ECE 470: Introduction to Robotics
Lecture 14

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Recap N-E Method: Acceleration

$$\vec{V}_1 = \vec{V}_{B,ORG} + \dot{x}\,\hat{i}_B + \dot{y}\,\hat{j}_B + \dot{z}\hat{k}_B + \vec{\omega} \times \vec{P}_{1/B}$$

Differentiate w.r.t. time

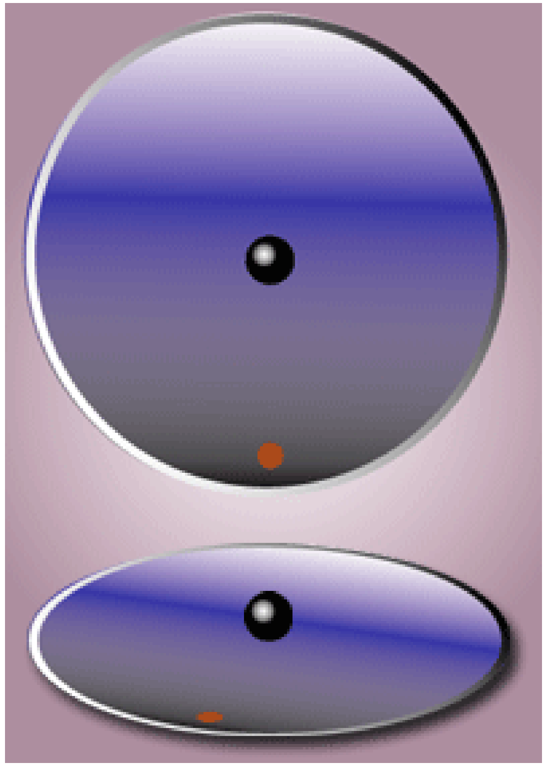
$$\dot{\vec{V}}_1 = \dot{\vec{V}}_{B,ORG} + \ddot{x}\,\hat{i}_B + \ddot{y}\,\hat{j}_B + \ddot{z}\hat{k}_B + \vec{\omega} \times \vec{V}_{1/B} + \vec{\omega} \times (\vec{V}_{1/B} + \vec{\omega} \times \vec{P}_{1/B}) + \dot{\vec{\omega}} \times \vec{P}_{1/B}$$

$$\dot{\vec{V}}_1 = \dot{\vec{V}}_{B,ORG} + \ddot{x}\,\hat{i}_B + \ddot{y}\,\hat{j}_B + \ddot{z}\hat{k}_B + 2\vec{\omega} \times \vec{V}_{1/B} + \vec{\omega} \times \vec{\omega} \times \vec{P}_{1/B} + \dot{\vec{\omega}} \times \vec{P}_{1/B}$$

coriolis acceleration

tangential acceleration

centrifugal acceleration



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Recap N-E Method: Iteration through Links

Outwards Iteration

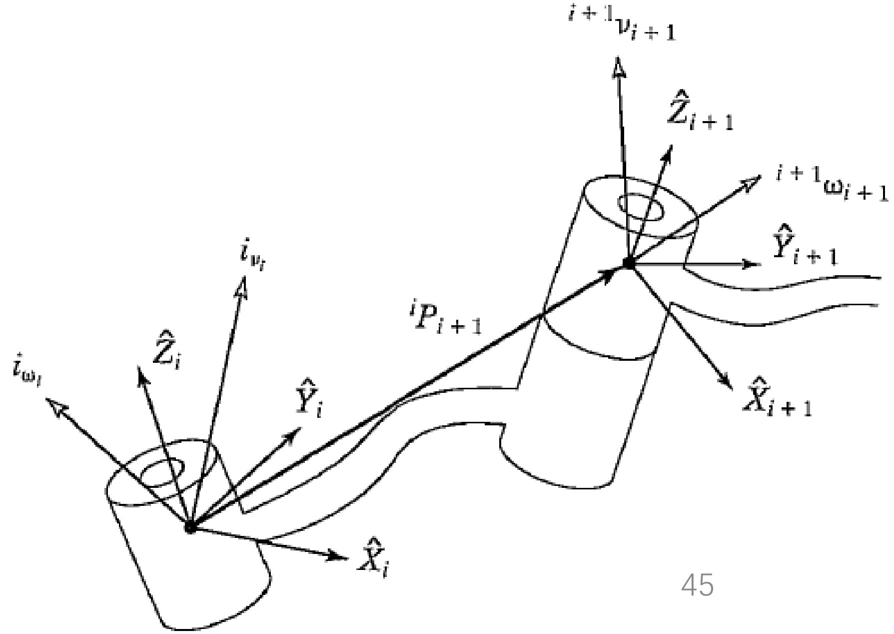
$$\boxed{{}^{i+1}\omega^{0}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega^{0}_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}\dot{\omega}^{0}_{i+1} = {}^{i+1}_{i}R\dot{\omega} + {}^{i+1}_{i}R\ {}^{i}\omega^{0}_{i} \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}v^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v^{0}_{i} + {}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right)}$$

$$\boxed{\begin{aligned}{}^{i+1}\dot{v}^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}^{0}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega^{0}_{i} \times \left({}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}^{0}_{i}\right) \\ +2\,{}^{i+1}\omega^{0}_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}\end{aligned}}$$

$$\boxed{{}^{i+1}\dot{v}^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}^{0}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega^{0}_{i} \times \left({}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}^{0}_{i}\right)}$$



Recap N-E Method: Iteration through Links

Kinematics

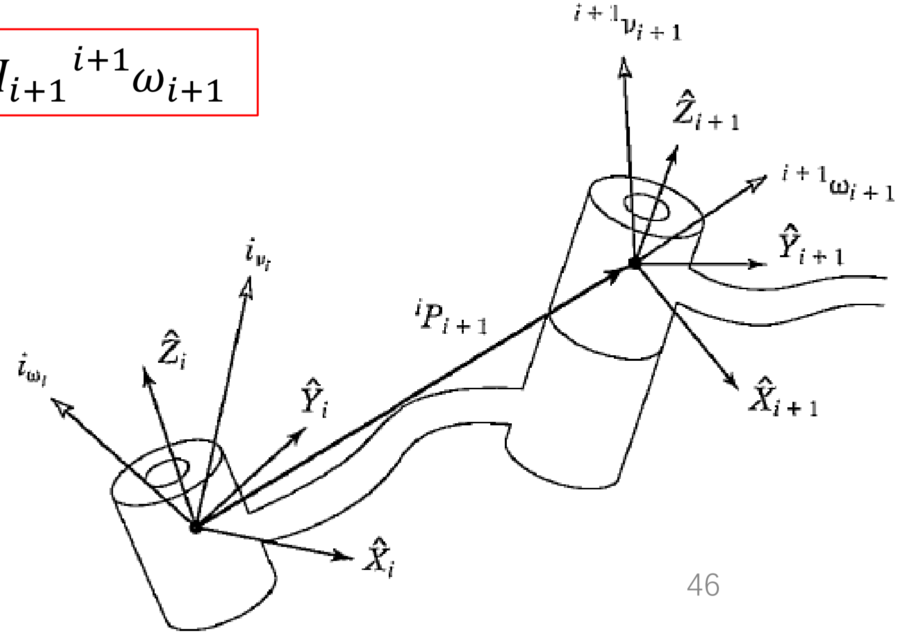
$$\boxed{{}^{i}\dot{v}^{0}_{Ci} = {}^{i}\dot{\omega}^{0}_{i} \times {}^{i}P_{Ci} + {}^{i}\omega^{0}_{i} \times \left({}^{i}\omega^{0}_{i} + {}^{i}P_{Ci} \right) + {}^{i}\dot{v}^{0}_{i}}$$

Newton Equation

$$\boxed{{}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{Ci+1}}$$

Euler Equation

$$\boxed{{}^{i+1}N_{i+1} = {}^{Ci+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{Ci+1}I_{i+1}\,{}^{i+1}\omega_{i+1}}$$



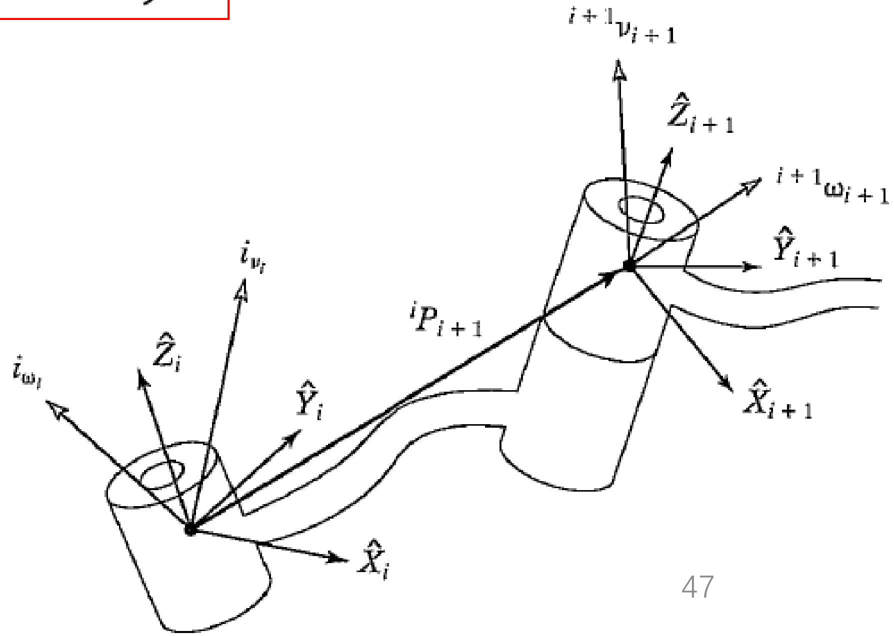
Recap N-E Method: Iteration through Links

Inwards Iteration

$$\boxed{{}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i}$$

$$\boxed{{}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times \left({}^{i}_{i+1}R\,{}^{i+1}f_{i+1}\right)}$$

$$\boxed{\tau_i = {}^{i}n_i^T\,{}^{i}\hat{Z}_i}$$



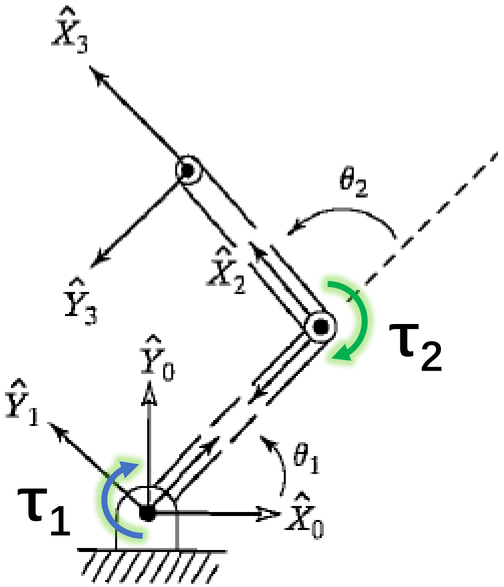
Recap: Dynamic Equation

$$\boxed{\tau = M(\Theta)\ddot{\Theta} + V\left(\Theta, \dot{\Theta}\right) + G(\Theta)}$$

From Example 5.2

Extracting the $\hat{Z}$ components of the ${}^{i}n_i$, we find the joint torques:

$$\tau_1 = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2(2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 \dot{\theta}_2^2$$
$$-2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1,$$
$$\tau_2 = m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2).$$



Recap: Dynamic Equation

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

From Example 5.2

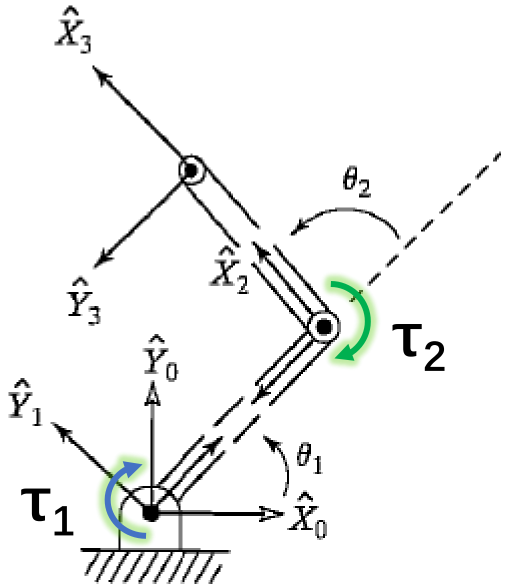
Extracting the $\hat{Z}$ components of the ${}^{i}n_i$, we find the joint torques:

$$\tau_1 = m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 l_2 c_2(2\ddot{\theta}_1 + \ddot{\theta}_2) + (m_1 + m_2) l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 \dot{\theta}_2^2$$
$$-2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 + m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1,$$
$$\tau_2 = m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2).$$

$$M(\Theta) = \begin{bmatrix} l_2^2 m_2 + 2 l_1 l_2 m_2 c_2 + l_1^2(m_1 + m_2) & l_2^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2^2 m_2 + l_1 l_2 m_2 c_2 & l_2^2 m_2 \end{bmatrix}$$

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$



Lagrangian Formulation

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Lagrangian Formulation

Method (1) Newton-Euler: “Force balance”

<u>Method (2) Lagrangian: Energy-based approach</u>

Lagrangian Formulation

Method (1) Newton-Euler: “Force balance”

Method (2) Lagrangian: Energy-based approach

Method (1) Newton-Euler is said to be “force balance” approach

Lagrangian is energy-based approach

Kinetic energy of the ith link is:

- $k_i = \frac{1}{2} m_i v_{C_i}^T v_{C_i} + \frac{1}{2}\ {}^i\omega_i^T\ {}^{C_i}I_i\ {}^i\omega_i$
- Must be positive

Kinetic energy of the manipulator is: $k = \sum_{i=1}^{n} k_i$

v_{C_i} and ${}^i\omega_i$ are functions of Θ and $\dot{\Theta}$

$k(\Theta, \dot{\Theta}) = \frac{1}{2} \dot{\Theta}^T M(\Theta)\, \dot{\Theta}$

- Potential energy of ith link:
  $$u_i = -m_i\ {}^0g^T\ {}^0P_{C_i} + u_{ref}$$
  - ${}^0g$ is 3 x 1 gravity vector
  - ${}^0P_{C_i}$ is the vector locating the center of mass of ith link
  - u_{ref} is the reference
- Total potential energy is: $u = \sum_{i=1}^{n} u_i$
- ${}^0P_{C_i}$ is a function of Θ, $\sum_{i=1}^{n} u_i = u(\Theta)$

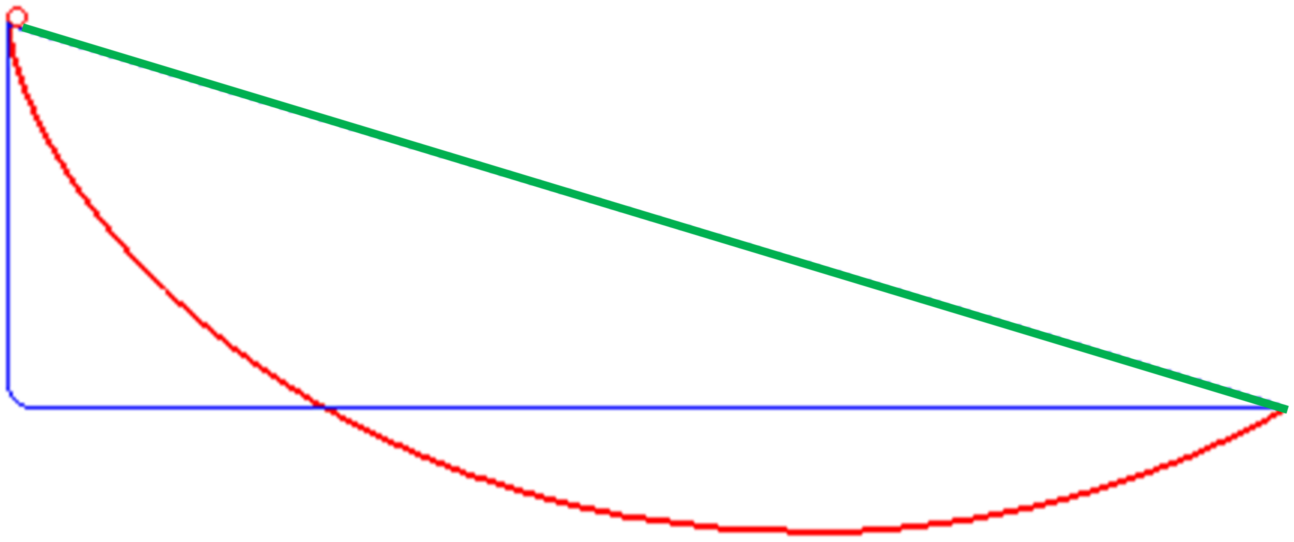
Lagrangian Formulation

Method (2) Lagrangian: Energy-based approach

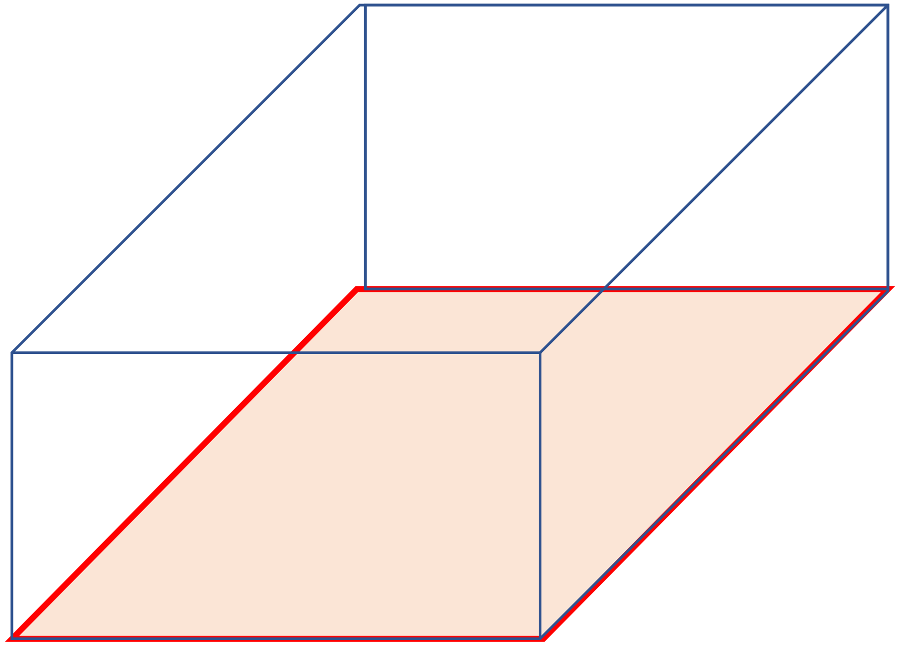
Based on Principle of Least Action

Short Break

Shortest Duration Path?



Shortest Path on Surface?



Lagrangian Formulation

Method (2) Lagrangian: Energy-based approach

Kinetic energy of the ith link is:

- $k_i = \frac{1}{2} m_i v_{C_i}^T \, v_{C_i} + \frac{1}{2} \; {}^i\omega_i^T \, {}^{C_i}I_i \; {}^i\omega_i$
- Must be positive

Kinetic energy of the manipulator is: $k = \sum_{i=1}^{n} k_i$

v_{C_i} and ${}^i\omega_i$ are functions of Θ and $\dot{\Theta}$

$k(\Theta, \dot{\Theta}) = \frac{1}{2} \dot{\Theta}^T \, M(\Theta) \, \dot{\Theta}$

Lagrangian Approach

Method (2) Lagrangian: Energy-based approach

Potential energy of ith link:

$$u_i = -m_i \, {}^0g^T \, {}^0P_{C_i} + u_{ref}$$

- ${}^0g$ is 3 x 1 gravity vector
- ${}^0P_{C_i}$ is the vector locating the center of mass of ith link
- u_{ref} is the reference

Total potential energy is: $u = \sum_{i=1}^{n} u_i$

${}^0P_{C_i}$ is a function of $\Theta, \sum_{i=1}^{n} u_i = u(\Theta)$

Lagrangian Approach

- Lagrangian dynamic formulation provides a means of deriving the equations of motion from a scalar function called the Lagrangian
- Lagrangian is defined as the difference between the kinetic and potential energy of a mechanical system
- $L(\Theta,\dot{\Theta}) \equiv k(\Theta,\dot{\Theta}) - u(\Theta)$
- Equations of motion are then given by:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\Theta}} - \frac{\partial L}{\partial \Theta} = \tau$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta} = \tau$$

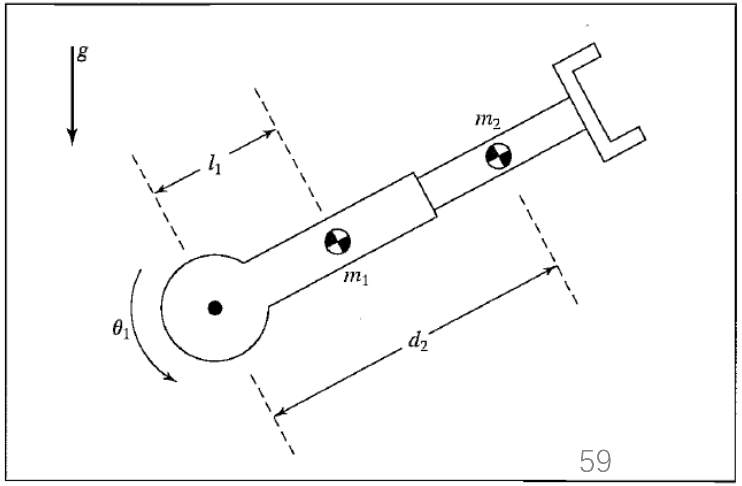
Ex. 6.1: Lagrangian Formulation

Example 6.5 in Textbook (J Craig, 3rd Ed.)

Given an R-P Manipulator of known inertia tensor

$$^{C_1}I_1 = \begin{bmatrix} I_{xx1} & 0 & 0 \\ 0 & I_{yy1} & 0 \\ 0 & 0 & I_{zz1} \end{bmatrix}, \quad ^{C_2}I_2 = \begin{bmatrix} I_{xx2} & 0 & 0 \\ 0 & I_{yy2} & 0 \\ 0 & 0 & I_{zz2} \end{bmatrix}$$

with total mass m_1 and m_2. Use Lagrangian dynamics to determine the equation of motion for this manipulator.



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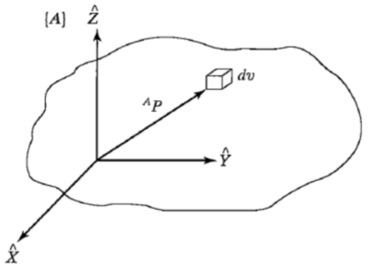
with total mass m_1 and m_2. Use Lagrangian dynamics to determine the equation of motion for this manipulator.

Mass Distribution
(/Inertia Tensor)

Inertia tensor:

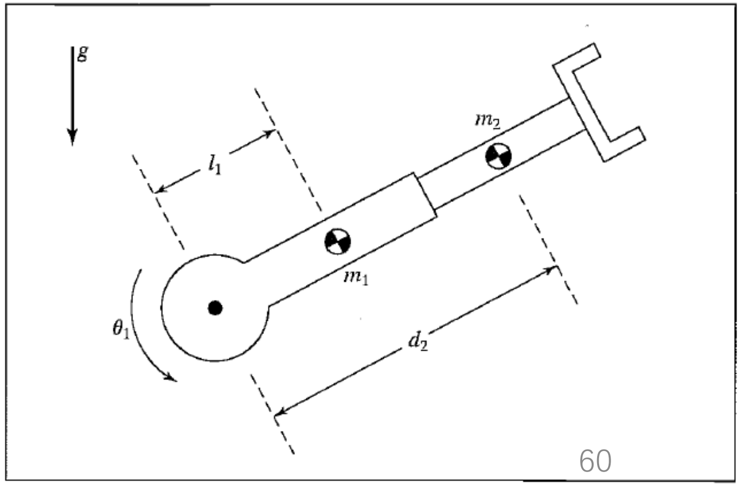
$$^{A}I = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{xy} & I_{yy} & -I_{yz} \\ -I_{xz} & -I_{yz} & I_{zz} \end{bmatrix}$$



where

- $I_{xx} = \iiint_V (y^2 + z^2)\,\rho dv$
- $I_{yy} = \iiint_V (x^2 + z^2)\,\rho dv$
- $I_{zz} = \iiint_V (x^2 + y^2)\,\rho dv$
- $I_{xy} = \iiint_V xy\,\rho dv$
- $I_{xz} = \iiint_V xz\,\rho dv$
- $I_{yz} = \iiint_V yz\,\rho dv$

I_{xx}, I_{yy} and I_{zz}: mass moments of inertia
I_{xy}, I_{xz} and I_{yz}: mass products of inertia



Ex. 6.1: Lagrangian Formulation

$$k_1 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} I_{zz1} \dot{\theta}_1^2$$

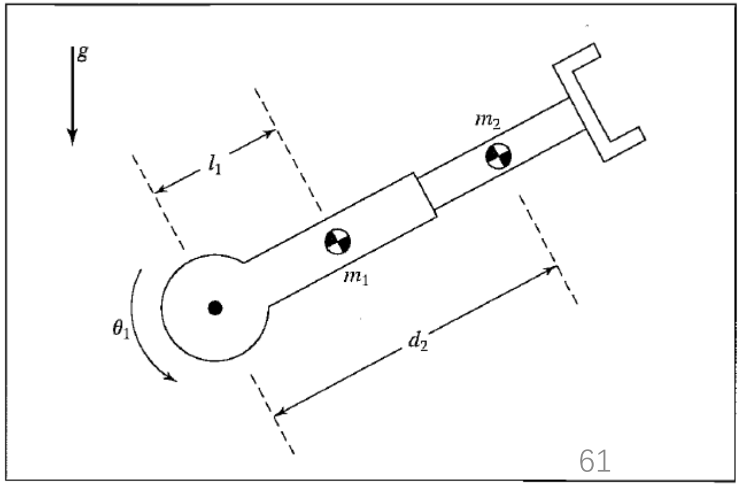
$$k_2 = \frac{1}{2} m_2 (d_2^2 \dot{\theta}_1^2 + \dot{d}_2^2) + \frac{1}{2} I_{yy2} \dot{\theta}_1^2$$

$$k(\Theta, \dot{\Theta}) = \frac{1}{2}(m_1 l_1^2 + I_{zz1} + I_{yy2} + m_2 d_2^2)\dot{\theta}_1^2 + \frac{1}{2} m_2 \dot{d}_2^2$$

$$u_1 = m_1 l_1 g \sin(\theta_1)$$

$$u_2 = m_2 g d_2 \sin(\theta_1)$$

$$u(\Theta) = g(m_1 l_1 + m_2 d_2) \sin \theta_1$$



Ex. 6.1: Lagrangian Formulation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\Theta}} - \frac{\partial L}{\partial \Theta} = \tau$$

$$\frac{d}{dt}\frac{\partial k}{\partial \dot{\Theta}} - \frac{\partial k}{\partial \Theta} + \frac{\partial u}{\partial \Theta} = \tau$$

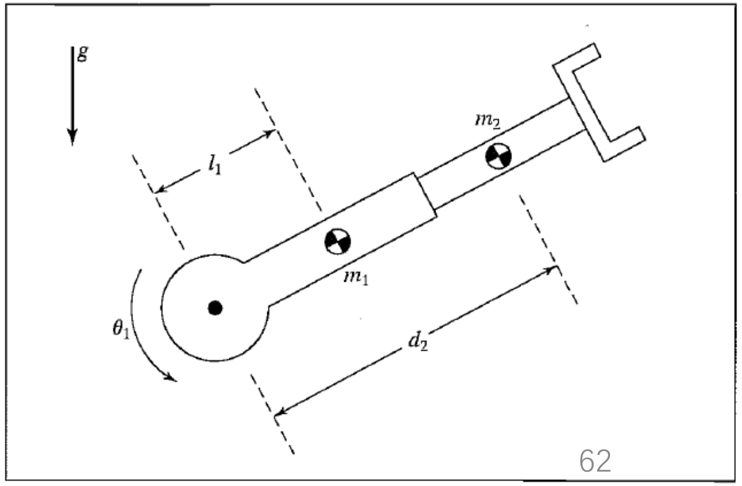
$$\frac{\partial k}{\partial \dot{\Theta}} = \begin{bmatrix} (m_1 l_1^2 + I_{zz1} + I_{yy2} + m_2 d_2^2)\dot{\theta}_1 \\ m_2 \dot{d}_2 \end{bmatrix}$$

$$\frac{\partial k}{\partial \Theta} = \begin{bmatrix} 0 \\ m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$\frac{\partial u}{\partial \Theta} = \begin{bmatrix} g(m_1 l_1 + m_2 d_2)\cos\theta_1 \\ g\ m_2 \sin\theta_1 \end{bmatrix}$$

$$\tau_1 = (m_1 l_1^2 + I_{zz1} + I_{yy2} + m_2 d_2^2)\ddot{\theta}_1 + 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 + g(m_1 l_1 + m_2 d_2)\cos\theta_1$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g \sin\theta_1$$



Ex. 6.1: Lagrangian Formulation

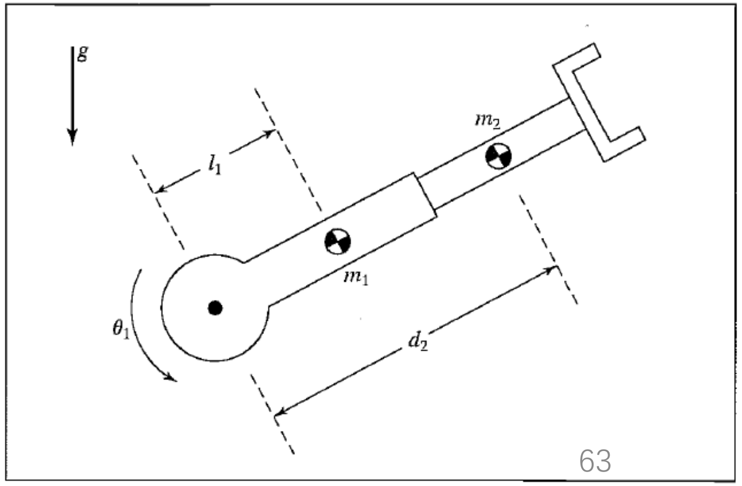
$$\tau_1 = (m_1 l_1^2 + I_{zz1} + I_{yy2} + m_2 d_2^2)\ddot{\theta}_1 + 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 + g(m_1 l_1 + m_2 d_2)\cos\theta_1$$

$$\tau_2 = m_2 \ddot{d}_2 - m_2 d_2 \dot{\theta}_1^2 + m_2 g \sin\theta_1$$

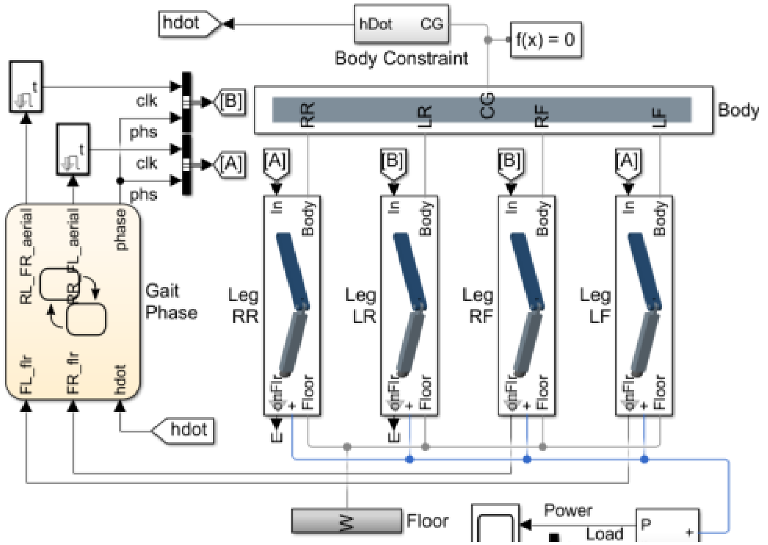
$$M(\Theta) = \begin{bmatrix} m_1 l_1^2 + I_{zz1} + I_{yy2} + m_2 d_2^2 & 0 \\ 0 & m_2 \end{bmatrix}$$

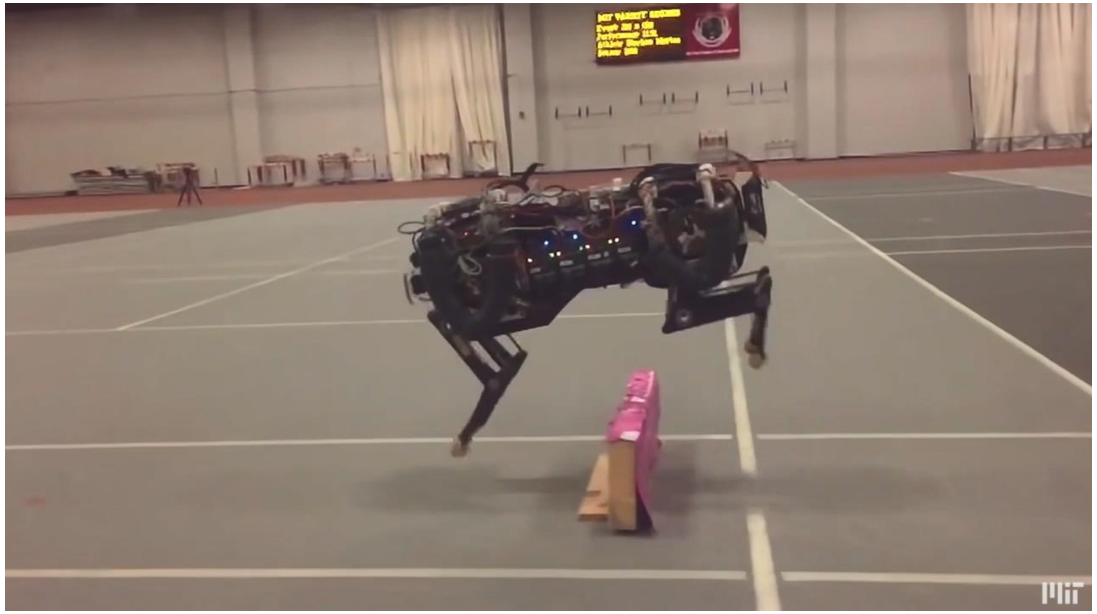
$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} 2m_2 d_2 \dot{\theta}_1 \dot{d}_2 \\ -m_2 d_2 \dot{\theta}_1^2 \end{bmatrix}$$

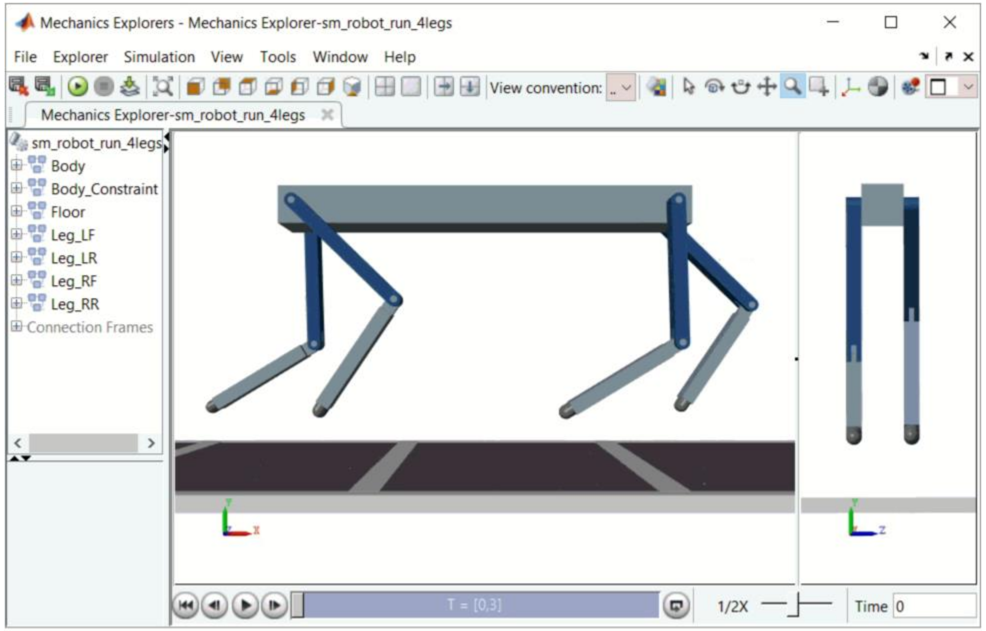
$$G(\Theta) = \begin{bmatrix} g(m_1 l_1 + m_2 d_2)\cos\theta_1 \\ m_2 g \sin\theta_1 \end{bmatrix}$$



Ex 6.2 Lagrangian Formulation

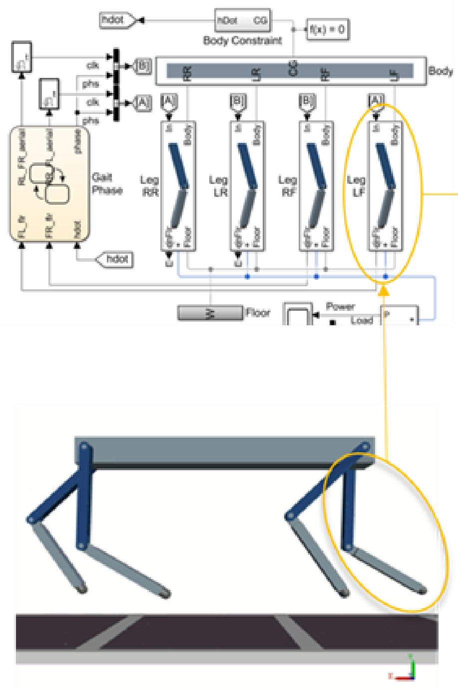


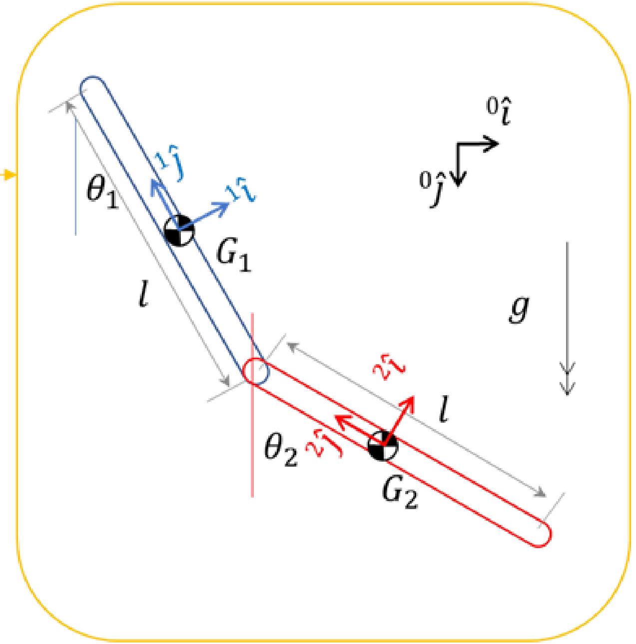


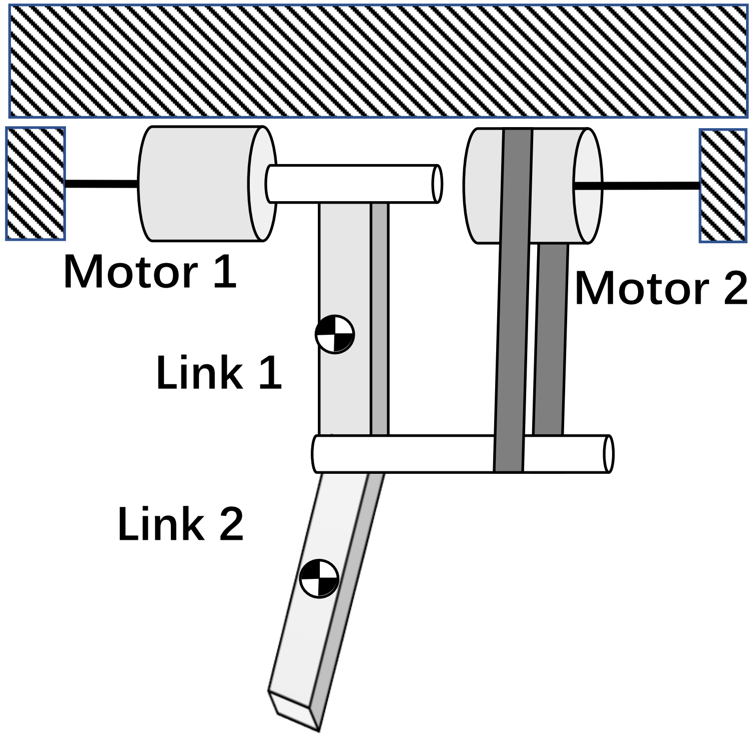


Ex 6.2 Lagrangian Formulation

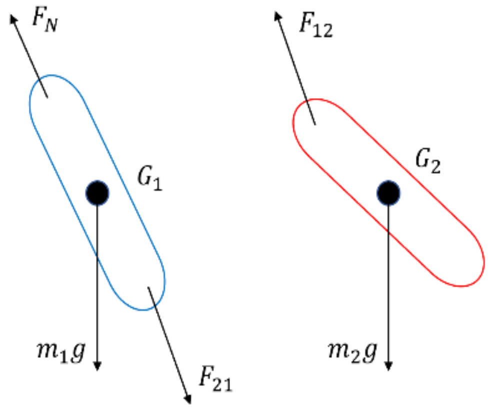
To model a dynamic system like the running four-legged robot, you decided to break down the system into subsystems and components.
You further decide to model each leg (when NOT contacting with ground) as a two-link serial manipulator and derive the equation of motion from first principal
Assuming both actuators are housed in the chassis the distal joint drive by a belt system







Ex 6.2 Lagrangian Formulation



$$V_{a1} = \frac{l}{2}\dot{\theta}_1 i$$

$$V_{a2} = l\dot{\theta}_1 i + \frac{l}{2}\dot{\theta}_2 j$$

$$T = \frac{1}{2}mV^2 + \frac{1}{2}I\omega^2$$

$$= \frac{1}{2}mV_{a1}{}^2 + \frac{1}{2}I_{a1}\omega_1{}^2 + \frac{1}{2}mV_{a2}{}^2 + \frac{1}{2}I_{a2}\omega_1{}^2$$

$$= \frac{1}{2}m\left(\frac{l^4}{4}\right)\dot{\theta}_1{}^2 + \frac{1}{2}\frac{1}{2}ml^2\dot{\theta}_1 l^2 + \frac{1}{2}m\left[l^2\dot{\theta}_1{}^2 + l^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1) + \frac{l^2}{4}\dot{\theta}_2{}^2\right] + \frac{1}{2}m\frac{ml^2}{12}\dot{\theta}_2{}^2$$

$$T = \frac{2}{3}ml^2\dot{\theta}_1{}^2 + \frac{1}{2}ml^2\frac{\partial y}{\partial x}\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1) + \frac{ml^2}{6}\dot{\theta}_2{}^2$$

Ex 6.2 Lagrangian Formulation

Lagrange Equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau$$

where $L = T - V$

$$L = \frac{2}{3}ml^2\dot{\theta}_1{}^2 + \frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\cos(\theta_2 - \theta_1) + \frac{1}{6}ml^2\dot{\theta}_2{}^2 - mg(\frac{l}{2}\cos\theta_1 l\cos\theta_1 + \frac{l}{2}\cos\theta_2)$$

When i=1,

$$\frac{\partial L}{\partial q_1} = -\frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1) + \frac{3}{2}mglsin\theta_1;$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = -\frac{4}{3}ml^2\ddot{\theta}_1 + \frac{1}{2}ml^2\ddot{\theta}_2\cos(\theta_2 - \theta_1)$$

When i=2,

$$\frac{\partial L}{\partial q_2} = -\frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2 - \theta_1) - \frac{mgl}{2}sin\theta_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = \frac{1}{2}ml^2\cos(\theta_2 - \theta_1)\ddot{\theta}_1 + \frac{1}{3}ml^2\ddot{\theta}_2$$

Ex 6.2 Lagrangian Formulation

When i=1,

$$\frac{\partial L}{\partial q_1} = -\frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)+\frac{3}{2}mglsin\theta_1;$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = -\frac{4}{3}ml^2\ddot{\theta}_1+\frac{1}{2}ml^2\ddot{\theta}_2\cos(\theta_2-\theta_1)$$

When i=2,

$$\frac{\partial L}{\partial q_2} = -\frac{1}{2}ml^2\dot{\theta}_1\dot{\theta}_2\sin(\theta_2-\theta_1)-\frac{mgl}{2}sin\theta_2$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_2} = \frac{1}{2}ml^2\cos(\theta_2-\theta_1)\,\ddot{\theta}_1+\frac{1}{3}ml^2\ddot{\theta}_2$$

Substituting L into Lagrange's Equation letting generalized coordinate be $\begin{pmatrix}\theta_1\\ \theta_2\end{pmatrix}$,

The generalized force can be written as $(\tau_a,\tau_b)^T$

$$\tau_1 = \frac{4}{3}ml^2\ddot{\theta}_1+\frac{1}{2}ml^2\ddot{\theta}_2\cos(\theta_2-\theta_1)-\frac{1}{2}ml^2\dot{\theta}_2^{\,2}\sin(\theta_2-\theta_1)+\frac{3}{2}mglsin\theta_1$$

$$\tau_2 = \frac{1}{2}ml^2\cos(\theta_2-\theta_1)\,\ddot{\theta}_1+\frac{1}{3}ml^2\ddot{\theta}_2+\frac{1}{2}ml^2\,\dot{\theta}_2\sin(\theta_2-\theta_1)+\frac{1}{2}mlgsin\theta_2$$

Robot Control

ECE 470 Introduction to Robotics