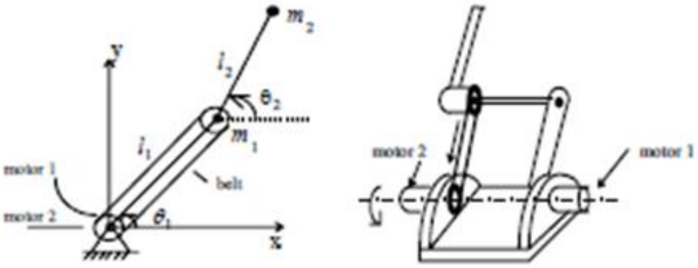
ECE 470: Introduction to Robotics Homework 3

Question 1.

A manipulator arm is designed as illustrated by the following figure. It can be assumed that the mass distributions of the links are insignificant and can be treated as lumped equivalent masses m_1 and m_2.



a) Write down the position of masses m_1 and m_2 in terms of θ_1 and θ_2 referenced from the given frame. *(2 marks)*

b) Obtain the velocities v_1 and v_2 of the mass m_1 and m_2, respectively. *(4 marks)*

c) Show that the total kinetic energy of the system, K can be written as

$$K = \frac{1}{2}\left(m_1 {l_1}^2 + m_2 {l_1}^2\right){\dot{\theta}_1}^2 + m_2 l_1 l_2 \cos(\theta_2 - \theta_1)\,\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2} m_2 {l_2}^2 {\dot{\theta}_2}^2$$ *(4 marks)*

d) Obtain the total potential energy of the system. *(3 marks)*

e) Write down the Lagrangian L. *(2 marks)*

f) Obtain the equation of motion. *(5 marks)*

Solution

a)

$$\begin{cases} x_1 = l_1 \cos\theta_1 \\ y_1 = l_1 \sin\theta_1 \end{cases} \qquad \begin{cases} x_2 = l_1 \cos\theta_1 + \ l_2 \cos\theta_2 \\ y_2 = \ l_1 \sin\theta_1 + \ l_2 \sin\theta_2 \end{cases}$$

b)

$$\begin{cases} \dot{x}_1 = -l_1 \sin\theta_1 \, \dot{\theta}_1 \\ \ \dot{y}_1 = l_1 \cos\theta_1 \dot{\theta}_1 \end{cases} \quad \begin{cases} \dot{x}_2 = -\left(l_1 \sin\theta_1 \dot{\theta}_1 + l_2 \sin\theta_2 \dot{\theta}_2\right) \\ \ \dot{y}_2 = l_1 \cos\theta_1 \dot{\theta}_1 + l_2 \cos\theta_2 \dot{\theta}_2 \end{cases}$$

$$v_1^2 = \dot{x}_1^2 + \dot{y}_1^2 = l_1^2 \dot{\theta}_1^2$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = l_1^2 \dot{\theta}_1^2 + 2 l_1 l_2 \cos(\theta_2 - \theta_1) \dot{\theta}_1 \dot{\theta}_2 + l_2^2 \dot{\theta}_2^2$$

c)

$$K = \frac{1}{2} m_1 {v_1}^2 + \frac{1}{2} m_2 {v_2}^2$$

$$= \frac{1}{2}\left(m_1 {l_1}^2 + m_2 {l_1}^2\right) {\dot{\theta}_1}^2 + m_2 l_1 l_2 \cos(\theta_2 - \theta_1) \, \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} m_2 {l_2}^2 {\dot{\theta}_2}^2$$

d)

$$U = [m_1 g l_1 \sin\theta_1 + m_2 g (l_1 \sin\theta_1 + l_2 \sin\theta_2)]$$

e)

Lagrangian $L = K - U$

$$L = \left[\frac{1}{2}\left(m_1 {l_1}^2 + m_2 {l_1}^2\right) {\dot{\theta}_1}^2 + m_2 l_1 l_2 \cos(\theta_2 - \theta_1) \, \dot{\theta}_1 \dot{\theta}_2 + \frac{1}{2} m_2 {l_2}^2 {\dot{\theta}_2}^2\right] - [m_1 g l_1 \sin\theta_1 + m_2 g (l_1 \sin\theta_1 + l_2 \sin\theta_2)]$$

f)

$$L = K - V$$

$(1\times 2)\times(2\times 2)$

1×2

2×2

$$L = \frac{1}{2}(m_1+m_2)l_1^2\dot{\theta}_1^2 + m_2 l_1 l_2 \cos(\theta_2-\theta_1)\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}m_2 l_2^2\dot{\theta}_2^2$$
$$- m_1 g l_1 \sin\theta_1 - m_2 g l_1 \sin\theta_1 - m_2 g l_2 \sin\theta_2$$

$$\tau_i = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}$$

$i=1$

$$\frac{\partial L}{\partial q_1} = m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_2-\theta_1) - (m_1+m_2) g l_1 \cos\theta_1$$

$i=2$

$$\frac{\partial L}{\partial q_2} = -m_2 l_1 l_2 \dot{\theta}_1\dot{\theta}_2 \sin(\theta_2-\theta_1) - m_2 g l_2 \cos\theta_2$$

$i=1,$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = (m_1 l_1^2 + m_2 l_1^2)\ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_2-\theta_1)\ddot{\theta}_2 \mp m_2 l_1 l_2 \sin(\theta_2-\theta_1)\dot{\theta}_2(\dot{\theta}_2-\dot{\theta}_1)$$
$$= (m_1+m_2) l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_2-\theta_1)\ddot{\theta}_2 + m_2 l_1 l_2 \sin(\theta_2-\theta_1)\dot{\theta}_1\dot{\theta}_2 - m_2 l_1 l_2 \sin(\theta_2-\theta_1)\dot{\theta}_2^2$$

$i=2$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2 \cos(\theta_2-\theta_1)\ddot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_2-\theta_1)(-1)(\dot{\theta}_2-\dot{\theta}_1)$$
$$= m_2 l_1 l_2 \cos(\theta_2-\theta_1)\ddot{\theta}_1 + m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2 \sin(\theta_2-\theta_1)\dot{\theta}_1^2 - m_2 l_1 l_2 \dot{\theta}_1 \sin(\theta_2-\theta_1)\dot{\theta}_1\dot{\theta}_2$$

$$\tau_1 = (m_1+m_2) l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \cos(\theta_2-\theta_1)\ddot{\theta}_2 - m_2 l_1 l_2 \sin(\theta_2-\theta_1)\dot{\theta}_2^2 + (m_1+m_2) g l_1 \cos\theta_1$$

$$\tau_2 = m_2 l_1 l_2 \cos(\theta_2-\theta_1)\ddot{\theta}_1 + m_2 l_2^2\ddot{\theta}_2 + m_2 l_1 l_2 \sin(\theta_2-\theta_1)\dot{\theta}_1^2 + m_2 g l_2 \cos\theta_2$$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = \begin{bmatrix}(m_1+m_2) l_1^2 & m_2 l_1 l_2 \cos(\theta_2-\theta_1)\\ m_2 l_1 l_2 \cos(\theta_2-\theta_1) & m_2 l_2^2 \cancel{\theta_2}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix}-m_2 l_1 l_2 \sin(\theta_2-\theta_1)\dot{\theta}_2^2\\ m_2 l_1 l_2 \sin(\theta_2-\theta_1)\dot{\theta}_1^2\end{bmatrix} + \begin{bmatrix}(m_1+m_2) g l_1 & \cos\theta_1\\ m_2 g l_2 & \cos\theta_2\end{bmatrix}$$