ZJU-UIUC Institute

Zhejiang University / University of Illinois at Urbana-Champaign Institute

ECE 470: Introduction to Robotics

Week 05

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Review on the Fundamentals

ECE 470: Introduction to Robotics

Review of the Fundamentals

- Covered mainly spatial representation, kinematics, static forces
- leading to more advanced topics in robot dynamics planning and control

Review of the Fundamentals

- The bigger picture of this course covers robot <u>mechanics, control and perception</u>
  - In alignment with the scope of robotics by definition: A <u>machine/agent</u> designed to execute <u>task</u>(s) while interacting with the <u>environment</u>

Agent

Tasks

Environment

Wk01 Impt Take Away: Homogenous Transformation Matrix

- Homogenous transformation matrix

$$ {}^{A}_{B}T = \begin{bmatrix} {}^{A}_{B}R & {}^{A}P_{BORG} \\ \overline{\boldsymbol{0}} & \mathbf{1} \end{bmatrix} $$

Coordinates of P in {B}

Coordinates of P in {A}

$$ {}^{A}P = {}^{A}_{B}R\ {}^{B}P + {}^{A}P_{BORG} $$

Coordinates of origin of {B} in {A}

$$ {}^{A}\tilde{P} = {}^{A}_{B}T\ {}^{B}\tilde{P} $$

$$ \begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc:c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hdashline \boxed{0 \quad 0 \quad 0} & & & \boxed{1} \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix} $$

Homogeneous Coordinates in {A}

Homogeneous Transformation

Homogeneous Coordinates in {B}

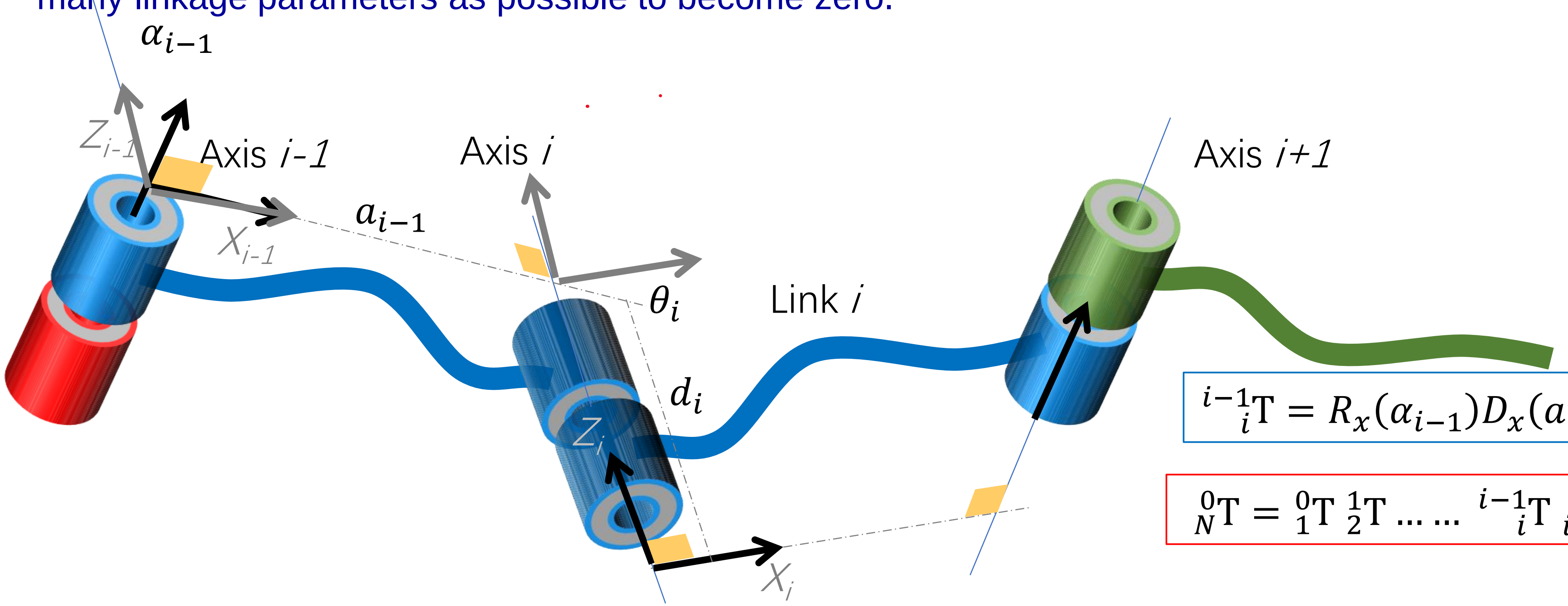
Can be seen as an multiplication operator of 4 x 4 matrix in 3D space

Wk01 Impt Take Away

- Kinematics: The science of motion (a branch of mechanics) without regards to the forces that cause it
  - Pose (/configuration) of the manipulator in static situations
  - Analyze (linear and angular) motion of bodies (/linkages)
- D-H Method: A systematic way to represent the configuration of the serial manipulator
  1. Frame assignment
  2. D-H parameters and tables
  3. Homogenous transformation matrix
- Forward Kinematics: <u>mapping from joint coordinates</u>, or robot configuration <u>to end-effector pose</u>
  - $^{0}_{\mathrm{E}}\mathrm{T} = {}^{0}_{1}\mathrm{T}(q_1) \cdot {}^{1}_{2}\mathrm{T}(q_2) \cdot {}^{2}_{3}\mathrm{T}(q_3) \cdot \cdots {}^{N-1}_{\mathrm{N}}\mathrm{T}(q_{\mathrm{N}}) \cdot {}^{N}_{\mathrm{E}}\mathrm{T}$

Wk02 Impt Take Away: D-H Method for Kinematic Analysis

1. Identify the joint axes and attach infinite lines along them. For neighboring pair (*i* and *i*+1)
2. Identify the common perpendicular or point of intersection. At the point of intersection, or at the point where the common perpendicular meets i^{th} axis, assign the link-frame origin.
3. Assign the Z_i axis pointing along the i^{th} joint axis.
4. Assign the X_i axis pointing along the direction normal to the two neighboring Z-axes.
5. Assign the Y_i axis to complete a right-hand coordinate system.
6. Assign {0} to match {1}. For {N}, choose an origin location and X direction freely, but generally so as to cause as many linkage parameters as possible to become zero.



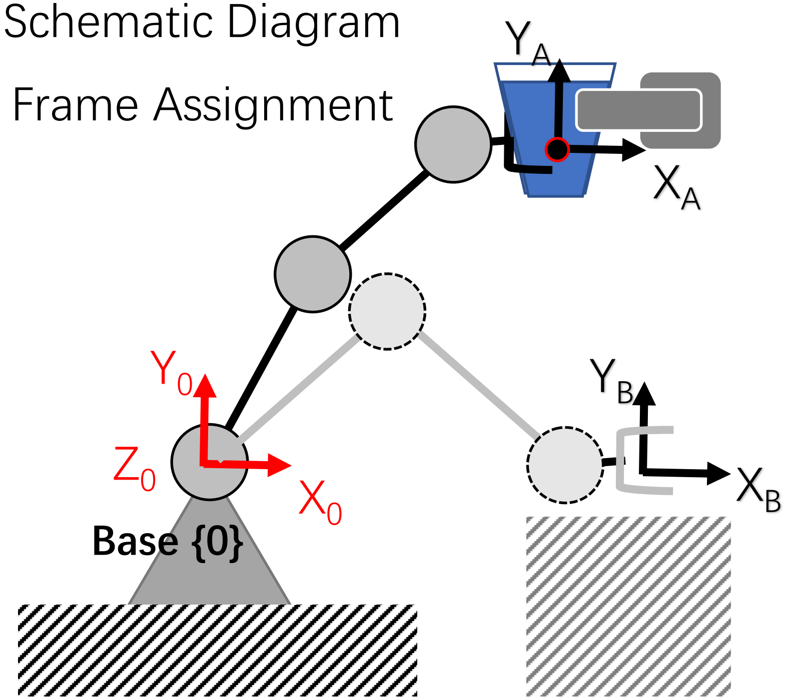
$$ {}^{i-1}_{i}T = R_x(\alpha_{i-1}) D_x(a_{i-1})\, R_z(\theta_i) D_z(d_i) $$

$$ {}^{0}_{N}T = {}^{0}_{1}T\, {}^{1}_{2}T \dots \dots {}^{i-1}_{i}T\, {}^{i}_{i+1}T \dots \dots {}^{N-2}_{N-1}T\, {}^{N-1}_{N}T $$

Wk02 Impt Take Away: Forward/ Inverse Kinematics

1. Schematic Diagram

2. Frame Assignment



3. DH Parameters & Table

	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	$Q1=\theta_1$	0
2	0	$L1$	$Q2=\theta_2$	0
3	0	$L2$	$Q3=\theta_3$	0

4. Homogenous Transformation

$$ {}^{i-1}_{i}\mathrm{T} = R_x(\alpha_{i-1})D_x(\alpha_{i-1})\ R_z(\theta_i)D_z(d_i) $$

$$ {}^{0}_{3}\mathrm{T} = {}^{0}_{1}\mathrm{T}\ {}^{1}_{2}\mathrm{T}\,{}^{2}_{3}\mathrm{T} $$

5. Forward Kinematics

$$ {}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & L_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & L_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}_{\mathrm{E}}T = \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{\mathrm{E}}\mathrm{T} = {}^{0}_{1}\mathrm{T}(\theta_1)\cdot{}^{1}_{2}\mathrm{T}(\theta_2)\cdot{}^{2}_{3}\mathrm{T}(\theta_3)\cdot{}^{3}_{\mathrm{E}}\mathrm{T} $$

$$ {}^{0}_{\mathrm{E}}\widetilde{\mathrm{P}} = {}^{0}_{1}\mathrm{T}(\theta_1)\cdot{}^{1}_{2}\mathrm{T}(\theta_2)\cdot{}^{2}_{3}\mathrm{T}(\theta_3)\cdot{}^{3}_{\mathrm{E}}\widetilde{\mathrm{P}} $$

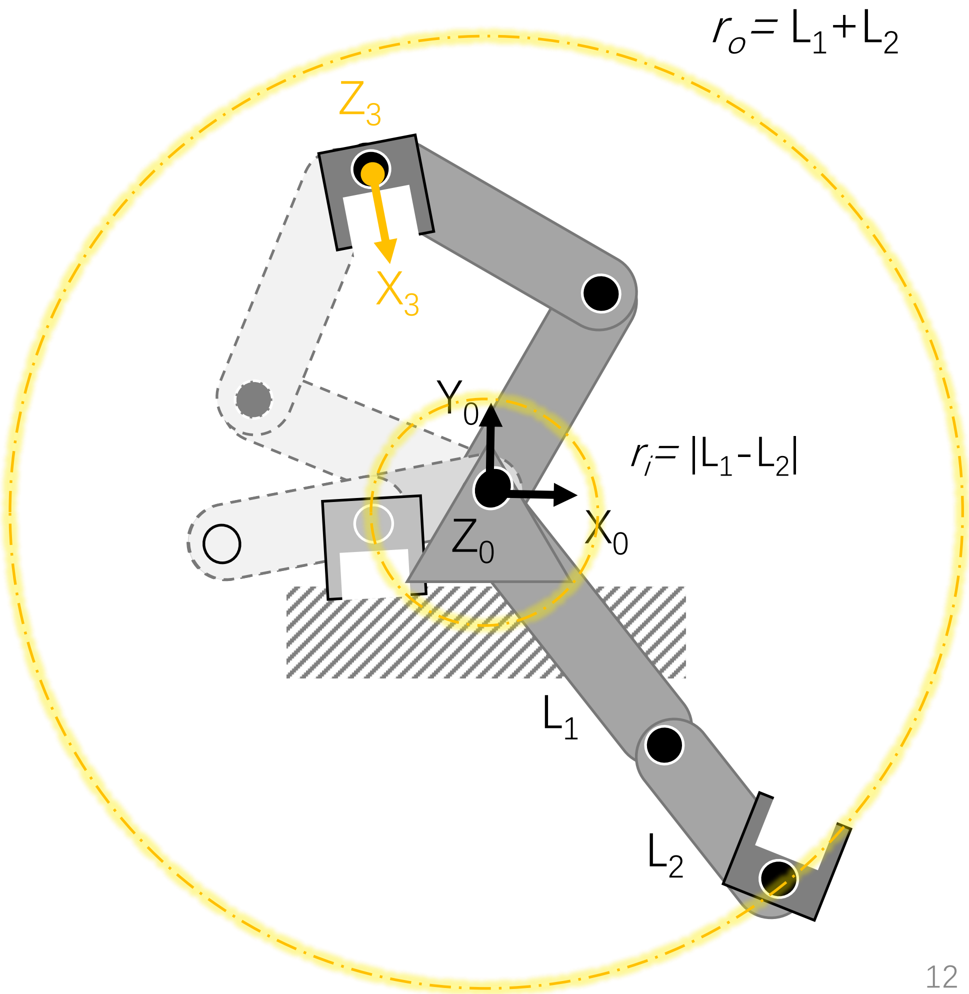
6. Inverse Kinematics

a) Solve $\theta_1\,\theta_2\,\theta_3$, such that ${}^{0}_{\mathrm{E}}\mathrm{T}={}^{0}_{\mathrm{A}}\mathrm{T}$

b) Solve $\theta_1\,\theta_2\,\theta_3$, such that ${}^{0}_{\mathrm{E}}\mathrm{T}={}^{0}_{\mathrm{B}}\mathrm{T}$

Wk03 Impt Take Away: Solvability

- Workspace
  - **Reachable**: Region where the end-effector can be located
  - **Dexterous**: Region where the end-effector can be located with all orientations
- Multiple solutions
  - For the same end-effector pose, there could be 2 possible solutions
- Approach to solutions:
  - Numerical
  - Closed-form



Wk03 Impt Take Away: Jacobian

- For an N-joint robot in 3D space,

Mapping of Velocity Coordinates

$$v_N = [J_1 \quad \dots J_i \dots \quad J_N] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_N \end{bmatrix}$$

Joint velocity is $\dot{\Theta}$ is N x 1,
Jacobian $J(\Theta)$ is 6 x N,
Cartesian velocity is ${}^{0}v = [\,{}^{0}\dot{\mathrm{P}} \quad {}^{0}\dot{\Theta}]^{\mathrm{T}}$ is 6 x 1
Column J_i represents motion contribution of Joint i

Jacobian in Force Domain

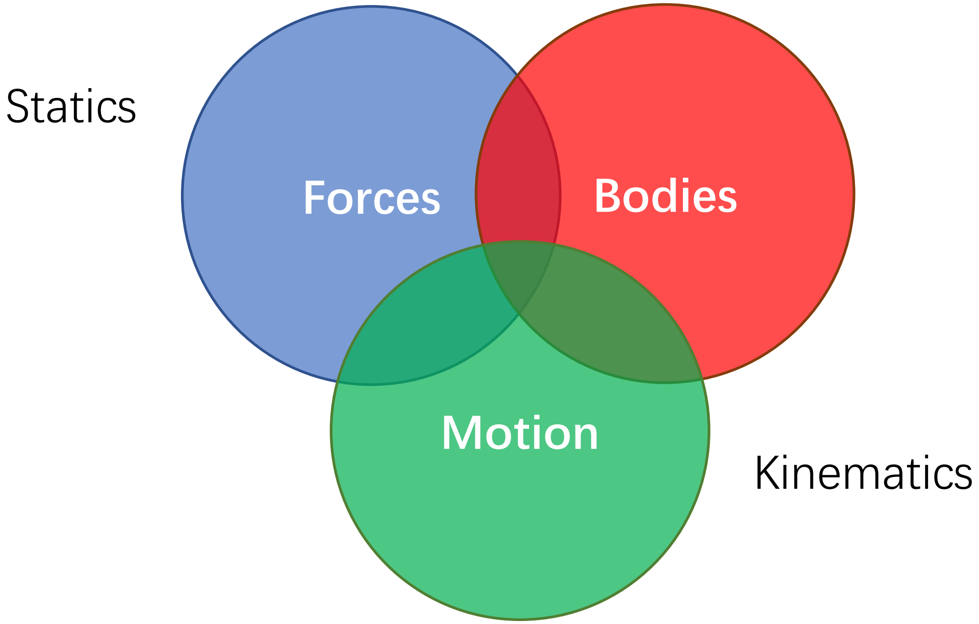
6-by-1 Cartesian Force-Moment Vector

6-by-1 torque/force at joints → $\boxed{\tau = J^{\mathrm{T}} F}$

N-by-6 Jacobian Transposed

Robot Mechanics

- **Kinematics**: The science of motion without regards to the forces that cause it
- **Statics**: Bodies in equilibrium and force (/moment) relationship
- **Dynamics**: Concern with the forces) on bodies that cause motion



HW 1: Spatial Representation/Transformation

Question 1.

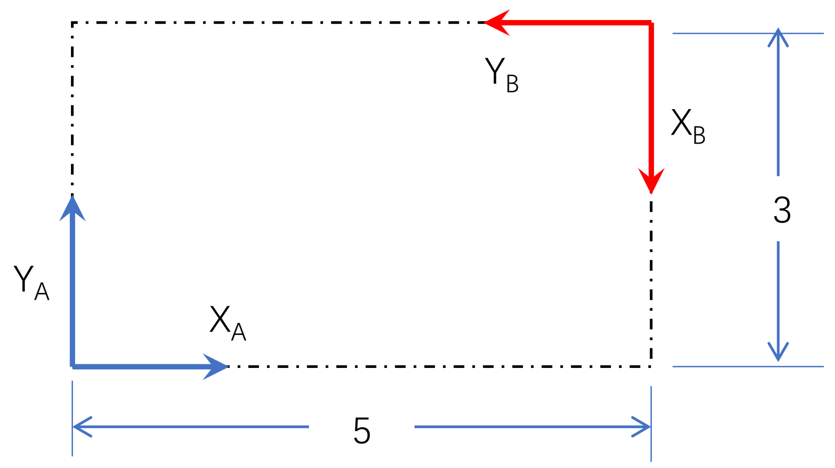
In Figure 1, Frame {A} and {B} are not connected.

a) Determine the transformation matrix ${}^{A}_{B1}T$ after {B} rotates 45° about its axis X_B to become {B1}.

b) Determine the inverse matrix ${}^{A}_{B1}T^{-1}$ in (a)

c) Determine the transformation matrix ${}^{A}_{B2}T$ if {B1} revolves 45° about Y_A to become {B2}.

d) Determine the transformation matrix ${}^{A1}_{B2}T$ if {A} rotates -90° about its X_A to become {A1}.



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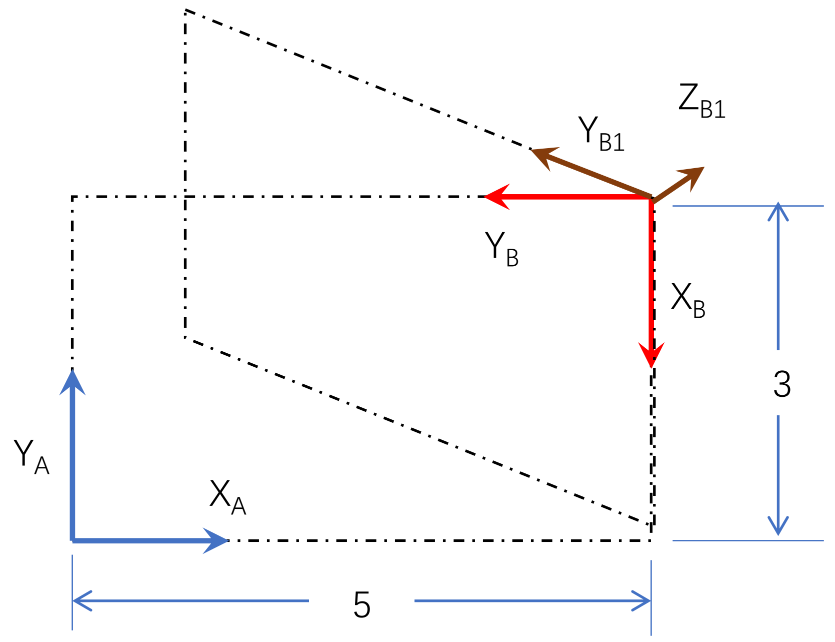

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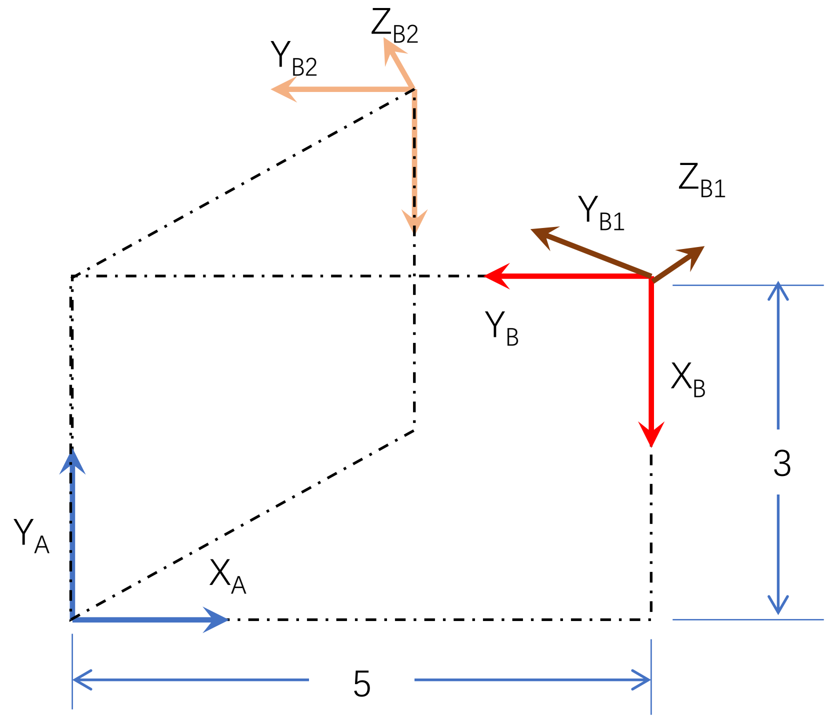

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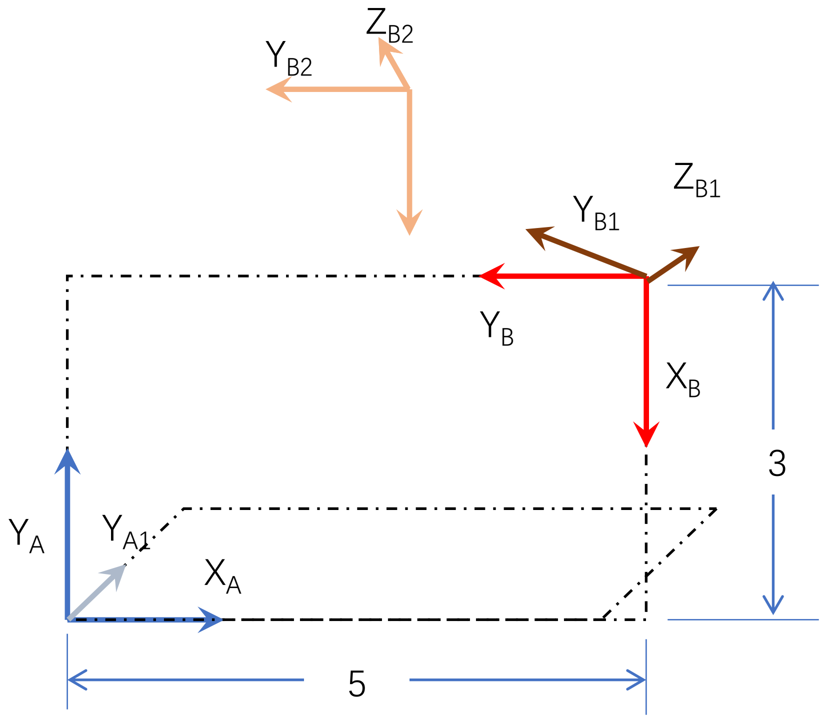

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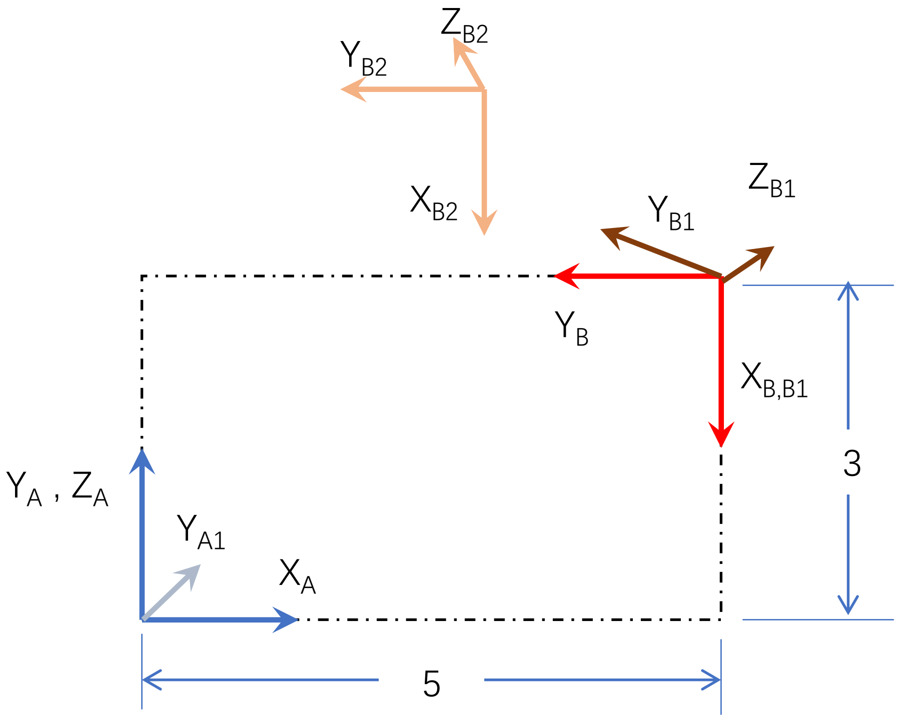

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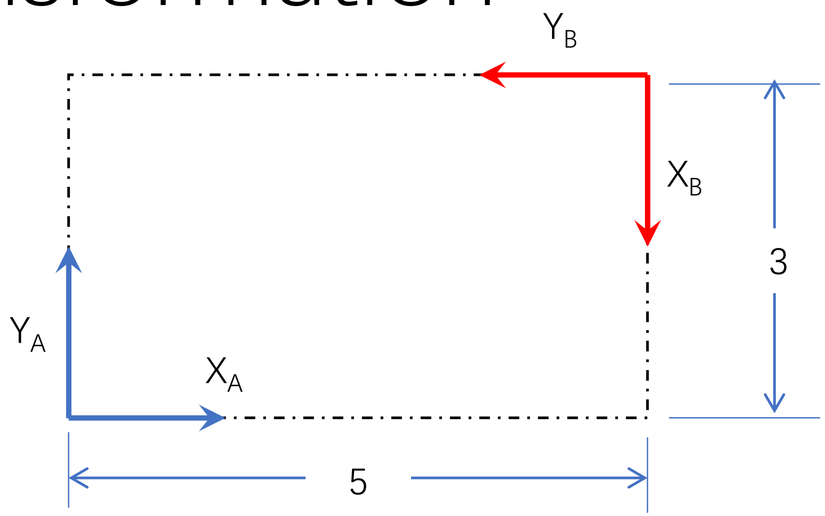

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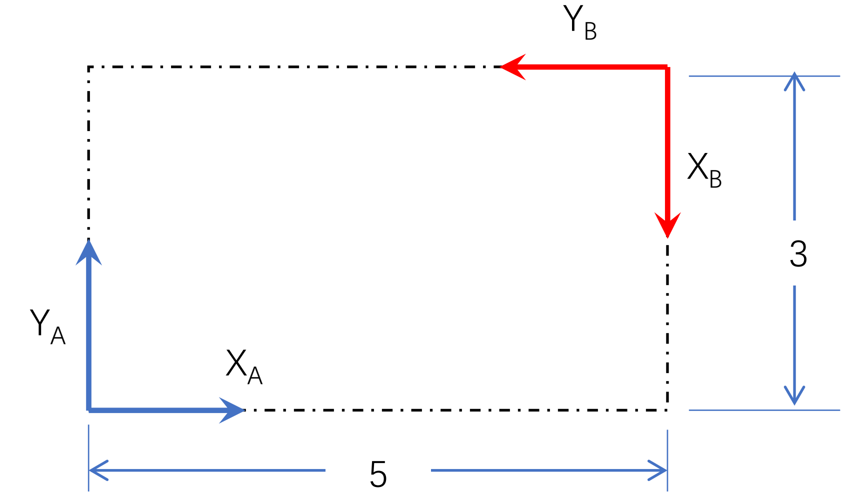


a)

$${}^{A}_{B1}T = {}^{A}_{B}T\,{}^{B}_{B1}T; \qquad {}^{A}_{B}T = \left(\begin{array}{ccc|c} 0 & -1 & 0 & 5 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad {}^{B}_{B1}T = rot_x(45) = \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & c45 & -s45 & 0 \\ 0 & s45 & c45 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$${}^{A}_{B1}T = \left(\begin{array}{ccc|c} 0 & -1 & 0 & 5 \\ -1 & 0 & 0 & 3 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)\left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & c45 & -s45 & 0 \\ 0 & s45 & c45 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 0 & -c45 & s45 & 5 \\ -1 & 0 & 0 & 3 \\ 0 & -s45 & -c45 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|c} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 5 \\ -1 & 0 & 0 & 3 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

(b)

$${}^{A}_{B1}T^{-1} = \begin{pmatrix} {}^{A}_{B1}R' & -{}^{A}_{B1}R'\,{}^{A}_{B1}P \\ 0 & 1 \end{pmatrix} = \left(\begin{array}{ccc|c} 0 & -1 & 0 & 3 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & \frac{5}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & -\frac{5}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{array}\right)$$

(c)

$${}^{A}_{B2}T = {}^{A}_{A2}T\,{}^{A2}_{B2}T; \qquad {}^{A}_{A2}T = rot_y(45) = \left(\begin{array}{ccc|c} c(45) & 0 & s(45) & 0 \\ 0 & 1 & 0 & 0 \\ -s(45) & 0 & c(45) & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad {}^{A2}_{B2}T = {}^{A}_{B1}T = \left(\begin{array}{ccc|c} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 5 \\ -1 & 0 & 0 & 3 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$${}^{A}_{B2}T = \left(\begin{array}{ccc|c} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 & 0 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)\left(\begin{array}{ccc|c} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 5 \\ -1 & 0 & 0 & 3 \\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} 0 & -1 & 0 & \frac{5}{\sqrt{2}} \\ -1 & 0 & 0 & 3 \\ 0 & 0 & -1 & \frac{-5}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(d)

$${}^{A1}_{B2}T = {}^{A1}_{A}T\,{}^{A}_{B2}T;$$

$${}^{A1}_{B2}T = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)\begin{pmatrix} 0 & -1 & 0 & \frac{5}{\sqrt{2}} \\ -1 & 0 & 0 & 3 \\ 0 & 0 & -1 & \frac{-5}{\sqrt{2}} \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & \frac{5}{\sqrt{2}} \\ 0 & 0 & 1 & \frac{5}{\sqrt{2}} \\ -1 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

HW 1: Spatial Representation/Transformation

Question 2.

A cuboid with Frame {M} and Frame {C} attached rigidly is shown in Figure 2. The universe frame of reference {U} serves as an absolute frame that is always fixed. The cuboid motion is described by the series of transformation operations.

1> Rotation about the z axis of Frame C by 30°, then
2> Translation of (1, 2, 3) along Frame C, then
3> Rotation about the x axis of Frame M by 45°, and then
4> Rotation about the y axis of Frame U by 60°.

Let ${}^{U}T_{C_i}$ and ${}^{U}T_{M_i}$ be the 4 × 4 homogeneous transformation matrices that describes the position and orientation of Frames C and M, respectively, in U after motion i.

Find

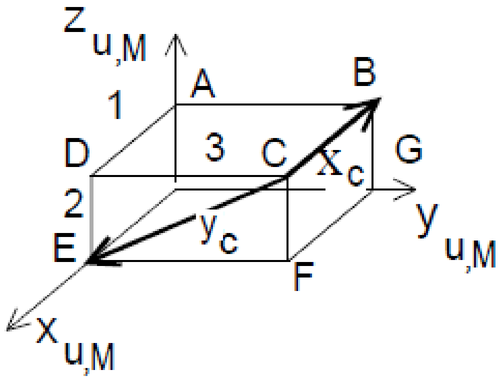
i. ${}^{U}T_{C_1}$

ii. ${}^{U}T_{C_2}$

iii. ${}^{U}T_{C_3}$

iv. ${}^{U}T_{C_4}$

v. ${}^{U}T_{M_4}$



line segment lengths:

AD=1

DC=3

DE=2

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iii. ${}^{U}T_{C_3}$
iv. ${}^{U}T_{C_4}$
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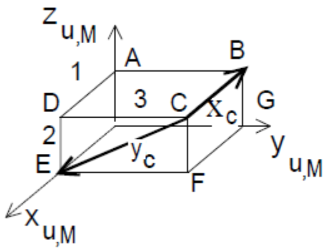


line segment lengths:
AD=1
DC=3
DE=2

$${}^{U}_{C0}T = [{}^{\widetilde{U}}_{C0}R \quad {}^{\widetilde{U}}_{C0}P] \text{ where } {}^{\widetilde{U}}_{C0}R = \begin{pmatrix} \begin{matrix}-1\\0\\0\end{matrix} & \widehat{\begin{pmatrix}0\\-3\\-2\end{pmatrix}} & \widehat{\begin{pmatrix}0\\-2\\3\end{pmatrix}} \\ & \mathbf{0} & \end{pmatrix} \text{ and } {}^{\widetilde{U}}_{C0}P = \begin{pmatrix}1\\3\\2\\1\end{pmatrix}$$

$${}^{U}_{C0}T = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -0.8321 & -0.5507 & 3 \\ 0 & -0.5507 & 0.8321 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(i)

$${}^{U}_{C1}T = {}^{U}_{C0}T\,{}^{C0}_{C1}T = {}^{U}_{C0}T\,rot_z(30)$$

$${}^{U}_{C1}T = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -0.8321 & -0.5507 & 3 \\ 0 & -0.5507 & 0.8321 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3}/2 & -0.5 & 0 & 0 \\ 0.5 & \sqrt{3}/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.866 & 0.5 & 0 & 1 \\ -0.416 & -0.721 & -0.555 & 3 \\ -0.277 & -0.48 & 0.832 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii)

$${}^{U}_{C2}T = {}^{U}_{C1}T\,{}^{C1}_{C2}T = {}^{U}_{C0}T\,trans(1,2,3)$$

$${}^{U}_{C2}T = \begin{bmatrix} -0.866 & 0.5 & 0 & 1 \\ -0.416 & -0.721 & -0.555 & 3 \\ -0.277 & -0.48 & 0.832 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -0.866 & 0.5 & 0 & 1.134 \\ -0.416 & -0.721 & -0.555 & -0.521 \\ -0.277 & -0.48 & 0.832 & 3.258 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii)

$${}^{U}_{C3}T = {}^{U}_{C2}T\,{}^{C2}_{M2}T\,{}^{M2}_{M3}T\,{}^{M3}_{C3}T = {}^{U}_{C2}T\,{}^{C}_{M}T\,rot_x(45)\,{}^{M}_{C}T$$

$$= \begin{bmatrix} -0.866 & 0.5 & 0 & 1.134 \\ -0.416 & -0.721 & -0.555 & -0.521 \\ -0.277 & -0.48 & 0.832 & 3.258 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -0.832 & -0.551 & 3 \\ 0 & -0.551 & 0.832 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & c45 & -s45 & 0 \\ 0 & s45 & c45 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -0.832 & -0.551 & 3 \\ 0 & -0.551 & 0.832 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -0.866 & 0.354 & 0.354 & 1.662 \\ -0.416 & -0.117 & -0.902 & -2.696 \\ -0.277 & -0.928 & 0.249 & 4.872 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

iv)

$${}^{U}_{C4}T = {}^{U}_{U4}T\,{}^{U4}_{C4}T = rot_y(60)\,{}^{U}_{C3}T$$

$${}^{U}_{C4}T = \begin{bmatrix} c60 & 0 & s60 & 0 \\ 0 & 1 & 0 & 0 \\ -s60 & 0 & c60 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -0.866 & 0.354 & 0.354 & 1.662 \\ -0.416 & -0.117 & -0.902 & -2.696 \\ -0.277 & -0.928 & 0.249 & 4.872 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

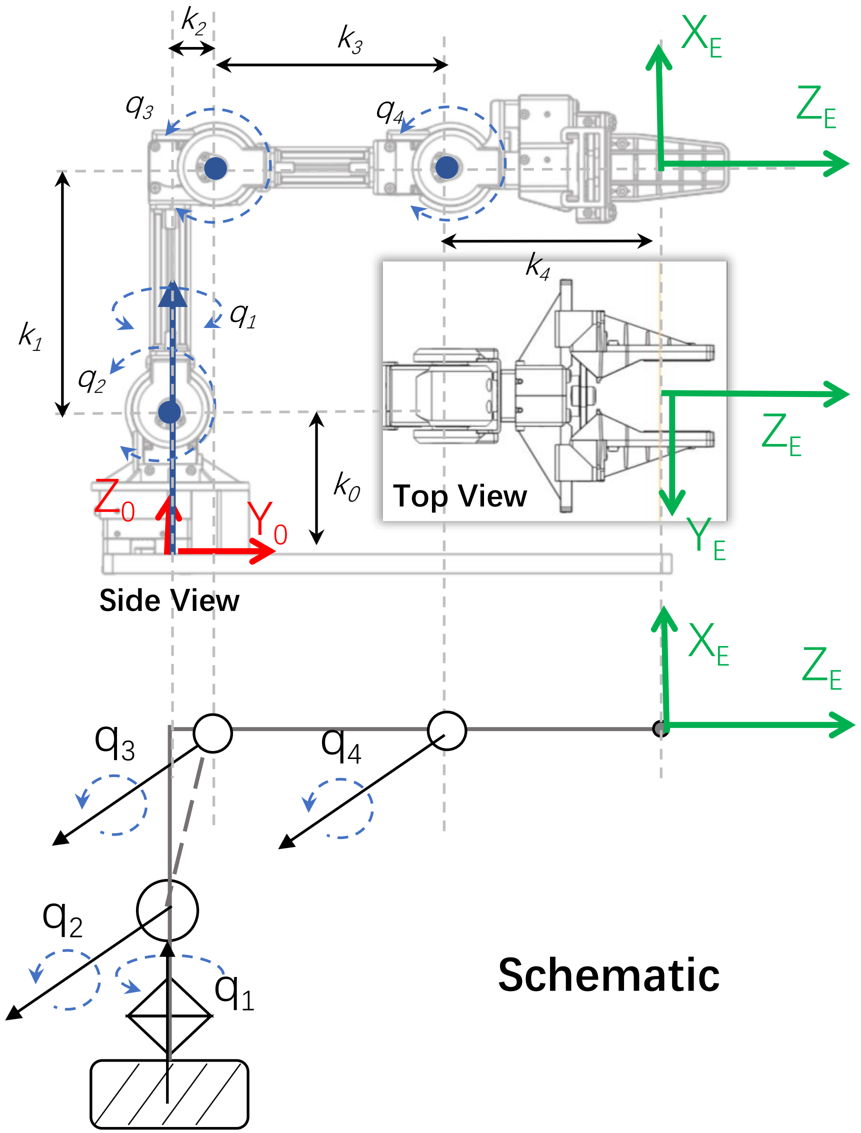
$$= \begin{bmatrix} -0.673 & -0.627 & 0.392 & 5.05 \\ -0.416 & -0.117 & -0.902 & -2.696 \\ 0.611 & -0.77 & -0.182 & 0.997 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

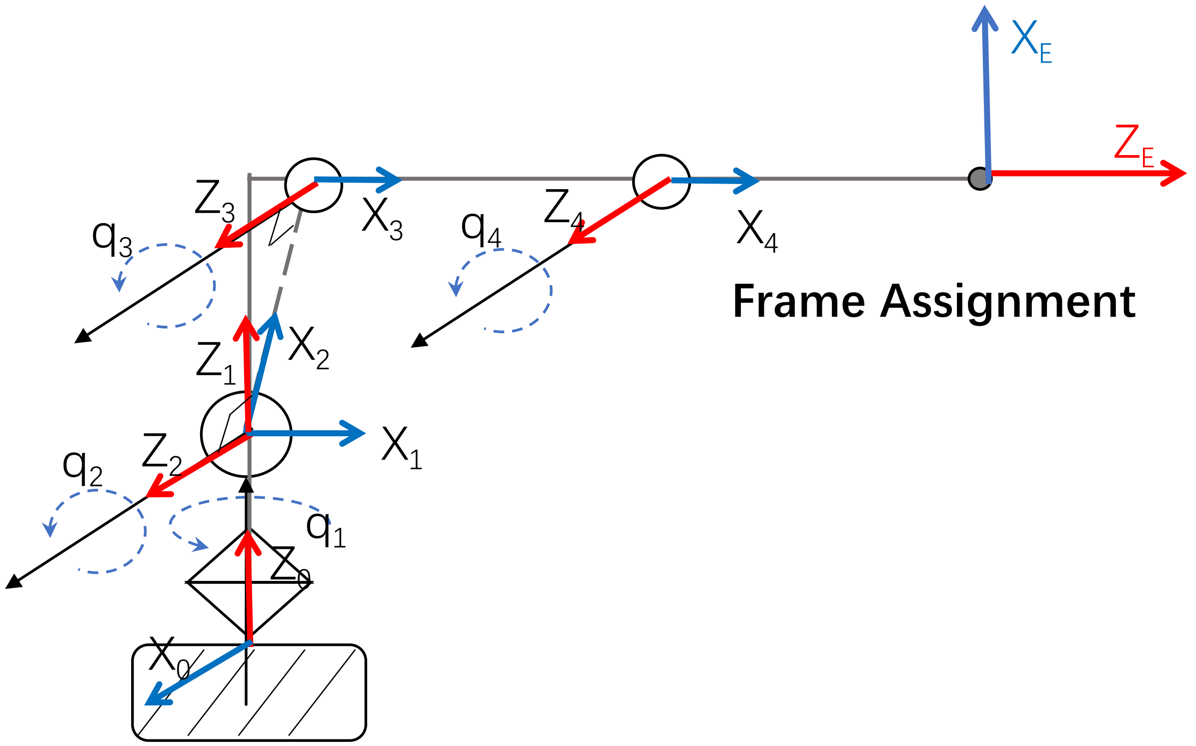
(v)

$${}^{U}_{M4}T = {}^{U}_{C4}T\,{}^{C4}_{M4}T = {}^{U}_{C4}T\,{}^{C0}_{U}T = \begin{bmatrix} 0.673 & 0.304 & 0.674 & 2.117 \\ 0.416 & 0.589 & -0.685 & -3.535 \\ -0.611 & 0.742 & 0.276 & -1.169 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

HW 2: Frame Assignment

Manipulator Model

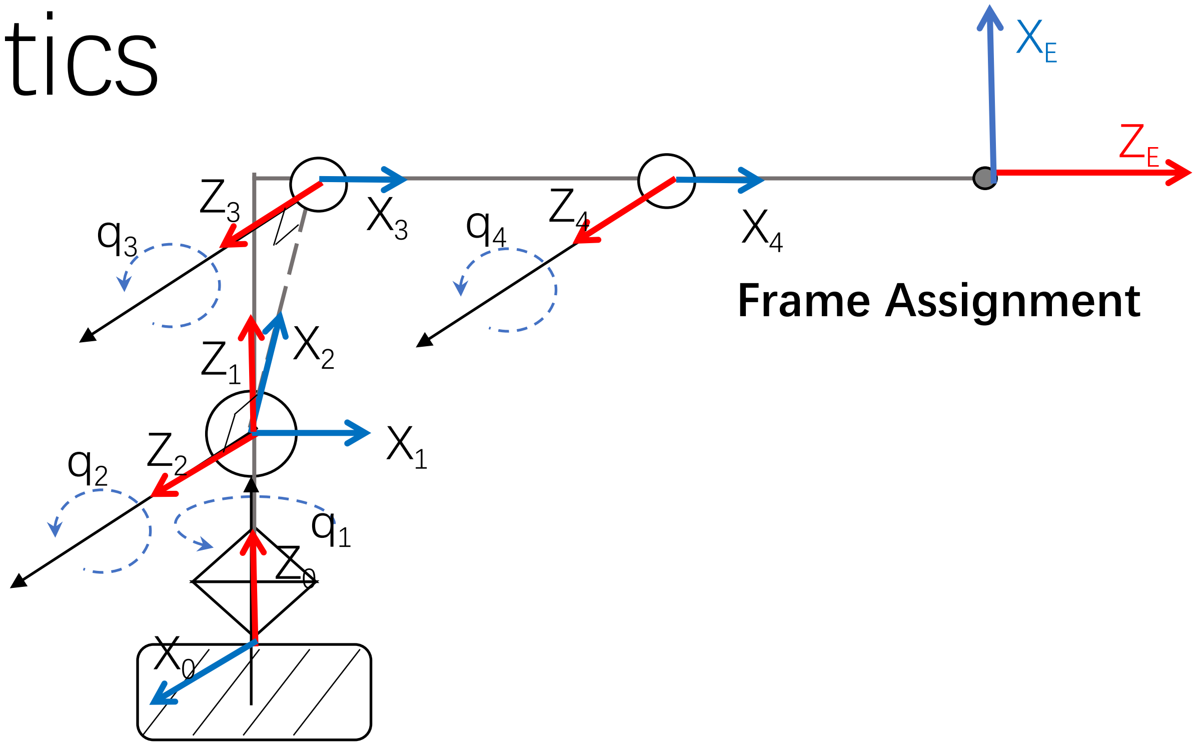




HW 2: Forward Kinematics

D-H Parameters

	α	a	θ	d
${}^{0}T_{1}$	0	0	$q_1 = 90^o$	k_0
${}^{1}T_{2}$	90	0	$q_2 = \text{atan2}(k_1, k_2)$	0
${}^{2}T_{3}$	0	$\sqrt{{k_1}^2 + {k_2}^2} = K_{12}$	$q_3 = -\text{atan2}(k_1, k_2)$	0
${}^{3}T_{4}$	0	k_3	$q_4 = 0$	0



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${}^3T_4$	0	k_3	$q_4 = 0$	0

(I) Obtain transformation between adjacent

$${}^{i-1}_{i}\mathrm{T} = R_x(\alpha_{i-1})D_x(a_{i-1})\,R_z(\theta_i)D_z(d_i)$$

$${}^0_1\mathrm{T} = [\mathrm{I}][\mathrm{I}]\begin{bmatrix} \cos q_1 & -\sin q_1 & 0 & 0 \\ \sin q_1 & \cos q_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} & 0 \\ \mathbf{I} & 0 \\ & k_0 \\ \mathbf{0} & 1 \end{bmatrix} = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & k_0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

$${}^1_2\mathrm{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}[\mathbf{I}]\begin{bmatrix} \cos q_2 & -\sin q_2 & 0 & 0 \\ \sin q_2 & \cos q_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}[\mathbf{I}] = \begin{bmatrix} c2 & -s2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

$${}^2_3\mathrm{T} = [\mathbf{I}]\begin{bmatrix} & K_{12} \\ \mathbf{I} & 0 \\ & 0 \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \cos q_3 & -\sin q_3 & 0 & 0 \\ \sin q_3 & \cos q_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}[\mathbf{I}] = \begin{bmatrix} c3 & -s3 & 0 & K_{12} \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

$${}^3_4\mathrm{T} = [\mathrm{I}]\begin{bmatrix} & K_3 \\ \mathbf{I} & 0 \\ & 0 \\ \mathbf{0} & 1 \end{bmatrix}\begin{bmatrix} \cos q_4 & -\sin q_4 & 0 & 0 \\ \sin q_4 & \cos q_4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}[\mathbf{I}] = \begin{bmatrix} c4 & -s4 & 0 & K_3 \\ s4 & c4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix};\ {}^4_E\mathrm{T} = \begin{bmatrix} 0 & 0 & 1 & K_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

(II) ${}^0_N\mathrm{T} = {}^0_1\mathrm{T}\,{}^1_2\mathrm{T} \ldots\ldots {}^{i-1}_{i}\mathrm{T}\,{}^{i}_{i+1}\mathrm{T} \ldots\ldots {}^{N-2}_{N-1}\mathrm{T}\,{}^{N-1}_{N}\mathrm{T}$

$${}^0_E\mathrm{T} = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & k_0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} c2 & -s2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} c3 & -s3 & 0 & K_{12} \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} c4 & -s4 & 0 & k_3 \\ s4 & c4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & k_4 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

$${}^0_4\mathrm{T} = \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & k_0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} c2 & -s2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} c3 & -s3 & 0 & K_{12} \\ s3 & c3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} c4 & -s4 & 0 & k_3 \\ s4 & c4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & k_0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} c2 & -s2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ s2 & c2 & 0 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} c_{34} & -s_{34} & 0 & c_3k_3 + K_{12} \\ s_{34} & c_{34} & 0 & s_3k_3 \\ 0 & 0 & 1 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & k_0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} c_{234} & -s_{234} & 0 & c_2(c_3k_3 + K_{12}) - s_2s_3k_3 \\ 0 & 0 & -1 & 0 \\ s_{234} & c_{234} & 0 & s_2(c_3k_3 + K_{12}) + c_2s_3k_3 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} c1 & -s1 & 0 & 0 \\ s1 & c1 & 0 & 0 \\ 0 & 0 & 1 & k_0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} c_{234} & -s_{234} & 0 & c_2K_{12} + c_{23}k_3 \\ 0 & 0 & -1 & 0 \\ s_{234} & c_{234} & 0 & s_2K_{12} + s_{23}k_3 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

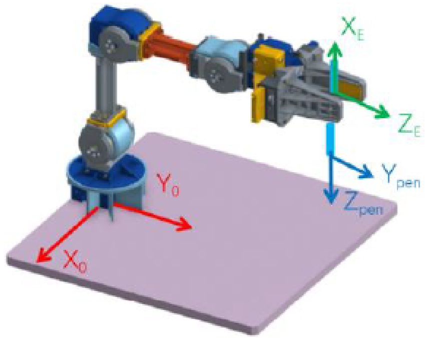
$$= \begin{bmatrix} c_1c_{234} & -c_1s_{234} & s_1 & c_1(c_2K_{12} + c_{23}k_3) \\ s_1c_{234} & s_1s_{234} & -c_1 & s_1(c_2K_{12} + c_{23}k_3) \\ s_{234} & c_{234} & 0 & s_2K_{12} + s_{23}k_3 + k_0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

$${}^0_\mathrm{E}\mathrm{T} = \begin{bmatrix} c_1c_{234} & -c_1s_{234} & s_1 & c_1(c_2K_{12} + c_{23}k_3) \\ s_1c_{234} & s_1s_{234} & -c_1 & s_1(c_2K_{12} + c_{23}k_3) \\ s_{234} & c_{234} & 0 & s_2K_{12} + s_{23}k_3 + k_0 \\ & \mathbf{0} & & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & K_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -c_1s_{234} & s_1 & c_1c_{234} & c_1(c_2K_{12} + c_{23}k_3) + c_1c_{234}k_4 \\ s_1s_{234} & -c_1 & s_1c_{234} & s_1(c_2K_{12} + c_{23}k_3) + s_1c_{234}k_4 \\ c_{234} & 0 & s_{234} & s_2K_{12} + s_{23}k_3 + k_0 + s_{234}k_4 \\ & \mathbf{0} & & 1 \end{bmatrix}$$

HW 2: Inverse Kinematics & Workspace

The serial manipulator arm is tasked to write on the board plane Z_o, with a pen attached to the gripper {E}. For the ink to flow, ${}^0Z_{pen}$ has to be $(0\ 0\ {-1})^T$ i.e. vertically downwards. As shown in the diagram, axis X_E and Z_E are parallel to Z_{pen} and Y_{pen}. respectively. The distance between Z_E and Y_{pen} is k_0.



State any assumption or condition while working on the following:

a) Write down the transformation matrix ${}^{E}_{pen}T$

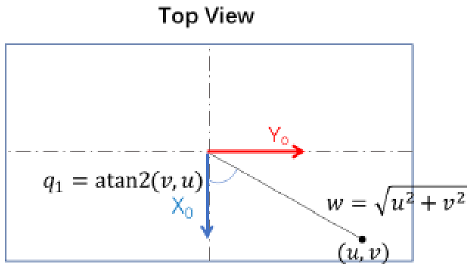
b) If the pen tip is to be place on the board with coordinates ${}^0(u,v)$, find the expressions describing the joint variable $\boldsymbol{q}$ in terms of $k_{0\text{-}4}$, u and v.

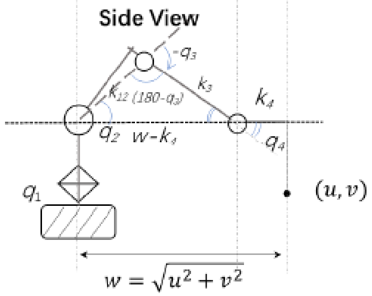
c) Describe the workspace of the writing task if the distance between Z_E and Y_{pen} is now change to $k_0/2$. Assume that q_2 can only move its link in a range of 0 to 180° from the plane.

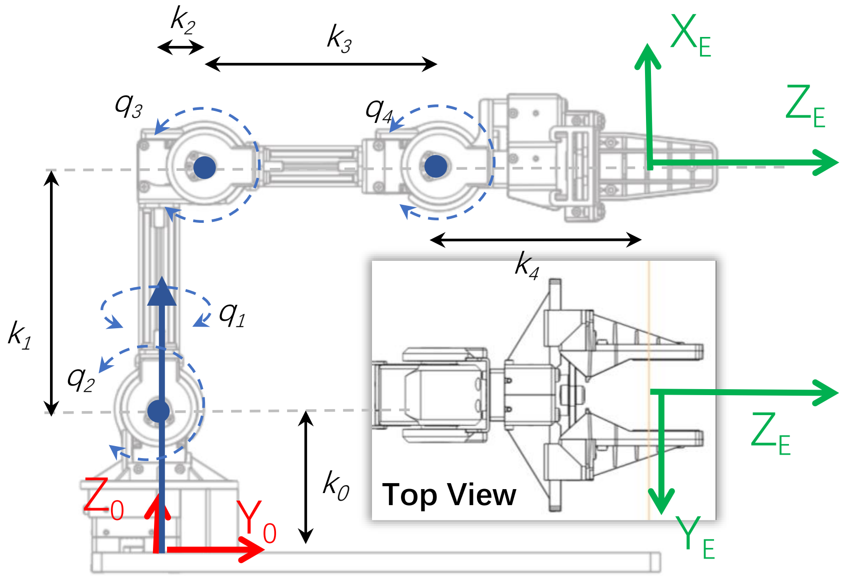
HW 2: Inverse Kinematics & Workspace

a) $ {}^{E}_{pen}T = \begin{bmatrix} 0 & 0 & -1 & -k_0 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

b) Assuming pen is pointing vertically downward.







From top view, $q_1 = \operatorname{atan2}(v, u)$

From side view,

Cosine rule:

$$\cos q_4 = \frac{{k_3}^2 + (w - k_4)^2 - {k_{12}}^2}{2k_3(w - k_4)}\ ;\ q_4 = \operatorname{acos}\left(\frac{{k_3}^2 + (w - k_4)^2 - {k_{12}}^2}{2k_3(w - k_4)}\right)$$

Sine rule:

$$\sin(180 - q_3) = \sin q_3 = \frac{\sin q_4}{k_{12}}(w - k_4);\ q_3 = \operatorname{asin}\left(\frac{\sin q_4}{k_{12}}(w - k_4)\right)$$

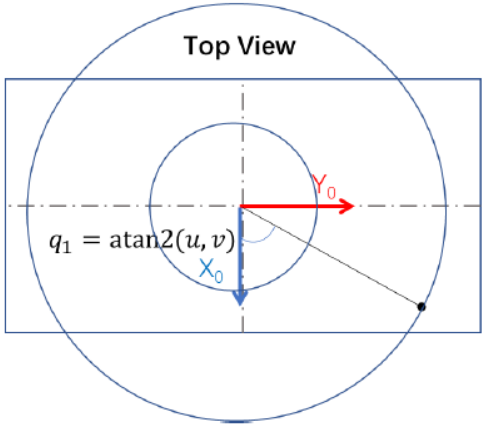
$$\sin q_2 = \frac{\sin q_4}{k_{12}} k_3;\ q_2 = \operatorname{asin}\left(\frac{\sin q_4}{k_{12}} k_3\right)$$

c) The shape of the writing taskspace has the following shape.

Initially when the distance between Z_E and Y_{pen} is k_0 :

If $k_1 = k_3$, the internal envelop will have a radius of k_4

Shortening the distance will shorten the radius of reach while pen maintain vertically downwards. Hence, when the distance between Z_E and Y_{pen} is halved, the outer boundary shrinks in radius while the internal circular envelop radius increases.



Practice 1: Spatial Description

Question 1. **Spatial Description**

Referring to Figure 1, determine the homogeneous transformation matrix describing frame B in frame A. Also determine the homogeneous transformation matrix that describes frame A in frame B. *(4 Points)*

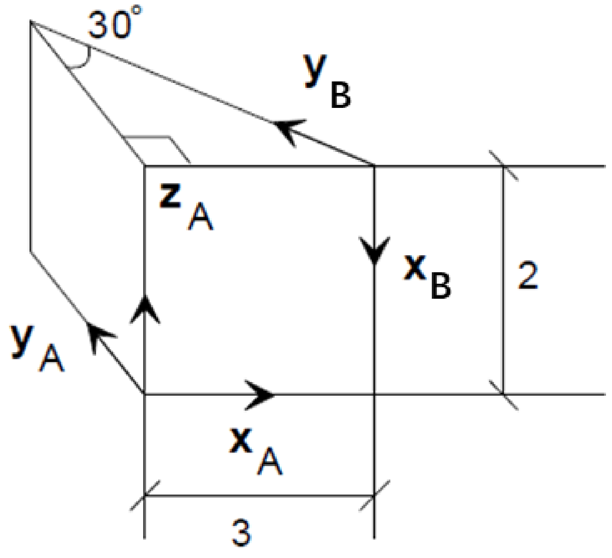


Figure 1

Solution

Spatial of Frame

$$
{}^{A}_{B}\mathrm{T} = \begin{bmatrix} 0 & -0.5 & 0.866 & 3 \\ 0 & 0.866 & 0.5 & 0 \\ -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Inverse of a Transformation

$$
{}^{B}_{A}\mathrm{T} = {}^{A}_{B}\mathrm{T}^{-1} = \begin{bmatrix} {}^{A}_{B}R^{\mathrm{T}} & -{}^{A}_{B}R^{\mathrm{T}}\,{}^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}
$$

$$
{}^{B}_{A}\mathrm{T} = \begin{bmatrix} 0 & 0 & -1 & 2 \\ -0.5 & 0.866 & 0 & 1.5 \\ 0.866 & 0.5 & 0 & -2.598 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Practice 1: Spatial Transformation

Question 2. **Spatial Transformation**

Frame C coincide with frame A initially in Figure 2. Frame C is then rotated 30^{o} about the vector described by the directed line segment from P to Q. Determine the position and orientation of the new frame C with respect to frame A in the form of a homogeneous transformation matrix. *(10 Points)*

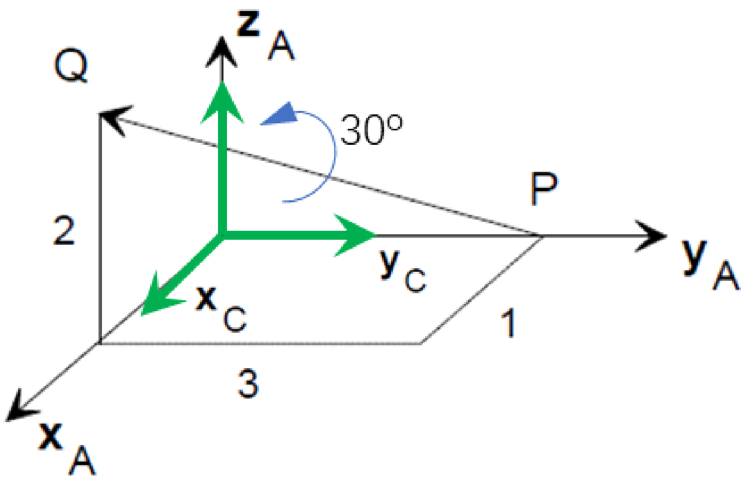


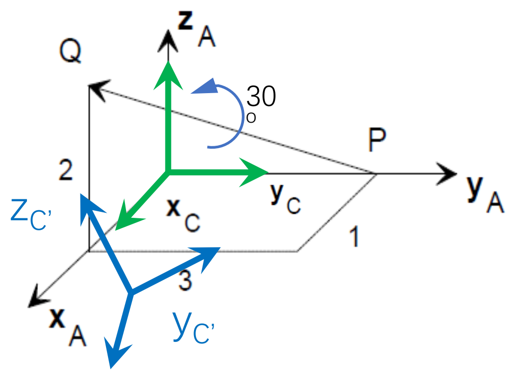
Figure 2

Practice 2: Spatial Transformation

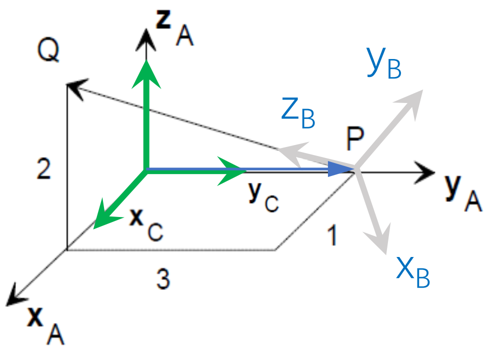
Question 2. **Spatial Transformation**

Frame C coincide with frame A initially in Figure 2. Frame C is then rotated 30° about the vector described by the directed line segment from P to Q. Determine the position and orientation of the new frame C with respect to frame A in the form of a homogeneous transformation matrix. *(10 Points)*

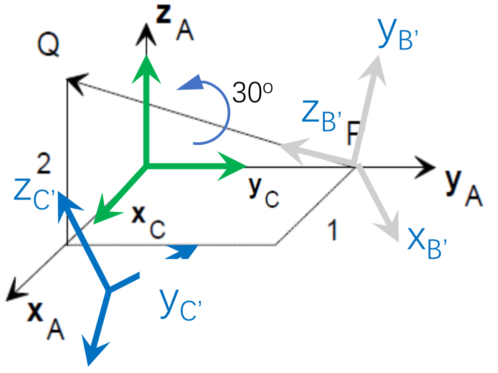
Rotate {C} about PQ



Introduce a frame {B} with origin at point P and Z_B-axis along PQ



Rotate {B} 30° about Z_B (i.e. PQ) with {C} rigidly attached



{C'} and {B'} denote the newly transformed frames

Practice 2: Spatial Transformation

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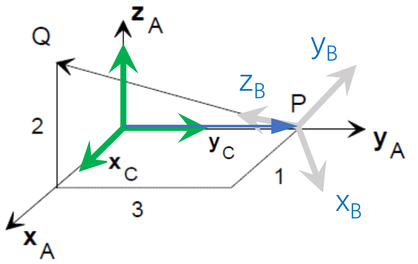

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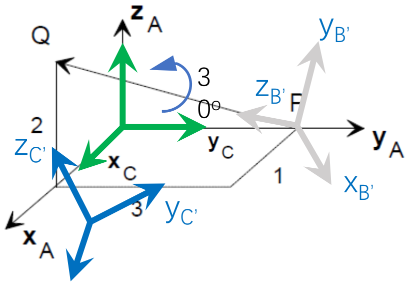

{B'} and {C'} denote the newly transformed frames

$$ {}^{A}_{C'}T = {}^{A}_{B}T \cdot {}^{B}_{B'}T \cdot {}^{B'}_{C'}T $$

$$ {}^{B}_{B'}T = \widetilde{R_z}\,(30^o) $$

$$ {}^{A}_{C'}T = {}^{A}_{B}T \cdot \widetilde{R_z}\,(30^o) \cdot {}^{A}_{B}T^{-1} $$

$$ \widetilde{R_z}\,(30^o) = \begin{bmatrix} \cos(30^o) & -\sin(30^o) & 0 & 0 \\ \sin(30^o) & \cos(30^o) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

${}^{B'}_{C'}T = {}^{B}_{A}T$ since {C'} is rigidly attached to {B'} the way {A} is to be {B}

$$ {}^{A}_{B}T = \begin{bmatrix} {}^{A}\boldsymbol{i}_B & {}^{A}\boldsymbol{j}_B & {}^{A}\boldsymbol{k}_B & {}^{A}P_B \\ & \mathbf{0} & & 1 \end{bmatrix} \qquad {}^{A}_{B}T = \begin{bmatrix} 13/\sqrt{182} & 0 & 1/\sqrt{14} & 0 \\ 3/\sqrt{182} & 2/\sqrt{13} & -3/\sqrt{14} & 3 \\ -2/\sqrt{182} & 3/\sqrt{13} & 2/\sqrt{14} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{A}P_B = {}^{A}\overrightarrow{OP} = \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} $$

$$ {}^{A}_{B}T^{-1} = \left[\begin{array}{c|c} \begin{matrix} {}^{A}\boldsymbol{i}_B{}^{T} \\ {}^{A}\boldsymbol{j}_B{}^{T} \\ {}^{A}\boldsymbol{k}_B{}^{T} \end{matrix} & -\begin{pmatrix} {}^{A}\boldsymbol{i}_B{}^{T} \\ {}^{A}\boldsymbol{j}_B{}^{T} \\ {}^{A}\boldsymbol{k}_B{}^{T} \end{pmatrix} {}^{A}P_B \\ \mathbf{0} & 1 \end{array}\right] $$

$$ {}^{A}_{B}R = [{}^{A}\boldsymbol{i}_B \quad {}^{A}\boldsymbol{j}_B \quad {}^{A}\boldsymbol{k}_B] $$

$$ {}^{A}\boldsymbol{k}_B = \widehat{\overrightarrow{PQ}} = \widehat{\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \qquad {}^{A}\boldsymbol{j}_B = \widehat{\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} = \widehat{\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}} = \frac{1}{\sqrt{13}}\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix} $$

$$ {}^{A}\boldsymbol{i}_B = {}^{A}\boldsymbol{j}_B \times {}^{A}\boldsymbol{k}_B = \widehat{\begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}} \times \widehat{\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix}} = \widehat{\begin{bmatrix} 13 \\ 3 \\ -2 \end{bmatrix}} = \frac{1}{\sqrt{182}}\begin{bmatrix} 13 \\ 3 \\ -2 \end{bmatrix} $$

Practice 3: Forward Kinematics

Question 3

The following two figures depict a two-link arm, with the right figure being the front view with axis assigned. Find ${}^{0}P$ in terms of q_1 and q_2. *(6 Points)*

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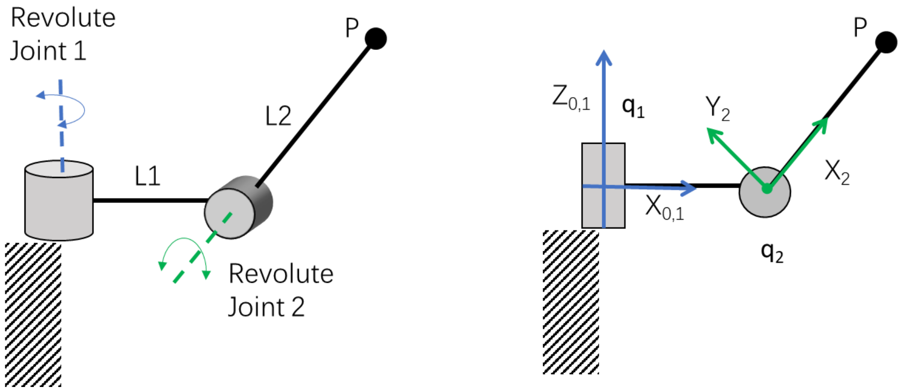


$$ {}^{0}\mathrm{P} = {}^{0}_{1}\mathbf{T}\,{}^{1}_{2}\mathbf{T}\;{}^{2}\mathrm{P} $$

$$ {}^{0}_{1}\mathbf{T} = \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{1}\mathbf{T} = \begin{bmatrix} c_2 & -s_2 & 0 & L_1 \\ 0 & 0 & -1 & 0 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}\mathrm{P} = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & L_1c_1 \\ s_1c_2 & -s_1s_2 & -c_1 & L_1s_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} L_1c_1 + L_2c_1c_2 \\ L_1s_1 + L_2s_1c_2 \\ L_2s_2 \\ 1 \end{bmatrix} $$

Practice 4: Manipulator Kinematics Analysis

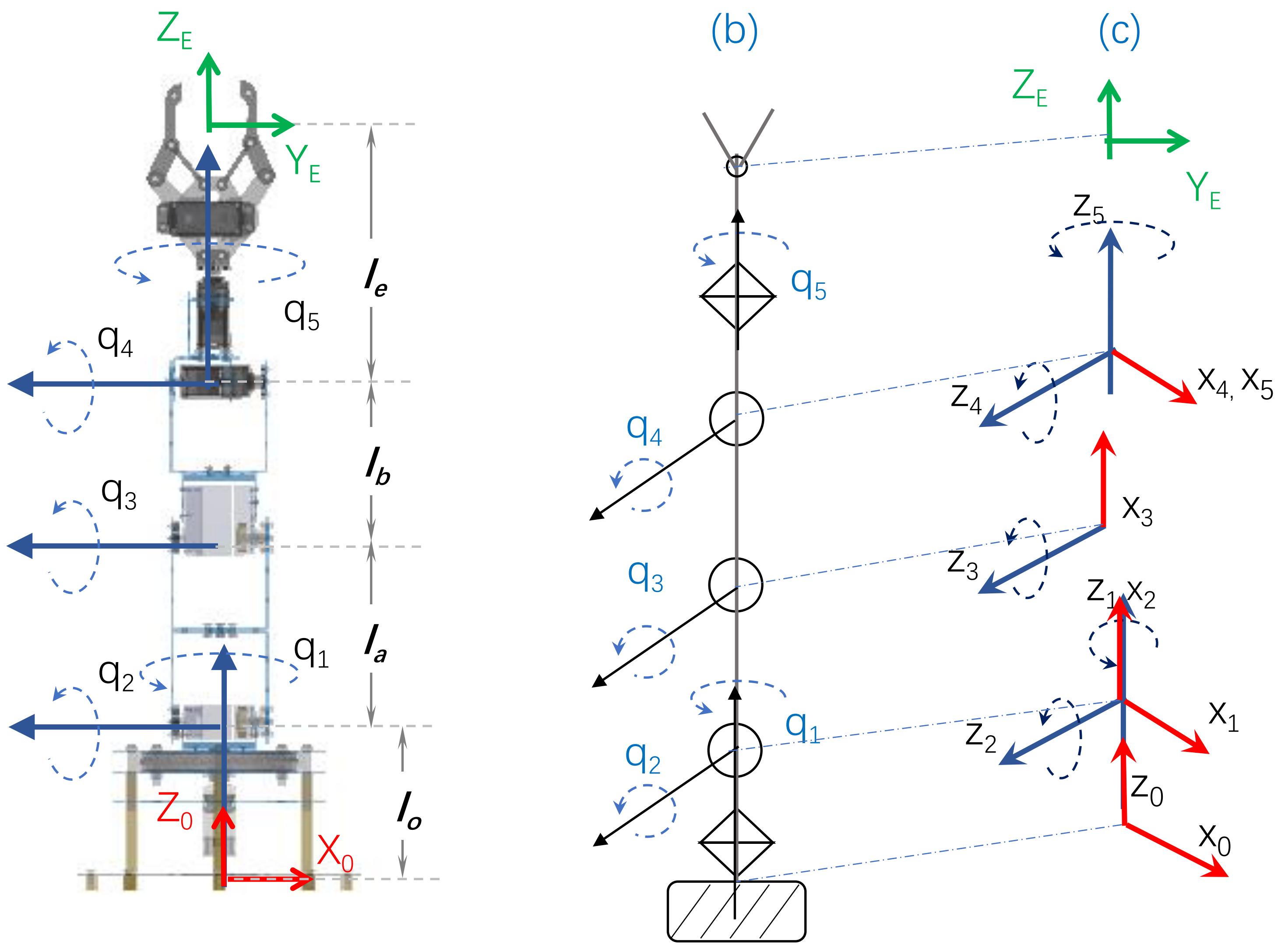
- **Question 1.**

The following figure depicts a robotic arm. All 5 joints are driven by rotational servo motors as indicated by the front view on the right (Ignored the 6^{th} motor for gripping at the end effector in this question).

a) Write down the joint coordinates of the manipulator
b) b) Sketch a schematic diagram showing the axis of rotation for each joint
c) c) Assign frames to the links using the D-H convention

d) Show the D-H notations in a D-H table

Practice 4: Manipulator Kinematics Analysis



(a) Joint coordinates are (q_1, q_2, q_3, q_4, q_5)

(d)

	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	$q_1 = 0$	l_o
2	$\frac{\pi}{2}$	0	$q_2 = \frac{\pi}{2}$	0
3	0	l_a	$q_3 = 0$	0
4	0	l_b	$q_4 = -\frac{\pi}{2}$	0
5	$-\frac{\pi}{2}$	0	$q_5 = 0$	0
E	0	0	0	l_e

Practice 5: Workspace Analysis

Question 2.

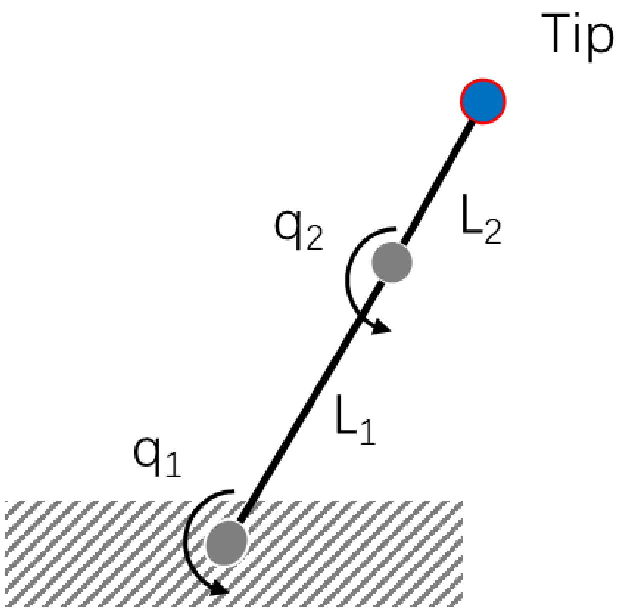
The following figure shows a two-link planar arm with rotary joints. For this arm, the second link L_2 is half as long as the first L_1. The joint range limits are as follows:

$$0 < q_1 < \pi$$

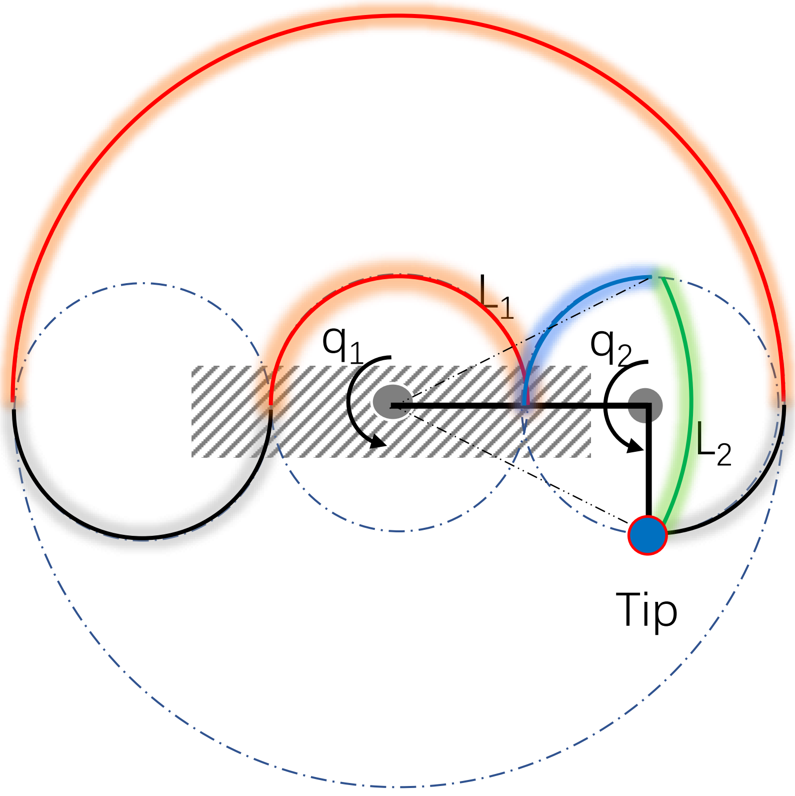
$$-\frac{\pi}{2} < q_2 < \pi$$

Sketch the reachable workspace of the tip of L_2. It can be assumed that the base box will not affect the movement of the links. (You may use Matlab to answer the question)

(4 Points)



Practice 5: Workspace Analysis



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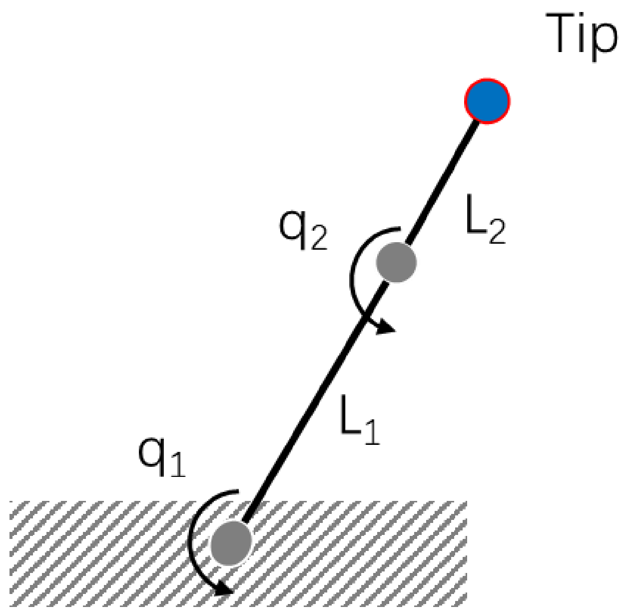
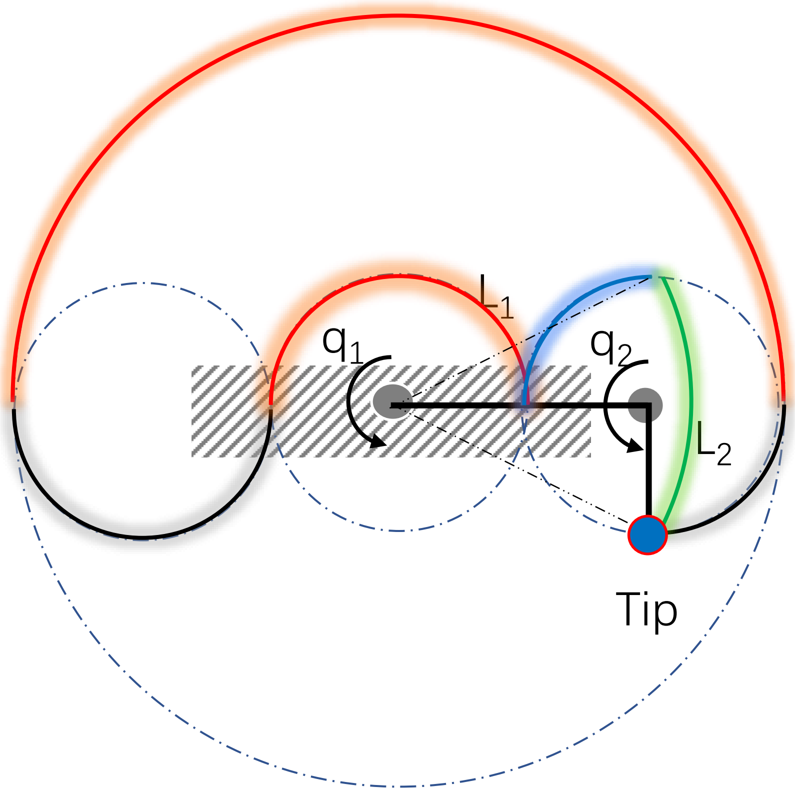
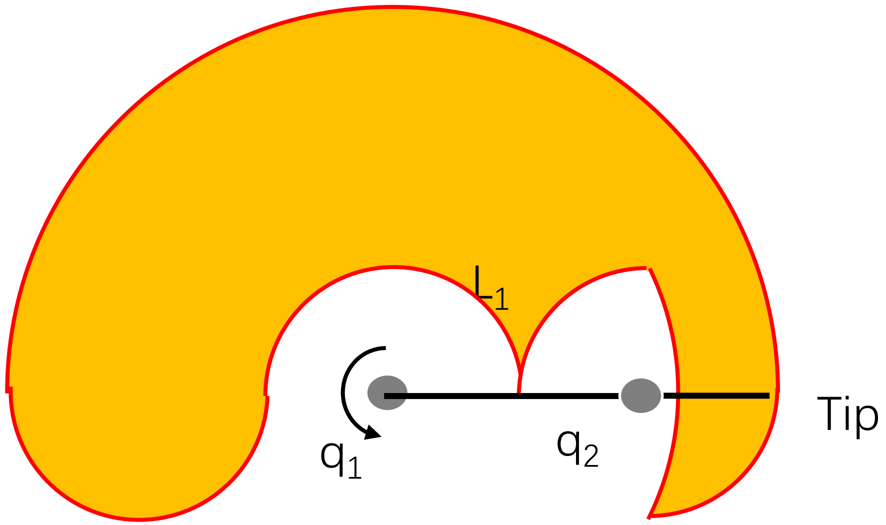
(4 Points)



Practice 5: Workspace Analysis





Practice 6: Inverse Kinematics

Solution

1. Forward kinematics

$$
{}^{0}P = \begin{bmatrix} c_1c_2 & -c_1s_2 & s_1 & L_1c_1 \\ s_1c_2 & -s_1s_2 & -c_1 & L_1s_1 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} L_2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} L_1c_1 + L_2c_1c_2 \\ L_1s_1 + L_2s_1c_2 \\ L_2s_2 \\ 1 \end{bmatrix}
$$

2. Inverse kinematics

Finding the joint coordinates to satisfy ${}^{0}P = (0.75, -0.75, 0.5)^T$.

$$
\begin{bmatrix} \boldsymbol{L_1c_1 + L_2c_1c_2} \\ \boldsymbol{L_1s_1 + L_2s_1c_2} \\ \boldsymbol{L_2s_2} \end{bmatrix} = \begin{bmatrix} \mathbf{0.75} \\ \mathbf{-0.75} \\ \mathbf{0.5} \end{bmatrix}
$$

The positional constraint forms a system of 3 Equations

(We need to bear in mind that there are only 2-independent variables, hence there might not be a solution that satisfy the 3D positional constraints.)

From the Z-coordinate $\theta_2 = 30^o$ or 150^o

Substituting into Y-coordinate

$$\boldsymbol{c_1}[\mathbf{1} + (\mathbf{0.866})] = \mathbf{0.75}$$

Substituting into X-coordinate

$$\boldsymbol{s_1}[\mathbf{1} + (\mathbf{0.866})] = \mathbf{-0.75}$$

We see that there is no set of joint coordinates that could satisfy the system of equations. Hence ${}^{0}P = (0.75, -0.75, 0.5)^T$ is NOT within the reachable workspace

Question 3

The following two figures depict a two-link arm, with the right figure being the front view with axis assigned. Given L1=L2=1, find q_1 and q_2 when ${}^{0}P = (0.75, -0.75, 0.5)^T$. You may use the result from Homework 1 directly. *(4 Points)*

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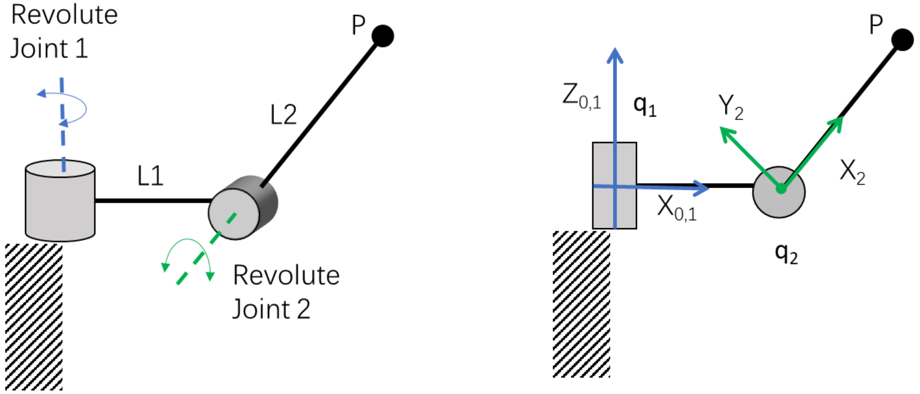
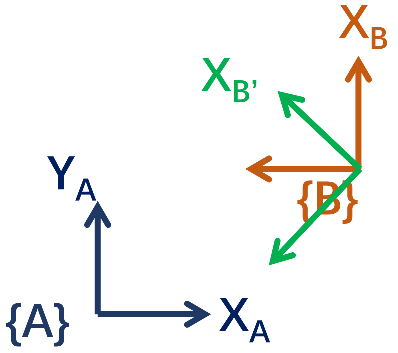


Rotation operation

Frame {A} is the absolute frame

Rotation about its own axis
Case 1: Rotate Frame {B} about it's own Z_B-axis to become frame {B1}
What is the new {B}, {B1} in absolute frame?
i.e. ${}^{A}T_{B1}$ =?



$${}^{A}T_{B'} = {}^{A}T_{B}\,{}^{B}T_{B'}$$

$${}^{A}T_{B'} = {}^{A}T_{B}\,[R_z(\theta)]$$ **Post-multiply by R**

Case 2: Rotate Frame {B} about an external axis Z_A
What is the new {B}, {B'} in absolute frame?
i.e. ${}^{A}T_{B'}$ =?

$${}^{A}T_{B'} = {}^{A}T_{A'}\,{}^{A'}T_{B'}$$

$${}^{A}T_{B'} = [R_z(\theta)]\,{}^{A'}T_{B'}$$

$${}^{A}T_{B'} = [R_z(\theta)]\,{}^{A}T_{B}$$ **Pre-multiply by R**