MONASH University

MONASH BUSINESS SCHOOL

ETC3550/ETC5550 Applied forecasting

Ch10. Dynamic regression models

OTexts.org/fpp3/

Outline

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Regression with ARIMA errors

Regression models

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t,$$

- y_t modeled as function of k explanatory variables $x_{1,t}, \dots, x_{k,t}$.
- In regression, we assume that ε_t was WN.
- Now we want to allow ε_t to be autocorrelated.

Regression with ARIMA errors

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- In regression, we assume that ε_t was WN.
- Now we want to allow ε_t to be autocorrelated.

Example: ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
$$(1 - \phi_1 B)(1 - B)\eta_t = (1 + \theta_1 B)\varepsilon_t,$$

where ε_t is white noise.

Residuals and errors

Example: η_t = ARIMA(1,1,1)

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$

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Residuals and errors

Example: η_t = ARIMA(1,1,1)

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$$(1-\phi_1 B)(1-B)\eta_t = (1+\theta_1 B)\varepsilon_t,$$

- Be careful in distinguishing η_t from ε_t.
- Only the errors ε_t are assumed to be white noise.
- In ordinary regression, η_t is assumed to be white noise and so $\eta_t = \varepsilon_t$.

Estimation

If we minimize $\sum \eta_t^2$ (by using ordinary regression):

1. Estimated coefficients $\hat{\beta}_0, \dots, \hat{\beta}_k$ are no longer optimal as some information ignored;
2. Statistical tests associated with the model (e.g., t-tests on the coefficients) are incorrect.
3. *p*-values for coefficients usually too small (“spurious regression’ ’).
4. AIC of fitted models misleading.

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4. AIC of fitted models misleading.

- Minimizing $\sum \varepsilon_t^2$ avoids these problems.
- Maximizing likelihood similar to minimizing $\sum \varepsilon_t^2$.

Stationarity

Regression with ARMA errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$

where η_t is an ARMA process.

- All variables in the model must be stationary.
- If we estimate the model while any of these are non-stationary, the estimated coefficients can be incorrect.
- Difference variables until all stationary.
- If necessary, apply same differencing to all variables.

Stationarity

Model with ARIMA(1,1,1) errors

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t,$$
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Equivalent to model with ARIMA(1,0,1) errors

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t,$$
$$(1 - \phi_1 B)\eta'_t = (1 + \theta_1 B)\varepsilon_t,$$

where $y'_t = y_t - y_{t-1}$, $x'_{t,i} = x_{t,i} - x_{t-1,i}$ and $\eta'_t = \eta_t - \eta_{t-1}$.

Regression with ARIMA errors

Any regression with an ARIMA error can be rewritten as a regression with an ARMA error by differencing all variables with the same differencing operator as in the ARIMA model.

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Original data

$$y_t = \beta_0 + \beta_1 x_{1,t} + \cdots + \beta_k x_{k,t} + \eta_t$$
$$\text{where} \quad \phi(B)(1-B)^d \eta_t = \theta(B)\varepsilon_t$$

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After differencing all variables

$$y'_t = \beta_1 x'_{1,t} + \cdots + \beta_k x'_{k,t} + \eta'_t.$$
$$\text{where} \quad \phi(B)\eta_t = \theta(B)\varepsilon_t$$
$$\text{and} \quad y'_t = (1-B)^d y_t$$

Model selection

- Fit regression model with automatically selected ARIMA errors. (R will take care of differencing before estimation.)
- Check that ε_t series looks like white noise.

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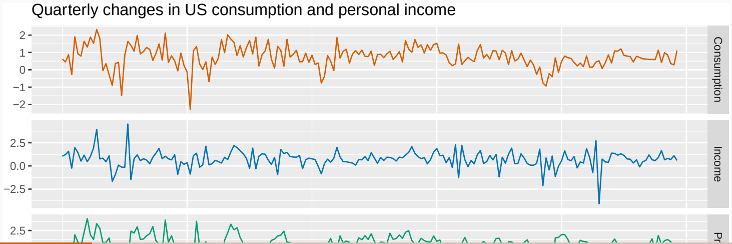
Selecting predictors

- AICc can be calculated for final model.
- Repeat procedure for all subsets of predictors to be considered, and select model with lowest AICc value.

US personal consumption and income

```
us_change %>%
  gather(key='variable', value='value') %>%
  ggplot(aes(y=value, x=Quarter, group=variabl
  geom_line() + facet_grid(variable ~ ., scale
  labs(y="",title ="Quarterly changes in US co
  guides(colour="none")
```



US personal consumption and income

- No need for transformations or further differencing.
- Increase in income does not necessarily translate into instant increase in consumption (e.g., after the loss of a job, it may take a few months for expenses to be reduced to allow for the new circumstances). We will ignore this for now.

US personal consumption and income

```
fit <- us_change %>% model(ARIMA(Consumption ~ Income))
report(fit)
```

```
## Series: Consumption
## Model: LM w/ ARIMA(1,0,2) errors
##
## Coefficients:
##          ar1      ma1     ma2   Income  intercept
##        0.707  -0.617  0.2066  0.1976      0.595
## s.e.   0.107   0.122  0.0741  0.0462      0.085
##
## sigma^2 estimated as 0.3113:  log likelihood=-163
## AIC=338   AICc=339   BIC=358
```

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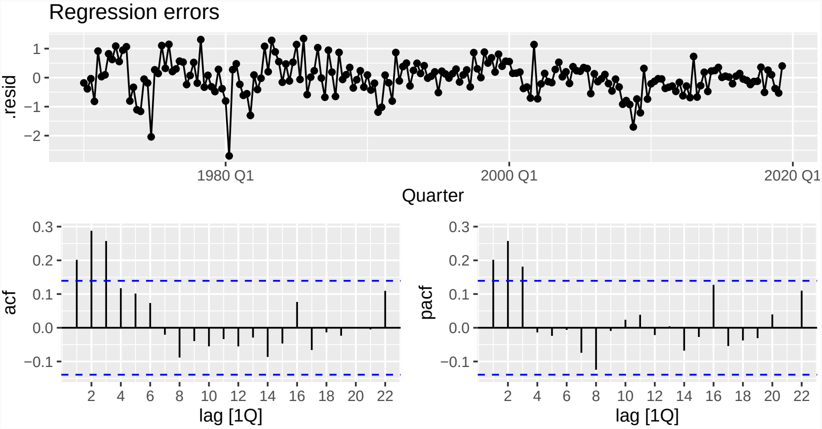
```

Write down the equations for the fitted model.

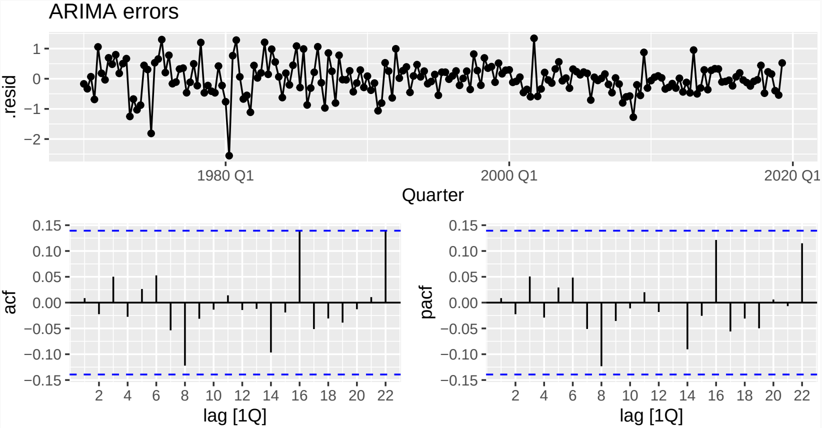
US personal consumption and income

```
residuals(fit, type='regression') %>%
  gg_tsdisplay(.resid, plot_type = 'partial') +
  labs(title = "Regression errors")
```



US personal consumption and income

```
residuals(fit, type='response') %>%
  gg_tsdisplay(.resid, plot_type = 'partial') +
  labs(title = "ARIMA errors")
```



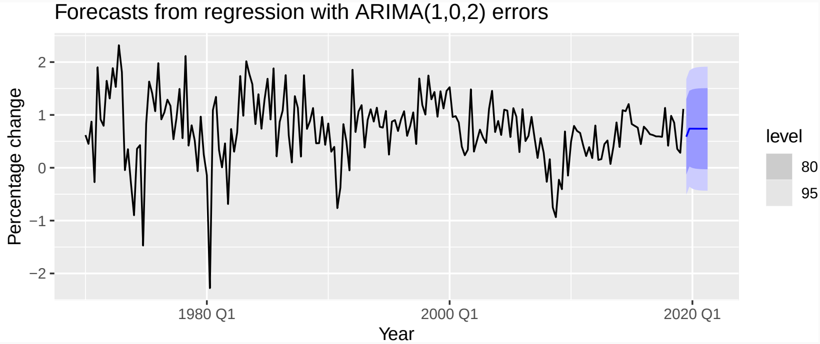
US personal consumption and income

```
augment(fit) %>%
  features(.resid, ljung_box, dof = 5, lag = 12)
```

```
## # A tibble: 1 x 3
##   .model                          lb_stat lb_pvalue
##   <chr>                             <dbl>     <dbl>
## 1 ARIMA(Consumption ~ Income)        5.54     0.595
```

US personal consumption and income

```
us_change_future <- new_data(us_change, 8) %>%
  mutate(Income = mean(us_change$Income))
forecast(fit, new_data = us_change_future) %>%
  autoplot(us_change) +
  labs(x = "Year", y = "Percentage change",
       title = "Forecasts from regression with ARIMA(1,0,2) errors")
```



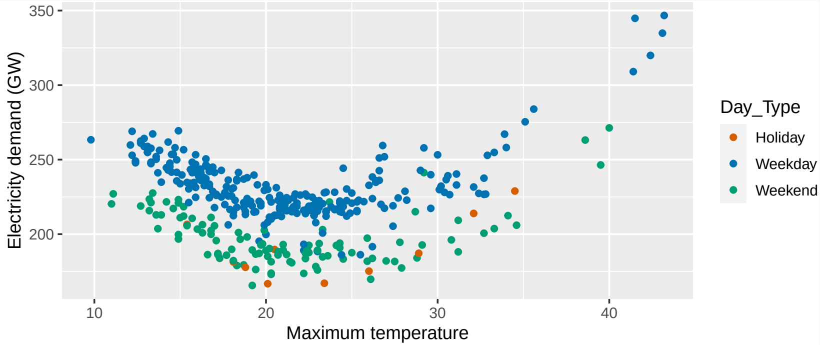
Forecasting

- To forecast a regression model with ARIMA errors, we need to forecast the regression part of the model and the ARIMA part of the model and combine the results.
- Some predictors are known into the future (e.g., time, dummies).
- Separate forecasting models may be needed for other predictors.
- Forecast intervals ignore the uncertainty in forecasting the predictors.

Daily electricity demand

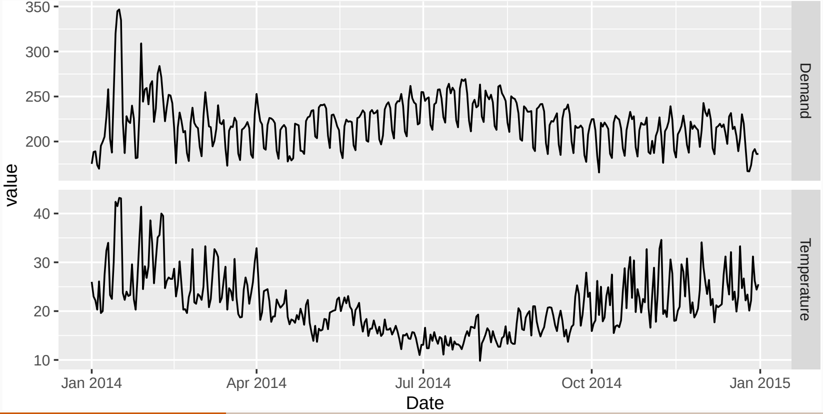
Model daily electricity demand as a function of temperature using quadratic regression with ARMA errors.

```
vic_elec_daily %>%
  ggplot(aes(x=Temperature, y=Demand, colour=Day_Type)) +
  geom_point() +
  labs(x = "Maximum temperature", y = "Electricity demand (GW)")
```



Daily electricity demand

```
vic_elec_daily %>%
  gather("var", "value", Demand, Temperature) %>%
  ggplot(aes(x = Date, y = value)) + geom_line() +
  facet_grid(vars(var), scales = "free_y")
```



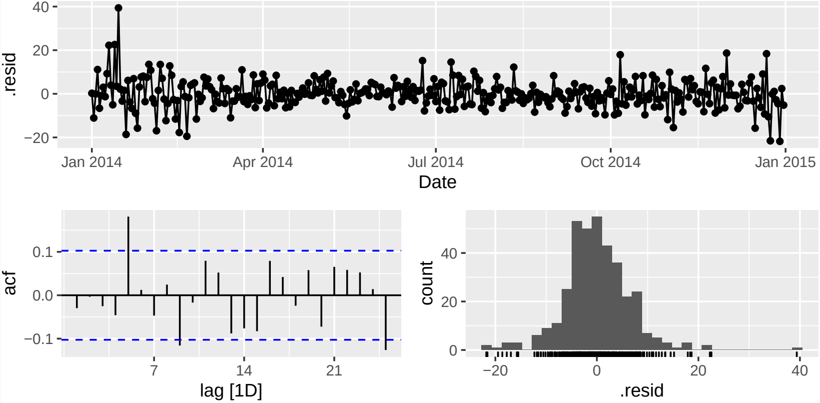
Daily electricity demand

```
fit <- vic_elec_daily %>%
  model(ARIMA(Demand ~ Temperature + I(Temperature^2) +
                (Day_Type=="Weekday")))
report(fit)
```

```
## Series: Demand
## Model: LM w/ ARIMA(2,1,2)(2,0,0)[7] errors
##
## Coefficients:
##           ar1     ar2      ma1      ma2    sar1   sar2
##       -0.1093  0.7226  -0.0182  -0.9381  0.1958  0.417
## s.e.   0.0779  0.0739   0.0494   0.0493  0.0525  0.057
##       Temperature  I(Temperature^2)
##            -7.614            0.1810
## s.e.        0.448            0.0085
##       Day_Type == "Weekday"TRUE
##                           30.40
## s.e.                       1.33
##
```

Daily electricity demand

```
gg_tsresiduals(fit)
```



Daily electricity demand

```r
augment(fit) %>%
  features(.resid, ljung_box, dof = 9, lag = 14)
```

```
## # A tibble: 1 x 3
##   .model                                            lb_stat lb_pvalue
##   <chr>                                               <dbl>     <dbl>
## 1 "ARIMA(Demand ~ Temperature + I(Tempera~    28.4 0.0000304
```

Daily electricity demand

```r
# Forecast one day ahead
vic_next_day <- new_data(vic_elec_daily, 1) %>%
  mutate(Temperature = 26, Day_Type = "Holiday")
forecast(fit, vic_next_day)
```

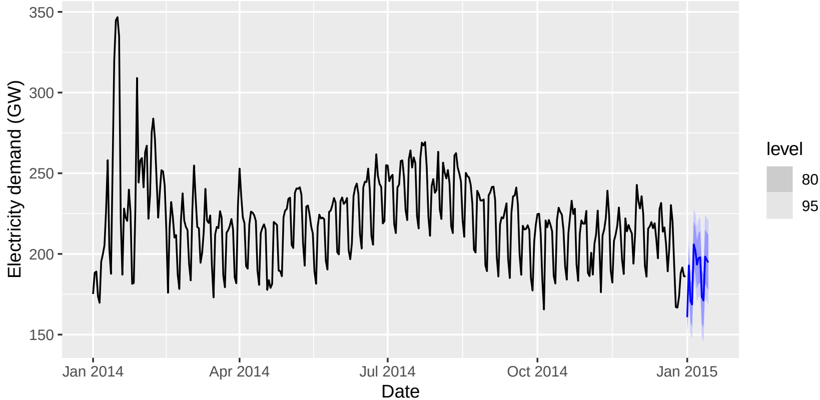
```
## # A fable: 1 x 6 [1D]
## # Key:     .model [1]
##   .model     Date             Demand .mean Temperature Day_Type
##   <chr>      <date>           <dist> <dbl>       <dbl> <chr>
## 1 "ARIMA(D~ 2015-01-01 N(161, 45)  161.          26 Holiday
```

Daily electricity demand

```r
vic_elec_future <- new_data(vic_elec_daily, 14) %>%
  mutate(
    Temperature = 26,
    Holiday = c(TRUE, rep(FALSE, 13)),
    Day_Type = case_when(
      Holiday ~ "Holiday",
      wday(Date) %in% 2:6 ~ "Weekday",
      TRUE ~ "Weekend"
    )
  )
```

Daily electricity demand

```
forecast(fit, new_data=vic_elec_future) %>%
  autoplot(vic_elec_daily) + labs(y="Electricity demand (GW)")
```



Outline

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARMA process.

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Stochastic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARIMA process with $d \geq 1$.

Stochastic & deterministic trends

Deterministic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARMA process.

Stochastic trend

$$y_t = \beta_0 + \beta_1 t + \eta_t$$

where η_t is ARIMA process with $d \geq 1$.

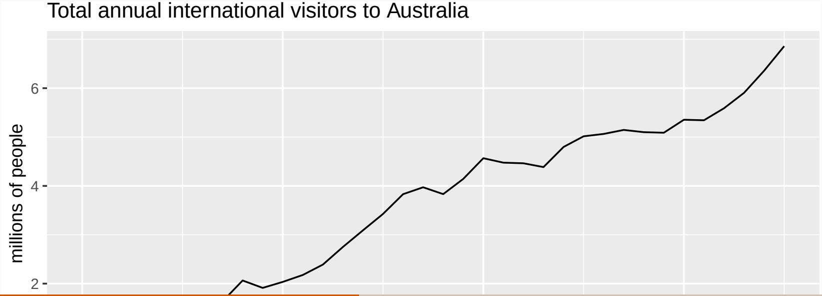
Difference both sides until η_t is stationary:

$$y'_t = \beta_1 + \eta'_t$$

where η'_t is ARMA process.

International visitors

```
aus_visitors <- as_tsibble(fpp2::austa)
aus_visitors %>%
  autoplot(value) +
  labs(x = "Year", y = "millions of people",
       title = "Total annual international vis
```



International visitors

Deterministic trend

```
fit_deterministic <- aus_visitors %>%
  model(Deterministic = ARIMA(value ~ trend() + pdq(d = 0)))
report(fit_deterministic)
```

```
## Series: value
## Model: LM w/ ARIMA(2,0,0) errors
##
## Coefficients:
##          ar1      ar2  trend()  intercept
##         1.11  -0.381   0.1710      0.416
## s.e.   0.16    0.159   0.0088      0.190
##
## sigma^2 estimated as 0.02979:  log likelihood=13.6
## AIC=-17.2   AICc=-15.2   BIC=-9.28
```

International visitors

Deterministic trend

```
fit_deterministic <- aus_visitors %>%
  model(Deterministic = ARIMA(value ~ trend() + pdq(d = 0)))
report(fit_deterministic)
```

```
## Series: value
## Model: LM w/ ARIMA(2,0,0) errors
##
## Coefficients:
##          ar1      ar2  trend()  intercept
##         1.11  -0.381   0.1710      0.416
## s.e.    0.16   0.159   0.0088      0.190
##
## sigma^2 estimated as 0.02979:  log likelihood=13.6
## AIC=-17.2   AICc=-15.2   BIC=-9.28
```

$$y_t = 0.42 + 0.17t + \eta_t$$
$$\eta_t = 1.11\eta_{t-1} - 0.38\eta_{t-2} + \varepsilon_t$$
$$\varepsilon_t \sim \text{NID}(0, 0.0298).$$

International visitors

Stochastic trend

```
fit_stochastic <- aus_visitors %>%
  model(Stochastic = ARIMA(value ~ pdq(d=1)))
report(fit_stochastic)
```

```
## Series: value
## Model: ARIMA(0,1,1) w/ drift
##
## Coefficients:
##          ma1  constant
##        0.301     0.173
## s.e.   0.165     0.039
##
## sigma^2 estimated as 0.03376:  log likelihood=10.6
## AIC=-15.2   AICc=-14.5   BIC=-10.6
```

International visitors

Stochastic trend

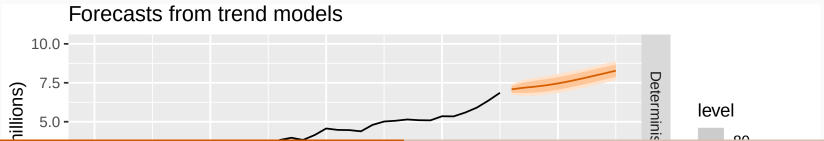
```
fit_stochastic <- aus_visitors %>%
  model(Stochastic = ARIMA(value ~ pdq(d=1)))
report(fit_stochastic)
```

```
## Series: value
## Model: ARIMA(0,1,1) w/ drift
##
## Coefficients:
##          ma1  constant
##        0.301     0.173
## s.e.   0.165     0.039
##
## sigma^2 estimated as 0.03376:  log likelihood=10.6
## AIC=-15.2   AICc=-14.5   BIC=-10.6
```

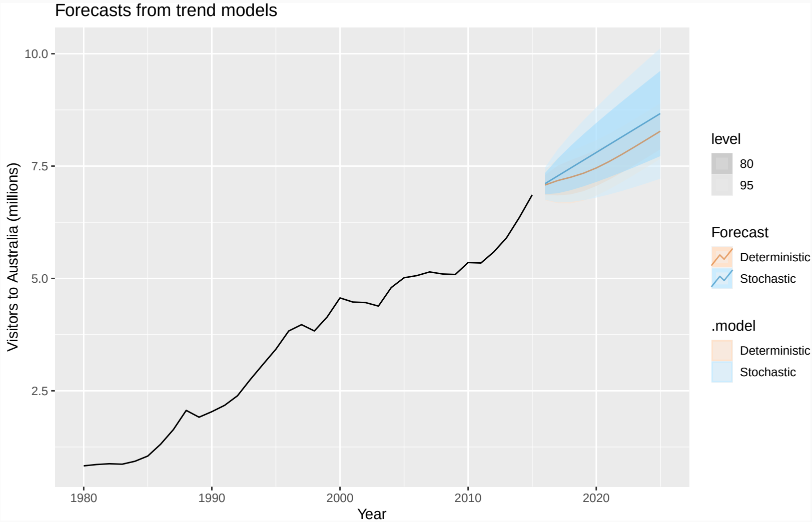
$$y_t - y_{t-1} = 0.173 + \varepsilon_t + 0.301\varepsilon_{t-1}$$
$$y_t = y_0 + 0.173t + \eta_t$$
$$\eta_t = \eta_{t-1} + 0.301\varepsilon_{t-1} + \varepsilon_t$$

International visitors

```
fc_deterministic <- forecast(fit_deterministic
fc_stochastic <- forecast(fit_stochastic, h =
rbind(fc_deterministic, fc_stochastic) %>%
  autoplot(aus_visitors) +
  facet_grid(vars(.model)) +
  labs(x = "Year", y = "Visitors to Australia
       title = "Forecasts from trend models")
  guides(colour = FALSE)
```



International visitors



Forecasting with trend

- Point forecasts are almost identical, but prediction intervals differ.
- Stochastic trends have much wider prediction intervals because the errors are non-stationary.
- Be careful of forecasting with deterministic trends too far ahead.

Outline

Dynamic harmonic regression

Combine Fourier terms with ARIMA errors

Advantages

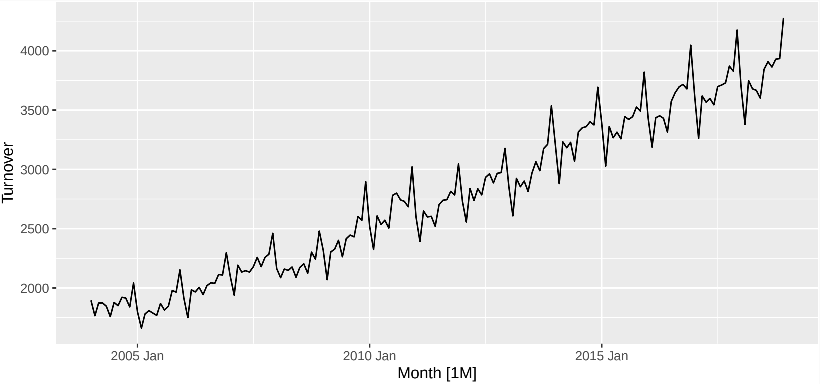
- it allows any length seasonality;
- for data with more than one seasonal period, you can include Fourier terms of different frequencies;
- the seasonal pattern is smooth for small values of K (but more wiggly seasonality can be handled by increasing K);
- the short-term dynamics are easily handled with a simple ARMA error.

Disadvantages

- seasonality is assumed to be fixed

Eating-out expenditure

```r
aus_cafe <- aus_retail %>% filter(
    Industry == "Cafes, restaurants and takeaway food services",
    year(Month) %in% 2004:2018
  ) %>% summarise(Turnover = sum(Turnover))
aus_cafe %>% autoplot(Turnover)
```



Eating-out expenditure

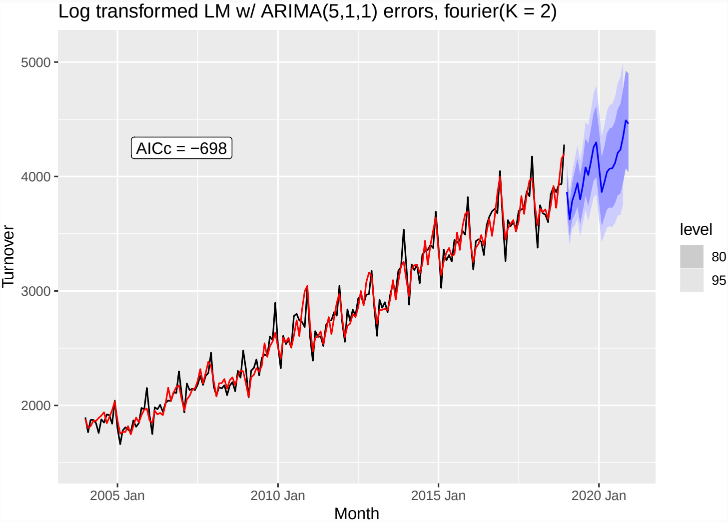
```
fit <- aus_cafe %>% model(
    `K = 1` = ARIMA(log(Turnover) ~ fourier(K = 1) + PDQ(0,0,0)),
    `K = 2` = ARIMA(log(Turnover) ~ fourier(K = 2) + PDQ(0,0,0)),
    `K = 3` = ARIMA(log(Turnover) ~ fourier(K = 3) + PDQ(0,0,0)),
    `K = 4` = ARIMA(log(Turnover) ~ fourier(K = 4) + PDQ(0,0,0)),
    `K = 5` = ARIMA(log(Turnover) ~ fourier(K = 5) + PDQ(0,0,0)),
    `K = 6` = ARIMA(log(Turnover) ~ fourier(K = 6) + PDQ(0,0,0)))
glance(fit)
```

.model	sigma2	log_lik	AIC	AICc	BIC
K = 1	0.002	317	-616	-615	-588
K = 2	0.001	362	-700	-698	-661
K = 3	0.001	394	-763	-761	-725
K = 4	0.001	427	-822	-818	-771
K = 5	0.000	474	-919	-917	-875
K = 6	0.000	474	-920	-918	-875

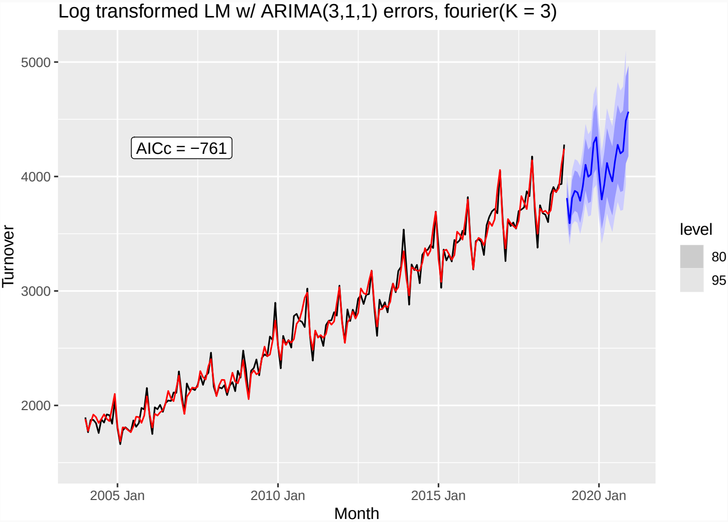
Eating-out expenditure

```
cafe_plot <- function(...){
  fit %>%
    select(...) %>%
    forecast() %>% autoplot(aus_cafe) +
    labs(title = sprintf("Log transformed %s,
    geom_label(
      aes(x = yearmonth("2007 Jan"), y = 4250,
      data = glance(select(fit,...))
    ) +
    geom_line(aes(y = .fitted), colour = "red"
    ylim(c(1500, 5100))
```

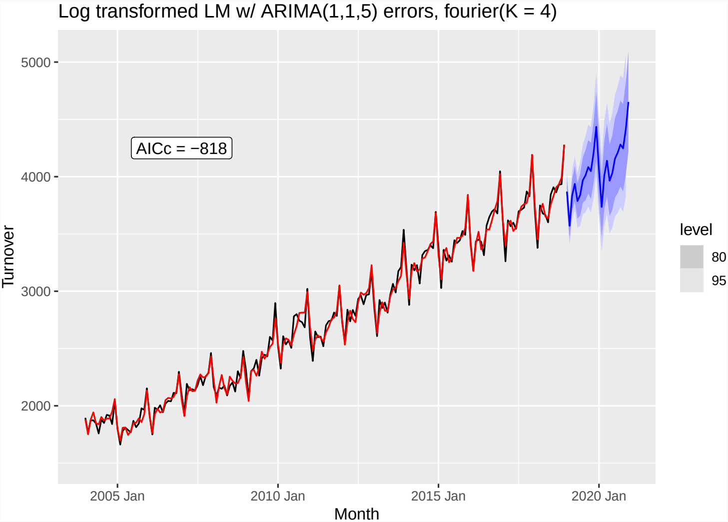
Eating-out expenditure



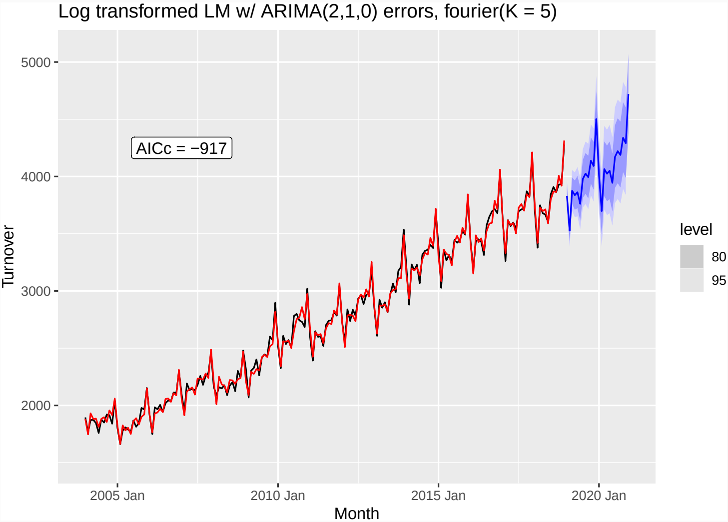
Eating-out expenditure



Eating-out expenditure



Eating-out expenditure



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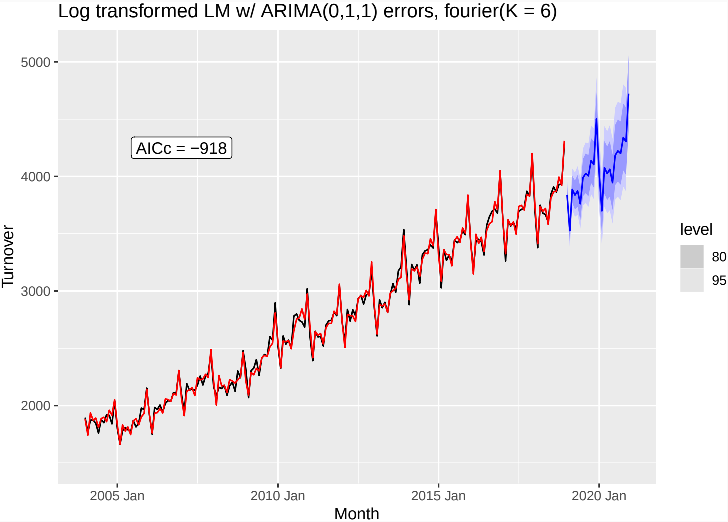


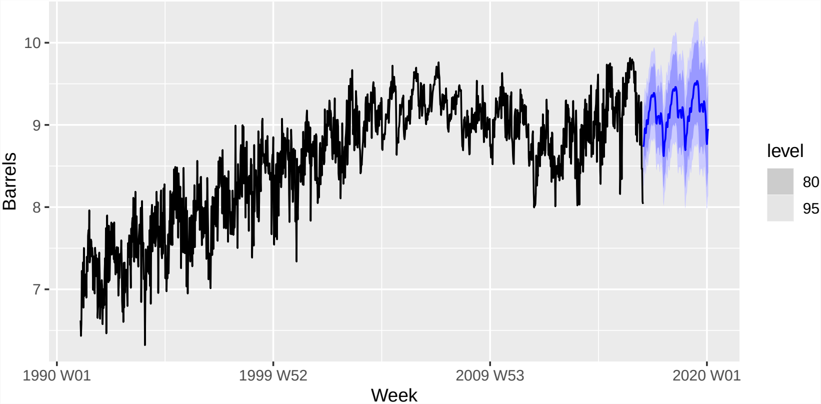
Example: weekly gasoline products

```r
fit <- us_gasoline %>%
  model(ARIMA(Barrels ~ fourier(K = 13) + PDQ(0,0,0)))
report(fit)
```

```
## Series: Barrels
## Model: LM w/ ARIMA(0,1,1) errors
##
## Coefficients:
##           ma1  fourier(K = 13)C1_52  fourier(K = 13)S1_52
##       -0.8934               -0.1121               -0.2300
## s.e.   0.0132                0.0123                0.0122
##       fourier(K = 13)C2_52  fourier(K = 13)S2_52
##                     0.0420                0.0317
## s.e.                0.0099                0.0099
##       fourier(K = 13)C3_52  fourier(K = 13)S3_52
##                     0.0832                0.0346
## s.e.                0.0094                0.0094
##       fourier(K = 13)C4_52  fourier(K = 13)S4_52
##                     0.0185                0.0398
## s.e.                0.0092                0.0092
##       fourier(K = 13)C5_52  fourier(K = 13)S5_52
##                    -0.0315                0.0009
## s.e.                0.0091                0.0091
##       fourier(K = 13)C6_52  fourier(K = 13)S6_52
```

Example: weekly gasoline products

```
forecast(fit, h = "3 years") %>%
  autoplot(us_gasoline)
```



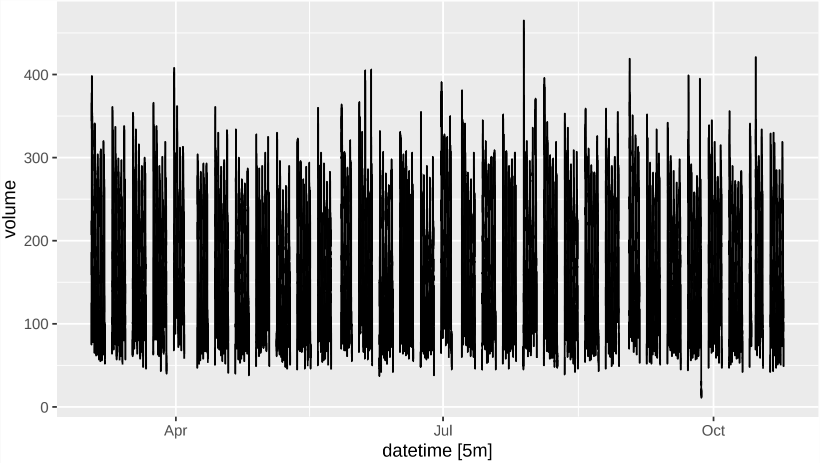
5-minute call centre volume

```
(calls <- readr::read_tsv("http://robjhyndman.com/data/callcenter.txt") %
  rename(time = X1) %>%
  pivot_longer(-time, names_to = "date", values_to = "volume") %>%
  mutate(
    date = as.Date(date, format = "%d/%m/%Y"),
    datetime = as_datetime(date) + time
  ) %>%
  as_tsibble(index = datetime))
```

```
## # A tsibble: 27,716 x 4 [5m] <UTC>
##    time   date       volume datetime
##    <time> <date>      <dbl> <dttm>
##  1 07:00  2003-03-03    111 2003-03-03 07:00:00
##  2 07:05  2003-03-03    113 2003-03-03 07:05:00
##  3 07:10  2003-03-03     76 2003-03-03 07:10:00
##  4 07:15  2003-03-03     82 2003-03-03 07:15:00
##  5 07:20  2003-03-03     91 2003-03-03 07:20:00
##  6 07:25  2003-03-03     87 2003-03-03 07:25:00
##  7 07:30  2003-03-03     75 2003-03-03 07:30:00
```

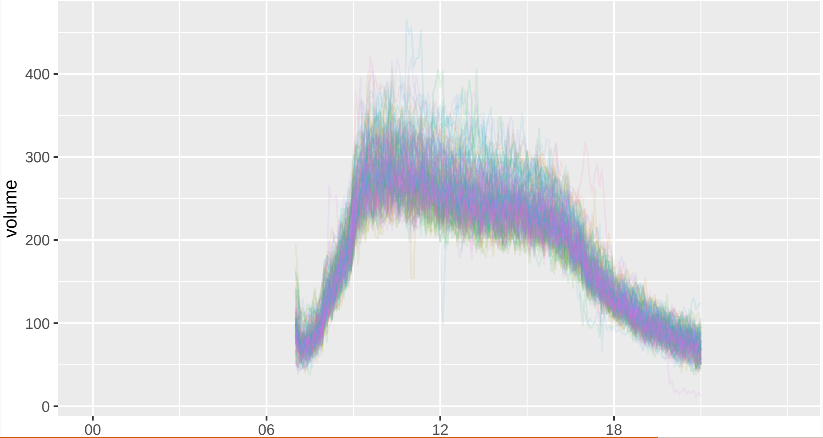
5-minute call centre volume

```
calls %>% fill_gaps() %>% autoplot(volume)
```



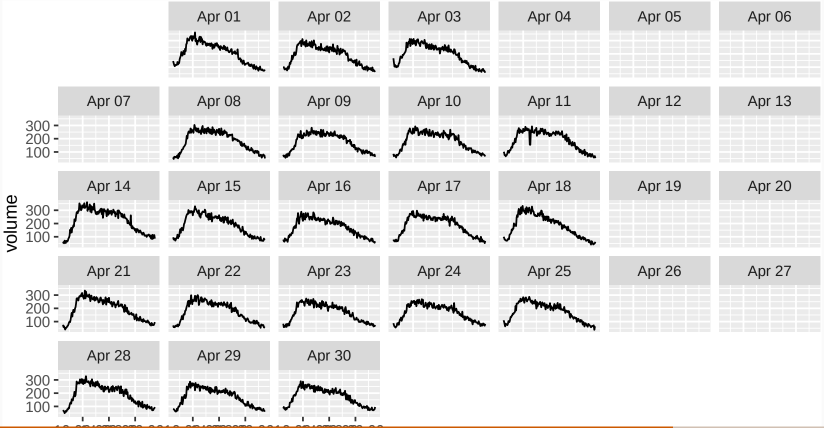
5-minute call centre volume

```
calls %>% fill_gaps() %>%
  gg_season(volume, period = "day", alpha = 0.1) +
  guides(colour = FALSE)
```



5-minute call centre volume

```
library(sugrrants)
calls %>% filter(month(date, label = TRUE) == "Apr") %>%
  ggplot(aes(x = time, y = volume)) +
  geom_line() + facet_calendar(date)
```



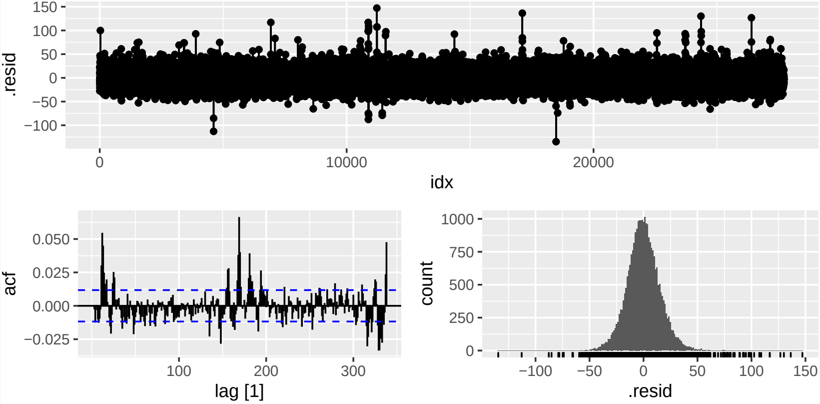
5-minute call centre volume

```
calls_mdl <- calls %>%
  mutate(idx = row_number()) %>%
  update_tsibble(index = idx)
fit <- calls_mdl %>%
  model(ARIMA(volume ~ fourier(169, K = 10) + pdq(d=0) + PDQ(0,0,0)))
report(fit)
```

```
## Series: volume
## Model: LM w/ ARIMA(1,0,3) errors
##
## Coefficients:
##          ar1      ma1      ma2      ma3  fourier(169, K = 10)C1_169
##        0.989  -0.7383  -0.0333  -0.0282                       -79.1
## s.e.   0.001   0.0061   0.0075   0.0060                         0.7
##        fourier(169, K = 10)S1_169  fourier(169, K = 10)C2_169
##                            55.298                     -32.361
## s.e.                        0.701                       0.378
##        fourier(169, K = 10)S2_169  fourier(169, K = 10)C3_169
##                            13.742                      -9.318
## s.e.                        0.379                       0.273
##        fourier(169, K = 10)S3_169  fourier(169, K = 10)C4_169
##                           -13.645                      -2.791
## s.e.                        0.273                       0.223
##        fourier(169, K = 10)S4_169  fourier(169, K = 10)C5_169
```

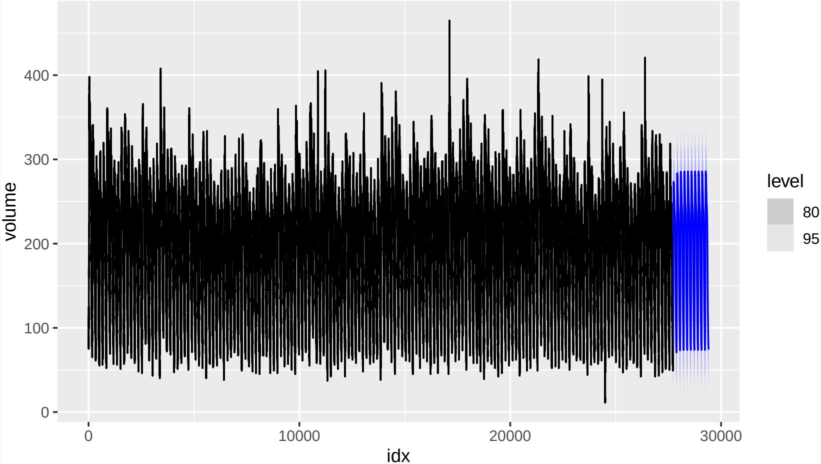
5-minute call centre volume

```
gg_tsresiduals(fit, lag = 338)
```



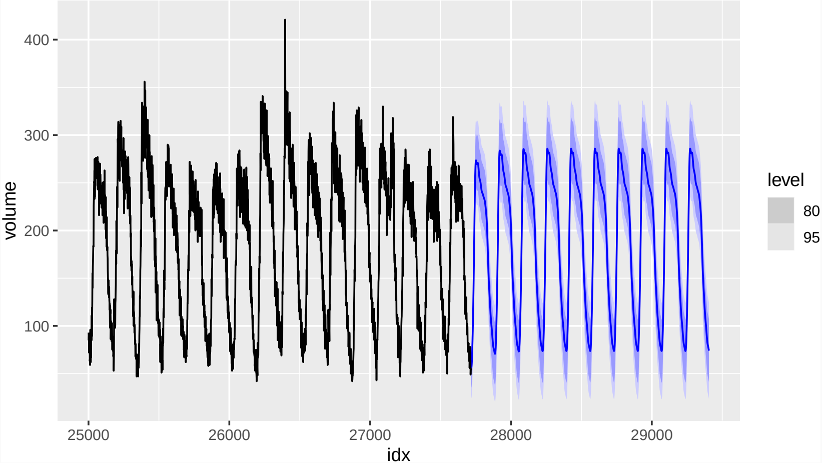
5-minute call centre volume

```r
fit %>% forecast(h = 1690) %>%
  autoplot(calls_mdl)
```



5-minute call centre volume

```
fit %>% forecast(h = 1690) %>%
  autoplot(filter(calls_mdl, idx > 25000))
```



Outline

Lagged predictors

Sometimes a change in x_t does not affect y_t instantaneously

Lagged predictors

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- y_t = sales, x_t = advertising.
- y_t = stream flow, x_t = rainfall.
- y_t = size of herd, x_t = breeding stock.

Lagged predictors

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- y_t = sales, x_t = advertising.
- y_t = stream flow, x_t = rainfall.
- y_t = size of herd, x_t = breeding stock.

- These are dynamic systems with input (x_t) and output (y_t).
- x_t is often a leading indicator.
- There can be multiple predictors.

Lagged predictors

The model include present and past values of predictor: $x_t, x_{t-1}, x_{t-2}, \ldots.$

$$y_t = a + \gamma_0 x_t + \gamma_1 x_{t-1} + \cdots + \gamma_k x_{t-k} + \eta_t$$

where η_t is an ARIMA process.

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where η_t is an ARIMA process.

Rewrite model as

$$\begin{aligned} y_t &= a + (\gamma_0 + \gamma_1 B + \gamma_2 B^2 + \cdots + \gamma_k B^k) x_t + \eta_t \\ &= a + \gamma(B) x_t + \eta_t. \end{aligned}$$

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The model include present and past values of predictor: $x_t, x_{t-1}, x_{t-2}, \ldots$.

$$y_t = a + \gamma_0 x_t + \gamma_1 x_{t-1} + \cdots + \gamma_k x_{t-k} + \eta_t$$

where η_t is an ARIMA process.

Rewrite model as

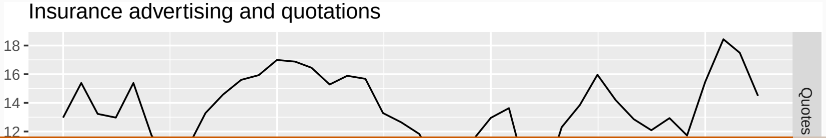
$$\begin{aligned} y_t &= a + (\gamma_0 + \gamma_1 B + \gamma_2 B^2 + \cdots + \gamma_k B^k) x_t + \eta_t \\ &= a + \gamma(B) x_t + \eta_t. \end{aligned}$$

- $\gamma(B)$ is called a *transfer function* since it describes how change in x_t is transferred to y_t.
- x can influence y, but y is not allowed to influence x.

Example: Insurance quotes and TV adverts

```r
insurance <- as_tsibble(fpp2::insurance, pivot
  rename(Month = index)

insurance %>%
  pivot_longer(c(Quotes, TV.advert)) %>%
  ggplot(aes(x = Month, y = value)) + geom_lin
  facet_grid(vars(name), scales = "free_y") +
  labs(x = "Year", y = NULL, title = "Insuranc
```



Example: Insurance quotes and TV adverts

```
insurance %>%
  mutate(
    lag1 = lag(TV.advert),
    lag2 = lag(lag1)
  ) %>%
  as_tibble() %>%
  select(-Month) %>%
  rename(lag0 = TV.advert) %>%
  pivot_longer(-Quotes, names_to="Lag", values
  ggplot(aes(x = TV_advert, y = Quotes)) + geo
  facet_grid(. ~ Lag) +
```

Example: Insurance quotes and TV adverts

```
fit <- insurance %>%
  # Restrict data so models use same fitting period
  mutate(Quotes = c(NA,NA,NA,Quotes[4:40])) %>%
  # Estimate models
  model(
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert),
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert)),
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert) +
                lag(TV.advert, 2)),
    ARIMA(Quotes ~ pdq(d = 0) + TV.advert + lag(TV.advert) +
                lag(TV.advert, 2) + lag(TV.advert, 3))
  )
```

Example: Insurance quotes and TV adverts

```
glance(fit)
```

Lag order	sigma2	log_lik	AIC	AICc	BIC
0	0.265	-28.3	66.6	68.3	75.0
1	0.209	-24.0	58.1	59.9	66.5
2	0.215	-24.0	60.0	62.6	70.2
3	0.206	-22.2	60.3	65.0	73.8

Example: Insurance quotes and TV adverts

```r
fit <- insurance %>%
  model(ARIMA(Quotes ~ pdq(3, 0, 0) + TV.advert + lag(TV.advert)))
report(fit)
```

```
## Series: Quotes
## Model: LM w/ ARIMA(3,0,0) errors
##
## Coefficients:
##         ar1     ar2    ar3  TV.advert  lag(TV.advert)  intercept
##        1.41  -0.932  0.359     1.2564          0.1625      2.039
## s.e.   0.17   0.255  0.159     0.0667          0.0591      0.993
##
## sigma^2 estimated as 0.2165:  log likelihood=-23.9
## AIC=61.8   AICc=65.3   BIC=73.6
```

Example: Insurance quotes and TV adverts

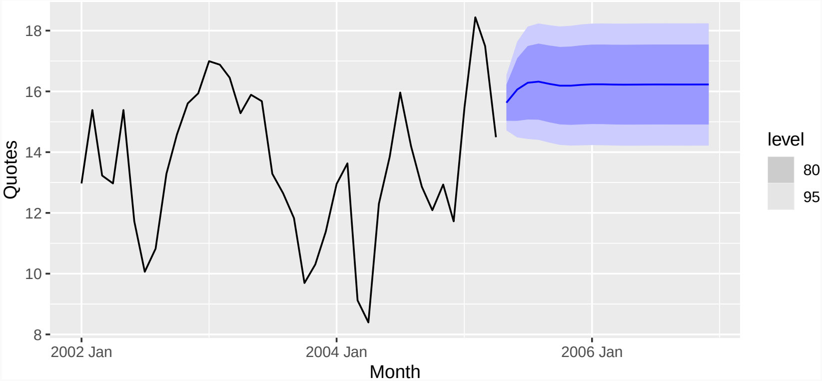
```r
fit <- insurance %>%
  model(ARIMA(Quotes ~ pdq(3, 0, 0) + TV.advert + lag(TV.advert)))
report(fit)
```

```
## Series: Quotes
## Model: LM w/ ARIMA(3,0,0) errors
##
## Coefficients:
##         ar1     ar2    ar3  TV.advert  lag(TV.advert)  intercept
##        1.41  -0.932  0.359     1.2564          0.1625      2.039
## s.e.   0.17   0.255  0.159     0.0667          0.0591      0.993
##
## sigma^2 estimated as 0.2165:  log likelihood=-23.9
## AIC=61.8   AICc=65.3   BIC=73.6
```

$$y_t = 2.04 + 1.26x_t + 0.16x_{t-1} + \eta_t,$$
$$\eta_t = 1.41\eta_{t-1} - 0.932\eta_{t-2} + 0.359\eta_{t-3} + \varepsilon_t$$

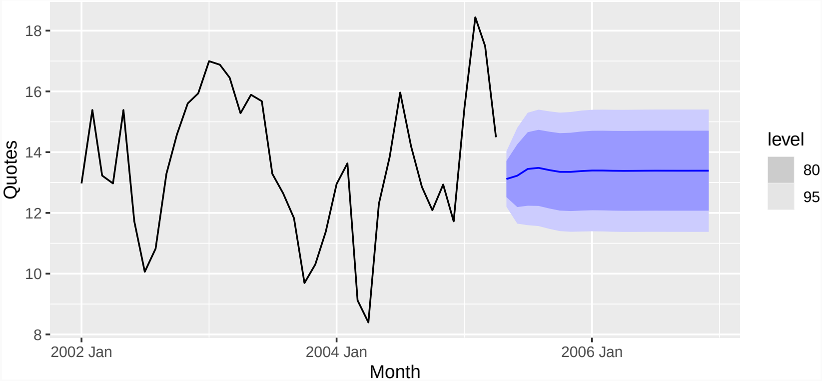
Example: Insurance quotes and TV adverts

```
advert_a <- new_data(insurance, 20) %>%
  mutate(TV.advert = 10)
forecast(fit, advert_a) %>% autoplot(insurance)
```



Example: Insurance quotes and TV adverts

```
advert_b <- new_data(insurance, 20) %>%
  mutate(TV.advert = 8)
forecast(fit, advert_b) %>% autoplot(insurance)
```



Example: Insurance quotes and TV adverts

```
advert_c <- new_data(insurance, 20) %>%
  mutate(TV.advert = 6)
forecast(fit, advert_c) %>% autoplot(insurance)
```