equivalent to extrapolating a line drawn between first and last observations.

# Some simple forecasting methods

## Drift method



# Model fitting

The `model()` function trains models to data.

```
brick_fit <- aus_production %>%  
  filter(!is.na(Bricks)) %>%  
  model(  
    Seasonal_naive = SNAIVE(Bricks),  
    Naive = NAIVE(Bricks),  
    Drift = RW(Bricks ~ drift()),  
    Mean = MEAN(Bricks)  
  )
```

```
## # A mable: 1 x 4  
##   Seasonal_naive   Naive      Drift    Mean  
##           <model> <model>    <model> <model>  
## 1           <SNAIVE> <NAIVE> <RW w/ drift> <MEAN>
```

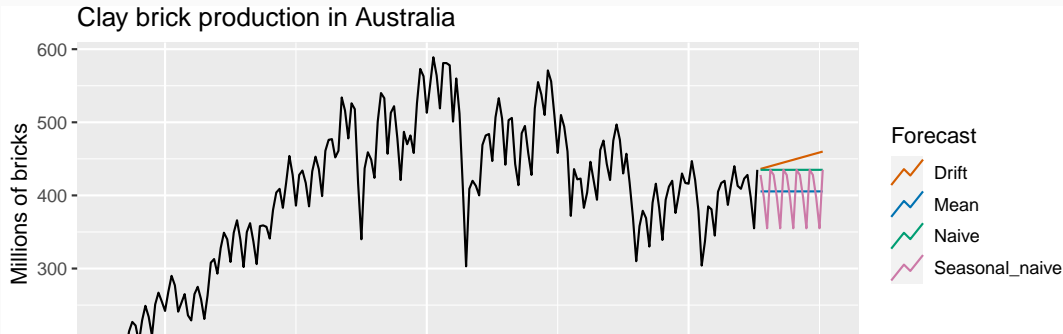
# Producing forecasts

```
brick_fc <- brick_fit %>%  
  forecast(h = "5 years")
```

```
## # A fable: 80 x 4 [1Q]  
## # Key:      .model [4]  
##   .model      Quarter      Bricks .mean  
##   <chr>        <qtr>        <dist> <dbl>  
## 1 Seasonal_naive 2005 Q3 N(428, 2336) 428  
## 2 Seasonal_naive 2005 Q4 N(397, 2336) 397  
## 3 Seasonal_naive 2006 Q1 N(355, 2336) 355  
## 4 Seasonal_naive 2006 Q2 N(435, 2336) 435  
## # ... with 76 more rows
```

# Visualising forecasts

```
brick_fc %>%  
  autoplot(aus_production, level = NULL) +  
  labs(title = "Clay brick production in Australia",  
       y = "Millions of bricks") +  
  guides(colour = guide_legend(title = "Forecast"))
```



# Facebook closing stock price

```
# Extract training data
fb_stock <- gafa_stock %>%
  filter(Symbol == "FB") %>%
  mutate(trading_day = row_number()) %>%
  update_tsibble(index=trading_day, regular=TRUE)

# Specify, estimate and forecast
fb_stock %>%
  model(
    Mean = MEAN(Close),
    Naive = NAIVE(Close),
    Drift = RW(Close ~ drift())
  ) %>%
  forecast(h=42) %>%
  autoplot(fb_stock, level = NULL) +
  labs(title = "Facebook closing stock price", y="$US") +
  guides(colour=guide_legend(title="Forecast"))
```

# Facebook closing stock price



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# Fitted values

- $\hat{y}_{t|t-1}$  is the forecast of  $y_t$  based on observations  $y_1, \dots, y_{t-1}$ .
- We call these “fitted values”.
- Sometimes drop the subscript:  $\hat{y}_t \equiv \hat{y}_{t|t-1}$ .
- Often not true forecasts since parameters are estimated on all data.

## For example:

- $\hat{y}_t = \bar{y}$  for average method.
- $\hat{y}_t = y_{t-1} + (y_T - y_1)/(T - 1)$  for drift method.

# Forecasting residuals

**Residuals in forecasting:** difference between observed value and its fitted value:  $e_t = y_t - \hat{y}_{t|t-1}$ .

# Forecasting residuals

**Residuals in forecasting:** difference between observed value and its fitted value:  $e_t = y_t - \hat{y}_{t|t-1}$ .

## Assumptions

- 1  $\{e_t\}$  uncorrelated. If they aren't, then information left in residuals that should be used in computing forecasts.
- 2  $\{e_t\}$  have mean zero. If they don't, then forecasts are biased.

# Forecasting residuals

**Residuals in forecasting:** difference between observed value and its fitted value:  $e_t = y_t - \hat{y}_{t|t-1}$ .

## Assumptions

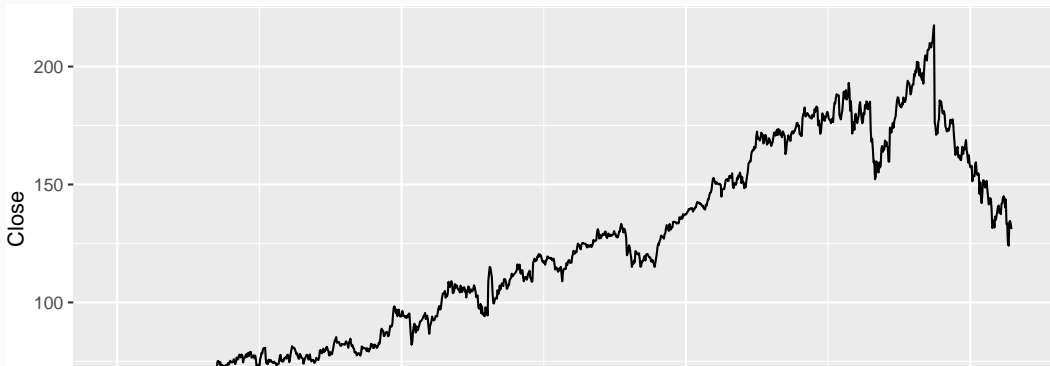
- 1  $\{e_t\}$  uncorrelated. If they aren't, then information left in residuals that should be used in computing forecasts.
- 2  $\{e_t\}$  have mean zero. If they don't, then forecasts are biased.

## Useful properties (for distributions & prediction intervals)

- 3  $\{e_t\}$  have constant variance.
- 4  $\{e_t\}$  are normally distributed.

# Facebook closing stock price

```
fb_stock <- gafa_stock %>%  
  filter(Symbol == "FB") %>%  
  mutate(trading_day = row_number()) %>%  
  update_tsibble(index = trading_day, regular = TRUE)  
fb_stock %>% autoplot(Close)
```



# Facebook closing stock price

```
fit <- fb_stock %>% model(NAIVE(Close))
augment(fit)
```

```
## # A tsibble: 1,258 x 7 [1]
## # Key:      Symbol, .model [1]
##   Symbol .model      trading_day Close .fitted .resid .innov
##   <chr>   <chr>          <int> <dbl>   <dbl>   <dbl>   <dbl>
## 1 FB     NAIVE(Close)         1  54.7    NA     NA     NA
## 2 FB     NAIVE(Close)         2  54.6    54.7 -0.150 -0.150
## 3 FB     NAIVE(Close)         3  57.2    54.6  2.64   2.64
## 4 FB     NAIVE(Close)         4  57.9    57.2  0.720  0.720
## 5 FB     NAIVE(Close)         5  58.2    57.9  0.310  0.310
## 6 FB     NAIVE(Close)         6  57.2    58.2 -1.01  -1.01
## 7 FB     NAIVE(Close)         7  57.9    57.2  0.720  0.720
## 8 FB     NAIVE(Close)         8  55.9    57.9 -2.03  -2.03
## 9 FB     NAIVE(Close)         9  57.7    55.9  1.83   1.83
## 10 FB    NAIVE(Close)        10  57.6    57.7 -0.140 -0.140
## # ... with 1,248 more rows
```

# Facebook closing stock price

```
fit <- fb_stock %>% model(NAIVE(Close))  
augment(fit)
```

```
## # A tsibble: 1,258 x 7 [1]  
## # Key:      Symbol, .model [1]  
##   Symbol .model      trading_day Close .fitted .resid .innov  
##   <chr>   <chr>          <int> <dbl>   <dbl>   <dbl>   <dbl>  
## 1 FB     NAIVE(Close)         1  54.7    NA     NA     NA  
## 2 FB     NAIVE(Close)         2  54.6    54.7  -0.150 -0.150  
## 3 FB     NAIVE(Close)         3  57.2    54.6   2.64   2.64  
## 4 FB     NAIVE(Close)         4  57.9    57.2   0.720  0.720  
## 5 FB     NAIVE(Close)         5  58.2    57.9   0.310  0.310  
## 6 FB     NAIVE(Close)         6  57.2    58.2  -1.01  -1.01  
## 7         NAIVE(Close)         7  57.9    57.2   0.720  0.720  
## 8         NAIVE(Close)         8  55.9    57.9  -2.03  -2.03  
## 9         NAIVE(Close)         9  57.7    55.9   1.83   1.83  
## 10        NAIVE(Close)        10  57.6    57.7  -0.140 -0.140
```

 $\hat{y}_{t|t-1}$  $e_t$ 

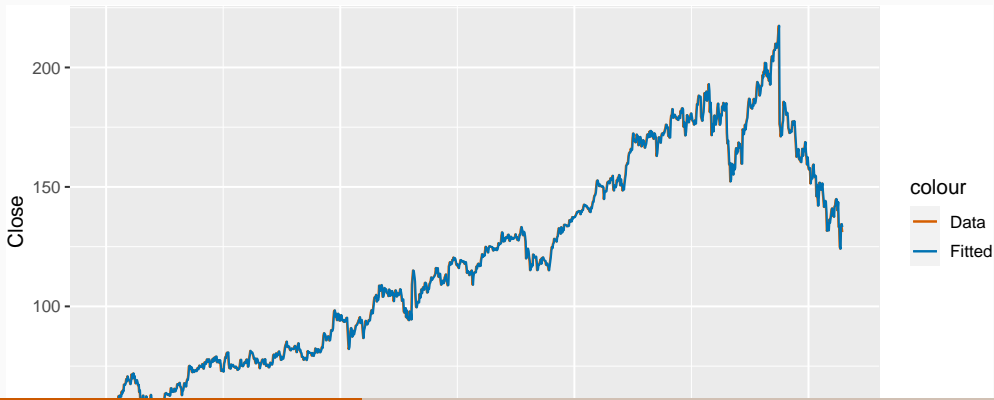
## Naïve forecasts:

$$\hat{y}_{t|t-1} = y_{t-1}$$

$$e_t = y_t - \hat{y}_{t|t-1} = y_t - y_{t-1}$$

# Facebook closing stock price

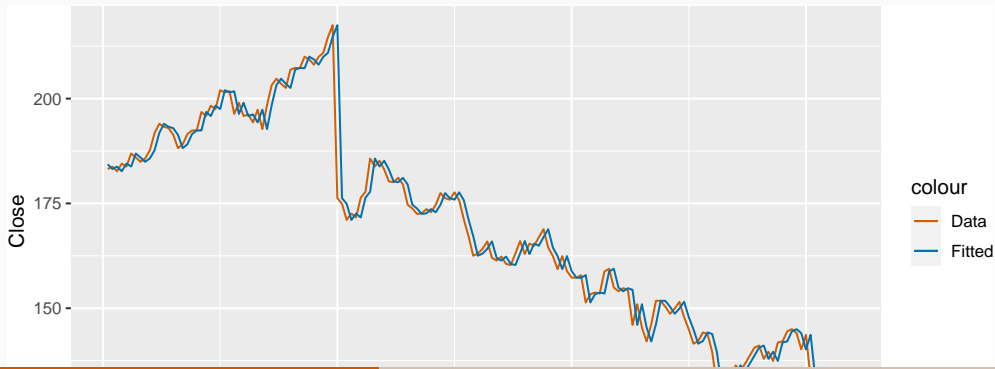
```
augment(fit) %>%  
  ggplot(aes(x = trading_day)) +  
  geom_line(aes(y = Close, colour = "Data")) +  
  geom_line(aes(y = .fitted, colour = "Fitted"))
```





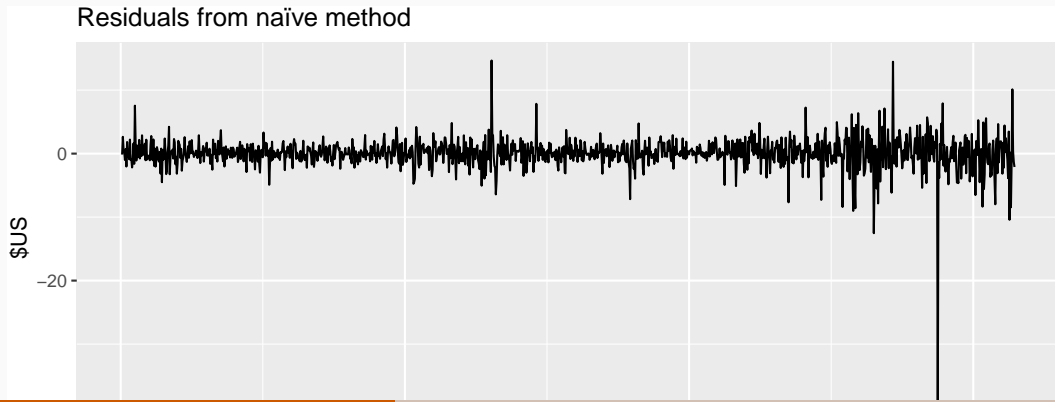
# Facebook closing stock price

```
augment(fit) %>%  
  filter(trading_day > 1100) %>%  
  ggplot(aes(x = trading_day)) +  
  geom_line(aes(y = Close, colour = "Data")) +  
  geom_line(aes(y = .fitted, colour = "Fitted"))
```



# Facebook closing stock price

```
augment(fit) %>%  
  autoplot(.resid) +  
  labs(y = "$US",  
       title = "Residuals from naïve method")
```



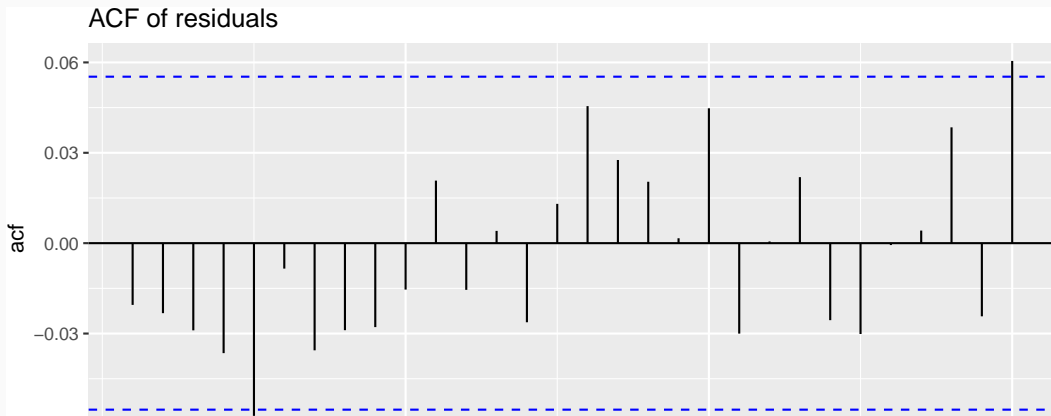
# Facebook closing stock price

```
augment(fit) %>%  
  ggplot(aes(x = .resid)) +  
  geom_histogram(bins = 150) +  
  labs(title = "Histogram of residuals")
```



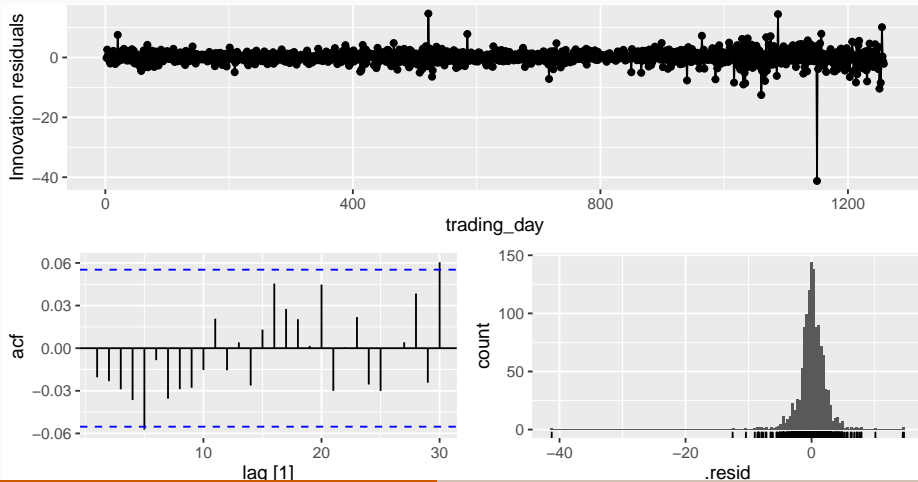
# Facebook closing stock price

```
augment(fit) %>%  
  ACF(.resid) %>%  
  autoplot() + labs(title = "ACF of residuals")
```



# gg\_tsresiduals() function

```
gg_tsresiduals(fit)
```



## ACF of residuals

- We assume that the residuals are white noise (uncorrelated, mean zero, constant variance). If they aren't, then there is information left in the residuals that should be used in computing forecasts.
- So a standard residual diagnostic is to check the ACF of the residuals of a forecasting method.
- We *expect* these to look like white noise.

# Portmanteau tests

Consider a *whole set* of  $r_k$  values, and develop a test to see whether the set is significantly different from a zero set.

# Portmanteau tests

Consider a *whole set* of  $r_k$  values, and develop a test to see whether the set is significantly different from a zero set.

## Box-Pierce test

$$Q = T \sum_{k=1}^{\ell} r_k^2$$

where  $\ell$  is max lag being considered and  $T$  is number of observations.

- If each  $r_k$  close to zero,  $Q$  will be **small**.
- If some  $r_k$  values large (positive or negative),  $Q$  will be **large**.



# Portmanteau tests

Consider a *whole set* of  $r_k$  values, and develop a test to see whether the set is significantly different from a zero set.

## Ljung-Box test

$$Q^* = T(T+2) \sum_{k=1}^{\ell} (T-k)^{-1} r_k^2$$

where  $\ell$  is max lag being considered and  $T$  is number of observations.

- My preferences:  $\ell = 10$  for non-seasonal data,  $h = 2m$  for seasonal data.
- Better performance, especially in small samples.

# Portmanteau tests

- If data are WN,  $Q^*$  has  $\chi^2$  distribution with  $(\ell - K)$  degrees of freedom where  $K$  = no. parameters in model.
- When applied to raw data, set  $K = 0$ .
- $\text{lag} = \ell$ ,  $\text{dof} = K$

```
augment(fit) %>%  
  features(.resid, lbjung_box, lag=10, dof=0)
```

```
## # A tibble: 1 x 4  
##   Symbol .model      lb_stat lb_pvalue  
##   <chr>   <chr>      <dbl>    <dbl>  
## 1 FB     NAIVE(Close)    12.1     0.276
```

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# Forecast distributions

- A forecast  $\hat{y}_{T+h|T}$  is (usually) the mean of the conditional distribution  $y_{T+h} \mid y_1, \dots, y_T$ .
- Most time series models produce normally distributed forecasts.
- The forecast distribution describes the probability of observing any future value.

# Forecast distributions

Assuming residuals are normal, uncorrelated,  $\text{sd} = \hat{\sigma}$ :

**Mean:**  $\hat{y}_{T+h|T} \sim N(\bar{y}, (1 + 1/T)\hat{\sigma}^2)$

**Naïve:**  $\hat{y}_{T+h|T} \sim N(y_T, h\hat{\sigma}^2)$

**Seasonal naïve:**  $\hat{y}_{T+h|T} \sim N(y_{T+h-m(k+1)}, (k+1)\hat{\sigma}^2)$

**Drift:**  $\hat{y}_{T+h|T} \sim N(y_T + \frac{h}{T-1}(y_T - y_1), h\frac{T+h}{T}\hat{\sigma}^2)$

where  $k$  is the integer part of  $(h - 1)/m$ .

Note that when  $h = 1$  and  $T$  is large, these all give the same approximate forecast variance:  $\hat{\sigma}^2$ .

# Prediction intervals

- A prediction interval gives a region within which we expect  $y_{T+h}$  to lie with a specified probability.
- Assuming forecast errors are normally distributed, then a 95% PI is

$$\hat{y}_{T+h|T} \pm 1.96\hat{\sigma}_h$$

where  $\hat{\sigma}_h$  is the st dev of the  $h$ -step distribution.

- When  $h = 1$ ,  $\hat{\sigma}_h$  can be estimated from the residuals.

# Prediction intervals

```
brick_fc %>% hilo(level = 95)
```

```
## # A tsibble: 80 x 5 [1Q]
## # Key:           .model [4]
##   .model      Quarter    Bricks .mean      `95%`
##   <chr>        <qtr>      <dist> <dbl>      <hilo>
## 1 Seasonal_naive 2005 Q3 N(428, 2336)  428 [333, 523]95
## 2 Seasonal_naive 2005 Q4 N(397, 2336)  397 [302, 492]95
## 3 Seasonal_naive 2006 Q1 N(355, 2336)  355 [260, 450]95
## 4 Seasonal_naive 2006 Q2 N(435, 2336)  435 [340, 530]95
## 5 Seasonal_naive 2006 Q3 N(428, 4672)  428 [294, 562]95
## 6 Seasonal_naive 2006 Q4 N(397, 4672)  397 [263, 531]95
## 7 Seasonal_naive 2007 Q1 N(355, 4672)  355 [221, 489]95
## 8 Seasonal_naive 2007 Q2 N(435, 4672)  435 [301, 569]95
## 9 Seasonal_naive 2007 Q3 N(428, 7008)  428 [264, 592]95
```

# Prediction intervals

- Point forecasts often useless without a measure of uncertainty (such as prediction intervals).
- Prediction intervals require a stochastic model (with random errors, etc).
- For most models, prediction intervals get wider as the forecast horizon increases.
- Use `level` argument to control coverage.
- Check residual assumptions before believing them.
- Usually too narrow due to unaccounted uncertainty.

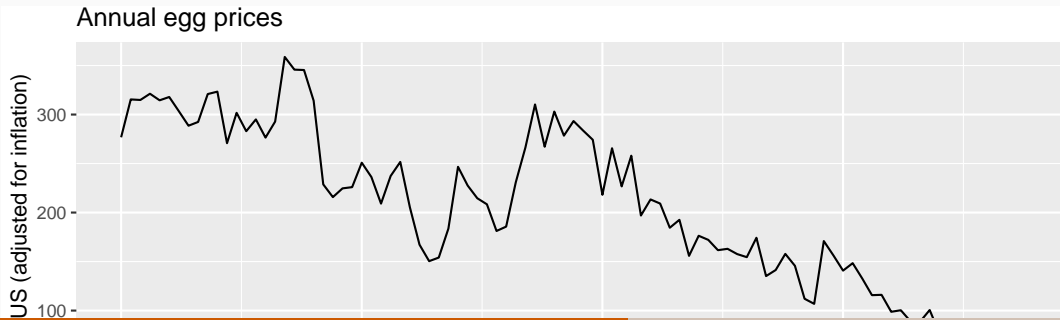


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# Modelling with transformations

```
eggs <- prices %>%  
  filter(!is.na(eggs)) %>% select(eggs)  
eggs %>% autoplot() +  
  labs(title="Annual egg prices",  
        y="$US (adjusted for inflation)")
```



# Modelling with transformations

Transformations used in the left of the formula will be automatically back-transformed. To model log-transformed egg prices, you could use:

```
fit <- eggs %>%  
  model(RW(log(eggs) ~ drift()))  
fit
```

```
## # A mable: 1 x 1  
##   `RW(log(eggs) ~ drift())`  
##                               <model>  
## 1                               <RW w/ drift>
```

# Forecasting with transformations

```
fc <- fit %>%  
  forecast(h = 50)  
fc
```

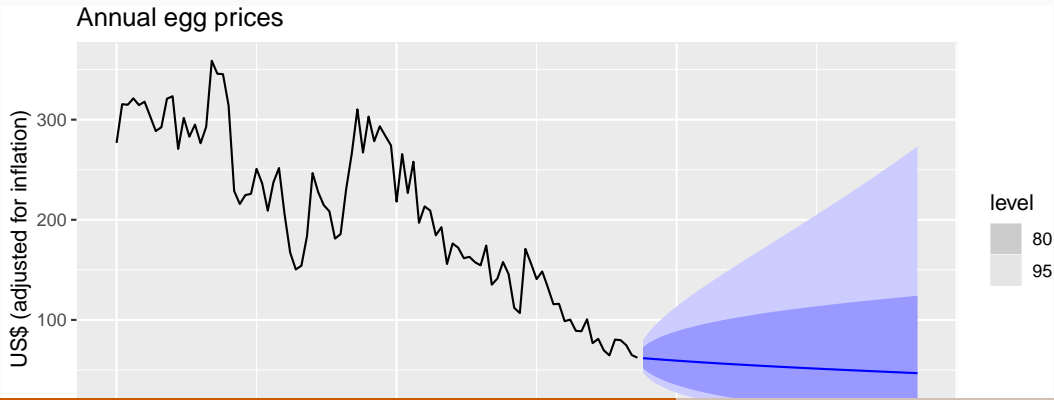
```
## # A fable: 50 x 4 [1Y]
```

```
## # Key:      .model [1]
```

##	.model	year	eggs	.mean
##	<chr>	<dbl>	<dist>	<dbl>
##	1 RW(log(eggs) ~ drift())	1994	t(N(4.1, 0.018))	61.8
##	2 RW(log(eggs) ~ drift())	1995	t(N(4.1, 0.036))	61.4
##	3 RW(log(eggs) ~ drift())	1996	t(N(4.1, 0.054))	61.0
##	4 RW(log(eggs) ~ drift())	1997	t(N(4.1, 0.073))	60.5
##	5 RW(log(eggs) ~ drift())	1998	t(N(4.1, 0.093))	60.1
##	6 RW(log(eggs) ~ drift())	1999	t(N(4, 0.11))	59.7
##	7 RW(log(eggs) ~ drift())	2000	t(N(4, 0.13))	59.3

# Forecasting with transformations

```
fc %>% autoplot(eggs) +  
  labs(title="Annual egg prices",  
        y="US$ (adjusted for inflation)")
```



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- Back-transformed point forecasts are medians.
- Back-transformed PI have the correct coverage.

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Let  $f(x)$  be back-transformation function, and  $Y = f(X)$ .

Taylor series expansion about  $\mu$ :

$$f(X) = f(\mu) + (X - \mu)f'(\mu) + \frac{1}{2}(X - \mu)^2f''(\mu).$$

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$$E[Y] = E[f(X)] = f(\mu) + \frac{1}{2}\sigma^2f''(\mu)$$



# Bias adjustment

**Box-Cox back-transformation:**

$$y_t = \begin{cases} \exp(w_t) & \lambda = 0; \\ (\lambda W_t + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

$$f(x) = \begin{cases} e^x & \lambda = 0; \\ (\lambda x + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

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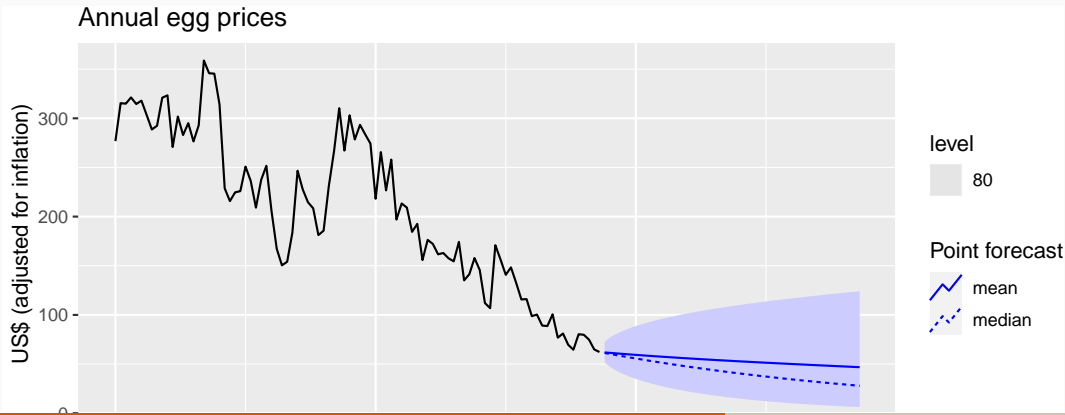
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$$E[Y] = \begin{cases} e^{\mu} \left[ 1 + \frac{\sigma^2}{2} \right] & \lambda = 0; \\ (\lambda \mu + 1)^{1/\lambda} \left[ 1 + \frac{\sigma^2(1-\lambda)}{2(\lambda \mu + 1)^2} \right] & \lambda \neq 0. \end{cases}$$

# Bias adjustment

```
fc %>%  
  autoplot(eggs, level = 80, point_forecast = lst(mean, median)) +  
  labs(title="Annual egg prices",  
       y="US$ (adjusted for inflation)")
```



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# Forecasting and decomposition

$$y_t = \hat{S}_t + \hat{A}_t$$

- $\hat{A}_t$  is seasonally adjusted component
  - $\hat{S}_t$  is seasonal component.
- 
- Forecast  $\hat{S}_t$  using SNAIVE.
  - Forecast  $\hat{A}_t$  using non-seasonal time series method.
  - Combine forecasts of  $\hat{S}_t$  and  $\hat{A}_t$  to get forecasts of original data.

# US Retail Employment

```
us_retail_employment <- us_employment %>%  
  filter(year(Month) >= 1990, Title == "Retail Trade") %>%  
  select(-Series_ID)  
us_retail_employment
```

```
## # A tsibble: 357 x 3 [1M]  
##      Month Title      Employed  
##      <mtm> <chr>      <dbl>  
##  1 1990 Jan Retail Trade 13256.  
##  2 1990 Feb Retail Trade 12966.  
##  3 1990 Mar Retail Trade 12938.  
##  4 1990 Apr Retail Trade 13012.  
##  5 1990 May Retail Trade 13108.  
##  6 1990 Jun Retail Trade 13183.  
##  7 1990 Jul Retail Trade 13170.  
##  8 1990 Aug Retail Trade 13160.
```

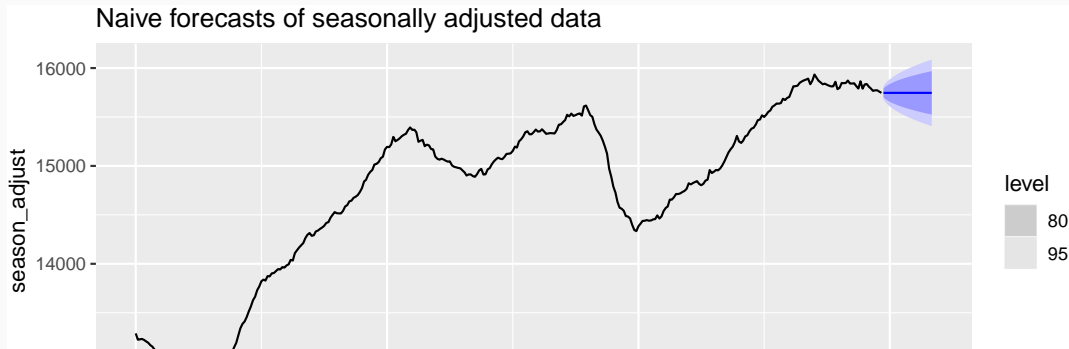
# US Retail Employment

```
dcmp <- us_retail_employment %>%  
  model(STL(Employed)) %>%  
  components() %>% select(-.model)  
dcmp
```

```
## # A tsibble: 357 x 6 [1M]  
##      Month Employed trend season_year remainder season_adjust  
##      <mth>      <dbl> <dbl>      <dbl>      <dbl>      <dbl>  
## 1 1990 Jan    13256. 13288.      -33.0        0.836    13289.  
## 2 1990 Feb    12966. 13269.     -258.       -44.6    13224.  
## 3 1990 Mar    12938. 13250.     -290.       -22.1    13228.  
## 4 1990 Apr    13012. 13231.     -220.         1.05    13232.  
## 5 1990 May    13108. 13211.     -114.        11.3    13223.  
## 6 1990 Jun    13183. 13192.     -24.3       15.5    13207.  
## 7 1990 Jul    13170. 13172.     -23.2       21.6    13193.  
## 8 1990 Aug    13160. 13151.      -9.52      17.8    13169.
```

# US Retail Employment

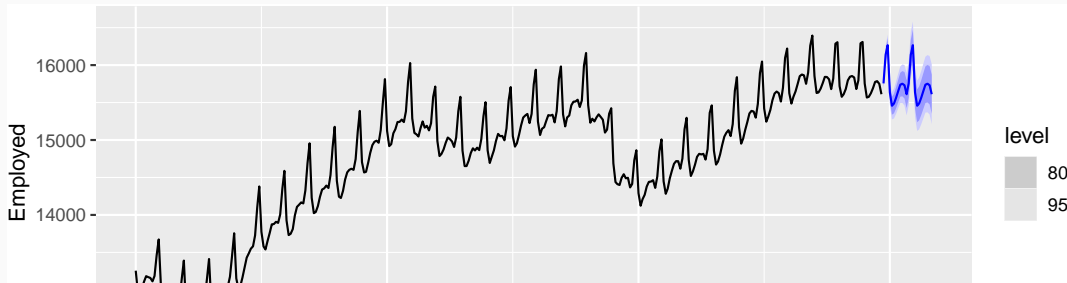
```
dcmp %>%  
  model(NAIVE(season_adjust)) %>%  
  forecast() %>%  
  autoplot(dcmp) +  
  labs(title = "Naive forecasts of seasonally adjusted data")
```





# US Retail Employment

```
us_retail_employment %>%  
  model(stlf = decomposition_model(  
    STL(Employed ~ trend(window = 7), robust = TRUE),  
    NAIVE(season_adjust)  
  )) %>%  
  forecast() %>%  
  autoplot(us_retail_employment)
```



# Decomposition models

`decomposition_model()` creates a decomposition model

- You must provide a method for forecasting the `season_adjust` series.
- A seasonal naive method is used by default for the `seasonal` components.
- The variances from both the seasonally adjusted and seasonal forecasts are combined.

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# Training and test sets



- A model which fits the training data well will not necessarily forecast well.
- A perfect fit can always be obtained by using a model with enough parameters.
- Over-fitting a model to data is just as bad as failing to identify a systematic pattern in the data.
- The test set must not be used for *any* aspect of model development or calculation of forecasts.
- Forecast accuracy is based only on the test set.

# Forecast errors

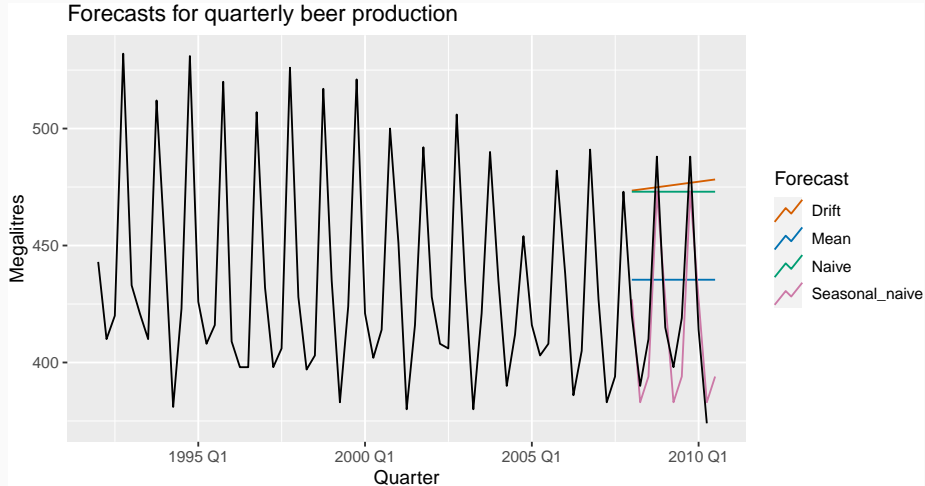
Forecast “error”: the difference between an observed value and its forecast.

$$e_{T+h} = y_{T+h} - \hat{y}_{T+h|T},$$

where the training data is given by  $\{y_1, \dots, y_T\}$

- Unlike residuals, forecast errors on the test set involve multi-step forecasts.
- These are *true* forecast errors as the test data is not used in computing  $\hat{y}_{T+h|T}$ .

# Measures of forecast accuracy



# Measures of forecast accuracy

$y_{T+h}$  =  $(T + h)$ th observation,  $h = 1, \dots, H$

$\hat{y}_{T+h|T}$  = its forecast based on data up to time  $T$ .

$$e_{T+h} = y_{T+h} - \hat{y}_{T+h|T}$$

$$\text{MAE} = \text{mean}(|e_{T+h}|)$$

$$\text{MSE} = \text{mean}(e_{T+h}^2)$$

$$\text{MAPE} = 100\text{mean}(|e_{T+h}|/|y_{T+h}|)$$

$$\text{RMSE} = \sqrt{\text{mean}(e_{T+h}^2)}$$

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$$\text{MAPE} = 100\text{mean}(|e_{T+h}|/|y_{T+h}|)$$

- MAE, MSE, RMSE are all scale dependent.
- MAPE is scale independent but is only sensible if  $y_t \gg 0$  for all  $t$ , and  $y$  has a natural zero.



# Measures of forecast accuracy

## Mean Absolute Scaled Error

$$\text{MASE} = \text{mean}(|e_{T+h}|/Q)$$

where  $Q$  is a stable measure of the scale of the time series  $\{y_t\}$ .

Proposed by Hyndman and Koehler (IJF, 2006).

For non-seasonal time series,

$$Q = (T - 1)^{-1} \sum_{t=2}^T |y_t - y_{t-1}|$$

works well. Then MASE is equivalent to MAE relative to a naïve method.

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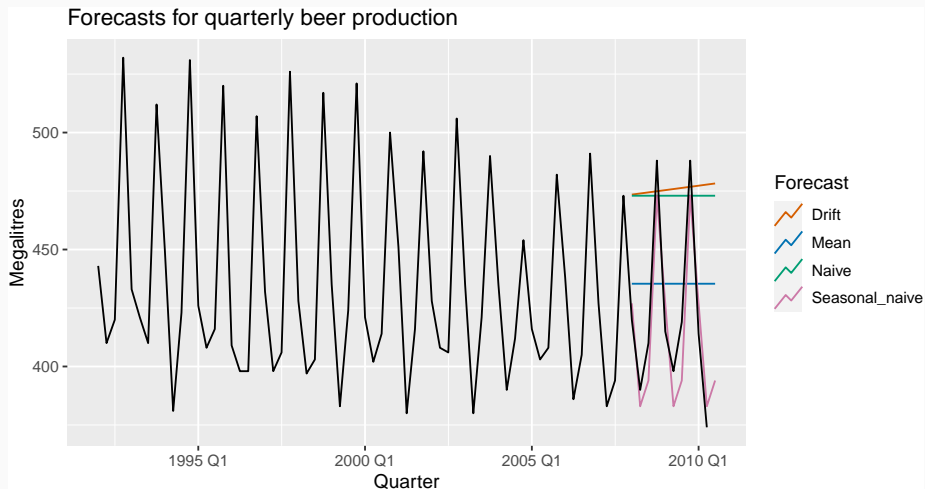
Proposed by Hyndman and Koehler (IJF, 2006).

For seasonal time series,

$$Q = (T - m)^{-1} \sum_{t=m+1}^T |y_t - y_{t-m}|$$

works well. Then MASE is equivalent to MAE relative to a seasonal naïve method.

# Measures of forecast accuracy



# Measures of forecast accuracy

```
recent_production <- aus_production %>%  
  filter(year(Quarter) >= 1992)  
train <- recent_production %>%  
  filter(year(Quarter) <= 2007)  
beer_fit <- train %>%  
  model(  
    Mean = MEAN(Beer),  
    Naive = NAIVE(Beer),  
    Seasonal_naive = SNAIVE(Beer),  
    Drift = RW(Beer ~ drift())  
  )  
beer_fc <- beer_fit %>%  
  forecast(h = 10)
```

# Measures of forecast accuracy

```
accuracy(beer_fit)
```

```
## # A tibble: 4 x 6
##   .model      .type    RMSE    MAE    MAPE    MASE
##   <chr>      <chr>    <dbl> <dbl> <dbl> <dbl>
## 1 Drift      Training  65.3   54.8  12.2   3.83
## 2 Mean       Training  43.6   35.2   7.89   2.46
## 3 Naive      Training  65.3   54.7  12.2   3.83
## 4 Seasonal_naive Training  16.8   14.3   3.31   1
```

```
accuracy(beer_fc, recent_production)
```

```
## # A tibble: 4 x 6
##   .model      .type    RMSE    MAE    MAPE    MASE
##   <chr>      <chr>    <dbl> <dbl> <dbl> <dbl>
## 1 Drift      Test     64.9   58.9  14.6   4.12
## 2 Mean       Test     38.4   34.8   8.28   2.44
## 3 Naive      Test     62.7   57.4  14.2   4.01
```

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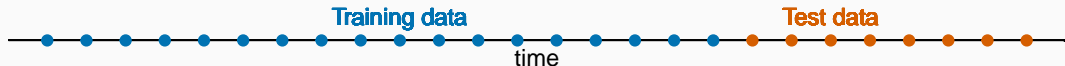
# Time series cross-validation

## Traditional evaluation

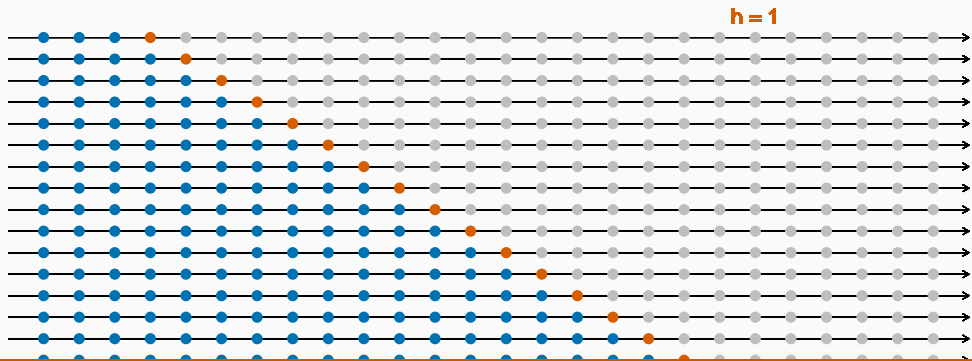


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## Traditional evaluation



## Time series cross-validation



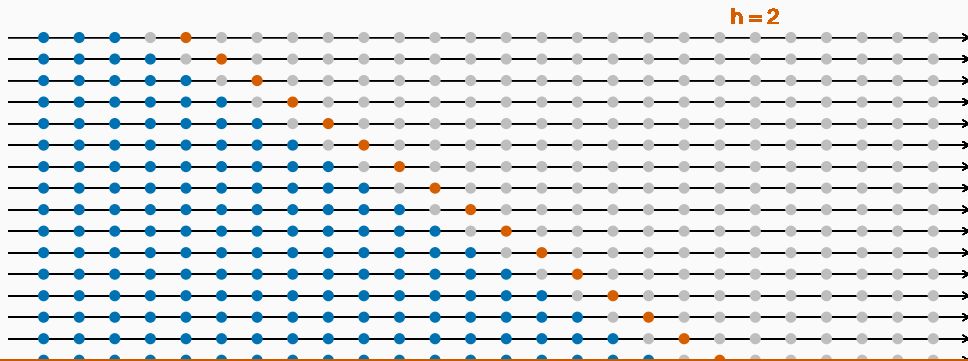


# Time series cross-validation

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## Time series cross-validation

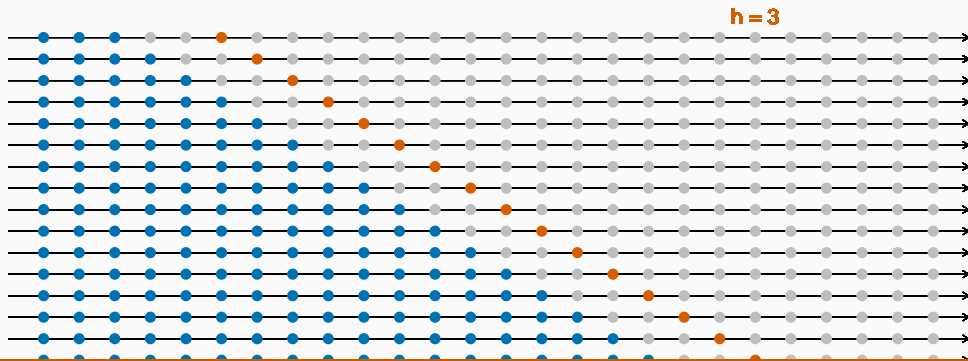


# Time series cross-validation

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## Time series cross-validation

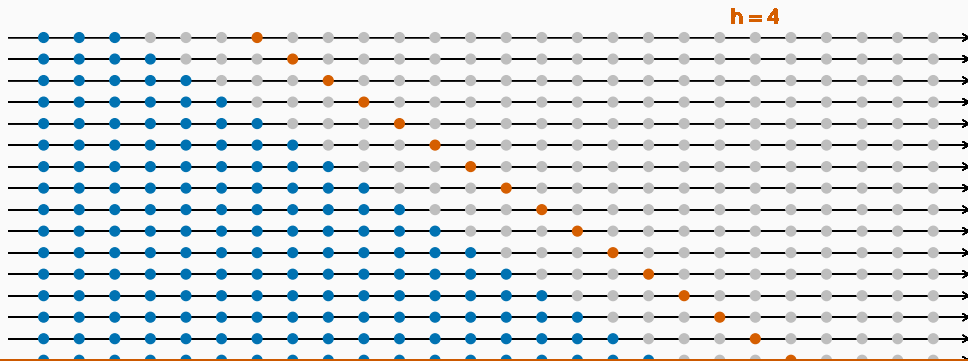


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation

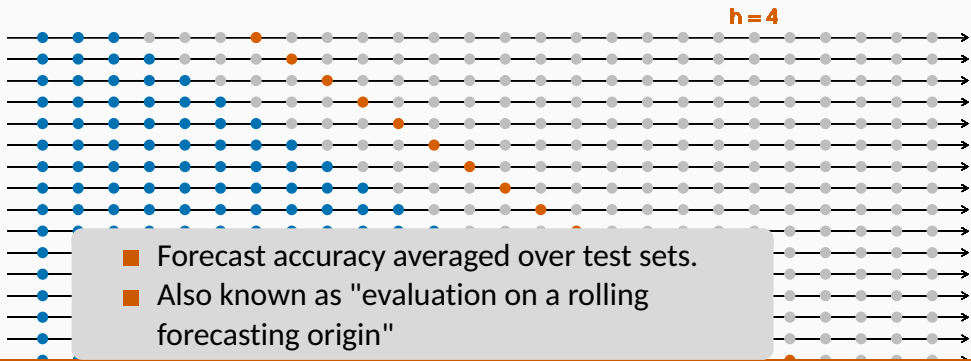


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation



# Time series cross-validation

Stretch with a minimum length of 3, growing by 1 each step.

```
fb_stretch <- fb_stock %>%  
  stretch_tsibble(.init = 3, .step = 1) %>%  
  filter(.id != max(.id))
```

```
## # A tsibble: 790,650 x 4 [1]  
## # Key:      .id [1,255]  
##   Date      Close trading_day  .id  
##   <date>    <dbl>      <int> <int>  
## 1 2014-01-02  54.7          1     1  
## 2 2014-01-03  54.6          2     1  
## 3 2014-01-06  57.2          3     1  
## 4 2014-01-02  54.7          1     2  
## 5 2014-01-03  54.6          2     2  
## 6 2014-01-06  57.2          3     2
```

# Time series cross-validation

Estimate RW w/ drift models for each window.

```
fit_cv <- fb_stretch %>%  
  model(RW(Close ~ drift()))
```

```
## # A mable: 1,255 x 3  
## # Key:      .id, Symbol [1,255]  
##   .id Symbol `RW(Close ~ drift())`  
##   <int> <chr>                <model>  
## 1      1 FB                  <RW w/ drift>  
## 2      2 FB                  <RW w/ drift>  
## 3      3 FB                  <RW w/ drift>  
## 4      4 FB                  <RW w/ drift>  
## # ... with 1,251 more rows
```

# Time series cross-validation

Produce one step ahead forecasts from all models.

```
fc_cv <- fit_cv %>%  
  forecast(h=1)
```

```
## # A tibble: 1,255 x 5  
## #   Key:      .id, Symbol [1,255]  
##   .id Symbol trading_day      Close .mean  
##   <int> <chr>      <dbl>      <dist> <dbl>  
## 1     1  FB          4 N(58, 3.9)  58.4  
## 2     2  FB          5 N(59, 2)   59.0  
## 3     3  FB          6 N(59, 1.5)  59.1  
## 4     4  FB          7 N(58, 1.8)  57.7  
## # ... with 1,251 more rows
```

# Time series cross-validation

```
# Cross-validated  
fc_cv %>% accuracy(fb_stock)  
# Training set  
fb_stock %>% model(RW(Close ~ drift())) %>% accuracy()
```

	RMSE	MAE	MAPE
Cross-validation	2.418	1.469	1.266
Training	2.414	1.465	1.261

A good way to choose the best forecasting model is to find the model with the smallest RMSE computed using time series cross-validation.