

# ETC3550/ETC5550

## Applied forecasting

Ch7. Regression models

[OTexts.org/fpp3/](https://OTexts.org/fpp3/)



# Outline

- 1 The linear model with time series
- 2 Some useful predictors for linear models
- 3 Residual diagnostics
- 4 Selecting predictors and forecast evaluation
- 5 Forecasting with regression
- 6 Matrix formulation
- 7 Correlation, causation and forecasting

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# Multiple regression and forecasting

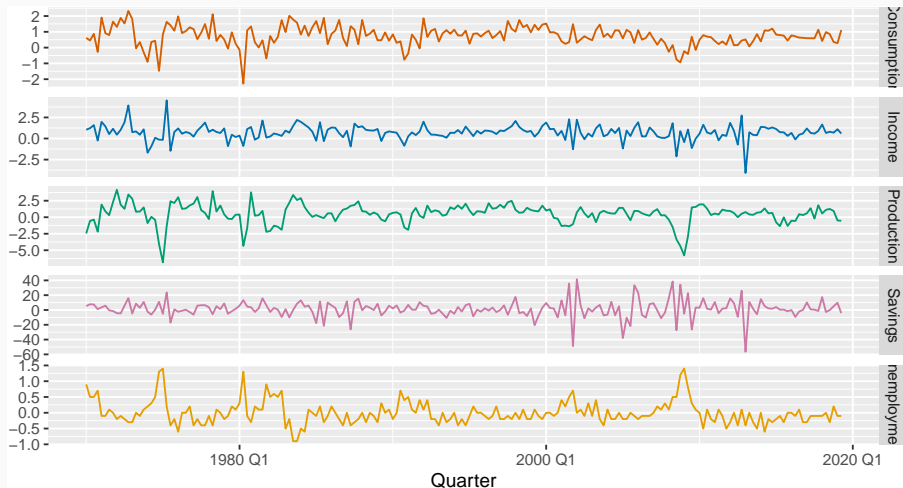
$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \cdots + \beta_k x_{k,t} + \varepsilon_t.$$

- $y_t$  is the variable we want to predict: the “response” variable
- Each  $x_{j,t}$  is numerical and is called a “predictor”. They are usually assumed to be known for all past and future times.
- The coefficients  $\beta_1, \dots, \beta_k$  measure the effect of each predictor after taking account of the effect of all other predictors in the model.

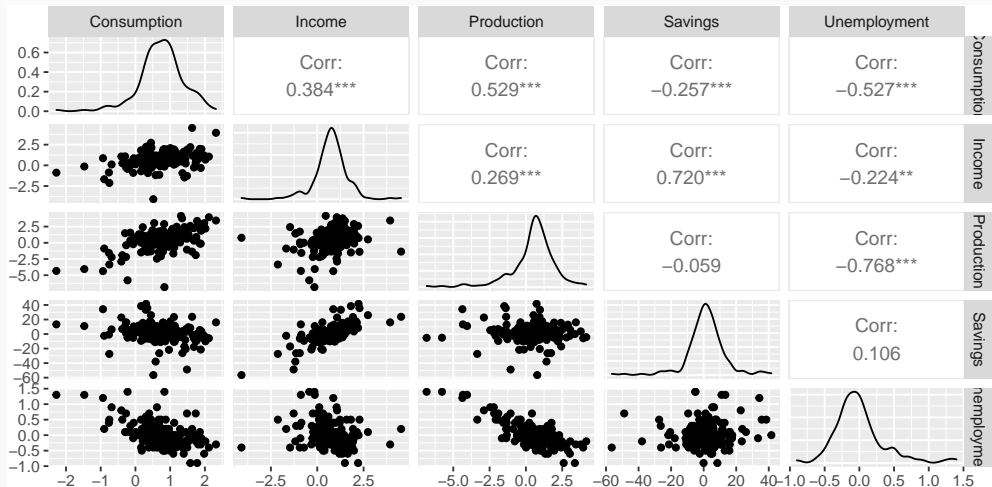
That is, the coefficients measure the **marginal effects**.

- $\varepsilon_t$  is a white noise error term

# Example: US consumption expenditure



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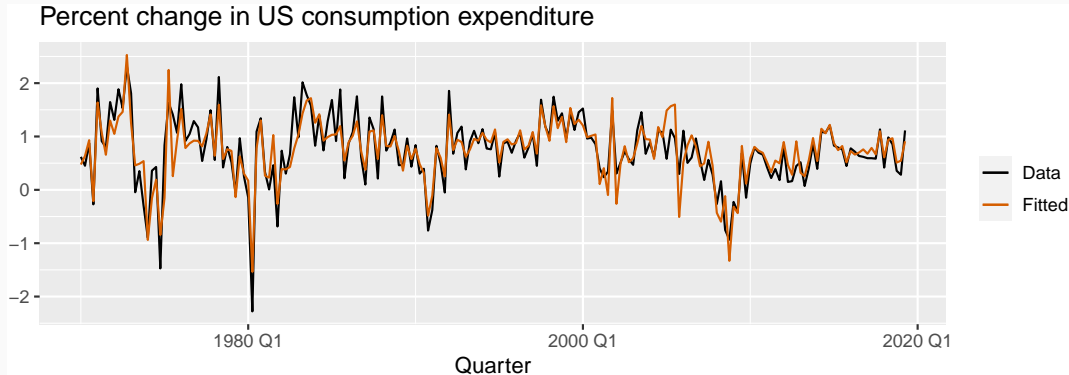


# Example: US consumption expenditure

```
fit_consMR <- us_change %>%  
  model(lm = TSLM(Consumption ~ Income + Production + Unemployment + Savings))  
report(fit_consMR)
```

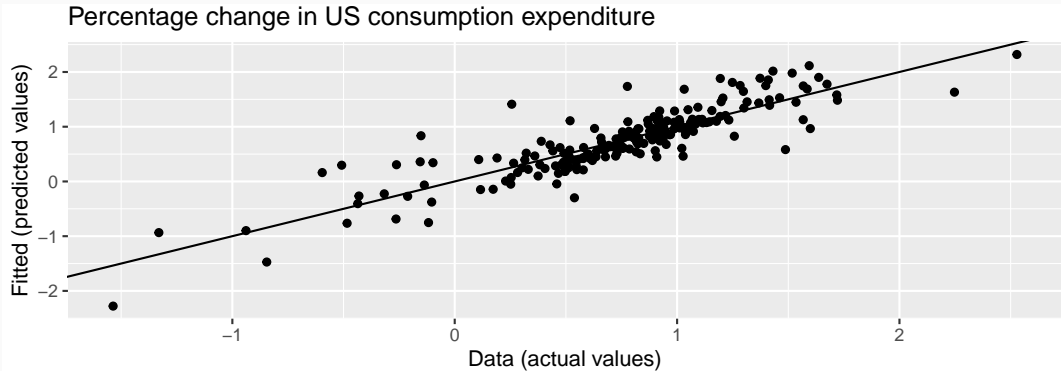
```
## Series: Consumption  
## Model: TSLM  
##  
## Residuals:  
##      Min       1Q   Median       3Q      Max  
## -0.906 -0.158 -0.036  0.136  1.155  
##  
## Coefficients:  
##              Estimate Std. Error t value Pr(>|t|)  
## (Intercept)   0.25311    0.03447   7.34 5.7e-12 ***  
## Income        0.74058    0.04012  18.46 < 2e-16 ***  
## Production    0.04717    0.02314   2.04  0.043 *  
## Unemployment -0.17469    0.09551  -1.83  0.069 .  
## Savings       -0.05289    0.00292 -18.09 < 2e-16 ***  
## ---  
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
##  
## Residual standard error: 0.31 on 193 degrees of freedom  
## Multiple R-squared:  0.768,    Adjusted R-squared:  0.763  
## F-statistic: 160 on 4 and 193 DF, p-value: <2e-16
```

# Example: US consumption expenditure



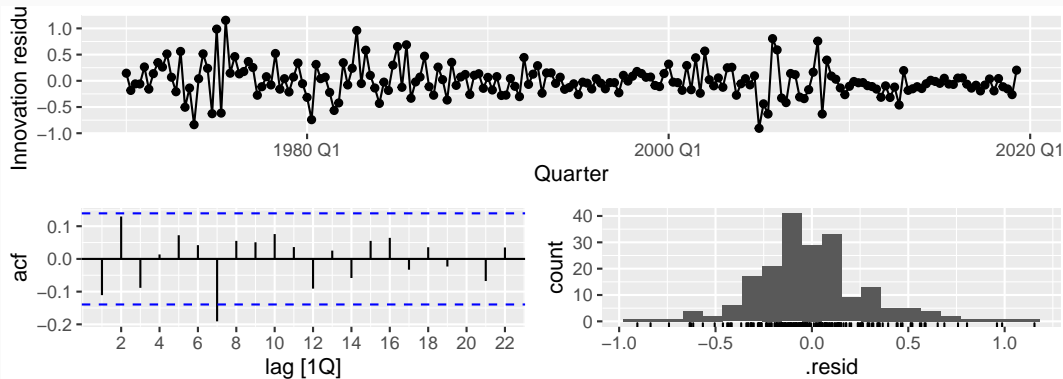


# Example: US consumption expenditure



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```
fit_consMR %>% gg_tsresiduals()
```



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## Linear trend

$$x_t = t$$

- $t = 1, 2, \dots, T$
- Strong assumption that trend will continue.

# Nonlinear trend

## Piecewise linear trend with bend at $\tau$

$$x_{1,t} = t$$

$$x_{2,t} = \begin{cases} 0 & t < \tau \\ (t - \tau) & t \geq \tau \end{cases}$$

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## Quadratic or higher order trend

$$x_{1,t} = t, \quad x_{2,t} = t^2, \quad \dots$$

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## Quadratic or higher order trend

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**NOT RECOMMENDED!**

# Dummy variables

If a categorical variable takes only two values (e.g., 'Yes' or 'No'), then an equivalent numerical variable can be constructed taking value 1 if yes and 0 if no. This is called a **dummy variable**.

	A	B
1	Yes	1
2	Yes	1
3	No	0
4	Yes	1
5	No	0
6	No	0
7	Yes	1
8	Yes	1
9	No	0
10	No	0
11	No	0
12	No	0
13	Yes	1
14	No	0
...		



# Dummy variables

If there are more than two categories, then the variable can be coded using several dummy variables (one fewer than the total number of categories).

	A	B	C	D	E
1	Monday	1	0	0	0
2	Tuesday	0	1	0	0
3	Wednesday	0	0	1	0
4	Thursday	0	0	0	1
5	Friday	0	0	0	0
6	Monday	1	0	0	0
7	Tuesday	0	1	0	0
8	Wednesday	0	0	1	0
9	Thursday	0	0	0	1
10	Friday	0	0	0	0
11	Monday	1	0	0	0
12	Tuesday	0	1	0	0
13	Wednesday	0	0	1	0
14	Thursday	0	0	0	1
15	Friday	0	0	0	0

# Beware of the dummy variable trap!

- Using one dummy for each category gives too many dummy variables!
- The regression will then be singular and inestimable.
- Either omit the constant, or omit the dummy for one category.
- The coefficients of the dummies are relative to the omitted category.

# Uses of dummy variables

## Seasonal dummies

- For quarterly data: use 3 dummies
- For monthly data: use 11 dummies
- For daily data: use 6 dummies
- What to do with weekly data?

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## Outliers

- If there is an outlier, you can use a dummy variable to remove its effect.

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## Outliers

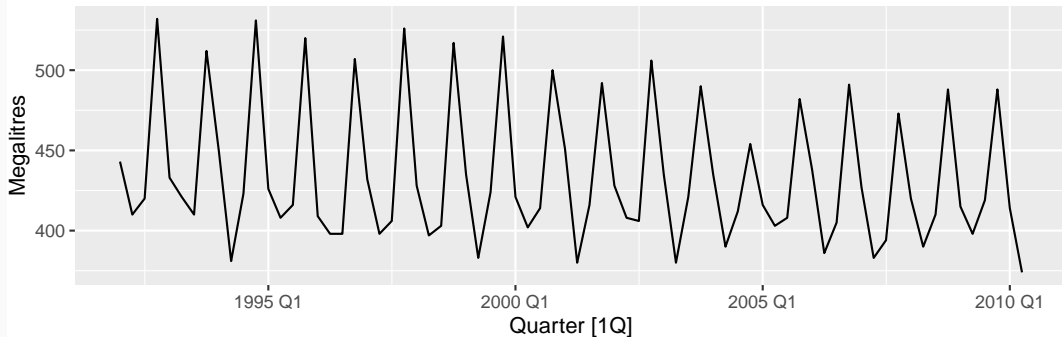
- If there is an outlier, you can use a dummy variable to remove its effect.

## Public holidays

- For daily data: if it is a public holiday,  $\text{dummy}=1$ , otherwise  $\text{dummy}=0$ .

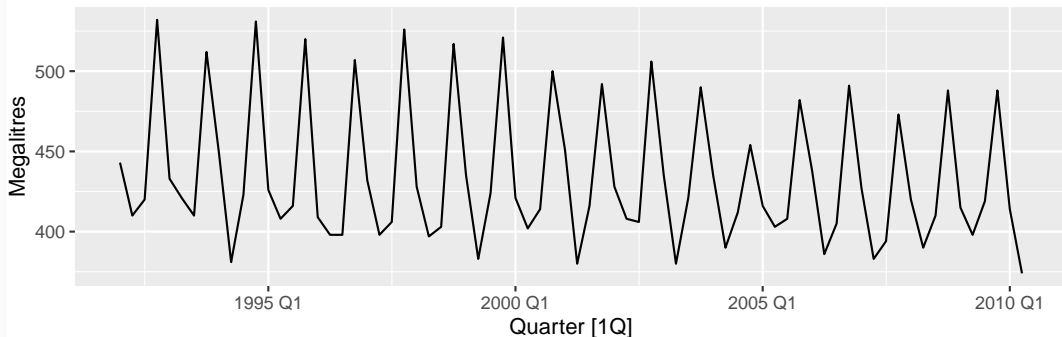
# Beer production revisited

Australian quarterly beer production



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Australian quarterly beer production



## Regression model

$$y_t = \beta_0 + \beta_1 t + \beta_2 d_{2,t} + \beta_3 d_{3,t} + \beta_4 d_{4,t} + \varepsilon_t$$

■  $d_{i,t} = 1$  if  $t$  is quarter  $i$  and 0 otherwise.

# Beer production revisited

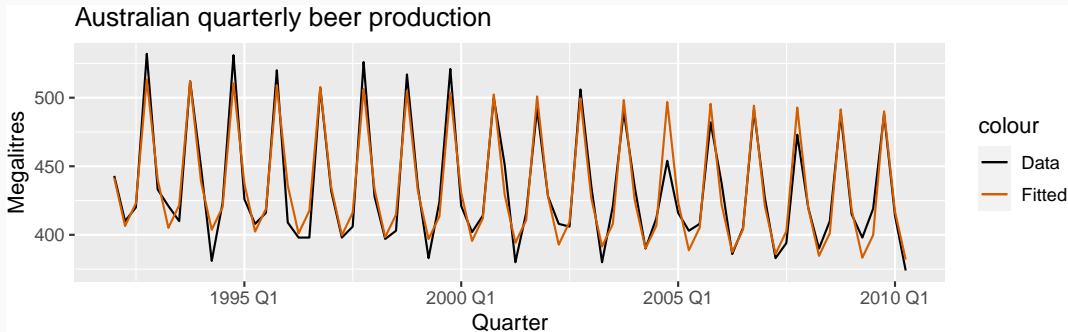
```
fit_beer <- recent_production %>% model(TSLM(Beer ~ trend() + season()))
report(fit_beer)
```

```
## Series: Beer
## Model: TSLM
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -42.9    -7.6    -0.5     8.0    21.8
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept)   441.8004     3.7335  118.33 < 2e-16 ***
## trend()        -0.3403     0.0666   -5.11 2.7e-06 ***
## season()year2 -34.6597     3.9683   -8.73 9.1e-13 ***
## season()year3 -17.8216     4.0225   -4.43 3.4e-05 ***
## season()year4  72.7964     4.0230   18.09 < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 12.2 on 69 degrees of freedom
```



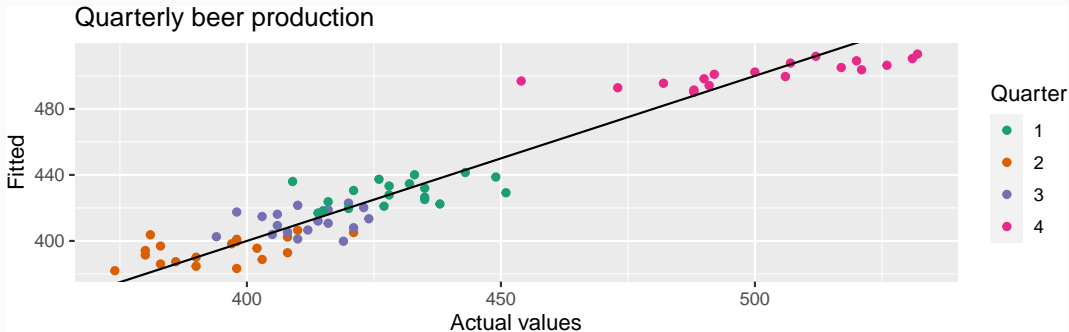
# Beer production revisited

```
augment(fit_beer) %>%  
  ggplot(aes(x = Quarter)) +  
  geom_line(aes(y = Beer, colour = "Data")) +  
  geom_line(aes(y = .fitted, colour = "Fitted")) +  
  labs(y="Megalitres", title="Australian quarterly beer production") +  
  scale_colour_manual(values = c(Data = "black", Fitted = "#D55E00"))
```



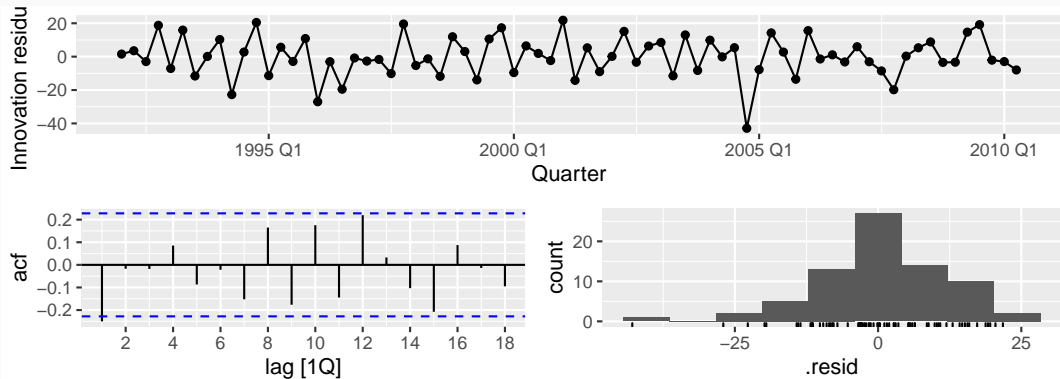
# Beer production revisited

```
augment(fit_beer) %>%  
  ggplot(aes(x=Beer, y=.fitted, colour=factor(quarter(Quarter)))) +  
    geom_point() +  
    labs(y="Fitted", x="Actual values", title = "Quarterly beer production") +  
    scale_colour_brewer(palette="Dark2", name="Quarter") +  
    geom_abline(intercept=0, slope=1)
```



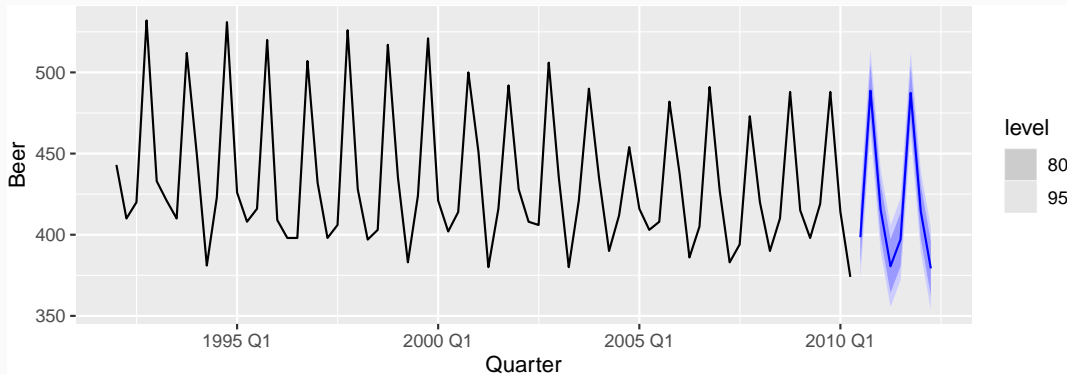
# Beer production revisited

```
fit_beer %>% gg_tsresiduals()
```



# Beer production revisited

```
fit_beer %>% forecast %>% autoplot(recent_production)
```



# Fourier series

Periodic seasonality can be handled using pairs of Fourier terms:

$$s_k(t) = \sin\left(\frac{2\pi kt}{m}\right) \quad c_k(t) = \cos\left(\frac{2\pi kt}{m}\right)$$

$$y_t = a + bt + \sum_{k=1}^K [\alpha_k s_k(t) + \beta_k c_k(t)] + \varepsilon_t$$

- Every periodic function can be approximated by sums of sin and cos terms for large enough  $K$ .
- Choose  $K$  by minimizing AICc.
- Called “harmonic regression”

```
TSLM(y ~ trend() + fourier(K))
```

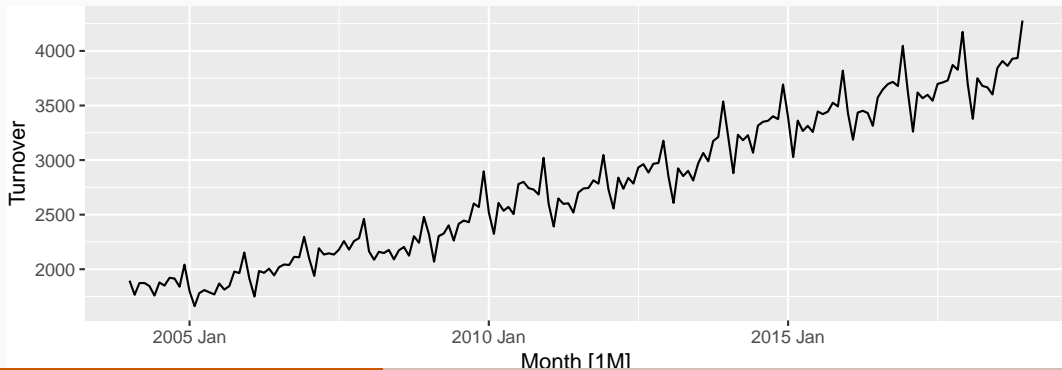
# Harmonic regression: beer production

```
fourier_beer <- recent_production %>% model(TSLM(Beer ~ trend() + fourier(K=2)))  
report(fourier_beer)
```

```
## Series: Beer  
## Model: TSLM  
##  
## Residuals:  
##      Min      1Q  Median      3Q      Max  
## -42.9   -7.6    -0.5     8.0    21.8  
##  
## Coefficients:  
##              Estimate Std. Error t value Pr(>|t|)  
## (Intercept)    446.8792     2.8732  155.53 < 2e-16 ***  
## trend()        -0.3403     0.0666   -5.11 2.7e-06 ***  
## fourier(K = 2)C1_4  8.9108     2.0112    4.43 3.4e-05 ***  
## fourier(K = 2)S1_4 -53.7281     2.0112  -26.71 < 2e-16 ***  
## fourier(K = 2)C2_4 -13.9896     1.4226   -9.83 9.3e-15 ***  
## ---  
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1  
##  
## Residual standard error: 12.2 on 69 degrees of freedom
```

# Harmonic regression: eating-out expenditure

```
aus_cafe <- aus_retail %>% filter(  
  Industry == "Cafes, restaurants and takeaway food services",  
  year(Month) %in% 2004:2018  
) %>% summarise(Turnover = sum(Turnover))  
aus_cafe %>% autoplot(Turnover)
```



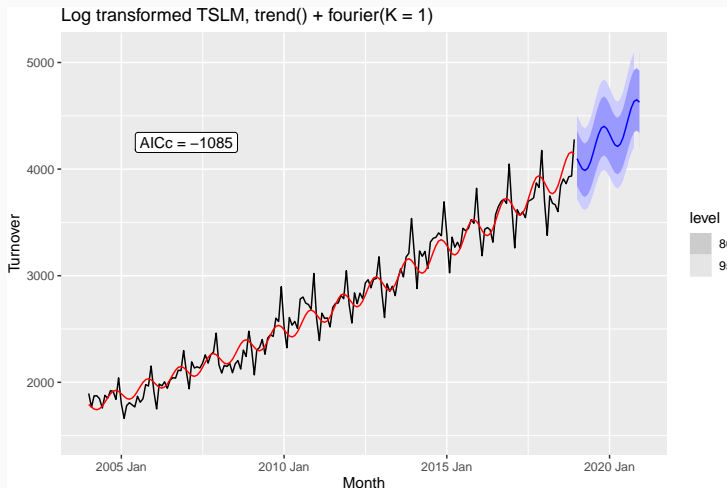
# Harmonic regression: eating-out expenditure

```
fit <- aus_cafe %>%  
  model(K1 = TSLM(log(Turnover) ~ trend() + fourier(K = 1)),  
        K2 = TSLM(log(Turnover) ~ trend() + fourier(K = 2)),  
        K3 = TSLM(log(Turnover) ~ trend() + fourier(K = 3)),  
        K4 = TSLM(log(Turnover) ~ trend() + fourier(K = 4)),  
        K5 = TSLM(log(Turnover) ~ trend() + fourier(K = 5)),  
        K6 = TSLM(log(Turnover) ~ trend() + fourier(K = 6)))  
glance(fit) %>% select(.model, r_squared, adj_r_squared, AICc)
```

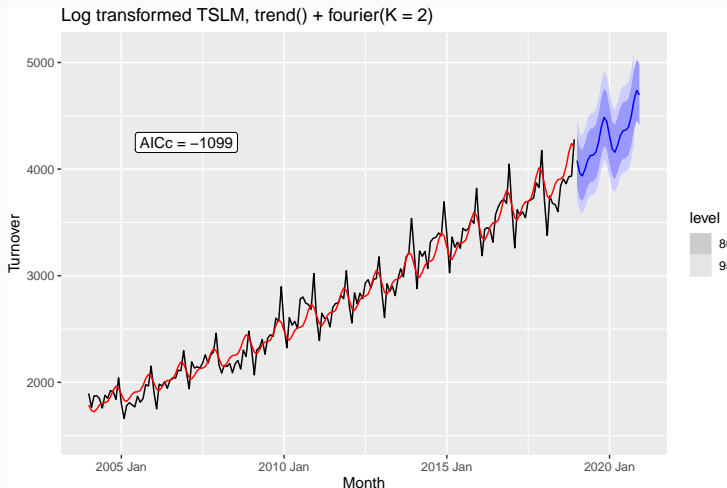
```
## # A tibble: 6 x 4  
##   .model r_squared adj_r_squared AICc  
##   <chr>      <dbl>      <dbl> <dbl>  
## 1 K1        0.962        0.962 -1085.  
## 2 K2        0.966        0.965 -1099.  
## 3 K3        0.976        0.975 -1160.  
## 4 K4        0.980        0.979 -1183.  
## 5 K5        0.985        0.984 -1234.  
## 6 K6        0.985        0.984 -1232.
```



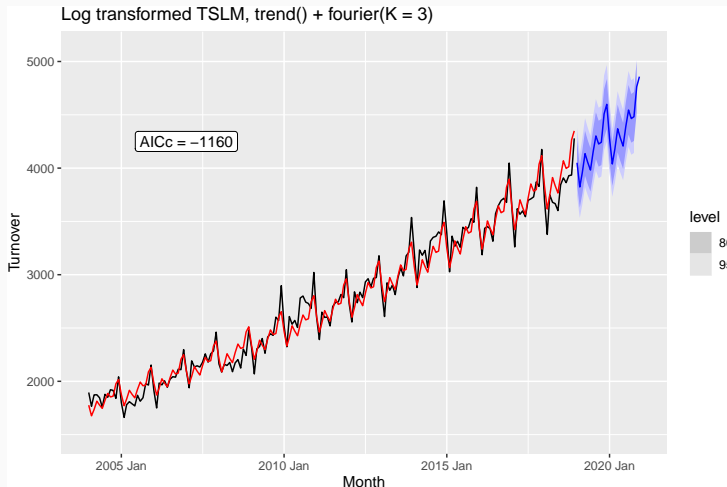
# Harmonic regression: eating-out expenditure



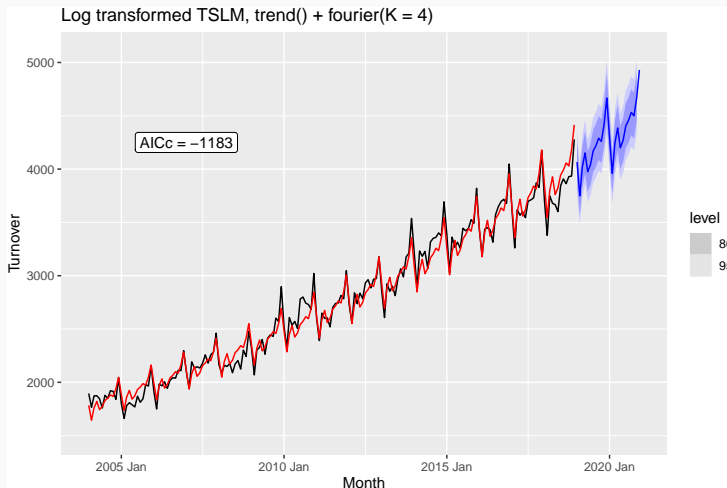
# Harmonic regression: eating-out expenditure



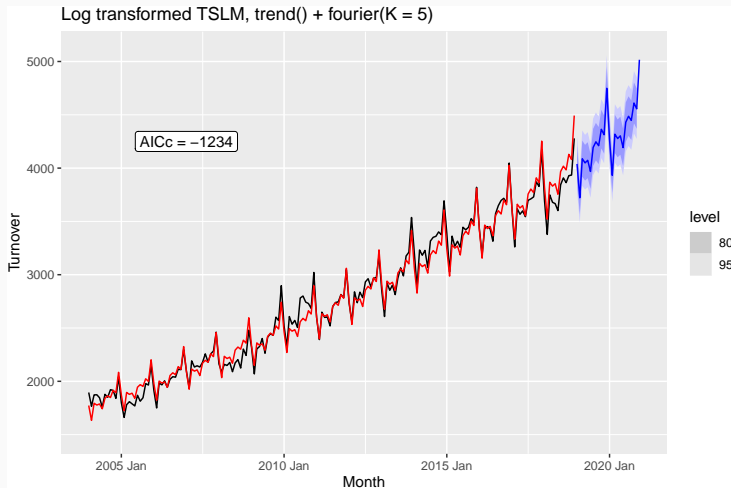
# Harmonic regression: eating-out expenditure



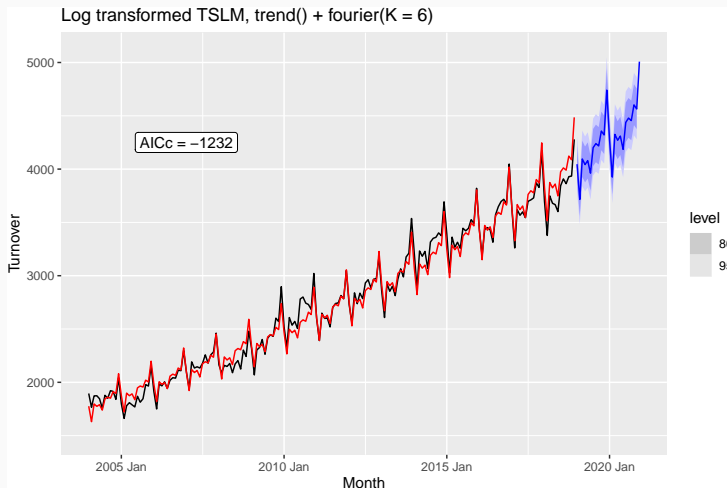
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# Intervention variables

## Spikes

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## Steps

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## Steps

- Variable takes value 0 before the intervention and 1 afterwards.

## Change of slope

- Variables take values 0 before the intervention and values  $\{1, 2, 3, \dots\}$  afterwards.

## For monthly data

- Christmas: always in December so part of monthly seasonal effect
- Easter: use a dummy variable  $v_t = 1$  if any part of Easter is in that month,  $v_t = 0$  otherwise.
- Ramadan and Chinese new year similar.

# Distributed lags

Lagged values of a predictor.

Example:  $x$  is advertising which has a delayed effect

$x_1$  = advertising for previous month;

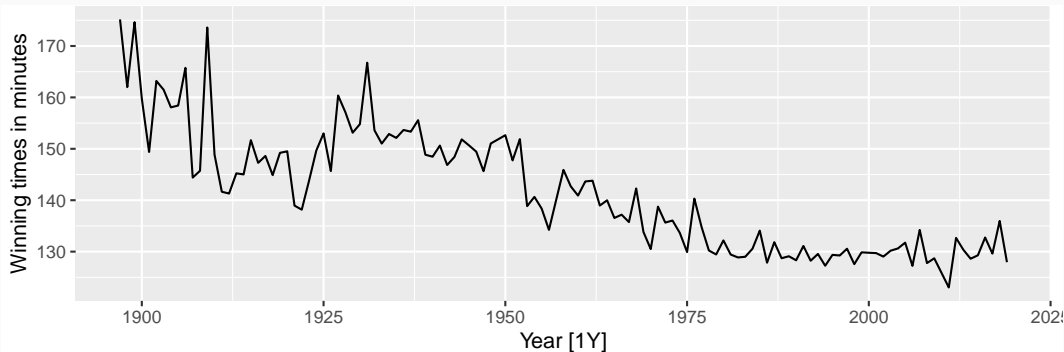
$x_2$  = advertising for two months previously;

$\vdots$

$x_m$  = advertising for  $m$  months previously.

# Example: Boston marathon winning times

```
marathon <- boston_marathon %>%  
  filter(Event == "Men's open division") %>%  
  select(-Event) %>%  
  mutate(Minutes = as.numeric(Time)/60)  
marathon %>% autoplot(Minutes) + labs(y="Winning times in minutes")
```



# Example: Boston marathon winning times

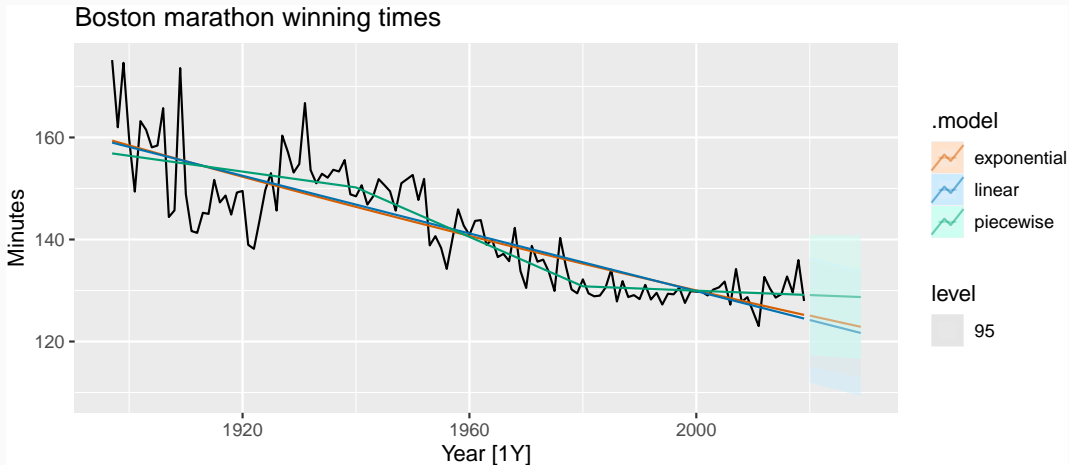
```
fit_trends <- marathon %>%  
  model(  
    # Linear trend  
    linear = TSLM(Minutes ~ trend()),  
    # Exponential trend  
    exponential = TSLM(log(Minutes) ~ trend()),  
    # Piecewise linear trend  
    piecewise = TSLM(Minutes ~ trend(knots = c(1940, 1980)))  
  )
```

```
fit_trends
```

```
## # A mable: 1 x 3  
##   linear exponential piecewise  
##   <model>      <model>    <model>  
## 1  <TSLM>      <TSLM>    <TSLM>
```

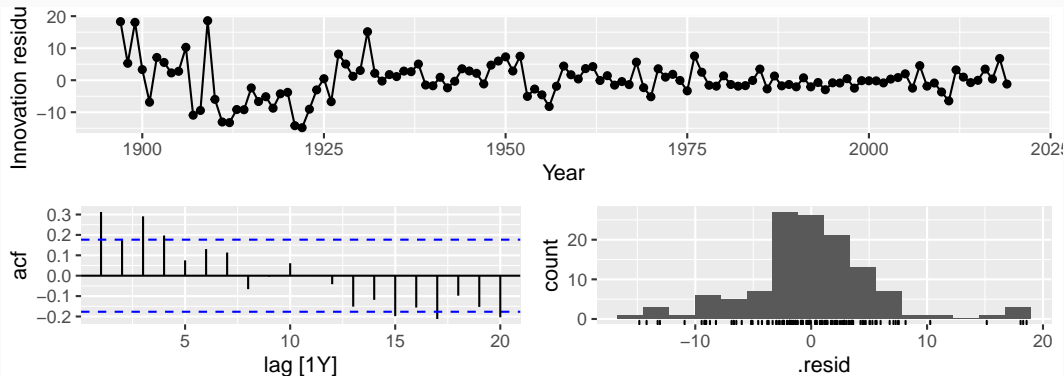
# Example: Boston marathon winning times

```
fit_trends %>% forecast(h=10) %>% autoplot(marathon)
```



# Example: Boston marathon winning times

```
fit_trends %>%  
  select(pieewise) %>%  
  gg_tsresiduals()
```



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For forecasting purposes, we require the following assumptions:

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- $\varepsilon_t$  are uncorrelated with each  $x_{j,t}$ .

It is **useful** to also have  $\varepsilon_t \sim N(0, \sigma^2)$  when producing prediction intervals or doing statistical tests.

# Residual plots

Useful for spotting outliers and whether the linear model was appropriate.

- Scatterplot of residuals  $\varepsilon_t$  against each predictor  $x_{j,t}$ .
- Scatterplot residuals against the fitted values  $\hat{y}_t$
- Expect to see scatterplots resembling a horizontal band with no values too far from the band and no patterns such as curvature or increasing spread.

# Residual patterns

- If a plot of the residuals vs any predictor in the model shows a pattern, then the relationship is nonlinear.
- If a plot of the residuals vs any predictor **not** in the model shows a pattern, then the predictor should be added to the model.
- If a plot of the residuals vs fitted values shows a pattern, then there is heteroscedasticity in the errors. (Could try a transformation.)

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# Comparing regression models

Computer output for regression will always give the  $R^2$  value. This is a useful summary of the model.

- It is equal to the square of the correlation between  $y$  and  $\hat{y}$ .
- It is often called the “coefficient of determination”.
- It can also be calculated as follows:

$$R^2 = \frac{\sum(\hat{y}_t - \bar{y})^2}{\sum(y_t - \bar{y})^2}$$

- It is the proportion of variance accounted for (explained) by the predictors.

# Comparing regression models

However ...

- $R^2$  does not allow for “degrees of freedom”.
- Adding *any* variable tends to increase the value of  $R^2$ , even if that variable is irrelevant.

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To overcome this problem, we can use *adjusted*  $R^2$ :

$$\bar{R}^2 = 1 - (1 - R^2) \frac{T - 1}{T - k - 1}$$

where  $k$  = no. predictors and  $T$  = no. observations.



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where  $k$  = no. predictors and  $T$  = no. observations.

**Maximizing  $\bar{R}^2$  is equivalent to minimizing  $\hat{\sigma}^2$ .**

$$\hat{\sigma}^2 = \frac{1}{T - k - 1} \sum_{t=1}^T \varepsilon_t^2$$

# Akaike's Information Criterion

$$\text{AIC} = -2 \log(L) + 2(k + 2)$$

where  $L$  is the likelihood and  $k$  is the number of predictors in the model.

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- AIC penalizes terms more heavily than  $\bar{R}^2$ .
- Minimizing the AIC is asymptotically equivalent to minimizing MSE via **leave-one-out cross-validation** (for any linear regression).

## Corrected AIC

For small values of  $T$ , the AIC tends to select too many predictors, and so a bias-corrected version of the AIC has been developed.

$$\text{AIC}_C = \text{AIC} + \frac{2(k+2)(k+3)}{T-k-3}$$

As with the AIC, the  $\text{AIC}_C$  should be minimized.

# Bayesian Information Criterion

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- BIC penalizes terms more heavily than AIC
- Also called SBIC and SC.
- Minimizing BIC is asymptotically equivalent to leave- $v$ -out cross-validation when  $v = T[1 - 1/(\log(T) - 1)]$ .

# Leave-one-out cross-validation

For regression, leave-one-out cross-validation is faster and more efficient than time-series cross-validation.

- Select one observation for test set, and use *remaining* observations in training set. Compute error on test observation.
- Repeat using each possible observation as the test set.
- Compute accuracy measure over all errors.

# Cross-validation

## Traditional evaluation



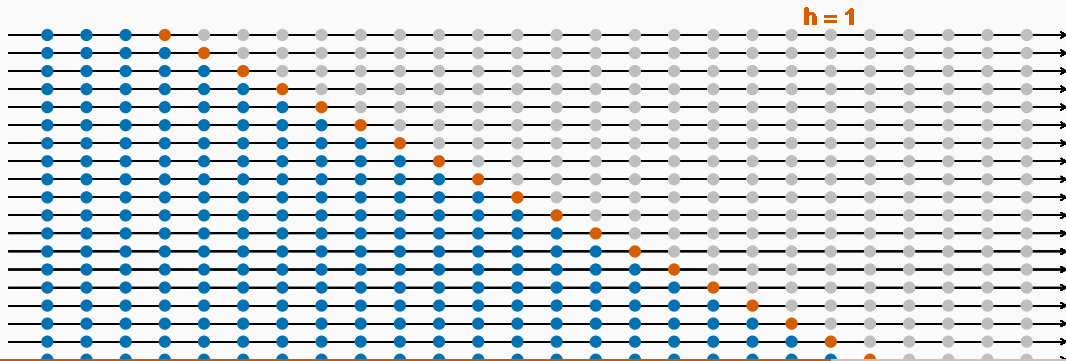


# Cross-validation

## Traditional evaluation



## Time series cross-validation

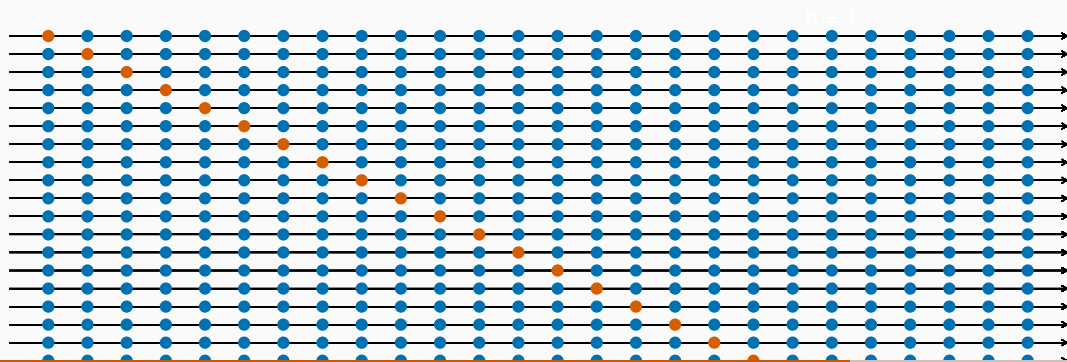


# Cross-validation

## Traditional evaluation



## Leave-one-out cross-validation

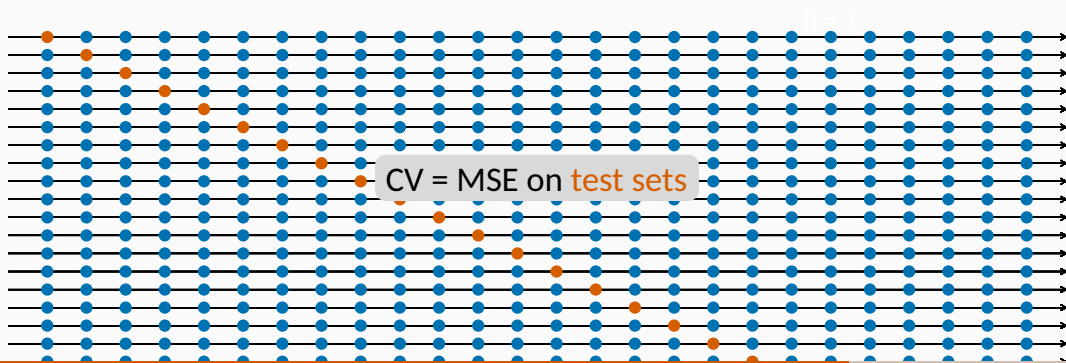


# Cross-validation

## Traditional evaluation



## Leave-one-out cross-validation



# Choosing regression variables

## Best subsets regression

- Fit all possible regression models using one or more of the predictors.
- Choose the best model based on one of the measures of predictive ability (CV, AIC, AICc).

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## Warning!

- If there are a large number of predictors, this is not possible.
- For example, 44 predictors leads to 18 trillion possible models!

# Choosing regression variables

## Backwards stepwise regression

- Start with a model containing all variables.
- Try subtracting one variable at a time. Keep the model if it has lower CV or AICc.
- Iterate until no further improvement.

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## Notes

- Stepwise regression is not guaranteed to lead to the best possible model.
- Inference on coefficients of final model will be wrong.

# Outline

- 1 The linear model with time series
- 2 Some useful predictors for linear models
- 3 Residual diagnostics
- 4 Selecting predictors and forecast evaluation
- 5 Forecasting with regression
- 6 Matrix formulation
- 7 Correlation, causation and forecasting



# Ex-ante versus ex-post forecasts

- *Ex ante forecasts* are made using only information available in advance.
  - ▶ require forecasts of predictors
- *Ex post forecasts* are made using later information on the predictors.
  - ▶ useful for studying behaviour of forecasting models.
- trend, seasonal and calendar variables are all known in advance, so these don't need to be forecast.

# Scenario based forecasting

- Assumes possible scenarios for the predictor variables
- Prediction intervals for scenario based forecasts do not include the uncertainty associated with the future values of the predictor variables.

# Building a predictive regression model

- If getting forecasts of predictors is difficult, you can use lagged predictors instead.

$$y_t = \beta_0 + \beta_1 x_{1,t-h} + \cdots + \beta_k x_{k,t-h} + \varepsilon_t$$

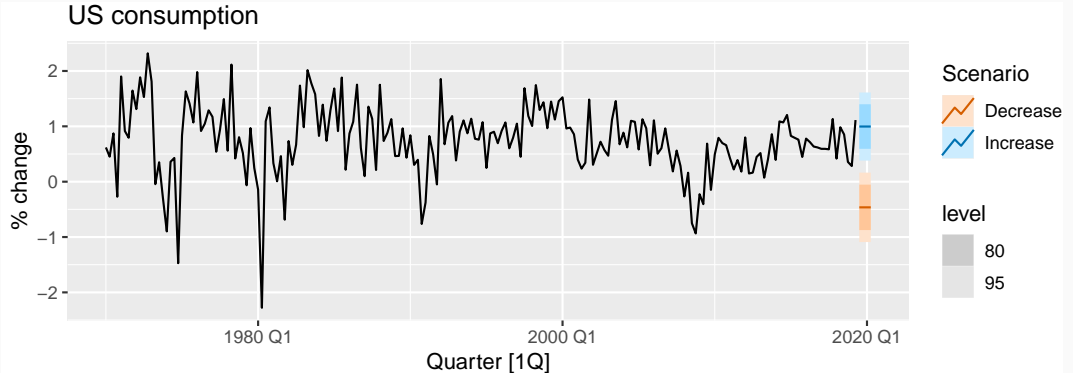
- A different model for each forecast horizon  $h$ .

# US Consumption

```
fit_consBest <- us_change %>%  
  model(  
    TSLM(Consumption ~ Income + Savings + Unemployment)  
  )  
  
future_scenarios <- scenarios(  
  Increase = new_data(us_change, 4) %>%  
    mutate(Income=1, Savings=0.5, Unemployment=0),  
  Decrease = new_data(us_change, 4) %>%  
    mutate(Income=-1, Savings=-0.5, Unemployment=0),  
  names_to = "Scenario")  
  
fc <- forecast(fit_consBest, new_data = future_scenarios)
```

# US Consumption

```
us_change %>% autoplot(Consumption) +  
  labs(y="% change in US consumption") +  
  autolayer(fc) +  
  labs(title = "US consumption", y = "% change")
```



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# Matrix formulation

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Let  $\mathbf{y} = (y_1, \dots, y_T)'$ ,  $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_T)'$ ,  $\boldsymbol{\beta} = (\beta_0, \beta_1, \dots, \beta_k)'$  and

$$\mathbf{X} = \begin{bmatrix} 1 & x_{1,1} & x_{2,1} & \dots & x_{k,1} \\ 1 & x_{1,2} & x_{2,2} & \dots & x_{k,2} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_{1,T} & x_{2,T} & \dots & x_{k,T} \end{bmatrix}.$$



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Then

$$\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}.$$

# Matrix formulation

## Least squares estimation

Minimize:  $(\mathbf{y} - \mathbf{X}\beta)'(\mathbf{y} - \mathbf{X}\beta)$

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$$\hat{\sigma}^2 = \frac{1}{T - k - 1}(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\hat{\boldsymbol{\beta}})$$

**Note:** If you fall for the dummy variable trap,  $(\mathbf{X}'\mathbf{X})$  is a singular matrix. 64

# Likelihood

If the errors are iid and normally distributed, then

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**So MLE = OLS.**

# Multiple regression forecasts

## Optimal forecasts

$$\hat{y}^* = E(y^* | \mathbf{y}, \mathbf{X}, \mathbf{x}^*) = \mathbf{x}^* \hat{\boldsymbol{\beta}} = \mathbf{x}^* (\mathbf{X}'\mathbf{X})^{-1} \mathbf{X}'\mathbf{y}$$

where  $\mathbf{x}^*$  is a row vector containing the values of the predictors for the forecasts (in the same format as  $\mathbf{X}$ ).

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## Forecast variance

$$\text{Var}(y^* | \mathbf{X}, \mathbf{x}^*) = \sigma^2 \left[ 1 + \mathbf{x}^* (\mathbf{X}'\mathbf{X})^{-1} (\mathbf{x}^*)' \right]$$

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- This ignores any errors in  $\mathbf{x}^*$ .
- 95% prediction intervals assuming normal errors:

$$\hat{y}^* \pm 1.96 \sqrt{\text{Var}(y^* | \mathbf{X}, \mathbf{x}^*)}.$$

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# Correlation is not causation

- When  $x$  is useful for predicting  $y$ , it is not necessarily causing  $y$ .
- e.g., predict number of drownings  $y$  using number of ice-creams sold  $x$ .
- Correlations are useful for forecasting, even when there is no causality.
- Better models usually involve causal relationships (e.g., temperature  $x$  and people  $z$  to predict drownings  $y$ ).

# Multicollinearity

In regression analysis, multicollinearity occurs when:

- Two predictors are highly correlated (i.e., the correlation between them is close to  $\pm 1$ ).
- A linear combination of some of the predictors is highly correlated with another predictor.
- A linear combination of one subset of predictors is highly correlated with a linear combination of another subset of predictors.

# Multicollinearity

If multicollinearity exists...

- the numerical estimates of coefficients may be wrong (worse in Excel than in a statistics package)
- don't rely on the  $p$ -values to determine significance.
- there is no problem with model *predictions* provided the predictors used for forecasting are within the range used for fitting.
- omitting variables can help.
- combining variables can help.