ETC3550/ETC5550
Applied forecasting

Ch7. Regression models

OTexts.org/fpp3/

Outline

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Multiple regression and forecasting

$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t.$$

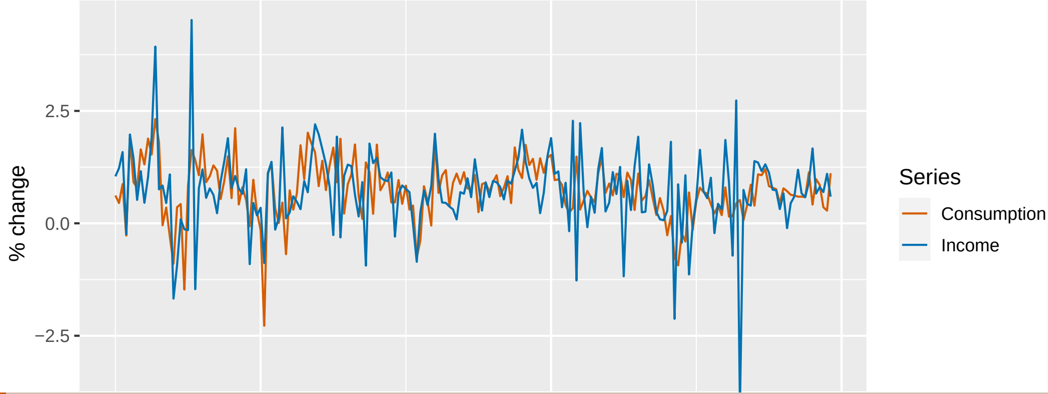
- y_t is the variable we want to predict: the "response" variable
- Each $x_{j,t}$ is numerical and is called a "predictor". They are usually assumed to be known for all past and future times.
- The coefficients $\beta_1, \dots, \beta_k$ measure the effect of each predictor after taking account of the effect of all other predictors in the model.

That is, the coefficients measure the **marginal effects**.

- ε_t is a white noise error term

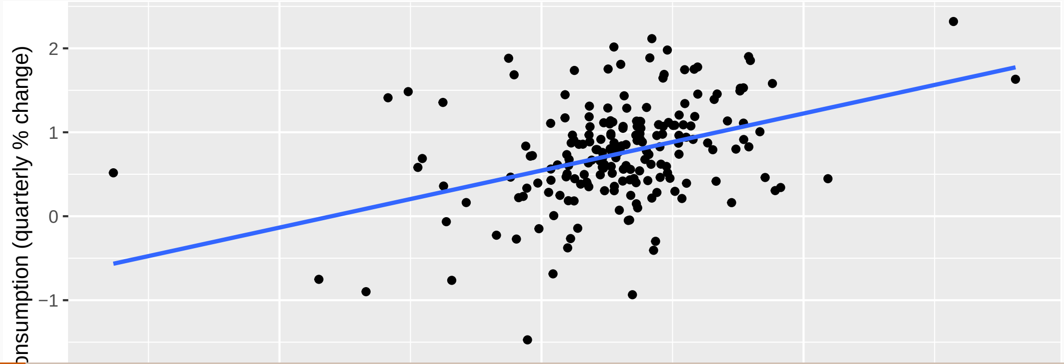
Example: US consumption expenditure

```
us_change %>%
  pivot_longer(c(Consumption, Income), names_to="Series") %>%
  autoplot(value) +
  labs(y="% change")
```



Example: US consumption expenditure

```
us_change %>%
  ggplot(aes(x = Income, y = Consumption)) +
    labs(y = "Consumption (quarterly % change)",
         x = "Income (quarterly % change)") +
    geom_point() + geom_smooth(method = "lm", se = FALSE)
```



Example: US consumption expenditure

```
fit_cons <- us_change %>%
  model(lm = TSLM(Consumption ~ Income))
report(fit_cons)
```

```
## Series: Consumption
## Model: TSLM
##
## Residuals:
##     Min      1Q  Median      3Q     Max
## -2.582 -0.278   0.019   0.323   1.422
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)   0.5445     0.0540   10.08  < 2e-16 ***
## Income        0.2718     0.0467    5.82  2.4e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.591 on 196 degrees of freedom
```

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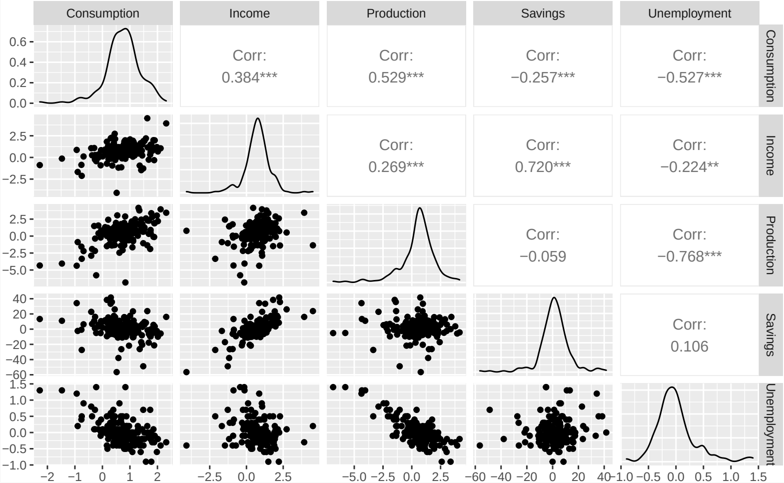
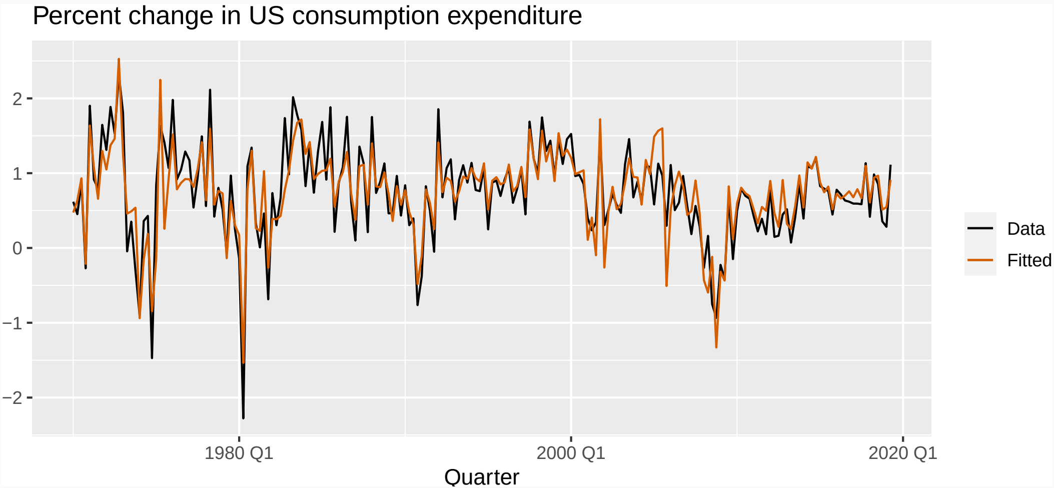


Example: US consumption expenditure

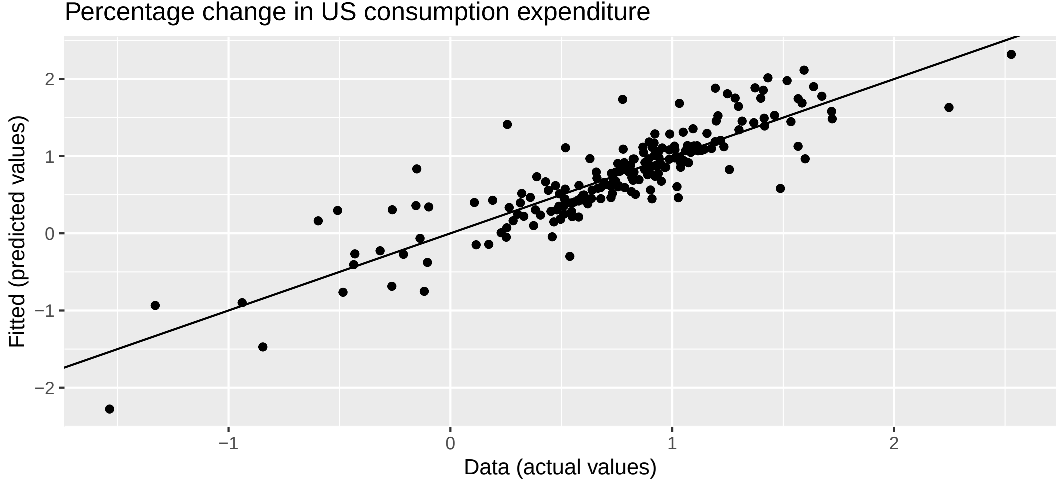
```r
fit_consMR <- us_change %>%
  model(lm = TSLM(Consumption ~ Income + Production + Unemployment + Savings))
report(fit_consMR)
```

```
## Series: Consumption
## Model: TSLM
##
## Residuals:
##    Min     1Q Median     3Q    Max
## -0.906 -0.158 -0.036  0.136  1.155
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   0.25311    0.03447    7.34  5.7e-12 ***
## Income        0.74058    0.04012   18.46  < 2e-16 ***
## Production    0.04717    0.02314    2.04    0.043 *
## Unemployment -0.17469    0.09551   -1.83    0.069 .
## Savings      -0.05289    0.00292  -18.09  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.31 on 193 degrees of freedom
## Multiple R-squared: 0.768,    Adjusted R-squared: 0.763
```

Example: US consumption expenditure

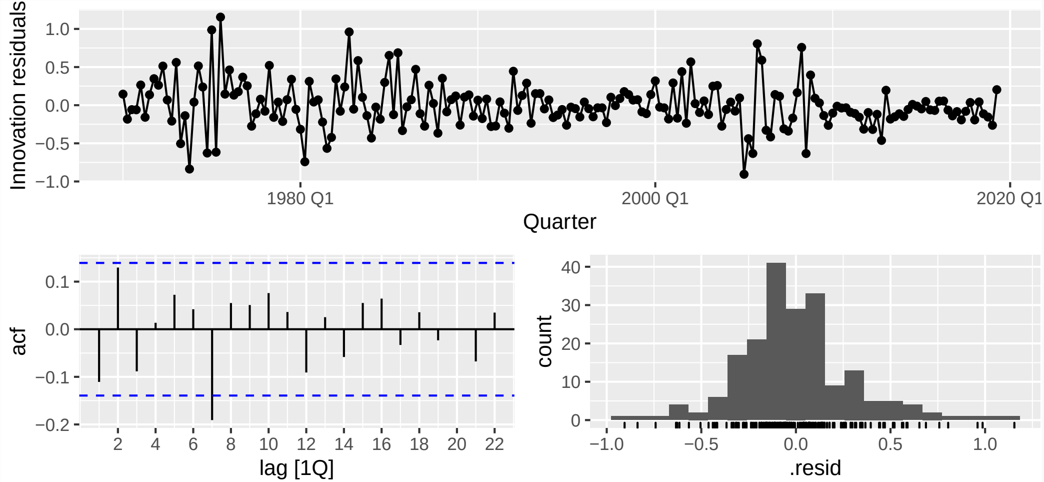


Example: US consumption expenditure



Example: US consumption expenditure

```
fit_consMR %>% gg_tsresiduals()
```



Outline

Trend

Linear trend

$$x_t = t$$

- $t = 1, 2, \ldots, T$
- Strong assumption that trend will continue.

Dummy variables

If a categorical variable takes only two values (e.g., ‘Yes’ or ‘No’), then an equivalent numerical variable can be constructed taking value 1 if yes and 0 if no. This is called a **dummy variable**.

	A	B
1	Yes	1
2	Yes	1
3	No	0
4	Yes	1
5	No	0
6	No	0
7	Yes	1
8	Yes	1
9	No	0
10	No	0
11	No	0
12	No	0
13	Yes	1
14	No	0

Dummy variables

If there are more than two categories, then the variable can be coded using several dummy variables (one fewer than the total number of categories).

	A	B	C	D	E
1	Monday	1	0	0	0
2	Tuesday	0	1	0	0
3	Wednesday	0	0	1	0
4	Thursday	0	0	0	1
5	Friday	0	0	0	0
6	Monday	1	0	0	0
7	Tuesday	0	1	0	0
8	Wednesday	0	0	1	0
9	Thursday	0	0	0	1
10	Friday	0	0	0	0
11	Monday	1	0	0	0
12	Tuesday	0	1	0	0
13	Wednesday	0	0	1	0
14	Thursday	0	0	0	1
15	Friday	0	0	0	0

Beware of the dummy variable trap!

- Using one dummy for each category gives too many dummy variables!
- The regression will then be singular and inestimable.
- Either omit the constant, or omit the dummy for one category.
- The coefficients of the dummies are relative to the omitted category.

Uses of dummy variables

Seasonal dummies

- For quarterly data: use 3 dummies
- For monthly data: use 11 dummies
- For daily data: use 6 dummies
- What to do with weekly data?

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- What to do with weekly data?

Outliers

- If there is an outlier, you can use a dummy variable to remove its effect.

Uses of dummy variables

Seasonal dummies

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- For daily data: use 6 dummies
- What to do with weekly data?

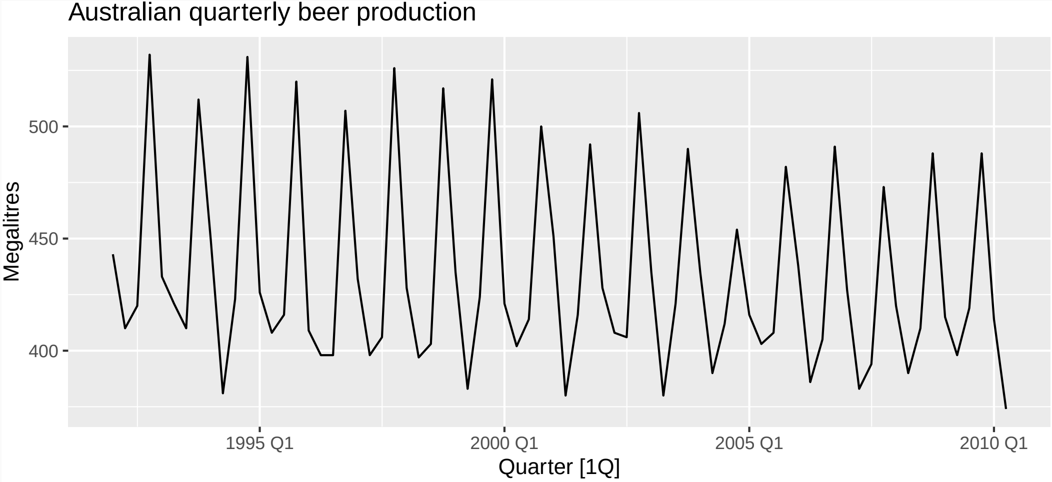
Outliers

- If there is an outlier, you can use a dummy variable to remove its effect.

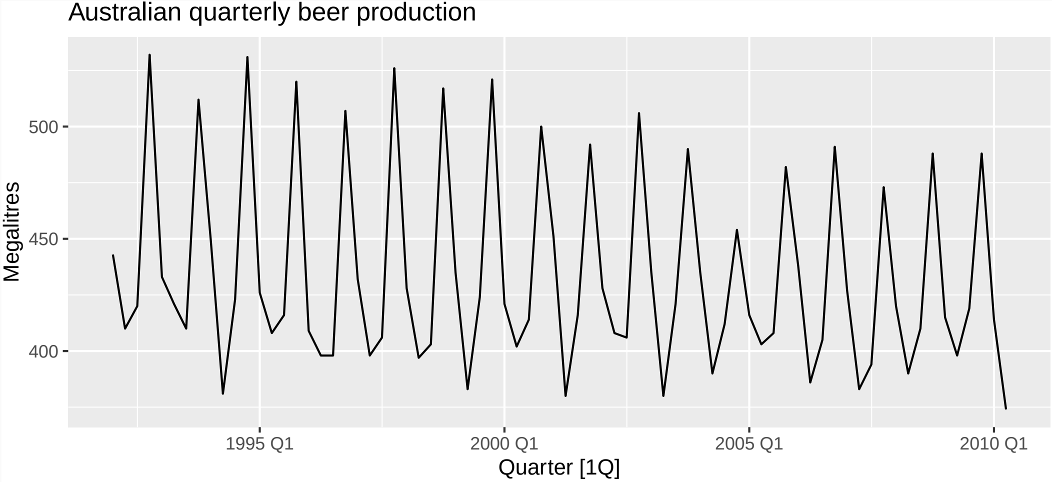
Public holidays

- For daily data: if it is a public holiday, dummy=1, otherwise dummy=0.

Beer production revisited



Beer production revisited



Regression model

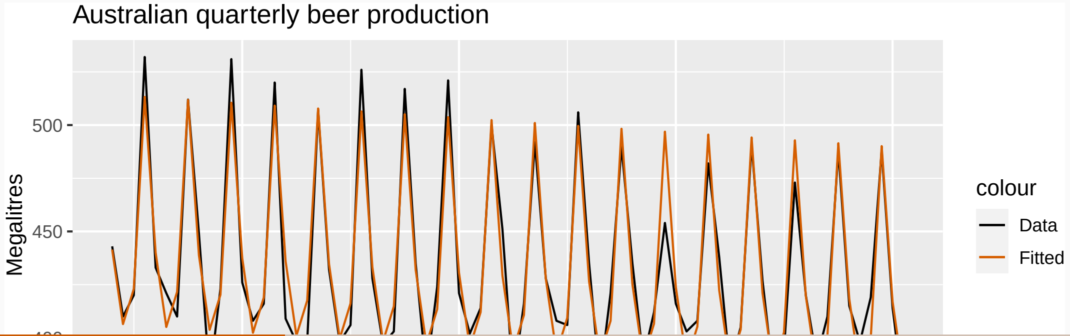
Beer production revisited

```r
fit_beer <- recent_production %>% model(TSLM(Beer ~ trend() + season()))
report(fit_beer)
```

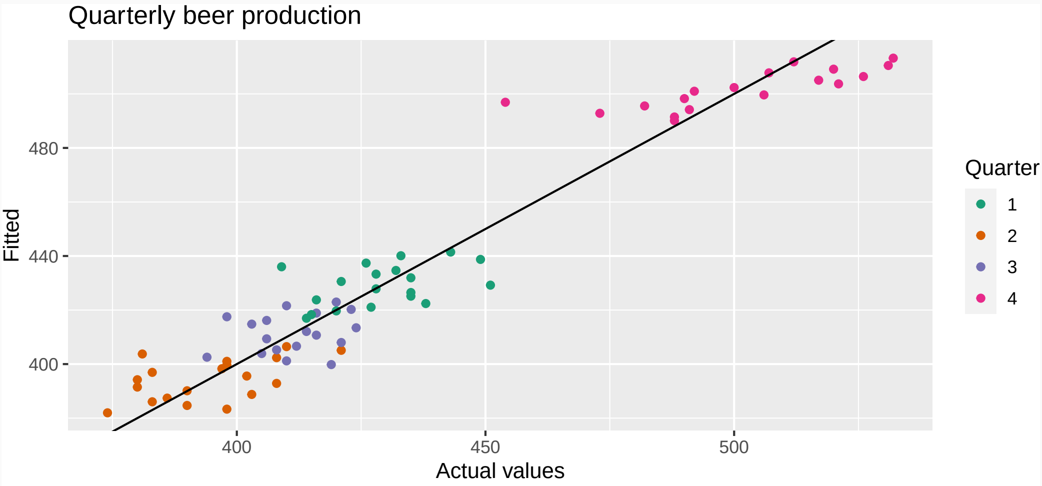
```
## Series: Beer
## Model: TSLM
##
## Residuals:
##     Min      1Q  Median      3Q     Max
##  -42.9    -7.6    -0.5     8.0    21.8
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    441.8004     3.7335  118.33  < 2e-16 ***
## trend()         -0.3403     0.0666   -5.11  2.7e-06 ***
## season()year2  -34.6597     3.9683   -8.73  9.1e-13 ***
## season()year3  -17.8216     4.0225   -4.43  3.4e-05 ***
## season()year4   72.7964     4.0230   18.09  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 12.2 on 69 degrees of freedom
## Multiple R-squared: 0.924,    Adjusted R-squared: 0.92
```

Beer production revisited

```
augment(fit_beer) %>%
  ggplot(aes(x = Quarter)) +
  geom_line(aes(y = Beer, colour = "Data")) +
  geom_line(aes(y = .fitted, colour = "Fitted")) +
  labs(y="Megalitres",title ="Australian quarterly beer production") +
  scale_colour_manual(values = c(Data = "black", Fitted = "#D55E00"))
```

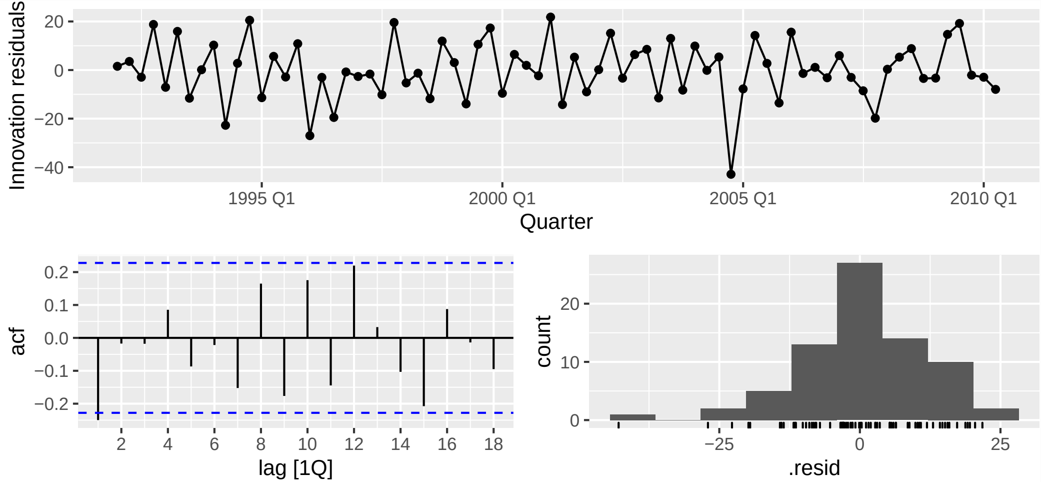


Beer production revisited



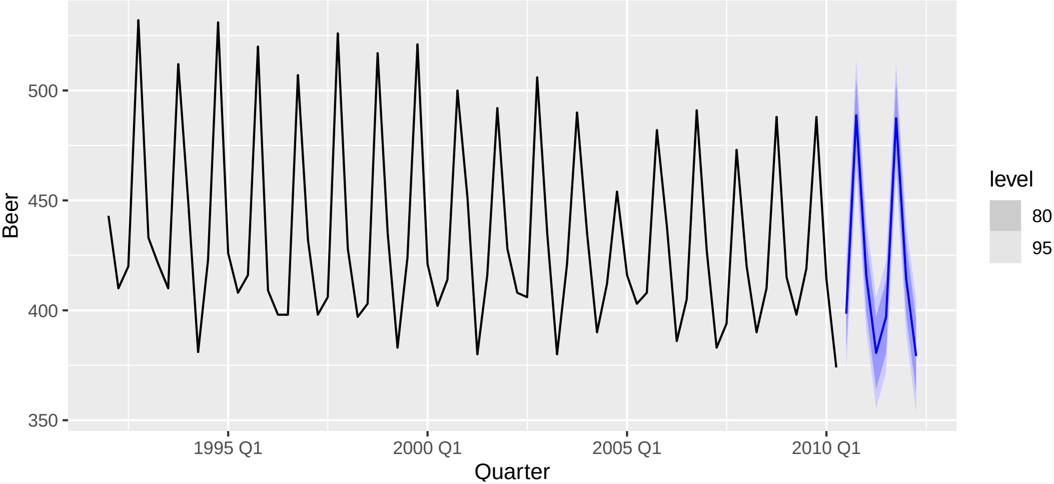
Beer production revisited

```
fit_beer %>% gg_tsresiduals()
```



Beer production revisited

```
fit_beer %>% forecast %>% autoplot(recent_production)
```



Fourier series

Periodic seasonality can be handled using pairs of Fourier terms:

$$s_k(t) = \sin\left(\frac{2\pi kt}{m}\right) \qquad c_k(t) = \cos\left(\frac{2\pi kt}{m}\right)$$

$$y_t = a + bt + \sum_{k=1}^{K}\left[\alpha_k s_k(t) + \beta_k c_k(t)\right] + \varepsilon_t$$

- Every periodic function can be approximated by sums of sin and cos terms for large enough K.
- Choose K by minimizing AICc.
- Called "harmonic regression"

```
TSLM(y ~ trend() + fourier(K))
```

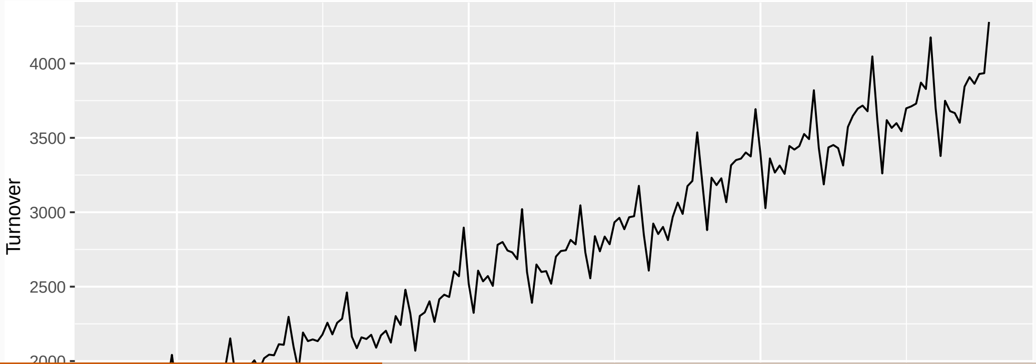
Harmonic regression: beer production

```
fourier_beer <- recent_production %>% model(TSLM(Beer ~ trend() + fourier(K=2)))
report(fourier_beer)
```

```
## Series: Beer
## Model: TSLM
##
## Residuals:
##     Min      1Q  Median      3Q     Max
##   -42.9    -7.6    -0.5     8.0    21.8
##
## Coefficients:
##                     Estimate Std. Error t value Pr(>|t|)
## (Intercept)         446.8792     2.8732  155.53  < 2e-16 ***
## trend()              -0.3403     0.0666   -5.11  2.7e-06 ***
## fourier(K = 2)C1_4    8.9108     2.0112    4.43  3.4e-05 ***
## fourier(K = 2)S1_4 -53.7281     2.0112  -26.71  < 2e-16 ***
## fourier(K = 2)C2_4 -13.9896     1.4226   -9.83  9.3e-15 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 12.2 on 69 degrees of freedom
## Multiple R-squared: 0.924,   Adjusted R-squared: 0.92
```

Harmonic regression: eating-out expenditure

```r
aus_cafe <- aus_retail %>% filter(
    Industry == "Cafes, restaurants and takeaway food services",
    year(Month) %in% 2004:2018
  ) %>% summarise(Turnover = sum(Turnover))
aus_cafe %>% autoplot(Turnover)
```

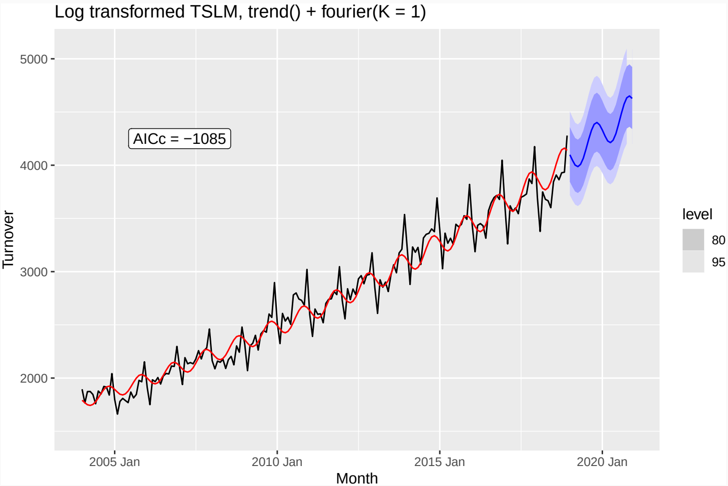


Harmonic regression: eating-out expenditure

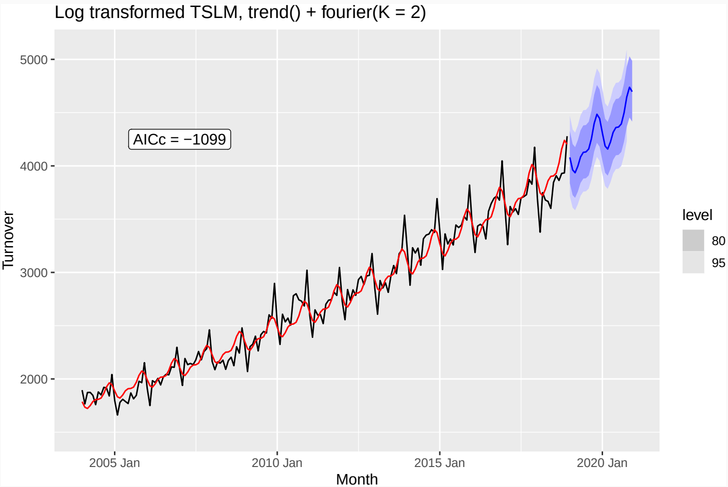
```r
fit <- aus_cafe %>%
  model(K1 = TSLM(log(Turnover) ~ trend() + fourier(K = 1)),
        K2 = TSLM(log(Turnover) ~ trend() + fourier(K = 2)),
        K3 = TSLM(log(Turnover) ~ trend() + fourier(K = 3)),
        K4 = TSLM(log(Turnover) ~ trend() + fourier(K = 4)),
        K5 = TSLM(log(Turnover) ~ trend() + fourier(K = 5)),
        K6 = TSLM(log(Turnover) ~ trend() + fourier(K = 6)))
glance(fit) %>% select(.model, r_squared, adj_r_squared, AICc)
```

```
## # A tibble: 6 x 4
##   .model r_squared adj_r_squared   AICc
##   <chr>      <dbl>         <dbl>  <dbl>
## 1 K1         0.962         0.962 -1085.
## 2 K2         0.966         0.965 -1099.
## 3 K3         0.976         0.975 -1160.
## 4 K4         0.980         0.979 -1183.
## 5 K5         0.985         0.984 -1234.
## 6 K6         0.985         0.984 -1232.
```

Harmonic regression: eating-out expenditure



Harmonic regression: eating-out expenditure



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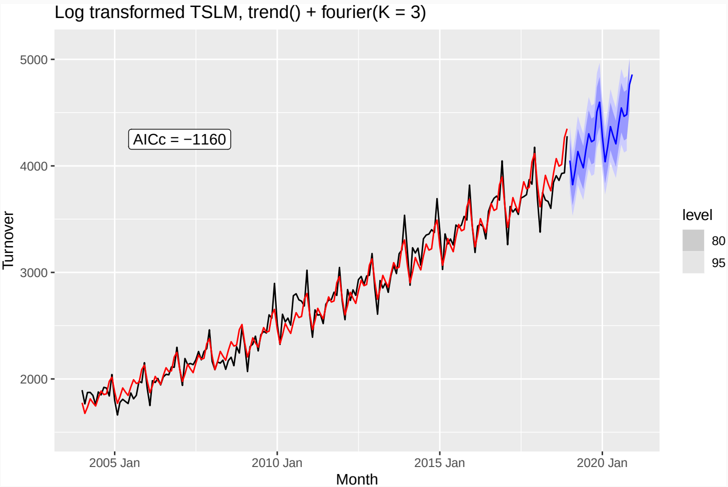
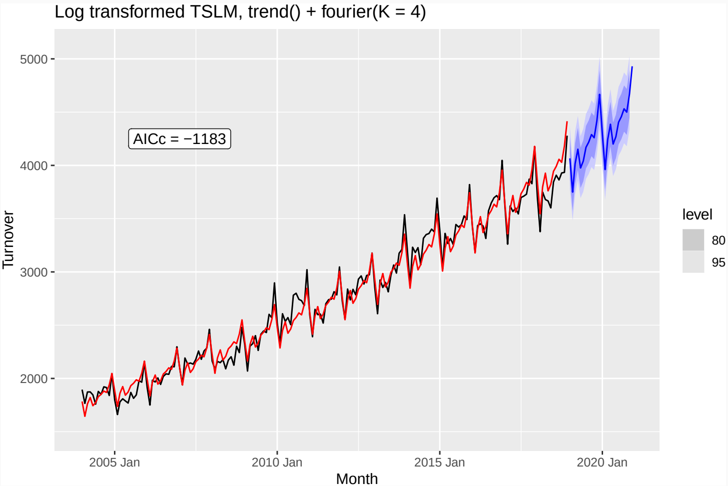
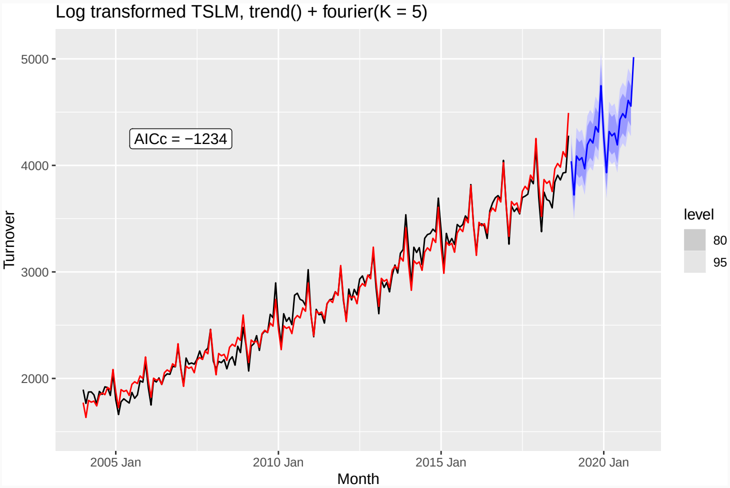


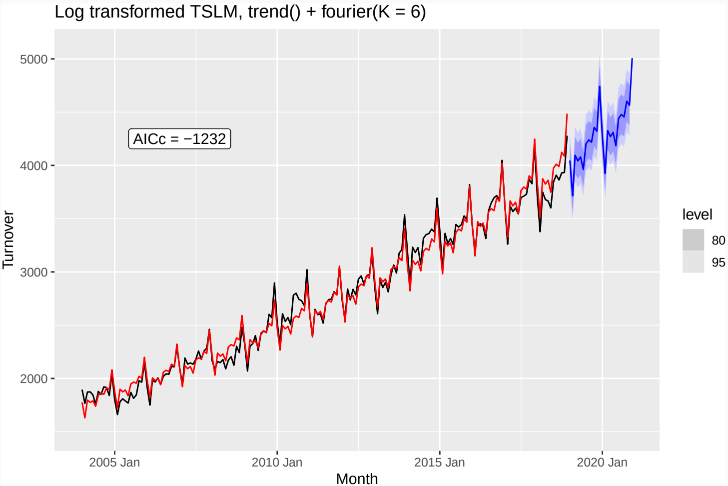
Harmonic regression: eating-out expenditure



Harmonic regression: eating-out expenditure



Harmonic regression: eating-out expenditure



Intervention variables

Spikes

- Equivalent to a dummy variable for handling an outlier.

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Steps

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Change of slope

- Variables take values 0 before the intervention and values $\{1, 2, 3, \ldots\}$ afterwards.

Holidays

For monthly data

- Christmas: always in December so part of monthly seasonal effect
- Easter: use a dummy variable $v_t = 1$ if any part of Easter is in that month, $v_t = 0$ otherwise.
- Ramadan and Chinese new year similar.

Trading days

With monthly data, if the observations vary depending on how many different types of days in the month, then trading day predictors can be useful.

$$
\begin{aligned}
z_1 &= \text{\# Mondays in month;} \\
z_2 &= \text{\# Tuesdays in month;} \\
&\vdots \\
z_7 &= \text{\# Sundays in month.}
\end{aligned}
$$

Distributed lags

Lagged values of a predictor.

Example: x is advertising which has a delayed effect

$$
\begin{aligned}
x_1 &= \text{advertising for previous month;} \\
x_2 &= \text{advertising for two months previously;} \\
&\vdots \\
x_m &= \text{advertising for } m \text{ months previously.}
\end{aligned}
$$

Nonlinear trend

Piecewise linear trend with bend at τ

$$x_{1,t} = t$$

$$x_{2,t} = \begin{cases} 0 & t < \tau \\ (t - \tau) & t \geq \tau \end{cases}$$

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Quadratic or higher order trend

$$x_{1,t} = t, \quad x_{2,t} = t^2, \quad \dots$$

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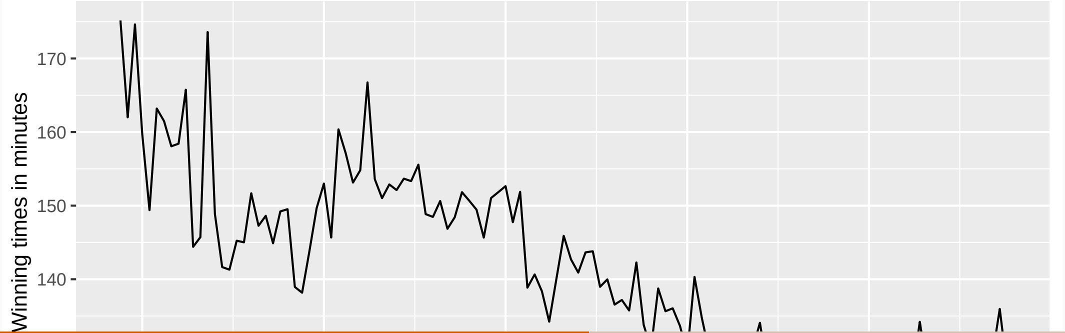
Quadratic or higher order trend

$$x_{1,t} = t, \quad x_{2,t} = t^2, \quad \dots$$

NOT RECOMMENDED!

Example: Boston marathon winning times

```
marathon <- boston_marathon %>%
  filter(Event == "Men's open division") %>%
  select(-Event) %>%
  mutate(Minutes = as.numeric(Time)/60)
marathon %>% autoplot(Minutes) +
  labs(y="Winning times in minutes")
```



Example: Boston marathon winning times

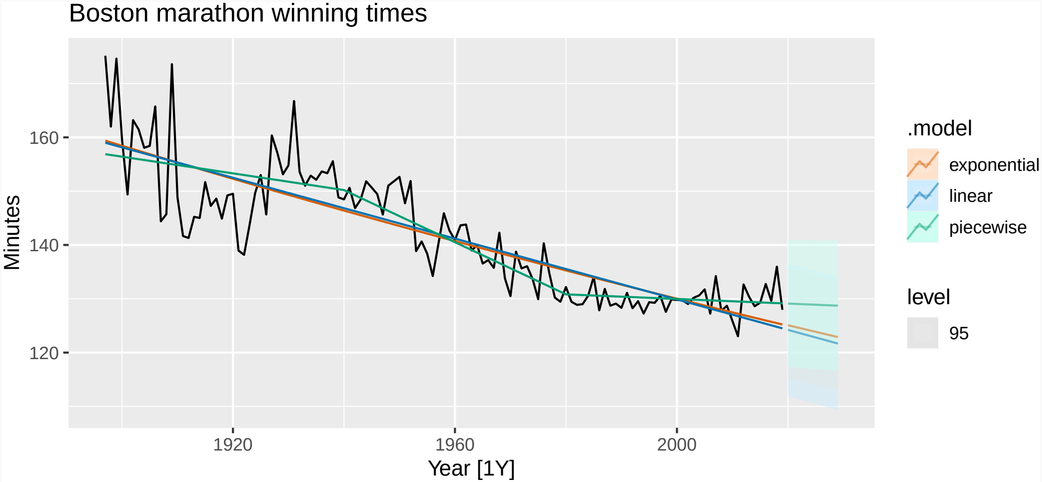
```r
fit_trends <- marathon %>%
  model(
    # Linear trend
    linear = TSLM(Minutes ~ trend()),
    # Exponential trend
    exponential = TSLM(log(Minutes) ~ trend()),
    # Piecewise linear trend
    piecewise = TSLM(Minutes ~ trend(knots = c(1940, 1980)))
  )
```

```r
fit_trends
```

```
## # A mable: 1 x 3
##    linear exponential piecewise
##   <model>     <model>   <model>
## 1  <TSLM>      <TSLM>    <TSLM>
```

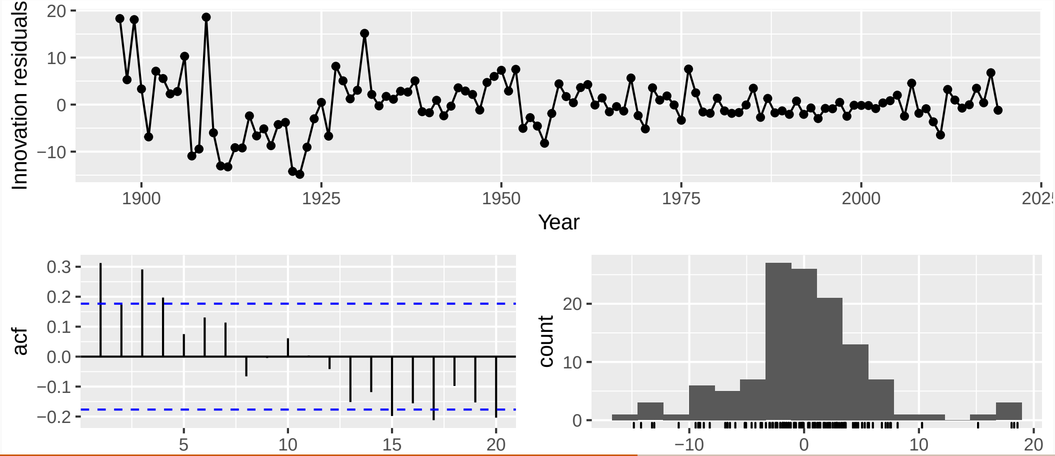
Example: Boston marathon winning times

```
fit_trends %>% forecast(h=10) %>% autoplot(marathon)
```



Example: Boston marathon winning times

```
fit_trends %>% select(piecewise) %>%
  gg_tsresiduals()
```



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Multiple regression and forecasting

For forecasting purposes, we require the following assumptions:

- ε_t are uncorrelated and zero mean
- ε_t are uncorrelated with each $x_{j,t}$.

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It is **useful** to also have $\varepsilon_t \sim \text{N}(0, \sigma^2)$ when producing prediction intervals or doing statistical tests.

Residual plots

Useful for spotting outliers and whether the linear model was appropriate.

- Scatterplot of residuals ε_t against each predictor $x_{j,t}$.
- Scatterplot residuals against the fitted values $\hat{y}_t$
- Expect to see scatterplots resembling a horizontal band with no values too far from the band and no patterns such as curvature or increasing spread.

Residual patterns

- If a plot of the residuals vs any predictor in the model shows a pattern, then the relationship is nonlinear.
- If a plot of the residuals vs any predictor **not** in the model shows a pattern, then the predictor should be added to the model.
- If a plot of the residuals vs fitted values shows a pattern, then there is heteroscedasticity in the errors. (Could try a transformation.)

Outline

Comparing regression models

Computer output for regression will always give the R^2 value. This is a useful summary of the model.

- It is equal to the square of the correlation between y and $\hat{y}$.
- It is often called the "coefficient of determination' '.
- It can also be calculated as follows:

$$R^2 = \frac{\sum(\hat{y}_t - \bar{y})^2}{\sum(y_t - \bar{y})^2}$$

- It is the proportion of variance accounted for (explained) by the predictors.

Comparing regression models

However . . .

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$$\bar{R}^2 = 1 - (1 - R^2)\frac{T-1}{T-k-1}$$

where k = no. predictors and T = no. observations.

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$$\bar{R}^2 = 1 - (1 - R^2)\frac{T - 1}{T - k - 1}$$

where k = no. predictors and T = no. observations.

Maximizing $\bar{R}^2$ is equivalent to minimizing $\hat{\sigma}^2$.

$$\hat{\sigma}^2 = \frac{1}{T - k - 1}\sum_{t=1}^{T}\varepsilon_t^2$$

Akaike's Information Criterion

$$\text{AIC} = -2\log(L) + 2(k+2)$$

where L is the likelihood and k is the number of predictors in the model.

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- AIC penalizes terms more heavily than $\bar{R}^2$.
- Minimizing the AIC is asymptotically equivalent to minimizing MSE via **leave-one-out cross-validation** (for any linear regression).

Corrected AIC

For small values of T, the AIC tends to select too many predictors, and so a bias-corrected version of the AIC has been developed.

$$\text{AIC}_\text{C} = \text{AIC} + \frac{2(k+2)(k+3)}{T-k-3}$$

As with the AIC, the AIC_C should be minimized.

Bayesian Information Criterion

$$\mathrm{BIC} = -2\log(L) + (k+2)\log(T)$$

where L is the likelihood and k is the number of predictors in the model.

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where L is the likelihood and k is the number of predictors in the model.

- BIC penalizes terms more heavily than AIC
- Also called SBIC and SC.
- Minimizing BIC is asymptotically equivalent to leave-v-out cross-validation when $v = T[1 - 1/(log(T) - 1)]$.

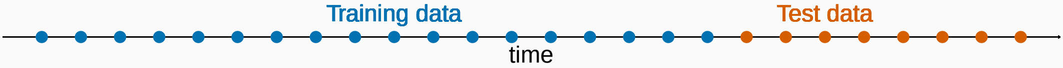
Leave-one-out cross-validation

For regression, leave-one-out cross-validation is faster and more efficient than time-series cross-validation.

- Select one observation for test set, and use *remaining* observations in training set. Compute error on test observation.
- Repeat using each possible observation as the test set.
- Compute accuracy measure over all errors.

Cross-validation

Traditional evaluation



Cross-validation

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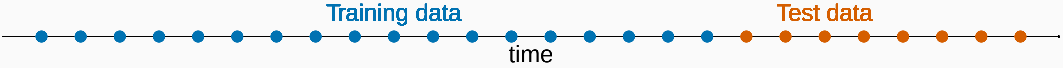
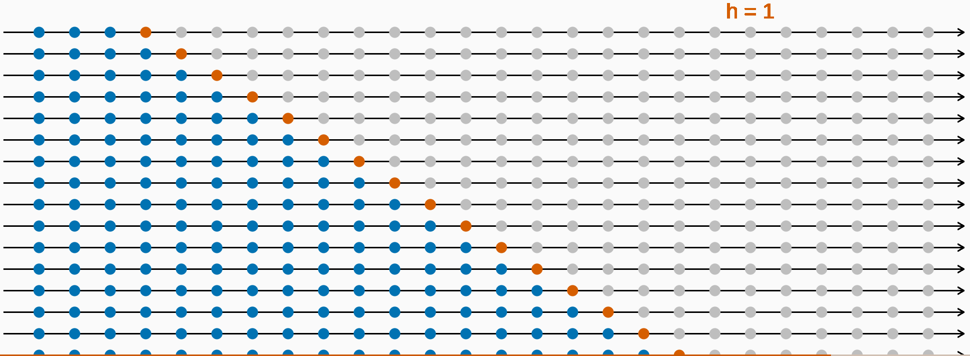


Time series cross-validation



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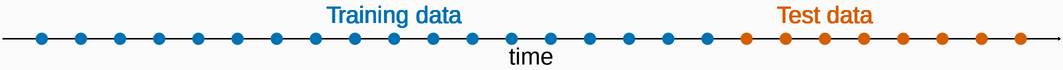
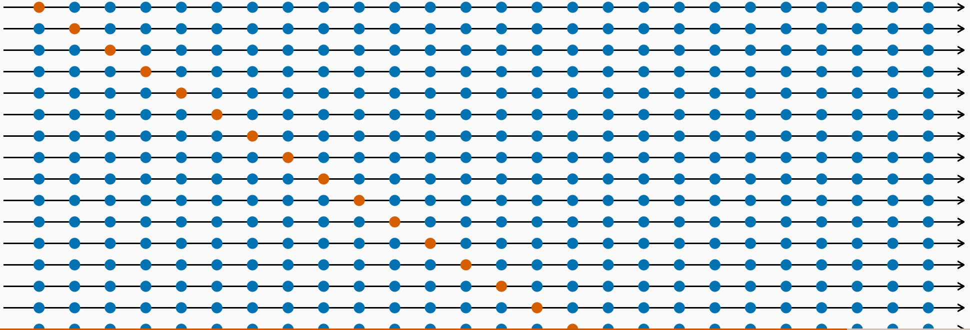


Leave-one-out cross-validation



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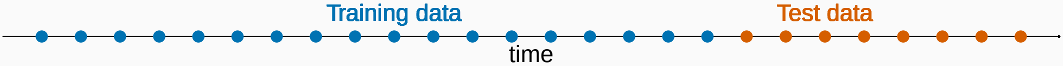

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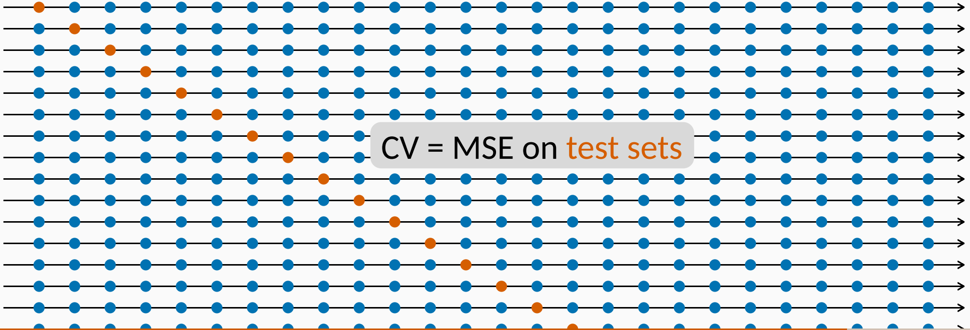

CV = MSE on test sets

Comparing regression models

```
glance(fit_trends) %>%
  select(.model, r_squared, adj_r_squared, AICc, CV)
```

```
## # A tibble: 3 x 5
##   .model       r_squared adj_r_squared  AICc       CV
##   <chr>            <dbl>         <dbl> <dbl>    <dbl>
## 1 linear           0.728         0.726  452. 39.1
## 2 exponential      0.744         0.742 -779.  0.00176
## 3 piecewise        0.767         0.761  438. 34.8
```

- Be careful making comparisons when transformations are used.

Choosing regression variables

Best subsets regression

- Fit all possible regression models using one or more of the predictors.
- Choose the best model based on one of the measures of predictive ability (CV, AIC, AICc).

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Warning!

- If there are a large number of predictors, this is not possible.
- For example, 44 predictors leads to 18 trillion possible models!

Choosing regression variables

Backwards stepwise regression

- Start with a model containing all variables.
- Try subtracting one variable at a time. Keep the model if it has lower CV or AICc.
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Notes

- Stepwise regression is not guaranteed to lead to the best possible model.
- Inference on coefficients of final model will be wrong.

Outline

Ex-ante versus ex-post forecasts

- *Ex ante forecasts* are made using only information available in advance.
  - require forecasts of predictors
- *Ex post forecasts* are made using later information on the predictors.
  - useful for studying behaviour of forecasting models.
- trend, seasonal and calendar variables are all known in advance, so these don't need to be forecast.

Scenario based forecasting

- Assumes possible scenarios for the predictor variables
- Prediction intervals for scenario based forecasts do not include the uncertainty associated with the future values of the predictor variables.

Building a predictive regression model

- If getting forecasts of predictors is difficult, you can use lagged predictors instead.

$$y_t = \beta_0 + \beta_1 x_{1,t-h} + \cdots + \beta_k x_{k,t-h} + \varepsilon_t$$

- A different model for each forecast horizon h.

US Consumption

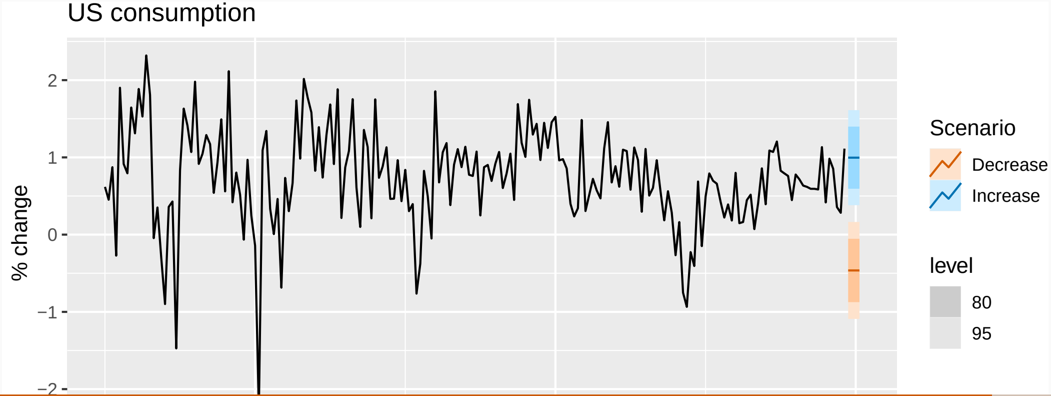
```
fit_consBest <- us_change %>%
  model(
    TSLM(Consumption ~ Income + Savings + Unemployment)
  )

future_scenarios <- scenarios(
  Increase = new_data(us_change, 4) %>%
    mutate(Income=1, Savings=0.5, Unemployment=0),
  Decrease = new_data(us_change, 4) %>%
    mutate(Income=-1, Savings=-0.5, Unemployment=0),
  names_to = "Scenario")

fc <- forecast(fit_consBest, new_data = future_scenarios)
```

US Consumption

```
us_change %>% autoplot(Consumption) +
  labs(y="% change in US consumption") +
  autolayer(fc) +
  labs(title = "US consumption", y = "% change")
```



Outline

Matrix formulation

$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t.$$

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Let $\boldsymbol{y} = (y_1, \dots, y_T)'$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_T)'$, $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_k)'$ and

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_{1,1} & x_{2,1} & \dots & x_{k,1} \\ 1 & x_{1,2} & x_{2,2} & \dots & x_{k,2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{1,T} & x_{2,T} & \dots & x_{k,T} \end{bmatrix}.$$

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Then

$$\boldsymbol{y} = \boldsymbol{X}\beta + \boldsymbol{\varepsilon}.$$

Matrix formulation

Least squares estimation

Minimize: $(\boldsymbol{y} - \boldsymbol{X}\beta)'(\boldsymbol{y} - \boldsymbol{X}\beta)$

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(The “normal equation”.)

$$\hat{\sigma}^2 = \frac{1}{T-k-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\beta})'(\boldsymbol{y} - \boldsymbol{X}\hat{\beta})$$

Note: If you fall for the dummy variable trap, $(\boldsymbol{X}'\boldsymbol{X})$ is a singular matrix.

Likelihood

If the errors are iid and normally distributed, then

$$\boldsymbol{y} \sim \mathrm{N}(\boldsymbol{X}\beta, \sigma^2 \boldsymbol{I}).$$

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So MLE = OLS.

Multiple regression forecasts

Optimal forecasts

$$\hat{y}^* = \text{E}(y^*|\boldsymbol{y}, \boldsymbol{X}, \boldsymbol{x}^*) = \boldsymbol{x}^*\hat{\boldsymbol{\beta}} = \boldsymbol{x}^*(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$$

where $\boldsymbol{x}^*$ is a row vector containing the values of the predictors for the forecasts (in the same format as $\boldsymbol{X}$).

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Forecast variance

$$\text{Var}(y^*|\boldsymbol{X}, \boldsymbol{x}^*) = \sigma^2\left[1 + \boldsymbol{x}^*(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{x}^*)'\right]$$

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Forecast variance

$$\text{Var}(y^*|\boldsymbol{X}, \boldsymbol{x}^*) = \sigma^2\left[1 + \boldsymbol{x}^*(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{x}^*)'\right]$$

- This ignores any errors in $\boldsymbol{x}^*$.
- 95% prediction intervals assuming normal errors:

$$\hat{y}^* \pm 1.96\sqrt{\text{Var}(y^*|\boldsymbol{X}, \boldsymbol{x}^*)}.$$

Outline

Correlation is not causation

- When x is useful for predicting y, it is not necessarily causing y.
- e.g., predict number of drownings y using number of ice-creams sold x.
- Correlations are useful for forecasting, even when there is no causality.
- Better models usually involve causal relationships (e.g., temperature x and people z to predict drownings y).

Multicollinearity

In regression analysis, multicollinearity occurs when:

- Two predictors are highly correlated (i.e., the correlation between them is close to ± 1).
- A linear combination of some of the predictors is highly correlated with another predictor.
- A linear combination of one subset of predictors is highly correlated with a linear combination of another subset of predictors.

Multicollinearity

If multicollinearity exists. . .

- the numerical estimates of coefficients may be wrong (worse in Excel than in a statistics package)
- don't rely on the *p*-values to determine significance.
- there is no problem with model *predictions* provided the predictors used for forecasting are within the range used for fitting.
- omitting variables can help.
- combining variables can help.