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IMAGE THEORY FOR THE STRUCTURE OF QUANTITATIVE VARIATES*

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A universe of infinitely many quantitative variables is considered, from which a sample of n variables is arbitrarily selected. Only linear least-squares regressions are considered, based on an infinitely large population of individuals or respondents. In the sample of variables, the predicted value of a variable x from the remaining $n - 1$ variables is called the partial image of x, and the error of prediction is called the partial anti-image of x. The predicted value of x from the entire universe, or the limit of its partial images as $n \to \infty$, is called the total image of x, and the corresponding error is called the total anti-image. Images and anti-images can be used to explain "why" any two variables x_j and x_k are correlated with each other, or to reveal the structure of the intercorrelations of the sample and of the universe. It is demonstrated that image theory is related to common-factor theory but has greater generality than common-factor theory, being able to deal with structures other than those describable in a Spearman-Thurstone factor space. A universal computing procedure is suggested, based upon the inverse of the correlation matrix.

1. *The Multiple-Correlation Approach to the Notion of "Commonness"*

There are two ways in which it is conventional to try to explain "why" statistical variables are intercorrelated. One is based on multiple correlation and the other on partial correlation.

The partial-correlation approach has been utilized to develop a theory to explain all intercorrelations simultaneously within a set of variates, namely, the theory of *common factors*. Spearman's celebrated hypothesis was that mental tests were intercorrelated because they had a single general factor in common; if this factor were partialed out, no correlations would remain. The generalization to multiple common factors by Spearman, Thurstone, and others remains a partial-correlation approach. If m variables can be found such that when they are partialed out the observed inter-test correlations vanish, then these m variables are said to constitute a set of m common factors, and to represent what the original tests have in common (4).

Common-factor theory is still beset with several different kinds of

*This paper introduces one of three new structural theories, each of which generalizes common-factor analysis in a different direction. *Nodular* theory extends common-factor analysis to qualitative data and to data with curvilinear regressions (6). *Order-factor* theory introduces the notions of *order* among the observed variables and of *separable* factors (7). The present *image* theory is relevant also to the other two.

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problems of indeterminacy (among them the problems of communalities, of rotation of axes, and of estimating factor scores) arising from the fact that the m variables to be partialed out are hypothetical in the first instance. Many controversies exist as to how to make these variables concrete, and many scientists are sceptical of the validity of the basic premises.

It is interesting that hitherto only the partial-correlation approach—using controversial hypothetical variables—has been used for a structural analysis of a set of variates, despite the fact that the more concrete notions involved in the multiple-correlation approach seem older and more widely accepted. Apparently no systematic attempt has been made previously to capitalize on the structural possibilities of the multiple-correlation approach. Such an investigation is the purpose of the present paper.

We shall show how the intercorrelations within a set of variables all can be simultaneously interpreted or explained by means of their mutual multiple regressions, the latter determining, in a certain unambiguous manner, what the observations have in common.

We treat here the case of quantitative variables with linear least-squares regressions. Elsewhere we shall treat qualitative cases [as in (5)].

For the multiple-correlation approach, we need introduce no hypothetical variables. If we are given a set of n observed variables x_i , we can consider directly the multiple regression of each variable on all the remaining $n - 1$ variables. If r_i is the resulting coefficient of multiple correlation for x_i , then traditionally r_i^2 has been called the "index* of determination" of x_i from the remaining variables, and $\sqrt{1 - r_i^2}$ the "coefficient of alienation."

Indeed, r_i^2 represents the proportion of the total variance of x_i that is dependent on the remaining variables, and in a real sense expresses how much x_i has in common with other variables. If $r_i^2 = 0$, then x_i has nothing in common with the other variables; in fact, it also then correlates zero with each separately. If $r_i^2 = 1$, then x_i is linearly dependent on the remaining variables, so that whatever could be done with x_i could be done as well without it; the remaining variables contain all the relevant information for any problem. Values of r_i^2 intermediate between zero and unity, then, express intermediate degrees of commonness between x_i and all the remaining variables.

This can be seen further by studying the classical normal equations from which one computes the multiple regression coefficients. According to these equations, we break x_i up into two parts, say p_i and e_i , where p_i is the predicted value and e_i is the error of prediction:

$$x_i = p_i + e_i .$$

*Although there is no standard usage in the literature, we shall systematically use the word "index" to refer to the *square* of a correlation coefficient, to distinguish the square from the coefficient itself.

The equations state in particular that e_j correlates zero with p_j : the prediction and the error of prediction are uncorrelated. But more important for us, the equations state more generally that e_j correlates zero with each predictor x_k separately, or

$$r_{e_j x_k} = 0; \qquad (j \neq k).$$

Thus x_j is broken up into two parts. One part, p_j , is perfectly dependent on the remaining $n - 1$ variables; the other part, e_j , correlates zero with each and every one of the $n - 1$ predictors, and hence with any possible (linear) combination of them. The multiple correlation of p_j on the predictors is perfect; the multiple correlation of e_j on the predictors is zero.

Multiple correlation gives us this simple and profound property of breaking each variable into two parts, one of which is determined entirely by the remaining variables, and the other of which has no relation with the remaining variables.

The study of the common and alien parts of the observed variates, as defined by multiple correlation, we propose to call *image analysis*, a name suggested by the n-dimensional geometry of the situation (**11**).

Paradoxically, the alien parts can play a role in the observed inter-test correlations, which is one of the major points analyzed in the present paper, especially in §8 below. Indeed, in a sense, the "alien" parts are more basic than the "common" parts, as shown in the final §11 below.

2. *Relationship to Common-Factor Theory*

It is of interest to inquire as to what relationship image analysis has to common-factor analysis in the Spearman-Thurstone sense. It turns out that image analysis is the more basic and inclusive approach; it includes common-factor theory as a special case. That this might be so could possibly be surmised by considering the respective properties of partial correlation and multiple correlation. A partial correlation coefficient in general can either increase or decrease as the number of variables eliminated increases; but common-factor theory is concerned only with a *special* kind of circumstance wherein partial correlations tend only to zero. On the other hand, a multiple correlation can never decrease as the number of predictors increases; in general, the correlation increases. This nondecreasing property is all that is required by image theory; so no restrictions at all are involved, and the theory is universally appropriate.

Because of its universality, image theory throws considerable light on common-factor theory, as well as on order-factor theory (7) and on any other special theory. It shows under what special circumstances a universe of data admits of a common-factor structure at all, regardless of the number of common factors. This we shall see in the present paper. In a later paper, we shall see how image theory explains why the problem of communalities has not been solved yet in the Spearman-Thurstone theory, and how a universal solution is impossible; it will also be shown there how misleading present computing routines can sometimes be that are based on "extracting" common factors (10).

A new, universal computing routine will be suggested that will help distinguish for a given set of data whether they have a finite common-factor structure (no matter what the finite number of common factors may be), an order-factor structure (simplex, circumplex, etc.), or some other kind of structure.

3. *The Observed Correlation Matrix*

We are concerned with a *universe of content** of indefinitely many quantitative variables, which is defined in advance of any statistical analysis. We assume that each variable has a finite population variance, and that all regressions are linear.†

It is essential to distinguish between the universe and any finite samples of n variables that may be selected from it. We assume nothing about *random* sampling of variables but that we can arrange the universe in an arbitrary order in which our particular sample will be the first n variables.

With respect to the population of individuals observed, it too is assumed indefinitely large. We are not concerned here with samples of people; throughout we treat only population parameters.

If x_j denotes the jth variable from the universe, then let x_{ji} be the score of person i on this variable. As usual, we can set the population mean of each variable equal to zero, and the variance equal to unity. Thus,

$$\underset{i}{E}\, x_{ji} \cdots = 0, \qquad \underset{i}{E}\, x_{ji}^2 \cdots = 1, \qquad (j = 1, 2, \cdots), \tag{1}$$

where the notation $\underset{i}{E}$ denotes the expected or mean value over the infinite population of individuals. Then the population correlation coefficient, r_{jk}, between any two observed variables is simply their covariance,

$$r_{jk} = \underset{i}{E}\, x_{ji}x_{ki}\ ; \qquad (j, k = 1, 2, \cdots). \tag{2}$$

If we are dealing with a finite number of n variables, then the values r_{jk} can be expressed as a Gramian matrix of order n which we shall denote by R_n,

$$R_n = \|\, r_{jk} \,\|; \qquad (j, k = 1, 2, \cdots, n). \tag{3}$$

The entries in the main diagonal of R_n are each unity, according to (2) and (1), indicating the total self-correlations.

*A term originating in the context of scale analysis, but appropriate more generally.

†Peculiarly, the theorems below do not depend on the *true* regressions; these may be curvilinear. Of course, the *meaning* of our results is fullest if the true regressions actually are linear; and even more, if the zero correlations imply complete statistical independence.

As more variables from the universe are added to the initial set of n in (3), nothing happens to the initial entries r_{jk} except that more rows and columns surround them. An observed correlation coefficient as in (2) between any two variables is not a function of n; it does not depend on which other variables are in the set. Therefore, if we inquire what happens to R_n in the limit as $n \to \infty$, we can state that *there always is a limiting matrix*, which we shall denote by R_∞,

$$R_\infty = \lim_{n \to \infty} R_n \, . \tag{4}$$

R_∞ is an infinite Gramian matrix, and represents the correlations between all variables in the infinite universe of content.

4. *The Inverse of the Correlation Matrix and Its Problem of Limits*

The *inverse* of the observed correlation matrix plays a central role in our analysis. We shall usually assume that, for a given set of n variables, R_n is nonsingular and possesses an inverse. This will be true, for instance, if all observed variables are experimentally independent and have retest reliabilities less than unity. The assumption of nonsingularity is usually correct in practice.

The inverse of R_n will be denoted as usual by R_n^{-1}. In contrast to the elements of R_n, the elements of R_n^{-1} are functions of n and change as additional variables are added to the set. As is well known, the elements of R_n^{-1} can be expressed in terms of minor determinants of R_n. Let

$$\Delta^{(n)} = \text{the determinant of } R_n \, ,$$

and let

$$\Delta_{jk}^{(n)} = \text{the cofactor of } r_{jk} \text{ in } R_n \, .$$

Then:

$$R_n^{-1} = \left\| \frac{\Delta_{jk}^{(n)}}{\Delta^{(n)}} \right\| . \tag{5}$$

$\Delta^{(n)}$ clearly varies as n varies; and for fixed j and k, $\Delta_{jk}^{(n)}$ also varies with n.

The sample of n variables is studied in order to yield inferences about the universe of content. We must ask what will happen if we increase the size of the sample. If nothing definite happens in the limit as $n \to \infty$, then surely we cannot infer much about the universe, and any structural theory we may have will be unfounded. The differences among finite common-factor structures, order-factor structures, and other kinds of structures will be seen to depend largely on what happens in the limit of R_n^{-1} as $n \to \infty$.

First we note that, in dealing with infinite matrices, the algebra of finite matrices need not at all hold. If there is a definite limit to R_n^{-1}, then in general it is *not* the same as the inverse of R_∞; that is, in general,

$$R_\infty^{-1} \neq \lim_{n \to \infty} R_n^{-1}, \tag{6}$$

even if both members exist. Indeed, the right member of (6) may exist and hence be uniquely defined; but at the same time R_∞^{-1} need not exist, or alternatively R_∞^{-1} may represent more than one matrix. Even if R_n^{-1} converges to something definite, we have no assurance that there exists a matrix R_∞^{-1} such that $R_\infty R_\infty^{-1} = I_\infty$. Even if such an R_∞^{-1} exists, it may be only a *right* inverse and not a *left* inverse, so that $R_\infty^{-1} R_\infty \neq I_\infty$; or there may be more than one such inverse to R_∞. These are paradoxes of infinity. For finite matrices, right and left inverses always are identical and unique.

The importance of inequality (6) is illustrated by common-factor theory. A number of years ago it was proved that the foundations of the Spearman-Thurstone approach rest essentially on the proposition that inequality (6) holds; in particular that $\lim_{n\to\infty} R_n^{-1}$ exists and is a *diagonal* matrix (2) or that *nondiagonal elements vanish in the limit,*

$$\lim_{n\to\infty} \frac{\Delta_{jk}^{(n)}}{\Delta^{(n)}} = 0; \qquad (j \neq k). \tag{7}$$

Such a diagonal matrix clearly *cannot* be an inverse for R_∞. This hitherto little-noticed theorem has most practical consequences, for it provides an entirely new way of testing empirical data for the *possible existence of common actors.* Given an observed matrix R_n, compute R_n^{-1} and see if the nondiagonal elements are all close to zero. Such a criterion requires no preliminary determination of "communalities" nor "fitting" of factor loadings, nor specification of the *number* of common factors. If the criterion (7) is not satisfied, then it is usually futile to attempt to "fit" any common-factor space of finite rank to the data. Image theory will enable us to improve on and to generalize this criterion, as is shown in §11 below.

One example where criterion (7) is not satisfied can be shown to be the *simplex* matrix, where the correlations have the law of formation,

$$r_{jk} = a_j b_k, \qquad (j < k),$$

a_j and b_j being two certain parameters belonging to x_j. It is futile to attempt to find any finite number of common factors for such a matrix as $n \to \infty$. Actually, a much simpler theory than that of finite common factors holds (7).

So much for what is for the moment a digression, to emphasize the importance of *proving the possible existence* of any quantities we may want to hypothesize. It is not to be regarded merely as a matter of mathematical pedantry.

5. *Partial Images and Total Images*

A sample of n variables from the universe of content defines a *partial image* for each variable, namely, its predicted values from the remaining

$n - 1$ variables. The limit of the partial images of x_j as n becomes infinite will be called the *total image* of x_j in the universe of content. Let $p_{ji}^{(n)}$ denote the *predicted value* of x_{ji} from the remaining $n - 1$ variables in the sample, and let $p_{ji}^{(\infty)}$ denote the limit as $n \to \infty$:

$$p_{ji}^{(\infty)} = \lim_{n \to \infty} p_{ji}^{(n)}. \tag{8}$$

We assume for the moment that the limit in (8) exists; in a later paper (**9**) we shall examine this assumption. Then $p_{ji}^{(n)}$ is the partial image score for person i for variable x_j, and $p_{ji}^{(\infty)}$ is his total image score for x_j.

It is well-known how to compute $p_{ji}^{(n)}$ from the observed data. If we let $w_{jk}^{(n)}$ denote the weight of x_k in the multiple regression for predicting x_j, then (cf. **1**, 306),

$$w_{jk}^{(n)} = \frac{-\Delta_{jk}^{(n)}}{\Delta_{jj}^{(n)}}, \qquad (j \neq k), \tag{9}$$

provided the denominator on the right does not vanish. Notice that (9) does not define a value for $j = k$, for this would imply a weight for predicting the test from itself. It will be convenient, however, apparently to include the test itself in its own regression by the artifice of giving it a regression weight of zero. So we define the value for $j = k$ as follows:

$$w_{jj}^{(n)} \equiv 0, \tag{10}$$

for all j and n. With this artifice, we can express the partial image scores of the x_j in the following convenient form,

$$p_{ji}^{(n)} = \sum_{k=1}^{n} w_{jk}^{(n)} x_{ki}. \tag{11}$$

The total image scores of x_j are then, from (8) and (11),

$$p_{ji}^{(\infty)} = \lim_{n \to \infty} \sum_{k=1}^{n} w_{jk}^{(n)} x_{ki}. \tag{12}$$

In the right member of (12), not only the number of terms being summed depends on n, but also the values of the terms themselves, for the regression weights $w_{jk}^{(n)}$ depend on all the variables used as predictors.

Formula (9) holds for the regression weights, *provided* the denominator $\Delta_{jj}^{(n)}$ does not vanish. If R_n is nonsingular, the denominator can never vanish for any j; but if R_n is singular, (9) may not hold and the regression weights may not be uniquely defined. Regardless, however, the partial image *scores* are always uniquely defined, whether R_n is singular or nonsingular. This is well-known, but it may be well to restate here the fundamentals involved. This we do in the next section.

6. The Fundamental Equation For Least-Squares Images.

Consider the weights $w_{jk}^{(n)}$ in (11) as unknowns to be solved for—except for the self-weights which are always zero as in (10). Let $e_{ji}^{(n)}$ be the errors of estimate according to the prediction (11), so that

$$x_{ji} = p_{ji}^{(n)} + e_{ji}^{(n)}. \tag{13}$$

Let the variance of the errors for x_j be denoted by σ_{nj}^2 ,

$$\sigma_{jn}^2 = \underset{i}{E}\, [e_{ji}^{(n)}]^2. \tag{14}$$

Then we wish to determine the $w_{jk}^{(n)}$ so as to minimize (14). Differentiating the right member of (14) with respect to the $w_{jk}^{(n)}$, using relations (13) and (11), shows that a necessary and sufficient condition for attaining a minimum is that the following fundamental equation holds:

$$\underset{i}{E}\, e_{ji}^{(n)} x_{ki} = 0, \qquad (j \neq k), \tag{15}$$

or that the *errors be uncorrelated with each predictor separately.* Fundamental equation (15) expresses the classical normal equations of linear least squares. There is a unique minimum to (14), obtained by unique values of $e_{ji}^{(n)}$, and hence of $p_{ji}^{(n)}$. If R_n is singular, it may be that the x_{ki} in (11) are linearly dependent in such a fashion that more than one set of $w_{jk}^{(n)}$ will yield the same best prediction $p_{ji}^{(n)}$, but the best prediction itself is uniquely determined regardless. If R_n is nonsingular, then also the best $w_{jk}^{(n)}$ are uniquely determined by the data, namely by formula (9).

More generally, then, we can regard (15) as our basic equation for determining the $w_{jk}^{(n)}$—uniquely or not. Equation (15) is *the basic equation of image analysis, from which all other results follow.* Together with definitions (11) and (13), equation (15) uniquely determines the partial images, and invests them with all their subsequent meaning and properties.

The errors of prediction from the partial images play a prominent role in our theory. We shall call the $e_{ji}^{(n)}$ the *partial anti-image* scores of x_j . Then, parallel to (12), the *total anti-image* scores will be denoted by $e_{ji}^{(\infty)}$ and

$$e_{ji}^{(\infty)} = \lim_{n \to \infty} e_{ji}^{(n)}, \tag{16}$$

assuming the limit on the right exists.

One immediate consequence of basic equation (15) is that the partial image and anti-image of each x_j correlate zero with each other,

$$\underset{i}{E}\, e_{ji}^{(n)} p_{ji}^{(n)} = 0. \tag{17}$$

This well-known result follows by multiplying each member of (11) by $e_{ji}^{(n)}$, taking expectations over i and using (15) and (10).

Let ρ_{jn} be the multiple correlation coefficient of x_j on the remaining $n - 1$ variables. Since the variance of x_j is unity, then—as is well-known—ρ_{jn} is also the standard deviation of the $p_{ji}^{(n)}$, or,

$$\rho_{jn}^2 = \underset{i}{E}\, [p_{ji}^{(n)}]^2. \tag{18}$$

A well-known consequence of (17) is, then, that

$$\rho_{jn}^2 + \sigma_{jn}^2 = 1. \tag{19}$$

It is ρ_{jn}^2 that has traditionally been called the "index of determination," and σ_{jn}^2 the "index of alienation." To avoid possible notions of determination in the sense of causation, and to use a more convenient terminology for our purposes, we shall call ρ_{jn}^2 the *partial norm* of x_j, and σ_{jn}^2 the *partial antinorm.*

Geometrically, the n variables x_j can be described as unit vectors with a common origin, defining an n-dimensional Euclidean space. A correlation r_{jk} is the cosine of the angle between x_j and x_k. The image variable of x_j is then represented by the projection of the vector x_j on the $(n - 1)$-dimensional space defined by the remaining vectors; and ρ_{jn} is the cosine of the angle between x_j and its projection, as well as being the *length* of the projection vector. σ_{jn} is the distance between the termini of the vectors of x_j and its projection, as well as being the cosine of one of the angles involved.

It is interesting that this geometry of image theory was known long before the advent of common-factor theory, which uses a similar geometry (cf. **11**).

A norm, then, is the square of the length of a test vector's projection; and an antinorm is the square of the distance between the termini of a vector and its projection. Equation (19) expresses the Pythagorean theorem for the right triangle defined by the vector of x_j and its projection or image.

The *total norm* of x_j will be defined as the limit of its partial norms, and will be denoted by $\rho_{j\infty}^2$,

$$\rho_{j\infty}^2 = \lim_{n\to\infty} \rho_{jn}^2 . \tag{20}$$

A similar definition holds for the total antinorm, denoted by $\sigma_{j\infty}^2$,

$$\sigma_{j\infty}^2 = \lim_{n\to\infty} \sigma_{jn}^2 . \tag{21}$$

Obviously, if the limits in (20) and (21) exist, then from (19),

$$\rho_{j\infty}^2 + \sigma_{j\infty}^2 = 1. \tag{22}$$

That total norms and antinorms always exist is easily established. It is well-known that a multiple correlation coefficient cannot decrease as more variables are added to the regression. Therefore, for each j, ρ_{jn}^2, is a monotonically increasing function of n. Being bounded above by unity, it follows that there must exist a limit to ρ_{jn}^2 as $n \to \infty$, by the usual theorem on bounded monotone functions. Similarly, σ_{jn}^2 always has a limit as $n \to \infty$. These results we shall state as:

THEOREM 1: *Total norms and antinorms,* $\rho_{j\infty}^2$ *and* $\sigma_{j\infty}^2$ *, always exist for each variable* x_j *in an infinite universe of content (where each* x_j *has unit variance).*

Further problems of existence of limits—with respect to individual image scores and parameters associated with them—will be treated in a separate paper (9).

7. *The Fundamental Identity for Least-Squares Images*

The purpose of any structural analysis is to provide a framework for comprehending the interrelationships among observations. Our present problem is to "explain" the correlation coefficients r_{jk} . For this purpose, image analysis has a universally applicable "explanation," as stated in the following fundamental theorem:

THEOREM 2: *The correlation between any two different observed variables (with unit variances) from a given set of* n *variables is the difference between the covariance of their respective partial images and the covariance of their respective partial anti-images. That is, if we let* $g_{jk}^{(n)}$ *and* $\gamma_{jk}^{(n)}$ *be the covariance between the partial images and anti-images, respectively, for* x_j *and* x_k *, or,*

$$g_{jk}^{(n)} = \underset{i}{E}\, p_{ji}^{(n)} p_{ki}^{(n)} \tag{23}$$

and

$$\gamma_{jk}^{(n)} = \underset{i}{E}\, e_{ji}^{(n)} e_{ki}^{(n)}, \tag{24}$$

then the following identity always holds:

$$r_{jk} = g_{jk}^{(n)} - \gamma_{jk}^{(n)}; \qquad (j \neq k). \tag{25}$$

To establish the theorem, we first multiply both members of (13) by x_{ki} , take expectations over i—remembering (2)—and arrive at

$$r_{jk} = \underset{i}{E}\, p_{ji}^{(n)} x_{ki} + \underset{i}{E}\, e_{ji}^{(n)} x_{ki} \,. \tag{26}$$

Now the second term on the right vanishes for $j \neq k$, according to (15). Hence (26) becomes

$$r_{jk} = \underset{i}{E}\, p_{ji}^{(n)} x_{ki} \,; \qquad (j \neq k). \tag{27}$$

The left member is symmetric in j and k. Hence we can rewrite the right member by interchanging j and k without altering the result:

$$r_{jk} = \underset{i}{E}\, p_{ki}^{(n)} x_{ji} \,; \qquad (j \neq k). \tag{28}$$

Equations (28) and (27) state that x_j has the same covariance with x_k as with the partial image of x_k, and vice versa.

Multiply both members of (13) by $p_{ki}^{(n)}$, take expectations over i, and use definition (23):

$$\underset{i}{E}\, p_{ki}^{(n)} x_{ji} = g_{jk}^{(n)} + \underset{i}{E}\, p_{ki}^{(n)} e_{ji}^{(n)}. \tag{29}$$

Multiply both members of (13) by $e_{ki}^{(n)}$, take expectations over i, and use definition (24):

$$\underset{i}{E}\, e_{ki}^{(n)} x_{ji} = \underset{i}{E}\, p_{ji}^{(n)} e_{ki}^{(n)} + \gamma_{jk}^{(n)}. \tag{30}$$

Now the left member of (30) vanishes for $j \neq k$, according to (15). Hence, from (30),

$$-\gamma_{jk}^{(n)} = \underset{i}{E}\, p_{ji}^{(n)} e_{ki}^{(n)} = \underset{i}{E}\, p_{ki}^{(n)} e_{ji}^{(n)}; \qquad (j \neq k). \tag{31}$$

The last member is obtained from the middle member by interchanging sub scripts j and k, which is permissible by virtue of the symmetry of the first member.

Using (31) in (29), and then (29) in (28) establishes that (25) is correct, or Theorem 2 holds.

It should be remarked that there are no assumptions whatsoever in establishing equation (25); *it is a universal identity.* We have not used here the assumption that R_n is nonsingular, or that the $w_{jk}^{(n)}$ are uniquely defined as by (9). Only the basic normal equations (15) have been used, which assure unique values for images and anti-images even when R_n is singular and (9) does not hold.

8. *Interpretation of the Fundamental Identity*

According to identity (25), any correlation coefficient can be regarded as the difference between two covariances, one from the common parts of the two variables and the other from the alien parts.

Students of common-factor theory may be puzzled at first by the fact that the alien parts should be correlated and affect the observed correlation r_{jk}. They are accustomed to the notion of "specific" or "unique" parts which are mutually uncorrelated and do *not* affect the r_{jk}, $(j \neq k)$. They may be tempted to take the point of view that the $\gamma_{jk}^{(n)}$ represent "errors of fit" of the image covariances $g_{jk}^{(n)}$ to the observed correlations r_{jk}. We shall see that such a point of view is correct, *provided a finite common-factor space really exists* (regardless of what dimensionality) *for the entire universe* as $n \rightarrow \infty$.

But we shall also see that such a point of view is very specialized. Let us first take the most general view of the situation, and then we shall see how various specializations can occur.

The fundamental equation (15) which led to the fundamental identity (25) states that the anti-image of x_j is orthogonal to (uncorrelated with) each of the remaining x_k. But, paradoxically, this anti-image is *not* necessarily orthogonal to the anti-images of the x_k; $\gamma_{jk}^{(n)}$ is not necessarily zero for any pair of subscripts. *An anti-image is always orthogonal to a total predictor, but not necessarily to parts of that predictor.*

It is indeed peculiar that $e_j^{(n)}$ should always be orthogonal to x_k, $(j \neq k)$, but not necessarily to $e_k^{(n)}$ or to $p_k^{(n)}$. It will seem less peculiar if we examine the meaning of $\gamma_{jk}^{(n)}$ more closely. We shall show now that *$\gamma_{jk}^{(n)}$ is intimately related to the partial correlation between* x_j *and* x_k, *holding constant the remaining* $n - 2$ *variables.*

In order to avoid details unnecessary to the main argument, let us assume R_n to be nonsingular, so that we can use formula (9) for the various regression weights, as well as further convenient determinantal formulas. Let $\pi_{jk}^{(n)}$ denote the partial correlation between x_j and x_k, eliminating the $n - 2$ remaining variables. This means that x_j and x_k first are predicted separately from the $n - 2$ remaining variables (where now x_j is *not* used in the regression for x_k, nor x_k in the regression for x_j, so that different weights are involved from those of the respective partial-image regressions on $n - 1$ variables) and then the resulting errors of prediction are correlated (1) to define the partial correlation coefficient $\pi_{jk}^{(n)}$. The well-known determinantal formula for this partial correlation is (1)

$$\pi_{jk}^{(n)} = \frac{-\Delta_{jk}^{(n)}}{\sqrt{\Delta_{jj}^{(n)}\Delta_{kk}^{(n)}}}; \qquad (j \neq k). \tag{32}$$

Now if we multiply both members of (11) by $e_{li}^{(n)}$, take expectations over i, and use (31), (15), and (10), we have

$$-\gamma_{jl}^{(n)} = w_{jk}^{(n)} \mathop{E}_{i} x_{li} e_{li}^{(n)}; \qquad (j \neq l). \tag{33}$$

But

$$\mathop{E}_{i} x_{ji} e_{ji}^{(n)} = \sigma_{jn}^2, \tag{34}$$

as can be seen by multiplying both members of (13) by $e_{ji}^{(n)}$, taking expectations over i and using (17). Hence, from (33) and (34), revising the subscripts, we have

$$-\gamma_{jk}^{(n)} = w_{jk}^{(n)}\sigma_{kn}^2; \qquad (j \neq k). \tag{35}$$

The determinantal formula for σ_{kn}^2 is (1)

$$\sigma_{kn}^2 = \frac{\Delta^{(n)}}{\Delta_{kk}^{(n)}}. \tag{36}$$

Therefore, using (36) and (9) in (35) we arrive at the *determinantal formula for the covariance between any two anti-images,*

$$\gamma_{jk}^{(n)} = \frac{\Delta_{jk}^{(n)}\Delta^{(n)}}{\Delta_{jj}^{(n)}\Delta_{kk}^{(n)}}\,; \qquad (j \neq k). \tag{37}$$

For our purposes, a more striking and important way of expressing $\gamma_{jk}^{(n)}$ is arrived at by using (36) and (32) in (37) to obtain:

$$\gamma_{jk}^{(n)} = -\pi_{jk}^{(n)}\sigma_{jn}\sigma_{kn}\,; \qquad (j \neq k). \tag{38}$$

Identity (38) shows precisely how $\gamma_{jk}^{(n)}$ is related to $\pi_{jk}^{(n)}$. This identity affords an explanation for the paradox of possibly correlated alien parts, which we shall state here as a theorem.

THEOREM 3. *If* R_n *is nonsingular, and if* $\rho_{e_je_k}^{(n)}$ *is the correlation between partial anti-images* $e_j^{(n)}$ *and* $e_k^{(n)}$, *then this anti-image intercorrelation is equal to the negative of the corresponding partial-correlation*:

$$\rho_{e_je_k}^{(n)} = -\pi_{jk}^{(n)}\,; \qquad (j \neq k).$$

The covariance $\gamma_{jk}^{(n)}$ *vanishes if and only if* $\pi_{jk}^{(n)}$ *vanishes* $(j \neq k)$.

This theorem follows directly from (38), by recognizing that $\rho_{e_je_k}^{(n)} = \gamma_{jk}^{(n)}/\sigma_{jn}\sigma_{kn}$ from the usual product-moment formula for a correlation coefficient.

That $\rho_{e_je_k}^{(n)}$ should be equal and opposite in sign to $\pi_{jk}^{(n)}$ is "obvious" from the geometric picture involved (cf. 11). $\pi_{jk}^{(n)}$ is the cosine of the angle, say θ, between two hyperplanes, while $\rho_{e_je_k}^{(n)}$ is the cosine of the angle between perpendiculars to these two hyperplanes, or of an angle equal to $180° - \theta$. Theorem 3 thus boils down to be a special version of the trigonometric identity that $\cos(180° - \theta) = -\cos\theta$.

According to Theorem 3, after we have subtracted out the common part—the partial image—from x_j, the alien part that remains behaves toward x_k almost as if x_k were *not* in the regression for predicting x_j. Subtracting out the common-parts of the n variables still leaves room for *pairwise linkages* to remain between them of the kind described by their partial correlations.

We can now interpret our fundamental identity (25) by rewriting it, using (38), as

$$r_{jk} = g_{jk}^{(n)} + \pi_{jk}^{(n)}\sigma_{jn}\sigma_{kn}\,; \qquad (j \neq k). \tag{39}$$

According to (39), an observed total correlation r_{jk} can be regarded as arising from two sources: (a) the covariance between the common parts of the two variables, and (b) a special pairwise linkage that may remain between the two variables after the remaining $n - 2$ variables are partialed out.

9. *Comparison with Common-Factor Theory*

The possible pairwise linkages in identity (39) are of profound importance for structural analysis. Different patterns of these linkages give rise to different kinds of order-factor theories. For the theory of mental activity, these linkages make possible some hypotheses as to the physiological workings of the nervous system (**7**).

The Spearman-Thurstone common-factor theory is a special—indeed, degenerate—type which specifies *zero* pairwise linkages. [More generally, it is an orderless theory, which is one reason why the problem of rotation of axes arises (**7**).]

In common-factor theory, it is hypothesized that each x_{ji} can be expressed as the sum of a common part, say c_{ji}, and a unique part, say u_{ji},

$$x_{ji} = c_{ji} + u_{ji} , \tag{40}$$

where the rank of c_{ji} is of basic importance. If the rank is m, then there are m common factors—expressible with unit variances—say y_f $(f = 1, 2, \cdots, m)$, such that

$$c_{ji} = \sum_{f=1}^{m} a_{jf} y_{fi} , \tag{41}$$

where a_{jf} are weights for the common factors. It is assumed that the unique parts are orthogonal to the common parts and are also orthogonal to each other,

$$\underset{i}{E}\, u_{ji} c_{ki} = 0, \tag{42}$$

and

$$\underset{i}{E}\, u_{ji} u_{ki} = 0; \qquad (j \neq k). \tag{43}$$

Hypothesis (42) holds for $j = k$ as well as $j \neq k$; for $j = k$ it is analogous to identity (17). If (42) and (41) hold, it easily follows that each u_j is orthogonal to each common factor y_f separately, which is the more traditional way of presenting the hypotheses of common-factor theory. We are not concerned here with the y_f separately, however, and do not hypothesize anything special about them as to whether they are orthogonal to each other or not, nor how they are to be located within the common-factor space. It is the *common-factor space as a whole* that is of present concern, and this is represented by the c_j . The c_j are invariant under any nonsingular transformation of the y_f .

In particular, the variance $\sigma_{c_j}^2$ of c_j is an invariant of the common-factor space, and is called the *communality* of x_j (**12**). Similarly $\sigma_{u_j}^2$, the variance of u_j , is an invariant and is called the *uniqueness* of x_j . Furthermore, the total variance of x_j , taken as unity, is the sum of the variances of its common and unique parts:

$$\sigma_{c_j}^2 + \sigma_{u_j}^2 = 1. \tag{44}$$

Equation (44) parallels identity (19), with the communality playing the role of the partial norm and the uniqueness the role of the partial antinorm.

A further parallel to an identity of image theory is the equation

$$\underset{i}{E}\, u_{ji} x_{ki} = 0; \qquad (j \neq k). \tag{45}$$

That (45) follows from the common-factor hypotheses can be seen by multiplying both members of (40) by u_{ki}, taking expectations over i, using (42) and (43), and revising subscripts. Equation (45) is parallel to the fundamental equation (15) of image analysis. Indeed, (45) can be used as a starting point for the common-factor hypotheses in place of (42). Equation (42) can be derived directly from (45) and (44) for $j \neq k$; and if $m < n$, it can also be seen that (42) holds for $j = k$. Starting with (45) in place of (42) will be a convenient way for us to compare common-factor theory with image theory.

The way common-factor theory "explains" the observed intercorrelations r_{jk} is by means of its fundamental factor equation,

$$r_{jk} = \underset{i}{E}\, c_{ji}c_{ki}\,, \qquad (j \neq k), \tag{46}$$

which follows from (40), (42), and (43). Common-factor analysts traditionally expand the right member of (46) in terms of some set of y_r, using (41), but this is irrelevant to the present discussion.

We can now summarize and compare the bases and consequences of image theory and of common-factor theory as in Table 1.

TABLE 1

Comparison of Characteristics of Image Theory and Common-Factor Theory

		Image Theory	Common-Factor Theory
Basic Partition		$x_{ji} = p_{ji}^{(n)} + e_{ji}^{(n)}$	$x_{ji} = c_{ji} + u_{ji}$
Basic Definition		$p_{ji}^{(n)} = \sum_{k=1}^{n} w_{jk}^{(n)} x_{ki}$	. . .
Basic Restrictions	(a):	$\underset{i}{E}\, e_{ji}^{(n)} x_{ki} = 0;\quad (j \neq k)$	$\underset{i}{E}\, u_{ji}x_{ki} = 0;\quad (j \neq k)$
	(b):	. . .	$\underset{i}{E}\, u_{ji}u_{ki} = 0;\quad (j \neq k)$
Consequences	(a):	. . .	$\underset{i}{E}\, u_{ji}c_{ki} = 0;\quad (j \neq k)$
	(b):	$\underset{i}{E}\, e_{ji}^{(n)} p_{ji}^{(n)} = 0$	$\underset{i}{E}\, u_{ji}c_{ji} = 0;\quad (m < n)$
	(c):	$\rho_{jn}^2 + \sigma_{jn}^2 = 1$	$\sigma_{c_j}^2 + \sigma_{u_j}^2 = 1$
	(d):	$r_{jk} = g_{jk}^{(n)} - \gamma_{jk}^{(n)};\quad (j \neq k)$	$r_{jk} = \underset{i}{E}\, c_{ij}c_{ik}\,;\quad (j \neq k)$
	(e):	$r_{jk} = g_{jk}^{(n)} + \pi_{jk}^{(n)}\sigma_{jn}\sigma_{kn}$ $(j \neq k)$	. . .

10. *The Special Case of Determininate Common Factors*

According to Table 1, common-factor theory *lacks a basic definition* for its common parts c_{ji}, and has two restrictions on the deviant parts compared to only one restriction for image theory. The single *restriction* of image theory can always be satisfied, making the *basic definition* unique, so that the *consequences* are all identities or tautologies—they are universally true. In contrast, the restrictions of common-factor theory do not generally suffice to define any particular partition of the observations; more than one common-factor partition can be found in general—with different u_{ji} and c_{ji}—to satisfy the restrictions. For example, if a set of c_{ji} of rank m can be found to satisfy the restrictions on the u_{ji}, then certainly a set of rank $m + 1$ can also be found, yielding new u_{ji} which also satisfy the restrictions. Or more than one set of c_{ji} can usually be found with the same rank m. Two different sets satisfying the restrictions *cannot* be obtained from each other by rotations within one of their common-factor spaces, for any set c_{ji} is invariant under rotations.

This highlights one of the basic problems of indeterminacy of common-factor theory. More than one total common-factor space can satisfy the same data. (To repeat, this problem of indeterminacy is entirely different from that of rotation of axes, which takes place *within* a given common-factor space.)

This indeterminacy can be removed by introducing the notation of a *determinate* common-factor space. Such a space is one in which there is a perfect regression for each common factor y_f on the observed x_j. For finite n, a common-factor space is in general indeterminate; the common-factor scores can only be *estimated* from the x_j, with positive variances of errors of estimate. As n increases, the errors of estimate decrease; if the limit of the errors of estimate is zero, then the common factors are perfectly determinate in the limit. General conditions under which a common-factor space is determinate have been established elsewhere (2); essentially they are that there exist a limit to R_n^{-1}, which is a diagonal matrix.

Instead of dealing with separate common-factors y_f here, let us define the determinateness of a total common-factor space of rank m as follows. Let $b_{jk}^{(n)}$ be the regression coefficient of x_k for predicting c_j in the multiple regression of c_j on n observed variables. Then the common-factor space of the c_j will be called determinate if and only if, for all j*,

$$c_{ji} = \lim_{n \to \infty} \sum_{k=1}^{n} b_{jk}^{(n)} x_{ki} \,. \tag{47}$$

Condition (47) can be seen to be parallel to our definition of total images in (12). If we now fill in condition (47) as a *basic definition* in the table of the previous section, it easily follows that only basic restriction (a) in the

*Except possibly for a zero proportion of the population.

table is now needed to prove that*

$$c_{ji} = p_{ji}^{(\infty)}, \tag{48}$$

or *the common parts of images and of common-factor theory are identical in the limit**. For the proof of (48), consider the quantity δ_{jn}^2 defined by

$$\delta_{jn}^2 = \underset{i}{E}\, [c_{ji} - p_{ji}^{(n)}]^2 = \underset{i}{E}\, [u_{ji} - e_{ji}^{(n)}]^2. \tag{49}$$

That the last member equals the middle member follows directly from the basic partitions. Expanding the last member shows that

$$\delta_{jn}^2 = \sigma_{u_j}^2 - 2\, \underset{i}{E}\, u_{ji} e_{ji}^{(n)} + \sigma_{jn}^2\,. \tag{50}$$

To evaluate the middle term on the right, multiply both members of (13) by u_{ji} and take expectations over i, remembering (11), (10), and (45), whence

$$\underset{i}{E}\, u_{ji} e_{ji}^{(n)} = \underset{i}{E}\, u_{ji} x_{ji} = \sigma_{u_j}^2\,. \tag{51}$$

That the last member of (51) is equal to the middle member is well-known and can be seen by multiplying (40) through by u_{ji}, taking expectations, and using (42). Using (51) in (50) shows that

$$\delta_{jn}^2 = \sigma_{jn}^2 - \sigma_{u_j}^2 = \sigma_{c_j}^2 - \rho_{jn}^2\,, \tag{52}$$

the last member following from the middle member by virtue of consequences (19) and (44).

Since the first member of (52) is nonnegative, the last member provides us with a new proof of a previously known theorem that *a communality of* x_j *is always an upper bound to the square of the multiple correlation coefficient or partial norm of* x_j (2, 92-93.) For a given set of n variables, more than one set of communalities can exist, but in all cases these communalities cannot be less than the corresponding—uniquely defined—partial norms. The closer a communality of x_j to the partial norm of x_j, then—according to (49) and (52)—the closer the c_{ji} to the $p_{ji}^{(n)}$, and the closer the u_{ji} to the $e_{ji}^{(n)}$.

Now, the c_{ji} and u_{ji}—hence also $\sigma_{c_j}^2$ and $\sigma_{u_j}^2$—do not depend on n. It is assumed that there is a fixed number m of common factors in the entire universe of content, and these will appear in any sample of n variables where $n > m$. In contrast, the image partition changes with n, with the partial norms always increasing (or at worst remaining constant). Our Theorem 1 above states that limiting or total norms always exist. We can now state further what the limiting values are if (47) holds; for it has been shown in (2) that if (47) holds, then also

$$\sigma_{c_j}^2 = \rho_{j\infty}^2\,, \tag{53}$$

*Except possibly for a zero proportion of the population.

or any communality equals the total norm. Taking the limit in (52) as $n \rightarrow \infty$ and using (53) shows that

$$\lim_{n \rightarrow \infty} \delta_{jn}^2 = 0, \tag{54}$$

which then establishes (48). These results can be stated as:

THEOREM 4: *If a common-factor space of rank* m *is determinate for an infinitely large universe of content, then there is no other determinate common-factor space possible for the same universe—whether of rank* m *or of any other rank. The communalities are uniquely determined and are equal to the corresponding total norms. The common-factor scores are the total image scores, and the unique factor scores are the total anti-image scores.*

We have now completed the demonstration of how common-factor theory is a special case of image theory. For a common-factor theory to be useful, it should be determinate; otherwise there is no uniquely defined common-factor space; and furthermore, common-factor scores cannot be estimated closely for use in practice. It the theory is determinate, it becomes a special case of image theory, with the special restriction that total anti-images are uncorrelated.

11. *A Universal Computing Procedure*

The fundamental identity (25) provides a computing procedure to analyze the structure of the interrelationships of n variables from any universe of content. Let Γ_n be the Gramian matrix of the covariances $\gamma_{jk}^{(n)}$, and let S_n^2 be the diagonal matrix defined by the partial antinorms. These matrices can both be computed easily by first computing R_n^{-1}, according to (5) and (36). Then, considering also (37), we have the working formula,

$$\Gamma_n = S_n^2 R_n^{-1} S_n^2 , \tag{55}$$

or Γ_n is computed from R_n^{-1} merely by premultiplication and postmultiplication with the diagonal matrix S_n^2 .

Once Γ_n is computed, it is easy to compute G_n , the Gramian matrix of the covariances $g_{jk}^{(n)}$; for by referring to (25), we can write

$$G_n = R_n + \Gamma_n - 2S_n^2 . \tag{56}$$

Only matric addition is needed to compute G_n according to (56). The diagonal matrix $2S_n^2$ has been subtracted in the right of (56) to make the main diagonals consistent; (25) does not define the main diagonals, which have to be considered separately.

In the special case of determinate common factors, the nondiagonal elements of Γ_n should all be close to zero, for their limit as $n \rightarrow \infty$ is hypothesized to be zero.

If not all nondiagonal elements are close to zero, one is led to reject the hypothesis that a determinate common-factor space of rank less than n holds for the universe. One could then examine Γ_n to see if any special order exists among the nonvanishing pairwise linkages. It is best, of course, to have a preliminary order theory *before* one examines the empirical evidence. Examples of preliminary hypotheses are the simplex and circumplex (7). Ultimately, special theories may have to be developed for each special kind of content.

An important paradox is that the anti-image covariances are more basic to the structural analysis of R_n than are the image covariances. If we know Γ_n, we know R_n, for from (55) we have immediately:

$$R_n = S_n^2 \Gamma_n^{-1} S_n^2 , \tag{57}$$

This is not the case with G_n; knowing G_n is not sufficient for determining R_n. In this sense, R_n is determined by the *alien* parts, rather than by the common parts.

Another way of stating this paradox is to express G_n itself as a function of Γ_n. From (56) and (57), we have

$$G_n = S_n^2 \Gamma_n^{-1} S_n^2 + \Gamma_n - 2S_n^2 . \tag{58}$$

Given Γ_n, we can compute G_n through (58). The converse is not true; G_n by itself does not determine Γ_n. In the general case, then, structural theories should be based on the anti-images, rather than on the images.

In the later paper, we shall discuss the general theory of the matrices of linear least-squares image analysis and show some further intimate connections between Γ_n and G_n as well as with other matrices that occur naturally in the theory (8).

If a determinate common-factor theory holds, so that Γ_n tends to a diagonal matrix (namely S_∞^2), then we have only G_n to deal with and no initial clues as to its structure. An order-theory may still hold within G_n regardless. If not, one is up against the problem of rotation of axes that is traditional to common-factor theory. But at least the indeterminacies of the communalities and of the common-factor space have been removed.

Empirically, it has often been found that multiple correlations tend to taper off rapidly as the number of predictors increases. It may often be expected in practice, then, that with n greater than 10 or 15, say, partial images and norms will be so close to their total images and norms that the differences will be negligible, and n can be regarded as virtually "infinite." Some order structures like the circumplex may require a larger number of tests than others in order to piece out the necessary details.

On the other hand, it is well-known that the sampling errors associated with multiple regressions can be quite enormous if the sample of people used is small. Only large samples of people can be reliably used if n is substantial, and this is as it ought to be. Any structural analysis of a universe requires a large sample of people. Small samples may be adequate for rejecting null hypotheses of zero relationships, but they are not adequate for estimating the details of involved nonzero relationships. In the future, the required sampling theory will undoubtedly be forthcoming which will indicate whether 500, 3,000, or some larger sample of people is needed in order adequately to study the structure of a given universe of content on the basis of a sample of n variables.

REFERENCES

1. Guttman, Louis. A note on the derivation of formulae for multiple and partial correlation. *Ann. math. Statist.*, 1938, 9, 305-308.
2. Guttman, Louis. Multiple rectilinear prediction and the resolution into components. *Psychometrika*, 1940, 5, 75-99.
3. Guttman, Louis, and Cohen, Jozef. Multiple rectilinear prediction and the resolution into components: II. *Psychometrika*, 1943, 8, 169-183.
4. Guttman, Louis. Review of Thurstone's *Multiple-factor analysis*. J. Amer. statist. Ass., 1947, 42, 651-656.
5. Guttman, Louis. The Israel alpha technique for scale analysis: a preliminary statement (stenciled). The Israel Institute of Applied Social Research, Jerusalem, Israel, 1951.
6. Guttman, Louis. The theory of nodular structures. (In preparation)
7. Guttman, Louis. A new approach to factor analysis: The Radex. In Paul F. Lazarsfeld, Mathematical thinking in the social sciences. New York: Columbia Univ. Press, 1953.
8. Guttman, Louis. The matrices of least-squares image analysis. (In preparation)
9. Guttman, Louis. The existence of total least-squares images and anti-images (In preparation)
10. Guttman, Louis. A reanalysis of factor analysis. (In preparation)
11. Jackson, Dunham. The trigonometry of correlation. *Amer. math. Monthly*, 1924, 31, 275-280.
12. Thurstone, L. L. Multiple-factor analysis. Chicago: Univ. Chicago Press, 1947.