ETC3550/ETC5550 Applied forecasting

Ch7. Regression models

OTexts.org/fpp3/

Outline

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Multiple regression and forecasting

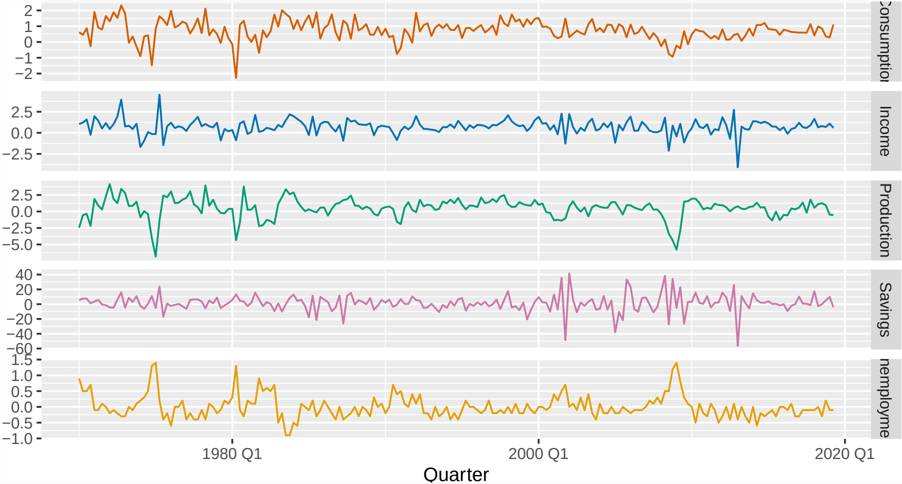
$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t.$$

- y_t is the variable we want to predict: the "response" variable
- Each $x_{j,t}$ is numerical and is called a "predictor". They are usually assumed to be known for all past and future times.
- The coefficients $\beta_1, \dots, \beta_k$ measure the effect of each predictor after taking account of the effect of all other predictors in the model.

That is, the coefficients measure the **marginal effects**.

- ε_t is a white noise error term

Example: US consumption expenditure



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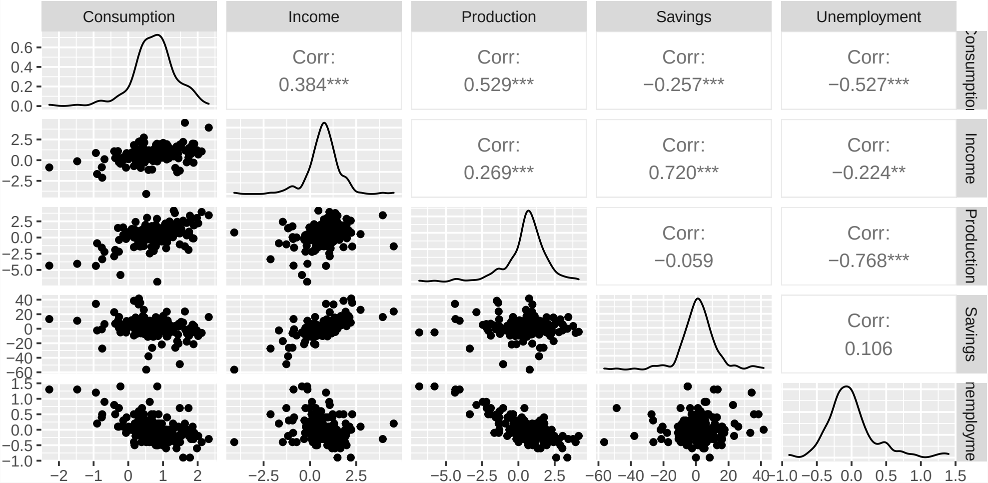
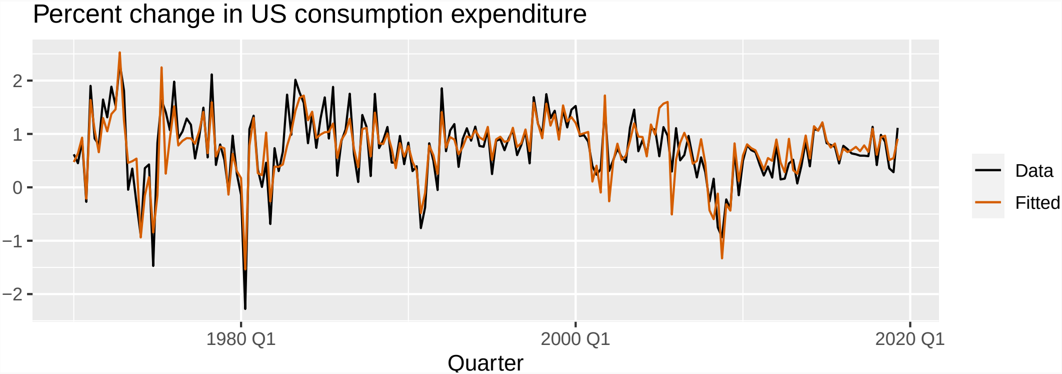


Example: US consumption expenditure

```r
fit_consMR <- us_change %>%
  model(lm = TSLM(Consumption ~ Income + Production + Unemployment + Savings))
report(fit_consMR)
```

```
## Series: Consumption
## Model: TSLM
##
## Residuals:
##     Min      1Q Median      3Q     Max
## -0.906 -0.158 -0.036  0.136  1.155
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   0.25311    0.03447    7.34  5.7e-12 ***
## Income        0.74058    0.04012   18.46  < 2e-16 ***
## Production    0.04717    0.02314    2.04    0.043 *
## Unemployment -0.17469    0.09551   -1.83    0.069 .
## Savings      -0.05289    0.00292  -18.09  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.31 on 193 degrees of freedom
## Multiple R-squared: 0.768,   Adjusted R-squared: 0.763
## F-statistic:  160 on 4 and 193 DF, p-value: <2e-16
```

Example: US consumption expenditure



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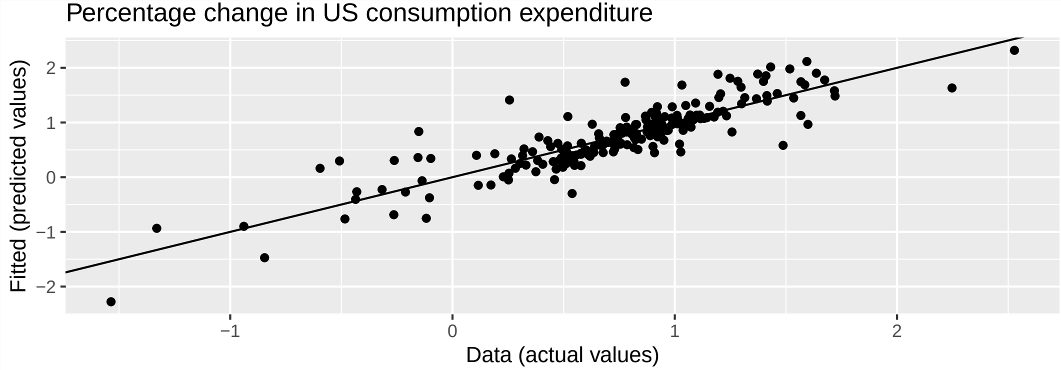
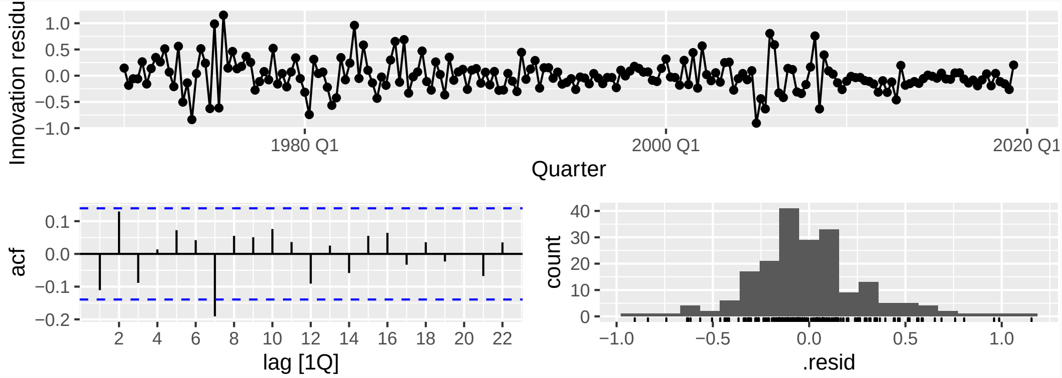


Example: US consumption expenditure

```
fit_consMR %>% gg_tsresiduals()
```



Outline

Trend

Linear trend

$$x_t = t$$

- $t = 1, 2, \ldots, T$
- Strong assumption that trend will continue.

Nonlinear trend

Piecewise linear trend with bend at τ

$$x_{1,t} = t$$

$$x_{2,t} = \begin{cases} 0 & t < \tau \\ (t - \tau) & t \geq \tau \end{cases}$$

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Quadratic or higher order trend

$$x_{1,t} = t, \quad x_{2,t} = t^2, \quad \dots$$

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Quadratic or higher order trend

$$x_{1,t} = t, \quad x_{2,t} = t^2, \quad \dots$$

NOT RECOMMENDED!

Dummy variables

If a categorical variable takes only two values (e.g., ‘Yes’ or ‘No’), then an equivalent numerical variable can be constructed taking value 1 if yes and 0 if no. This is called a **dummy variable**.

	A	B
1	Yes	1
2	Yes	1
3	No	0
4	Yes	1
5	No	0
6	No	0
7	Yes	1
8	Yes	1
9	No	0
10	No	0
11	No	0
12	No	0
13	Yes	1
14	No	0

Dummy variables

If there are more than two categories, then the variable can be coded using several dummy variables (one fewer than the total number of categories).

	A	B	C	D	E
1	Monday	1	0	0	0
2	Tuesday	0	1	0	0
3	Wednesday	0	0	1	0
4	Thursday	0	0	0	1
5	Friday	0	0	0	0
6	Monday	1	0	0	0
7	Tuesday	0	1	0	0
8	Wednesday	0	0	1	0
9	Thursday	0	0	0	1
10	Friday	0	0	0	0
11	Monday	1	0	0	0
12	Tuesday	0	1	0	0
13	Wednesday	0	0	1	0
14	Thursday	0	0	0	1
15	Friday	0	0	0	0

Beware of the dummy variable trap!

- Using one dummy for each category gives too many dummy variables!
- The regression will then be singular and inestimable.
- Either omit the constant, or omit the dummy for one category.
- The coefficients of the dummies are relative to the omitted category.

Uses of dummy variables

Seasonal dummies

- For quarterly data: use 3 dummies
- For monthly data: use 11 dummies
- For daily data: use 6 dummies
- What to do with weekly data?

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- What to do with weekly data?

Outliers

- If there is an outlier, you can use a dummy variable to remove its effect.

Uses of dummy variables

Seasonal dummies

- For quarterly data: use 3 dummies
- For monthly data: use 11 dummies
- For daily data: use 6 dummies
- What to do with weekly data?

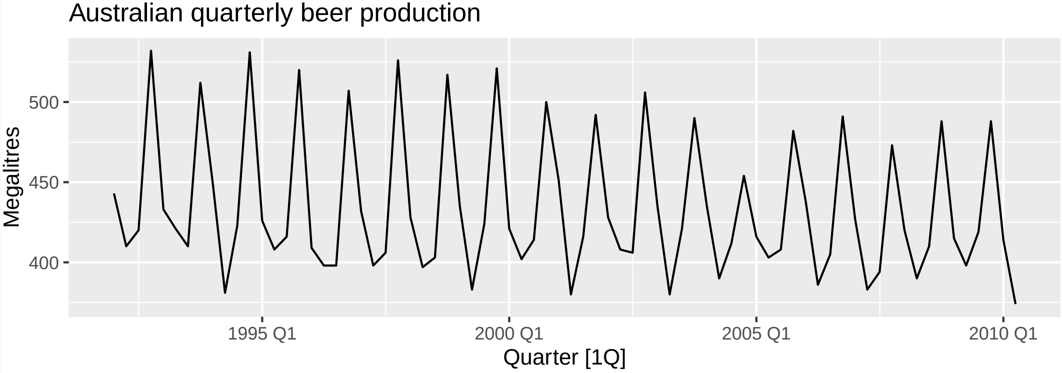
Outliers

- If there is an outlier, you can use a dummy variable to remove its effect.

Public holidays

- For daily data: if it is a public holiday, dummy=1, otherwise dummy=0.

Beer production revisited



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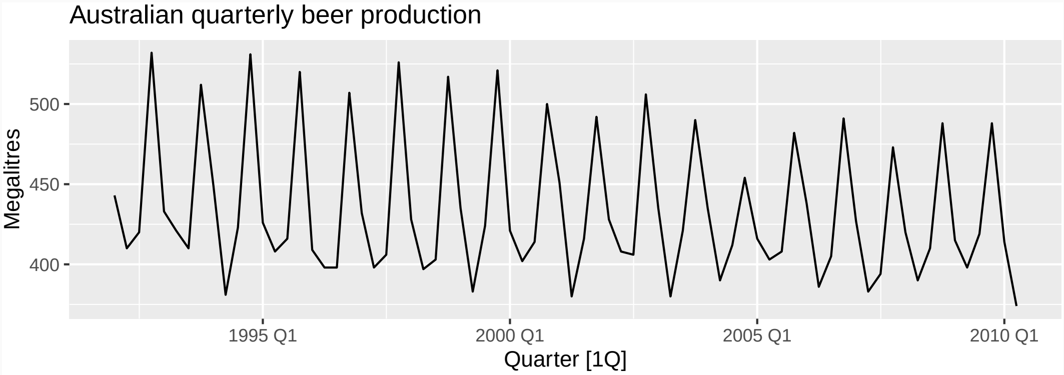


Regression model

$$y_t = \beta_0 + \beta_1 t + \beta_2 d_{2,t} + \beta_3 d_{3,t} + \beta_4 d_{4,t} + \varepsilon_t$$

- $d_{i,t} = 1$ if t is quarter i and 0 otherwise.

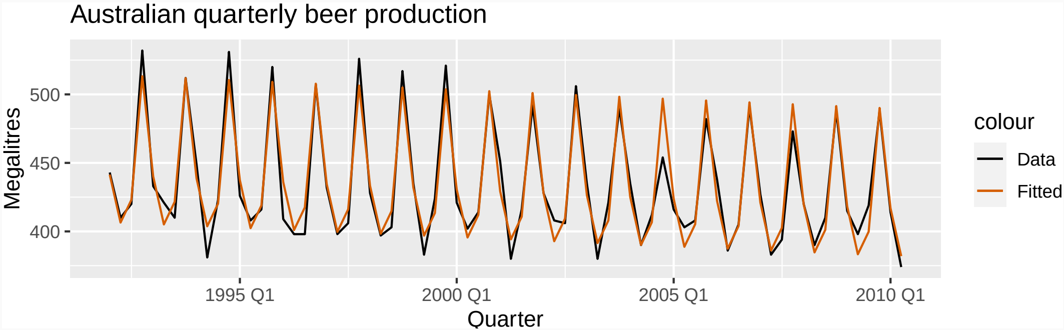
Beer production revisited

```r
fit_beer <- recent_production %>% model(TSLM(Beer ~ trend() + season()))
report(fit_beer)
```

```
## Series: Beer
## Model: TSLM
##
## Residuals:
##     Min      1Q Median      3Q     Max
##   -42.9    -7.6   -0.5     8.0    21.8
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept)   441.8004     3.7335  118.33  < 2e-16 ***
## trend()        -0.3403     0.0666   -5.11  2.7e-06 ***
## season()year2 -34.6597     3.9683   -8.73  9.1e-13 ***
## season()year3 -17.8216     4.0225   -4.43  3.4e-05 ***
## season()year4  72.7964     4.0230   18.09  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 12.2 on 69 degrees of freedom
## Multiple R-squared: 0.924,    Adjusted R-squared: 0.92
```

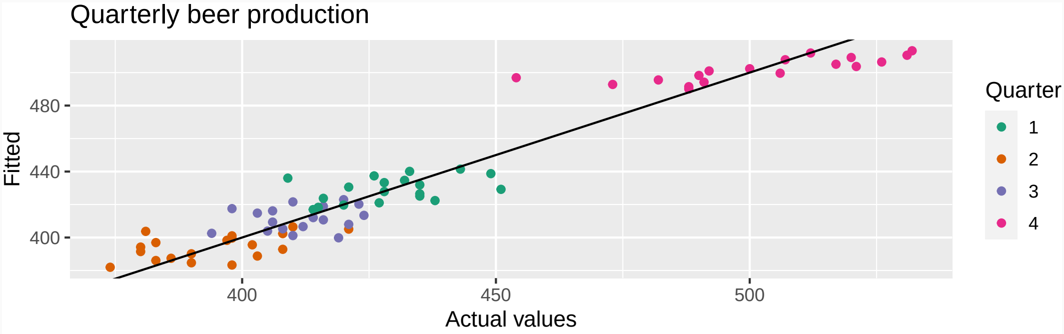
Beer production revisited

```
augment(fit_beer) %>%
  ggplot(aes(x = Quarter)) +
  geom_line(aes(y = Beer, colour = "Data")) +
  geom_line(aes(y = .fitted, colour = "Fitted")) +
  labs(y="Megalitres",title ="Australian quarterly beer production") +
  scale_colour_manual(values = c(Data = "black", Fitted = "#D55E00"))
```



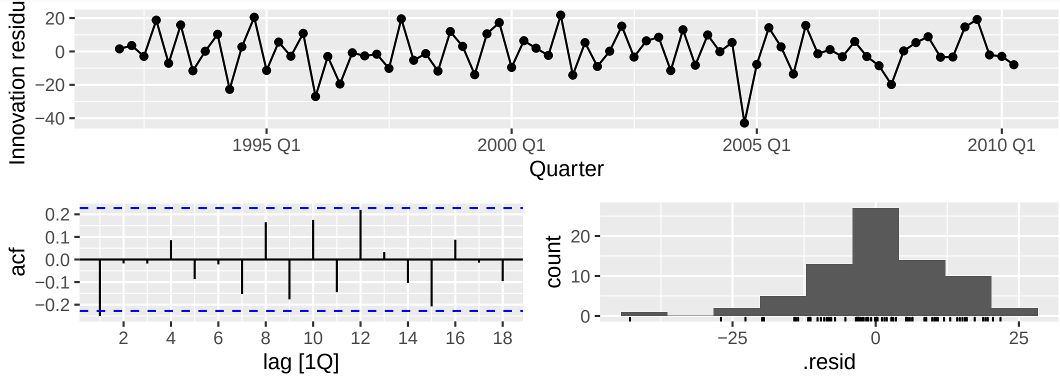
Beer production revisited

```
augment(fit_beer) %>%
  ggplot(aes(x=Beer, y=.fitted, colour=factor(quarter(Quarter)))) +
    geom_point() +
    labs(y="Fitted", x="Actual values", title = "Quarterly beer production") +
    scale_colour_brewer(palette="Dark2", name="Quarter") +
    geom_abline(intercept=0, slope=1)
```



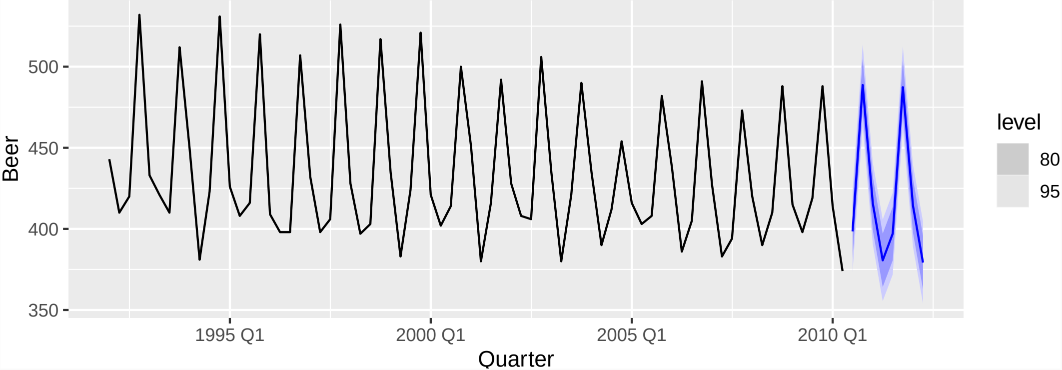
Beer production revisited

```
fit_beer %>% gg_tsresiduals()
```



Beer production revisited

```
fit_beer %>% forecast %>% autoplot(recent_production)
```



Fourier series

Periodic seasonality can be handled using pairs of Fourier terms:

$$s_k(t) = \sin\left(\frac{2\pi kt}{m}\right) \qquad c_k(t) = \cos\left(\frac{2\pi kt}{m}\right)$$

$$y_t = a + bt + \sum_{k=1}^{K} [\alpha_k s_k(t) + \beta_k c_k(t)] + \varepsilon_t$$

- Every periodic function can be approximated by sums of sin and cos terms for large enough K.
- Choose K by minimizing AICc.
- Called "harmonic regression"

```
TSLM(y ~ trend() + fourier(K))
```

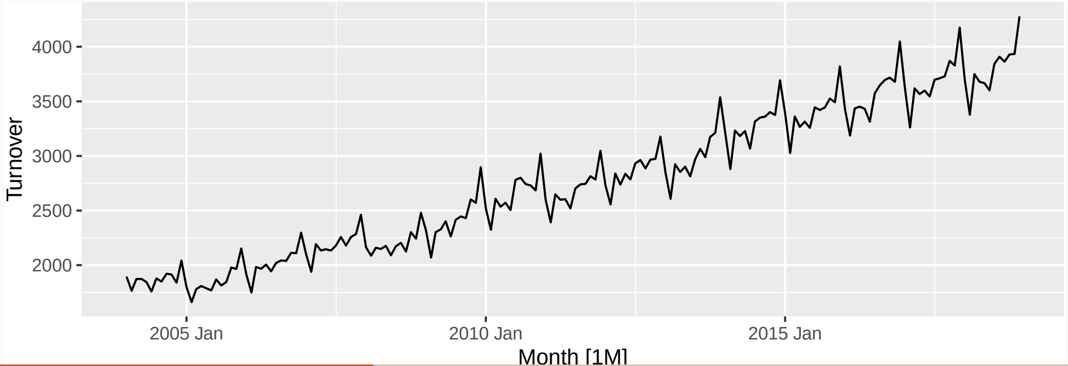
Harmonic regression: beer production

```r
fourier_beer <- recent_production %>% model(TSLM(Beer ~ trend() + fourier(K=2)))
report(fourier_beer)
```

```
## Series: Beer
## Model: TSLM
##
## Residuals:
##     Min      1Q  Median      3Q     Max
##   -42.9    -7.6    -0.5     8.0    21.8
##
## Coefficients:
##                      Estimate Std. Error t value Pr(>|t|)
## (Intercept)          446.8792     2.8732  155.53  < 2e-16 ***
## trend()               -0.3403     0.0666   -5.11  2.7e-06 ***
## fourier(K = 2)C1_4    8.9108     2.0112    4.43  3.4e-05 ***
## fourier(K = 2)S1_4 -53.7281     2.0112  -26.71  < 2e-16 ***
## fourier(K = 2)C2_4 -13.9896     1.4226   -9.83  9.3e-15 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 12.2 on 69 degrees of freedom
## Multiple R-squared: 0.924,    Adjusted R-squared: 0.92
```

Harmonic regression: eating-out expenditure

```r
aus_cafe <- aus_retail %>% filter(
    Industry == "Cafes, restaurants and takeaway food services",
    year(Month) %in% 2004:2018
  ) %>% summarise(Turnover = sum(Turnover))
aus_cafe %>% autoplot(Turnover)
```

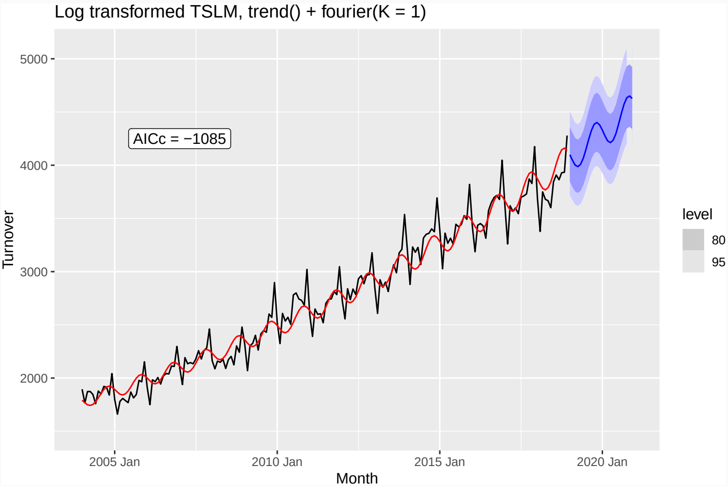


Harmonic regression: eating-out expenditure

```r
fit <- aus_cafe %>%
  model(K1 = TSLM(log(Turnover) ~ trend() + fourier(K = 1)),
        K2 = TSLM(log(Turnover) ~ trend() + fourier(K = 2)),
        K3 = TSLM(log(Turnover) ~ trend() + fourier(K = 3)),
        K4 = TSLM(log(Turnover) ~ trend() + fourier(K = 4)),
        K5 = TSLM(log(Turnover) ~ trend() + fourier(K = 5)),
        K6 = TSLM(log(Turnover) ~ trend() + fourier(K = 6)))
glance(fit) %>% select(.model, r_squared, adj_r_squared, AICc)
```

```
## # A tibble: 6 x 4
##   .model r_squared adj_r_squared   AICc
##   <chr>      <dbl>         <dbl>  <dbl>
## 1 K1         0.962         0.962 -1085.
## 2 K2         0.966         0.965 -1099.
## 3 K3         0.976         0.975 -1160.
## 4 K4         0.980         0.979 -1183.
## 5 K5         0.985         0.984 -1234.
## 6 K6         0.985         0.984 -1232.
```

Harmonic regression: eating-out expenditure



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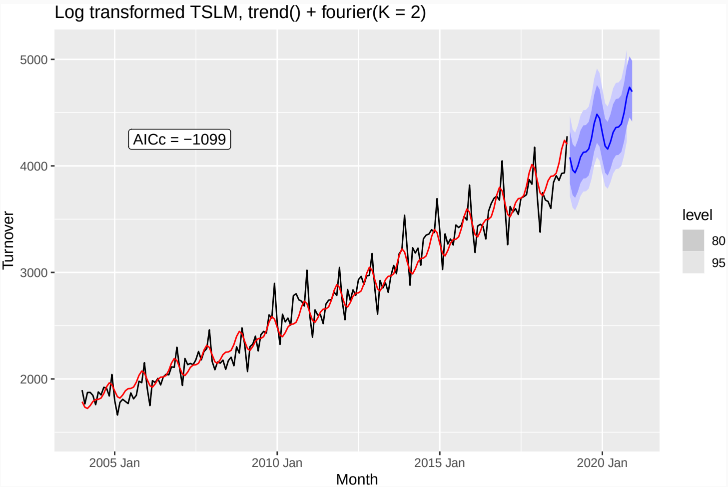

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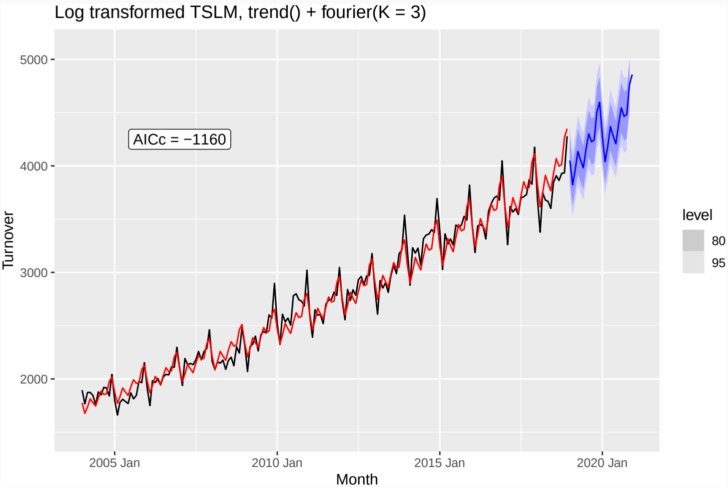

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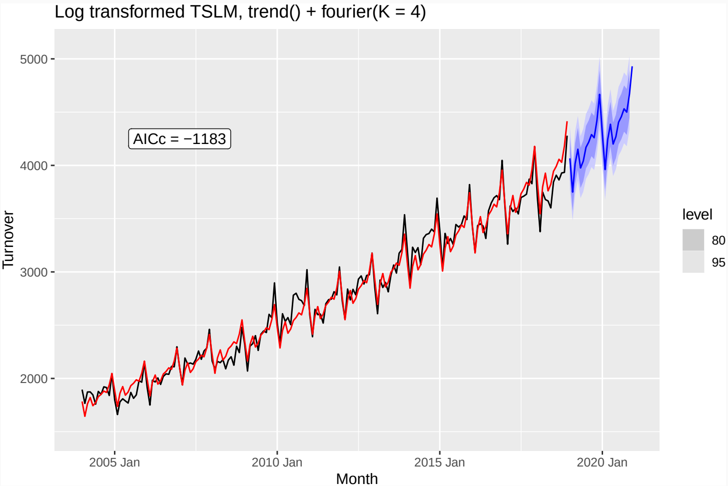

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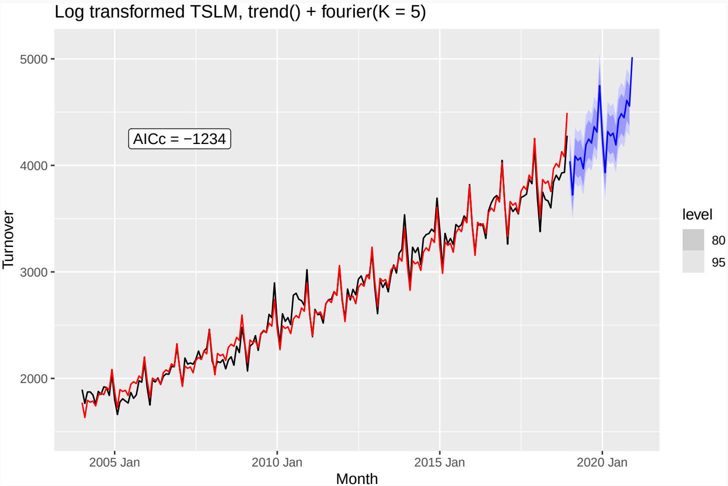

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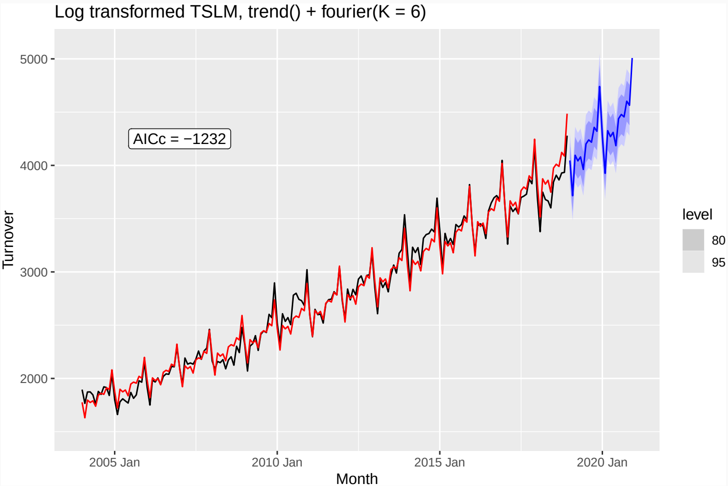

Intervention variables

Spikes

- Equivalent to a dummy variable for handling an outlier.

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Steps

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Steps

- Variable takes value 0 before the intervention and 1 afterwards.

Change of slope

- Variables take values 0 before the intervention and values $\{1, 2, 3, \ldots\}$ afterwards.

Holidays

For monthly data

- Christmas: always in December so part of monthly seasonal effect
- Easter: use a dummy variable $v_t = 1$ if any part of Easter is in that month, $v_t = 0$ otherwise.
- Ramadan and Chinese new year similar.

Distributed lags

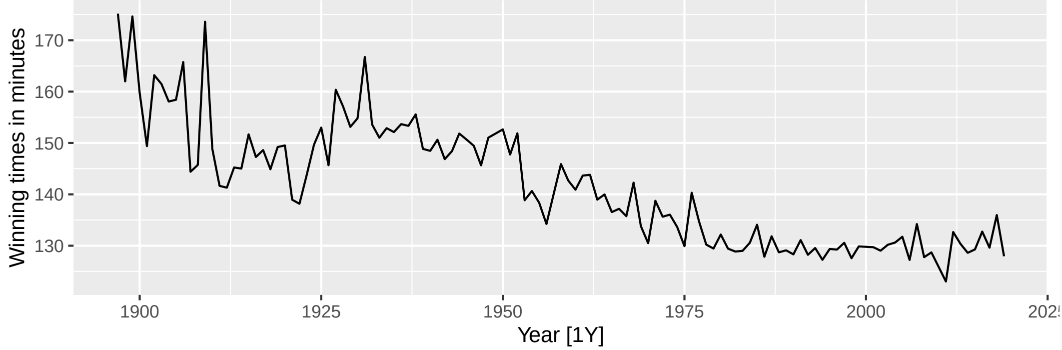
Lagged values of a predictor.

Example: x is advertising which has a delayed effect

$$\begin{aligned}
x_1 &= \text{advertising for previous month;} \\
x_2 &= \text{advertising for two months previously;} \\
&\vdots \\
x_m &= \text{advertising for } m \text{ months previously.}
\end{aligned}$$

Example: Boston marathon winning times

```
marathon <- boston_marathon %>%
  filter(Event == "Men's open division") %>%
  select(-Event) %>%
  mutate(Minutes = as.numeric(Time)/60)
marathon %>% autoplot(Minutes) + labs(y="Winning times in minutes")
```



Example: Boston marathon winning times

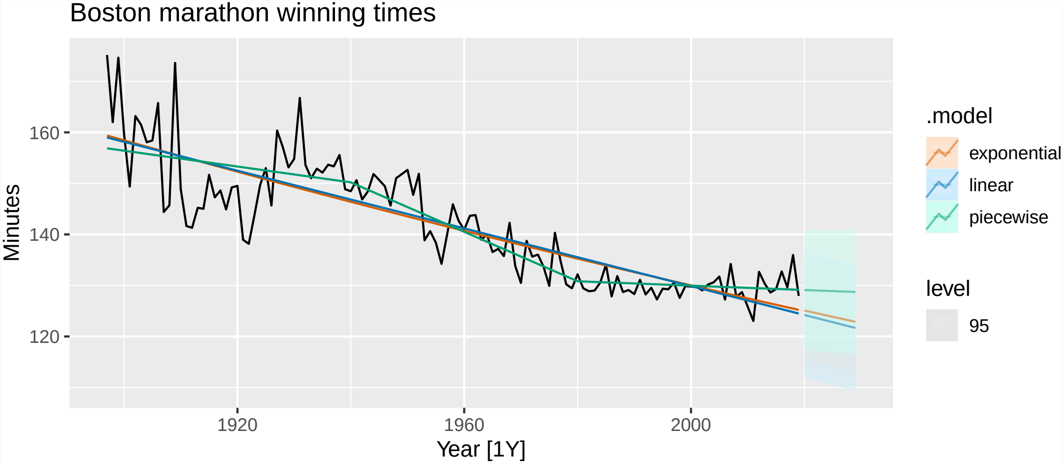
```
fit_trends <- marathon %>%
  model(
    # Linear trend
    linear = TSLM(Minutes ~ trend()),
    # Exponential trend
    exponential = TSLM(log(Minutes) ~ trend()),
    # Piecewise linear trend
    piecewise = TSLM(Minutes ~ trend(knots = c(1940, 1980)))
  )
```

```
fit_trends
```

```
## # A mable: 1 x 3
##    linear exponential piecewise
##   <model>     <model>   <model>
## 1  <TSLM>      <TSLM>    <TSLM>
```

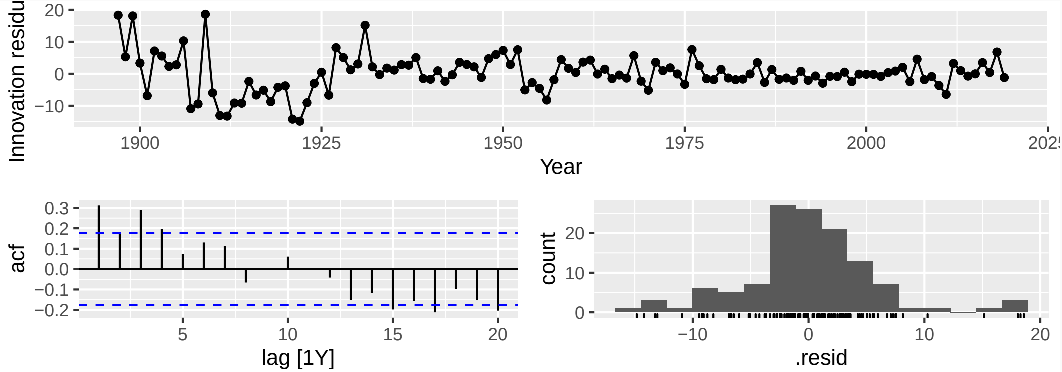
Example: Boston marathon winning times

```
fit_trends %>% forecast(h=10) %>% autoplot(marathon)
```



Example: Boston marathon winning times

```
fit_trends %>%
  select(piecewise) %>%
  gg_tsresiduals()
```



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Multiple regression and forecasting

For forecasting purposes, we require the following assumptions:

- ε_t are uncorrelated and zero mean
- ε_t are uncorrelated with each $x_{j,t}$.

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It is **useful** to also have $\varepsilon_t \sim \text{N}(0, \sigma^2)$ when producing prediction intervals or doing statistical tests.

Residual plots

Useful for spotting outliers and whether the linear model was appropriate.

- Scatterplot of residuals ε_t against each predictor $x_{j,t}$.
- Scatterplot residuals against the fitted values $\hat{y}_t$
- Expect to see scatterplots resembling a horizontal band with no values too far from the band and no patterns such as curvature or increasing spread.

Residual patterns

- If a plot of the residuals vs any predictor in the model shows a pattern, then the relationship is nonlinear.
- If a plot of the residuals vs any predictor **not** in the model shows a pattern, then the predictor should be added to the model.
- If a plot of the residuals vs fitted values shows a pattern, then there is heteroscedasticity in the errors. (Could try a transformation.)

Outline

Comparing regression models

Computer output for regression will always give the R^2 value. This is a useful summary of the model.

- It is equal to the square of the correlation between y and $\hat{y}$.
- It is often called the "coefficient of determination' '.
- It can also be calculated as follows:
$$R^2 = \frac{\sum(\hat{y}_t - \bar{y})^2}{\sum(y_t - \bar{y})^2}$$
- It is the proportion of variance accounted for (explained) by the predictors.

Comparing regression models

However …

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- Adding *any* variable tends to increase the value of R^2, even if that variable is irrelevant.

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To overcome this problem, we can use *adjusted* R^2:

$$\bar{R}^2 = 1 - (1 - R^2)\frac{T - 1}{T - k - 1}$$

where k = no. predictors and T = no. observations.

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$$\bar{R}^2 = 1 - (1 - R^2)\frac{T - 1}{T - k - 1}$$

where k = no. predictors and T = no. observations.

Maximizing $\bar{R}^2$ is equivalent to minimizing $\hat{\sigma}^2$.

$$\hat{\sigma}^2 = \frac{1}{T - k - 1}\sum_{t=1}^{T}\varepsilon_t^2$$

Akaike's Information Criterion

$$\text{AIC} = -2\log(L) + 2(k+2)$$

where L is the likelihood and k is the number of predictors in the model.

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- AIC penalizes terms more heavily than $\bar{R}^2$.
- Minimizing the AIC is asymptotically equivalent to minimizing MSE via **leave-one-out cross-validation** (for any linear regression).

Corrected AIC

For small values of T, the AIC tends to select too many predictors, and so a bias-corrected version of the AIC has been developed.

$$\text{AIC}_{\text{C}} = \text{AIC} + \frac{2(k+2)(k+3)}{T-k-3}$$

As with the AIC, the AIC_{C} should be minimized.

Bayesian Information Criterion

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- BIC penalizes terms more heavily than AIC
- Also called SBIC and SC.
- Minimizing BIC is asymptotically equivalent to leave-v-out cross-validation when $v = T[1 - 1/(log(T) - 1)]$.

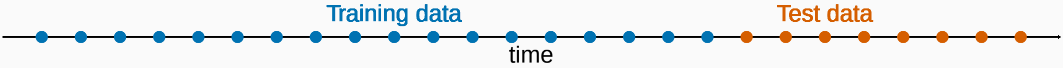
Leave-one-out cross-validation

For regression, leave-one-out cross-validation is faster and more efficient than time-series cross-validation.

- Select one observation for test set, and use *remaining* observations in training set. Compute error on test observation.
- Repeat using each possible observation as the test set.
- Compute accuracy measure over all errors.

Cross-validation

Traditional evaluation



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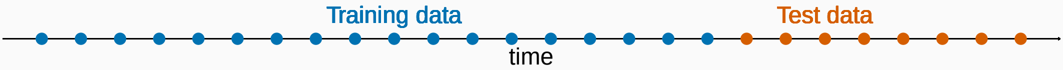
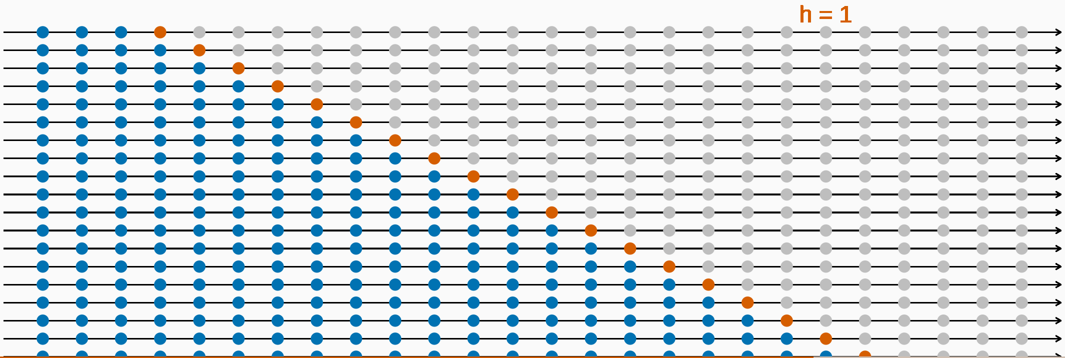
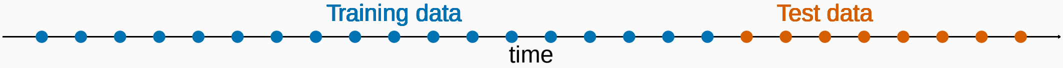


Time series cross-validation

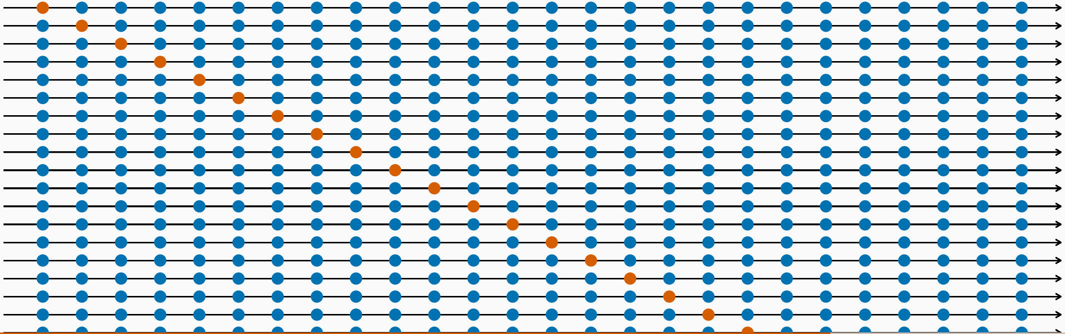


Cross-validation

Traditional evaluation



Leave-one-out cross-validation



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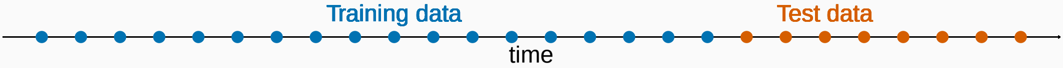

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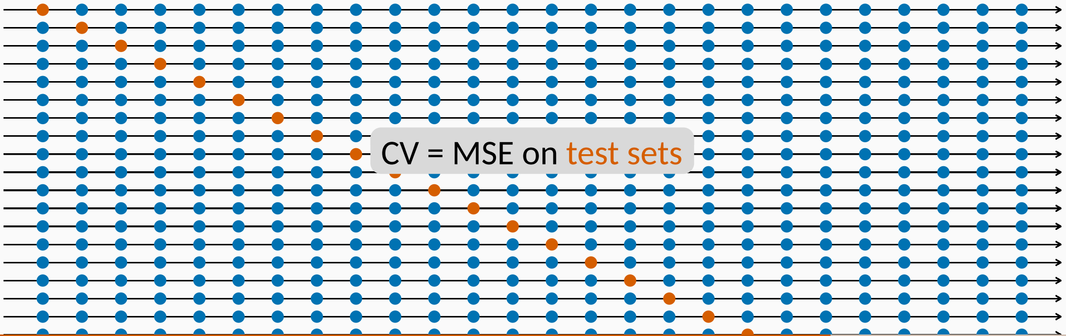

Comparing regression models

```
glance(fit_trends) %>%
  select(.model, r_squared, adj_r_squared, AICc, CV)
```

```
## # A tibble: 3 x 5
##   .model      r_squared adj_r_squared  AICc      CV
##   <chr>           <dbl>         <dbl> <dbl>   <dbl>
## 1 linear          0.728         0.726  452. 39.1
## 2 exponential     0.744         0.742 -779.  0.00176
## 3 piecewise       0.767         0.761  438. 34.8
```

- Be careful making comparisons when transformations are used.

Choosing regression variables

Best subsets regression

- Fit all possible regression models using one or more of the predictors.
- Choose the best model based on one of the measures of predictive ability (CV, AIC, AICc).

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Warning!

- If there are a large number of predictors, this is not possible.
- For example, 44 predictors leads to 18 trillion possible models!

Choosing regression variables

Backwards stepwise regression

- Start with a model containing all variables.
- Try subtracting one variable at a time. Keep the model if it has lower CV or AICc.
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Notes

- Stepwise regression is not guaranteed to lead to the best possible model.
- Inference on coefficients of final model will be wrong.

Outline

Ex-ante versus ex-post forecasts

- *Ex ante forecasts* are made using only information available in advance.
  - require forecasts of predictors
- *Ex post forecasts* are made using later information on the predictors.
  - useful for studying behaviour of forecasting models.
- trend, seasonal and calendar variables are all known in advance, so these don't need to be forecast.

Scenario based forecasting

- Assumes possible scenarios for the predictor variables
- Prediction intervals for scenario based forecasts do not include the uncertainty associated with the future values of the predictor variables.

Building a predictive regression model

- If getting forecasts of predictors is difficult, you can use lagged predictors instead.

$$y_t = \beta_0 + \beta_1 x_{1,t-h} + \cdots + \beta_k x_{k,t-h} + \varepsilon_t$$

- A different model for each forecast horizon h.

US Consumption

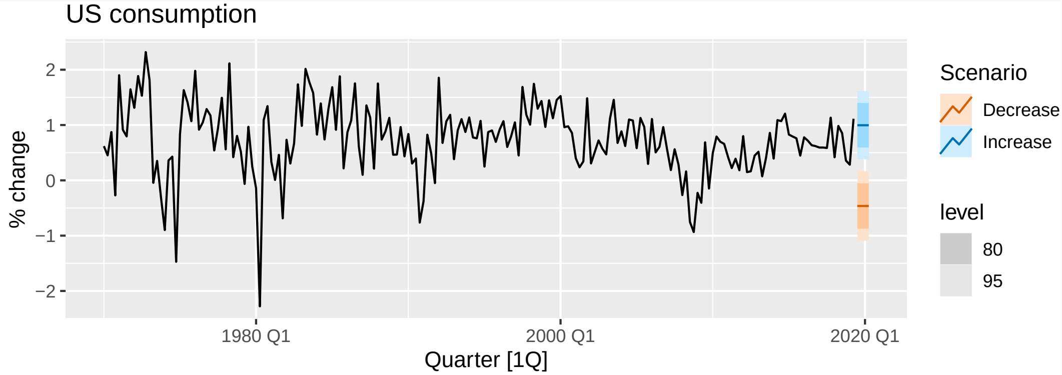
```
fit_consBest <- us_change %>%
  model(
    TSLM(Consumption ~ Income + Savings + Unemployment)
  )

future_scenarios <- scenarios(
  Increase = new_data(us_change, 4) %>%
    mutate(Income=1, Savings=0.5, Unemployment=0),
  Decrease = new_data(us_change, 4) %>%
    mutate(Income=-1, Savings=-0.5, Unemployment=0),
  names_to = "Scenario")

fc <- forecast(fit_consBest, new_data = future_scenarios)
```

US Consumption

```
us_change %>% autoplot(Consumption) +
  labs(y="% change in US consumption") +
  autolayer(fc) +
  labs(title = "US consumption", y = "% change")
```



Outline

Matrix formulation

$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t.$$

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Let $\boldsymbol{y} = (y_1, \dots, y_T)'$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_T)'$, $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_k)'$ and

$$\boldsymbol{X} = \begin{bmatrix} 1 & x_{1,1} & x_{2,1} & \dots & x_{k,1} \\ 1 & x_{1,2} & x_{2,2} & \dots & x_{k,2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{1,T} & x_{2,T} & \dots & x_{k,T} \end{bmatrix}.$$

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Then

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}.$$

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Least squares estimation

Minimize: $(\boldsymbol{y} - \boldsymbol{X}\beta)'(\boldsymbol{y} - \boldsymbol{X}\beta)$

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(The “normal equation”.)

$$\hat{\sigma}^2 = \frac{1}{T-k-1}(\boldsymbol{y} - \boldsymbol{X}\hat{\beta})'(\boldsymbol{y} - \boldsymbol{X}\hat{\beta})$$

Note: If you fall for the dummy variable trap, $(\boldsymbol{X}'\boldsymbol{X})$ is a singular matrix.

Likelihood

If the errors are iid and normally distributed, then

$$\boldsymbol{y} \sim \mathrm{N}(\boldsymbol{X}\beta, \sigma^2 \boldsymbol{I}).$$

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So MLE = OLS.

Multiple regression forecasts

Optimal forecasts

$$\hat{y}^* = \text{E}(y^*|\boldsymbol{y}, \boldsymbol{X}, \boldsymbol{x}^*) = \boldsymbol{x}^*\hat{\boldsymbol{\beta}} = \boldsymbol{x}^*(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$$

where $\boldsymbol{x}^*$ is a row vector containing the values of the predictors for the forecasts (in the same format as $\boldsymbol{X}$).

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Forecast variance

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Forecast variance

$$\text{Var}(y^*|\boldsymbol{X}, \boldsymbol{x}^*) = \sigma^2\left[1 + \boldsymbol{x}^*(\boldsymbol{X}'\boldsymbol{X})^{-1}(\boldsymbol{x}^*)'\right]$$

- This ignores any errors in $\boldsymbol{x}^*$.
- 95% prediction intervals assuming normal errors:

$$\hat{y}^* \pm 1.96\sqrt{\text{Var}(y^*|\boldsymbol{X}, \boldsymbol{x}^*)}.$$

Outline

Correlation is not causation

- When x is useful for predicting y, it is not necessarily causing y.
- e.g., predict number of drownings y using number of ice-creams sold x.
- Correlations are useful for forecasting, even when there is no causality.
- Better models usually involve causal relationships (e.g., temperature x and people z to predict drownings y).

Multicollinearity

In regression analysis, multicollinearity occurs when:

- Two predictors are highly correlated (i.e., the correlation between them is close to ± 1).
- A linear combination of some of the predictors is highly correlated with another predictor.
- A linear combination of one subset of predictors is highly correlated with a linear combination of another subset of predictors.

Multicollinearity

If multicollinearity exists…

- the numerical estimates of coefficients may be wrong (worse in Excel than in a statistics package)
- don't rely on the *p*-values to determine significance.
- there is no problem with model *predictions* provided the predictors used for forecasting are within the range used for fitting.
- omitting variables can help.
- combining variables can help.