ECE 470: Introduction to Robotics

Lecture 12

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Lecture Outline

- Review Quiz
- Recap Structure of Dynamic Equation/Equation of Motion (EOM); Acceleration of Rigid Body
- Newton-Euler Formulation

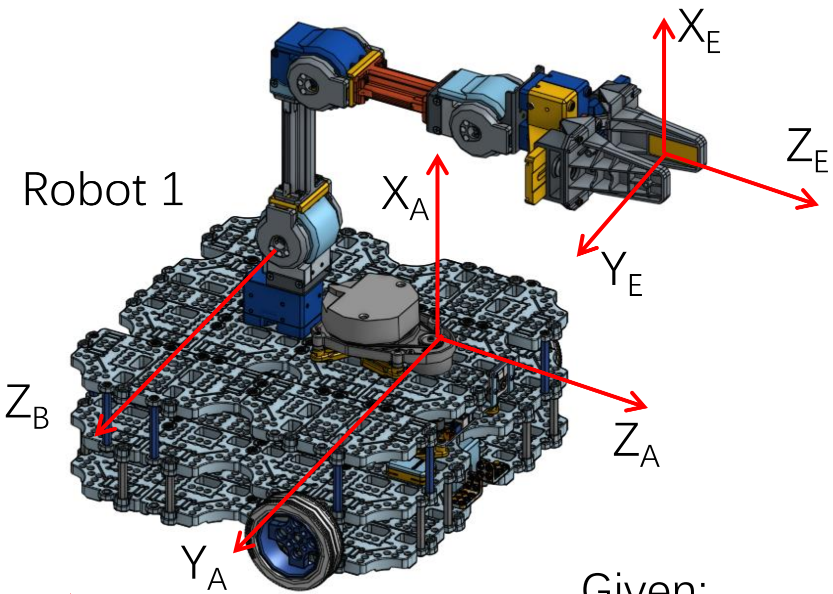
Review Quiz I

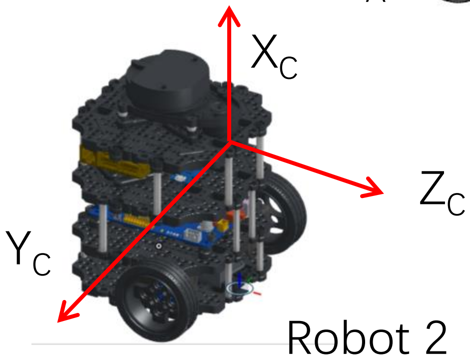
Robot 1 and Robot 2 went through some transformations in the following order

1. Robot 2 moves around Robot 1 such that {C} rotates 90° about axis X_A to become {C1}.
2. Robot 2 rotates about itself such that {C1} rotates 90° about axis X_{c1} to become {C2}.
3. Robot 1 moves forward such that {A}, {B} and {E} translate along the vector ${}^{A}(0, 0, 2)'$ to become {A3}, {B3} and {E3}
4. The arm on Robot 1 moves such that {E3} rotates 90° about axis Z_B to become {E4}

Obtain the expression for

i. ${}^{A}T_{C1}$ *(3 Points)*
ii. ${}^{A}T_{C2}$ *(3 Points)*
iii. ${}^{A3}T_{C2}$ *(3 Points)*
iv. ${}^{E3}T_{C2}$ *(3 Points)*
v. ${}^{C2}T_{E4}$ *(4 Points)*



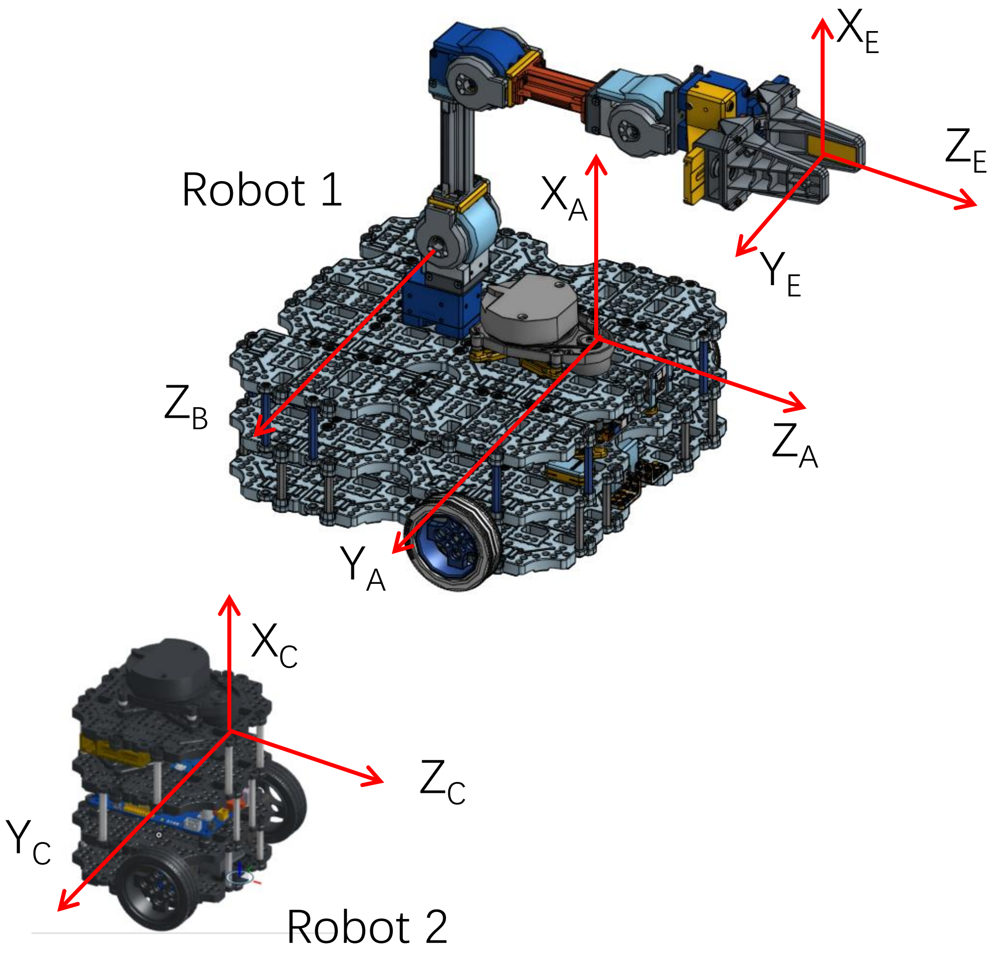


Given:

$$ {}^{B}T_{E} = \begin{bmatrix} 0 & 0 & 1 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}T_{B} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^{A}T_{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Review Quiz I



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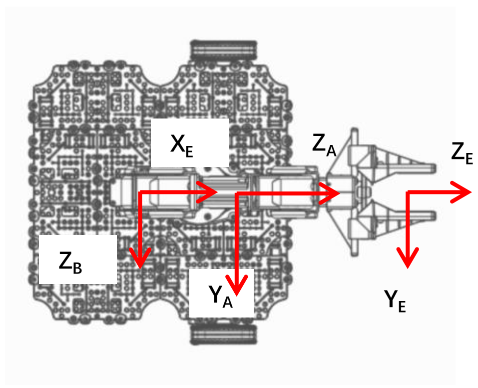

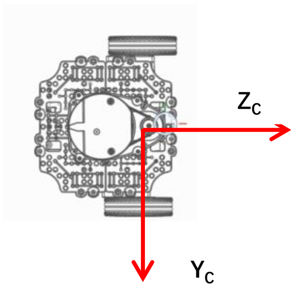

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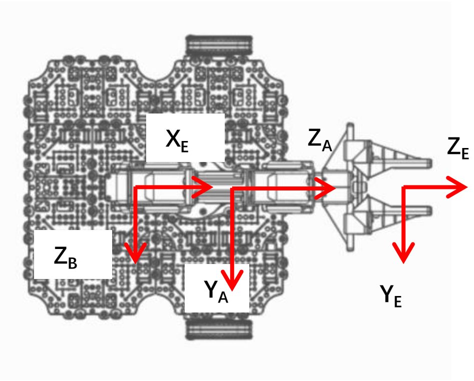

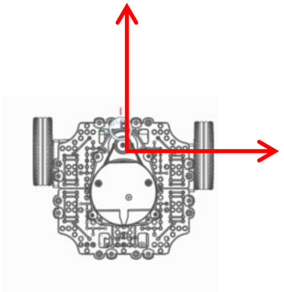

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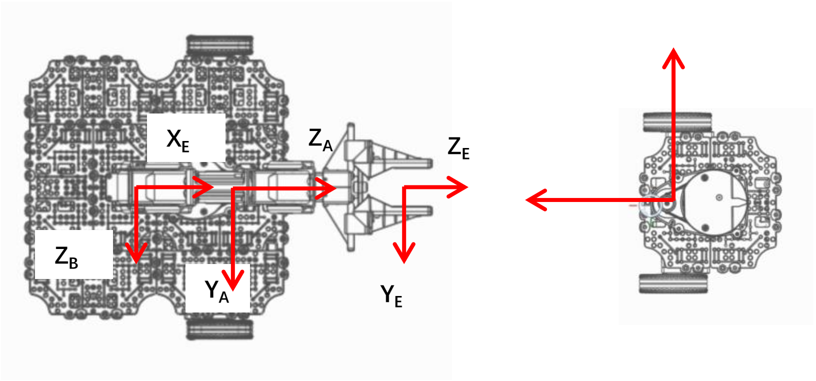

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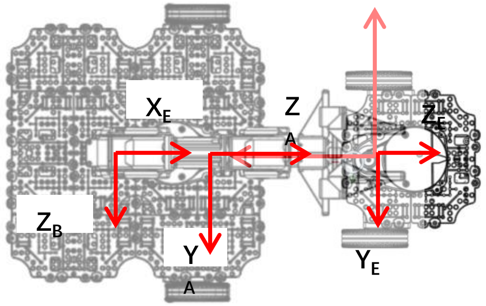

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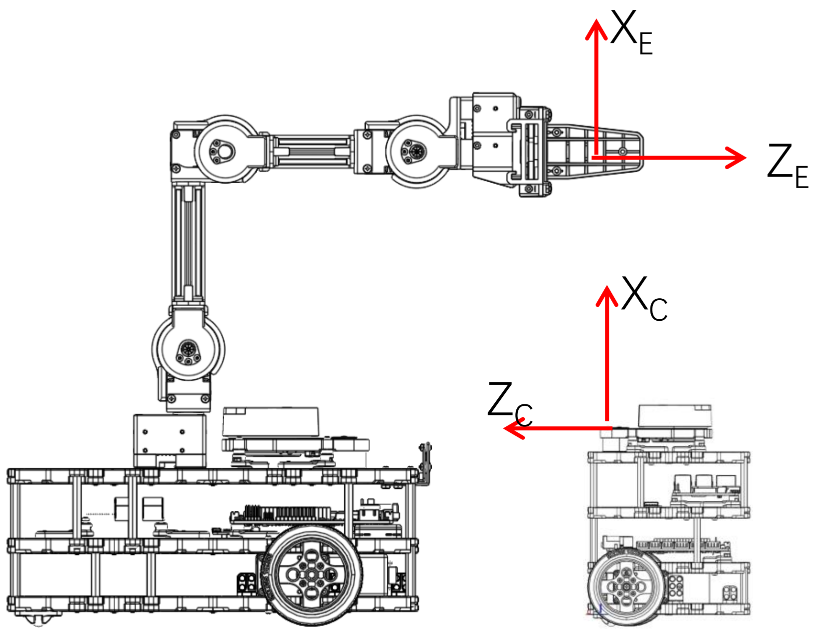

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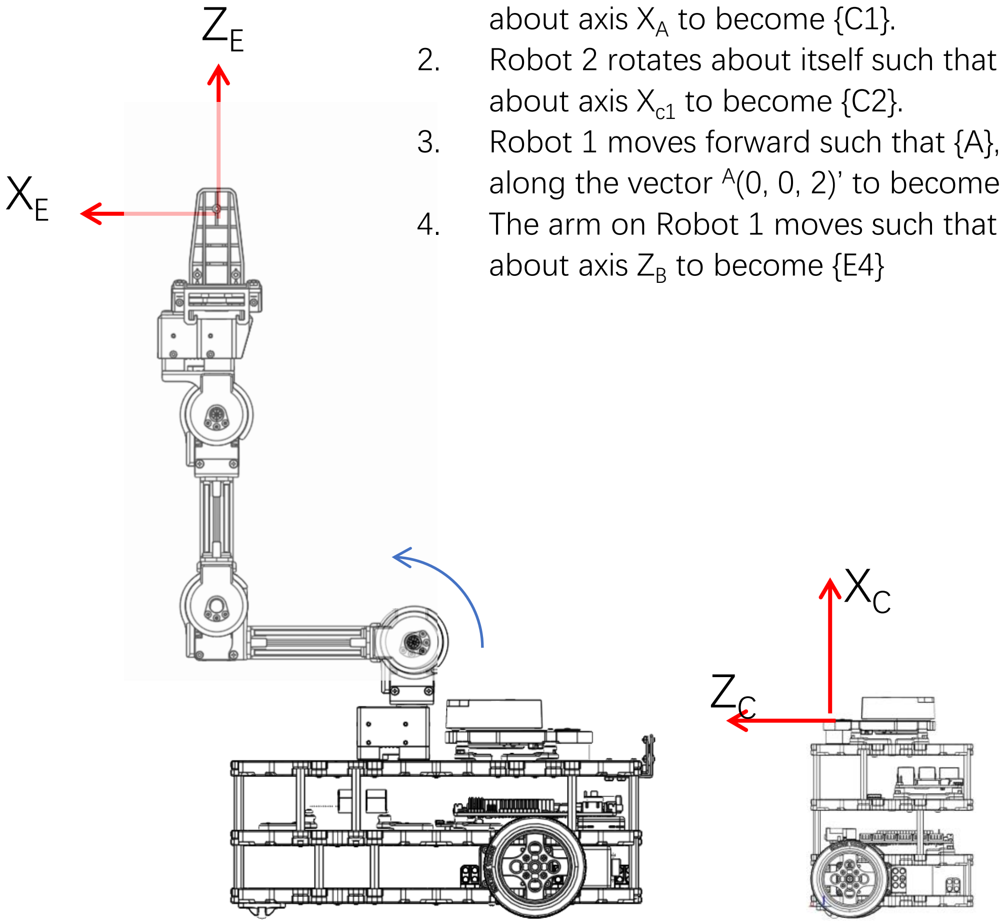
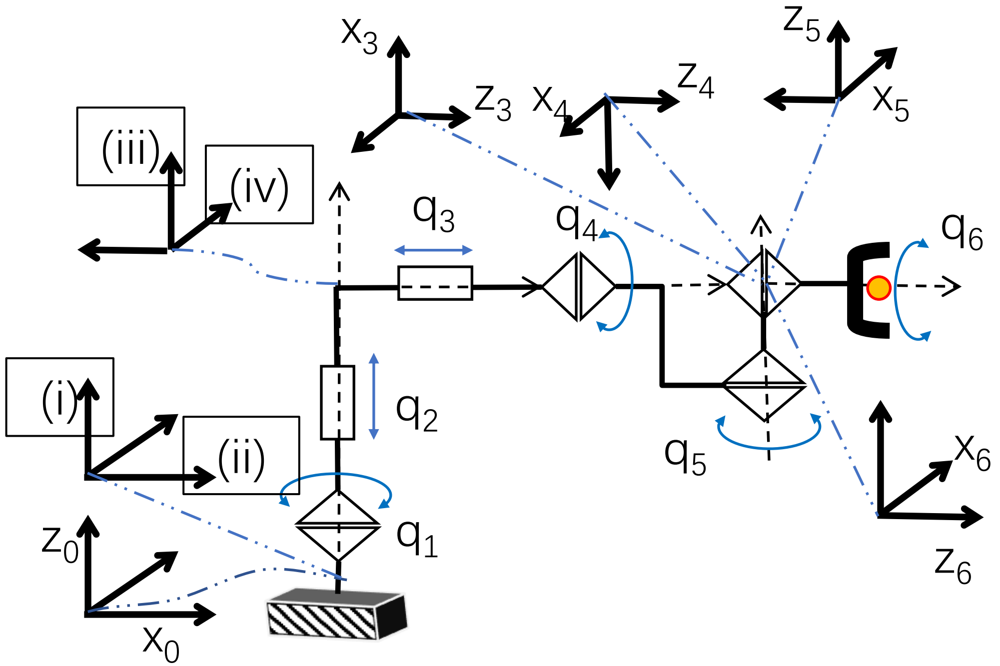


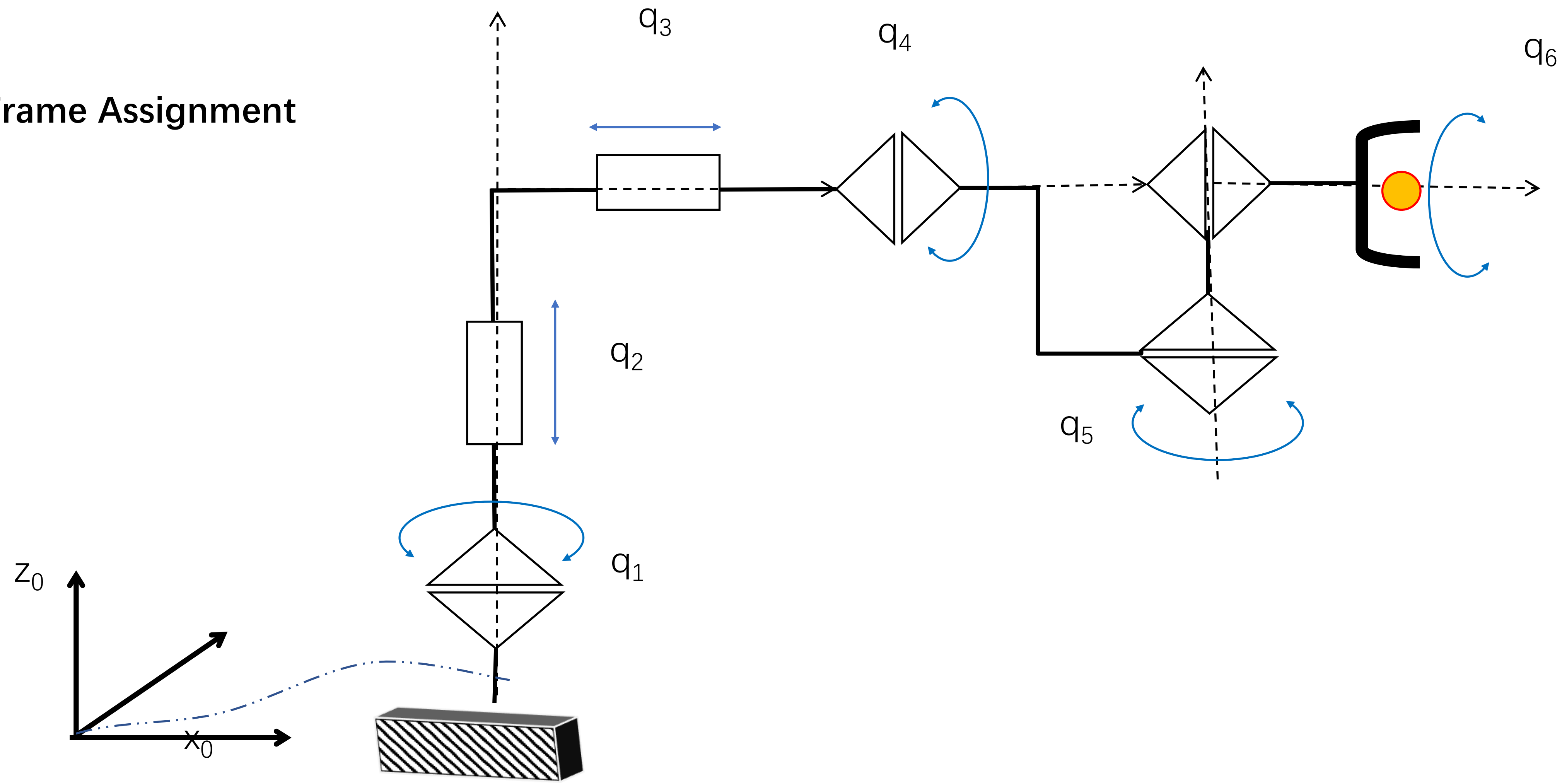
Review Quiz I

	Link Twist α_{i-1}	Link Length a_{i-1}	Joint Angle θ_i	Link offset d_i
${}^0_1T$	0	0	$q_1=0$	0
${}^1_2T$	0	0	90°	$q_2=d_2$
${}^2_3T$	90°	0	(v)______?	(vi)______
${}^3_4T$	(vii)______?	(viii)______?	(ix)______?	0
${}^4_5T$	(x)______?	(xi)______?	$q_5=180°$	(xii)______
${}^5_6T$	90°_	0	$q_6=0$	0



Review Quiz I

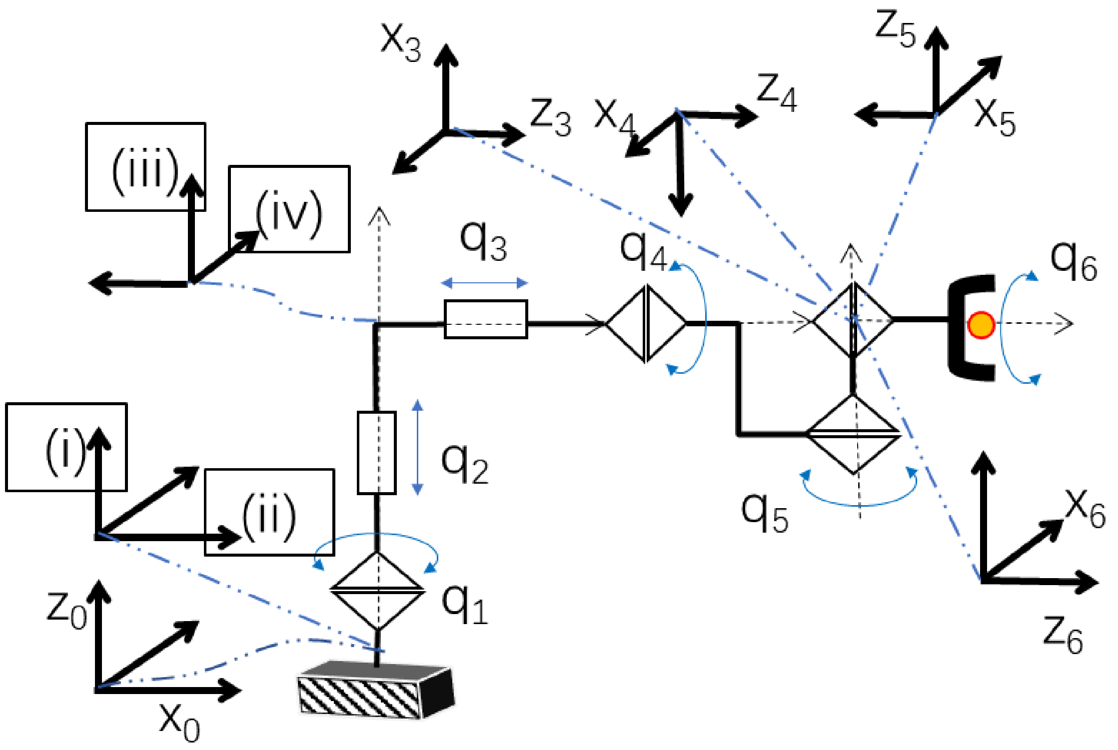
1. Frame Assignment



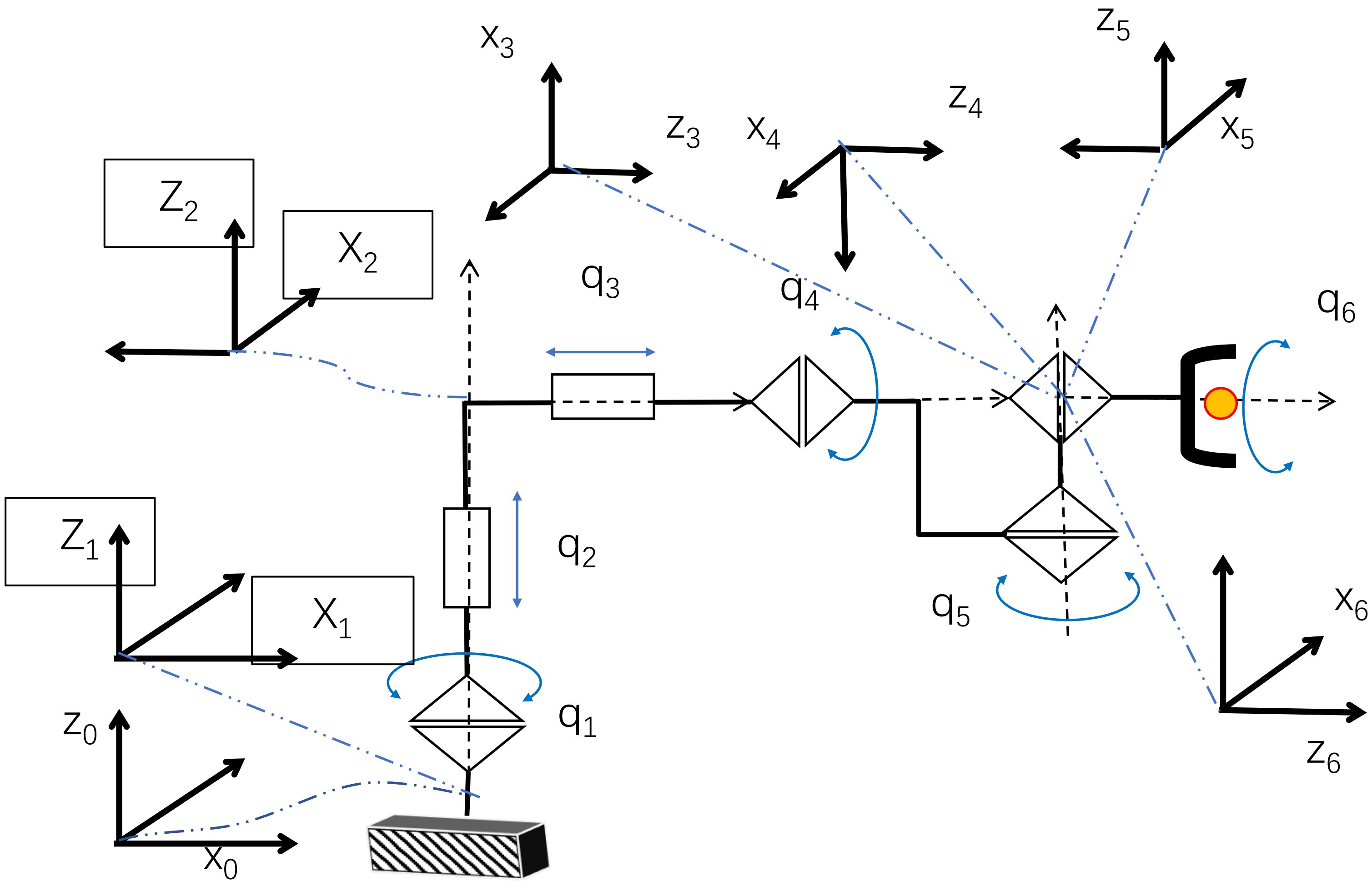
Review Quiz I

2. D-H Table

	Link Twist α_{i-1}	Link Length a_{i-1}	Joint Angle θ_i	Link offset d_i
${}^0T_1$				
${}^1T_2$				
${}^2T_3$				
${}^3T_4$				
${}^4T_5$				
${}^5T_6$				

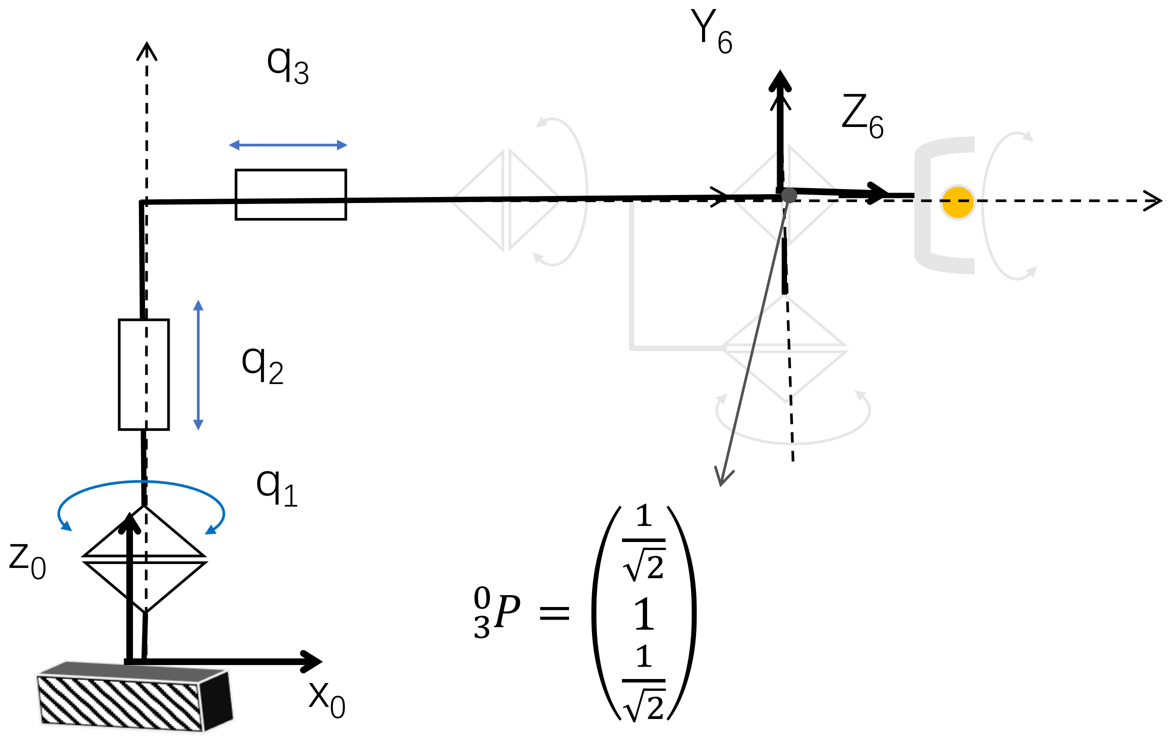


Review Quiz I



Review Quiz I

Forward kinematics:

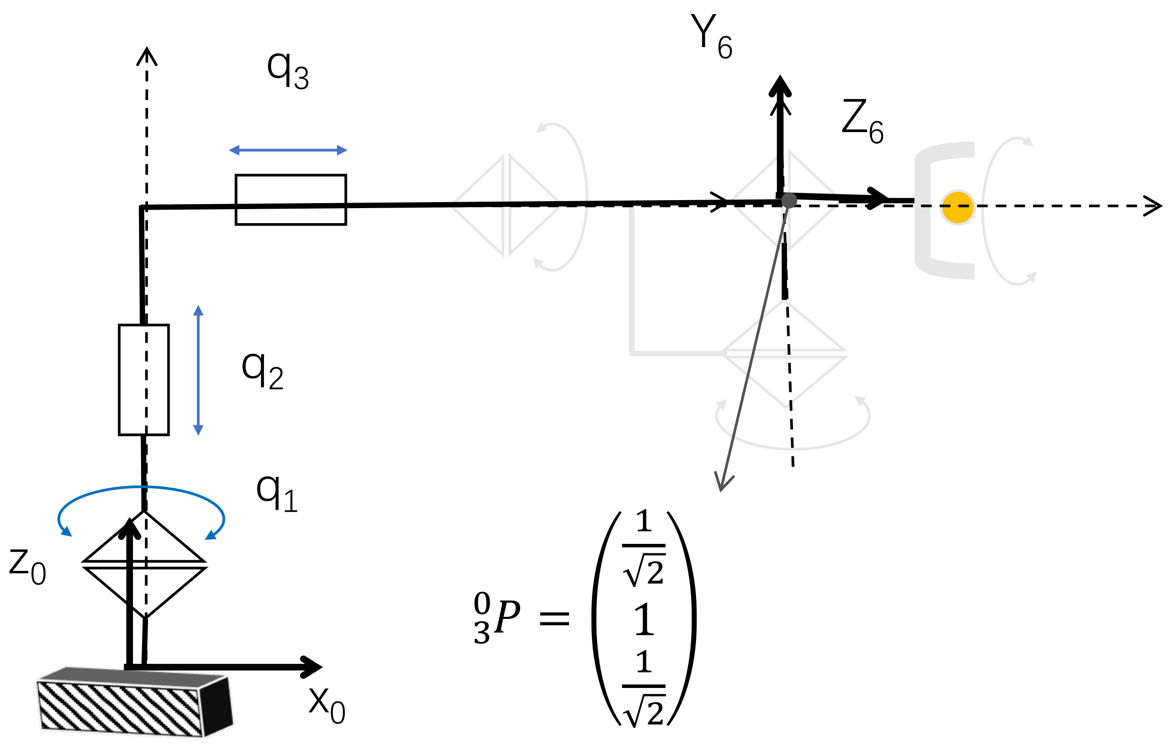


$$ {}^{0}_{3}P = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} $$

Inverse kinematics:

Review Quiz I

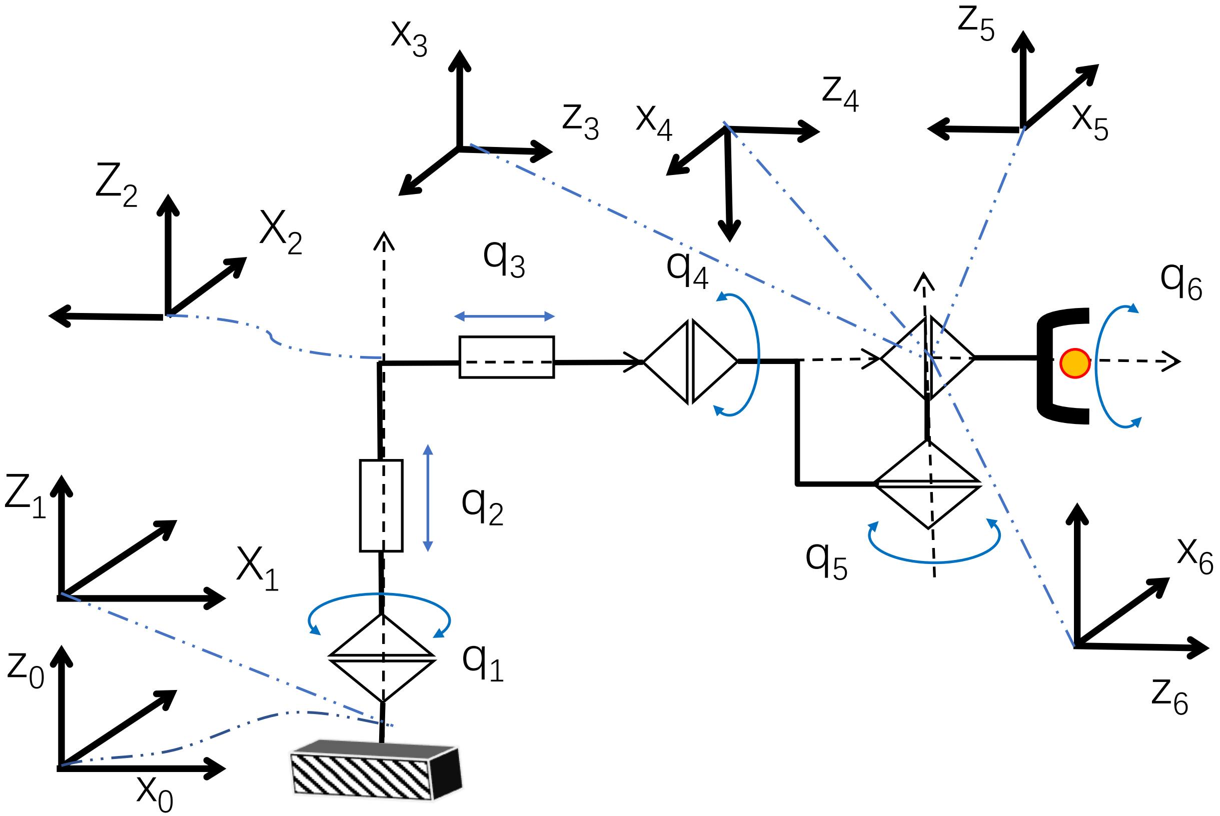
Workspace Analysis



$$ {}^{0}_{3}P = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ 1 \\ \frac{1}{\sqrt{2}} \end{pmatrix} $$

Review Quiz I

Manipulability



Recap: Acceleration

$$\vec{P}_1 = \vec{P}_{B,ORG} + x\,\hat{i}_B + y\,\hat{j}_B + z\hat{k}_B$$

Differentiating with respecting to time,

$$\vec{V}_1 = \vec{V}_{B,ORG} + \dot{x}\,\hat{i}_B + \dot{y}\,\hat{j}_B + \dot{z}\hat{k}_B + x\dot{\hat{i}}_B + y\dot{\hat{j}}_B + z\dot{\hat{k}}_B$$

$$\vec{V}_1 = \vec{V}_{B,ORG} + \dot{x}\,\hat{i}_B + \dot{y}\,\hat{j}_B + \dot{z}\hat{k}_B + \vec{\omega} \times \vec{P}_{1/B}$$

Differentiate w.r.t. time

$$\dot{\vec{V}}_1 = \dot{\vec{V}}_{B,ORG} + \ddot{x}\,\hat{i}_B + \ddot{y}\,\hat{j}_B + \ddot{z}\hat{k}_B + \vec{\omega} \times \vec{V}_{1/B} + \vec{\omega} \times \left(\vec{V}_{1/B} + \vec{\omega} \times \vec{P}_{1/B}\right) + \dot{\vec{\omega}} \times \vec{P}_{1/B}$$

$$\dot{\vec{V}}_1 = \dot{\vec{V}}_{B,ORG} + \ddot{x}\,\hat{i}_B + \ddot{y}\,\hat{j}_B + \ddot{z}\hat{k}_B + 2\vec{\omega} \times \vec{V}_{1/B} + \vec{\omega} \times \vec{\omega} \times \vec{P}_{1/B} + \dot{\vec{\omega}} \times \vec{P}_{1/B}$$

coriolis acceleration

tangential acceleration

centrifugal acceleration

Recap: Acceleration for “Propagation” from link to link

$$^{0}\omega_{i+1} = {}^{0}\omega_i + {}_{i+1}^{0}R\ \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

Differentiate wrt to time

$$^{0}\dot{\omega}^{0}{}_{i+1} = {}^{0}\dot{\omega}_i + {}_{i+1}^{0}\dot{R}\ \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + {}_{i+1}^{0}R\ \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

$$= {}^{0}\dot{\omega}_i + {}_{i+1}^{0}\dot{R}\ {}_{i+1}^{0}R^T\ {}_{i+1}^{0}R\ \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + {}_{i+1}^{0}R\ \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

$$= {}^{0}\dot{\omega}_i + {}^{0}\Omega_i\ {}_{i+1}^{0}R\ \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + {}_{i+1}^{0}R\ \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

$$= {}^{0}\dot{\omega}_i + {}^{0}\omega_i \times {}_{i+1}^{0}R\ \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + {}_{i+1}^{0}R\ \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

$$\boxed{{}^{i+1}\dot{\omega}^{0}{}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\dot{\omega}_i + {}^{i+1}_{i}R\ \underbrace{{}^{i}\omega_i \times \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}}_{0}}$$

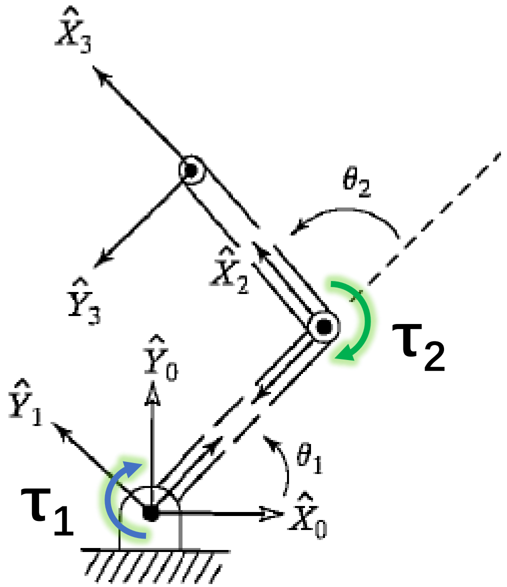
- For prismatic joint, ${}^{i+1}\dot{\omega}^{0}{}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\dot{\omega}^{0}{}_i$ 0

Since $\dot{\vec{V}}_1 = \dot{\vec{V}}_{B,ORG} + \ddot{x}\,\hat{i}_B + \ddot{y}\,\hat{j}_B + \ddot{z}\hat{k}_B + 2\vec{\omega} \times \vec{V}_{1/B} + \vec{\omega} \times \vec{\omega} \times \vec{P}_{1/B} + \dot{\vec{\omega}} \times \vec{P}_{1/B}$

$$\boxed{{}^{i+1}\dot{v}^{0}{}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{v}^{0}{}_i + 0 + 0 + {}^{i}\omega^{0}{}_i \times {}^{i}\omega^{0}{}_i \times {}^{i}P_{i+1} + {}^{i}\dot{\omega}^{0}{}_i \times {}^{i}P_{i+1}\right)}$$

Q5.1 Example of acceleration

- Given the following 2-link planar manipulator in Q3.4, determine for each link the a) absolute angular acceleration, and b) joint absolute linear acceleration. Express the answer in their own frame.
- $^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad ^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $\quad ^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Assume acceleration due to gravity to be g
  - *i.e.* $^0\dot{v}_0 = g\hat{Y}_0$



$$^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ ^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ ^0\dot{\omega}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ ^0\dot{v}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$$

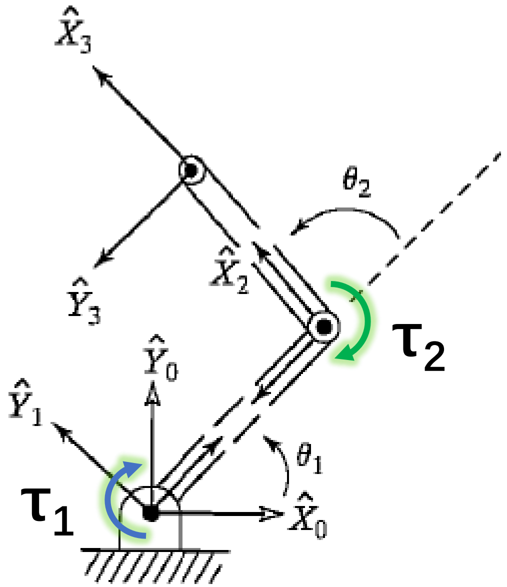
Q5.1 Example of acceleration

Assume ${}^0\dot{v}_0 = g\,\hat{Y}_0$

$$ {}^0_1R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^1_2R = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^0\dot{\omega}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^0\dot{v}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} $$



Q5.1 Example of acceleration

$$\boxed{{}^{i+1}\omega^{0}_{i+1} = {}^{i+1}_{i}R\;{}^{i}\omega^{0}_{i} + \dot{\theta}_{i+1}\;{}^{i+1}\hat{Z}_{i+1}}$$

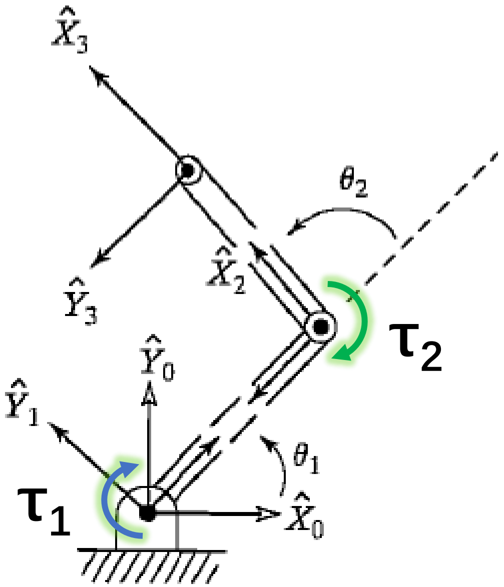
$$\boxed{{}^{i+1}\dot{\omega}^{0}_{i+1} = {}^{i+1}_{i}R\;{}^{i}\dot{\omega}_{i} + {}^{i+1}_{i}R\;{}^{i}\omega_{i} \times \dot{\theta}_{i+1}\;{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\;{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}v^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v^{0}_{i} + {}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right)}$$

$$\boxed{{}^{i+1}\dot{v}^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{v}^{0}_{i} + {}^{i}\Omega_{i}\;{}^{i}\Omega_{i}\;{}^{i}P_{i+1} + {}^{i}\dot{\Omega}_{i}\;{}^{i}P_{i+1}\right)}$$

- i=0

$$
{}^{1}\omega_{1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \quad
{}^{1}\dot{\omega}_{1} = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}, \quad
{}^{1}\dot{v}_{1} = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} g\,s_1 \\ g\,c_1 \\ 0 \end{bmatrix}
$$



Q5.1 Example of acceleration

$$\boxed{{}^{i+1}\omega^{0}{}_{i+1} = {}^{i+1}_{\ \ i}R\ {}^{i}\omega^{0}{}_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}\dot{\omega}^{0}{}_{i+1} = {}^{i+1}_{\ \ i}R\ {}^{i}\dot{\omega}_{i} + {}^{i+1}_{\ \ i}R\ {}^{i}\omega_i \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}v^{0}{}_{i+1} = {}^{i+1}_{\ \ i}R\left({}^{i}v^{0}{}_{i} + {}^{i}\omega^{0}{}_{i} \times {}^{i}P_{i+1}\right)}$$

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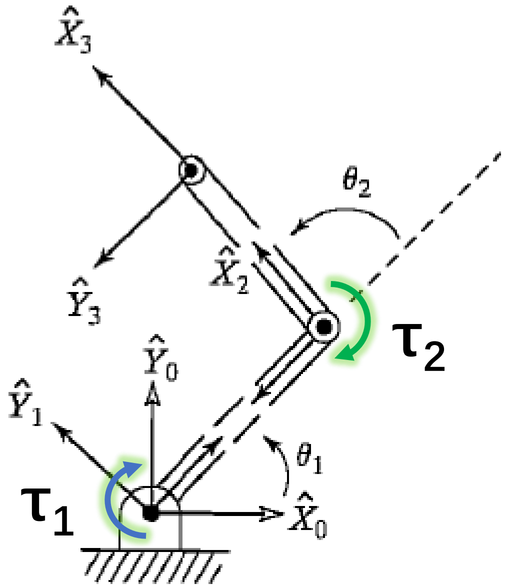
- i=1

$$ {}^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix},\quad {}^{2}\dot{\omega}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}, $$

$$ {}^{2}\dot{v}_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_1\dot{\theta}_1^2 + g\,s_1 \\ l_1\ddot{\theta}_1 + g\,c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\ddot{\theta}_1^2 s_2 - l_1\dot{\theta}_1^2\,c_2 + g\,s_{12} \\ l_1\ddot{\theta}_1^2 c_2 + l_1\dot{\theta}_1^2\,s_2 + g\,c_{12} \\ 0 \end{bmatrix} $$



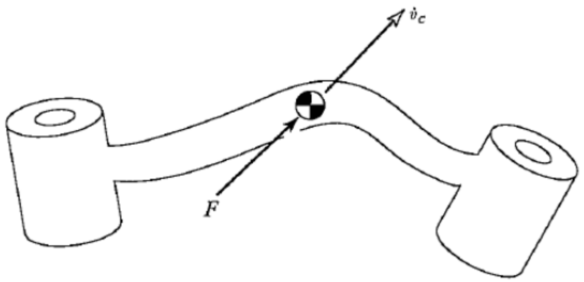
Newton-Euler Formulation

ECE 470: Introduction to Robotics

Newton’s Law of Motion

Newton’s 2nd Law

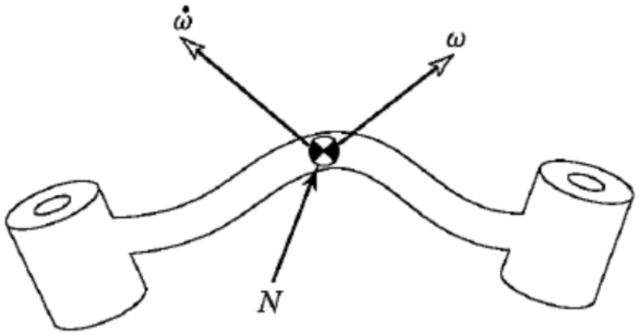
$$\boxed{F_i = m\dot{v}_{C_i}}$$



Euler’s equation (Torque)

$$\boxed{N_i = {}^{C_i}I\,\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i}$$

Frame {C} is located at the center of mass



Force and Torque

Notation:

- f_i is the force exerted on link i by link $i-1$
- n_i is the torque exerted on link i by link $i-1$
- F_i is the net force exerted on the CG
- N_i is the net torque exerted on the CG

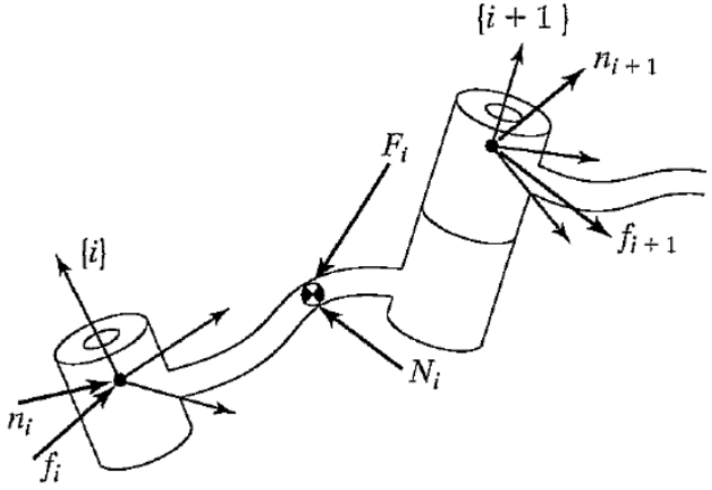
- Summing forces acting on link *i*,

$$ {}^{i}F_i = {}^{i}f_i - {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} $$

$$ \boxed{{}^{i}f_i = {}^{i}_{i+1}R\,{}^{i+1}f_{i+1} + {}^{i}F_i} $$

- Summing torques about CM of link *i*,

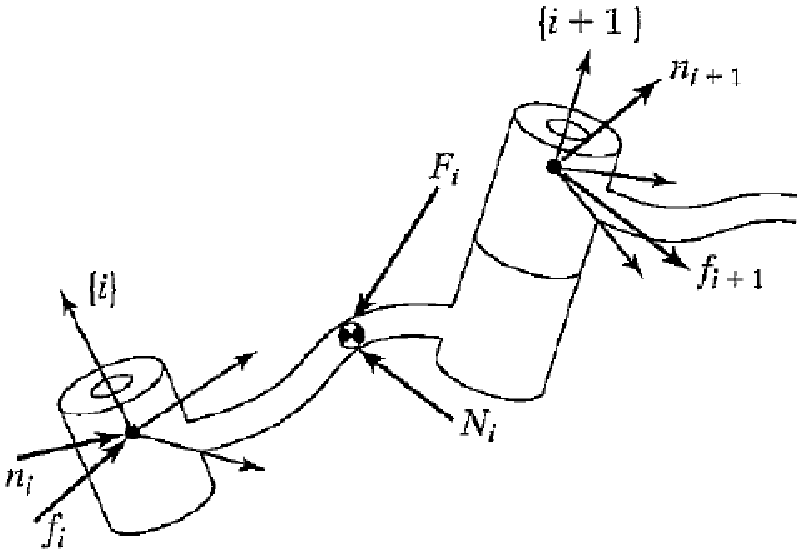
- $$ \begin{aligned} {}^{i}N_i &= {}^{i}n_i - {}^{i}n_{i+1} + (-\,{}^{i}P_{C_i}) \times {}^{i}f_i - ({}^{i}P_{i+1} - {}^{i}P_{C_i}) \times {}^{i}f_{i+1} \\ &= {}^{i}n_i - {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} - {}^{i}P_{C_i} \times ({}^{i}f_i - {}^{i}_{i+1}R\,{}^{i+1}f_{i+1}) \\ &\quad - {}^{i}P_{i+1} \times {}^{i}f_{i+1} \end{aligned} $$



$$ \boxed{{}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\,{}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times ({}^{i}_{i+1}R\,{}^{i+1}f_{i+1})} $$

Force and Torque

- Torque required by motor:
  - $\tau_i = {}^i n_i^T \; {}^i\hat{Z}_i$ (ie dot product of the two vectors)
    - Dot product because the rest are reaction forces
- In the case of prismatic joint, force required by actuator:
  - $F_i = {}^i f_i^T \; {}^i\hat{Z}_i$



Recall in Wk 04: **Jacobian**

For a rotational joint,

$$J_i = \begin{bmatrix} Z_{i-1} \times (P_N - P_{i-1}) \\ Z_{i-1} \end{bmatrix}$$

For a translational joint,

$$J_i = \begin{bmatrix} Z_{i-1} \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Formulation

Outwards

$$\boxed{{}^{i+1}\omega^{0}{}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega^{0}{}_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}\dot{\omega}^{0}{}_{i+1} = {}^{i+1}_{i}R\dot{\omega} + {}^{i+1}_{i}R\ {}^{i}\omega^{0}{}_{i} \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}v^{0}{}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v^{0}{}_{i} + {}^{i}\omega^{0}{}_{i} \times {}^{i}P_{i+1}\right)}$$

$$\boxed{\begin{aligned}{}^{i+1}\dot{v}^{0}{}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}^{0}{}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega^{0}{}_{i} \times \left({}^{i}\omega^{0}{}_{i} \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}^{0}{}_{i}\right)\\ +2\,{}^{i+1}\omega^{0}{}_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}\end{aligned}}$$

$$\boxed{{}^{i+1}\dot{v}^{0}{}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}^{0}{}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega^{0}{}_{i} \times \left({}^{i}\omega^{0}{}_{i} \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}^{0}{}_{i}\right)}$$

Iterative Newton-Euler Formulation

- Newton and Euler

$$\boxed{{}^{i}\dot{v}^{0}{}_{Ci} = {}^{i}\dot{\omega}^{0}{}_{i} \times {}^{i}P_{Ci} + {}^{i}\omega^{0}{}_{i} \times \left({}^{i}\omega^{0}{}_{i} + {}^{i}P_{Ci} \right) + {}^{i}\dot{v}^{0}{}_{i}}$$

$$\boxed{{}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{Ci+1}}$$

$$\boxed{{}^{i+1}N_{i+1} = {}^{Ci+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{Ci+1}I_{i+1}\,{}^{i+1}\omega_{i+1}}$$

Iterative Newton-Euler Formulation

- Inwards

$$\boxed{{}^{i}f_i = {}^{i}_{i+1}R\; {}^{i+1}f_{i+1} + {}^{i}F_i}$$

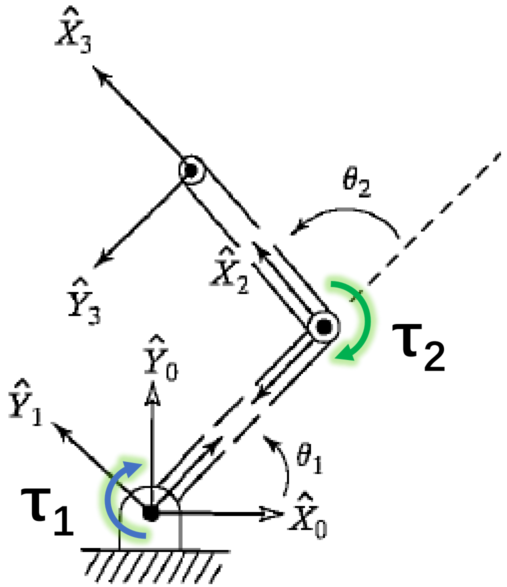
$$\boxed{{}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\; {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times \left({}^{i}_{i+1}R\; {}^{i+1}f_{i+1}\right)}$$

$$\boxed{\tau_i = {}^{i}n_i^T \; {}^{i}\hat{Z}_i}$$

Q5.2 Example of Dynamics

- Given the following 2-link planar manipulator in Q3.4, Given the following two-link planar manipulator, and assuming all the mass exists as a point mass at the distal end of each link, determine the torque required by each motor.

- Previously,
- $${}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



- Assume acceleration due to gravity to be g
  - *i.e.* ${}^0\dot{v}_0 = g\hat{Y}_0$

$${}^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^0\dot{\omega}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^0\dot{v}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$$

Q5.2 Example of Dynamics

$$ {}^{0}_{1}R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}R = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^{0}v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad {}^{0}\dot\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, $$

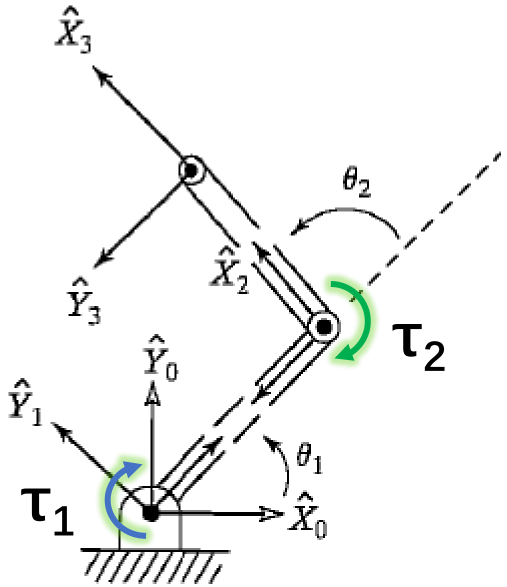
$$ {}^{0}\dot v_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} $$

$$ \boxed{{}^{i+1}\omega^{0}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega^{0}_{i} + \dot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1}} $$

$$ \boxed{{}^{i+1}\dot\omega^{0}_{i+1} = {}^{i+1}_{i}R\dot\omega + {}^{i+1}_{i}R\,{}^{i}\omega^{0}_{i} \times \dot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1} + \ddot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1}} $$

$$ \boxed{{}^{i+1}v^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v^{0}_{i} + {}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right)} $$

$$ \boxed{{}^{i+1}\dot v^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot\omega^{0}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega^{0}_{i} \times \left({}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right) + {}^{i}\dot v^{0}_{i}\right)} $$



i=0

$$ {}^{1}\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 \end{bmatrix}, \quad {}^{1}\dot\omega_1 = \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_1 \end{bmatrix}, \quad {}^{1}\dot v_1 = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} $$

i=1

$$ {}^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix}, \quad {}^{2}\dot\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_1 + \ddot\theta_2 \end{bmatrix}, \quad {}^{2}\dot v_2 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_1\dot\theta_1^{\,2} + gs_1 \\ -l_2\ddot\theta_1 + gc_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\ddot\theta_1^{\,2}s_2 - l_1\dot\theta_1^{\,2}c_2 + gs_{12} \\ l_1\ddot\theta_1^{\,2}c_2 - l_1\dot\theta_1^{\,2}s_2 + gc_{12} \\ 0 \end{bmatrix} $$

Q5.2 Example of Dynamics

$$\boxed{{}^{i}\dot{v}^{0}_{Ci} = {}^{i}\dot{\omega}^{0}_{i} \times {}^{i}P_{Ci} + {}^{i}\omega^{0}_{i} \times \left({}^{i}\omega^{0}_{i} + {}^{i}P_{Ci} \right) + {}^{i}\dot{v}^{0}_{i}}$$

$$\boxed{{}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{Ci+1}}$$

$$\boxed{{}^{i+1}N_{i+1} = {}^{Ci+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{Ci+1}I_{i+1}\,{}^{i+1}\omega_{i+1}}$$

For i=0,

$$ {}^{1}\dot{v}_{C_1} = \begin{bmatrix} g s_1 \\ g c_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -l_1\dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ l_1\ddot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1\dot{\theta}_1^2 + g s_1 \\ l_1\ddot{\theta}_1 + g c_1 \\ 0 \end{bmatrix} $$

$$ {}^{1}F_1 = \begin{bmatrix} -m_1 l_1\dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1\ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix} $$

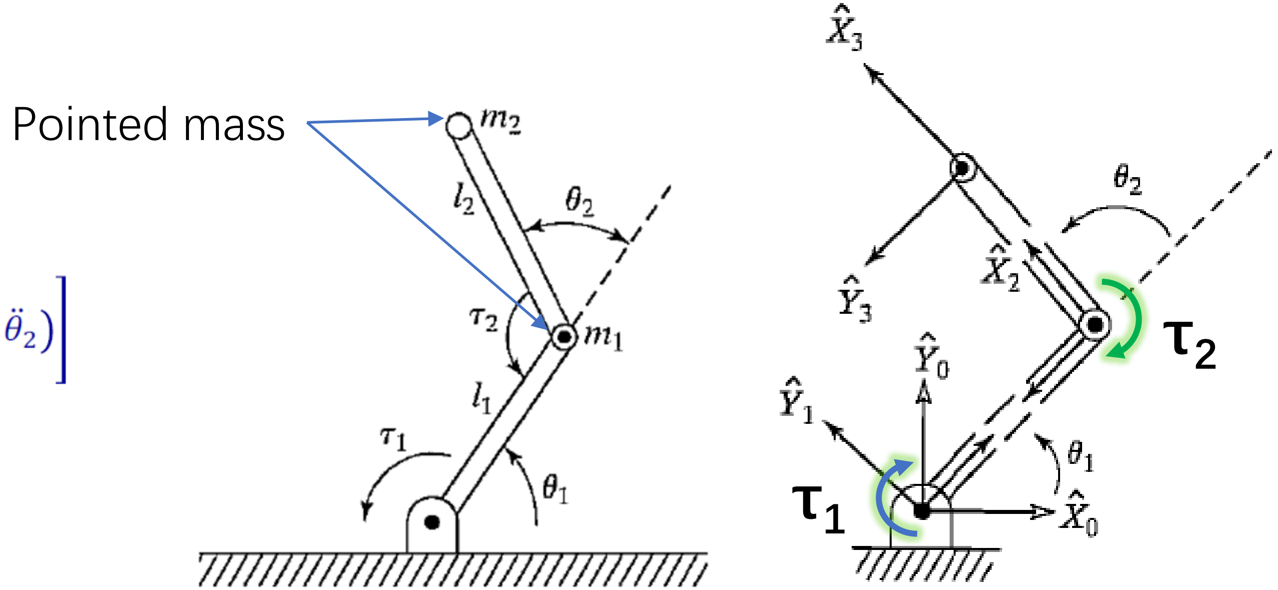
$$ {}^{1}N_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$

For i=1,

$$ {}^{2}\dot{v}_{C_2} = \begin{bmatrix} l_1\ddot{\theta}_1^2 s_2 - l_1\dot{\theta}_1^2\ c_2 + g\ s_{12} \\ l_1\ddot{\theta}_1^2 c_2 + l_1\dot{\theta}_1^2\ s_2 + g\ c_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} -l_2(\dot{\theta}_1\ + \dot{\theta}_2\ )^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} $$

$$ {}^{2}F_2 = \begin{bmatrix} m_2 l_1\ddot{\theta}_1 s_2 - m_2 l_1\dot{\theta}_1^2 c_2 + m_2 g s_{12} - m_2 l_2(\dot{\theta}_1\ + \dot{\theta}_2\ )^2 \\ m_2 l_1\ddot{\theta}_1 c_2 + m_2 l_1\dot{\theta}_1^2 s_2 + m_2 g c_{12} + m_2 l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} $$

$$ {}^{2}N_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} $$



Q5.2 Example of Dynamics

$$\boxed{{}^{i}f_i = {}^{i}_{i+1}R\ {}^{i+1}f_{i+1} + {}^{i}F_i}$$

$$\boxed{{}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\ {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times \left({}^{i}_{i+1}R\ {}^{i+1}f_{i+1}\right)}$$

$$\boxed{\tau_i = {}^{i}n_i^T\ {}^{i}\hat{Z}_i}$$

For i =2,

$$ {}^{2}f_2 = {}^{2}F_2 $$

$$ {}^{2}n_2 = \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 \ddot{\theta}_1 c_2 + m_2 l_1 l_2 \dot{\theta}_1^2 s_2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix} $$

Torque required by motor 2

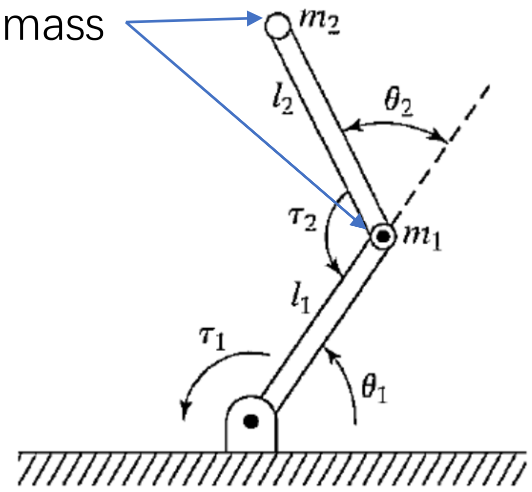
For i =1,

$$ {}^{1}f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 l_1 s_2 \ddot{\theta}_1 - m_2 l_1 c_2 \dot{\theta}_1^2 + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 c_2 \ddot{\theta}_1 + m_2 l_1 s_2 \dot{\theta}_1^2 + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -m_1 l_1 \dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}, $$

$$ {}^{1}n_1 = \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 g c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ m_2 l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 l_1 g s_2 s_{12} \\ + m_2 l_1 l_2 c_2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 g c_2 c_{12} \end{bmatrix}. $$

Torque required by motor 1

Pointed mass



Dynamic Equation

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

$M(\Theta)$ is n x n mass matrix of the manipulator

$V(\Theta, \dot{\Theta})$ is an n x 1 vector of centrifugal and Coriolis terms

$G(\Theta)$ is an n x 1 vector of gravity terms

Dynamic Equation

Using the previous case as the example, $\tau_2 = m_2 l_1 l_2 \ddot{\theta}_1 c_2 + m_2 l_1 l_2 \dot{\theta}_1^2 s_2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2)$

$$M(\Theta) = \begin{bmatrix} l_2^2 m_2 + 2 l_1 l_2 m_2 c_2 + l_1^2(m_1 + m_2) & l_2^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2^2 m_2 + l_1 l_2 m_2 c_2 & l_2^2 m_2 \end{bmatrix}$$

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

Lagrangian Approach

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