Quantum teleportation: an operator algebraic perspective

by

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To the memory of my father, who fostered my love of science and mathematics. To my friends and family, who supported and believed in me throughout this demanding academic endeavour.

Abstract

We show that quantum teleportation is intimately connected to Jones' basic construction. This connection leads to generalized, or hybrid teleportation protocols to simultaneously transmit classical and quantum information through finite dimensional von Neumann algebras. We present two operationally concrete teleportation protocols, *Hybrid teleportation* and our so-called *Scaffolding teleportation.* Under some mild symmetry assumptions, we establish that any teleportation scheme in the scaffolding framework is equivalent to a teleportation scheme of a special form.

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Contents

Chapter 1

Introduction

In 1993, Bennett, et al. published their seminal paper *Teleporting an Unknown Quantum State via Dual Classical and Einstein-Podolsky-Rosen Channels* [4], in which they introduced a protocol capable of transferring quantum information between spatially separated parties using only quantum operations local to each party and classical communication, hence the term *quantum teleportation*. At the time of writing this, the storage and transfer of quantum information is delicate and unstable making it prone to errors induced by its environment. This means that quantum communication (the sending and receiving of quantum states over a quantum network) over any appreciable distance is impossible without robust error-correcting methods. Since in quantum teleportation the transfer of quantum information is done in the absence of a quantum network, it promises the feasibility of quantum communication over large distances without the need for quantum error-correcting. The caveat is that the quantum teleportation protocol requires a shared entangled quantum state between the two parties involved, which means sending and receiving of quantum states must still occur in order for quantum teleportation to work. The imperfect nature of quantum networks means that the "strength" of entanglement between particles is difficult to maintain over the length of the network. However, there exist entanglement distillation protocols [26, §12.5.3] capable of transforming a set of weakly entangled particles into a smaller set of strongly entangled particles, establishing the resource required for quantum teleportation. So, quantum teleporation is a viable method for reliably transferring quantum information and has been proven experimentally [31] [30].

The Schrödinger and Heisenberg representations of quantum mechanics [27] offer different interpretations of the quantum teleportation protocol. The protocol presented by Bennett, et al. in [4] is given in the Schrödinger picture, where teleportation happens at the level of states. In the Heisenberg picture, however, the interpretation is that teleportation happens at the level of operators. There is a natural reformulation of quantum teleportation in the Heisenberg picture into an operator algebraic perspective that we present in §4.5.

Quantum teleportation of finite dimensional systems identified by the n-by-n matrices $M_n(\mathbb{C})$ was generalized by R. F. Werner in [34]. Rather than teleporting full systems, one can imagine the desire to teleport subsystems, which are identified by subalgebras of $M_n(\mathbb{C})$. This would arise in the context of subsystem error-correcting codes such as in [6], or hybrid codes used for the simultaneous transmission of classical and quantum information [16]. Specifically, codes in the finite dimensional setting are identified by finite dimensional von Neumann algebras that up to unitary equivalence can be decomposed as $N = \oplus_{i=1}^{d} 1_{m_i} \otimes M_{n_i}(\mathbb{C})$ [24]. Note that N is contained inside the full matrix algebra $M_n(\mathbb{C})$ where $n = \sum_{i=1}^{d} m_i n_i$. Then, we could of course employ a teleportation protocol for $M_n(\mathbb{C})$ and simply restrict the input to the desired subalgebra N, however this is an inefficient use of the entanglement resources since we are only interested in teleporting a subsystem.

In this thesis, we present two subsystem teleportation protocols in an operator algebraic framework that require minimal entanglement resources. The first, given in Theorem 4.6.4, *Hybrid quantum teleportation*, is capable of teleporting any subsystem using minimal entanglement resources. A draw-back to this protocol is that it has an inconvenient physical implementation as the sender and receiver would require custom quantum systems for each subsystem they would like to teleport. Our second protocol, given in Theorem 5.0.5, *Scaffolding protocol*, solves the physical implementation issue encountered by our hybrid quantum teleportation scheme by first fixing a quantum system shared by the sender and receiver, then defining a teleportation scheme with respect to the subsystem of interest. Specifically, we show that if a subsystem satisfies a certain normalizer condition, then there exists a scaffolding teleportation protocol for that subsystem. We show that there exists a class of subsystems in Proposition 5.0.9 that fail the normalizer condition.

In [34], R. F. Werner shows that if a quantum teleportation protocol for a system identified by a full matrix algebra, $M_n(\mathbb{C})$, satisfies a certain cardinality condition on the number of its observables, known as the *tightness* condition, then the teleportation scheme must be of a very special form. We give a partial generalization of Werner's work in our scaffolding framework presented in Theorem 5.1.4, *Tight shift covariant teleportation*, where we show that under some mild symmetry assumptions on the subsystem that any scaffolding teleportation scheme that satisfies our shift covariant condition is equivalent to a scaffolding teleportation scheme of a very special form. It is worth noting that the symmetry assumptions are consistent with hybrid codes considered in [16].

Thesis structure

In Chapter 2, we establish definitions of various mathematical objects and introduce notation.

In Chapter 3, we discuss von Neumann algebra inclusions. We introduce constructions upon these inclusion known as the Gelfand-Naimark-Segal (GNS) construction and Jones' basic construction, and present objects given by these two constructions.

This chapter's primary references are [15] [20] [17] [2].

In Chapter 4, we begin with introducing objects and structures relevant to the study of quantum information. We then present the quantum teleportation protocol given in [4] and proceed to reformulate it into a von Neumann algebraic setting. Finally, we present our hybrid teleportation protocol.

In Chapter 5, we define a fixed quantum system and present our scaffolding teleportation protocol. We show that a certain class of von Neumann algebra inclusions fail to satisfy the conditions required for a scaffolding teleportation protocol. We then define the tightness conditions, inspired by [34], and shift covariant teleportation. Finally, we present our tight shift covariant teleportation protocol.

Chapter 6 concludes the thesis and discusses direction of future work.

Chapter 2

Mathematical preliminaries and notation

Let $\mathcal{H}$ be a finite-dimensional Hilbert space, that is, a finite-dimensional vector-space over $\mathbb{C}$ with a sesquilinear inner product, let $\mathcal{L}(\mathcal{H})$ denote the algebra of bounded linear operators on $\mathcal{H}$, and let $\mathcal{M}_n$ denote the n-by-n matrices over $\mathbb{C}$. Every Hilbert space mentioned throughout this paper should be taken to be of finite dimension whether mentioned or not.

Tensor products

We introduce the tensor product for vectors and operators in this section. The tensor product is key to the definition of entanglement.

Definition 2.0.1. The tensor product between $\psi \in \mathcal{H}_1$ and $\phi \in \mathcal{H}_2$, is denoted by $\psi \otimes \phi$ and satisfies

- $\psi \otimes (\phi + \phi') = (\psi \otimes \phi) + (\psi \otimes \phi')$, for all $\psi \in \mathcal{H}_1$ and $\phi, \phi' \in \mathcal{H}_2$,
- $(\psi + \psi') \otimes \phi = (\psi \otimes \phi) + (\psi' + \phi)$, for all $\psi, \psi' \in \mathcal{H}_1$ and $\phi \in \mathcal{H}_2$,
- $c(\psi \otimes \phi) = c\psi \otimes \phi = \psi \otimes c\phi$, for all $c \in \mathbb{C}$

The tensor product $\psi \otimes \phi$ for $\psi \in \mathcal{H}_1$ and $\phi \in \mathcal{H}_2$ is called a *simple tensor*. The span of all simple tensors between elements of $\mathcal{H}_1$ and $\mathcal{H}_2$ forms a new Hilbert space, denoted $\mathcal{H}_1 \otimes \mathcal{H}_2$, with inner product on simple tensors defined by

$$\langle \psi \otimes \phi,\ \psi' \otimes \phi' \rangle_{\mathcal{H}_1 \otimes \mathcal{H}_2} \equiv \langle \psi, \psi' \rangle_{\mathcal{H}_1} \cdot \langle \phi, \phi' \rangle_{\mathcal{H}_2}$$

To gain a feel for the mechanics of the tensor product we introduce the *Kronecker product.* It offers an explicit representation of the tensor product.

Definition 2.0.2. Let ψ be an element of the n-dimensional Hilbert space $\mathbb{C}^n$ and ϕ be an element of an m-dimensional Hilbert space $\mathbb{C}^m$. Then the *Kronecker product*, which represents the tensor product, is defined by

$$\psi \otimes \phi = \begin{pmatrix} \psi_1 \\ \vdots \\ \psi_n \end{pmatrix} \otimes \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_m \end{pmatrix} = \begin{pmatrix} \psi_1 \cdot \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_m \end{pmatrix} \\ \vdots \\ \psi_n \cdot \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_m \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \psi_1 \cdot \phi_1 \\ \vdots \\ \psi_1 \cdot \phi_m \\ \vdots \\ \vdots \\ \psi_n \cdot \phi_1 \\ \vdots \\ \psi_n \cdot \phi_m \end{pmatrix} \in \mathbb{C}^{nm}.$$

Remark 2.0.3. The following are some practical facts about tensor products:

Let $\mathcal{H}_1, \mathcal{H}_2$ be n- and m-dimensional Hilbert spaces, respectively. Let $\{e_i\}_{i=1}^n$ and $\{f_i\}_{i=1}^m$ be orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. Then

- $\{e_i \otimes f_j\}_{i,j=1}^{n,m}$ is an orthonormal basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$;

- $\dim(\mathcal{H}_1 \otimes \mathcal{H}_2) = \dim(\mathcal{H}_1) \cdot \dim(\mathcal{H}_2)$;

- $\mathcal{H}_1 \otimes \mathcal{H}_2 \equiv \left\{ \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} e_i \otimes f_j \mid a_{ij} \in \mathbb{C} \right\}$.

Definition 2.0.4. Let $A : \mathcal{H}_1 \to \mathcal{H}_1$ and $B : \mathcal{H}_2 \to \mathcal{H}_2$ be linear operators. The tensor product of A and B, denoted $A \otimes B$, is the unique linear operator $A \otimes B : \mathcal{H}_1 \otimes \mathcal{H}_2 \to \mathcal{H}_1 \otimes \mathcal{H}_2$ satisfying

$$(A \otimes B)(\psi \otimes \phi) \equiv A\psi \otimes B\phi \quad \psi \in \mathcal{H}_1, \phi \in \mathcal{H}_2.$$

Definition 2.0.5. We denote by A^* the *adjoint* of $A \subseteq \mathcal{L}(\mathcal{H})$, defined by

$$\langle A\xi, \eta \rangle = \langle \xi, A^* \eta \rangle \quad \xi, \eta \in \mathcal{H}.$$

Definition 2.0.6. An operator A in $\mathcal{L}(\mathcal{H})$ is *self-adjoint* if $\langle A\xi, \eta \rangle = \langle \xi, A\eta \rangle$ for all $\xi, \eta \in \mathcal{H}$, i.e. $A = A^*$.

Remark 2.0.7. Any element x of a von Neumann algebra M can be written as a sum of self-adjoint operators in M ([17]):

$$x = \frac{x + x^*}{2} + i\frac{x - x^*}{2}.$$

Note that if A, B are self-adjoint operators, then $A \otimes B$ is a self-adjoint operator.

We denote by $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ the set of linear operators $T : \mathcal{H}_1 \to \mathcal{H}_2$. The notation is compressed to $\mathcal{L}(\mathcal{H}_1)$ for the set of linear operators $T : \mathcal{H}_1 \to \mathcal{H}_1$. Tensor products of linear operators hold similar properties to those of tensor products of vectors shown in Definition 2.0.1 (just swap vectors for linear operators), with the added property $(A \otimes B)(A' \otimes B') = AA' \otimes BB'$. It follows that, the span of tensor products of elements in $\mathcal{L}(\mathcal{H}_1)$ and $\mathcal{L}(\mathcal{H}_2)$, denoted $\mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_2)$, forms the set of linear operators on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, i.e. $\mathcal{L}(\mathcal{H}_1) \otimes \mathcal{L}(\mathcal{H}_2) = \mathcal{L}(\mathcal{H}_1 \otimes \mathcal{H}_2)$.

Definition 2.0.8. The set of eigenvalues $\{\lambda_1, \lambda_2, ..., \lambda_n\}$ of an operator $A \in \mathcal{L}(H)$ is the *spectrum* of A.

Definition 2.0.9. An operator $A \in \mathcal{L}(\mathcal{H})$ is *normal* if it commutes with its adjoint, $A^* A = AA^*$.

Given a normal matrix A and a function $f : \sigma(A) \to \mathbb{C}$, where $\sigma(A)$ denotes the spectrum of A, it is possible to make sense of $f(A)$. Let $A = \sum_\lambda \lambda P_\lambda$ be the spectral decomposition of the normal matrix A. Then define

$$f(A) \equiv \sum_\lambda f(\lambda) P_\lambda.$$

We observe that when f is a polynomial,

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n,$$

then

$$f(A) = a_0 + a_1 A + \cdots + a_n A^n.$$

We can use this idea to define the square root of a positive operator. Suppose we have a positive operator E with spectral decomposition $E = \sum_\lambda \lambda P_\lambda$, then we may define

$$\sqrt{E} = \sum_\lambda \sqrt{\lambda} P_\lambda,$$

as each eigenvalue $\lambda \geq 0$.

For example, suppose we have an operator A where

$$A = \operatorname{diag}(\lambda_1, ..., \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix},$$

with $\lambda_i \geq 0$, then

$$\sqrt{A} = \operatorname{diag}(\sqrt{\lambda_1}, ..., \sqrt{\lambda_n}) = \begin{pmatrix} \sqrt{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & \sqrt{\lambda_n} \end{pmatrix}.$$

Note that an operator A is positive if and only if there exists B such that $A = B^* B$. Then $A^* = \left(B^* B\right)^* = B^* B = A$, thus positive implies self-adjoint. In this case, one may take $B = \sqrt{A}$, since

$$B^* B = \sqrt{A}^* \sqrt{A} = \sqrt{A}\sqrt{A} = A.$$

Proposition 2.0.10. *Let $A \in \mathcal{L}(H)$, then there exists a unitary $U \in \mathcal{L}(H)$ such that*

$$A = U\sqrt{A^* A}, \tag{2.1}$$

[18].

Definition 2.0.11. Equation (2.1) is known as the *polar decomposition* of A.

Dirac's "bra-ket" notation

In later portions of this paper, we will make use of Dirac's "bra-ket" notation when convenient. This notation takes advantage of the fact that for any $T \in \mathcal{L}(\mathcal{H}, \mathbb{C})$ there exists a unique $\psi \in \mathcal{H}$ such that $T(\eta) \equiv \langle \psi,\, \eta \rangle_{\mathcal{H}}$. Dirac uses the shorthand notation $T = \langle \psi |$, where the $\langle \cdot |$ is referred to as a *bra*. Dirac notation expresses the column vector $\psi \in \mathcal{H}$ as $|\psi\rangle$, where the $|\cdot\rangle$ is known as a *ket*. Then,

$$T(\phi) = \langle \psi || \phi \rangle = \langle \psi | \phi \rangle \equiv \langle \psi, \phi \rangle_{\mathcal{H}}.$$

So, $|\psi\rangle$ is the column vector $\psi \in \mathcal{H}$, and $\langle \psi |$ represents the conjugate row vector $|\psi\rangle^*$.

Chapter 3

Von Neumann algebra inclusions

In this chapter, we introduce the mathematical structure known as a von Neumann algebra and its properties. This chapter is designed to introduce important objects related to von Neumann algebras for the study of quantum teleportation in the von Neumann algebraic setting. Anytime we write "algebra" without specificity as to what sort of algebra we are referring to, should be taken to mean von Neumann algebra.

We begin with defining a von Neumann algebra, discuss some of its underlying structures, and introduce the notion inclusions; an algebra/subalgebra pair, $N \subseteq M$. We then give the famous fundamental theorem about von Neumann algebras called the bicommutant theorem (or sometimes the double commutant theorem), in §3.1, Theorem 3.1.5. The inclusion matrix for an inclusion is introduced next in §3.2 and we discuss traces in §3.3. In §3.4, we show that from an inclusion $N \subseteq M$ we can construct a pair of Hilbert spaces $\mathcal{H}_N \subseteq \mathcal{H}_M$, known as the GNS construction. This leads to the notion of an orthogonal projection from $\mathcal{H}_M$ onto $\mathcal{H}_N$ known as the Jones projection, which we present in §3.5. The Jones projection is then shown to induce a map from M onto N that is a completely positive, trace preserving bimodule map with respect to the subalgebra N, known as the conditional expectation. Next, in §3.6 we present Jones' basic construction which allows us to produce an algebra M_1 from an inclusion $N \subseteq M$ such that $N \subseteq M \subseteq M_1$. In §3.8 we introduce the notion of a Pimsner-Popa basis. We then show in §3.9 that there exists a class of inclusions $N \subseteq M$ on which one may iterate Jones' basic construction to obtain a tower of inclusions, $N \subseteq M \subseteq M_1 \subseteq M_2$, and we introduce an important isomorphism, known as the canonical shift, between certain relative commutants of such a tower, which is key to our presentation of quantum teleportation in the operator algebraic setting. Finally, we end the chapter by giving examples of each of the constructions and objects presented throughout the chapter for the inclusions $\mathbb{C}1_n \subseteq M_n(\mathbb{C})$ in §3.10 and $D_n \subseteq M_n(\mathbb{C})$ in §3.11.

Definition 3.0.12. A *von Neumann algebra on* $\mathcal{H}$ is a subalgebra $M \subseteq \mathcal{L}(\mathcal{H})$, with $1_{\mathcal{L}(\mathcal{H})} \in M$, that is closed under the adjoint operation.

Some examples of finite-dimensional von Neumann algebras include the d-by-d matrices, $\mathcal{M}_d$, on a d-dimensional Hilbert space, and the n-dimensional diagonal matrices, $\mathcal{D}_n$, on a n-dimensional Hilbert space.

For the following definitions, let M be a von Neumann algebra in $\mathcal{L}(\mathcal{H})$.

Definition 3.0.13. The *commutant of* M, denoted by M', is the set of elements in $\mathcal{L}(\mathcal{H})$ which commute with each element in M:

$$M' = \{x \in \mathcal{L}(\mathcal{H}) \mid xy = yx, \forall y \in M\}.$$

Definition 3.0.14. The *centre of* M, denoted by $Z(M)$, is defined as $M \cap M'$.

It is a straightforward series of calculations to check that both the commutant and the centre are themselves von Neumann algebras.

Definition 3.0.15. If M and K are von Neumann algebras, an *algebra homomorphism* is a map $\Phi : M \to K$ such that, for all $\mu \in \mathbb{C}$ and all $x, y \in M$:

- $\Phi(\mu x + y) = \mu\Phi(x) + \Phi(y)$
- $\Phi(xy) = \Phi(x)\Phi(y)$.

Definition 3.0.16. A $*$-*homomorphism* is an algebra homomorphism $\Phi : M \to K$, such that $\Phi(x)^* = \Phi(x^*)$, for all $x \in M$.

Definition 3.0.17. A $*$-*isomomorphism* is a bijective $*$-homomorphism. Two von Neumann algebras M and K are $*$-isomomorphic, denoted $M \cong K$, if and only if there exists a $*$-isomorphism $\Phi : M \to K$.

Definition 3.0.18. Given Hilbert spaces $\mathcal{H}_1, ..., \mathcal{H}_d$, the *Hilbert space direct sum* denoted by $\bigoplus_{i=1}^d \mathcal{H}_i$, is the usual vector space direct sum with inner product $\langle \xi, \eta \rangle = \sum_{i=1}^d \langle \xi_i, \eta_i \rangle$ and norm $\|\xi\|^2 = \sum_{i=1}^d \|\xi_i\|^2$. If $\mathcal{H}_i = \mathcal{H}$ for all i, we write $\mathcal{H}^d$.

Definition 3.0.19 (Operator norm)**.** The norm of $x \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is given by

$$\|x\| = \max\{\|x\xi\| \mid \xi \in \mathcal{H}_1 \text{ and } \|\xi\| \leq 1\}.$$

Definition 3.0.20. Given von Neumann algebras $M_1, ..., M_d$, the *von Neumann algebra direct sum* is the von Neumann algebra denoted by $\bigoplus_{i=1}^d M_i$, with norm $\|x\| = \sup_i \|x_i\|$, where $x_i \in M_i$, for all i.

Proposition 3.0.21. *Let $\mathcal{H}$ be a Hilbert space. Then for each $d \in \mathbb{N}$ there are canonical algebra isomorphisms*

$$\mathcal{L}(\mathcal{H}^d) \cong M_d(\mathcal{L}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H}) \otimes \mathcal{M}_d.$$

Proof. For $1 \leq i \leq d$ let $\phi_i : \mathcal{H}^d \to \mathcal{H}$ be the projection onto the i^{th} component, and for $1 \leq j \leq d$ let $\epsilon_j : \mathcal{H} \to \mathcal{H}^d$ be the injection such that

$$\phi_i \circ \epsilon_j = \delta_{ij}, \quad (\epsilon_i = \phi_i^T) \tag{3.1}$$

on $\mathcal{H}$, for all i. Given $x \in \mathcal{L}(\mathcal{H}^d)$, define a matrix $[x_{ij}]_{i,j=1}^d \in M_d(\mathcal{L}(\mathcal{H}))$ by $x_{ij} = \phi_i x \epsilon_j$. Then, for $x, y \in \mathcal{L}(\mathcal{H}^d)$, and $c \in \mathbb{C}$ we have

- $\phi_i(cx)\epsilon_j = c\phi_i x \epsilon_j$
- $\phi_i(x+y)\epsilon_j = \phi_i x \epsilon_j + \phi_i y \epsilon_j$
- $xy \mapsto [\phi_i(xy)\varepsilon_j]_{i,j=1}^d = [\sum_{t=1}^d x_{it}y_{tj}]_{i,j=1}^d = [x_{ij}]_{i,j=1}^d [y_{ij}]_{i,j=1}^d = (x \mapsto [\phi_i x \varepsilon_j]_{i,j=1}^d)(y \mapsto [\phi_i y \varepsilon_j]_{i,j=1}^d)$
- $(\phi_i x \epsilon_j)^* = \epsilon_j^* x^* \phi_i^* = \phi_i x^* \epsilon_i$ (by (3.1)),

so the map $x \mapsto [x_{ij}]$ is a $*$-homomorphism. It follows that $x = \sum_{i,j=1}^d \epsilon_i x_{ij} \phi_j$ is the inverse. This gives us the first isomorphism.

Let $\{e_{ij}\}_{ij}^d$ denote the standard basis in $\mathcal{M}_d$, and define the homomorphism

$$M_d(\mathcal{L}(\mathcal{H})) \ni [x_{ij}] \mapsto \sum_{i,j=1}^d x_{ij} \otimes e_{ij} \in \mathcal{L}(\mathcal{H}) \otimes \mathcal{M}_d.$$

To show that this is a homomorphism, consider $[x_{ij}], [y_{ij}] \in M_d(\mathcal{L}(\mathcal{H}))$. Then, $[x_{ij}] + [y_{ij}] = [x_{ij} + y_{ij}]$ so that

$$[x_{ij} + y_{ij}] \mapsto \sum_{i,j=1}^d (x_{ij} + y_{ij}) \otimes e_{ij} = \sum_{i,j=1}^d x_{ij} \otimes e_{ij} + \sum_{i,j=1}^d y_{ij} \otimes e_{ij}.$$

Multiplication of elements in $M_d(\mathcal{L}(\mathcal{H}))$ yields

$$[x_{ij}][y_{ij}] = [z_{ij} = \sum_{t=1}^d x_{it} y_{tj}] \mapsto \sum_{i,j=1}^d z_{ij} \otimes e_{ij}.$$

The multiplication of two image elements gives us

$$
\begin{aligned}
(\sum_{i,t=1}^{d} x_{it} \otimes e_{it})(\sum_{s,j=1}^{d} y_{sj} \otimes e_{sj}) &= \sum_{i,t=1}^{d} \sum_{s,j=1}^{d} x_{it} y_{sj} \otimes e_{it} e_{sj} \quad ; \quad e_{it}e_{sj} = \begin{cases} e_{ij}, & \text{if } t = s \\ 0, & \text{otherwise.} \end{cases} \\
&= \sum_{i,t=1}^{d} \sum_{s,j=1}^{d} x_{it} y_{sj} \otimes \delta_{ts} e_{ij} \\
&= \sum_{t=1}^{d} \sum_{i,j=1}^{d} x_{it} y_{tj} \otimes e_{ij} \\
&= \sum_{i,j=1}^{d} z_{ij} \otimes e_{ij}.
\end{aligned}
$$

as required. The map is clearly bijective, hence,

$$M_d(\mathcal{L}(\mathcal{H})) \cong \mathcal{L}(\mathcal{H}) \otimes \mathcal{M}_d.$$

□

For any subalgebra A of $\mathcal{L}(\mathcal{H})$, we have the following congruence $A \otimes \mathcal{M}_d \cong M_d(A)$ in $\mathcal{L}(\mathcal{H}^d)$, which follows from the proposition above. We identify $A \otimes 1_d$ with the set of block diagonal matrices where each diagonal block is the same element of A; an element of $A \otimes 1_d$ is identified by

$$\operatorname{diag}(a) = \begin{bmatrix} a & & & \\ & a & & \\ & & \ddots & \\ & & & a \end{bmatrix} \in M_d(A) \subset M_d(\mathcal{L}(\mathcal{H})).$$

3.1 The bicommutant theorem

The bicommutant theorem (or sometimes the double commutant theorem) states that the double commutant of a von Neumann algebra is equal to itself. This is a key feature of von Neumann algebras. In fact, in the literature a von Neumann algebra is sometimes (equivalently) defined as a C^*-algebra that is equal its own double commutant [20]. We begin with the factor setting.

Lemma 3.1.1. *Let $\mathcal{H}$ be a Hilbert space and let A be a von Neumann algebra on $\mathcal{H}$. Then,*

(a) the commutant of $A \otimes 1_d$ in $\mathcal{L}(\mathcal{H}^d)$ is $A' \otimes \mathcal{M}_d = M_d(A')$;

(b) the commutant of $A \otimes \mathcal{M}_d$ in $\mathcal{L}(\mathcal{H}^d)$ is $A' \otimes 1_d$.

Proof. (a) Since

$$(a \otimes 1_d)\sum_{i,j=1}^{d} x_{ij} \otimes e_{ij} = \sum_{i,j=1}^{d} a x_{ij} \otimes e_{ij}$$

and

$$(\sum_{i,j=1}^{d} x_{ij} \otimes e_{ij})(a \otimes 1_d) = \sum_{i,j=1}^{d} x_{ij} a \otimes e_{ij},$$

it follows that $\sum_{i,j=1}^{d} x_{ij} \otimes e_{ij}$ is in the commutant of $A \otimes 1_d$ if and only if $x_{ij} \in A'$, for all i, j.

(b) Suppose $x = \sum_{i,j=1}^{d} x_{ij} \otimes e_{ij}$ commutes with $A \otimes \mathcal{M}_d$, then in particular it commutes with matrix units $1_A \otimes e_{\mu\nu}$ for all μ, ν. Recalling that $e_{\mu\nu} e_{ij} = \delta_{\nu i} e_{\mu j}$ we have

$$\begin{aligned}\sum_{j=1}^{d} x_{\nu j} \otimes e_{\mu j} &= \sum_{i,j=1}^{d} x_{ij} \otimes e_{\mu\nu} e_{ij} \\ &= (1_d \otimes e_{\mu\nu})x \\ &= x(1_d \otimes e_{\mu\nu}) \\ &= \sum_{i,j=1}^{d} x_{ij} \otimes e_{ij} e_{\mu\nu} \\ &= \sum_{i=1}^{d} x_{i\mu} \otimes e_{i\nu},\end{aligned}$$

which implies that $x_{\nu j} = 0$ for $j \neq \nu$ and $x_{i\mu} = 0$ for $i \neq \mu$ and $x_{\nu\nu} = x_{\mu\mu}$ for all μ, ν. So $x = \sum_{i=1}^{d} y \otimes e_{ii} = y \otimes 1_d$, for some $y \in \mathcal{L}(\mathcal{H})$. Since x commutes with $A \otimes 1_d$, it follows that $y \in A'$, which implies that

$$(A \otimes \mathcal{M}_d)' \subset A' \otimes 1_d.$$

On the otherhand, any element of $A' \otimes 1_d$ has the form $y \otimes 1_d$ with $y \in A'$, so commutes with $A \otimes \mathcal{M}_d$ and

$$A' \otimes 1_d \subseteq (A \otimes \mathcal{M}_d)'.$$

Hence $(A \otimes \mathcal{M}_d)' = A' \otimes 1_d$. □

Definition 3.1.2. A von Neumann algebra M on $\mathcal{H}$ is a *factor* if its centre only contains scalar multiples of the identity, that is $Z(M) \cong \mathbb{C}$.

If M is a factor in $\mathcal{L}(\mathcal{H})$ then there is a tensor decomposition of $\mathcal{H}$ given by $\mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_{\bar{M}}$ such that $M = \mathcal{L}(\mathcal{H}_M) \otimes 1_{\bar{M}}$ [17, Theorem A.5]. In particular, if M

is a factor then there exists a Hilbert space $\mathcal{H}_M$ for which $M \cong \mathcal{L}(\mathcal{H}_M)$. Throughout this paper, we occasionally describe a von Neumann algebra, M, as a factor without referencing a relative fixed Hilbert space. In this case, one should take $M = \mathcal{L}(\mathcal{H}_M)$ for some Hilbert space $\mathcal{H}_M$.

Lemma 3.1.3. *Suppose M is a factor and N be a subfactor of M, i.e. we have an inclusion of von Neumann algebras, $N \subseteq M$, where both M and N are factors. Then,*

(a) N' is a factor;

(b) $N \otimes N' \cong M$.

Proof. Let M and N be as above. Then $M \cong \mathcal{L}(\mathcal{H})$ and $N \cong \mathcal{L}(\mathcal{H}_A)$, where $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$ with $N = \mathcal{L}(\mathcal{H}_A) \otimes 1_{\bar{A}}$ on $\mathcal{H}$. By the Lemma 3.1.1 we have

(a) $N' = (\mathcal{L}(\mathcal{H}_A) \otimes 1_{\bar{A}})' = 1_A \otimes \mathcal{L}(\mathcal{H}_{\bar{A}})$;

(b) $N \otimes N' = \mathcal{L}(\mathcal{H}_A) \otimes (1_{\bar{A}} \otimes 1_A) \otimes \mathcal{L}(\mathcal{H}_{\bar{A}}) \cong \mathcal{L}(\mathcal{H}_A) \otimes \mathcal{L}(\mathcal{H}_{\bar{A}}) \cong \mathcal{L}(\mathcal{H}) \cong M$.

□

Factors are very nice to work with, as exemplified by the above. However, we must broaden our perspective to general von Neumann algebras, in order to discuss the hybrid quantum system. The following theorem shows that any finite-dimensional von Neumann algebra is a direct sum of factors.

Definition 3.1.4. A *projection* is an operator $P \in \mathcal{L}(\mathcal{H})$ such that $P = P^2 = P^*$.

Theorem 3.1.5 (Bicommutant theorem)**.** *For any von Neumann algebra M on $\mathcal{H}$, we have $M'' \equiv (M')' = M$.*

Proof. Instead of considering M directly, we extend M to a von Neumann algebra $M \otimes 1_n$ on $\mathcal{H} \otimes \mathcal{H}$, where $n = \dim \mathcal{H}$. Then, we can view $\mathcal{L}(\mathcal{H} \otimes \mathcal{H})$ as the set of n-by-n block matrices, $\mathcal{M}_n \otimes \mathcal{M}_n$. By Lemma 3.1.1, $(M \otimes 1_n)' = M' \otimes \mathcal{M}_n$, and $(M' \otimes \mathcal{M}_n)' = M'' \otimes 1_n$.

Consider $v \in \mathcal{H} \otimes \mathcal{H}$. Define a subspace $V \subseteq \mathcal{H} \otimes \mathcal{H}$ by $V \equiv (M \otimes 1_n)v$, and let p_v, be the projection onto V. Then, p_v commutes with all elements in $(M \otimes 1_n)$, implying that $p_v \in M' \otimes \mathcal{M}_n$. Then, p_v commutes with all $M'' \otimes 1_n$. So, any element $z \otimes 1_n \in M'' \otimes 1_n$ must preserve V. In particular, this means that $(z \otimes 1_n)v = (x \otimes 1_n)v$ for some $x \in M$. But, if we choose $v = \bigoplus_{i=1}^n v_i$ for some basis v_i of $\mathcal{H}$, then $z = x$. Thus $M'' \subseteq M$, and since $M \subseteq M''$ by definition, it follows that $M'' = M$. □

Definition 3.1.6. Given positive operators x and y in a von Neumann algebra M, define the partial order on M denoted $x \leq y$ when $y - x$ is positive semi-definite, i.e. $y - x \geq 0$. If p and q are projections in M, then we write $p < q$ if $p \leq q$ and $\operatorname{rank}(p) < \operatorname{rank}(q)$.

Lemma 3.1.7. *Let M be a von Neumann algebra on $\mathcal{H}$, and consider $x \in M$. If $0 \leq x \leq p$ for some projection $p \in M$, then $x = pxp$.*

Proof. Since $x \geq 0$, then $\sqrt{x} \in M$. For any $\xi \in \mathcal{H}$,

$$\|\sqrt{x}(1-p)\|^2 = \langle x(1-p)\xi, (1-p)\xi\rangle \leq \langle p(1-p)\xi, (1-p)\xi\rangle = 0.$$

Thus, $\sqrt{x}(1-p)\xi = 0$, which implies $x(1-p)\xi = 0$. Then, $x\xi = x(p\xi + (1-p)\xi) = xp\xi$, which gives us $x = xp$. Since $x = x^* = (xp)^* = px$, it follows that $x = pxp$. □

Definition 3.1.8. Let M be a von Neumann algebra. A non-zero projection $p \in M$ is a *minimal central projection* if p is contained inside $Z(M)$ such that for all projections $q \in Z(M)$, $q \leq p$ implies $q = 0$ or $p = q$.

Lemma 3.1.9. *Let M be a von Neumann algebra in $\mathcal{L}(\mathcal{H})$. Then, there exists a unique set of orthogonal projections $p_1, ..., p_m \in M$ that are minimal in $Z(M)$, and sum to $1_M \in M$.*

Proof. Clearly, 1_M is a projection in $Z(M)$. If M is a factor, then $Z(M) = \mathbb{C}$ and 1_M is minimal in $Z(M)$ and the proof is done. If M is not a factor, then $Z(M)$ contains elements other than multiples of the identity. Suppose x is one of such elements. Then, let $X = \text{span}(x) \subseteq Z(M)$ and let $v \in \mathcal{H}$. Then $Y = Xv$ is a subspace of $\mathcal{H}$ giving us the decomposition $\mathcal{H} = Y \oplus Y^\perp$. This implies the existence of an orthogonal projection $p_v : \mathcal{H} \to Y$. It follows that $p_v \leq 1_M$. Then, there exists a non-zero minimal projection $p_1 \in Z(M)$ such that $p_1 \leq p_v$. Similarly, there exists a non-zero minimal projection $p_2 \leq 1_M - p_1$ in $Z(M)$. Repeating this process until the projection $1_M - \sum_{i=1}^{m-1} p_i$ is minimal in $Z(M)$, gives us a set of orthogonal, non-zero, minimal projections $\{p_i\}_{i=1}^m$ in $Z(M)$, with $\sum_{i=1}^m p_i = 1_M$.

To prove uniqueness, suppose $\{q_j\}_{j=1}^n$ is an orthogonal set of non-zero minimal central projections such that $\sum_{j=1}^n q_j = 1_M$. Then, $p_i = \sum_{j=1}^m p_i q_j$ implying that $p_i q_j \neq 0$ for some $j = 1, ..., m$. Fix j so that $p_i q_j \neq 0$. Then, $p_i\mathcal{H} \cap q_j\mathcal{H} \neq \{0\}$. Then, there exists an orthogonal projection $s : \mathcal{H} \to (p_i\mathcal{H} \cap q_j\mathcal{H})$ with $s < p_i$, contradicting the minimality of p_i. Hence, $s = p_i$ so that $(p_i\mathcal{H} \cap q_j\mathcal{H}) = p_i\mathcal{H}$, and hence $q_j = p_i$. It follows that $\{p_i\}_{i=1}^m$ is unique (up to indexing). □

Given a von Neumann algebra M, we will refer to the set of orthogonal minimal projections in $Z(M)$ as *the minimal central projections of* M.

Lemma 3.1.10. *The centre of a von Neumann algebra M is the span of its minimal central projections; $Z(M) = \text{span}(p_1, ..., p_m)$, where $p_1, ..., p_m$ are the minimal central projections of M.*

Proof. Consider $p_i Z(M) p_i$. Note that $\mathbb{C}p_i \subseteq p_i Z(M) p_i$ for each i. If $p_i Z(M) p_i$ contains any other operators, then there will be a non-trivial projection $q < p_i$. But this contradicts the minimality of p_i. Thus, $p_i Z(M) p_i = \mathbb{C}p_i$ for all i. Hence,

$$Z(M) = \big(\sum_{i=1}^{m} p_i\big) Z(M) = \sum_{i=1}^{m} p_i Z(M) = \sum_{i=1}^{m} p_i Z(M) p_i = \sum_{i=1}^{m} \mathbb{C}p_i,$$

where the first equality holds from the fact that $1_M = \sum_{i=1}^{m} p_i$. □

Theorem 3.1.11. *Let M be a finite dimensional von Neumann algebra on a Hilbert space $\mathcal{H}$. Then, M is a von Neumann algebra direct sum of factors.*

Proof. Let $p_1, .., p_m$ be the minimal central projections of M, with $\sum_{j=1}^{m} p_1 = 1$. Using this set of minimal central projections, we can decompose the Hilbert space, $\mathcal{H}$, into the Hilbert space direct sum

$$\mathcal{H} = \bigoplus_{i=1}^{m} p_i \mathcal{H}.$$

By centrality of the p_i's, the action of M on $\mathcal{H}$ decomposes as $\oplus_{i=1}^{m} p_i M$ on $\oplus_{i=1}^{m} p_i \mathcal{H}$. In particular, $p_i \mathcal{H}$ is invariant under the action of M for each i. It follows that $p_i M$ is a von Neumann algebra on $p_i \mathcal{H}$, for each i.

Now, suppose $p_i M$ has a non-trivial central element $c \in Z(p_i M)$. Then, $c \oplus 0_{(1-p_i)\mathcal{H}} \in M$, and we get

$$\begin{aligned} M(c \oplus 0_{(1-p_i)\mathcal{H}}) &= p_i M c \\ &= c p_i M \\ &= (c \oplus 0_{(1-p_i)\mathcal{H}}) M. \end{aligned}$$

Which implies $c \oplus 0_{(1-p_i)M} \in Z(M) = \text{span}(p_1, .., p_m)$. Then, $c \in \mathbb{C}p_i$. Hence, $Z(p_i M) = \mathbb{C}p_i$, and for each $i = 1, ..., m$, $p_i M$ is a factor, as required. □

We now present a few facts in order to prepare for the forthcoming Section 3.2, Inclusion matrix.

Definition 3.1.12. Let M be a von Neumann algebra. A *two-sided ideal* of M is a set $I \subseteq M$ such that for all $x \in M$ we have $xI \subseteq I$ and $Ix \subseteq I$.

Definition 3.1.13. A von Neumann algebra M is *simple* when its only two-sided ideals are 0 and M itself.

Lemma 3.1.14. *Let M be a factor inside $F = \mathcal{L}(\mathcal{H})$ and let $q \in M \cup M'$ be a non-zero projection. Then,*

(a) qMq is a factor in $\mathcal{L}(\mathcal{H})$;

(b) $(qMq)' = qM'q$.

Proof. **First assume that $q \in M$:**

(a) Since M is a factor we have the tensor factorization $\mathcal{H} = \mathcal{H}_M \otimes \mathcal{H}_{\bar{M}}$ such that $M = \mathcal{L}(\mathcal{H}_M) \otimes 1_{\bar{M}}$. It follows that $q = p \otimes 1_{\bar{M}}$ for some non-zero projection p in $\mathcal{L}(\mathcal{H}_M)$. Then,

$$qMq = q(\mathcal{L}(\mathcal{H}_M) \otimes 1_{\bar{M}})q = p\mathcal{L}(\mathcal{H}_M)p \otimes 1_{\bar{M}} = \mathcal{L}(p\mathcal{H}_M) \otimes 1_{\bar{M}},$$

and clearly $Z(qMq) = \mathbb{C}1_M \otimes 1_{\bar{M}}$. Hence, qMq is a factor.

(b) Consider first $x \in qMq \subset M$ and $y \in qM'q$. Choose $z \in M'$ so that $y = qzq$. Then we have

$$xy = xqzq = xzq = qxzq = qzxq = qzqx = yx,$$

and it follows that $qM'q \subset (qMq)'$.

Now consider $s \in (qM'q)'$ and $t \in M'$. Since $tq = qt$, we get

$$st = sqqt = sqtq = qtqs = tqqs = ts.$$

It follows that $(qM'q)' \subset qM''q = qMq$, where the equality holds by Lemma 3.1.3 (b) as M is a factor. Lemma 3.1.3 (a) tells us that M' is a factor, and part (a) of this proof gives us that $qM'q$ is also a factor. Taking the commutant inside of qFq of both sides of the previous inclusion flips the inclusion giving us

$$(qMq)' \subset (qM'q)'' = qM'q.$$

Hence, $(qMq)' = qM'q$.

Now assume that $q \in M'$:

(a) The map $\varphi : M \to qMq$ defined by qxq is a homomorphism of algebras. Since

$$1_M \mapsto q1_Mq = q, \tag{3.2}$$

which acts as the identity on qMq. The kernel of an algebra homomorphism forms a two-sided ideal in its domain. However, since M is a factor on $\mathcal{H}$ we have $M = \mathcal{L}(\mathcal{H}_M) \otimes 1_{\mathcal{H}_{\bar{M}}}$, which implies that

$$M \cong \mathcal{L}(\mathcal{H}_M) \cong \mathcal{M}_d,$$

where $d = \dim(\mathcal{H}_M)$. Then M is simple [10, p.832] and thus the kernel of the homomorphism φ is either 0 or all of M. By (3.2) we know φ is non-zero, then φ is an isomorphism. Hence, qMq is a factor.

(b) By replacing M by M', in the first part of the proof, have $(qM'q)' = qM''q$. But $qM''q = qMq$, which means $(qM'q)' = qMq$. Then,

$$(qM'q)'' = (qMq)' \quad \Rightarrow \quad qM'q = (qMq)',$$

as required. □

Proposition 3.1.15. *Let M be a von Neumann algebra contained in $\mathcal{L}(\mathcal{H})$ and let $q \in M \cup M'$ be a non-zero projection. Then*

(a) qMq is a von Neumann algebra;

(b) $(qMq)' = qM'q$.

Proof. (a) Let $p_1, ..., p_m$ be minimal central projections of M, chosen such that $\sum_{i=1}^m p_i = 1$. Set $q_i = p_i q$ and observe that $q = \sum_{i=1}^m q_i$. Moreover, note that q_i is a projection for all i since $q_i^2 = q_i = q_i^*$. By Lemma 3.1.14 (a), $q_i M q_i$ is a factor for all i. It follows from orthogonality of the q_i,

$$qMq = \bigoplus_{i=1}^{m} q_i M q_i$$

is a von Neumann algebra on the Hilbert space $\bigoplus_{i=1}^n q_i \mathcal{H}$.

(b) Note that $q_i q = q_i = q q_i$. Then,

$$(qMq)' = \bigoplus_{i=1}^{m} (qMq)' = \bigoplus_{i=1}^{m} (q_i M q_i)'.$$

We can identify M' with $\oplus_{i=1}^m (p_i M p_i)'$. Then, for q_j with $j \neq i$ we have

$$q_j((p_i M p_i)')q_j = 0.$$

Then,

$$qM'q = \bigoplus_{i=1}^{m} q_i M' q_i = \bigoplus_{i=1}^{m} q_i((p_i M p_i)')q_i.$$

By Theorem 3.1.11, $p_i M$ is a factor. Then

$$\begin{aligned}
\bigoplus_{i=1}^{m} (q_i M q_i)' &= \bigoplus_{i=1}^{m} (q_i (p_i M p_i) q_i)' \\
&= \bigoplus_{i=1}^{m} q_i (p_i M)' q_i \qquad \text{by Lemma 3.1.14 (b)} \\
&= \bigoplus_{i=1}^{m} q_i M' q_i \\
&= qM'q,
\end{aligned}$$

which is what we wanted to show. □

Let $N \subseteq M \subseteq \mathcal{L}(\mathcal{H})$ be an inclusion on $\mathcal{H}$, where N and M have minimal central projections $q_1, ..., q_n$ and $p_1, ..., p_m$, respectively. Since $p_i M$ is isomorphic to $\mathcal{L}(p_i \mathcal{H})$, we may identify each p_iM by $\mathcal{M}_{\mu_i}$, where $\mu_i = \dim p_i\mathcal{H}$. It follows that,

$$M = \bigoplus_{i=1}^{m} p_iM = \bigoplus_{i=1}^{m} \mathcal{M}_{\mu_i}.$$

We denote by $\vec{\mu}$ the m-tuple of dimensions $(\mu_1, ..., \mu_m)^T$. Then, giving N the same treatment, we obtain the identification

$$N = \bigoplus_{j=1}^{n} q_jN = \bigoplus_{j=1}^{n} \mathcal{M}_{\nu_j},$$

where $\nu_j = \dim q_j\mathcal{H}$, and we denote by $\vec{\nu}$ the n-tuple of dimensions $(\nu_1, ..., \nu_n)^T$.

Set $M_{ij} = p_iq_jMp_iq_j$ and $N_{ij} = p_iq_jNp_iq_j$. Clearly, we have $N_{ij} \subseteq M_{ij}$, and

$$\bigoplus_{j=1}^{n} N_{ij} \subseteq p_iM.$$

Since p_i acts as the identity on p_iM, it follows that $N_{ij} = N_j = \mathcal{M}_{\nu_j}$, for all $p_iq_j \neq 0$. Then, for the minimal central projections p_i, q_j, we have $(p_iq_j)^2 = p_iq_j$ and $(p_iq_j)^* = p_iq_j$, so p_iq_j is a projection in M, for all i, j. By Proposition 3.1.15, M_{ij} is a factor, which we identify by $\mathcal{M}_{\omega_{ij}}$. Thus, $N_{ij} \subseteq M_{ij}$ is an inclusion of factors on the Hilbert space $p_iq_j\mathcal{H}$. Then, given $p_iq_j \neq 0$,

$$N_{ij} = \mathcal{M}_{\nu_j} \otimes 1_{\lambda_{ij}} \subseteq \mathcal{M}_{\omega_{ij}},$$

with $\nu_j\lambda_{ij} = \omega_{ij}$. By orthogonality of the projections $q_1, ..., q_n$, it follows that

$$\bigoplus_{j=1}^{n} M_{\nu_j} \otimes 1_{\lambda_{ij}} \subseteq \mathcal{M}_{\mu_i} = p_iM,$$

which implies $\sum_j^n \nu_j\lambda_{ij} = \mu_i$ since the inclusion is unital, where $\lambda_{ij} = 0$ whenever $p_iq_j = 0$. By orthogonality of the projections $p_1, ..., p_m$, it follows that

$$N = \bigoplus_{i=1}^{m} \Big(\bigoplus_{j=1}^{n} \mathcal{M}_{\nu_j} \otimes 1_{\lambda_{ij}} \Big) \subseteq \bigoplus_{i=1}^{m} \mathcal{M}_{\mu_i} = M.$$

3.2 Inclusion matrix

In this section study inclusions of von Neumann algebras, $N \subseteq M$ via the so-called inclusion matrices. We begin by the simplest case, where N and M are both factors and introduce the notion of the index of an inclusion. With the help of the upcoming Lemma 3.1.14 and Proposition 3.1.15, we define the index for a general inclusion of von Neumann algebras. The idea of an inclusion matrix for a general inclusion is introduced next, along with some of its key properties.

For a von Neumann algebra inclusion $N \subseteq M$ the *index of N inside M* represents the multiplicity N inside M. To help elucidate the definition above, consider a factor inclusion $N \subseteq M$. The index of N inside M is defined to be

$$[M : N] = \frac{\dim(M)}{\dim(N)}.$$

Let $N \cong \mathcal{L}(\mathcal{H}_B)$ and $M \cong \mathcal{L}(\mathcal{H}_A)$. By Lemma 3.1.3, $M \cong N \otimes N' \cong \mathcal{L}(\mathcal{H}_B) \otimes \mathcal{L}(\mathcal{H}_{\bar{B}})$ so that

$$\dim(M) = \dim(N) \cdot \dim(N') = \dim(N) \cdot \dim(\mathcal{L}(\mathcal{H}_{\bar{B}})) = \dim(N) \cdot \dim(\mathcal{H}_{\bar{B}})^2.$$

Then, for the index of N inside M we have

$$[M : N] = \frac{\dim(M)}{\dim(N)} = \frac{\dim(N) \cdot \dim(\mathcal{H}_{\bar{B}})^2}{\dim(N)} = \dim(\mathcal{H}_{\bar{B}})^2.$$

So, given an inclusion of factors $N \subseteq M$, the index $[M : N]$ is the square of an integer and represents the *multiplicity* of N inside M, i.e. $N = \mathcal{M}_n \otimes 1_{\mathcal{H}_{\bar{B}}} \subseteq \mathcal{M}_m = M$, where $m = n \cdot \dim(\mathcal{H}_{\bar{B}})^2$.

Definition 3.2.1. The *inclusion matrix for $N \subseteq M$* is the matrix denoted by Λ_N^M with entries

$$\lambda_{ij} = [M_{ij} : N_{ij}]^{1/2} = \left(\frac{\dim(M_{ij})}{\dim(N_{ij})}\right)^{1/2},$$

with $\lambda_{ij} = 0$ whenever $p_i q_j = 0$.

The λ_{ij}'s are the multiplicities associated with each direct summand. For example, take the inclusion $N = \bigoplus_{j=1}^{n} \mathcal{M}_{n_j} \otimes 1_{m_j}$ inside $\mathcal{M}_d$. The inclusion matrix for this inclusion is the 1-by-n matrix with entries $[m_j]$.

Definition 3.2.2. For a a von Neumann algebra inclusion $N \subseteq M$, the *index of the inclusion*, denoted by $[M : N]$, is defined by the operator norm $[M : N] = \|\Lambda_N^M\|^2$.

Continuing with our example from above, the index of the inclusion $N \subseteq \mathcal{M}_d$, is

$$[M : N] = \|[m_1 \ ... \ m_n]\|^2 = \sum_{j=1}^{n} m_j^2 = \dim N'.$$

Proposition 3.2.3. *Let $N \subseteq M$ be the inclusion of von Neumann algebras, and the notation be as above. Then,*

$$\vec{\mu} = \Lambda_N^M \vec{\nu}.$$

Proof. Let the map $\pi_{ij} : q_jN \to M_{ij}$ be defined by $\pi_{ij}(x) = p_i x p_i$. Then π_{ij} is a *-homomorphism. In particular, π_{ij} is unital; $\pi_{ij}(q_j) = p_i q_j = 1_{M_{ij}}$. Since q_jN is a factor, in particular, q_jN is simple for all $j = 1, ..., n$, then $\ker(\pi_{ij})$ is either $\{0\}$ or q_jN which depends on whether or not $p_iq_j = 0$. If $p_iq_j \neq 0$, then

$$\mathcal{M}_{\nu_j} \cong q_jN \cong \pi_{ij}(q_jN) \subseteq M_{ij} \cong \mathcal{M}_{\omega_{ij}},$$

where the last congruence holds since M_{ij} is a factor. From this, together with the fact that π_{ij} is unital, it follows that $\pi_{ij}(q_jN) \cong \mathcal{M}_{\nu_j} \otimes 1_{\lambda_{ij}}$ which lies in $\mathcal{M}_{\omega_{ij}} \cong M_{ij}$, where $\nu_j\lambda_{ij} = \omega_{ij}$. Furthermore, observe that $p_iM \cong \oplus_{j=1}^n q_jp_iM = \oplus_{j=1}^n M_{ij}$, since $\sum_{j=1}^n q_j = 1$. Then, we have

$$\bigoplus_{j=1}^n \pi_{ij}(q_jN) \cong \bigoplus_{j=1}^n \mathcal{M}_{\nu_j} \otimes 1_{\lambda_{ij}} \subseteq \bigoplus_{j=1}^n \mathcal{M}_{\omega_{ij}} \cong \bigoplus_{j=1}^n M_{ij} \cong p_iM \cong \mathcal{M}_{\mu_i}.$$

It follows that,

$$\mu_i = \sum_{j=1}^n \omega_{ij} = \sum_{j=1}^n \nu_j\lambda_{ij}.$$

Hence, $\vec{\mu} = \Lambda_N^M \vec{\nu}$. □

Corollary 3.2.4. *Let $N \subseteq M \subseteq L$ be an inclusion of von Neumann algebras. Then*

$$\Lambda_N^L = \Lambda_M^L \Lambda_N^M.$$

Proof. Let $r_1, ..., r_\ell$ be the set of minimal central projections of L, and $p_1, ..., p_m$ and $q_1, ..., q_n$ be the set of minimal central projections for M and N respectively. We identify L, M and N by:

$$L = \bigoplus_{k=1}^{\ell} r_kL, \qquad M = \bigoplus_{i=1}^{m} p_iM, \qquad N = \bigoplus_{j=1}^{n} q_jN.$$

Let the inclusion matrices be described as follows: $\Lambda_N^L = [\tau_{kj}]$, $\Lambda_M^L = [\omega_{ki}]$, and $\Lambda_N^M = [\lambda_{ij}]$. Then $\Lambda_M^L \Lambda_N^M = [\sum_{i=1}^m \omega_{ki}\lambda_{ij}]$.

From Propostion 3.2.3 we have that $p_iq_jNp_i = q_jN \otimes 1_{\lambda_{ij}}$, so it follows that

$r_k p_i M r_k = p_i M \otimes 1_{\omega_{ki}}$ and $r_k q_j N r_k = q_j N \otimes 1_{\tau_{kj}}$. Then

$$\begin{aligned} r_k q_j N r_k &= \bigoplus_{i=1}^{m} r_k p_i q_j N p_i r_k \\ &= \bigoplus_{i=1}^{m} p_i q_j N p_i \otimes 1_{\omega_{ki}} \\ &= \bigoplus_{i=1}^{m} q_j N \otimes 1_{\lambda_{ij}} \otimes 1_{\omega_{ki}} \\ &= q_j N \otimes \big(\bigoplus_{i=1}^{m} 1_{\lambda_{ij}} \otimes 1_{\omega_{ki}} \big) \end{aligned}$$

which implies $\tau_{kj} = \sum_{i=1}^{m} \lambda_{ij}\omega_{ki}$. Thus, $\Lambda_N^L = [\tau_{kj}] = [\sum_{i=1}^{m} \omega_{ki}\lambda_{ij}] = \Lambda_M^L \Lambda_N^M$, as required. □

Lemma 3.2.5. *Let $F = \mathcal{L}(\mathcal{H})$ with subfactors M and N such that $N \subseteq M$. Then,*

$$[N' : M'] = [M : N].$$

Proof. From Lemma 3.1.3 we know that $F \cong M \otimes M'$ and $F \cong N \otimes N'$, which implies:

$$\dim(M') = \frac{\dim(F)}{\dim(M)} \quad \text{and} \quad \dim(N') = \frac{\dim(F)}{\dim(N)}.$$

Thus,

$$[N' : M'] = \frac{\dim(N')}{\dim(M')} = \frac{\dim(F)}{\dim(N)} \cdot \frac{\dim(M)}{\dim(F)} = \frac{\dim(M)}{\dim(N)} = [M : N].$$

□

Proposition 3.2.6. *Let $N \subseteq M$ be an inclusion of von Neumann algebras. Then*

$$\Lambda_{M'}^{N'} = (\Lambda_N^M)^T.$$

Proof. Observe that the pairs N and N', and M and M' have the same minimal central projections. Recall that $M' \subseteq N'$, for the inclusion $N \subseteq M$. Set $N'_{ji} = q_j p_i N' q_j p_i$, and $M'_{ji} = q_j p_i M' q_j p_i$. Then, the inclusion matrix $\Lambda_{M'}^{N'}$, has entries

$$\tilde{\lambda}_{ji} = [N'_{ji} : M'_{ji}]^{1/2} = \Big(\frac{\dim N'_{ji}}{\dim M'_{ji}}\Big)^{1/2}.$$

But, from our observation above we get,

$$N'_{ji} = q_j p_i N' q_j p_i = (p_i q_j N p_i q_j)' = (N_{ij})',$$

and similarly, $M'_{ji} = (M_{ij})'$.

Since each $N_{ij} \subseteq M_{ij}$ is a pair of factors, Lemma 3.2.5 tells us that

$$[M_{ij} : N_{ij}]^{1/2} = [(N_{ij})' : (M_{ij})']^{1/2},$$

thus

$$\tilde{\lambda}_{ji} = [N'_{ji} : M'_{ji}]^{1/2} = [(N_{ij})' : (M_{ij})']^{-1/2} = [M_{ij} : N_{ij}]^{1/2} = \lambda_{ij}.$$

Hence, $\Lambda_{M'}^{N'} = (\Lambda_N^M)^T$. □

3.3 Traces

Here, we introduce the notion of a *trace* on a von Neumann algebra and showcase its connection to the inclusion matrix of a von Neumann algebra inclusion. As in previous subsections, we begin by restricting ourselves to factors, then move on to studying general finite-dimensional von Neumann algebras.

Definition 3.3.1. Let M be a von Neumann algebra on $\mathcal{H}$. An element $A \in M$ is a *positive operator* if $\langle \psi, A\psi \rangle \geq 0$, for all $\psi \in \mathcal{H}$.

Definition 3.3.2. Let M and N be von Neumann algebras. A linear map $\Phi : M \to N$ is *positive* if and only if $\Phi(A)$ is positive for all positive $A \in M$. That is, it maps positive elements to positive elements.

Definition 3.3.3. Let M and N be von Neumann algebras. Let $\mathrm{id}_p \otimes \Phi : M_p(M) \to M_p(N)$ be defined by

$$\mathrm{id}_p \otimes \Phi((A_{ij})_{1 \leq i,j \leq p}) = (\Phi(A_{ij}))_{1 \leq i,j \leq p}.$$

Then, Φ is *completely positive* (CP) if and only if $\mathrm{id}_p \otimes \Phi$ is positive for all integers p.

Definition 3.3.4. Let M and N be von Neumann algebras. A map, $\phi : M \to N$ is *faithful* if $\phi(x^* x) \neq 0$ for any non-zero $x \in M$.

Definition 3.3.5. Any bijection $U \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ that preserves the inner product is a *unitary map.*

Note that, $U \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ is unitary if and only if for all $\psi, \phi \in \mathcal{H}$,

$$\langle U\psi, U\phi \rangle_K = \langle U^* U\psi, \phi \rangle_K = \langle \psi, \phi \rangle_{\mathcal{H}}.$$

Definition 3.3.6. A *trace* on a von Neumann algebra M, is a positive linear functional $\phi : M \to \mathbb{C}$ which satisfies $\phi(xy) = \phi(yx)$, for all $x, y \in M$.

Lemma 3.3.7. *Consider the factor M. Then there exists a unique map $\tau : M \to \mathbb{C}$ such that*

(i) *τ is a positive linear functional;*

(ii) *$\tau(xy) = \tau(yx)$, for all $x, y \in M$;*

(iii) *$\tau(1_n) = 1 \in \mathbb{C}$.*

Proof. Since M is a factor, we may identify it by $\mathcal{M}_n$. Let e_{ij} denote a matrix unit in $\mathcal{M}_n$, that is the $n \times n$ matrix with a 1 in the ij^{th} entry and zeros elsewhere. Then $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$, for any $i, j, k, \ell = 1, ..., n$. Let $\tau : \mathcal{M}_n \to \mathbb{C}$ satisfy properties (i)-(iii). Then (ii) gives us

$$\tau(e_{i\ell}) = \tau(e_{ij}e_{j\ell}) = \tau(e_{j\ell}e_{ij}) = \tau(0) = 0, \tag{3.3}$$

for all $j = 1, ..., n$ and for any $i \neq \ell$. From (iii) we know τ is non-zero, in particular $\tau(e_{ii})$ is non-zero for at least one $i = 1, ..., n$. But for any $i, j = 1, ...n$, (ii) gives us:

$$\tau(e_{ii}) = \tau(e_{ii}e_{ii}) = \tau(e_{ij}e_{ji}e_{ii}) = \tau(e_{ji}e_{ii}e_{ij}) = \tau(e_{ji}e_{ij}) = \tau(e_{jj}), \tag{3.4}$$

and from (iii) we get

$$1 = \sum_{i=0}^{n-1} \tau(e_{ii}) = n\tau(e_{11}),$$

It follows that $\tau(e_{ii}) = \tau(e_{11}) = 1/n$, for all $i = 1, ..., n$; this gives us uniqueness. For existence, consider the normalized trace on $\mathcal{M}_n$. That is, let $\tau : \mathcal{M}_n \to \mathbb{C}$ be the mapped defined by $\tau(x) = n^{-1}\mathrm{tr}(x)$, where $\mathrm{tr} : \mathcal{M}_n \to \mathbb{C}$ is defined by:

$$M \ni x = \sum_{i,j=1}^{n} x_{ij}e_{ij} \mapsto \sum_{i=1}^{n} x_{ii}.$$

Then, τ satisfies (i)-(iii). □

As we know, we may identify a von Neumann algebra M by

$$M = \bigoplus_{i=1}^{m} p_i M,$$

where $p_1, ..., p_m$ are the minimal central projections of M. In particular, we know that each $p_i M$ is a factor on $p_i\mathcal{H}$. Then, we associate to a trace τ on M a so-called *trace vector*, given by the row-vector

$$\vec{s} = (\tau(e_1), ..., \tau(e_m)) \in \mathbb{R}^m,$$

where e_i is a minimal projection in p_iM. By Equation (3.4), our choice of minimal projection in p_iM is inconsequential for all i. It follows from the positivity of traces that $s_i \geq 0$, for all i. (We denote by $\mathbb{R}_+$ the set of non-negative reals.) On the other hand, we can associate to any vector in $\mathbb{R}_+^m$ a unique trace on M.

For example, the trace $\sigma^{(i)}$ defined by $e_j \mapsto \delta_{ij}$ corresponds to the i^{th} vector of the canonical basis of $\mathbb{R}_+^m$. Then any vector $\vec{s} \in \mathbb{R}_+^m$ corresponds to a unique trace τ_s on M:

$$\tau_s = \sum_{i=1}^{m} s_i \sigma^{(i)} : M \to \mathbb{C},$$

with associated vector $\vec{s}$. For any $x \in M$ we have

$$\tau_s(x) = \sum_{i,j=1}^{m} \sum_{t=1}^{d_i} s_i x_{tt}^{(j)} \sigma^{(i)}(e_j) = \sum_{i,j=1}^{m} \sum_{t=1}^{d_i} s_i x_{tt}^{(j)} \delta_{ij} = \sum_{i=1}^{m} \sum_{t=1}^{d_i} s_i x_{tt}^{(i)},$$

giving us the fact that $\tau_s(x)$ is determined by the diagonal entries of x and the trace vector $\vec{s}$. It follows that if τ_s is faithful, then $s_i > 0$ for all i. We now show the striking relationships between traces, in particular their trace vectors, and inclusion matrices.

Proposition 3.3.8. *Let $N \subseteq M$ be an inclusion of von Neumann algebras with*

$$N = \bigoplus_{j=1}^{n} q_j N = \bigoplus_{j=1}^{n} \mathcal{M}_{\nu_j}, \qquad M = \bigoplus_{i=1}^{m} p_i M = \bigoplus_{i=1}^{m} \mathcal{M}_{\mu_i},$$

and with inclusion matrix Λ_N^M. Let σ be a trace on M with associated vector $\vec{s} \in \mathbb{R}_+^m$ and let τ be a trace on N corresponding to $\vec{t} \in \mathbb{R}_+^n$. Then σ extends τ if and only if $\vec{t} = \vec{s}\Lambda_N^M$. Moreover, if σ extends τ, then $\langle \vec{s}, \vec{\mu} \rangle = \langle \vec{t}, \vec{v} \rangle$, where $\vec{\mu} = (\mu_1, ..., \mu_m)^T$ and $\vec{\nu} = (\nu_1, ..., \nu_n)$.

Proof. $(\Rightarrow)$: Suppose σ extends τ. If f_j is a minimal projection of q_jN, then f_jp_i is the sum of λ_{ij} minimal projections in p_iM. The restriction of σ to N is described by $\vec{t'}$ with entries

$$t'_j = \sigma(f_j) = \sum_{i=1}^{m} \sigma(f_jp_i) = \sum_{i=1}^{m} \sigma(e_i)\lambda_{ij} = \sum_{i=1}^{m} s_i\lambda_{ij}.$$

So $\vec{t'} = \vec{s}\Lambda_N^M$, and since $\sigma|_N = \tau$ we have $\vec{t} = \vec{t'}$ by uniqueness of the trace/vector correspondence.

$(\Leftarrow)$: Let $\vec{t} = \vec{s}\Lambda_N^M$. Then,

$$\tau(f_j) = t_j = \sum_{i=1}^{m} s_i\lambda_{ij} = \sum_{i=1}^{m} \sigma(e_i)\lambda_{ij} = \sum_{i=1}^{m} \sigma(f_jp_i) = \sigma(f_j),$$

which means $\sigma|_N$ is described by $\vec{t}$. Thus, by uniqueness of the trace-vector, $\sigma|_N = \tau$. Hence, σ extends τ. Moreover,

$$\begin{aligned}
\langle \vec{s}, \vec{\mu} \rangle &= \sum_{i=1}^{m} s_i \mu_i \\
&= \sum_{i=1}^{m} \sigma(e_i) \mu_i \\
&= \sigma(1) \\
&= \tau(1) \quad ; \text{ since } 1 \in N \subseteq M \\
&= \sum_{j=1}^{n} \tau(f_j) v_j \\
&= \sum_{j=1}^{n} t_j v_j \\
&= \langle \vec{t}, \vec{v} \rangle,
\end{aligned}$$

as required. □

3.4 GNS construction

The Gelfand-Naimark-Segal (GNS) construction appears implicitly in teleportation, and this observation will allow us to study entanglement between two commuting subalgebras through what is known as the Jones projection. This commuting operator analogue of entanglement will be used as the resource for our teleportation scheme.

Let τ be a faithful trace on a d-dimensional von Neumann algebra M, with trace vector $\vec{s} = (s_1, ..., s_m)^t$ and minimal central projections $p_1, \ldots, p_m$, respectively. The map from $M \times M$ to $\mathbb{C}$ defined by $\langle x, y \rangle_\tau = \tau(y^* x)$ is an inner product on M. We denote the resulting Hilbert space by $L^2(M, \tau)$, and we denote by $\widehat{x} \in L^2(M, \tau)$ the canonical image of x in M. The left-representation of M on $L^2(M, \tau)$, denoted by π^ℓ, is such that $\pi^\ell(x)\widehat{y} = \widehat{xy}$, for all $x, y \in M$. Similarly, the right-representation of M on $L^2(M, \tau)$, denoted by π^r, is such that $\pi^r(x)\widehat{y} = \widehat{yx}$, for all $x, y \in M$. Note that $\widehat{1}$ is the *cyclic* vector, i.e. $\pi^\ell(M)\widehat{1} = L^2(M, \tau)$. These objects are known as the *GNS construction* applied to (M, τ).

More explicitly, consider $c_1, c_2 \in \mathbb{C}$ and $w, x, y, z \in M$. Then,

(i) $\langle w + x, y + z \rangle_\tau = \tau(y^* w + y^* x + z^* w + z^* x) = \langle w, y \rangle_\tau + \langle x, y \rangle_\tau + \langle w, z \rangle_\tau + \langle x, z \rangle_\tau$

(ii) $\langle \cdot, \cdot \rangle_\tau$ is sesquilinear (linear in the first):

$$\langle c_1 x, c_2 y \rangle_\tau = \tau(\bar{c}_2 y^* c_1 x) = c_1 \bar{c}_2 \tau(y^* x) = c_1 \bar{c}_2 \langle x, y \rangle_\tau.$$

(iii) $\langle \cdot, \cdot \rangle_\tau$ is conjugate symmetric:

$$\overline{\langle x, y \rangle_\tau} = \overline{\tau(y^* x)} = \tau(\overline{y^* x}) = \tau(y^t \bar{x}) = \tau((x^* y)^t) = \tau(x^* y) = \langle y, x \rangle_\tau$$

(iv) $\langle \cdot, \cdot \rangle_\tau$ is positive definite by faithfulness of τ.

Hence, the pair $(M, \langle \cdot, \cdot \rangle_\tau)$ forms a Hilbert space of dimension $\dim M$. It follows that, this Hilbert space is isomorphic to the $\dim(M)$-dimensional Hilbert space equipped with standard inner-product,

$$L^2(M, \tau) \cong \bigoplus_{i=1}^{n} L^2(\mathcal{M}_{n_i}, \tau) \cong \bigoplus_{i=1}^{n} \mathbb{C}^{n_i} \otimes \mathbb{C}^{n_i},$$

via the map

$$M \ni x \mapsto \widehat{x} = \bigoplus_{i=1}^{n} \sum_{t,k=1}^{n_i} \sqrt{s_i} x_{tk} e_t^{(i)} \otimes e_k^{(i)} \in L^2(M, \tau),$$

which is clearly bijective, and isometric as

$$\langle \widehat{x}, \widehat{y} \rangle = \sum_{i=1}^{m} \sum_{k,t=1}^{d_i} s_i \bar{y}_{kt}^{(i)} x_{kt}^{(i)} = \tau(y^* x) = \langle x, y \rangle_\tau.$$

3.5 Jones projection & conditional expectation

Suppose we have a von Neumann algebra inclusion $N \subseteq M$, with a faithful trace τ on M. Let $L^2(M, \tau)$ be the Hilbert space, with subspace $L^2(N, \tau)$, obtained by the GNS construction. The orthogonal projection from $L^2(M, \tau)$ onto $L^2(N, \tau)$ is called the *Jones projection* and is denoted by e_N. At the level of von Neumann algebras, the Jones projection defines a conditional expectation:

Definition 3.5.1. A *conditional expectation* for the inclusion $N \subseteq M$ is a map $E_N : M \to N$ which satisfies

1. $E_N^2 = E_N$
2. $\|E_N\| \leq 1$.

J. Tomiyama shows in [29, Theorem 1], that any projection of norm one from a von Neumann algebra onto a von Neumann subalgebra is a bimodule of the subalgebra and completely positive. Thus, any conditional expectation is a bimodule map, with respect to the subalgebra, and completely positive (CP). In particular, E_N is a CP N-bimodule map. That is, E_N is a CP map such that for all $x \in M$ and all $y_1, y_2 \in N$ we have

$$E_N(y_1 x y_2) = y_1 E_N(x) y_2.$$

The Jones projection, e_N, induces a map $E_N : M \to N$, such that $e_N\widehat{x} = \widehat{E_N(x)}$, for all $x \in M$. If $y \in N$, then $\widehat{E_N(y)} = e_N\widehat{y} = \widehat{y}$. Thus, E_N acts as the identity on N, in particular, E_N is unital. Then, we have:

- $\widehat{E_N^2(x)} = e_N^2\widehat{x} = e_N\widehat{x} = \widehat{E_N(x)}$
- $\tau(E_N(x)) = \langle\widehat{E_N(x)}, \widehat{1}\rangle_\tau = \langle e_N\widehat{x}, \widehat{1}\rangle_\tau = \langle\widehat{x}, e_N\widehat{1}\rangle_\tau = \tau(x)$.

So, E_N is an orthogonal projection from M into N, and τ-preserving.

Note that $L^2(M,\tau) = L^2(N,\tau) \oplus L^2(N,\tau)^\perp$, where orthogonality is taken with respect to the inner product $\langle\cdot,\cdot\rangle_\tau$. Let $y \in N$, then both $\pi^\ell(y)$ and $\pi^r(y)$ leave $L^2(M,\tau)$ invariant, hence $\pi^\ell(y)e_N = e_N\pi^\ell(y)e_N$ and $\pi^r(y)e_N = e_N\pi^r(y)e_N$. For any $\xi, \eta \in L^2(M,\tau)$, we have

$$\langle e_N\pi^\ell(y)(1-e_N)\xi, \eta\rangle = \langle(1-e_N)\xi, \pi^\ell(y)e_N\eta\rangle = 0,$$

for all $y \in N$. It follows that, $e_N\pi^\ell(y)(1-e_N) = 0$, and so

$$\pi^\ell(y)e_N = e_N\pi^\ell(y)e_N = e_N\pi^\ell(y)e_N + e_N\pi^\ell(y)(1-e_N) = e_N\pi^\ell(y).$$

Similarly, $\pi^r(y)e_N = e_N\pi^r(y)$. From this, we get the bi-module property of the E_N:

If $x \in M$, and $y, z \in N$, then

$$[\widehat{E_N(yxz)}] = e_N[\widehat{yxz}] = e_N\pi^\ell(y)\pi^r(z)[\widehat{x}] = \pi^\ell(y)\pi^r(z)e_N[\widehat{x}] = [y\widehat{E_N(x)}z],$$

and by injectivety of the embedding of M into $L^2(M,\tau)$, we get that $E_N(yxz) = yE_N(x)z$, for all $x \in M$, $y, z \in N$. Consequently, for all $x, y \in M$,

$$e_N\pi^\ell(x)e_N[\widehat{y}] = [E_N(\widehat{xE_N(y)})] = \pi^\ell(E_N(x))[\widehat{E_N(y)}] = \pi^\ell(E_N(x))e_N[\widehat{y}],$$

which implies that $e_N\pi^\ell(x)e_N = \pi^\ell(E_N(x))e_N$. Since the map $N \ni x \mapsto \pi^\ell(x)e_N \in \mathcal{L}(L^2(M,\tau))$ is an injective $*$-homomorphism, it is isometric [28, Corollary 5.4]. Thus, for any $x \in M$,

$$\|E_N(x)\| = \|\pi^\ell(E_N(x))e_N\| = \|e_N\pi^\ell(x)e_N\| \leq \|\pi^\ell(x)\| = \|x\|.$$

Thus, E_N is a projection of norm one and therefore CP by Tomiyama [29, Theorem 1]. Hence, E_N is a conditional expectation. If $E : M \to N$ is another τ-preserving conditional expectation, then by Tomiyama it is an N-bimodule map. Let $y \in N$ and $xN^\perp$, where by $N^\perp$ we mean the set of elements $\{x\} \subseteq M$ such that $\widehat{x} \in L^2(N,\tau)^\perp$. Then,

$$\langle\widehat{E(x)}, \widehat{y}\rangle = \tau(y^*E(x)) = \langle\widehat{x}, \widehat{y}\rangle = 0.$$

Thus, $E(N^{\perp}) = 0$ and $E|_N$ is the identity. So, $E = E_N$. Thus, E_N is the unique τ-preserving conditional expectation from M to N.

So, the Jones projection e_N for a von Neumann algebra inclusion, $N \subseteq M$ with faithful trace, induces a unique, faithful, completely positive, trace-preserving conditional expectation $E_N : M \to N$.

3.6 Jones' basic construction

Given an inclusion $N \subseteq M$ with faithful trace τ on M, the object generated by M and the Jones projection e_N is a von Neumann algebra $M_1 \subseteq \mathcal{L}(L^2(M, \tau))$. This method of generating a von Neumann algebra is known as the basic construction. Here, we present some of its properties and show that one can define the basic construction.

To begin, let $J : L^2(M, \tau) \to L^2(M, \tau)$ be the canonical anti-unitary map defined by $\widehat{x} \mapsto \widehat{x}^*$ for $x \in M$.

Proposition 3.6.1. *Let M be a von Neumann algebra, let $\pi^\ell(M)$ and $\pi^r(M)$ be the left and right representations of M on $L^2(M, \tau)$, and let J be as above. Then,*

(i) $\pi^r(M) = J\pi^\ell(M)J$

(ii) $J\pi^\ell(M)J = \pi^\ell(M)'$

(iii) $J\pi^r(M)J = \pi^r(M)'$

Proof. (i) For $x, y \in M$ we have $J\pi^\ell(x)J\widehat{y} = J\pi^\ell(x)\widehat{y^*} = J\widehat{xy^*} = \widehat{yx^*} = \pi^r(x^*)\widehat{y}$, so that $J\pi^\ell(M)J \subset \pi^r(M)$. The above computation in reverse order yields $\pi^r(x) = J\pi^\ell(x^*)J$, which implies $\pi^r(M) \subset J\pi^\ell(M)J$. Hence, $\pi^r(M) = J\pi^\ell(M)J$.

(ii) For $x, y, z \in M$, we have

$$\pi^r(x)\pi^\ell(y)\widehat{z} = \pi^r(x)\widehat{yz} = \widehat{yzx} = \pi^\ell(y)\pi^r(x)\widehat{z}.$$

Which implies that $\pi^r(M) \subset \pi^\ell(M)'$, and from part (i) we get that $J\pi^\ell(M)J \subset \pi^\ell(M)'$.

We claim that the trace defined by the map $x \mapsto \langle x\widehat{1}, \widehat{1}\rangle$ is a trace on $\pi^\ell(M)'$.

Let $x \in \pi^\ell(M)'$. We must first show that $Jx\widehat{1} = x^*\widehat{1}$. For $a \in M$, we have

$$\begin{aligned}
\langle Jx\widehat{1}, \widehat{a}\rangle &= \langle Jx\widehat{1}, \pi^\ell(a)\widehat{1}\rangle \\
&= \langle J\pi^\ell(a)\widehat{1}, JJx\widehat{1}\rangle \\
&= \langle \pi^\ell(a^*)\widehat{1}, x\widehat{1}\rangle \\
&= \langle x^*\pi^\ell(a^*)\widehat{1}, \widehat{1}\rangle \\
&= \langle \pi^\ell(a^*)x^*\widehat{1}, \widehat{1}\rangle \\
&= \langle x^*\widehat{1}, \pi^\ell(a)\widehat{1}\rangle \\
&= \langle x^*\widehat{1}, \widehat{a}\rangle,
\end{aligned}$$

so that $J x\widehat{1} = x^*\widehat{1}$. Now let $x, y \in \pi^\ell(M)'$. Then we have

$$\langle xy\widehat{1}, \widehat{1}\rangle = \langle y\widehat{1}, x^*\widehat{1}\rangle = \langle y\widehat{1}, Jx\widehat{1}\rangle = \langle x\widehat{1}, Jy\widehat{1}\rangle = \langle x\widehat{1}, y^*\widehat{1}\rangle = \langle yx\widehat{1}, \widehat{1}\rangle.$$

Hence, $x \mapsto \langle x\widehat{1}, \widehat{1}\rangle$ is a trace on $\pi^\ell(M)'$ which we will denote by φ.

The GNS construction on $(\pi^\ell(M)', \varphi)$ yields the Hilbert space $\mathcal{W} = L^2(\pi^\ell(M)', \varphi)$. Let K be the canonical anti-unitary on $\mathcal{W}$ defined by $Kx\widehat{1} = Kx^*\widehat{1}$, for $x \in \pi^\ell(M)'$. The first part of the proof for (ii) gives us

$$K\pi^\ell(M)'K \subset (\pi^\ell(M)')' = \pi^\ell(M).$$

From the first part of the claim above, we get that K and J coincide on $\mathcal{W}$. It follows that $J\pi^\ell(M)'J = K\pi^\ell(M)'K \subset \pi^\ell(M)$, and so $J\pi^\ell(M)J = \pi^\ell(M)'$.

The proof of (iii) is done similarly to that of (ii) by swapping π^ℓ for π^r. $\square$

It clearly follows from the proposition above that the commutant, taken with respect to $\mathcal{L}(L^2(M,\tau))$, of the left-representation of M on $L^2(M,\tau)$ is the right-representation of M on $L^2(M,\tau)$, $\pi^\ell(M)' = J\pi^\ell(M)J = \pi^r(M)$, and the opposite is true; $\pi^\ell(M) = \pi^r(M)'$.

Proposition 3.6.2. *Let $N \subseteq M$ be a von Neumann algebra inclusion and let e_N be the Jones projection from $L^2(M,\tau)$ into $L^2(N,\tau)$. Then*

(i) $J\pi^\ell(N)'J = \langle \pi^\ell(M), e_N\rangle$

(ii) $J\pi^r(N)'J = \langle \pi^r(M), e_N\rangle$

Proof. Clearly, $\pi^\ell(N) \subset \pi^\ell(M)$. For $y \in N$ and $x \in M$ we have

$$\pi^\ell(y)e_N\widehat{x} = \pi^\ell(y)\widehat{E_N(x)} = y\widehat{E_N(x)} = \widehat{E_N(yx)} = e_N\widehat{yx} = e_N\pi^\ell(y)\widehat{x},$$

so $e_N\pi^\ell(y) = \pi^\ell(y)e_N$, for all $y \in N$. Which implies $\pi^\ell(N) \subset \pi^\ell(M) \cap \{e_N\}'$. Consider $\pi^\ell(x) \in \pi^\ell(M) \cap \{e_N\}'$. Then

$$\widehat{E(x)} = e_N\widehat{x} = e_N\pi^\ell(x)\widehat{1} = \pi^\ell(x)e_N\widehat{1} = \pi^\ell(x)\widehat{1} = \widehat{x},$$

so $E_N(x) = x$ for all $\pi^\ell(x) \in \pi^\ell(M) \cap \{e_N\}'$. It follows that $x \in N$ for all $\pi^\ell(x) \in \pi^\ell(M) \cap \{e_N\}'$, and so $\pi^\ell(M) \cap \{e_N\}' \subset \pi^\ell(N)$. Therefore, $\pi^\ell(N) = \pi^\ell(M) \cap \{e_N\}'$. Then,

$$\pi^\ell(N)' = (\pi^\ell(M) \cap \{e_N\}')' = \langle \pi^\ell(M)', \{e_N\}''\rangle = \langle \pi^\ell(M)', \{e_N\}\rangle.$$

Note that $Je_NJ = e_N$, which follows from the fact the E_N is $*$-preserving. Then,

$$J\pi^\ell(N)'J = J\langle \pi^\ell(M)', e_N\rangle J = \langle J\pi^\ell(M)'J, Je_NJ\rangle = \langle \pi^\ell(M), e_N\rangle,$$

giving us (i).

The proof of (ii) is done similarly to that of (i) by swapping π^ℓ for π^r. $\square$

Corollary 3.6.3. *For an inclusion $N \subseteq M$, the algebra M_1, given by the basic construction, is the $\mathbb{C}$-vector space generated by elements of the form $\pi^\ell(x)e_N\pi^\ell(y)$, $x, y \in M$. That is, $M_1 = \pi^\ell(M)e_N\pi^\ell(M)$.*

Proof. Clearly, $\mathcal{I} = \mathrm{span}_{\mathbb{C}}(Me_NM)$ is a subalgebra of $\langle M, e_N\rangle$ and a 2-sided ideal of the von Neumann algebra generated by M and e_N. Moreover,

$$\mathcal{I}L^2(M) = Me_NL^2(M) = ML^2(N) \supset M\widehat{1}.$$

Since $\mathcal{I}$ is non-degenerate, we get $\mathcal{I} = \langle M, e_N\rangle$. $\square$

In the final portion of this subsection, we relate the basic construction to inclusion matrices. In particular, Proposition 3.6.6 states that the inclusion matrix for the inclusion $N \subseteq M$ is the transpose of the inclusion matrix for the inclusion $M \subseteq M_1$.

Lemma 3.6.4. *Let $N \subseteq M$ and $\bar{N} \subseteq \bar{M}$ be pairs of von Neumann algebras. If there exists an isomorphism $\theta : M \to \bar{M}$ with $\theta(N) = \bar{N}$, then*

$$\Lambda_N^M = \Lambda_{\bar{N}}^{\bar{M}}.$$

Proof. Let $\Lambda_N^M = [\lambda_{ij}]$ and $\Lambda_{\bar{N}}^{\bar{M}} = [\bar{\lambda}_{ij}]$. Let $\bar{p}_1, ..., \bar{p}_m$ and $\bar{q}_1, ..., \bar{q}_n$ be minimal central projections in $\bar{M}$ and $\bar{N}$, respectively, indexed so that we have $\theta(p_iM) = \bar{p}_i\bar{M}$ and $\theta(q_jN) = \bar{q}_j\bar{N}$ for each i and j; these equalities hold since $\theta : M \to \bar{M}$ is an isomorphism with $\theta(N) = \bar{N}$. Clearly, $q_jN \cong \bar{q}_j\bar{N}$ so that $q_jN \otimes 1_{\lambda_{ij}} \cong \bar{q}_j\bar{N} \otimes 1_{\bar{\lambda}_{ij}}$ for all i and j. Then p_iM contains $\bigoplus_{j=1}^n(q_jN \otimes 1_{\lambda_{ij}})$ which is isomorphic to $\bigoplus_{j=1}^n(\bar{q}_j\bar{N} \otimes 1_{\bar{\lambda}_{ij}})$ in $\bar{p}_i\bar{M}$, and since $p_iM \cong \bar{p}_i\bar{M}$, we have that $\lambda_{ij} = \bar{\lambda}_{ij}$. Thus,

$$\Lambda_N^M = [\lambda_{ij}] = [\bar{\lambda}_{ij}] = \Lambda_{\bar{N}}^{\bar{M}}.$$

$\square$

Definition 3.6.5. The *opposite algebra* of an algebra N is an algebra with the same elements as N where multiplication is done in the opposite order, and is denoted by N^{opp}. That is, N and N^{opp} are anti-isomorphic algebras via the anti-isomorphism defined by

$$xy \mapsto yx, \quad x, y \in N.$$

Proposition 3.6.6. *Let $N \subseteq M \subseteq M_1$ be an inclusion of von Neumann algebras, where M_1 was obtained by the basic construction on $N \subseteq M$. Then the inclusion matrix $\Lambda_M^{M_1}$ is the transpose of Λ_N^M.*

Proof. For the pair of inclusions $N \subseteq M$ and $N^{opp} \subseteq M^{opp}$ we clearly have $\Lambda_N^M = \Lambda_{N^{opp}}^{M^{opp}}$. Let $\pi^r : M \to \mathcal{L}(L^2(M, \tau))$ be the anti-representation of M on $L^2(M, \tau)$

defined by $\pi^r(x)\widehat{y} = \widehat{yx}$. It follows that M^{opp} is isomorphic to $\pi^r(M) \subseteq \mathcal{L}(L^2(M,\tau))$, and hence N^{opp} is isomorphic to $\pi^r(N)$. Then, by Lemma 3.6.4, $\Lambda_{N^{opp}}^{M^{opp}} = \Lambda_{\pi^r(N)}^{\pi^r(M)}$. The immediate implications of Proposition 3.6.1 and Proposition 3.6.2 are $\pi^\ell(M) = \pi^r(M)'$ and $\pi^r(N)' = M_1$, respectively. It follows that,

$$\Lambda_M^{M_1} = \Lambda_{\pi^r(M)'}^{\pi^r(N)'} = \left(\Lambda_{\pi^r(N)}^{\pi^r(M)}\right)^T = \left(\Lambda_N^M\right)^T,$$

where the second equality holds by Proposition 3.2.6. □

Corollary 3.6.7. *Let $N \subseteq M \subseteq M_1$ be an inclusion of von Neumann algebras with faithful trace τ on M, where M_1 is obtained by the basic construction (§3.6). We identify N, M, and M_1 by their direct sum decompositions:*

$$N \cong \bigoplus_{j=1}^{n} q_j N, \quad M \cong \bigoplus_{i=1}^{m} p_i M, \quad M_1 \cong \bigoplus_{j=1}^{n} \pi^\ell(q_j) M_1.$$

Suppose there is a faithful conditional expectation $E_N : M \to N$ and associated orthogonal projection $e_N : L^2(M,\tau) \to L^2(N,\tau)$. Then, the $\mathbb{C}$-linear map $\varphi : N \to e_N M_1 e_N$ defined by $\varphi(y) = \pi^\ell(y) e_N$ is an isomorphism of algebras.

Proof. Firstly, we show that φ is a morphism of algebras: For $x, y, z \in N$, we have

$$\varphi(xy) = \pi^\ell(xy)e_N = \pi^\ell(x)\pi^\ell(y)e_N^2 = \pi^\ell(x)e_N\pi^\ell(y)e_N = \varphi(x)\varphi(y)$$

and

$$\varphi(x) + \varphi(y) = (\pi^\ell(x) + \pi^\ell(y))e_N = \pi^\ell(x+y)e_N = \varphi(x+y).$$

If $\varphi(x) = 0$, then $x = \varphi(x)\widehat{1} = 0$ which implies φ is injective. From Corollary 3.6.3 we can write elements of M_1 as $\pi^\ell(x)e_N\pi^\ell(y)$, with $x, y \in M$. Then any elements of $e_N M_1 e_N$ looks like

$$e_N\pi^\ell(x)e_N\pi^\ell(y)e_N = \pi^\ell(E_N(x))\pi^\ell(E_N(y))e_N = \pi^\ell(x'y')e_N = \varphi(x'y'),$$

so φ is surjective. Hence, φ is an isomorphism. □

3.7 Markov traces

Let $N \subseteq M$ be a pair of von Neumann algebras with a faithful trace τ on M. Let $E_N : M \to N$ be the faithful τ-preserving conditional expectation, and let $e_N : L^2(M,\tau) \to L^2(N,\tau)$ be the associated Jones projection. By Corollary 3.6.3, the algebra M_1 obtained by the basic construction is generated by $\pi^\ell(x)e_N\pi^\ell(y)$, where $x, y \in M$ and $\pi^\ell : M \to \mathcal{L}(L^2(M,\tau))$ is the left-representation of M on $L^2(M,\tau)$. Let $\mathrm{Tr} : M_1 \to \mathbb{C}$ be a trace on M_1. Then, for all $x, y \in M$,

$$\mathrm{Tr}(\pi^\ell(x)e_N\pi^\ell(y)) = \mathrm{Tr}(\pi^\ell(yx)e_N) = \mathrm{Tr}(e_N\pi^\ell(yx)e_N) = \mathrm{Tr}(\pi^\ell(E_N(yx))e_N).$$

From this, we see that Tr is determined by its values on elements $\pi^\ell(z)e_N$ for $z \in N$. The trace τ is a *Markov trace of modulus* β if there exists a trace Tr on M_1 such that, for all $x \in M$,

$$\mathrm{Tr}(\pi^\ell(x)) = \beta \mathrm{Tr}(\pi^\ell(x)e_N) = \tau(x),$$

for some $\beta \in \mathbb{C}$. An inclusion $N \subseteq M$ with trace τ that satisfies these properties is called *Markov.*

Lemma 3.7.1. *Consider the von Neumann algebra inclusions $N \subseteq M \subseteq M_1$, the Jones projection e_N and the (left) representation of M given by $\pi^\ell(M)$ from above. If f_j is a minimal projection in N, then $\pi^\ell(f_j)e_N$ is minimal in M_1.*

Proof. The map $\varphi : N \to e_N M_1 e_N$ defined by by $\varphi(y) = \pi^\ell(y)e_N$ is an isomorphism and so is its restriction $\varphi_j : q_j N \to \pi^\ell(q_j)e_N M_1 e_N$. It follows that $\varphi_j(f_j) = \pi^\ell(f_j)e_N$ is a minimal projection in $\pi^\ell(q_j)e_N M_1 e_N$. But if e is a minimal projection in M_1 such that $e \leq \pi^\ell(f_j)e_N \leq \pi^\ell(q_j)e_N$, then

$$e = \pi^\ell(q_j)e_N e \pi^\ell(q_j)e_N,$$

which is contained inside $\pi^\ell(q_j)e_N M_1 e_N$. Then, by minimality of e and $\pi^\ell(f_j)e_N$ in $\pi^\ell(q_j)e_N M_1 e_N$, it follows that $e = \pi^\ell(f_j)e_N$. Hence, $\pi^\ell(f_j)e_N$ is minimal in M_1. □

Lemma 3.7.2. *There exists **at most** one trace* Tr *on M_1 such that $\beta\mathrm{Tr}(\pi^\ell(y)e_N) = \tau(y)$, for all $y \in N$. If such a trace exists, then $\beta\mathrm{Tr}(\pi^\ell(x)e_N) = \tau(x)$, for all $x \in M$. Moreover, if such a trace exists, and $\vec{r}$ describes* Tr *and $\vec{t}$ describes $\tau|_N$, then $\beta\vec{r} = \vec{t}$.*

Proof. Suppose there exists such a trace Tr on M_1. Then for $x \in M$ we have

$$\beta\mathrm{Tr}(\pi^\ell(x)e_N) = \beta\mathrm{Tr}(e_N\pi^\ell(x)e_N) = \beta\mathrm{Tr}(\pi^\ell(E_N(x))e_N) = \tau(E_N(x)) = \tau(x).$$

Let Tr on M_1 correspond to the vector $\vec{r}$. Let $\tau|_N$ correspond to the vector $\vec{t}$, and let f_j be a minimal projection in N so that $\tau(f_j) = t_j$. By 3.7.1 we have that $\pi^\ell(f_j)e_N$ is a minimal projection in M_1. Then

$$\beta\vec{r}_j = \beta\mathrm{Tr}(\pi^\ell(f_j)e_N) = \tau(f_j) = t_j.$$

Hence, $\beta\vec{r} = \vec{t}$.

Since $\vec{t}$ corresponds to a unique trace, then so too does $\beta^{-1}\vec{t}$. Hence, Tr is unique. □

Proposition 3.7.3. *Let $N \subseteq M \subseteq M_1$ be an inclusion of von Neumann algebras where M_1 is obtained by the basic construction, and let Λ denote the inclusion matrix for $N \subseteq M$. Let τ be a trace on M with trace vectors $\vec{s} \in \mathbb{R}^m_+$ describing τ on M and $\vec{t} \in \mathbb{R}^n_+$ describing $\tau|_N$. Then, the following are equivalent:*

(i) τ is a Markov trace of modulus β;

(ii) $\vec{s}\Lambda\Lambda^T = \beta\vec{s}$ *and* $\vec{t}\Lambda^T\Lambda = \beta\vec{t}$*.*

Proof. (i) $\Rightarrow$ (ii) Suppose τ is a Markov trace of modulus β with associated trace Tr on M_1 such that

$$\mathrm{Tr}(\pi^\ell(x)) = \tau(x) \quad \text{and} \quad \beta\mathrm{Tr}(\pi^\ell(x)e_N) = \tau(x),$$

for all $x \in M$, and let $\vec{r} \in \mathbb{C}^n$ be the vector corresponding to Tr on M_1. By Proposition 3.6.6 we know that the inclusion matrix for $M \subseteq M_1$ is Λ^T, and by Proposition 3.3.8 we have $\vec{s} = \vec{r}\Lambda^T$. From Lemma 3.7.2, we have $\vec{t} = \beta\vec{r}$, which gives us

$$\beta\vec{t} = \beta\vec{s}\Lambda = \beta\vec{r}\Lambda^T\Lambda = \vec{t}\Lambda^T\Lambda.$$

Thus,

$$\vec{s}\Lambda\Lambda^T = \vec{t}\Lambda^T = \beta\vec{r}\Lambda^T = \beta\vec{s},$$

as required.

(ii) $\Rightarrow$ (i) Suppose we have that $\vec{s}\Lambda\Lambda^T = \beta\vec{s}$ and $\vec{t}\Lambda^T\Lambda = \beta\vec{t}$. Set $\vec{r} = \beta^{-1}\vec{t}$ and let $\mathrm{Tr} : M_1 \to \mathbb{C}$ be the associated trace. Then Tr extends τ on M since

$$\vec{r}\Lambda^T = \beta^{-1}\vec{t}\Lambda^T = \beta^{-1}\vec{s}\Lambda\Lambda^T = \vec{s}.$$

Next, we consider the linear map $\tilde{\tau} : N \to \mathbb{C}$ defined by $\tilde{\tau}(y) = \beta\mathrm{Tr}(\pi^\ell(y)e_N)$. This map is a trace because for $x, y \in N$ we have

$$\begin{aligned}
\tilde{\tau}(xy) &= \beta\mathrm{Tr}(\pi^\ell(xy)e_N) \\
&= \beta\mathrm{Tr}(e_N\pi^\ell(x)\pi^\ell(y)) \\
&= \beta\mathrm{Tr}(\pi^\ell(x)e_N\pi^\ell(y)) \\
&= \beta\mathrm{Tr}(\pi^\ell(y)\pi^\ell(x)e_N) \\
&= \beta\mathrm{Tr}(\pi^\ell(yx)e_N) \\
&= \tilde{\tau}(yx).
\end{aligned}$$

For minimal projections $f_j \in q_jN$ we have

$$\tilde{\tau}(f_j) = \beta\mathrm{Tr}(\pi^\ell(f_j)e_N) = \beta r_j = t_j,$$

for all $j = 1, ..., n$, where the first equality holds by 3.7.1, which implies that $\tilde{\tau} = \tau|_N$. That is, Tr satisfies $\beta\mathrm{Tr}(\pi^\ell(y)e_N) = \tau(y)$, for all $y \in N$. Thus, by Lemma 3.7.2, Tr satisfies the Markov condition $\beta\mathrm{Tr}(\pi^\ell(x)e_N) = \tau(x)$, for all $x \in M$. □

Corollary 3.7.4. *Let $N \subseteq M \subseteq M_1$ be an inclusion of von Neumann algebras where M_1 is obtained by the basic construction, and where $N \subseteq M$ is Markov with Markov trace τ on M. Then, $M \subseteq M_1$ is a Markov inclusion. Moreover, if* Tr *is the Markov extension of τ on M_1, then* Tr *is the Markov trace for the inclusion $M \subseteq M_1$.*

Proof. Let Λ_N^M and $\Lambda_M^{M_1}$ denote the inclusion matrix for $N \subseteq M$ and $M \subseteq M_1$, respectively. Let $\vec{s} \in \mathbb{R}_+^m$ be the vector that describes τ on M and let $\vec{t} \in \mathbb{R}_+^n$ describe $\tau|_N$. Let $\vec{r} \in \mathbb{R}_+^n$ be the vector that describes the Markov extension of τ, Tr, on M_1. By Proposition 3.7.3, we must show that

$$\vec{r}\Lambda_M^{M_1}\left(\Lambda_M^{M_1}\right)^T = [M_1 : M]\vec{r} \quad \text{and} \quad \vec{s}\left(\Lambda_M^{M_1}\right)^T\Lambda_M^{M_1} = [M_1 : M]\vec{s}.$$

Recall that we have $\Lambda_M^{M_1} = \left(\Lambda_N^M\right)^T$, by Proposition 3.6.6. Note that,

$$[M_1 : M] = \|\Lambda_M^{M_1}\left(\Lambda_M^{M_1}\right)^T\| = \|\left(\Lambda_N^M\right)^T\Lambda_N^M\| = \|\left(\Lambda_N^M\right)^T\|^2 = \|\Lambda_N^M\|^2 = [M : N].$$

Then,

$$\vec{s}\left(\Lambda_M^{M_1}\right)^T\Lambda_M^{M_1} = \vec{s}\Lambda_N^M\left(\Lambda_N^M\right)^T = [M : N]\vec{s} = [M_1 : M]\vec{s},$$

and,

$$\vec{r}\Lambda_M^{M_1}\left(\Lambda_M^{M_1}\right)^T = \vec{r}\left(\Lambda_N^M\right)^T\Lambda_N^M = [M : N]\vec{t}\left(\Lambda_N^M\right)^T\Lambda_N^M = [M : N][M : N]\vec{t} = [M_1 : M]\vec{r},$$

where the second equality holds by Lemma 3.7.2. Hence, $M \subseteq M_1$ is a Markov inclusion with Markov trace Tr. □

We now relate the moduli of the Markov traces to indices of inclusions. For this we require some preliminaries on irreducible matrices.

Definition 3.7.5. A matrix is *reducible* if it can be placed into block upper-triangular form by conjugation of permutations. A square matrix that is not reducible is *irreducible.*

Remark 3.7.6. A matrix $A \in M_k(\mathbb{R}_+)$ is irreducible if and only if, for each $i, j \in \{1, ..., k\}$, there exists an integer p (depending on i and j) such that the ij^{th} entry of A^p is strictly positive [15, p. 12].

Definition 3.7.7. A *Perron-Frobenius* vector for an irreducible matrix $A \in M_k(\mathbb{R}_+)$, is an eigenvector $\xi \in \mathbb{R}_+^k$ with strictly positive entries.

Remark 3.7.8. Such a vector always exists for such a matrix, it is unique up to positive scalar multiplication, and corresponds to an eigenvalue which is simple and is equal to the spectral radius of A, i.e. the largest absolute value of its eigenvalues, denoted by $\rho(A)$ [14, §XIII]. For any square matrix A, $\|A\| = \sqrt{\rho(A^*A)}$, [13, §6.10, Corollary 1].

Lemma 3.7.9. *Let $A \in M_k(\mathbb{R}_+)$ be self-adjoint. Then, $\|A\| = \rho(A)$.*

Proof. If $A \in M_k(\mathbb{R}_+)$ is self-adjoint, it follows that $\rho(A^*A) = \rho(A^2) = \rho(A)^2$. Then, $\|A\| = \sqrt{\rho(A^*A)} = \rho(A)$. □

Definition 3.7.10. Let $A \in M_{m,n}(\mathbb{R}_+)$. Then, A is *pseudo-equivalent to* $X \in M_{m,n}(\mathbb{R}_+)$ if the rows and columns of A can be permuted in such a way that $A = X$.

It follows from this definition, that an inclusion matrix Λ_N^M, for a pair of von Neumann algebras $N \subseteq M$, is *pseudo-equivalent to* $X \in M_{m,n}(\mathbb{R}_+)$ if the minimal central projections of M and N can be indexed in such a way that $\Lambda_N^M = X$.

Definition 3.7.11. A matrix is *irredundant* if none of its rows and columns are zero. An irredundant matrix $X \in M_{m,n}(\mathbb{R})$ is *decomposable* if there exist integers $m', m'', n', n'' \geq 1$ with $m = m' + m''$ and $n = n' + n''$, as well as $X' \in M_{m',n'}(\mathbb{R})$ and $X'' \in M_{m'',n''}(\mathbb{R})$ such that X is pseudo-equivalent to $\begin{bmatrix} X' & 0 \\ 0 & X'' \end{bmatrix}$. A matrix is *indecomposable* if it is irredundant and not decomposable.

Definition 3.7.12. An an inclusion $N \subseteq M$ is *connected* if and only if $Z(M) \cap Z(N) = \mathbb{C}$.

Lemma 3.7.13. *The inclusion matrix, Λ_N^M, is indecomposable if and only if $N \subseteq M$ is a connected inclusion.*

Proof. If $\dim(Z(M) \cap Z(N)) > 1$, then there is a non-trivial projection $r \in Z(M) \cap Z(N)$. Choosing an appropriate indexing for the minimal central projections of M and N, one gets

$$p_1, ..., p_{m'}; \quad q_1, ..., q_{n'} \in rMr,$$

and

$$p_{m'+1}, ..., p_m; \quad q_{n'+1}, ..., q_n \in (1-r)M(1-r),$$

for some m', n' with $0 < m' < m$ and $0 < n' < n$. It follows that, for any entry, λ_{ij}, of the inclusion matrix, λ_N^M, we have $\lambda_{ij} = 0$, unless $1 \leq i \leq m'$ and $1 \leq j \leq n'$ or $m' + 1 \leq i \leq m$ and $n' + 1 \leq j \leq n$. Hence, Λ_N^M is decomposable.

On the other hand, if Λ_N^M is decomposable, say $\lambda_{ij} = 0$, unless $1 \leq i \leq m'$ and $1 \leq j \leq n'$ or $m' + 1 \leq i \leq m$ and $n' + 1 \leq j \leq n$, for some m', n' with $0 < m' < m$ and $0 < n' < n$. Then, we get the following equalities

$$\sum_{i=1}^{m'} p_i = \sum_{j=1}^{n'} q_j \quad \text{and} \quad \sum_{i=m'+1}^{m} p_i = \sum_{j=1}^{n} q_j.$$

It follows that each sum above is contained inside $Z(M) \cap Z(N)$ so that $Z(M) \cap Z(N) \neq \mathbb{C}$. □

Theorem 3.7.14. *Let $N \subseteq M$ be a connected von Neumann algebra inclusion with inclusion matrix Λ. Then, there exists a Markov trace of modulus β on M if and only if $\beta = [M : N] = \|\Lambda\|^2 = \|\Lambda\Lambda^T\|$. Any two Markov traces on M are proportional. Consequently there exists a unique tracial state.*

Proof. Since $Z(M) \cap Z(N) = \mathbb{C}$, it follows that Λ is indecomposable and $\Lambda\Lambda^T$ is irreducible [15, Lemma 1.3.2.b] and self-adjoint. Set $\beta = \|\Lambda\Lambda^T\| = \rho(\Lambda\Lambda^T)$ (by Lemma 3.7.9). By Perron-Frobenius theory, there exist $\vec{s}$ with only positive real entries such that $\vec{s}\Lambda\Lambda^T = \beta\vec{s}$, [14, §XIII]. It follows that $\vec{s}$ is a faithful trace on M, [15], and $\vec{t} = \vec{s}\Lambda$ corresponds to that trace's faithful restriction to N. Then, $\vec{t}\Lambda^T\Lambda = \vec{s}\Lambda\Lambda^T\Lambda = \beta\vec{s}\Lambda = \beta\vec{t}$. By Proposition 3.7.3, $\vec{s}$ corresponds to a trace of modulus β.

If tr is a Markov trace of modulus β, then it follows that the corresponding vector $\vec{s}$ is a Perron-Forbenius eigenvector of $\Lambda\Lambda^T$. In particular, by Remark 3.7.8 and Lemma 3.7.9 we have

$$\beta = \rho(\Lambda\Lambda^T) = \|\Lambda\Lambda^T\| = [M : N].$$

□

Lemma 3.7.15. *Let τ be a Markov trace of modulus β on the von Neumann algebra inclusion $N \subseteq M$, and set $M_1 = \langle M, e_N \rangle$. Let $\tau_1 : M_1 \to \mathbb{C}$ be the Markov extension of τ to a trace on M_1, let E_M be the conditional expectation from M_1 to M, and e_N be the Jones projection from $L^2(M, \tau)$ to $L^2(N, \tau)$. Then $E_M(e_N) = \beta^{-1}\mathrm{id}$.*

Proof. Let $x, y \in M_1$ and $\widehat{x}, \widehat{y} \in L^2(M_1, \tau_1)$. We may identify $L^2(M_1, \tau_1)$ by $L^2(N, \tau_1) \oplus L^2(N, \tau_1)^\perp$, where orthogonality is taken with respect to the bilinear form $\langle \cdot, \cdot \rangle_{\tau_1}$ on $L^2(M_1, \tau_1)$. Then, for all $x \in N$, one has

$$\begin{aligned}
\tau_1(E_M(e_N)x - \beta^{-1}x) &= \tau_1(E_M(e_N)x) - \beta^{-1}\tau_1(x) \\
&= \tau_1(E_M(xe_N)) - \beta^{-1}\tau(x) \\
&= \tau_1(e_M x e_N) - \tau_1(xe_N) \\
&= \tau_1(xe_N) - \tau_1(xe_N) \\
&= 0.
\end{aligned}$$

So, $E_M(e_N) - \beta^{-1}\mathrm{id}$ is in $N^\perp$. But, $E_M(e_N) - \beta^{-1}\mathrm{id}$ is also in N. Therefore, $E_N(e_N) - \beta^{-1}\mathrm{id} = 0$. Hence, $E_M(e_N) = \beta^{-1}\mathrm{id}$. □

3.8 Pimsner-Popa basis

In this section, we present the Pimsner-Popa basis [2] for a von Neumann algebra inclusion.

We will write x for $\pi^\ell(x)$, $x \in M$, when convenient.

Lemma 3.8.1. *If $x_1 \in M_1$, then there exists a unique element $x_0 \in M$ such that $x_1 e_N = x_0 e_N$, this element is given by $x_0 = [M : N]E_M(x_1 e_N)$*

Proof. See Lemma 4.3.1 (i) in [20] □

Proposition 3.8.2. *The following are equivalent for a finite set $\{\lambda\}_{i \in \mathcal{I}} \subset M$:*

(i) $1 = \sum_{i\in\mathcal{I}} \lambda_i^* e_N \lambda_i$

(ii) $x = \sum_{i\mathcal{I}} E_N(x\lambda_i^*)\lambda_i$, *for all* $x \in M$.

(iii) $x = \sum_{i\in\mathcal{I}} \lambda_i^* E_N(\lambda_i x)$, *for all* $x \in M$

Proof. We denote by $\Lambda_\tau(x) \in L^2(M,\tau)$ the canonical image of x in M. (i) $\Rightarrow$ (iii). We assume (i) holds. Let $x \in M$. Then,

$$\Lambda_\tau(x) = \sum_i \lambda_i^* e_N \lambda_i \Lambda_\tau(x) = \Lambda_\tau(\sum_i \lambda_i^* E_N(\lambda_i x)).$$

By injectivety of $\Lambda_\tau(\cdot)$, we get (iii).

(iii) $\Rightarrow$ (ii).

We assume (iii) holds. Let $x \in M$. Then,

$$x^* = \sum_i E_N(\lambda_i x)^* \lambda_i = \sum_i E_N(x^* \lambda^*)\lambda_i.$$

Which gives us (ii).

(ii) & (iii) $\Rightarrow$ (i).

We assume (iii) holds. Let $x, y \in M$. Then,

$$\begin{aligned}(xe_N y)\Big(\sum_{i\in\mathcal{I}} \lambda_i^* e_N \lambda_i\Big) &= x \sum_{i\in\mathcal{I}} e_N y \lambda_i^* e_N \lambda_i \\ &= x \sum_{i\in\mathcal{I}} E_N(y\lambda_i^*) e_N \lambda_i \\ &= xe_N \sum_{i\in\mathcal{I}} E_N(y\lambda_i^*)\lambda_i \\ &= xe_N y \quad \text{(by (ii))}.\end{aligned}$$

Recall from Corollary 3.6.3 that the von Neumann algebra obtained from the basic construction on the inclusion $N \subseteq M$ is the linear span $\{xe_N y : x, y \in M\}$. It follows that, $\sum_{i\in\mathcal{I}} \lambda_i^* e_N \lambda_i = 1$. □

Definition 3.8.3. We call any finite set $\{\lambda_i\}_{i\in\mathcal{I}} \subset M$ which satisfies the conditions in Proposition 3.8.2 a *Pimsner-Popa basis* for M/N.

A Pimsner-Popa basis for a von Neumann algebra inclusion is a basis in the following sense. Let $N \subseteq M$ be a Markov inclusion, τ a Markov trace on M, and $E_N : M \to N$ the unique τ-preserving conditional expectation. Then,

$$\langle x, y\rangle_N \equiv E_N(xy^*),$$

defines a sesquilinear N-valued form $\langle \cdot, \cdot \rangle_N : M \times M \to N$, turning M into a left

Hilbert C^*-module [22] with respect to left multiplication by N on M. The Pimnser-Popa basis condition says that for any $x \in M$

$$x = \sum_i E_N(x\lambda_i^*)\lambda_i = \sum_i \langle x, \lambda_i \rangle_N \lambda_i,$$

so that $\{\lambda_i\}$ forms a "basis" for the left Hilbert C^*-module M (over N), analogous to a basis decomposition in a Hilbert space.

Definition 3.8.4. A Pimsner-Popa basis $\{\lambda_i\}_{i\in\mathcal{I}}$ for a Markov inclusion $N \subseteq M$ is orthonormal when

$$\delta_{ij} 1_M \langle \lambda_i, \lambda_j \rangle_N.$$

Remark 3.8.5. Let $N \subseteq \mathcal{M}_d$ be a von Neumann algebra inclusion and let $\{\lambda_i\}_{i\in\mathcal{I}}$ be a Pimsner-Popa basis for the inclusion. Then,

$$[\mathcal{M}_d : N]1_d = \sum_{i\in\mathcal{I}} \lambda_i^* \lambda_i = |\mathcal{I}| 1_d.$$

Note that this is independent of choice of the Pimsner-Popa basis. [19, Definition 2.6]

To show existence of a Pimsner-Popa basis for a given inclusion $N \subseteq M$, we must first establish some facts about projections in M and a bit of notation to go along with it.

Given $x \in M$, the smallest projection $e \in M$ with $ex = x$ is the *left support of* x and denoted by $s_\ell(x)$. The *right support of* x, denoted by $s_r(x)$, is the smallest projection $f \in M$ with $xf = x$. The *central support* $z(e)$ of a projection $e \in M$ is the smallest central projection in M majorizing e. Two projections $p, q \in M$ are *equivalent* if there exists $v \in M$ such that $v^*v = p$ and $vv^* = q$. We denote equivalent projections by $p \sim q$.

Lemma 3.8.6. *Given two projections $e \sim f$ in M with $v^*v = e$ and $vv^* = f$ for some $v \in M$, then $ve = v$ and $v^*f = v^*$.*

Proof. Let $g = ve - v$. Then,

$$g^*g = e^3 - e^2 - e^2 - e = 0,$$

which implies $g = 0$. Similarly, if $h = v^*f - v^*$, then we find that $h^*h = 0$, which implies $h = 0$. Hence, $ve = v$ and $v^*f = v^*$. $\square$

Lemma 3.8.7. *For two projections e and f of a von Neumann algebra M the following are equivalent:*

- *$z(e)$ and $z(f)$ are not orthogonal*
- *eMf is not zero*

- *There exist nonzero projections $e_1 \leq e$ and $f_1 \leq f$ in M such that $e_1 \sim f_1$.*

Proof. See Lemma 1.7 in [28]. □

Theorem 3.8.8. *If $N \subseteq M$ is a Markov inclusion, then there exists a Pimsner-Popa basis for M/N.*

Proof. Let $e = e_N$. Since e has central support $z(e) = 1$ in $M_1 = \langle M, e_N \rangle$, then we may apply Lemma 3.8.7 to e and any projection in $\langle M, e_N \rangle$. In particular, we may apply the lemma to e and $f = 1 - e$, to obtain v_1 such that $v_1^* v_1 = e_1 \leq e$ and $v_1 v_1^* = f_1 \leq f$. The second inequality implies that $1 \leq e_N + v_1 v_1^*$. If $1 = e_N + v_1 v_1^*$, then we stop here. Otherwise, repeat the application of Lemma 3.8.7 to $e = e_N$ and $f = 1 - e_N - v_1 v_1^*$. We can continue this process until we fill the space, i.e. $1 = \sum_{i=0}^{d-1} v_i v_i^*$, with $v_0 = e_N$.

Since $v_i^* v_i \leq e_N$, we have $v_i^* v_i = e_N v_i^* v_i e_N = v_i^* v_i e_N = e_N v_i^* v_i$. Then,

$$\sum_{i=0}^{d-1} v_i e_N v_i^* = \sum_{i=0}^{d-1} (v_i v_i^* v_i) e_N v_i^* = \sum_{i=0}^{d-1} v_i (v_i^* v_i e_N) v_i^* = \sum_{i=0}^{d-1} v_i v_i^* v_i v_i^* = \sum_{i=0}^{d-1} v_i v_i^* = 1_{L^2(M)}.$$

Since $v_i \in M_1$ for all i, by Lemma 3.8.1 we have that for each v_i there exists a $\lambda_i \in M$ such that $v_i e_N = \lambda_i^* e_N$. Hence, $\{\lambda_i\}_{i=0}^{d-1}$ is a basis for M/N. □

3.9 The canonical shift & iterating the basic construction

In this section, we introduce an isomorphism between relative commutants, known as the cannonical shift. This isomorphism is the basis for quantum teleportation in the von Neumann algebraic setting which will become clear in §4.5. However, we begin with showing that given a Markov inclusion $N \subseteq M$ and the von Neumann algebra M_1 obtained by the basic construction so that $N \subseteq M \subseteq M_1$, then we may apply a second iteration of the basic construction to the inclusion $M \subseteq M_1$.

Iterating the basic construction

Let $N \subseteq M \subseteq M_1$ be a set of inclusions where M_1 is the result of the basic construction applied to the Markov inclusion $N \subseteq M$ with Markov trace τ. Then, by Corollary 3.7.4, $M \subseteq M_1$ is a Markov inclusion where the Markov trace is the extension of τ on M_1, τ_1. Then, we may apply the basic construction to the inclusion $M \subseteq M_1$ to obtain M_2. Since $M \subseteq M_1$ is Markov, then so is $M_1 \subseteq M_2$ by Corollary 3.7.4. Then, the basic construction can be performed for the inclusion $M \subseteq M_1$ obtaining the von Neumann algebra M_3, and Markov inclusion $M_2 \subseteq M_3$. This process can be repeated indefinitely, creating what is known in the literature as a Jones tower [20],

$$N \subseteq M \subseteq M_1 \subseteq M_2 \subseteq ... \subseteq M_n \subseteq M_{n+1} \subseteq ...$$

However, for the purposes of this paper, we restrict ourselves to the first two iterations, i.e. up to M_2. For more on the Jones tower see [20]. In what follows we write N, N', M, M' to mean $\pi^\ell(N)$, $\pi^\ell(N)'$, $\pi^\ell(M)$, and $\pi^\ell(M)'$, respectively, when convenient.

The canonical shift

There is a canonical isomorphism between $N' \cap M$ and $M_1' \cap M_2$. Let $J_M : L^2(M,\tau) \to L^2(M,\tau)$ denote the anti-unitary map $J_M\widehat{x} = \widehat{x^*}$, for all $x \in M$, and let $J_{M_1} : L^2(M_1,\tau_1) \to L^2(M_1,\tau_1)$ be the anti-unitary map $J_{M_1}\widehat{y} = \widehat{y^*}$, for all $y \in M_1$. It follows from Proposition 3.6.1 and Proposition 3.6.2 that, $J_M(N' \cap M)J_M = M_1 \cap M'$. Similarly, we have $J_{M_1}(M' \cap M_1)J_{M_1} = M_2 \cap M_1'$. Let $\gamma_1 : N' \cap M \to M' \cap M_1$ and $\gamma_2 : M' \cap M_1 \to M_1' \cap M_2$ be the anti-isomorphisms defined by $\gamma_1(x) = J_M x^* J_M$ and $\gamma_2(y) = J_{M_1} y^* J_{M_1}$.

Definition 3.9.1. The isomorphism defined by the composite map

$$\Gamma = \gamma_2 \circ \gamma_1 : N' \cap M \to M_1' \cap M_2$$

is referred to as the *canonical shift*.

Lemma 3.9.2. *The following two statements are equivalent for any $x \in M$:*

(i) $x \in N' \cap M$

(ii) $\pi^\ell(x)e_N = \gamma_1(x)e_N$.

Proof. First, let $x \in N' \cap M$ and $y \in M$. Then

$$\begin{aligned}
\pi^\ell(x)e_N\widehat{y} &= \pi^\ell(x)\widehat{E(y)} \\
&= \widehat{xE(y)} \\
&= J_M\widehat{E(y^*)x^*} \\
&= J_M\widehat{x^*E(y^*)} \\
&= J_M\pi^\ell(x)^*\widehat{E(y^*)} \\
&= J_M\pi^\ell(x)^*J_M\widehat{E(y)} \\
&= J_M\pi^\ell(x)^*J_M e_N\widehat{y} \\
&= \gamma_1(x)e_N\widehat{y},
\end{aligned}$$

Hence, (i) implies (ii). On the other hand, if $x \in M$, and $\pi^\ell(x)e_N\widehat{y} = \gamma_1(x)e_N\widehat{y}$, then

$$\begin{aligned}
\pi^\ell(x)e_N\widehat{y} &= \gamma_1(x)e_N\widehat{y} \\
\pi^\ell(x)\widehat{E(y)} &= J_M\pi^\ell(x)^*J_M\widehat{E(y)} \\
\widehat{xE(y)} &= J_M\pi^\ell(x)^*\widehat{E(y^*)} \\
\widehat{xE(y)} &= J_M\widehat{x^*E(y^*)} \\
\widehat{xE(y)} &= \widehat{E(y)x},
\end{aligned}$$

so $x \in N' \cap M$. Hence, (ii) implies (i). □

Lemma 3.9.2 implies that any unit vector $\psi \in L^2(N,\tau)$ is a perfectly correlated Einstein-Podolski-Rosen (EPR) state with respect to the commuting algebras $N' \cap M$ and $M' \cap M_1$, which means that any self-adjoint $x \in N' \cap M$ has an "EPR double" $x' \in M' \cap M_1$ for which

$$\langle (x - x')^2\psi, \psi\rangle = 0.$$

Indeed,

$$\langle x\psi, \psi\rangle = \langle xe_N\psi, \psi\rangle = \langle \gamma_1(x)e_N\psi, \psi\rangle = \langle \gamma_1(x)\psi, \psi\rangle,$$

which implies the doubling condition. For details on perfect correlation see [1] for the type I case and [33] for the general von Neumann algebraic setting (both works of course building on the seminal paper [11] of Einstein–Podolski–Rosen).

More generally, if a unitary u belongs to the normaliser

$$\mathcal{N}_M(N' \cap M) := \{u \in M \mid u(N' \cap M)u^* = N' \cap M\}$$

of the relative commutant $N' \cap M$, then $u^*\psi$ is also an EPR state with respect to the same commuting algebras:

$$\gamma_1(uxu^*)u^*\psi = u^*\gamma_1(uxu^*)\psi = u^*\gamma_1(uxu^*)e_N\psi = u^*(uxu^*)\psi = xu^*\psi, \quad x \in N' \cap M. \tag{3.5}$$

In other words, the EPR double of $x \in N' \cap M$ is $\gamma_1(uxu^*) \in M' \cap M_1$. Moreover, the restricted vector state $\omega_{u^*\psi}|_{N'\cap M}$ is tracial, which is often viewed as a form of maximal entanglement in the commuting operator framework (see e.g., [21, §V.A] or [9, §6]).

3.10 Example: $\mathbb{C}1_n \subseteq \mathcal{M}_n$

In this subsection, we begin with the inclusion $N = \mathbb{C}1_n \subseteq \mathcal{M}_n = M$, and explicitly describe the corresponding constructions and objects we've discussed so far.

Inclusion matrix and trace

Since $\mathbb{C}1_n$ and $\mathcal{M}_n$ are factors, it follows that the inclusion matrix Λ_N^M is a 1-by-1 matrix, where the single entry is given by

$$\mu = \Big(\frac{\dim(\mathcal{M}_n)}{\dim(\mathbb{C}1_n)}\Big)^{1/2} = n,$$

that is, $\Lambda_N^M = \big[n\big]$.

A minimal projection in M is the diagonal matrix unit e_{11}, so the trace vector associated to τ on M is the one-tuple $\vec{s} = (\tau(e_{11})) = (n^{-1})$. The unique minimal projection in $\mathbb{C}1_n$ is 1_n. So, the trace vector $\vec{t}$ associated to $\tau|_{\mathbb{C}1_n}$ is the singleton $\vec{t} = (\tau(1_n)) = (1)$. Note that $\vec{t} = \vec{s}\Lambda_N^M$.

GNS construction

The trace vector associated to τ on M is the singleton $\vec{s} = (s) = (n^{-1})$. Let $L^2(M,\tau)$ and $L^2(N,\tau)$ be the Hilbert spaces obtained by the GNS construction. From Section 3.4, we get for any $x, y \in M$,

$$\langle \widehat{x}, \widehat{y}\rangle_\tau = \sum_{i,j=1}^{n} s\bar{y}_{ji}x_{ij} = s\sum_{i,j=1}^{n} \bar{y}_{ji}x_{ij} = n^{-1}\langle \widehat{x}, \widehat{y}\rangle,$$

where $\langle\cdot,\cdot\rangle$ is the standard inner-product on $\mathbb{C}^n \otimes \mathbb{C}^n$. It follows that,

$$\widehat{x} \mapsto \frac{1}{\sqrt{n}}\sum_{i,j=1}^{n} x_{ij}e_i \otimes e_j \in \mathbb{C}^n \otimes \mathbb{C}^n.$$

So we have,

$$L^2(M,\tau) \cong (\mathbb{C}^n \otimes \mathbb{C}^n, n^{-1}\langle\cdot,\cdot\rangle).$$

Recall from Section 3.4, that the GNS construction for N with faithful trace $\tau|_N$ has the following congruencies

$$L^2(N,\tau) = (\widehat{N}, \langle\cdot,\cdot\rangle_\tau) \cong (\widehat{tN}, \langle\cdot,\cdot\rangle),$$

where t is the sum of products between the square root of the entries of the trace vectors $\vec{t}$ and the corresponding minimal central projection of N,

$$t = \sqrt{1}\cdot 1_n = 1_n.$$

It follows that, the Hilbert space $L^2(N,\tau)$ is isomorphic to the Hilbert space $\mathbb{C}$ with standard inner product.

The left-representation, $\pi^\ell : M \to \mathcal{L}(L^2(M,\tau))$, is such that

$$\pi^\ell(x)\widehat{y} = \widehat{xy}, \quad x, y \in M.$$

Then, for $x, y \in M$ where $x = \sum_{i,j=1}^n x_{ij}|e_i\rangle\langle e_j|$ and $y = \sum_{u,v=1}^n y_{uv}|e_u\rangle\langle e_v|$, we have

$$\begin{aligned}
\pi^\ell(x)\widehat{y} &= \widehat{xy} \\
&= \sum_{i,j,u,v=1}^n x_{ij}y_{uv}\langle e_j|e_u\rangle\widehat{|e_i\rangle\langle e_v|} \\
&= n^{-1/2}\sum_{i,j,u,v=1}^n x_{ij}y_{uv}\langle e_j|e_u\rangle|e_i\rangle \otimes |e_v\rangle \\
&= n^{-1/2}\sum_{i,j,u,v=1}^n x_{ij}y_{uv}(|e_i\rangle\langle e_j| \otimes 1)(|e_u\rangle \otimes |e_v\rangle) \\
&= (x \otimes 1)\widehat{y}.
\end{aligned}$$

Hence, $\pi^\ell(M) = \mathcal{M}_n \otimes 1_n \subseteq \mathcal{L}(L^2(M,\tau))$.

The right-representation of M on $L^2(M,\tau)$, gives us

$$\begin{aligned}
\pi^r(x)\widehat{y} &= \widehat{yx} \\
&= \sum_{u,v,i,j=1}^{n-1} y_{uv}x_{ij}\langle e_v|e_i\rangle\widehat{|e_u\rangle\langle e_j|} \\
&= \sum_{u,v,i,j=1}^{n-1} y_{uv}x_{ij}\langle e_v|e_i\rangle|e_u\rangle \otimes |e_j\rangle \\
&= \sum_{u,v,i,j=1}^{n-1} y_{uv}x_{ij}\langle e_i|e_v\rangle|e_u\rangle \otimes |e_j\rangle \\
&= \sum_{u,v,i,j=1}^{n-1} y_{uv}x_{ij}(1 \otimes |e_j\rangle\langle e_i|)(|e_u\rangle \otimes |e_v\rangle) \\
&= (1 \otimes x^t)\widehat{y}
\end{aligned}$$

Thus, $\pi^r(x) = 1 \otimes x^t$ for all $x \in M$.

Jones projection

We claim that the Jones projection is given by

$$e_N = |\psi_n\rangle\langle\psi_n| = n^{-1}\sum_{t,k=1}^n e_{tk} \otimes e_{tk}.$$

For any $x \in M$, we get

$$
\begin{aligned}
|\psi_n\rangle\langle\psi_n|\widehat{x} &= |\psi_n\rangle\langle\psi_n| \cdot 1/\sqrt{n} \cdot \sum_{i,j=1}^{n} x_{ij} e_i \otimes e_j \\
&= (n\sqrt{n})^{-1} \cdot \sum_{i,j,t,k=1}^{n} x_{ij} e_{tk} e_i \otimes e_{tk} e_j \\
&= (n\sqrt{n})^{-1} \cdot \sum_{i,j,t,k=1}^{n} x_{ij} \delta_{ki} e_t \otimes \delta_{kj} e_t \\
&= (n\sqrt{n})^{-1} \cdot \sum_{t,k=1}^{n} x_{kk} \delta_{ki} e_t \otimes \delta_{kj} e_t \quad (i = k = j) \\
&= \big(n^{-1} \cdot \sum_{k=1}^{n} x_{kk}\big) \cdot 1/\sqrt{n} \cdot \sum_{t=1}^{n} e_t \otimes e_t \\
&= \tau(x) \cdot 1/\sqrt{n} \cdot \sum_{t=1}^{n} e_t \otimes e_t \\
&= \tau(x)\widehat{1}_n,
\end{aligned}
$$

which is contained in $L^2(N, \tau)$. Clearly, $|\psi_n\rangle\langle\psi_n|$ acts as the identity on $\mathbb{C}\widehat{1}_n$. Hence, the Jones projection $e_N : L^2(M, \tau) \to L^2(N, \tau)$, is defined by $e_N = |\psi_n\rangle\langle\psi_n|$.

Conditional expectation

The conditional expectation $E_N : M \to N$ is induced by the Jones projection such that $e_N\widehat{x} = \widehat{E_N(x)}$, for all $x \in M$. It follows that, E_N is defined by $x \mapsto \tau(x) \cdot 1_n$,

$$e_N\widehat{x} = \tau(x)\widehat{1}_n = \widehat{\tau(x)1_n} = \widehat{E_N(x)}.$$

Basic construction

Recall that $\widehat{1}_n$ is cyclic for $M \otimes 1_n = \pi^\ell(M)$. So, $Me_NM = \operatorname{span}\{(x \otimes 1_n)|\psi_n\rangle\langle\psi_n|(y \otimes 1_n)\}$ contains all rank-1 operators $|\xi\rangle\langle\eta|$, and hence all of $\mathcal{L}(L^2(M, \tau))$. Thus, $M_1 = \mathcal{L}(L^2(M, \tau)) = \mathcal{M}_n \otimes \mathcal{M}_n$.

Markov trace

The inclusion $\mathbb{C}1_n \subseteq \mathcal{M}_n$ is connected, so it has Markov trace, which is necessarily the unique one with Markov extension $\tau_1 = \tau \otimes \tau$ on $\mathcal{M}_n \otimes \mathcal{M}_n$.

Pimsner-Popa basis

The generalized Pauli-X and Pauli-Z operators on a n-dimensional Hilbert space are defined by the unitary n-by-n matrices

$$X = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{bmatrix} \qquad Z = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & \alpha & 0 & \dots & 0 & 0 \\ 0 & 0 & \alpha^2 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha^{n-2} & 0 \\ 0 & 0 & 0 & \dots & 0 & \alpha^{n-1} \end{bmatrix}.$$

where α is the n^{th}-root of unity, $\alpha = \exp(2\pi i/n)$. Consider the set of unitaries $\{X^r Z^s\}_{r,s=1}^n$ in M. Then,

$$\begin{aligned}
\sum_{r,s=0}^{n-1} (X^r Z^s \otimes 1)^* e_N (X^r Z^s \otimes 1) &= n^{-1} \sum_{r,s,i,j=1}^{n} (X^r Z^s \otimes 1)^* |e_i\rangle\langle e_j| \otimes |e_i\rangle\langle e_j| (X^r Z^s \otimes 1) \\
&= n^{-1} \sum_{r,s,i,j=1}^{n} Z^{-s} X^{-r} |e_i\rangle\langle e_j| X^r Z^s \otimes |e_i\rangle\langle e_j| \\
&= n^{-1} \sum_{r,s,i,j=1}^{n} Z^{-s} |e_{i-r}\rangle\langle e_{j-r}| Z^s \otimes |e_i\rangle\langle e_j| \\
&= n^{-1} \sum_{r,s,i,j=1}^{n} \omega^{-(i-r-1)s} \omega^{(j-r-1)s} |e_{i-r}\rangle\langle e_{j-r}| \otimes |e_i\rangle\langle e_j| \\
&= n^{-1} \sum_{r,s,i,j=1}^{n} \omega^{(j-r-1-i+r+1)s} |e_{i-r}\rangle\langle e_{j-r}| \otimes |e_i\rangle\langle e_j| \\
&= n^{-1} \sum_{r,s,i,j=1}^{n} \omega^{(j-i)s} |e_{i-r}\rangle\langle e_{j-r}| \otimes |e_i\rangle\langle e_j| \\
&= n^{-1} \sum_{i,j=1}^{n} \underbrace{\Big(\sum_{s=1}^{n} \omega^{(j-i)s} \Big)}_{\text{is 0 if } i \neq j \text{, and is } n \text{ if } i=j} \Big(\sum_{r=1}^{n} |e_{i-r}\rangle\langle e_{j-r}| \Big) \otimes |e_i\rangle\langle e_j| \\
&= n^{-1} n \sum_{i=1}^{n} \sum_{r=1}^{n} |e_{i-r}\rangle\langle e_{i-r}| \otimes |e_i\rangle\langle e_i| \\
&= 1_{n^2}
\end{aligned}$$

Thus, $\{X^r Z^s\}_{r,s=1}^n$ forms a Pimsner-Popa basis for $\mathcal{M}_n/\mathbb{C}1_n$.

Canonical shift & iterating the basic construction

Let $L^2(M_1, \tau_1)$ and $L^2(M, \tau_1)$ be the Hilbert spaces obtained by the GNS construction for the inclusion $M \subseteq M_1$, and where τ_1 is as above. They are both equipped with the sesquilinear inner-product

$$\begin{aligned} \langle w \otimes x, y \otimes z \rangle_{\tau_1} &= \tau_1(y^* w \otimes z^* x) \\ &= \tau(y^* w)\tau(z^* x) \\ &= \langle \widehat{w}, \widehat{y} \rangle_{n^2} \langle \widehat{x}, \widehat{z} \rangle_{n^2} \\ &= \langle \widehat{w} \otimes \widehat{x}, \widehat{y} \otimes \widehat{z} \rangle_{n^4}. \end{aligned}$$

Thus, $L^2(M_1, \tau_1) \cong \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}^n$. It follows that,

$$L^2(M, \tau_1) \cong \mathbb{C}^n \otimes \mathbb{C}^n \otimes e_N(\mathbb{C}^n \otimes \mathbb{C}^n) = \mathbb{C}^n \otimes \mathbb{C}^n \otimes \mathbb{C}\psi_n.$$

The Jones projection $e_M : L^2(M_1, \tau_1) \to L^2(M, \tau_1)$ is given by $e_M = 1_n \otimes 1_n \otimes e_N$. The natural embedding of M_1 in $L^2(M_1, \tau_1)$ is defined by the map

$$\widehat{\widehat{x}} = \sum_{i,j,t,k=1}^{n} x_{ijtk} \widehat{|e_i\rangle\langle e_j|} \otimes \widehat{|e_t\rangle\langle e_k|} = \sum_{i,j,t,k=1}^{n} x_{ijtk} |e_i\rangle \otimes |e_j\rangle \otimes |e_t\rangle \otimes |e_k\rangle.$$

It follows that, $M_2 = \mathcal{M}_n \otimes \mathcal{M}_n \otimes \mathcal{M}_n$.

3.11 Example: $\mathcal{D}_n \subseteq \mathcal{M}_n$

Consider the inclusion $N = \mathcal{D}_n \subseteq \mathcal{M}_n = M$, where $\mathcal{D}_n$ denotes the n-by-n diagonal matrices over $\mathbb{C}$, and describe the corresponding objects discussed above.

Inclusion matrix & Markov trace

The central minimal projections, $q_1, ..., q_n$, of $\mathcal{D}_n$ are given by $q_j = e_{jj}$, where e_{jj} is the j^{th} diagonal matrix unit. Since M is a factor it only possesses one minimal central projection, namely, the identity matrix 1_n. The inclusion matrix, Λ, for $\mathcal{D}_n \subseteq \mathcal{M}_n$ is the 1-by-n matrix with all entries equal to 1; $\Lambda = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix} \in M_{1\times n}$.

The Markov trace is the normalized trace, τ, on $\mathcal{M}_n$, with trace vector $\vec{s} = (1/n)$. The trace vector associated to $\tau|_{\mathcal{D}_n}$ is the n-tuple $\vec{t} = \vec{s}\Lambda = \begin{pmatrix} 1/n & 1/n & \dots & 1/n \end{pmatrix}$.

GNS construction

The GNS construction yields the following Hilbert spaces

$$\mathcal{L}(M, \tau) \cong \mathbb{C}^n \otimes \mathbb{C}^n, \text{ and}$$

$$\mathcal{L}(N,\tau) \cong \bigoplus_{j=1}^{n} \mathcal{L}(\mathbb{C},\tau) \cong \bigoplus_{j=1}^{n} \mathbb{C},$$

with the embedding map for M into $L^2(M,\tau)$

$$M_n \ni x \mapsto \widehat{x} = \frac{1}{\sqrt{n}} \sum_{i,j=1}^{n} x_{ij} |e_i\rangle \otimes |e_j\rangle \in L^2(M,\tau), \text{ and}$$

$$\mathcal{D}_n \ni y \mapsto \widehat{y} = \sum_{j=1}^{n} y_j |e_j\rangle \otimes |e_j\rangle \in L^2(N,\tau).$$

The left and right representation of $\mathcal{M}_n$ on $L^2(\mathcal{M}_n,\tau)$ are $\pi^\ell(\mathcal{M}_n) = \mathcal{M}_n \otimes 1_n$ and $\pi^r(\mathcal{M}_n) = 1_n \otimes \mathcal{M}_n$, where $\pi^\ell(x) = x \otimes 1_n$ and $\pi^r(x) = 1_n \otimes x^t$, for all $x \in M$ (see Example 3.10 for details).

Jones projection & conditional expectation

The Jones projection $e_N : L^2(\mathcal{M}_n,\tau) \to L^2(\mathcal{D}_n,\tau)$ is

$$e_N = \sum_{k=1}^{n} |e_k\rangle\langle e_k| \otimes |e_k\rangle\langle e_k|,$$

and induces the unique τ-preserving conditional expectation $E_N : \mathcal{M}_n \to \mathcal{D}_n$ defined by

$$E_N(x) = \sum_{j=1}^{n} q_j x q_j, \quad x \in \mathcal{M}_n.$$

Basic construction

Let M_1 denote the von Neumann algebra resulting from the basic construction. Note that $N' \cap M = \mathcal{D}_n$ and $\mathcal{L}(L^2(M,\tau)) = \mathcal{M}_n \otimes \mathcal{M}_n$. We know that

$$N' \cap M \cong M' \cap M_1 \subseteq 1_n \otimes \mathcal{M}_n.$$

Then, $M' \cap M_1 = 1_n \otimes \mathcal{D}_n$. It follows that, $M_1 = \mathcal{M}_n \otimes \mathcal{D}_n$.

Pimsner-Popa basis

The set $\{\lambda_i\}_{i=1}^n = \{X^{i-1}\}_{i=1}^n$, where X is the generalized Pauli-X operator in $\mathcal{M}_n$ (see Example 3.10 for more details), is a Pimsner-Popa basis for $\mathcal{M}_n/\mathcal{D}_n$. Note that the action of X^{i-1} on diagonal elements of $\mathcal{M}_n$ is a cyclic shift of the diagonal entries by $i-1$, for all $i = 1, ..., n$. This observation will be important for Theorem 5.1.4.

Chapter 4

Quantum teleportation

Quantum teleportation was first developed by Bennett, et al. in the celebrated 1993 paper *Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels* [4]. In this chapter, we recast Bennett, et al.'s original teleportation identity into a von Neumann algebraic setting using the constructions and objects presented in previous chapters of the present paper.

We begin by introducing the mathematics of quantum systems necessary to discuss quantum teleportation (§4.1). We then introduce the notion of maps consisting of local operations and classical communication (§4.2) in preparation to discuss the quantum teleportation protocol presented in [4] (§4.3). The teleportation protocol we represent is then repackaged into the Heisenberg representation, [27], of quantum mechanics (§4.4), and finally, we recast the teleporation protocol into a von Neumann algebra framework (§4.5) using the constructions presented in previous chapters.

4.1 Quantum systems

With every quantum system, there is an associated complex Hilbert space $\mathcal{H}$, known as the *state space* of the system. The state of a quantum system can be completely described by its state vector, which is a unit vector in $\mathcal{H}$. The set of vector states $|\psi_i\rangle \in \mathcal{H}$ one may observe with respective probability p_i is denoted by $\{p_i, |\psi_i\rangle\}$, and is referred to as an *ensemble of pure states*. A useful object in the study of the ensemble $\{p_i,\, |\psi_i\rangle\}_{i=0}^{n-1}$ is its *density operator*, defined by

$$\rho \equiv \sum_{i=0}^{n-1} p_i |\psi_i\rangle(|\psi_i\rangle)^* = \sum_{i=0}^{n-1} p_i |\psi_i\rangle\langle\psi_i|.$$

Note that pure states correspond to rank-1 projection density operators. The set of all possible *states* of a quantum system are all positive linear trace densities in $\rho \in \mathcal{L}(\mathcal{H})$. The evolution of a closed quantum system is described by a unitary transformation.

For example in the Schrödinger picture of quantum mechanics, if a quantum system $\mathcal{H}$ is initially in state $|\psi_{\text{in}}\rangle$, then the state of the system after time t is given by $|\psi_t\rangle = U_t|\psi_{\text{in}}\rangle$ for some unitary $U_t \in \mathcal{L}(\mathcal{H})$ determined by t.

Definition 4.1.1. A *positive operator-valued measure* (POVM) is a finite set of positive operators $\{F_j\}_{j=1}^n \subset \mathcal{L}(\mathcal{H})$ such that $\sum\limits_{j=1}^{n} F_j = \text{id}_{\mathcal{H}}$.

Definition 4.1.2. *Quantum measurements* are described by a collection of *measurement operators* $\{K_j\}_{j=1}^n \subseteq \mathcal{L}(\mathcal{H})$, called a *measurement system*, that satisfy the completeness relation $\sum\limits_{j=1}^{n} K_j^* K_j = \text{id}_{\mathcal{H}}$. The index j refers to the potential measurement outcomes in an experiment.

Note that if $\{K_j\}_{j=1}^n$ is a measurement system, then $\{F_j\}_{j=1}^n$ is a POVM, where $F_j = K_j^* K_j$. Conversely, if $\{F_j\}_{j=1}^n$ is a POVM, then $\{\sqrt{F_j}\}_{j=1}^n$ is a measurement system. For a given state ρ, $p_j = \text{tr}(F_j\rho)$ forms a probability distribution associated to the outcomes of the POVM $\{F_j\}_{j=1}^n$ in the state ρ.

Definition 4.1.3. A *Projector Valued Measure* (PVM) is a POVM $\{F_j\}_{j=1}^n \subset \mathcal{L}(\mathcal{H})$ such that each F_j is an orthogonal projection.

Definition 4.1.4. An *observable* is a self-adjoint operator $F \in \mathcal{L}(\mathcal{H})$.

By the spectral theorem, any observable F can be written as $F = \sum_{j=1}^n \lambda_j |e_j\rangle\langle e_j|$, where λ_j is an eigenvalue of F and $\{|e_j\rangle\}_{j=1}^n$ is an orthonormal basis of eigenvectors of F. Then $\{F_j\}_{j=1}^n = \{|e_j\rangle\langle e_j|\}_{j=1}^n$ is the PVM associated to the observable F.

If $\mathcal{H}_A$ and $\mathcal{H}_B$ are Hilbert spaces corresponding to two distinct quantum systems (e.g. two spatially distant parties, Alice and Bob), then the combined (bipartite) quantum system has corresponding Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. One might expect that local operations performed on one portion of the bipartite system, e.g. $\mathcal{H}_A$, would leave the other portion, e.g. $\mathcal{H}_B$, unaffected. However, if Alice and Bob are in possession of a pre-shared *entangled* state, then the action of a local operator on Alice's system is equal to the action of a local operator on Bob's system.

Definition 4.1.5. An element $\psi \in \mathcal{H}_A \otimes \mathcal{H}_B$ is *separable* if there exists $\xi \in \mathcal{H}_A$, and $\eta \in \mathcal{H}_B$ such that $\psi = \xi \otimes \eta$. If ψ is not of this form, then it is *non-separable.*

Definition 4.1.6. An *entangled* state is a density operator corresponding to a non-separable vector state, and a separable vector state corresponds to a *non-entangled* state.

Definition 4.1.7. Let $\mathcal{H}_A$ and $\mathcal{H}_B$ be Hilbert spaces of equal dimension, d, with orthonormal bases $\{e_j\}_{j=1}^d$ and $\{f_j\}_{j=1}^d$. The vector state $|\psi_d\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ defined by

$$|\psi_d\rangle \equiv \frac{1}{\sqrt{d}} \sum_{j=1}^{d} |e_j\rangle \otimes |f_j\rangle,$$

is a maximally entangled state vector in $\mathcal{H}_A \otimes \mathcal{H}_B$, with corresponding maximally entangled state

$$\rho_\psi = |\psi_d\rangle\langle\psi_d| \equiv \frac{1}{d}\sum_{i,j=1}^{d} |e_i\rangle\langle e_j| \otimes |f_i\rangle\langle f_j|,$$

in $\mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Remark 4.1.8. It is shown in [32, G. Vidal, 2000] that there exists a unique measure of entanglement (satisfying a fixed set of axioms) for a bipartite system, $\mathcal{H}_A \otimes \mathcal{H}_B$, and is given by the von Neumann entropy, S, of the reduced density operator $(\mathrm{id} \otimes \mathrm{tr}_B)(\rho^{AB}) = \rho^A$;

$$S = -\mathrm{tr}(\rho^A \ln \rho^A).$$

The reduced density operator of ρ_ψ in the definition above is

$$\begin{aligned}\rho_\psi^A = (\mathrm{id} \otimes \mathrm{tr}_B)(\rho_\psi) &= \frac{1}{d}\sum_{i,j=1}^{d} (\mathrm{id} \otimes \mathrm{tr}_B)\big(|e_i\rangle\langle e_j| \otimes |f_i\rangle\langle f_j|\big)\\ &= \frac{1}{d}\sum_{i,j=1}^{d} \delta_{ij}|e_i\rangle\langle e_j| = \frac{1}{d}\sum_{i=1}^{d} |e_i\rangle\langle e_i|.\end{aligned}$$

The von Neumann entropy of the reduced density operator ρ_ψ^A is

$$S = -\mathrm{tr}(\rho_\psi^A \ln \rho_\psi^A) = -\frac{1}{d}\sum_{i=1}^{d} \ln\frac{1}{d} = \ln d,$$

which is the maximal entropy [26]. Hence the term, *maximally* entangled.

In the next section, we define local operation and classical communication maps in preparation to discuss how two spatially separated parties may use a shared entangled state as a resource for, among other things, quantum communication using only local quantum operations and classical communication.

4.2 LOCC

Local operations and classical communications (LOCC) refers to a procedure in which spatially separated parties each with their own quantum systems perform quantum operations on their respective quantum systems and communicate via classical means in order to achieve a common goal using entanglement as a physical resource [7]. Several uses for LOCC procedures have been developed such as shared entangled state transformations [25] where an entangled state between the two parties can be transformed into a new desired state conditional on [25, Theorem 1]. There exist LOCC protocols for entanglement distillation where dilute entanglement present in a shared state is “distilled” into a more concentrated form of entanglement across a shared state[26, §12.5.2], quantum cryptography such as the celebrated BB84 protocol developed by C. H. Bennett and G. Brassard [3], and quantum error-correction [26, §12.5.3]. In other words, LOCC procedures are a useful tool in quantum communication and computing. The quantum teleportation protocol developed by Bennett, et al. in the celebrated paper [4] is a quintessential example of an LOCC operation.

Definition 4.2.1. Let $\mathcal{H}_1, \mathcal{H}_2$ be state spaces. A *quantum channel* is a completely positive, trace preserving (CPTP) map $\Psi : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$.

Proposition 4.2.2. *$\Phi : \mathcal{L}(\mathcal{H}_1) \to \mathcal{L}(\mathcal{H}_2)$ is a quantum channel if and only if there exists a set of operators $\{W_i\}_{i=1}^n$ such that $\Phi(x) = \sum_{i=1}^n W_i^*(x)W_i$, for all $x \in \mathcal{L}(\mathcal{H}_1)$, with $\sum_{i=1}^n W_iW_i^* = \mathrm{id}_{\mathcal{H}_1}$.*

Proof. $(\Leftarrow)$:
Due to Choi's work in [8], we know that Φ is completely positive. To see that Φ is trace preserving we have

$$\begin{aligned}\mathrm{tr}(\Phi(x)) &= \mathrm{tr}\Big(\sum_{i=1}^n W_i^*(x)W_i\Big) = \sum_{i=1}^n \mathrm{tr}\big(W_i^*(x)W_i\big)\\ &= \sum_{i=1}^n \mathrm{tr}\big(W_iW_i^*x\big) = \mathrm{tr}\Big(\sum_{i=1}^n W_iW_i^*x\Big) = \mathrm{tr}(x).\end{aligned}$$

$(\Rightarrow)$:
By Choi, we know that exist operators $\{W_i\}_{i=1}^n$ such that

$$\Phi(x) = \sum_{i=1}^n W_i^*(x)W_i.$$

Then, for $x \in \mathcal{L}(\mathcal{H}_1)$, we have

$$\mathrm{tr}(x) = \mathrm{tr}(\Phi(x)) = \mathrm{tr}\Big(\sum_{i=1}^n W_i^*(x)W_i\Big) = \mathrm{tr}\Big(\Big(\sum_{i=1}^n W_iW_i^*\Big)x\Big).$$

Since this holds for any $x \in \mathcal{L}(\mathcal{H}_1)$, it follows that $\sum_{i=1}^n W_iW_i^* = \mathrm{id}_{\mathcal{H}_1}$. ☐

The decomposition of a CPTP map Φ, as in the lemma above, is known as the *Kraus decomposition* of Φ.

Definition 4.2.3. A quantum channel $\Psi : \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B) \to \mathcal{L}(\mathcal{H}_A \otimes \mathcal{H}_B)$ is one-way right *LOCC* if there exists a measurement system $\{K_i^A\}_{i=1}^m$ on $\mathcal{L}(\mathcal{H}_A)$ and channels $\{\Phi_i^B\}_{i=1}^m$ on $\mathcal{L}(\mathcal{H}_B)$ such that

$$\Psi(\rho) = \sum_{i=1}^m \sum_{j=1}^{n_i} (K_i^A \otimes W_{ij}^B)^* \ \rho \ (K_i^A \otimes W_{ij}^B),$$

where $\{W_{ij}^B\}_{j=1}^{n_i}$ are the Kraus operators of Φ_i^B.

We will only consider one-way right LOCC so, we will omit the "one-way right" terminology from now on.

Equivalently, one could define this at the level of state vectors, i.e. if ρ has state vector $|\xi\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$, then the state vector of $\Psi(\rho)$, denoted $|\xi'\rangle$, is given by

$$|\xi'\rangle = \sum_{i=1}^{m} \sum_{j=1}^{n_i} (K_j^A \otimes W_{ij}^B)^* |\xi\rangle.$$

In what follows, we show how LOCC and entanglement come together to allow for quantum teleportation.

4.3 Standard teleporation scheme

We begin by describing the typical (Schrödinger picture [27]) example of quantum teleporation at the level of state vectors originlly developped by Bennett, et al. in their celebrated paper *Teleporting an unknown quantum state via dual classical and Einstein–Podolsky–Rosen channels* [4]. We will refer to state vectors as *states*, and state spaces as *systems* for this section.

In this LOCC set-up, the sender, Alice, is in possession of the system $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$ and the receiver, Bob, is in possession of the system $\mathcal{H}_B$, where $\mathcal{H}_{A_1}, \mathcal{H}_{A_2}, \mathcal{H}_B = \mathbb{C}^2$. The total system shared by Alice and Bob is $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2} \otimes \mathcal{H}_B$. Alice and Bob first create a maximally entangled state, say $|\beta_{11}\rangle = \sqrt{2}^{-1}(|e_1e_1\rangle + |e_2e_2\rangle)$, on the system $\mathcal{H}_{A_2} \otimes \mathcal{H}_B$, where by $|e_ie_i\rangle$ we mean $|e_i\rangle \otimes |e_i\rangle$. The state $|\phi\rangle = a|e_1\rangle + b|e_2\rangle$, which Alice would like to send to Bob, resides in $\mathcal{H}_{A_1}$. Then, the total state of the system is given by

$$|\Psi\rangle = |\phi\rangle \otimes |\beta_{11}\rangle. \tag{4.1}$$

The set of maximally entangled vectors on $\mathbb{C}^2 \otimes \mathbb{C}^2$ below are known as the *Bell states* (or sometimes *EPR states* or *EPR pairs*) [26]:

$$\begin{aligned}
|\beta_{11}\rangle &= \frac{1}{\sqrt{2}}(|e_1e_1\rangle + |e_2e_2\rangle) \\
|\beta_{12}\rangle &= \frac{1}{\sqrt{2}}(|e_1e_1\rangle - |e_2e_2\rangle) = (Z \otimes 1)|\beta_{11}\rangle \\
|\beta_{21}\rangle &= \frac{1}{\sqrt{2}}(|e_2e_1\rangle + |e_1e_2\rangle) = (X \otimes 1)|\beta_{11}\rangle \\
|\beta_{22}\rangle &= \frac{1}{\sqrt{2}}(|e_1e_2\rangle - |e_2e_1\rangle) = (ZX \otimes 1)|\beta_{11}\rangle
\end{aligned}$$

From these equations, it follows that

$$\begin{aligned}
|e_1e_1\rangle &= \frac{1}{\sqrt{2}}(|\beta_{11}\rangle + |\beta_{12}\rangle)\\
|e_1e_2\rangle &= \frac{1}{\sqrt{2}}(|\beta_{21}\rangle + |\beta_{22}\rangle)\\
|e_2e_1\rangle &= \frac{1}{\sqrt{2}}(|\beta_{21}\rangle - |\beta_{22}\rangle)\\
|e_2e_2\rangle &= \frac{1}{\sqrt{2}}(|\beta_{11}\rangle - |\beta_{12}\rangle)
\end{aligned}$$

Using the equations above, we can rewrite the state of the system as

$$\begin{aligned}
|\Psi\rangle &= |\phi\rangle|\beta_{11}\rangle\\
&= a|e_1\rangle|\beta_{11}\rangle + b|e_2\rangle|\beta_{11}\rangle\\
&= \frac{a}{\sqrt{2}}(|e_1e_1\rangle|e_1\rangle + |e_1e_2\rangle|e_2\rangle) + \frac{b}{\sqrt{2}}(|e_2e_1\rangle|e_1\rangle + |e_2e_2\rangle|e_2\rangle)\\
&= \frac{a}{2}((|\beta_{11}\rangle + |\beta_{12}\rangle)|e_1\rangle + (|\beta_{21}\rangle + |\beta_{22}\rangle)|e_2\rangle) + \frac{b}{2}((|\beta_{21}\rangle - |\beta_{22}\rangle)|e_1\rangle\\
&+ (|\beta_{11}\rangle - |\beta_{22}\rangle)|e_2\rangle)\\
&= \frac{a}{2}(|\beta_{11}\rangle|e_1\rangle + |\beta_{12}\rangle|e_1\rangle + |\beta_{21}\rangle|e_2\rangle + |\beta_{22}\rangle|e_2\rangle) + \frac{b}{2}(|\beta_{21}\rangle|e_1\rangle - |\beta_{22}\rangle|e_1\rangle\\
&+ |\beta_{11}\rangle|e_2\rangle - |\beta_{22}\rangle|e_2\rangle)\\
&= \frac{1}{2}(|\beta_{11}\rangle|\phi\rangle + |\beta_{12}\rangle(a|e_1\rangle - b|e_2\rangle) + |\beta_{21}\rangle(b|e_1\rangle + a|e_2\rangle) + |\beta_{22}\rangle(a|e_2\rangle - b|e_1\rangle))\\
&= \frac{1}{2}(|\beta_{11}\rangle|\phi\rangle + |\beta_{12}\rangle Z|\phi\rangle + |\beta_{21}\rangle X|\phi\rangle + |\beta_{22}\rangle XZ|\phi\rangle). \qquad (4.2)
\end{aligned}$$

Alice then measures her portion of the system, $\mathcal{H}_{A_1} \otimes \mathcal{H}_{A_2}$, in the Bell basis and observes the classical output (ij) which she sends to Bob via classical means. Bob then applies the transformation $Z^{j-1}X^{i-1}$ to his system $\mathcal{H}_B$ to recover $|\phi\rangle$. The total state of the system is then given by $|\beta_{ij}\rangle \otimes |\phi\rangle$. So, Alice transferred a quantum state to Bob using only local operations and classical communication with pre-shared entanglement, motivating the term "*quantum* teleportation." Mathematically, the LOCC map for this protocol is written as

$$|\Psi\rangle = \sum_{i,j=1}^{2}(K_{ij} \otimes Z^{j-1}X^{i-1})|\phi\rangle \otimes |\beta_{11}\rangle, \qquad (4.3)$$

where $K_{ij} = |\beta_{ij}\rangle\langle\beta_{ij}| = (Z^{j-1}X^{i-1} \otimes 1_2)|\beta_{11}\rangle\langle\beta_{11}|(X^{i-1}Z^{j-1} \otimes 1_2)$. Note that the first tensor product in the summands of equation (4.3) is between systems A_2 and B, and the second is between A_1 and A_2.

4.4 Heisenberg picture of teleportation

Here, we repackage the Schrödigner representation of Bennett, et al.'s teleportation identity in (4.3) into the Heisenberg picture of quantum mechanics [27].

We begin by representing the states $|\phi\rangle$ and $|\beta_{11}\rangle$, from the discussion above, by their density matrices $\rho = |\phi\rangle\langle\phi|$ and $\omega = |\beta_{11}\rangle\langle\beta_{11}|$ where $\rho \in \mathcal{M}_2$ and $\omega \in \mathcal{M}_2 \otimes \mathcal{M}_2$. Let $U_{ij} = X^{i-1}Z^{j-1} \in \mathcal{M}_2$. Line (4.2) implies that the initial state of the system, before any local operations have taken place, is

$$\rho \otimes \omega = \sum_{i,j=1}^{2} (U_{ij}^* \otimes 1)\omega(U_{ij} \otimes 1) \otimes (U_{ij}^* \rho U_{ij}). \tag{4.4}$$

In this picture, the total system lies inside $S = \mathcal{M}_2 \otimes \mathcal{M}_2 \otimes \mathcal{M}_2$, where Alice is in possession of the first two terms of the tensor product and Bob has the third term.

Alice's measurement system, $F = \{F_{ij}\}_{i,j=1}^2$, is a collection of observables, F_{ij}, which map S to itself and are defined by

$$F_{ij}(\theta) = (M_{ij} \otimes 1)\theta(M_{ij} \otimes 1),$$

for all $\theta \in S$, where $M_{ij} = (U_{ij}^* \otimes 1)\omega(U_{ij} \otimes 1)$. Note that F_{ij} only acts non-trivially on the first two legs of S, i.e. Alice's system. Then, Alice's application of the observable, F, to the initial state of the system, $\rho \otimes \omega$, yields:

$$F(\rho \otimes \omega) = \sum_{i,j=1}^{2} F_{ij}(\rho \otimes \omega) = \sum_{i,j=1}^{2} (M_{ij} \otimes 1)\rho \otimes \omega(M_{ij} \otimes 1). \tag{4.5}$$

Next, we must apply Bob's operations, $B_{ij} : \mathcal{M}_2 \to \mathcal{M}_2$, corresponding to the observation (ij), which are defined by $B_{ij}(\varepsilon) = U_{ij}^* \varepsilon U_{ij}$. Applying this to (4.5), gives us

$$B \circ F(\rho \otimes \omega) = \sum_{i,j=1}^{2} B_{ij} \circ F_{ij}(\rho \otimes \omega) = \sum_{i,j=1}^{2} (M_{ij} \otimes U_{ij}^*)(\rho \otimes \omega)(M_{ij} \otimes U_{ij}).$$

Finally, Bob applies a non-zero orthogonal projection $A \in \mathcal{M}_2$ of his choice. Then, the teleportation scheme applied to the density $\rho \in \mathcal{M}_2$ is described by the following equation:

$$\Phi(\rho) = \sum_{i,j=1}^{2} (M_{ij} \otimes AU_{ij}^*)(\rho \otimes \omega)(M_{ij} \otimes U_{ij}A).$$

x The probability of Alice obtaining the result (ij) and Bob obtaining a result "yes" on A is

$$\begin{aligned}\mathrm{tr}((M_{ij} \otimes AU_{ij}^*)(\rho \otimes \omega)(M_{ij} \otimes U_{ij}A)) &= \mathrm{tr}((\rho \otimes \omega)(M_{ij} \otimes U_{ij}A)(M_{ij} \otimes AU_{ij}^*)) \\ &= \mathrm{tr}((\rho \otimes \omega)(M_{ij}M_{ij} \otimes U_{ij}AU_{ij}^*)) \\ &= \mathrm{tr}((\rho \otimes \omega)(F_{ij} \otimes T_{ij}(A))),\end{aligned}$$

where $T_{ij} : \mathcal{M}_2 \to \mathcal{M}_2$ is Bob's operation defined by $T_{ij}(x) = U_{ij}xU_{ij}^*$ for all $x \in \mathcal{M}_2$. Then, a teleportation scheme is said to be *successful* if the probability of observing A, given the state ρ, is unaffected by the application of the teleportation scheme. That is, a teleportation scheme is successful if

$$\sum_{i,j=1}^{2} \mathrm{tr}((\rho \otimes \omega)(F_{ij} \otimes T_{ij}(A))) = \mathrm{tr}(\rho A), \tag{4.6}$$

for any projection $A \in \mathcal{M}_2$.

The fact that $\mathcal{M}_2$ is a von Neumann algebra means that $\mathcal{M}_2$ is linearly spanned by its projections [23, Theorem 4.1.11 (1)]. Then, by linearity of Equation (4.6), it follows that if a teleportation scheme is successful then Equation (4.6) holds for all $x \in \mathcal{M}_2$;

$$\sum_{i,j=1}^{2} \mathrm{tr}((\rho \otimes \omega)(F_{ij} \otimes T_{ij}(x))) = \mathrm{tr}(\rho x), \quad x \in \mathcal{M}_2.$$

In particular, a teleportation scheme is successful if for all $x \in \mathcal{M}_2$

$$x = \sum_{i,j=1}^{2} (\mathrm{id} \otimes \mathrm{tr}_2 \otimes \mathrm{tr}_2)\big((1_2 \otimes \omega)(F_{ij} \otimes T_{ij}(x))\big).$$

The teleporation scheme that we have been discussing generalizes to $\mathbb{C}1_d \subseteq \mathcal{M}_d$ for arbitrary d in a canonical way. If the Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$, and $\mathcal{H}_3$ are all of dimension d, then we simply replace $|\beta_{11}\rangle$ by the typical maximally entangled state on $\mathcal{H}_2 \otimes \mathcal{H}_3$, which we denote by $|\psi_d\rangle$, so that $\omega \equiv |\psi_d\rangle\langle\psi_d| \in \mathcal{M}_d \otimes \mathcal{M}_d$. The input ρ lies in $\mathcal{M}_d$, and we replace the Pauli operators X and Z by the d-by-d generalized Pauli-X and Pauli-Z operators (see §3.10). Then, a teleportation scheme is successful if

$$x = \sum_{i,j=1}^{d} (\mathrm{id} \otimes \mathrm{tr}_d \otimes \mathrm{tr}_d)\big((1_d \otimes \omega)(F_{ij} \otimes T_{ij}(x))\big), \tag{4.7}$$

for all $x \in \mathcal{M}_d$.

Next we use (4.7) to recast the teleportation scheme in terms of von Neumann algebra inclusions.

4.5 Recast teleportation into von Neumann algebraic setting

Below, we show that each object in (4.7) can be obtained by the GNS and basic constructions beginning with the inclusion $\mathbb{C}1_d \subseteq \mathcal{M}_d$ (§3.10). We rewrite (4.7) using the Jones projections, trace-preserving conditional expectations, and Pimsner-Popa basis for $\mathcal{M}_d/\mathbb{C}1_d$ obtained by the GNS and basic constructions, which yields a description of quantum teleportation in a von Neumann algebraic setting.

Let us begin by recalling the identities of each object in (4.7). For convenience we include (4.7) below:

A teleportation scheme on $\mathcal{M}_d \otimes \mathcal{M}_d \otimes \mathcal{M}_d$ is successful if

$$x = \sum_{i,j=1}^{d} (\mathrm{id} \otimes \mathrm{tr}_d \otimes \mathrm{tr}_d)\big((1_d \otimes \omega)(F_{ij} \otimes T_{ij}(x))\big), \quad x \in \mathcal{M}_d, \tag{4.8}$$

where

(a.1) $\omega = |\psi_d\rangle\langle\psi_d|$, where ψ_d is the maximally entangled vector state

$$|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |e_i\rangle \otimes |e_i\rangle.$$

(a.2) $F_{ij} = (U_{ij}^* \otimes 1_d)\omega(U_{ij} \otimes 1_d)$, where $U_{ij} = X^{i-1}Y^{j-1}$, where X and Y are the generalized Pauli-X and -Y operators in $\mathcal{M}_d$, respectively.

Now, recall from §3.10 that from the inclusion $\mathbb{C}1_d \otimes \mathcal{M}_d$ we obtained the following inclusions by iterating the basic construction:

$$\overbrace{\mathbb{C}1_d}^{N} \subseteq \overbrace{\mathcal{M}_d}^{M} \subseteq \overbrace{\mathcal{M}_d \otimes \mathcal{M}_d}^{M_1} \subseteq \overbrace{\underbrace{\mathcal{M}_d \otimes \mathcal{M}_d}_{\text{Alice}} \otimes \underbrace{\mathcal{M}_d}_{\text{Bob}}}^{M_2}.$$

We also obtained the following objects via the GNS construction:

(b.1) The Jones projection $e_N : L^2(M, \tau_d) \to L^2(N, \tau_d)$, where τ_d is the normalized trace on $\mathcal{M}_d$, defined by

$$e_N = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |e_i\rangle \otimes |e_i\rangle.$$

(b.2) A Pimsner-Popa basis $\{\lambda_{ij}\}_{i,j=1}^{d}$ for $\mathcal{M}_d/\mathbb{C}1_d$ where $\lambda_{ij} = X^{i-1}Y^{j-1}$, where X and Y are the generalized Pauli-X and -Y operators in $\mathcal{M}_d$, respectively.

(b.3) The τ-preserving conditional expectation $E_{M_1\to M}$ defined by $(\mathrm{id}_d \otimes \tau_d)$ on M_1.

Then (a.1) and (b.1) give us that $\omega = e_N$, and (a.2) and (b.2) give us that $\{U_{ij}\}_{i,j=1}^d$ is the basis for $\mathcal{M}_d/\mathbb{C}1_d$ given by $\{\lambda_{ij}\}_{i,j=1}^d$. Then we may rewrite (4.8) as

$$x = \sum_{i,j=1}^{d} (\mathrm{id} \otimes \mathrm{tr}_d \otimes \mathrm{tr}_d)\big((1_d \otimes e_N)((\lambda_{ij}^* \otimes 1_d)e_N(\lambda_{ij} \otimes 1_d) \otimes T_{ij}(x))\big). \tag{4.9}$$

It is straightforward to show that the Jones projection $e_M : L^2(M_1, \phi) \to L^2(M, \phi)$, where $\phi = (\mathrm{id}_d \otimes \tau_d)$ is the Markov extension of τ_d on M_1, is given by $e_M = (1_d \otimes e_N)$. This induces the ϕ-preserving conditional expectation $E_{M_2\to M_1}$ defined by $(\mathrm{id}_d \otimes \mathrm{id}_d \otimes \tau_d)$.

Let E_M denote the τ-preserving conditional expectation defined by

$$E_M \equiv E_{M_1\to M} \circ E_{M_2\to M_1} = (\mathrm{id}_d \otimes \tau_d \otimes \tau_d), \quad \text{on } M_2.$$

Finally, it has been implicit throughout this section that $x \in \mathcal{M}_d$ in (4.8) and (4.9) lives on the left-most tensor leg of M_2, which is described by the relative commutant $N' \cap M$. However, each summand of the equations in (4.8) and (4.9) are operating on x in the right-most leg of M_2, which is described by the relative commutant $M_1' \cap M_2$. Recall from §3.9 the isomorphism $\Gamma : N' \cap M \to M_1' \cap M_2$ known as the canonical shift. Then, from (4.9), a teleportation scheme on M_2 is successful if

$$\Gamma^{-1}(x) = \sum_{i,j=1}^{d} E_M\big(e_M((\lambda_{ij} \otimes 1_d)e_N(\lambda_{ij} \otimes 1_d) \otimes T_{ij}(x))\big), \quad x \in M_1' \cap M_2.$$

From the above identity we see that quantum teleportation is an LOCC implementation of the canonical shift. In the next section we present a generalization of this identity in order to teleport arbitrary von Neumann algebras with optimal resource requirements.

4.6 Hybrid quantum teleportation

In this section, we present an operationally concrete operator algebraic teleportation scheme, which we named *Hybrid quantum teleportation*, via Theorem 4.6.4. This scheme is capable of teleporting any von Neumann algebra, and can be viewed as a generalization of the standard example of teleportation described in §4.5.

The idea behind our hybrid quantum teleportation scheme is as follows. We begin with the von Neumann algebra inclusion $\mathbb{C}1_M \subseteq M$ and Markov trace τ, where M is an arbitrary von Neumann algebra, and apply two iterations of the basic construction to obtain von Neumann algebras M_1 and M_2 (§3.6) with the series of inclusions

$$\overbrace{\mathbb{C}1_M}^{N} \subseteq \overbrace{\bigoplus_{j=1}^{n} \mathcal{M}_{n_j}}^{M} \subseteq \overbrace{\bigoplus_{j=1}^{m} \mathcal{M}_{n_j} \otimes \mathcal{M}_{n_j}}^{M_1} \subseteq \overbrace{\underbrace{\Big(\bigoplus_{j=1}^{n} \mathcal{M}_{n_j} \otimes \mathcal{M}_{n_j}\Big)}_{\text{Alice}} \otimes \underbrace{\Big(\bigoplus_{k=1}^{n} 1_{n_k} \otimes \mathcal{M}_{n_k}\Big)}_{\text{Bob}}}^{M_2} \tag{4.10}$$

The operational interpretation this scheme offers is as follows: Alice is in possession of the system described by M_1 and Bob is in possession of the system described by the relative commutant $M_1' \cap M_2$. As shown in line (4.10), these two systems are distinctly separated by a tensor, corresponding to two spatially separated quantum systems. Alice and Bob share the Jones projection (§3.4) $e_M : L^2(M_1, \tau_1) \to L^2(M, \tau_1)$, where τ_1 is the Markov extension of τ on M_1, which serves as the entanglement resource for the teleportation protocol. Alice's local measurements and Bob's local operations are then defined summand-wise on the their respective systems. This leads to the interpretation that this protocol is really just the "direct sum" of the standard protocol applied each summand $\mathcal{M}_{n_j}$ of M, and is reflected in the structure of all objects involved; the entangled resource state e_M, Alice's POVM and Bob's operations.

In preparation for the proof of Theorem 4.6.4, we establish notation and a few helpful results. We denote by $\mathcal{N}_M(N)$ the elements of M which are in the normalizer of N, i.e.

$$\mathcal{N}_M(N) = \{x \in M \mid xNx^* \subseteq N\},$$

and by $\mathcal{U}(\mathcal{N}_M(N))$ the set of unitaries in $\mathcal{N}_M(N)$.

Definition 4.6.1. Given an inclusion $N \subseteq M$, the normalizer of N inside M is denoted $\mathcal{N}_M(N)$ and defined by

$$\mathcal{N}_M(N) = \{x \in M \mid xNx^* \subseteq N\}.$$

We denoted by $\mathcal{U}(\mathcal{N}_M(N))$ the set of unitaries in $\mathcal{N}_M(N)$.

Lemma 4.6.2. *Given an inclusion $N \subseteq M$, it is true that*

$$\mathcal{U}(\mathcal{N}_M(N)) = \mathcal{U}(\mathcal{N}_M(N' \cap M)).$$

Proof. Let $\lambda \in \mathcal{U}(\mathcal{N}_M(N))$, $x \in N' \cap M$, and $y \in N$. Then,

$$(\lambda x \lambda^*)y = \lambda x(\lambda^* y \lambda)\lambda^* = \lambda(\lambda^* y \lambda)x\lambda^* = y(\lambda x \lambda^*),$$

implying $\lambda x \lambda^*$ lies in $N' \cap M$. Let $x_1, x_2 \in N' \cap M$ such that $\lambda^* x_1 \lambda = \lambda^* x_2 \lambda$ for some unitary $\lambda \in \mathcal{U}(\mathcal{N}_M(N))$. Then, $\lambda\lambda^* x_1 \lambda\lambda^* = \lambda\lambda^* x_2 \lambda\lambda^*$, so $x_1 = x_2$. Hence, $\mathcal{U}(\mathcal{N}_M(N))$ is contained in $\mathcal{U}(\mathcal{N}_M(N' \cap M))$.

On the other hand, if we choose any unitary $u \in \mathcal{U}(\mathcal{N}_M(N' \cap M))$, then for any $y \in N$ and any $x \in N' \cap M$, we get

$$(u^*yu)x = u^*y(uxu^*)u = u^*(uxu^*)yu = u^*ux(u^*yu) = x(u^*yu),$$

implying u^*yu lies in N. Since u is unitary, it follows that for any $y_1, y_2 \in N$ with $u^*y_1u = u^*y_2u$ implies $y_1 = y_2$. Hence, $\mathcal{U}(\mathcal{N}_M(N' \cap M)) \subseteq \mathcal{U}(\mathcal{N}_M(N))$.

Hence, $\mathcal{U}(\mathcal{N}_M(N' \cap M)) = \mathcal{U}(\mathcal{N}_M(N))$, as required. □

Lemma 4.6.3. *Consider the inclusion $N \subseteq M$ and the Jones projection e_N obtained by the GNS construction. Let M_1 be the von Neumann algebra obtained by the basic construction and $\gamma_1 : N' \cap M \to M' \cap M_1$ be the shift operation from §3.9. Then, for any $\lambda \in \mathcal{U}(\mathcal{N}_M(N' \cap M))$ we have*

$$e_N\lambda x = e_N\lambda\gamma_1(\lambda x\lambda^*), \quad x \in N' \cap M. \tag{4.11}$$

Proof. Recall from Lemma 3.9.2 we have $\gamma_1(x)e_N = xe_N$, for all $x \in N' \cap M$. Then, if $\lambda \in \mathcal{U}(\mathcal{N}_M(N' \cap M))$, it follows that

$$x^*\lambda^*e_N = \lambda^*(\lambda x^*\lambda^*)e_N = \gamma_1(\lambda x^*\lambda^*)\lambda^*e_N,$$

for all $x \in N' \cap M$. Taking the adjoint of both sides yields:

$$e_N\lambda x = e_N\gamma_1(\lambda x\lambda^*)\lambda = e_N\lambda\gamma_1(\lambda x\lambda^*).$$

□

In the following theorem we denote by M_1 and M_2 von Neumann algebras for the basic construction applied to $\mathbb{C}1_M \subseteq M$.

We are now ready to state our hybrid quantum teleportation theorem.

Theorem 4.6.4 (Hybrid quantum teleportation)**.** *Consider the inclusion from line (4.10), $\mathbb{C}1_M \subseteq M \subseteq M_1 \subseteq M_2$. There exists a POVM $\{F_i\}_{i\in\mathcal{I}}$ in M_1, and automorphisms $\Phi_i : (M_1' \cap M_2) \to (M_1' \cap M_2)$, with $|\mathcal{I}| = \dim M$, for which*

$$x = \dim(M)E_M\Big(\sum_{i\in\mathcal{I}} F_i \otimes \Phi_i(\Gamma(x))e_M\Big), \quad x \in N' \cap M,$$

where $E_M \equiv E_{M_1\to M} \circ E_{M_2\to M_1}$ is the unique trace preserving conditional expectation (§3.5) from M_2 to M, and where Γ is the canonical shift (§3.9) from $N' \cap M$ to $M_1' \cap M_2$.

Proof. Let $\tau : M \to \mathbb{C}$ be the Markov trace defined by

$$\tau = \frac{1}{\dim M}\sum_{j=1}^{n} n_j \mathrm{tr}_{n_j},$$

where tr_{n_j} is the standard trace operation on M_{n_j}. Next, let $W_j(z_j) = W_j(x_j, y_j) = X_{n_j}^{x_j} Z_{n_j}^{y_j}$, where X_{n_j}, Z_{n_j} are generalized Pauli-operators in $\mathcal{U}(M_{n_j})$, with $x_j, y_j \in \mathbb{Z}_{n_j}$. Then,

$$W_j : \mathbb{Z}_{n_j} \times \mathbb{Z}_{n_j} \to \mathcal{U}(M_{n_j}),$$

and $\{W_j(z_j)\}$ is an orthonormal basis of $\mathcal{M}_{n_j}$ with respect to the normalized trace τ_{n_j} - the unique tracial state on $\mathcal{M}_{n_j}$. Let $\ddot{W}_j$ be defined by

$$\ddot{W}_j(z_j) = 0 \oplus ... \oplus 0 \oplus W_j(z_j) \oplus 0 \oplus ... \oplus 0,$$

where the only non-zero component, $W_j(z_j)$, is in the j^{th} slot. It follows that the set $\{\ddot{W}_j(z_j) \mid j = 1, ..., n,\ z_j \in \mathbb{Z}_{n_j}^2\}$ is an orthogonal basis of M,

$$\begin{aligned}
\tau(\ddot{W}_j(z_j)^* \ddot{W}_i(z_i)) &= \delta_{ij} \tau(\ddot{W}_j(z_j)^* \ddot{W}_j(z_j')) \\
&= \delta_{ij} \frac{1}{\dim M} n_j^2 \tau_{n_j}(W_j(z_j)^* W_j(z_j')) \\
&= \delta_{ij} \frac{n_j^2}{\dim M} \tau_{n_j}(\delta_{z_j z_j'} 1_d) \\
&= \delta_{ij} \frac{n_j^2}{\dim M} \delta_{z_j z_j'}.
\end{aligned}$$

Applying the basic construction to the inclusion $\mathbb{C}1_M \subseteq M$ yields the factor $M_1 = \mathcal{L}(L^2(M, \tau))$,

$$M_1 = \langle M, e_{\mathbb{C}} \rangle = \bigoplus_{j=1}^{n} \mathcal{M}_{n_j} \otimes \mathcal{M}_{n_j},$$

where $e_{\mathbb{C}}$ is the Jones projection from $L^2(M, \tau)$ onto $L^2(\mathbb{C}1_M, \tau)$. The standard representation of N' on $L^2(M, \tau)$ is given by

$$\pi^{\ell}(M) = \bigoplus_{j=1}^{n} \mathcal{M}_{n_j} \otimes 1_{n_j},$$

which implies that the relative commutant $(\pi^{\ell}(M))' \cap M_1$ is the subalgebra

$$\pi^{\ell}(M)' \cap M_1 = \bigoplus_{j=1}^{n} 1_{n_j} \otimes \mathcal{M}_{n_j}.$$

Note that for $x \in M$, the representation $\pi^{\ell}(x)$ is equivalent to a summand-wise application of the representations $\pi_{n_j}^{\ell} : \mathcal{M}_{n_j} \to \mathcal{M}_{n_j} \otimes 1_{n_j}$ corresponding to the inclusions $\mathbb{C}1_{n_j} \subseteq \mathcal{M}_{n_j}$,

$$\pi^{\ell}(x) = \bigoplus_{j=1}^{n} x_{n_j} \otimes 1_{n_j} = \bigoplus_{j=1}^{n} \pi_{n_j}^{\ell}(x_j),$$

where $x_j \in \mathcal{M}_{n_j}$. Also note that the anti-unitary map $J_M : L^2(M,\tau) \to L^2(M,\tau)$ defined by $J_M\widehat{x} = \widehat{x^*}$ coincides with $\bigoplus_{j=1}^n J_{\mathcal{M}_{n_j}}$,

$$J_M\widehat{x} = \widehat{x^*} = \bigoplus_{j=1}^{n} \widehat{x_j^*} = \bigoplus_{j=1}^{n} J_{\mathcal{M}_{n_j}}\widehat{x}_j.$$

Then, the shift operation $\gamma_1 : N' \to \pi^\ell(M)' \cap M_1$ can also be applied summand-wise using $\bigoplus_{j=1}^n \gamma_1^{(j)}$, where

$$\gamma_1^{(j)} : \mathcal{M}_{n_j} \otimes 1_{n_j} \to 1_{n_j} \otimes \mathcal{M}_{n_j},$$

for any $x \in M$,

$$\gamma_1(x) = J_M\pi^\ell(x)^* J_M = \bigoplus_{j=1}^{n} J_{\mathcal{M}_{n_j}}\pi^\ell_{n_j}(x_j)^* J_{\mathcal{M}_{n_j}} = \bigoplus_{j=1}^{n} 1_{n_j} \otimes x_j^t.$$

A second iteration of the basic construction yields

$$M_2 = M_1 \otimes \pi^\ell(M)' = M_1 \otimes \Big(\bigoplus_{j=1}^{n} 1_{n_j} \otimes \mathcal{M}_{n_j}\Big).$$

The standard representation of M_1 in M_2 is simply $\pi_1^\ell(M_1) = M_1 \otimes 1_{M_1}$. The shift operation $\gamma_2 : \pi^\ell(M)' \cap M_1 \to M_1' \cap M_2$ is defined by

$$\gamma_2(y) = J_{M_1}\pi_1^\ell(y)^* J_{M_1} = J_{M_1}(y \otimes 1_d)^* J_{M_1} = 1_{M_1} \otimes y^t,$$

for $y \in \pi^\ell(M)' \cap M_1$. The canonical shift $\Gamma : M \to M_1' \cap M_2$ defined by $\Gamma = \gamma_2 \circ \gamma_1$ is the map

$$x = \Big(\bigoplus_{j=1}^{n} x_j \otimes 1_{n_j}\Big) \otimes 1_{M_1} \ \mapsto\ 1_{M_1} \otimes \Big(\bigoplus_{j=1}^{n} 1_{n_j} \otimes x_{n_j}\Big).$$

Let $|\psi_{n_j}\rangle$ denote the maximally entangled state in $\mathbb{C}^{n_j} \otimes \mathbb{C}^{n_j}$ so that $|\psi_{n_j}\rangle\langle\psi_{n_j}|$ is the Jones projection from $L^2(M_{n_j}, \tau_{n_j})$ to $L^2(\mathbb{C}, \tau_{n_j})$ (see §3.10). Then, the set $\{W_j(z_j) \mid z_j \in \mathbb{Z}^2_{n_j}\}$ is a basis for $\mathcal{M}_{n_j}/\mathbb{C}1_{n_j}$. Let F_{j,z_j} be the element in the j^{th} summand of M_1 defined by

$$F_{j,z_j} = (W_j(z_j)^* \otimes 1_{n_j})|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j) \otimes 1_{n_j}),$$

for $j = 1, ...n$, and $z_j \in \mathbb{Z}^2_{n_j}$. It follows that, $F_j = \{F_{j,z_j} \mid z_j \in \mathbb{Z}^2_{n_j}\}$ is a POVM in

$\mathcal{M}_{n_j} \otimes \mathcal{M}_{n_j}$, since

$$\begin{aligned}
&\sum_{z_j \in \mathbb{Z}_{n_j}^2} (W_j(z_j)^* \otimes 1_{n_j})|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j) \otimes 1_{n_j}) \\
&= \sum_{z_j \in \mathbb{Z}_{n_j}^2} (W_j(z_j)^* \otimes 1_{n_j})|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j) \otimes 1_{n_j}) \\
&= 1_{n_j} \otimes 1_{n_j}
\end{aligned}$$

for all $j = 1, ..., n$. So, $\{F_{j,z_j} \mid j = 1, ..., n, z_j \in \mathbb{Z}_{n_j}^2\}$ is a POVM in M_1 with cardinality $\dim M$. Note that $\{W_j(z_j) \mid z_j \in \mathbb{Z}_{n_j}^2\}$ forms a basis for $\mathcal{M}_{n_j}/\mathbb{C}1_{n_j}$, for all $j = 1, ..., n$. Let

$$\tilde{W}_j(z_j) = 1_{n_1} \oplus ... \oplus 1_{n_{j-1}} \oplus W_j(z_j) \oplus 1_{n_{j+1}} \oplus ... \oplus 1_{n_n},$$

which is unitary in M, and define the automorphism $\Phi_{j,z_j} : M_1' \cap M_2 \to M_1' \cap M_2$ by

$$\Phi_{j,z_j}(y) = \Gamma(\tilde{W}_j(z_j))y\Gamma(\tilde{W}_j(z_j)^*),$$

for $y \in M_1' \cap M_2$.

Let $x \in M$ and consider the following expression of our identity

$$\dim(M)E_M\Big(\bigoplus_{j=1}^{n} \sum_{z_j \in \mathbb{Z}_{n_j}^2} (F_{j,z_j} \otimes \Phi_{j,z_j}(\Gamma(x)))e_M\Big). \tag{4.12}$$

Recall that $\Gamma : M \to M_1' \cap M_2$ is an isomorphism, so we may write any $y \in M_1' \cap M_2$ as $\Gamma(x)$, for a unique $x \in M$. Then, for any $x \in M$ we have

$$\Phi_{j,z_j}(\Gamma(x)) = \Gamma(\tilde{W}_j(z_j))\Gamma(x)\Gamma(\tilde{W}_j(z_j)^*) = \Gamma\Big(\tilde{W}_j(z_j)x\tilde{W}_j(z_j)^*\Big).$$

Let $x \in M$, and consider the $\Phi_{j,z_j}(\Gamma(x))e_M$ term in (4.12). Expanding this term gives us

$$\Phi_{j,z_j}(\Gamma(x))e_M = \Gamma\Big(\tilde{W}_j(z_j)x\tilde{W}_j(z_j)^*\Big)e_{N'} = \gamma_2 \circ \gamma_1\Big(\tilde{W}_j(z_j)x\tilde{W}_j(z_j)^*\Big)e_M.$$

By Lemma 3.9.2, the equation above becomes

$$\begin{aligned}\gamma_2 \circ \gamma_1\Big(\tilde{W}_j(z_j)x\tilde{W}_j(z_j)^*\Big)\cdot e_M &= \Big(\gamma_1\Big(\tilde{W}_j(z_j)x\tilde{W}_j(z_j)^*\Big)\otimes 1_{M_1}\Big)\cdot e_M\\ &= \Big(\gamma_1\Big(\bigoplus_{i=1}^{n} W_i(z_i)^{\delta_{ij}}x_iW_i(z_i)^{*\delta_{ij}}\Big)\otimes 1_{M_1}\Big)\cdot e_M\\ &= \Big(\bigoplus_{i=1}^{n}\gamma_1^{(i)}\Big(W_i(z_i)^{\delta_{ij}}x_iW_i(z_i)^{*\delta_{ij}}\Big)\otimes 1_{M_1}\Big)\cdot e_M,\end{aligned}$$

for each $j = 1, ..., n$ and $z_j \in \mathbb{Z}^2_{n_j}$. Inserting this into (4.12) yields

$$\begin{aligned}&\dim(M)E_M\Big(\bigoplus_{j=1}^{n}\sum_{z_j\in\mathbb{Z}^2_{n_j}} F_{j,z_j}\otimes\Phi_{j,z_j}(\Gamma(x))e_M\Big)\\ &= \dim(M)E_M\Big(\bigoplus_{j=1}^{n}\sum_{z_j\in\mathbb{Z}^2_{n_j}} F_{j,z_j}\bigoplus_{i=1}^{n}\gamma_1^{(i)}\Big(W_i(z_i)^{\delta_{ij}}x_iW_i(z_i)^{*\delta_{ij}}\Big)\otimes 1_{M_1}\cdot e_M\Big)\\ &= \dim(M)E_M\Big(\bigoplus_{j=1}^{n}\sum_{z_j\in\mathbb{Z}^2_{n_j}} F_{j,z_j}\gamma_1^{(j)}\Big(W_j(z_j)x_jW_j(z_j)^*\Big)\otimes 1_{M_1}\cdot e_M\Big). \qquad (4.13)\end{aligned}$$

Turning our attention the term $F_{j,z_j}\gamma_1^{(n_j)}\Big(W_j(z_j)x_jW_j(z_j)^*\Big)$ in the equation above, we see that

$$\begin{aligned}&F_{j,z_j}\gamma_1^{(j)}\Big(W_j(z_j)x_jW_j(z_j)^*\Big)\\ &= \Big((W_j(z_j)^*\otimes 1_{n_j})|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j)\otimes 1_{n_j})\Big)\gamma_1^{(j)}\Big(W_j(z_j)x_jW_j(z_j)^*\Big)\\ &= \big((W_j(z_j)^*\otimes 1_{n_j})|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j)\otimes 1_{n_j})\big)\big(1_{n_j}\otimes\bar{W}_j(z_j)x_j^tW_j(z_j)^t\big).\end{aligned}$$

Note that $(1_{n_j}\otimes\bar{W}_j(z_j)x_jW_j(z_j)^t)$ is the image of the shift operation from $\mathcal{M}_{n_j}\otimes 1_{n_j}$ to $1_{n_j}\otimes\mathcal{M}_{n_j}$, corresponding to the inclusion $\mathbb{C}1_{n_j}\subseteq\mathcal{M}_{n_j}$. We will denote this shift by $\tilde{\gamma}_1^{(j)}$. Continuing the calculation above we get

$$\begin{aligned}&(W_j(z_j)^*\otimes 1_{n_j})|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j)\otimes 1_{n_j})(1_{n_j}\otimes\bar{W}_j(z_j)x_j^tW_j(z_j)^t)\\ &= (W_j(z_j)^*\otimes 1_{n_j})|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j)\otimes 1_{n_j})\tilde{\gamma}_1^{(j)}\big(W_j(z_j)x_jW_j(z_j)^*\big)\big). \qquad (4.14)\end{aligned}$$

From (4.11), we know that

$$|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j)\otimes 1_{n_j})\tilde{\gamma}_1^{(j)}\big(W_j(z_j)x_jW_j(z_j)^*\big) = |\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j)\otimes 1_{n_j})(x_j\otimes 1_{n_j}).$$

Then, continuing line (4.14), we get

$$
\begin{aligned}
&(W_j(z_j)^* \otimes 1_{n_j})|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j) \otimes 1_{n_j})\tilde{\gamma}_1^{(j)}(W_j(z_j)x_jW_j(z_j)^*)\big) \\
&= \big((W_j(z_j)^* \otimes 1_{n_j})|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j) \otimes 1_{n_j})(x_j \otimes 1_{n_j})\big) \\
&= (W_j(z_j)^* \otimes 1_{n_j})|\psi_{n_j}\rangle\langle\psi_{n_j}|(W_j(z_j) \otimes 1_{n_j})(x_j \otimes 1_{n_j}) \\
&= F_{j,z_j}(x_j \otimes 1_{n_j}).
\end{aligned}
$$

Continuing line (4.13), with this in hand, takes us to our desired outcome:

$$
\begin{aligned}
&\dim(M)E_M\Big(\bigoplus_{j=1}^{n}\sum_{z_j\in\mathbb{Z}_{n_j}^2} F_{j,z_j}\gamma_1^{(j)}\Big(W_j(z_j)x_jW_j(z_j)^* \otimes 1_{m_j}\Big) \otimes 1_{M_1} \cdot e_M\Big) \\
&= \dim(M)E_M\Big(\bigoplus_{j=1}^{n}\sum_{z_j\in\mathbb{Z}_{n_j}^2} F_{j,z_j}(x_j \otimes 1_{m_j} \otimes 1_{n_j} \otimes 1_{m_j}) \otimes 1_{M_1} \cdot e_M\Big) \\
&= \dim(M)E_M\Big(\bigoplus_{j=1}^{n}(x_j \otimes 1_{m_j} \otimes 1_{n_j} \otimes 1_{m_j}) \otimes 1_{M_1} \cdot e_M\Big) \\
&= \dim(M)E_M\big((x \otimes 1_{M_1}) \cdot e_M\big) \\
&= x\dim(M)E_M(e_M) \\
&= x,
\end{aligned}
$$

as stated. □

However, the system Hilbert space in the above hybrid teleportation scheme is "designed" to make the scheme work. This is physically equivalent to requiring a custom shared quantum system for each von Neumann algebra one would like to teleport. A more realistic scenario is that Alice and Bob are in possession of two parts of a fixed tripartite system, say $\mathcal{M}_d \otimes \mathcal{M}_d \otimes \mathcal{M}_d$, and implement a teleportation scheme capable of teleportin a subalgebra of $\mathcal{M}_d$ using LOCC relative to the fixed kinematical system. One could of course do this using the teleportation identity presented in §4.5 and restrict the input to a desired subalgebra of $\mathcal{M}_d$. However, as we show in the next chapter, for a certain class of inclusions $N \subseteq \mathcal{M}_d$ that approach is inefficient with respect to the entanglement resources.

Chapter 5

Scaffolding framework

Our second (and final) operational example is the most concrete in the sense that we first fix a quantum system shared by two spatially separated parties, and then perform operator algebraic teleportation across the two portions of the system. We refer to this as a *scaffolding* approach and this will be the subject of this chapter. In section §5.1 we present a result via Theorem 5.1.4 that partially generalizes work by R. F. Werner in [34] on so-called *tight* teleportation schemes.

We now describe the fixed quantum system shared by the spatially separated parties, Alice and Bob. We begin with the Markov inclusion $\mathbb{C}1_d \subseteq \mathcal{M}_d$, with Markov trace $\tau = d^{-1}\mathrm{tr}_d$, where tr_d is the standard matrix trace on $\mathcal{M}_d$. Two iterations of the basic construction yields the tower

$$\mathbb{C}1_d \subseteq \mathcal{M}_d \subseteq \mathcal{M}_d \otimes \mathcal{M}_d \subseteq \mathcal{M}_d \otimes \mathcal{M}_d \otimes \mathcal{M}_d.$$

Recall from Subsection 3.9, we have the anti-isomorphisms $\gamma_1 : \mathcal{M}_d \to 1_d \otimes \mathcal{M}_d$, and $\gamma_2 : 1_d \otimes \mathcal{M}_d \to 1_d \otimes 1_d \otimes \mathcal{M}_d$ which we can compose, creating the canonical shift $\Gamma : \mathcal{M}_d \to 1_d \otimes 1_d \otimes \mathcal{M}_d$. The maps γ_1 and γ_2 are defined as follows:

$$\begin{aligned}\gamma_1(x) &= Jx^*J\\ \gamma_2(y) &= J_1y^*J_1,\end{aligned}$$

for $x \in \mathcal{M}_d$ and $y \in 1_d \otimes \mathcal{M}_d$, where $J : L^2(\mathcal{M}_d, \tau) \to L^2(\mathcal{M}_d, \tau)$ and $J_1 : L^2(\mathcal{M}_d \otimes \mathcal{M}_d, \tau \otimes \tau) \to L^2(\mathcal{M}_d \otimes \mathcal{M}_d, \tau \otimes \tau)$ are the anti-unitaries defined by $J\widehat{w} = \widehat{w^*}$ and $J_1\widehat{z} = \widehat{z^*}$, for $w \in \mathcal{M}_d$ and $z \in \mathcal{M}_d \otimes \mathcal{M}_d$.

Theorem 5.0.5. *Given the objects described above, and an inclusion $N \subseteq \mathcal{M}_d$ with a unitary Pimsner-Popa basis $\{\lambda_i\}_{i\in I}$ in $\mathcal{N}_{\mathcal{M}_d}(N')$, there exist automorphisms Φ_i of $1_d \otimes 1_d \otimes N'$ for which*

$$x = [M : N]E_{\mathcal{M}_d}\left(\sum_{i\in I} \lambda_i^* e_N \lambda_i \Phi_i(\Gamma(x))(1_d \otimes e_N)\right), \tag{5.1}$$

where $E_{\mathcal{M}_d} : \mathcal{M}_d \otimes \mathcal{M}_d \otimes \mathcal{M}_d \to \mathcal{M}_d$ *is defined by* $\mathrm{id} \otimes \tau \otimes \tau$.

Proof. Let $N \subseteq \mathcal{M}_d$ be an inclusion with unitary basis $\{\lambda_i\}_{i \in I}$ for $\mathcal{M}_d/N$ that normalizes N. We denote the Jones projection corresponding to this inclusion by e_N. The basic construction gives us the von Neumann algebra $\mathcal{M}_d \otimes N'$, which is contained inside $\mathcal{M}_d \otimes \mathcal{M}_d$.

Now, consider the inclusion $\mathcal{M}_d \otimes N \subseteq \mathcal{M}_d \otimes \mathcal{M}_d$. Note that the Jones projection corresponding to this inclusion is $(1_d \otimes e_N)$. Applying the basic construction, we get the von Neumann algebra $\mathcal{M}_d \otimes \mathcal{M}_d \otimes N'$, contained inside of $\mathcal{M}_d \otimes \mathcal{M}_d \otimes \mathcal{M}_d$.

Note that $\gamma_1(N') = 1_d \otimes N'$, and that $\gamma_2(1_d \otimes N') = 1_d \otimes 1_d \otimes N'$. It follows that, $\Gamma(N') = 1_d \otimes 1_d \otimes N'$. The von Neumann algebras $1_d \otimes N' \otimes 1_d$ and $1_d \otimes 1_d \otimes N'$ are adjacent relative commutants of the basic construction applied to the inclusion $\mathcal{M}_d \otimes N \subseteq \mathcal{M}_d \otimes \mathcal{M}_d$, where the anti-isomorphism $\gamma : 1_d \otimes N' \to 1_d \otimes 1_d \otimes N'$ is simply $\gamma_2|_{1_d \otimes N'}$. Then, $\gamma(x)(1_d \otimes e_N) = \gamma_2(x)(1_d \otimes e_N)$. But, by Lemma 3.9.2, we know that $\gamma(x)(1_d \otimes e_N) = x(1_d \otimes e_N)$, for all $x \in 1_d \otimes N'$. It follows that, $\gamma_2(x)(1_d \otimes e_N) = x(1_d \otimes e_N)$, for any $x \in 1_d \otimes N'$.

Now, let $\Phi_i : 1_d \otimes 1_d \otimes N' \to 1_d \otimes 1_d \otimes N'$ be the automorphism defined by

$$x \mapsto \Gamma(\lambda_i) x \Gamma(\lambda_i^*).$$

Let $x \in N' \cap \mathcal{M}_d$. Note, $\Phi_i(\Gamma(x)) = \Gamma(\lambda_i)\Gamma(x)\Gamma(\lambda_i^*) = \Gamma(\lambda_i x \lambda_i^*) = \gamma_2 \circ \gamma_1(\lambda_i x \lambda_i^*)$. Then, each summand in (5.1) becomes

$$\lambda_i^* e_N \lambda_i \Phi_i(\Gamma(x))(1_d \otimes e_N) = \lambda_i^* e_N \lambda_i \gamma_2 \circ \gamma_1(\lambda_i x \lambda_i^*)(1_d \otimes e_N).$$

Note that γ_2 vanishes because of its adjacency with the Jones projection $(1_d \otimes e_N)$, giving us

$$\lambda_i^* e_N \lambda_i \gamma_1(\lambda_i x \lambda_i^*)(1_d \otimes e_N).$$

Next, by Equation (4.11), we know that $e_N \lambda_i \gamma_1(\lambda_i x \lambda_i^*) = e_N \lambda_i x$. So, each summand in (5.1) can be written as

$$\lambda_i^* e_N \lambda_i x(1_d \otimes e_N).$$

Then the sum in (5.1) is

$$\sum_{i \in I} \lambda_i^* e_N \lambda_i x(1_d \otimes e_N) = x(1_d \otimes e_N).$$

Inserting this into the right hand side of (5.1) yields,

$$\begin{aligned}[\mathcal{M}_d : N]E_{\mathcal{M}_d}\Big(\sum_{i\in I}\lambda_i^* e_N \lambda_i \Phi_i(\Gamma(x))(1_d\otimes e_N)\Big) &= [\mathcal{M}_d : N]E_{\mathcal{M}_d}\big(x(1_d\otimes e_N)\big)\\ &= x[\mathcal{M}_d : N](\mathrm{id}\otimes\tau\otimes\tau)\big(1_d\otimes e_N\big)\\ &= x[\mathcal{M}_d : N](\tau\otimes\tau)(e_N) \qquad (5.2)\end{aligned}$$

Note that $(\tau\otimes\tau)(e_N) = d^{-2}\dim(N)$ as $\mathrm{rank}(e_N) = \dim N$. We also know that $1_d\otimes 1_d = \sum_{i\in\mathcal{I}}(\lambda_i^*\otimes 1_d)e_N(\lambda_i\otimes 1_d)$, and taking the trace on both sides yields $d^2 = |\mathcal{I}|\dim(N)$. But, $\{\lambda_i\}_{i\in\mathcal{I}}$ is a unitary Pimsner-Popa basis and necessarily $|\mathcal{I}| = [\mathcal{M}_d : N]$ as

$$[\mathcal{M}_d : N]1_d = \sum_{i\in\mathcal{I}}\lambda_i^*\lambda_i = |\mathcal{I}|1_d.$$

Hence, $d^{-2}\dim(N) = [\mathcal{M}_d : N]^{-1}$. Then, line (5.2) is equal to x. $\square$

Remark 5.0.6. The portion of the proof above between line (5.2) and the end of the proof implies that $[\mathcal{M}_d : N]e_N$ is a $\tau\otimes\tau$ density; $(\tau\otimes\tau)([\mathcal{M}_d : N]e_N) = 1$.

With the initial setup $\mathcal{M}_d\otimes\mathcal{M}_d\otimes\mathcal{M}_d$ from above, one could of course teleport any subalgebra N of $\mathcal{M}_d$ using as resources the Jones projections $e_{\mathbb{C}} : L^2(\mathcal{M}_d,\tau)\to\mathcal{L}(\mathbb{C}1_d,\tau)$ and $e_M : L^2(M_1,\tau_1)\to L^2(\mathcal{M}_d,\tau_1)$, where $e_M = 1_d\otimes e_{\mathbb{C}}$, just by restricting the input to elements of N, since this scheme would be capable of teleporting the full algebra $\mathcal{M}_d$. The Jones projections $e_{\mathbb{C}}$ and e_M are analogues of maximal entanglement between the algebras (systems) $\mathcal{M}_d\otimes 1_d$ and $1_d\otimes\mathcal{M}_d$. In the scaffolding scheme, however, e_N and $(1_d\otimes e_N)$ represent maximal entanglement between the algebras (systems) $N'\otimes 1_d$ and $1_d\otimes N'$. Theorem 5.0.5 provides a method to teleport an algebra N' using LOCC operations with respect to $\mathcal{M}_d\otimes\mathcal{M}_d\otimes\mathcal{M}_d$ by using the less expensive resource e_N.

Further work can done to determine viable methods of quantifying the entanglement offered by a Jones projection. But certainly, the dimension of the algebras being entangled plays a key role. So for now, we use this as justification for the claim that the scaffold scheme is more resource efficient than using a scheme capable of teleporting the full algebra $\mathcal{M}_d$.

The main assumption in Theorem 5.0.5 above is the existence of a unitary Pimsner-Popa basis $\{u_i\}$ for M/N inside the normaliser $\mathcal{N}_{\mathcal{M}_d}(N')$. When $N = \mathbb{C}1_n \subseteq \mathcal{M}_n = M$, the normaliser assumption holds trivially. As we shall see in the next section, under some mild symmetry assumptions, any $\mathcal{M}_d$ scaffold N'-teleportation scheme necessitates the exsitence of a unitary basis for $\mathcal{M}_d/N$ in the normalizer of N'. We have found a class of inclusions that do not admit a Pimsner-Popa basis in the normalizer of N'.

Definition 5.0.7. A von Neumann algebra $N\cong\bigoplus_{j=1}^n\mathcal{M}_{\nu_j}$ is *homogeneous* if $\nu_i=\nu_j$ for all i,j. Otherwise, N is *non-homogeneous*.

Definition 5.0.8. A *multiplicity-free inclusion* is a von Neumann algebra inclusion where are no multiplicities associated to the summands of the representation of N inside M, e.g. $N = \bigoplus_{j=1}^{n} \mathcal{M}_{n_j} \subseteq M$.

Proposition 5.0.9. *Any basis for a multiplicity-free von Neumann inclusion $N \subseteq M$ where M is a factor and N is non-homogeneous does not lie inside the normalizer of N'.*

To build some intuition before going into the proof, we begin with presenting the example which inspired the result in the first place.

Example 5.0.10. Given the inclusion $N = \mathbb{C} \oplus \mathcal{M}_2 \subseteq \mathcal{M}_3 = M$, with normalized trace τ, the GNS construction yields Hilbert spaces $L^2(M, \tau) = \mathbb{C}^3 \otimes \mathbb{C}^3$ and $L^2(N, \tau) = \mathbb{C} \oplus (\mathbb{C}^2 \otimes \mathbb{C}^2)$ and the natural embedding of M into $L^2(M, \tau)$ defined by $x \mapsto \widehat{x}$. The basic construction gives us $M_1 = M \otimes N'$, and the standard representation of M on $L^2(M, \tau)$, $\pi^\ell : M \to \mathcal{L}(L^2(M))$, defined by $\pi^\ell(x) = x \otimes 1_3$. More explicitly, the natural embedding maps 3-by-3 matrices into a 9-dimensional column vector as follows

$$M \ni x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \mapsto \frac{1}{\sqrt{3}} \begin{bmatrix} x_{11} & x_{21} & x_{31} & x_{12} & x_{22} & x_{31} & x_{13} & x_{23} & x_{33} \end{bmatrix}^T \in L^2(M).$$

Then for $y \in N$ this embedding map looks like

$$y = \begin{bmatrix} y_{11} & 0 & 0 \\ 0 & y_{22} & y_{23} \\ 0 & y_{32} & y_{33} \end{bmatrix} \mapsto \frac{1}{\sqrt{3}} \begin{bmatrix} y_{11} & 0 & 0 & 0 & y_{22} & y_{32} & 0 & y_{23} & y_{33} \end{bmatrix}^T.$$

From here, it's easy to see that the Jones projection $e_N : L^2(M) \to L^2(N)$ must be the orthogonal projection

$$e_N = \begin{bmatrix} 1 &&&&&&&& \\ & 0 &&&&&&& \\ && 0 &&&&&& \\ &&& 0 &&&&& \\ &&&& 1 &&&& \\ &&&&& 1 &&& \\ &&&&&& 0 && \\ &&&&&&& 1 & \\ &&&&&&&& 1 \end{bmatrix}.$$

Note that N' has two minimal central projections, namely

$$q_1 = \begin{bmatrix} 1 & & \\ & 0 & \\ & & 0 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 0 & & \\ & 1 & \\ & & 1 \end{bmatrix}.$$

Then, we may identify the Jones projection, e_N, by

$$e_N = \begin{bmatrix} q_1 & & \\ & q_2 & \\ & & q_2 \end{bmatrix} = q_1 \oplus (q_2 \otimes 1_2).$$

Let $\{\lambda_i\}_{i\in I}$ be a basis for M/N, and suppose it is in the normalizer of N'. Then,

$$\begin{aligned} 1_9 = 1_3 \otimes 1_3 &= \sum_{i\in I} (\lambda_i \oplus \lambda_i \otimes 1_2)^* e_N (\lambda_i \oplus \lambda_i \otimes 1_2) \\ &= \sum_{i\in I} (\lambda_i^* \oplus \lambda_i^* \otimes 1_2)(q_1 \oplus (q_2 \otimes 1_2))(\lambda_i \oplus \lambda_i \otimes 1_2) \\ &= \sum_{i\in I} \lambda_i^* q_1 \lambda_i \oplus \lambda_i^* q_2 \lambda_i \otimes 1_2 \\ &= \sum_{i\in I} \begin{bmatrix} \lambda_i^* q_1 \lambda_i & & \\ & \lambda_i^* q_2 \lambda_i & \\ & & \lambda_i^* q_2 \lambda_i \end{bmatrix}. \end{aligned}$$

In particular, $\sum_{i\in I} \lambda_i^* q_1 \lambda_i = 1_3$. Note that $N' = \mathbb{C} \oplus \mathbb{C}1_2$. If $\{\lambda_i\}_{i\in I}$ normalizes N', then $\lambda_i^* q_1 \lambda_i$ must be non-trivial in either $\mathbb{C}$ or $\mathbb{C}1_2$. But, $\mathbb{C}1_2$ only contains non-trivial elements of rank-2, and since q_1 is the minimal central projection of smallest rank, i.e. $\text{rank}(q_1) = 1$, it follows that $\lambda_i^* q_1 \lambda_i$ is of rank-1. Hence, there must be a $t \in I$ for which λ_t lies outside of the normalizer of N'.

Remark 5.0.11. Let M be a von Neumann algebra on a Hilbert space $\mathcal{H}$. Let $x, y \in M$. It's clear that $\ker(y) \subseteq \ker(xy)$, so by rank nullity we have $\text{rank}(xy) \leq \text{rank}(y)$.

Proof of Proposition 5.0.9. Suppose we have the multiplicity-free inclusion

$$N = \bigoplus_{j=1}^{n} \mathcal{M}_{\nu_j} \subseteq \mathcal{M}_d = M,$$

where N is non-homogeneous, i.e. $\nu_i \neq \nu_j$ for some $i \neq j$. Let $q_1, ..., q_n$ be the minimal central projections of N, indexed such that $q_j N \cong M_{\nu_j}$. Since there are no multiplicities associated to the direct summands in N, it follows that $q_j M q_j = \mathcal{M}_{\nu_j}$ for all j, which implies that $q_j \mathbb{C}^d \cong \mathbb{C}^{\nu_j}$. Applying the GNS construction to the inclusion $N \subseteq M$ yields Hilbert spaces $L^2(M, \tau) = \mathbb{C}^d \otimes \mathbb{C}^d$ and $L^2(N, \tau) = \bigoplus_{j=1}^n L^2(\mathcal{M}_{\nu_j}) = \bigoplus_{j=1}^n \mathbb{C}^{\nu_j} \otimes \mathbb{C}^{\nu_j}$, and the Jones projection $e_N : L^2(M, \tau) \to L^2(N, \tau)$. The basic construction gives us the von Neumann algebra $M_1 = M \otimes N' = \bigoplus_{j=1}^n \mathcal{M}_d \otimes 1_{\nu_j}$. Let q denote the orthogonal projection in M_1 given by $q = \bigoplus_{j=1}^n q_j \otimes 1_{\nu_j}$. Then,

$$\text{rank } q = \sum_{j=1}^n \text{rank}(q_j \otimes 1_{\nu_j}) = \sum_{j=1}^n \nu_j^2 = \dim L^2(N,\tau)$$

and

$$\begin{aligned} qL^2(N,\tau) &= \Big(\bigoplus_{j=1}^n q_j \otimes 1_{\nu_j}\Big)\Big(\bigoplus_{j=1}^n \mathbb{C}^{\nu_j} \otimes \mathbb{C}^{\nu_j}\Big) \\ &= \Big(\bigoplus_{j=1}^n q_j \otimes 1_{\nu_j}\Big)\Big(\bigoplus_{j=1}^n q_j\mathbb{C}^{d} \otimes \mathbb{C}^{\nu_j}\Big) \\ &= \Big(\bigoplus_{j=1}^n q_j\mathbb{C}^{d} \otimes \mathbb{C}^{\nu_j}\Big) \\ &= \Big(\bigoplus_{j=1}^n \mathbb{C}^{\nu_j} \otimes \mathbb{C}^{\nu_j}\Big) \\ &= L^2(N,\tau). \end{aligned}$$

It follows that, $q = e_N$. Let $\{\lambda_i\}_{i\in I}$ be a basis for M/N and suppose this basis normalizes N'. Note that $1_d \otimes 1_d$ is the identity in M_1. Then,

$$\begin{aligned} 1_d \otimes 1_d &= \sum_{i\in I} (\lambda_i \otimes 1_d)^* e_N (\lambda_i \otimes 1_d) \\ &= \sum_{i\in I} (\lambda_i^* \otimes 1_d)\Big(\bigoplus_{j=1}^n q_j \otimes 1_{\nu_j}\Big)(\lambda_i \otimes 1_d) \\ &= \bigoplus_{j=1}^n \sum_{i\in I} \lambda_i^* q_j \lambda_i \otimes 1_{\nu_j}, \end{aligned}$$

which implies that $\sum_{i\in I} \lambda_i^* q_j \lambda_i = 1_d$, for all $j = 1, ..., n$.

Note that $N' = \text{span}\{q_1, ..., q_n\}$. If $x \in N'$ is non-zero, then there exists $q_r \in \{q_1, ..., q_n\}$ such that $q_r x$ is non-zero scalar multiple of q_r. Choose $q_k \in \{q_1, ..., q_n\}$ such that $\text{rank}(q_k) = \min_{j=1,...,n}\{\text{rank}(q_j)\}$, and let q_t be of larger than minimal rank. Suppose $\{\lambda_i\}_{i\in\mathcal{I}}$ normalizes N'. Since $1_d = \sum_{i\in\mathcal{I}} \lambda_i^* q_k \lambda_i$, then $q_t(\lambda_i^* q_k \lambda_i) \neq 0$ for at least one $i \in \{1, ..., n\}$. By the observation above, this implies that $q_t(\lambda_i^* q_k \lambda_i)$ is a non-zero scalar multiple of q_t since $\lambda_i^* q_k \lambda_i \in N'$. But,

$$\text{rank}\big(q_t(\lambda_i^* q_k \lambda_i)\big) \leq \text{rank}(\lambda_i^* q_k \lambda_i) \leq \text{rank}(q_k) < \text{rank}(q_t).$$

Thus, $q_t \lambda_i^* q_k \lambda_i$ cannot be a multiple of q_t. Hence, $\{\lambda_i\}_{i\in\mathcal{I}}$ does not normalize N'. $\square$

5.1 Shift covariant teleportation

We begin with a brief description of the dense coding scheme described in the 1992 paper by C. H. Bennett and S. J. Wiesner *Communication via one- and two-particle operators on Einstein–Podolsky–Rosen states* [5] : Alice and Bob share an entangled state. Bob applies one of four unitary operators to his half of the entangled state, and then sends it to Alice. By measuring the two particles jointly, Alice can reliably learn which operator Bob used.

Work by R. F. Werner [34, 2001], determines a specific class quantum teleporation schemes. His work was inspired in part by the observation that the teleportation scheme presented by C. H. Bennett, et al. in [4] also worked as a dense coding scheme [26, §2.3] ; the sender and receiver just have to swap equipment. From this, Werner hypothesized that there was a 1-to-1 relationship between all possible teleportation and dense coding schemes. However, it turns out that this fails in general. The parameters for the number of possible classical signals and the size of the quantum state to be sent play opposite roles in teleporation and dense coding. Werner shows that for a specific ratio between these two parameters that, there indeed exists a 1-to-1 correspondence between teleporation and dense schemes. Moreover, he shows that all teleportation schemes with this ratio of the parameters are of a specific form; this is what is of particular interest to us. He refers to these schemes as *tight* teleportation schemes. We have developed a class of teleportation schemes with similar constraints to Werner's tight teleportation which generalize his tight teleportation schemes to homogeneous algebras.

Definition 5.1.1. Given a Markov inclusion $N \subseteq M$ with the unique trace-preserve conditional expectation E_N, the *normalizer* of E_N, denoted $\mathcal{N}_M(E_N)$ is defined by

$$\mathcal{N}_M(E_N) = \{x \in M \mid xE_N(y)x^* = E_N(xyx^*),\ y \in \mathcal{M}_d\}.$$

Lemma 5.1.2. *Let N be a subalgebra of $\mathcal{M}_d$, and let u be a unitary in $\mathcal{N}_{\mathcal{M}_d}(E_N)$. Then*

$$(u\otimes 1_d)e_N(u^*\otimes 1_d) = (1_d\otimes u^t)e_N(1_d\otimes \bar{u}) \quad \textit{and} \quad (\bar{u}\otimes 1_d)e_N(u^t\otimes 1_d) = (1_d\otimes u^*)e_N(1_d\otimes u).$$

Proof. For any $y \in \mathcal{M}_d$,

$$\begin{aligned}(u \otimes 1_d)e_N(u^* \otimes 1_d)\widehat{[y]} &= [u\widehat{E_N(u^*y)}] = [uE_N\widehat{(u^*y)}u^*u] = [E_N\widehat{(yu^*)}u] \\ &= \pi^r(u)[\widehat{E_N(yu^*)}] = (1_d \otimes u^t)e_N\widehat{[yu^*]} \\ &= (1 \otimes u^t)e_N(1 \otimes \bar{u})\widehat{[y]}.\end{aligned}$$

This proves the first equality. The second follows from the fact that $\mathcal{N}_{\mathcal{M}_d}(E_N)$ is conjugation invariant. □

Definition 5.1.3. A *homogenous inclusion* is a von Neumann algebra inclusion $N \subseteq$

M, where each summand of N have equal multiplicities and equal dimensions, e.g. $N = \bigoplus_{j=1}^{k} 1_n \otimes \mathcal{M}_m \subseteq M$.

Let $N \subseteq \mathcal{M}_d$ be a homogeneous inclusion. Then, $N \cong \mathcal{D}_k \otimes 1_n \otimes \mathcal{M}_m$, with $knm = d$. Let $u_{pqr} = X_k^{p-1} \otimes X_n^{q-1} Y_n^{r-1} \otimes 1_m$, where X_k is the generalized Pauli-X operator in $\mathcal{M}_k$, and where X_n, Y_n are the generalized Pauli-X and Pauli-Y operators in $\mathcal{M}_n$. Then, the set $\{u_{pqr} \mid p = 1, ..., k,\ q, r = 1, ..., n\}$ forms an orthonormal unitary basis for $\mathcal{M}_d/N$ (see Example 3.10 and 3.11). Note that $N' \cap \mathcal{M}_d$ is just N'. By Theorem 5.0.5, we have

$$x = [\mathcal{M}_d : N] E_M \Big(\sum_{p=1}^{k} \sum_{q,r=1}^{n} F_{pqr} \otimes \Phi_{pqr}(\Gamma(x))(1_d \otimes e_N) \Big), \quad x \in N' \cap \mathcal{M}_d,$$

where $F_{pqr} = (u_{pqr}^* \otimes 1_d) e_N (u_{pqr} \otimes 1_d)$ and $\Phi_{pqr}(\cdot) = u_{pqr}(\cdot) u_{pqr}^*$. Let $\lambda_p = X_k^{p-1} \otimes 1_n \otimes 1_m$. Then, $\lambda_p u_{pqr} = u_{pqr} \lambda_p$, for all p, q, r. We observe the following properties, for all p, q, r:

i) $(\lambda_p^* \otimes 1_d) e_N (\lambda_p \otimes 1_d) = (1_d \otimes \bar{\lambda}_p) e_N (1_d \otimes \lambda_p^t)$ by Lemma 5.1.2,

ii) From i) and the fact that $\lambda_p u_{pqr} = u_{pqr} \lambda_p$ we get

$$(\lambda_p^* \otimes 1_d) F_{pqr} (\lambda_p \otimes 1_d) = (1_d \otimes \bar{\lambda}_p) F_{pqr} (1_d \otimes \lambda_p^t),$$

iii) From the fact that $\lambda_p u_{pqr} = u_{pqr} \lambda_p$ we get $\Phi(\lambda_p^* x \lambda_p) = \lambda_p^* \Phi(x) \lambda_p$.

iv) The cardinality of the p, q, r indexing is $kn^2 = \dim(N')$.

We call a scaffolding teleportation scheme which satisfies these properties i) - iii) a *shift covariant* teleportation scheme. The "shift" in shift covariant refers to the fact that conjugation by λ_p corresponds to cyclically shifting the blocks of N by p units. We refer to iv) as the *tightness* condition; the cardinality of the POVM and set of channels are equal to the dimension of the von Neumann algebra being teleported. This is a von Neumann algebraic analogue of Werner's tightness condition [34].

Theorem 5.1.4 (Tight shift covariant teleportation)**.** *Let $N \subseteq \mathcal{M}_d = M$ be a homogeneous inclusion such that there exists a shift covariant teleportation scheme satisfying the tightness condition consisting of*

- *a $\tau \otimes \tau$ density ρ in $N' \otimes N'$,*
- *a collection of channels $T_i : N' \to N'$,*
- *a POVM $\{F_i\}_{i \in \mathcal{I}}$ in $N' \otimes N'$,*

such that for all $x \in N'$,

$$x = E_M \Big(\sum_{i \in I} \big[F_i \otimes T_i(x) \big] \cdot \big[1_d \otimes \rho \big] \Big). \tag{5.3}$$

Then, there exists an equivalent scaffolding teleportation scheme with $\rho = [\mathcal{M}_d : N]e_N$, $F_i = (U_i^* \otimes 1_d)e_N(U_i \otimes 1_d)$, and $T_i(x) = U_i x U_i^*$, where $\{U_i\}_{i\in\mathcal{I}}$ is a unitary orthonormal basis for $\mathcal{M}_d/N$ in the normalizer of N', so that for any $x \in N'$,

$$x = [\mathcal{M}_d : N]E_M\Big(\sum_{i\in I}\big[(U_i^* \otimes 1_d)e_N(U_i \otimes 1_d) \otimes U_i(x)U_i^*\big] \cdot \big[1_d \otimes e_N\big]\Big). \tag{5.4}$$

Before we begin the proof of Theorem 5.1.4, we establish several preparatory results.

Lemma 5.1.5. *A set of D vectors $\xi_1, ..., \xi_D$ on a D-dimensional Hilbert space, $\mathcal{H}$, form an orthonormal basis if and only if*

$$\sum_{i=1}^{D} |\xi_i\rangle\langle\xi_i| = 1_{\mathcal{H}}.$$

Proof. [34, Lemma 2] □

The next lemma will be used to leverage the tightness condition in the proof of Theorem 5.1.4.

Definition 5.1.6. A the basis $\{\lambda_i\}_{i=1}^d$ for an inclusion $N \subseteq M$ is orthonormal when $E_N(\lambda_i\lambda_j^*) = \delta_{ij}1_M$.

Lemma 5.1.7. *Let $N \subseteq M$ be a Markov inclusion with Markov trace τ. A basis $\{\lambda_i\}_{i=1}^d$ for M/N is orthonormal if, and only if,*

$$d = \frac{\dim M}{\dim N}.$$

Proof. Assume $d = \dim(M)/\dim(N)$, and let $\{e_n \mid n = 1, ..., \dim(N)\}$ be an orthonormal basis for $L^2(N, \tau)$. Then,

$$e_N = \sum_{n=1}^{\dim(N)} |e_n\rangle\langle e_n|$$

so that

$$1_{L^2(M,\tau)} = \sum_{i=1}^{d} \lambda_i^* e_N \lambda_i = \sum_{i=1}^{d}\sum_{n=1}^{\dim(N)} \lambda_i^* |e_n\rangle\langle e_n|\lambda_i.$$

Since $d \cdot \dim(N) = \dim(M) = \dim(L^2(M, \tau))$, by Lemma 5.1.5, the set $\{\lambda_i^*|e_n\rangle \mid i = 1, ..., d;\ n = 1, ..., \dim(N)\}$, is an orthonormal basis of $L^2(M, \tau)$. In particular, the subspaces

$$\lambda_i^* e_N L^2(M, \tau) = \operatorname{span}\{\lambda_i^*|e_n\rangle \mid n = 1, ..., \dim(N)\}$$

are orthogonal. This implies that $\{\lambda_i^* e_N \lambda_i\}_{i=1}^d$ are mutually orthogonal projections. Then,

$$\begin{aligned}\lambda_i^* \delta_{ij} e_N \lambda_i &= \delta_{ij} \lambda_i^* e_N \lambda_i \\ &= \lambda_i^* e_N \lambda_i \lambda_j^* e_N \lambda_j \\ &= \lambda_i^* E_N(\lambda_i \lambda_j^*) e_N \lambda_j.\end{aligned}$$

Since the map $N \ni y \to y e_N \in \mathcal{L}(L^2(M,\tau))$ is injective by Corollary 3.6.7, we get

$$E_N(\lambda_i \lambda_j^*) = \delta_{ij} 1_M.$$

Conversly, suppose $\{\lambda_i\}_{i=1}^d$ is orthonormal, i.e., $E_N(\lambda_i \lambda_j^*) = \delta_{ij} 1_M$. Then, $\lambda_i^* e_N \lambda_i$ are mutually orthogonal projections, and λ_i^* is isometric on $L^2(N,\tau)$;

$$\begin{aligned}\langle \lambda_i^* e_N \xi, \lambda_i^* e_N \eta \rangle &= \langle e_N \lambda_i \lambda_i^* e_N \xi, \eta \rangle \\ &= \langle E_N(\lambda_i \lambda_i^*) e_N \xi, \eta \rangle \\ &= \langle e_N \xi, \eta \rangle \\ &= \langle e_N \xi, e_N \eta \rangle,\end{aligned}$$

it follows that each projection $\lambda_i^* e_N \lambda_i$ has rank $= \dim(N)$. The identity $1_{L^2(M,\tau)} = \sum_{i=1}^d \lambda_i^* e_N \lambda_i$ implies that $\dim L^2(M,\tau) = d \cdot \dim(N)$. □

Lemma 5.1.8. *Let $\mathcal{H}$ be a finite dimensional Hilbert space, and let $\{T_\alpha\}_\alpha$ be a finite family of CP maps $T_\alpha : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H})$ such that $\sum_\alpha T_\alpha = \mathrm{id}_{\mathcal{L}(\mathcal{H})}$. Then, for each α, there exists $\mu_\alpha \in \mathbb{R}_+$, such that*

$$T_\alpha = \mu_\alpha \mathrm{id}_{\mathcal{L}(\mathcal{H})}.$$

Proof. Due to Choi's work in [8], we know that a map, $T_\alpha \in \mathcal{L}(\mathcal{H})$ is CP if and only if it admits an expression $T_\alpha(A) = \sum_\beta K_{\alpha\beta}^* A K_{\alpha\beta}$, where $K_{\alpha\beta} \in \mathcal{L}(\mathcal{H})$ for all α, β. Note, this implies that each map $T_{\alpha\beta}(A) = K_{\alpha\beta}^* A K_{\alpha\beta}$ is CP, and $\sum_{\alpha,\beta} T_{\alpha\beta} = \mathrm{id}$. So, we may as well assume that each $T_\alpha(A) = K_\alpha^* A K_\alpha$.

Choose $A = |\psi\rangle\langle\psi|$. Then,

$$K_\alpha^* |\psi\rangle\langle\psi| K_\alpha \leq \sum_\alpha K_\alpha^* |\psi\rangle\langle\psi| K_\alpha = |\psi\rangle\langle\psi|.$$

Which implies that $K_\alpha^* \psi = \lambda_\psi \psi$, with $\lambda_\psi \in \mathbb{C}$. Since this holds for any vector $\psi \in \mathcal{H}$, it follows that, every vector in $\mathcal{H}$ is an eigenvector of K_α^*, which is only possible when K_α^* is a multiple of the identity. □

Proposition 5.1.9. *Let M be a von Neumann algebra and let $\{T_\alpha\}$ be a family of CP maps $T_\alpha : M \to M$ such that $\sum_\alpha T_\alpha = \mathrm{id}_M$. If $M = \bigoplus_{i=1}^m p_i M$, where $p_1, ..., p_m$, are the minimal central projections of M, then for each α, i there exists $\mu_\alpha^i \in \mathbb{R}_+$ such* *that*

$$T_\alpha|_{p_iM} = \mu_\alpha^i \mathrm{id}_{p_iM}.$$

Proof. Let p a minimal projection of M. Then,

$$0 \leq T_\alpha(p) \leq \sum_\alpha T_\alpha(p) = p,$$

for all α. By Lemma 3.1.7

$$T_\alpha(p) = pT_\alpha(p)p \in pMp = \mathbb{C}p,$$

by minimality. It follows that $T_\alpha|_{p_iM}(p_iM) \subseteq p_iM$, for each i. Then for each i, $T_\alpha|_{p_iM} = p_iM \to p_iM$, and $\sum_\alpha T_\alpha|_{p_iM} = \mathrm{id}_{p_iM}$. By Lemma 5.1.8, each $T_\alpha|_{p_iM} = \mu_\alpha^i \mathrm{id}_{p_iM}$. □

Lemma 5.1.10. *Let $N \subseteq M \subseteq M_1$ be an inclusion of von Neumann algebras where M_1 was obtained by the basic construction on the Markov inclusion $N \subseteq M$ with Markov trace τ. Let $e_N : L^2(M,\tau) \to L^2(N,\tau)$ denote the Jones projection obtained by the GNS construction on $N \subseteq M$ and let $E_M : M_1 \to M$ be the conditional expectation obtained from the GNS construction on the Markov inclusion $M \subseteq M_1$. Then, for any $x \in M1$ we have*

$$e_n E_M(e_n x) = [M : N]^{-1} e_N x.$$

Proof. Let $x \in M_1$. Let $\widehat{x}$ denote the image of the natural embedding of M_1 into $L^2(M_1,\tau)$ (§3.4). Then,

$$e_N E_M(e_N x)\widehat{1_{M_1}} = e_N \widehat{E_M(e_N x)} = e_N e_M e_N x \widehat{1_{M_1}},$$

where e_M is the Jones projection $e_M : L^2(M_1,\tau_1) \to L^2(M,\tau_1)$, and where τ_1 is the Markov extension of τ on M_1. By [20, Proposition 3.3.2 (v)], $e_n e_M e_N = [M : N]^1 e_N x$.
Hence, $e_N E_M(e_N x) = [M : N]^{-1} e_N x$, for all $x \in M_1$. □

The next lemma will provide a uesful decomposition of the entangled resource state in the proof of Theorem 5.1.4.

Lemma 5.1.11. *Let $N \subseteq \mathcal{M}_d = M$ be a Markov inclusion which admits a basis $\{\lambda_\ell\}_{\ell \in L}$, in the normalizer of N'. Let ω be positive in $N' \otimes N'$. Then, there exist $\{W_\ell\}_{\ell \in L} \subseteq \mathcal{N}_M(N')$ such that*

$$\omega = \sum_{\ell \in L} (W_\ell^* \otimes 1_d) e_N (W_\ell \otimes 1_d).$$

Proof. Suppose ω is positive in $N' \otimes N'$. Let $\{\lambda_\ell\}_{\ell \in L}$ be a basis for M/N. Then, by positivity of ω, we get

$$\begin{aligned}\omega &= \sqrt{\omega}\sqrt{\omega} \\ &= \sqrt{\omega}\Big(\sum_{\ell\in L}(\lambda_\ell^*\otimes 1_d)e_N(\lambda_\ell\otimes 1_d)\Big)\sqrt{\omega} \\ &= \sum_{\ell\in L}\sqrt{\omega}(\lambda_\ell^*\otimes 1_d)e_N(\lambda_\ell\otimes 1_d)\sqrt{\omega} \\ &= \sum_{\ell\in L}\sqrt{\omega}(\lambda_\ell^*\otimes 1_d)e_N\big(\sqrt{\omega}(\lambda_\ell^*\otimes 1_d)e_N\big)^*.\end{aligned}$$

By Lemma 3.8.1, $\sqrt{\omega}(\lambda_\ell^*\otimes 1_d)e_N = [M:N]E_M\big(\sqrt{\omega}(\lambda_\ell^*\otimes 1_d)e_N\big)e_N$. Now, let

$$W_\ell^* = [M:N]E_M\big(\sqrt{\omega}(\lambda_\ell^*\otimes 1_d)e_N\big)\in M,$$

so that

$$\omega = \sum_{\ell\in L}(W_\ell^*\otimes 1_d)e_N\big((W_\ell^*\otimes 1_d)e_N\big)^* = \sum_{\ell\in L}(W_\ell^*\otimes 1_d)e_N(W_\ell\otimes 1_d).$$

Consider $x\in N'$. Then, $W_\ell^* x W_\ell = [M:N]E_M\big(\sqrt{\omega}(\lambda_\ell^*\otimes 1_d)e_N\big)xW_\ell$. By the bimodule property of the conditional expectation E_M, with the representation of M on $L^2(M)$ given by $M\otimes 1_d$, we get

$$W_\ell^* x W_\ell = [M:N]E_M\big(\sqrt{\omega}(\lambda_\ell^*\otimes 1_d)e_N(xW_\ell\otimes 1_d)\big) = [M:N]E_M\big(\sqrt{\omega}(\lambda_\ell^*\otimes 1_d)e_N(W_\ell\otimes x^t)\big),$$

where the last equality uses the entanglement of e_N, Lemma 3.9.2. Then, appealing to Lemma 5.1.10,

$$e_N(W_\ell\otimes 1_d) = [M:N]e_N E_M\big(e_N(\lambda_\ell\otimes 1_d)\sqrt{\omega}\big) = e_N(\lambda_\ell\otimes 1_d)\sqrt{\omega}.$$

Plugging this back in gives us

$$W_\ell^* x W_\ell = [M:N]E_M\big(\sqrt{\omega}(\lambda_\ell^*\otimes 1_d)e_N(\lambda_\ell\otimes 1_d)\sqrt{\omega}(1_d\otimes x^t)\big).$$

Then, since e_N and $\sqrt{\omega}$ are in $N'\otimes N'$, $\lambda_\ell\in\mathcal{N}_M(N')$, and $(1_d\otimes x^t)\in 1_d\otimes N'$, it follows that $W_\ell^* x W_\ell\in N'$. So, $W_\ell\in\mathcal{N}_M(N')$, as required. $\square$

Remark 5.1.12. Let $N\subseteq\mathcal{M}_d$ be a homogeneous inclusion with $N\cong\mathcal{D}_k\otimes 1_n\otimes\mathcal{M}_m\subseteq\mathcal{M}_d$. The basic construction yields $\langle\mathcal{M}_d, N\rangle = \mathcal{M}_d\otimes(\mathcal{D}_k\otimes\mathcal{M}_n\otimes 1_m)$ inside $\mathcal{M}_d\otimes\mathcal{M}_d$. Note that $\mathcal{M}_d\otimes\mathcal{M}_d$ is the von Neumann algebra obtained by the basic construction on the inclusion $\mathbb{C}1_d\subseteq\mathcal{M}_d$. We know the conditional expectation $E:\mathcal{M}_d\otimes\mathcal{M}_d\to\mathcal{M}_d$ is defined by $E(x) = (\mathrm{id}_d\otimes\tau)(x)$, $x\in\mathcal{M}_d\otimes\mathcal{M}_d$. The conditional expectation $E':\mathcal{M}_d\otimes(\mathcal{D}_k\otimes\mathcal{M}_n\otimes 1_m)\to\mathcal{M}_d$, is defined by $\mathrm{id}_d\otimes E_{\mathcal{D}_k\to\mathbb{C}1_k}\otimes E_{\mathcal{M}_n\to\mathbb{C}1_n}\otimes\tau_m$, where τ_m is the normalized trace on $\mathcal{M}_m$. The conditional expectations $E_{\mathcal{D}_k\to\mathbb{C}1_k}$ and $E_{\mathcal{M}_n\to\mathbb{C}1_n}$ are defined by $E_{\mathcal{D}_k\to\mathbb{C}1_k}(\cdot) = \tau_k(\cdot)1_k$ and by $E_{\mathcal{M}_n\to\mathbb{C}1_n}(\cdot) = \tau_n(\cdot)1_n$. So,

$E' = \mathrm{id}_d \otimes (\tau_k \otimes \tau_n \otimes \tau_m) = \mathrm{id}_d \otimes \tau = E$. In particular, $E'(e_N) = E(e_N)$, where e_N is the Jones projections from $L^2(\mathcal{M}_d, \tau)$ onto $L^2(N, \tau)$.

We are finally ready for the proof of Theorem 5.1.4.

Proof of Theorem 5.1.4. Let $N \subseteq \mathcal{M}_d$ be a homogeneous inclusion with orthonormal unitary basis $\{u_j\}_{j\in\mathcal{I}}$, and let there be a shift covariant teleporation scheme as described in the statement of Theorem 5.1.4. The $\tau \otimes \tau$-density ρ and each F_i of the POVM are positive in $N' \otimes N'$. By Lemma 5.1.11, we may make the following decompositions:

$$\rho = \sum_{\alpha\in\mathcal{I}} (W_\alpha^* \otimes 1_d) e_N (W_\alpha \otimes 1_d)$$

$$F_i = \sum_{\beta\in\mathcal{I}} (A_{i\beta}^* \otimes 1_d) e_N (A_{i\beta} \otimes 1_d),$$

where $W_\alpha^* = [M : N] E_M(\sqrt{\rho}(u_\alpha^* \otimes 1_d) e_N)$ and $A_{i\beta} = [M : N] E_M(\sqrt{F_i}(u_\beta^* \otimes 1_d) e_N)$, with $W_k, A_{i\beta} \in \mathcal{N}_M(N')$ for all i, α, β. Plugging this into (5.3) gives us

$$x = \sum_{i,\alpha,\beta\in\mathcal{I}} E_M\big([(A_{i\beta}^* \otimes 1_d) e_N (A_{i\beta} \otimes 1_d) \otimes T_i(x)] \cdot [(1_d \otimes (W_\alpha^* \otimes 1_d) e_N (W_\alpha \otimes 1_d) \otimes 1_d)]\big). \tag{5.5}$$

Let $\Psi_{i\alpha\beta} : N' \to N'$ be defined by

$$\Psi_{i\alpha\beta}(x) = E_M\big([(A_{i\beta}^* \otimes 1_d) e_N (A_{i\beta} \otimes 1_d) \otimes T_i(x)] \cdot [1_d \otimes (W_\alpha^* \otimes 1_d) e_N (W_\alpha \otimes 1_d) \otimes 1_d]\big).$$

Since $E_M \equiv \mathrm{id} \otimes \tau \otimes \tau$, then $\Psi_{i\alpha\beta}$ becomes

$$\Psi_{i\alpha\beta}(x) = (\mathrm{id} \otimes \tau \otimes \tau)\big([(A_{i\beta}^* \otimes W_\alpha) e_N (A_{i\beta} \otimes W_\alpha^*) \otimes T_i(x)] \cdot [1_d \otimes (1_d \otimes e_N)]\big), \tag{5.6}$$

where we use cyclicity of the trace, τ, on the second tensor-leg to cycle W_α. The von Neumann algebras $1_d \otimes N'$ and $1_d \otimes 1_d \otimes N'$ are adjacent relative commutants of the basic construction applied to the inclusion $\mathcal{M}_d \otimes N \subseteq \mathcal{M}_d \otimes \mathcal{M}_d$, where the anti-isomorphism $\gamma : 1_d \otimes N' \to 1_d \otimes 1_d \otimes N'$ is simply $\gamma_2|_{1_d\otimes N'}$. Then, $\gamma(x)(1_d \otimes e_N) = \gamma_2(x)(1_d \otimes e_N)$. But, by Lemma 3.9.2, we know that $\gamma(x)(1_d \otimes e_N) = x(1_d \otimes e_N)$, for all $x \in 1_d \otimes N'$. Then, $\gamma_2(x)(1_d \otimes e_N) = x(1_d \otimes e_N)$, for any $x \in 1_d \otimes N'$. It follows that

$$1_d \otimes T_i(x)(1_d \otimes e_N) = (T_i(x)^t \otimes 1_d)(1_d \otimes e_N).$$

So, we get

$$\Psi_{i\alpha\beta}(x) = (\mathrm{id} \otimes \tau \otimes \tau)\big([(A_{i\beta}^* \otimes W_\alpha) e_N (A_{i\beta} \otimes W_\alpha^* T_i(x)^t)] \cdot [1_d \otimes (1_d \otimes e_N)]\big).$$

Then we can bring in the right-most $(\mathrm{id} \otimes \tau)$ to get

$$\Psi_{i\alpha\beta}(x) = (\mathrm{id} \otimes \tau)\big((A_{i\beta}^* \otimes W_\alpha) e_N (A_{i\beta} \otimes W_\alpha^* T_i(x)^t)(1_d \otimes (\mathrm{id} \otimes \tau)(e_N))\big).$$

By Remark 5.1.12 and Lemma 3.7.15, we get

$$\begin{aligned}\Psi_{i\alpha\beta}(x) &= [\mathcal{M}_d : N]^{-1}(\mathrm{id}\otimes\tau)\big((A^*_{i\beta}\otimes W_\alpha)e_N(A_{i\beta}\otimes W^*_\alpha T_i(x)^t)\big)\\ &= (kn^2)^{-1}(\mathrm{id}\otimes\tau)\big((A^*_{i\beta}\otimes W_\alpha)e_N(A_{i\beta}\otimes W^*_\alpha T_i(x)^t)\big).\end{aligned}$$

We now use cyclicity of the trace once more to cycle $W^*_k T_i(x)^t$, giving us

$$\Psi_{i\alpha\beta}(x) = (kn^2)^{-1}(\mathrm{id}\otimes\tau)\big((A^*_{i\beta}\otimes W^*_\alpha T_i(x)^t W_\alpha)e_N(A_{i\beta}\otimes 1_d)\big).$$

Since W_α normalizes N', for all α, it follows that $W^*_\alpha T_i(\Gamma(x))^t W_\alpha \in N'$. Then we may move $W^*_\alpha T_i(\Gamma(x))^t W_\alpha$ to the first leg (by Lemma 3.9.2) at the cost of a transpose giving us

$$\begin{aligned}\Psi_{i\alpha\beta}(x) &= (kn^2)^{-1}(\mathrm{id}\otimes\tau)\big((A^*_{i\beta}W^t_\alpha T_i(x)\bar{W}_\alpha\otimes 1_d)e_N(A_{i\beta}\otimes 1_d)\big)\\ &= (kn^2)^{-1}(A^*_{i\beta}W^t_\alpha T_i(x)\bar{W}_\alpha)(\mathrm{id}\otimes\tau)\big(e_N\big)A_{i\beta}\\ &= (kn^2)^{-1}(\mathrm{id}\otimes\tau)\big(e_N\big)A^*_{i\beta}W^t_\alpha T_i(\Gamma(x))\bar{W}_\alpha A_{i\beta}\\ &= (kn^2)^{-1}[M : N]^{-1}A^*_{i\beta}W^t_\alpha T_i(x)\bar{W}_\alpha A_{i\beta}\\ &= (kn^2)^{-2}A^*_{i\beta}W^t_\alpha T_i(x)\bar{W}_\alpha A_{i\beta}. \qquad (5.7)\end{aligned}$$

As the transpose operation is defined with respect to the N'-block diagonal basis of $\mathbb{C}^d$, it follows that $W^t_\alpha N'\bar{W}_\alpha \subseteq N'$, which implies $\Psi_{i\alpha\beta}(x)\in N'$ for each i, α, β. By Proposition 5.1.9, for each j we have

$$\Psi_{i\alpha\beta}|_{\mathcal{M}_{n_j}\otimes 1_{m_j}} = \mu^{(j)}_{i\alpha\beta}\mathrm{id}_{\mathcal{M}_{n_j}\otimes 1_{m_j}}.$$

Since $\sum_{i,\alpha,\beta}\Psi_{i\alpha\beta} = \mathrm{id}_{N'}$, then for each j, $\sum_{i,\alpha,\beta}\mu^{(j)}_{i\alpha\beta} = 1$. Hence, for all i,α,β, there is a j such that $\mu^{(j)}_{i\alpha\beta} > 0$. By the shift covariance assumption, we have

$$\Psi_{i\alpha\beta}(\lambda^*_\ell(\cdot)\lambda_\ell) = \lambda^*_\ell\Psi_{i\alpha\beta}(\cdot)\lambda_\ell,$$

for all $\ell = 1,...,k$. Then for all i,α,β, we have $\mu^{(j-\ell)}_{i\alpha\beta} = \mu^{(j)}_{i\alpha\beta}$. It follows that the $\mu^{(j)}_{i\alpha\beta}$'s are independent of j. In particular, $\mu_{i\alpha\beta} > 0$, for all i,α,β, giving us

$$\Psi_{i\alpha\beta}|_{\mathcal{M}_{n_j}\otimes 1_{m_j}} = \mu_{i\alpha\beta}\mathrm{id}_{\mathcal{M}_{n_j}\otimes 1_{m_j}},$$

for all j. It follows that $\Psi_{i\alpha\beta} = \mu_{i\alpha\beta}\mathrm{id}_{N'}$ on N'. Then,

$$(kn^2)^{-2}A^*_{i\beta}W^t_\alpha\bar{W}_\alpha A_{i\beta} = (kn^2)^{-2}A^*_{i\beta}W^t_\alpha T_i(1)\bar{W}_\alpha A_{i\beta} = (kn^2)^{-2}\mu_{i\alpha\beta}1_d.$$

Absorbing $(kn^2)^{-2} > 0$ into $\mu_{i\alpha\beta}$, we get that

$$A^*_{i\beta}W^t_\alpha\bar{W}_\alpha A_{i\beta} = \mu_{i\alpha\beta}1_d.$$

Then, $\bar{W}_\alpha A_{i\beta}/\sqrt{\mu_{i\alpha\beta}}$ is an isometry in $\mathcal{M}_d$, and hence a unitary, as

$$\left(\frac{\bar{W}_\alpha A_{i\beta}}{\sqrt{\mu_{i\alpha\beta}}}\right)^*\left(\frac{\bar{W}_\alpha A_{i\beta}}{\sqrt{\mu_{i\alpha\beta}}}\right) = 1_d.$$

Necessarily, both $\bar{W}_\alpha$ and $A_{i\beta}$ are invertible. Let $U_{i\alpha\beta} = \bar{W}_\alpha A_{i\beta}/\sqrt{\mu_{i\alpha\beta}}$. Then,

$$\begin{aligned} A_{i\beta} &= \sqrt{\mu_{i\alpha\beta}} \cdot \bar{W}_\alpha^{-1} U_{i\alpha\beta} \\ &= \sqrt{\frac{\mu_{i\alpha\beta}}{\mu_{i\alpha\beta'}}}\left(\sqrt{\mu_{i\alpha\beta'}} \cdot \bar{W}_\alpha^{-1} U_{i\alpha\beta'}\right) U_{i\alpha\beta'}^* U_{i\alpha\beta} \\ &= A_{i\beta'} \cdot \sqrt{\frac{\mu_{i\alpha\beta}}{\mu_{i\alpha\beta'}}} \cdot U_{i\alpha\beta'}^* U_{i\alpha\beta}. \end{aligned}$$

Since, $\Psi_{i\alpha\beta} = A_{i\beta}^* W_\alpha^t T_i(\cdot)\bar{W}_\alpha A_{i\beta} = \mu_{i\alpha\beta}\mathrm{id}_{N'}$, it implies that $U_{i\alpha\beta}^* T_i(\cdot) U_{i\alpha\beta} = \mathrm{id}_{N'}$. So, for all $x \in N'$,

$$T_i(x) = U_{i\alpha\beta} x U_{i\alpha\beta}^*, \tag{5.8}$$

for all i, α, β. Then, $U_{i\alpha\beta} \in \mathcal{U}(N_M(N')) = \mathcal{U}(N_M(N))$, and from the independence of α and β in (5.8) we get

$$(U_{i\alpha'\beta'}^* U_{i\alpha\beta})x = x(U_{i\alpha'\beta'}^* U_{i\alpha\beta}), \tag{5.9}$$

for all $x \in N'$, and for all $\alpha, \alpha', \beta, \beta'$. Hence, for each i, $U_{i\alpha'\beta'}^* U_{i\alpha\beta} \in (N')' = N$. Let $V_{i\alpha\beta'\beta} = \sqrt{\frac{\mu_{i\alpha\beta}}{\mu_{i\alpha\beta'}}} U_{i\alpha\beta'}^* U_{i\alpha\beta}$ in N. Then, conjugating $x \in N'$ by $A_{i\beta}^*$ gives us

$$A_{i\beta}^* x A_{i\beta} = V_{i\alpha\beta'\beta}^* A_{i\beta'}^* x A_{i\beta'} V_{i\alpha\beta'\beta} = V_{i\alpha\beta'\beta}^* V_{i\alpha\beta'\beta} \cdot A_{i\beta'}^* x A_{i\beta'} = \frac{\mu_{i\alpha\beta}}{\mu_{i\alpha\beta'}} A_{i\beta'}^* x A_{i\beta'}.$$

Fix β_0 in $\mathcal{I}$. Then,

$$\begin{aligned} 1_d \otimes 1_d &= \sum_{i \in \mathcal{I}} F_i \\ &= \sum_{i,\beta \in \mathcal{I}} (A_{i\beta}^* \otimes 1_d) e_N (A_{i\beta} \otimes 1_d) \\ &= \sum_{i,\beta \in \mathcal{I}} \frac{\mu_{i\alpha\beta}}{\mu_{i\alpha\beta_0}} (A_{i\beta_0}^* \otimes 1_d) e_N (A_{i\beta_0} \otimes 1_d). \end{aligned}$$

This gives us a basis for M/N defined by $\{\sqrt{\mu_{i\alpha}/\mu_{i\alpha\beta_0}} A_{i\beta_0}\}_{i \in \mathcal{I}}$, where $\mu_{i\alpha} = \sum_{\beta \in \mathcal{I}} \mu_{i\alpha\beta}$, and by Lemma 5.1.7 and the assumption that $|\mathcal{I}| = \dim(M)/\dim(N)$, this basis is orthonormal.

It remains to show that W_{α_0} is the scalar multiple of a unitary. This will allow us to show that there exists a set of scalars $\{z_i\}_{\imath \in \mathcal{I}}$ such that $\{z_i U_{i\alpha_0\beta_0}\}_{i \in \mathcal{I}}$ forms a unitary orthonormal basis for $\mathcal{M}_d/N$ in the normalizer of N', completing the proof.

Recall that $A_{i\beta_0} = \sqrt{\mu_{i\alpha\beta_0}} \bar{W}_\alpha^{-1} U_{i\alpha\beta_0}$. Then,

$$\bar{W}_\alpha = \sqrt{\mu_{i\alpha\beta_0}} U_{i\alpha\beta} A_{i\beta_0}^{-1}.$$

Fix $\alpha_0 \in \mathcal{I}$. Then, we get the following sequence of equalities:

$$\begin{aligned}\bar{W}_{\alpha_0} &= \sqrt{\mu_{i\alpha_0\beta_0}} U_{i\alpha_0\beta_0} A_{i\beta_0}^{-1} \\ &= \sqrt{\mu_{i\alpha_0\beta_0}} U_{i\alpha_0\beta_0} \big(\sqrt{\mu_{i\alpha\beta_0}} \bar{W}_\alpha^{-1} U_{i\alpha\beta_0}\big)^{-1} \\ &= \sqrt{\frac{\mu_{i\alpha_0\beta_0}}{\mu_{i\alpha\beta_0}}} U_{i\alpha_0\beta_0} U_{i\alpha\beta_0}^* \bar{W}_\alpha.\end{aligned}$$

It follows that,

$$W_\alpha = \sqrt{\frac{\mu_{i\alpha\beta_0}}{\mu_{i\alpha_0\beta_0}}} \bar{U}_{i\alpha\beta_0} \bar{U}_{i\alpha_0\beta_0}^* W_{\alpha_0}.$$

Since $U_{i\alpha_0\beta_0} U_{i\alpha\beta_0}^* \in N$, then so is $\bar{U}_{i\alpha\beta_0} \bar{U}_{i\alpha_0\beta_0}^*$, this gives us the following equality for the $\tau \otimes \tau$-density ρ:

$$\begin{aligned}\rho &= \sum_{\alpha \in \mathcal{I}} (W_\alpha^* \otimes 1_d) e_N (W_\alpha \otimes 1_d) \\ &= \sum_{\alpha \in \mathcal{I}} \frac{\mu_{i\alpha\beta_0}}{\mu_{i\alpha_0\beta_0}} (W_{\alpha_0}^* \bar{U}_{i\alpha_0\beta_0} \bar{U}_{i\alpha\beta_0}^* \otimes 1_d) e_N (\bar{U}_{i\alpha\beta_0} \bar{U}_{i\alpha_0\beta_0}^* W_{\alpha_0} \otimes 1_d) \\ &= \sum_{\alpha \in \mathcal{I}} \frac{\mu_{i\alpha\beta_0}}{\mu_{i\alpha_0\beta_0}} (W_{\alpha_0}^* \otimes 1_d) e_N (W_{\alpha_0} \otimes 1_d).\end{aligned}$$

Note that $A_{i\beta_0} = \sqrt{\mu_{i\alpha_0\beta_0}} \bar{W}_{\alpha_0}^{-1} U_{i\alpha_0\beta_0}$ so, we can express the orthonormal basis $\{\sqrt{\mu_{i\alpha_0}/\mu_{i\alpha_0\beta_0}} A_{i\beta_0}\}_{i \in \mathcal{I}}$ by $\{\sqrt{\mu_{i\alpha_0}} \bar{W}_{\alpha_0}^{-1} U_{i\alpha_0\beta_0}\}_{i \in \mathcal{I}}$. Orthonormality of the basis gives us

$$\begin{aligned}\delta_{ij} 1_d &= E_N(\sqrt{\mu_{i\alpha_0}} \bar{W}_{\alpha_0}^{-1} U_{i\alpha_0\beta_0} U_{j\alpha_0\beta}^* (W_{\alpha_0}^t)^{-1} \sqrt{\mu_{j\alpha_0}}) \\ &= \mu_{i\alpha_0} E_N(\bar{W}_{\alpha_0}^{-1} U_{i\alpha_0\beta} U_{j\alpha_0\beta_0}^* (W_{\alpha_0}^t)^{-1}).\end{aligned} \tag{5.10}$$

When $i = j$, the term $U_{i\alpha_0\beta_0} U_{j\alpha_0\beta_0}^*$ vanishes, so that

$$E_N(W_{\alpha_0}^{-1} (W_{\alpha_0}^*)^{-1}) = \frac{1}{\mu_{i\alpha_0}} 1_d.$$

By Polar decomposition, we may write

$$W_{\alpha_0} = U_{\alpha_0} |W_{\alpha_0}| \equiv U_{\alpha_0} \sqrt{W_{\alpha_0}^* W_{\alpha_0}}.$$

From the normalization property of W_{α_0}, it follows that $W_{\alpha_0}^* W_{\alpha_0} \in N'$. Then, $|W_{\alpha_0}| \in$

N', and hence $|W_{\alpha_0}|^{-1} \in N'$. Then, $W_{\alpha_0}^{-1} = |W_{\alpha_0}|^{-1}U_{\alpha_0}^*$, and

$$\begin{aligned} \bar{W}_{\alpha_0}^{-1}U_{i\alpha_0\beta_0} &= |\bar{W}_{\alpha_0}|^{-1}U_{\alpha_0}^t U_{i\alpha_0\beta_0} \\ U_{i\alpha_0\beta_0}^*(W_{\alpha_0}^t)^{-1} &= U_{i\alpha_0\beta_0}^*\bar{U}_{\alpha_0}|\bar{W}_{\alpha_0}|^{-1} \end{aligned}$$

Let V_i be the unitary contained in the normalizer of N defined by $V_i = U_{\alpha_0}^t U_{i\alpha_0\beta_0}$. Then (5.10) becomes

$$\delta_{ij} = \mu_{i\alpha_0} E_N(|\bar{W}_{\alpha_0}|^{-1}V_iV_j^*|\bar{W}_{\alpha_0}|^{-1}).$$

Let $\langle x, y\rangle_{\alpha_0}$ be the N-valued inner product defined by

$$\langle x, y\rangle_{\alpha_0} \equiv E_N(|\bar{W}_{\alpha_0}|^{-1}xy^*|\bar{W}_{\alpha_0}|^{-1}) = \langle|\bar{W}_{\alpha_0}|^{-1}x,\ |\bar{W}_{\alpha_0}|^{-1}y\rangle_N.$$

To show that this inner product is faithful, consider $x \in M_d$ with

$$\langle x, x\rangle_{\alpha_0} = 0 = E_N(|\bar{W}_{\alpha_0}|^{-1}xx^*|\bar{W}_{\alpha_0}|^{-1}).$$

By faithfulness of E_N, this implies $|\bar{W}_{\alpha_0}|^{-1}xx^*|\bar{W}_{\alpha_0}|^{-1} = 0$. But $|\bar{W}_{\alpha_0}|^{-1}$ are invertible for all α_0, so we must have $xx^* = 0$. Hence, $\langle\cdot,\cdot\rangle_{\alpha_0}$ is a faithful N-valued inner product on M_d.

It follows that $\{\sqrt{\mu_{i\alpha_0}}V_i\}_{i\in\mathcal{I}}$ is an orthonormal basis for M/N with respect to the inner product $\langle\cdot,\cdot\rangle_{\alpha_0}$. To see this, recall the orthonormal basis $\{\sqrt{\mu_{i\alpha_0}}\bar{W}_{\alpha_0}^{-1}U_{i\alpha_0\beta_0}\}_{i\in\mathcal{I}}$. Then, for any $x \in M_d$, we have

$$\begin{aligned} x &= \sum_{i\in\mathcal{I}} E_N(xV_i^*|\bar{W}_{\alpha_0}|^{-1}\sqrt{\mu_{i\alpha_0}})\sqrt{\mu_{i\alpha_0}}|\bar{W}_{\alpha_0}|^{-1}V_i \\ &= \sum_{i\in\mathcal{I}} \mu_{i\alpha_0}|\bar{W}_{\alpha_0}|^{-1}E_N(xV_i^*|\bar{W}_{\alpha_0}|^{-1}\sqrt{\mu_{i\alpha_0}})V_i \\ &= \sum_{i\in\mathcal{I}} \mu_{i\alpha_0}|\bar{W}_{\alpha_0}|^{-1}\langle|\bar{W}_{\alpha_0}|x, V_i\rangle_{\alpha_0}V_i, \end{aligned}$$

where the second equality holds by the fact that $|\bar{W}_{\alpha_0}|^{-1} \in N'$. Multiplication on the left by $|\bar{W}_{\alpha_0}|$ on both sides of the equation above yields

$$|\bar{W}_{\alpha_0}|x = \sum_{i\in\mathcal{I}}\langle|\bar{W}_{\alpha_0}|x, \sqrt{\mu_{i\alpha_0}}V_i\rangle_{\alpha_0}\sqrt{\mu_{i\alpha_0}}V_i.$$

Since this holds for any $x \in \mathcal{M}_d$, it follows that

$$x = \sum_{i\in\mathcal{I}}\langle x, \sqrt{\mu_{i\alpha_0}}V_i\rangle_{\alpha_0}\sqrt{\mu_{i\alpha_0}}V_i.$$

Hence, $\{\sqrt{\mu_{i\alpha_0}}V_i\}_{i\in\mathcal{I}}$ is a basis for M/N, and by cardinality of $\mathcal{I}$, it is orthonormal by Lemma 5.1.7. In particular, it satisfies a completeness relation so that, for any

$x, y \in M_d$, we have

$$\langle x, y\rangle_{\alpha_0} = \sum_{i\in\mathcal{I}} \langle x, \sqrt{\mu_{i\alpha_0}}V_i\rangle_{\alpha_0}\langle\sqrt{\mu_{i\alpha_0}}V_i, y\rangle_{\alpha_0}. \tag{5.11}$$

Let the left hand side and right hand side of equation (5.11) be denoted by LHS and RHS, respectively. The final big step in this proof is to take the normalized trace, τ, of LHS and RHS. Then, through some algebraic manipulation and Schur's orthogonality relations, we will have that the W_{α_0}'s are scalar multiples of a unitary.

Let $x, y \in M_d$ be the rank-1 operators defined by $x = |\xi\rangle\langle\eta|$ and $y = |\varphi\rangle\langle\psi|$. The normalized trace, τ, of LHS gives us

$$\begin{aligned}\tau(LHS) &= \tau\big(E_N(|\bar{W}_{\alpha_0}|^{-1}xy^*|\bar{W}_{\alpha_0}|^{-1})\big)\\ &= \tau\big(|\bar{W}_{\alpha_0}|^{-1}|\xi\rangle\langle\eta|\psi\rangle\langle\varphi||\bar{W}_{\alpha_0}|^{-1}\big)\\ &= \langle\eta,\psi\rangle\tau\big(|\bar{W}_{\alpha_0}|^{-1}|\xi\rangle\langle\varphi||\bar{W}_{\alpha_0}|^{-1}\big)\\ &= \frac{1}{d}\langle\eta,\psi\rangle\langle|\bar{W}_{\alpha_0}|^{-1}\xi,\ |\bar{W}_{\alpha_0}|^{-1}\varphi\rangle.\end{aligned}$$

Applying τ to RHS

$$\begin{aligned}\tau(RHS) &= \sum_{i\in\mathcal{I}}\mu_{i\alpha_0}\tau\Big(E_N\big(|\bar{W}_{\alpha_0}|^{-1}xV_i^*|\bar{W}_{\alpha_0}|^{-1}\big)E_N\big(|\bar{W}_{\alpha_0}|^{-1}V_iy^*|\bar{W}_{\alpha_0}|^{-1}\big)\Big)\\ &= \sum_{i\in\mathcal{I}}\mu_{i\alpha_0}\tau\Big(E_N\big(|\bar{W}_{\alpha_0}|^{-1}V_iy^*|\bar{W}_{\alpha_0}|^{-1}\big)E_N\big(|\bar{W}_{\alpha_0}|^{-1}xV_i^*|\bar{W}_{\alpha_0}|^{-1}\big)\Big)\\ &= \sum_{i\in\mathcal{I}}\mu_{i\alpha_0}\tau\Big(E_N\big(E_N\big(|\bar{W}_{\alpha_0}|^{-1}V_iy^*|\bar{W}_{\alpha_0}|^{-1}\big)|\bar{W}_{\alpha_0}|^{-1}xV_i^*|\bar{W}_{\alpha_0}|^{-1}\big)\Big)\\ &= \sum_{i\in\mathcal{I}}\mu_{i\alpha_0}\tau\Big(E_N\big(|\bar{W}_{\alpha_0}|^{-1}V_iy^*|\bar{W}_{\alpha_0}|^{-1}\big)|\bar{W}_{\alpha_0}|^{-1}xV_i^*|\bar{W}_{\alpha_0}|^{-1}\Big)\\ &= \sum_{i\in\mathcal{I}}\mu_{i\alpha_0}\langle E_N(|\bar{W}_{\alpha_0}|^{-1}V_i|\psi\rangle\langle\varphi||\bar{W}_{\alpha_0}|^{-1})|\bar{W}_{\alpha_0}|^{-1}\xi,\ |\bar{W}_{\alpha_0}|^{-1}V_i\eta\rangle\\ &= \sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\langle(\tau_{n_j}\otimes\mathrm{id})(P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\psi\rangle\langle\varphi||\bar{W}_{\alpha_0}|^{-1}P_j)|\bar{W}_{\alpha_0}|^{-1}\xi,\ |\bar{W}_{\alpha_0}|^{-1}V_i\eta\rangle.\end{aligned} \tag{5.12}$$

Focusing on the term from the equation above given by

$$(\tau_{n_j}\otimes\mathrm{id})(P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\psi\rangle\langle\varphi||\bar{W}_{\alpha_0}|^{-1}P_j).$$

Let G_j be a finite group of unitaries in $\mathcal{M}_{n_j}$ that act irreducibly on $\mathbb{C}^{n_j}$, such that

$$\tau_{n_j}(\cdot)1_{n_j} = \frac{1}{|G_j|}\sum_{U\in G_j} U^*(\cdot)U$$

(For example, take G_j to be the group generated by the generalized Pauli-X and -Y in $\mathcal{M}_{n_j} \subseteq \mathcal{L}(\mathcal{H}_j)$ modulo the group $\{\pm 1_{\mathcal{H}_j}\}$). This, gives us

$$\begin{aligned}&(\tau_{n_j}\otimes \mathrm{id})(P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\psi\rangle\langle\varphi||\bar{W}_{\alpha_0}|^{-1}P_j)\\ &= \frac{1}{|G_j|}\sum_{U\in G_j}(U^*\otimes 1_{m_j})P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\psi\rangle\langle\varphi||\bar{W}_{\alpha_0}|^{-1}P_j(U\otimes 1_{m_j}),\end{aligned} \tag{5.13}$$

where $1_{m_j} = \sum_{\ell=1}^{m_j}|e_\ell^{(j)}\rangle\langle e_\ell^{(j)}|$ with the standard basis $\{e_\ell^{(j)} \mid \ell = 1, ..., m_j\}$ for $\mathbb{C}^{m_j}$. Note that

$$(\mathrm{id}\otimes|e_\ell^{(j)}\rangle\langle e_\ell^{(j)}|) = (\mathrm{id}\otimes\langle e_\ell^{(j)}|)^*(\mathrm{id}\otimes\langle e_\ell^{(j)}|),$$

where $\mathrm{id}\otimes\langle e_\ell^{(j)}| : \mathbb{C}^d\otimes\mathbb{C}^d \to \mathbb{C}^d$ is the slice map with dual functional $\varepsilon_{j\ell}^*(\psi) = \langle\psi, e_\ell^{(j)}\rangle$. Picking back up from line (5.13) yields

$$\frac{1}{|G_j|}\sum_{U\in G_j}\sum_{r,s=1}^{m_j}(U^*\otimes\varepsilon_{jr}^*\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\psi\rangle\langle\varphi||\bar{W}_{\alpha_0}|^{-1}P_j(U\otimes\varepsilon_{js}^*\varepsilon_{js}).$$

Inserting this into (5.12), we get

$$\begin{aligned}\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{|G_j|}\sum_{U\in G_j}\sum_{r,s=1}^{m_j}\langle(U^*\otimes\varepsilon_{jr}^*\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\psi\rangle\langle\varphi||\bar{W}_{\alpha_0}|^{-1}P_j(U\otimes\varepsilon_{js}^*\varepsilon_{js})|\bar{W}_{\alpha_0}|^{-1}\xi,\\ |\bar{W}_{\alpha_0}|^{-1}V_i\eta\rangle.\end{aligned}$$

Note that for the term in the first argument above given by $\langle\varphi||\bar{W}_{\alpha_0}|^{-1}P_j(U\otimes\varepsilon_{js}^*\varepsilon_{js})|\bar{W}_{\alpha_0}|^{-1}\xi$, we have

$$\langle\varphi||\bar{W}_{\alpha_0}|^{-1}P_j(U\otimes\varepsilon_{js}^*\varepsilon_{js})|\bar{W}_{\alpha_0}|^{-1}\xi = \langle\ |\bar{W}_{\alpha_0}|^{-1}P_j(U\otimes\varepsilon_{js}^*\varepsilon_{js})|\bar{W}_{\alpha_0}|^{-1}\xi,\ \varphi\rangle,$$

so we may treat it as a scalar and bring it out of the linear form:

$$\begin{aligned}\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{|G_j|}\sum_{U\in G_j}\sum_{r,s=1}^{m_j}\langle(U^*\otimes\varepsilon_{jr}^*\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\psi\rangle,\ |\bar{W}_{\alpha_0}|^{-1}V_i\eta\rangle\\ \cdot\langle|\bar{W}_{\alpha_0}|^{-1}P_j(U\otimes\varepsilon_{js}^*\varepsilon_{js})|\bar{W}_{\alpha_0}|^{-1}\xi,\ \varphi\rangle\end{aligned}$$

Note that P_j commutes with the operator $(U^*\otimes\varepsilon_{jr}^*\varepsilon_{jr})$ for all j, r. Then, we may leave one copy of P_j where it is and insert another copy of P_j on the other side of the inner product (this will be needed to appeal to Schur's orthogonality relations in subsequent steps of this calculation), so we get

$$\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{|G_j|}\sum_{U\in G_j}\sum_{r,s=1}^{m_j}\langle (U^*\otimes\varepsilon_{jr}^*\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\psi\rangle,\ P_j|\bar{W}_{\alpha_0}|^{-1}V_i\eta\rangle \\ \cdot\langle |\bar{W}_{\alpha_0}|^{-1}P_j(U\otimes\varepsilon_{js}^*\varepsilon_{js})P_j|\bar{W}_{\alpha_0}|^{-1}\xi,\ \varphi\rangle.$$

Next we bring $(U^*\otimes\varepsilon_{jr}^*)$ and $|\bar{W}_{\alpha_0}|^{-1}P_j(\mathrm{id}\otimes\varepsilon_{js}^*)$ to the second argument of the inner product at the cost of an adjoint,

$$=\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{|G_j|}\sum_{U\in G_j}\sum_{r,s=1}^{m_j}\langle (\mathrm{id}\otimes\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i\psi,\ (U\otimes\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i\eta\rangle \\ \cdot\langle (U\otimes\varepsilon_{js})P_j|\bar{W}_{\alpha_0}|^{-1}\varphi,\ (\mathrm{id}\otimes\varepsilon_{js})P_j|\bar{W}_{\alpha_0}|^{-1}\xi\rangle.$$

Applying conjugation twice to the first inner product in the sum above, yields

$$\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{|G_j|}\sum_{U\in G_j}\sum_{r,s=1}^{m_j}\langle (U\otimes\varepsilon_{js})P_j|\bar{W}_{\alpha_0}|^{-1}\xi,\ (\mathrm{id}\otimes\varepsilon_{js})P_j|\bar{W}_{\alpha_0}|^{-1}\varphi\rangle \\ \cdot\overline{\langle (U\otimes\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i\eta,\ (\mathrm{id}\otimes\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i\psi\rangle}.$$

The sum is now in the form

$$\frac{1}{|G_j|}\sum_{U\in G_j}\langle Uw_1,v_1\rangle\overline{\langle Uw_2,v_2\rangle},$$

where the vectors v_1,w_1,v_2,w_2 are in $\mathbb{C}^{n_j}$ since P_j first projects down to $\mathbb{C}^{n_j}\otimes\mathbb{C}^{m_j}$ then the slice maps $(1_{n_j}\otimes\varepsilon_{jr})$ ensure each argument is in $\mathbb{C}^{n_j}$. Thus we may appeal to Schur's orthogonality relations [12, Theorem 5.8] so that

$$\frac{1}{|G_j|}\sum_{U\in G_j}\langle Uw_1,v_1\rangle\overline{\langle Uw_2,v_2\rangle}=\frac{1}{n_j}\langle w_1,w_2\rangle\langle v_2,v_1\rangle.$$

By homogeneity of the inlcusion we have $n_j=n_\ell$ and $m_j=m_\ell$, for all $j,\ell=1,...,k$. Then, $\tau(\mathrm{RHS})$ becomes,

$$\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{n}\sum_{r,s=1}^{m_j}\langle (\mathrm{id}\otimes\varepsilon_{js})P_j|\bar{W}_{\alpha_0}|^{-1}\xi,\ (\mathrm{id}\otimes\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i\eta\rangle \\ \cdot\langle (\mathrm{id}\otimes\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i\psi,\ (\mathrm{id}\otimes\varepsilon_{js})P_j|\bar{W}_{\alpha_0}|^{-1}\varphi\rangle.$$

Now, shifting $(\mathrm{id} \otimes \varepsilon_{jr})$ and $(\mathrm{id} \otimes \varepsilon_{js})$ to the first argument:

$$\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{n}\sum_{r,s=1}^{m_j}\langle(\mathrm{id}\otimes\varepsilon_{jr}^{*}\varepsilon_{js})P_j|\bar{W}_{\alpha_0}|^{-1}\xi,\ P_j|\bar{W}_{\alpha_0}|^{-1}V_i\eta\rangle$$
$$\cdot\langle(\mathrm{id}\otimes\varepsilon_{js}^{*}\varepsilon_{jr})P_j|\bar{W}_{\alpha_0}|^{-1}V_i\psi,\ P_j|\bar{W}_{\alpha_0}|^{-1}\varphi\rangle$$
$$=\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{n}\sum_{r,s=1}^{m_j}\langle(\mathrm{id}\otimes|e_r^{(j)}\rangle\langle e_s^{(j)}|)P_j|\bar{W}_{\alpha_0}|^{-1}\xi,\ P_j|\bar{W}_{\alpha_0}|^{-1}V_i\eta\rangle$$
$$\cdot\langle(\mathrm{id}\otimes|e_s^{(j)}\rangle\langle e_r^{(j)}|)P_j|\bar{W}_{\alpha_0}|^{-1}V_i\psi,\ P_j|\bar{W}_{\alpha}|^{-1}\varphi\rangle$$
$$=\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{n}\sum_{r,s=1}^{m_j}\langle|\bar{W}_{\alpha_0}|^{-1}\xi,\ (\mathrm{id}\otimes|e_s^{(j)}\rangle\langle e_r^{(j)}|)P_j|\bar{W}_{\alpha_0}|^{-1}V_i\eta\rangle$$
$$\cdot\langle\psi,\ V_i^{*}|\bar{W}_{\alpha_0}|^{-1}P_j(\mathrm{id}\otimes|e_r^{(j)}\rangle\langle e_s^{(j)}|)|\bar{W}_{\alpha_0}|^{-1}\varphi\rangle.$$

Note that,

$$\sum_{r,s=1}^{m_j}\big[(\mathrm{id}\otimes|e_s^{(j)}\rangle\langle e_r^{(j)}|)\big]P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\eta\rangle\langle\psi|V_i^{*}|\bar{W}_{\alpha_0}|^{-1}P_j\big[(\mathrm{id}\otimes|e_r^{(j)}\rangle\langle e_s^{(j)}|)\big]$$
$$=(\mathrm{id}\otimes\mathrm{tr}_{m_j})(P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\eta\rangle\langle\psi|V_i^{*}|\bar{W}_{\alpha_0}|^{-1}P_j)\otimes 1_{m_j}$$

Then, $\tau(\mathrm{RHS})$ becomes:

$$\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{n}\langle|\bar{W}_{\alpha_0}|^{-1}\xi|(\mathrm{id}\otimes\mathrm{tr}_{m_j})(P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\eta\rangle\langle\psi|V_i^{*}|\bar{W}_{\alpha_0}|^{-1}P_j)\otimes 1_{m_j}||\bar{W}_{\alpha_0}|^{-1}\varphi\rangle$$
$$=\langle|\bar{W}_{\alpha_0}|^{-1}\xi|\left(\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{n}(\mathrm{id}\otimes\mathrm{tr}_{m_j})(P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\eta\rangle\langle\psi|V_i^{*}|\bar{W}_{\alpha_0}|^{-1}P_j)\otimes 1_{m_j}\right)||\bar{W}_{\alpha_0}|^{-1}\varphi\rangle$$

Recall that

$$\tau(\mathrm{LHS})=\frac{1}{d}\langle\eta,\psi\rangle\langle|\bar{W}_{\alpha_0}|^{-1}\xi,\ |\bar{W}_{\alpha_0}|^{-1}\varphi\rangle.$$

It follows that,

$$\frac{1}{d}\langle\eta,\psi\rangle 1_d=\sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{n}(\mathrm{id}\otimes\mathrm{tr}_{m_j})(P_j|\bar{W}_{\alpha_0}|^{-1}V_i|\eta\rangle\langle\psi|V_i^{*}|\bar{W}_{\alpha_0}|^{-1}P_j)\otimes 1_{m_j}.$$

Note that both $\tau(\mathrm{LHS})$ and $\tau(\mathrm{RHS})$ are linear with respect to rank-1 operators $|\eta\rangle\langle\psi|$.

Consider 1_d. Then, we get

$$\begin{aligned}\frac{1}{d}1_d &= \sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{n}(\mathrm{id}\otimes\mathrm{tr}_{m_j})(P_j|\bar{W}_{\alpha_0}|^{-1}V_i1_dV_i^*|\bar{W}_{\alpha_0}|^{-1}P_j)\otimes 1_{m_j}\\ &= \sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\mu_{i\alpha_0}\frac{1}{n}(\mathrm{id}\otimes\mathrm{tr}_{m_j})(P_j|\bar{W}_{\alpha_0}|_j^{-2}P_j)\otimes 1_{m_j},\end{aligned}$$

where $|\bar{W}_{\alpha_0}|^{-2} = \bigoplus_{j=1}^{k}|\bar{W}_{\alpha_0}|_j^{-2}\otimes 1_{m_j}$. So, we have

$$\frac{1}{d}1_d = \sum_{i\in\mathcal{I}}\sum_{j=1}^{k}\frac{m}{n}\mu_{i\alpha_0}|\bar{W}_{\alpha_0}|_j^{-2}\otimes 1_{m_j} = \frac{m}{n}\big(\sum_{i\in\mathcal{I}}\mu_{i\alpha_0}\big)|\bar{W}_{\alpha_0}|^{-2},$$

which implies $|\bar{W}_{\alpha_0}|^{-2}\in\mathbb{C}1_n$. Hence, $W_{\alpha_0}^*W_{\alpha_0} = |W_{\alpha_0}|\in\mathbb{C}1_n$. Hence, W_{α_0} is a scalar multiple of a unitary.

We now put all of the above derivations together to deduce the explicit structure of the teleportation scheme we started with.

Recall that $U_{i\alpha_0\beta_0} = \bar{W}_{\alpha_0}A_{i\beta_0}/\sqrt{\mu_{i\alpha_0\beta_0}}$. Then, we get the following equalities for $\Psi_{i\alpha_0\beta_0}$, starting from equation (5.6):

$$\begin{aligned}&E_M\big(\big[(A_{i\beta_0}^*\otimes W_{\alpha_0})e_N(A_{i\beta_0}\otimes W_{\alpha_0}^*)\otimes T_i(x)\big]\big[(1_d\otimes e_N)\big]\big)\\ &= (kn^2)^{-2}A_{i\beta_0}^*W_{\alpha_0}^tT_i(x)\bar{W}_{\alpha_0}A_{i\beta_0}\\ &= (kn^2)^{-2}\mu_{i\alpha_0\beta_0}U_{i\alpha_0\beta_0}^*T_i(x)U_{i\alpha_0\beta_0}\\ &= (kn^2)^{-1}\mu_{i\alpha_0\beta_0}(\mathrm{id}\otimes\tau)\big((U_{i\alpha_0\beta_0}^*T_i(x)\otimes 1_d)e_N(U_{i\alpha_0\beta_0}\otimes 1_d)\big)\\ &= \mu_{i\alpha_0\beta_0}(\mathrm{id}\otimes\tau\otimes\tau)\big((U_{i\alpha_0\beta_0}^*T_i(x)\otimes 1_d)e_N(U_{i\alpha_0\beta_0}\otimes e_N)\big)\\ &= \mu_{i\alpha_0\beta_0}E_M\big(\big[(U_{i\alpha_0\beta_0}^*\otimes 1_d)e_N(U_{i\alpha_0\beta_0}\otimes 1_d)\otimes T_i(x)\big]\big[(1_d\otimes e_N)\big]\big)\\ &= E_M\big(\big[(A_{i\beta_0}^*W_{\alpha_0}^t\otimes 1_d)e_N(\bar{W}_{\alpha_0}A_{i\beta_0}\otimes 1_d)\otimes T_i(x)\big]\big[(1_d\otimes e_N)\big]\big),\end{aligned}$$

where the first equality holds by (5.7). Going back to our identity, for any $x\in N'$ we have

$$\begin{aligned}
x &= \sum_{i,\alpha,\beta\in\mathcal{I}} E_M\Big(\big[(A_{i\beta}^*\otimes 1_d)e_N(A_{i\beta}\otimes 1_d)\otimes T_i(x)\big]\big[1_d\otimes(W_\alpha^*\otimes 1_d)e_N(1_d\otimes W_\alpha)\big]\Big)\\
&= \sum_{i,\alpha,\beta\in\mathcal{I}} \frac{\mu_{i\alpha_0\beta}}{\mu_{i\alpha_0\beta_0}}\cdot\frac{\mu_{i\alpha\beta_0}}{\mu_{i\alpha_0\beta_0}} E_M\Big(\big[(A_{i\beta_0}^*\otimes 1_d)e_N(A_{i\beta_0}\otimes 1_d)\otimes T_i(x)\big]\\
&\qquad\qquad\qquad\qquad\qquad\cdot\big[1_d\otimes(W_{\alpha_0}^*\otimes 1_d)e_N(W_{\alpha_0}\otimes 1_d)\big]\Big)\\
&= \sum_{i\in\mathcal{I}} \frac{\mu_{i\alpha_0}\mu_{i\beta_0}}{(\mu_{i\alpha_0\beta_0})^2} E_M\Big(\big[(A_{i\beta_0}^*\otimes 1_d)e_N(A_{i\beta_0}\otimes 1_d)\otimes T_i(x)\big]\\
&\qquad\qquad\qquad\qquad\qquad\cdot\big[1_d\otimes(W_{\alpha_0}^*\otimes 1_d)e_N(W_{\alpha_0}\otimes 1_d)\big]\Big)\\
&= \sum_{i\in\mathcal{I}} \frac{\mu_{i\alpha_0}\mu_{i\beta_0}}{(\mu_{i\alpha_0\beta_0})^2} E_M\Big(\big[(A_{i\beta_0}^*\otimes W_{\alpha_0})e_N(A_{i\beta_0}\otimes W_{\alpha_0}^*)\otimes T_i(x)\big]\big[(1_d\otimes e_N)\big]\Big)\\
&= \sum_{i\in\mathcal{I}} \frac{\mu_{i\alpha_0}\mu_{i\beta_0}}{(\mu_{i\alpha_0\beta_0})^2} E_M\Big(\big[(A_{i\beta_0}^*W_{\alpha_0}^t\otimes 1_d)e_N(\bar{W}_{\alpha_0}A_{i\beta_0}\otimes 1_d)\otimes T_i(x)\big]\big[(1_d\otimes e_N)\big]\Big) \qquad (5.14)\\
&= \sum_{i\in\mathcal{I}} \frac{\mu_{i\alpha_0}\mu_{i\beta_0}}{\mu_{i\alpha_0\beta_0}} E_M\Big(\big[(U_{i\alpha_0\beta_0}^*\otimes 1_d)e_N(U_{i\alpha_0\beta_0}\otimes 1_d)\otimes U_{i\alpha_0\beta_0}xU_{i\alpha_0\beta_0}^*\big]\big[(1_d\otimes e_N)\big]\Big) \qquad (5.15)
\end{aligned}$$

where (5.14) holds by Lemma 5.1.2. We know $\{\sqrt{\mu_{i\alpha_0}/\mu_{i\alpha_0\beta_0}}A_{i\beta_0}\}_{i\in\mathcal{I}}$ is a basis. So,

$$\begin{aligned}
&\sum_{i\in\mathcal{I}}\mu_{i\alpha_0}\big(U_{i\alpha_0\beta_0}^*\otimes 1_d\big)e_N\big(U_{i\alpha_0\beta_0}\otimes 1_d\big)\\
&= \sum_{i\in\mathcal{I}}\frac{\mu_{i\alpha_0}}{\mu_{i\alpha_0\beta_0}}\big(A_{i\beta_0}^*W_{\alpha_0}^t\otimes 1_d\big)e_N\big(\bar{W}_{\alpha_0}A_{i\beta_0}\otimes 1_d\big)\\
&= \sum_{i\in\mathcal{I}}\frac{\mu_{i\alpha_0}}{\mu_{i\alpha_0\beta_0}}\big(A_{i\beta_0}^*\otimes W_{\alpha_0}\big)e_N\big(A_{i\beta_0}\otimes W_{\alpha_0}^*\big)\\
&= (1_d\otimes W_{\alpha_0})\Big(\sum_{i\in\mathcal{I}}\frac{\mu_{i\alpha_0}}{\mu_{i\alpha_0\beta_0}}\big(A_{i\beta_0}^*\otimes 1_d\big)e_N\big(A_{i\beta_0}\otimes 1_d\big)\Big)(1_d\otimes W_{\alpha_0}^*)\\
&= 1_d\otimes W_{\alpha_0}W_{\alpha_0}^*\\
&= c1_d\otimes 1_d.
\end{aligned}$$

To identify c, we first recall that ρ is a $(\tau \otimes \tau)$-density. Then,

$$\begin{aligned}
1 &= (\tau \otimes \tau)(\rho) \\
&= \sum_{\alpha \in \mathcal{I}} (\tau \otimes \tau)\big((W_\alpha^* \otimes 1_d) e_N (W_\alpha \otimes 1_d)\big) \\
&= \sum_{\alpha \in \mathcal{I}} \frac{\mu_{i\alpha\beta_0}}{\mu_{i\alpha_0\beta_0}} (\tau \otimes \tau)\big((W_{\alpha_0}^* \otimes 1_d) e_N (W_{\alpha_0} \otimes 1_d)\big) \\
&= \frac{\mu_{i\alpha_0}}{\mu_{i\alpha_0\beta_0}} (\tau \otimes \tau)\big((W_{\alpha_0} W_{\alpha_0}^* \otimes 1_d) e_N\big) \\
&= \frac{\mu_{i\alpha_0}}{\mu_{i\alpha_0\beta_0}} (\tau \otimes \mathrm{id})\big((W_{\alpha_0} W_{\alpha_0}^* \otimes 1_d)(\mathrm{id} \otimes \tau)(e_N)\big) \\
&= \frac{\mu_{i\alpha_0}}{\mu_{i\alpha_0\beta_0}} (kn^2)^{-1} \tau\big(W_{\alpha_0} W_{\alpha_0}^*\big).
\end{aligned}$$

So,

$$c = \frac{\mu_{i\alpha_0\beta_0}(kn^2)}{\mu_{i\alpha_0}}.$$

It follows that, the set

$$\left\{ \sqrt{\frac{\mu_{i\alpha_0}\mu_{i\beta_0}}{\mu_{i\alpha_0\beta_0}(kn^2)}} U_{i\alpha_0\beta_0} \right\}_{i \in I}$$

is a basis for $\mathcal{M}_d/N$ and since it is indexed by $i \in \mathcal{I}$, by Lemma 5.1.7 it is orthonormal and so

$$1_d = \frac{\mu_{i\alpha_0}\mu_{i\beta_0}}{\mu_{i\alpha_0\beta_0}(kn^2)} E_N(U_{i\alpha_0\beta_0} U_{i\alpha_0\beta_0}^*) = \frac{\mu_{i\alpha_0}\mu_{i\beta_0}}{\mu_{i\alpha_0\beta_0}(kn^2)} 1_d \quad \Rightarrow \quad \frac{\mu_{i\alpha_0}\mu_{i\beta_0}}{\mu_{i\alpha_0\beta_0}(kn^2)} = 1.$$

Hence, $\{U_{i\alpha_0\beta_0}\}_{i \in \mathcal{I}}$ is a unitary orthonormal basis for $\mathcal{M}_d/N$ contained inside $\mathcal{N}_M(N')$. Picking up from line (5.15), we get

$$\begin{aligned}
x &= \sum_{i \in \mathcal{I}} \frac{\mu_{i\alpha_0}\mu_{i\beta_0}}{\mu_{i\alpha_0\beta_0}} E_M\Big((U_{i\alpha_0\beta_0}^* \otimes 1_d) e_N (U_{i\alpha_0\beta_0} \otimes 1_d) \otimes U_{i\alpha_0\beta_0} x U_{i\alpha_0\beta_0}^* (1_d \otimes e_N)\Big) \\
&= (kn^2) \sum_{i \in \mathcal{I}} \frac{\mu_{i\alpha_0}\mu_{i\beta_0}}{\mu_{i\alpha_0\beta_0}(kn^2)} E_M\Big((U_{i\alpha_0\beta_0}^* \otimes 1_d) e_N (U_{i\alpha_0\beta_0} \otimes 1_d) \otimes U_{i\alpha_0\beta_0} x U_{i\alpha_0\beta_0}^* (1_d \otimes e_N)\Big) \\
&= [M_d : N] \sum_{i \in \mathcal{I}} E_M\Big((U_{i\alpha_0\beta_0}^* \otimes 1_d) e_N (U_{i\alpha_0\beta_0} \otimes 1_d) \otimes U_{i\alpha_0\beta_0} x U_{i\alpha_0\beta_0}^* (1_d \otimes e_N)\Big),
\end{aligned}$$

for any $x \in N'$, as required. □

Chapter 6

Conclusion

The research presented in this paper explored properties of a generalized operator algebraic teleportation identity developed by J. Crann, D. W. Kribs, and R. H. Levene. One of the main goals in our research was to determine an operational interpretation of their teleportation identity. We were successful in doing so, demonstrated by our Hybrid quantum teleportation theorem (Theorem 4.6.4) and our Scaffolding teleportation theorem (Theorem 5.0.5). In the former, we showed that it is possible to build a teleportation scheme for any finite dimensional von Neumann algebra and that it can be seen as a "direct sum" generalization of the standard teleportation protocol presented in [4]. In the latter, we gave a operator algebraic teleportation identity on a fixed quantum system. The Scaffolding teleportation identity allowed us to achieve a second major goal of ours by relating operator algebraic teleportation to Werner's work on tight teleportation in Theorem 5.1.4 where, under some mild symmetry assumptions, we showed that any teleportation scheme was of a special form.

Portions of the proof of Theorem 5.1.4 hold without the symmetry assumption, however, we cannot guarantee positivity of the scalars, $\mu_{i\alpha\beta}$. Subsequent attempts were made at finding a connection to Werner's work using a combination of prior assumptions, however, without the symmetry assumption the path required seemed to be extremely non-trivial and it was unclear how to proceed.

A natural future topic to explore is dense coding in the operator algebraic regime. The most immediate question to answer is whether or not there is a "scaffolding" dense coding scheme. In particular, given a shift covariant teleportation scheme which satisfies the tightness condition, can we construct a dense coding scheme using the following equations:

$$\begin{aligned} \rho &= [\mathcal{M}_d, N]e_N \\ F_i &= (U_i^* \otimes 1_d)e_N(U_i \otimes 1_d) \\ T_i(x) &= U_i(x)U_i^*, \quad x \in N', \end{aligned}$$

where $\{U_i\}_{i\in\mathcal{I}} \subseteq \mathcal{N}_{\mathcal{M}_d}(N')$ is a unitary orthonormal basis for $\mathcal{M}_d/N$? If it is the case, this would give us an operator algebraic generalization of Werner's work on tight teleportation protocols in [34], further solidifying the role of the operator algebraic framework in the study of quantum teleportation.

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