ZJU-UIUC Institute

Zhejiang University / University of Illinois at Urbana-Champaign Institute

ECE 470: Introduction to Robotics

Lecture 03

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Recap

- Kinematics: The science of motion (a branch of mechanics) without regards to the forces that cause it
  - Pose (/configuration) of the manipulator in static situations
  - Analyze (linear and angular) motion of bodies (/linkages)
- D-H Method: A systematic way to represent the configuration of the serial manipulator
  1. Frame assignment
  2. D-H parameters and tables
  3. Homogenous transformation matrix
- Forward Kinematics: mapping from joint coordinates, or robot configuration to end-effector pose
  - $^{0}_{\mathrm{E}}\mathrm{T} = {}^{0}_{1}\mathrm{T}(q_1) \cdot {}^{1}_{2}\mathrm{T}(q_2) \cdot {}^{2}_{3}\mathrm{T}(q_3) \cdot \cdots {}^{N-1}_{\mathrm{N}}\mathrm{T}(q_{\mathrm{N}}) \cdot {}^{N}_{\mathrm{E}}\mathrm{T}$

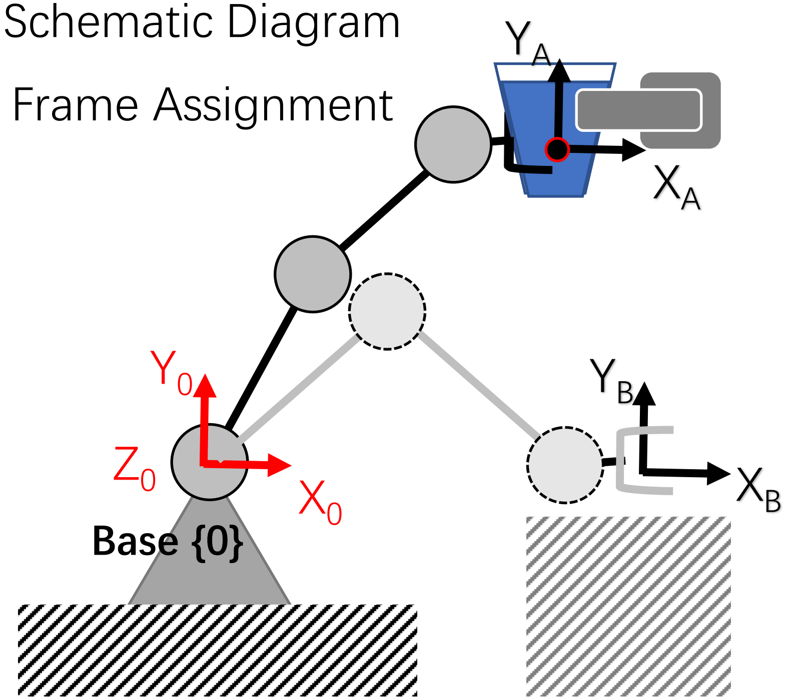
Recap on Lecture 03

- Kinematics: The science of motion (a branch of mechanics) without regards to the forces that cause it
  - Pose (/configuration) of the manipulator in static situations
  - Analyze (linear and angular) motion of bodies (/linkages)
- **Inverse kinematics** is concerned with obtaining the joint coordinates for a desired end-effector pose
- Workspace
  - **Reachable**: Region where end-effector can be located
  - **Dexterous**: Region where end-effector can be located with all orientations
- Solvability; Number of Solutions; Jacobian; Singularity

Recap: Forward/ Inverse Kinematics

1. Schematic Diagram

2. Frame Assignment



3. DH Parameters & Table

	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	$Q1=\theta_1$	0
2	0	$L1$	$Q2=\theta_2$	0
3	0	$L2$	$Q3=\theta_3$	0

4. Homogenous Transformation

$$^{i-1}_{i}\mathrm{T} = R_x(\alpha_{i-1})D_x(\alpha_{i-1})\ R_z(\theta_i)D_z(d_i)$$

$$^{0}_{3}\mathrm{T} = {}^{0}_{1}\mathrm{T}\ {}^{1}_{2}\mathrm{T}\,{}^{2}_{3}\mathrm{T}$$

5. Forward Kinematics

$$^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & L_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & L_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}_{\mathrm{E}}T = \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$^{0}_{\mathrm{E}}\mathrm{T} = {}^{0}_{1}\mathrm{T}(\theta_1)\cdot {}^{1}_{2}\mathrm{T}(\theta_2)\cdot {}^{2}_{3}\mathrm{T}(\theta_3)\cdot {}^{3}_{\mathrm{E}}\mathrm{T}$$

$$^{0}_{\mathrm{E}}\widetilde{\mathrm{P}} = {}^{0}_{1}\mathrm{T}(\theta_1)\cdot {}^{1}_{2}\mathrm{T}(\theta_2)\cdot {}^{2}_{3}\mathrm{T}(\theta_3)\cdot {}^{3}_{\mathrm{E}}\widetilde{\mathrm{P}}$$

6. Inverse Kinematics

a) Solve $\theta_1\,\theta_2\,\theta_3$, such that ${}^{0}_{\mathrm{E}}\mathrm{T} = {}^{0}_{\mathrm{A}}\mathrm{T}$

b) Solve $\theta_1\,\theta_2\,\theta_3$, such that ${}^{0}_{\mathrm{E}}\mathrm{T} = {}^{0}_{\mathrm{B}}\mathrm{T}$

Q3.1: RRR example for Inverse Kinematics

Forward Kinematics

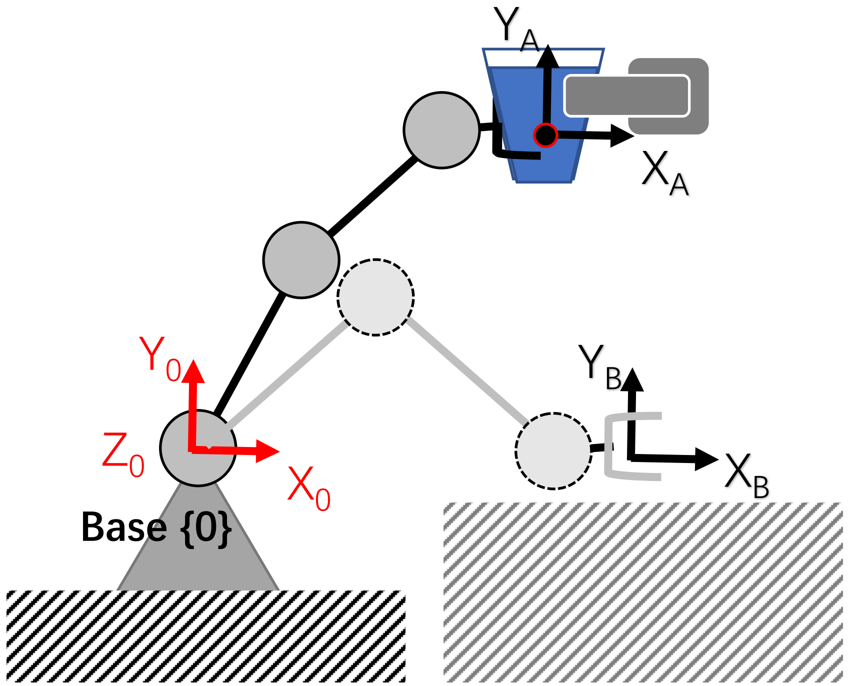
$${}^{0}_{E}T = {}^{0}_{1}T(\theta_1) \cdot {}^{1}_{2}T(\theta_2) \cdot {}^{2}_{3}T(\theta_3) \cdot {}^{3}_{E}T$$

$${}^{0}_{E}\widetilde{P} = {}^{0}_{1}T(\theta_1) \cdot {}^{1}_{2}T(\theta_2) \cdot {}^{2}_{3}T(\theta_3) \cdot {}^{3}_{E}\widetilde{P}$$

Inverse Kinematics

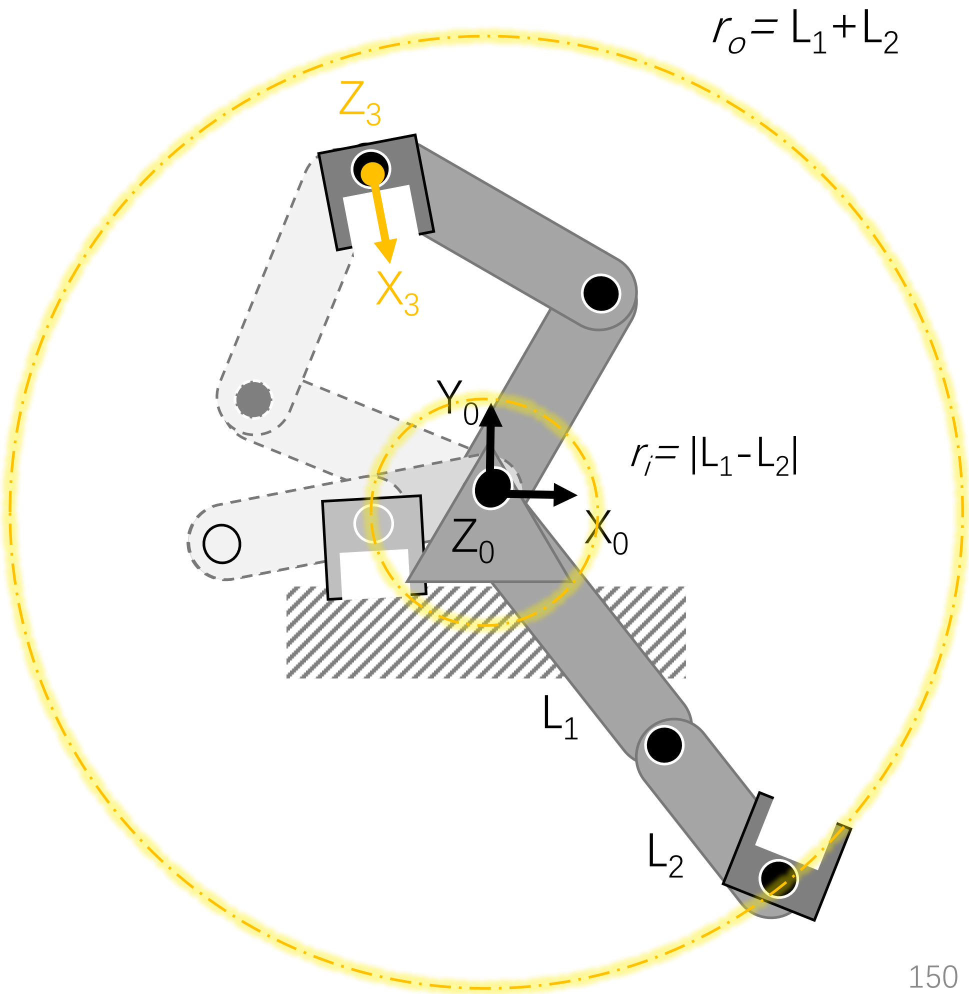
a) Solve $\theta_1\,\theta_2\,\theta_3$, such that ${}^{0}_{E}T = {}^{0}_{A}T$

b) Solve $\theta_1\,\theta_2\,\theta_3$, such that ${}^{0}_{E}T = {}^{0}_{B}T$



Recap: Solvability

- Workspace
  - **Reachable**: Region where the end-effector can be located
  - **Dexterous**: Region where the end-effector can be located with all orientations
- Multiple solutions
  - For the same end-effector pose, there could be 2 possible solutions
- Approach to solutions:
  - Numerical
  - Closed-form



Q3.1: RRR example for Inverse Kinematics

Inverse Kinematics

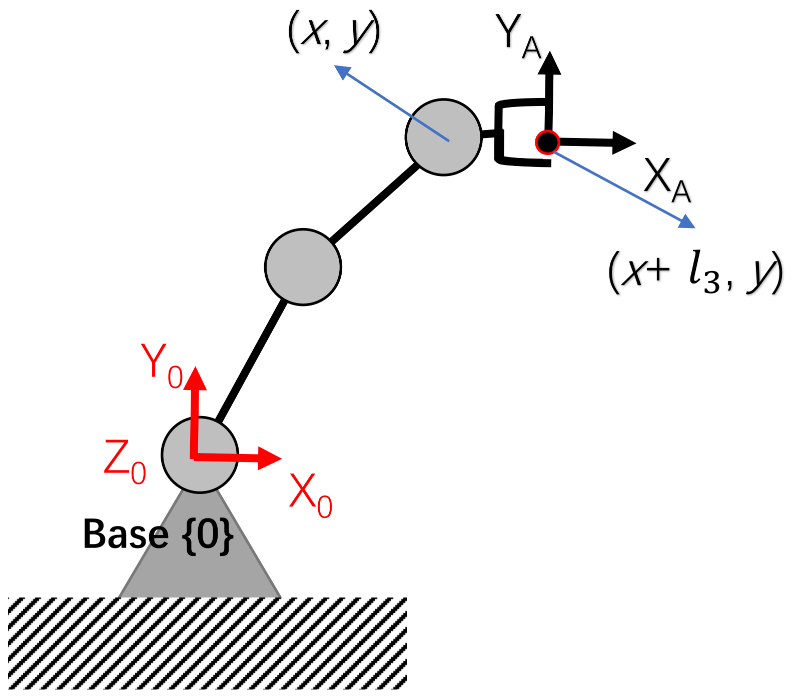
a) Solve $\theta_1\,\theta_2\,\theta_3$, such that ${}^{0}_{E}\mathrm{T} = {}^{0}_{A}\mathrm{T}$

a) Solve $\theta_1, \theta_2, \theta_3$, such that:

$$
{}^{0}_{3}\mathrm{T} = \begin{bmatrix} c_{123} & -s_{123} & 0 & \mathrm{L1}c_1 + \mathrm{L2}c_{12} \\ s_{123} & c_{123} & 0 & \mathrm{L1}s_1 + \mathrm{L2}s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & x \\ 0 & 1 & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Generally

$$
\begin{bmatrix} c_{123} & -s_{123} & 0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 & x \\ \sin(\varphi) & \cos(\varphi) & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$



See textbook Section 4.4

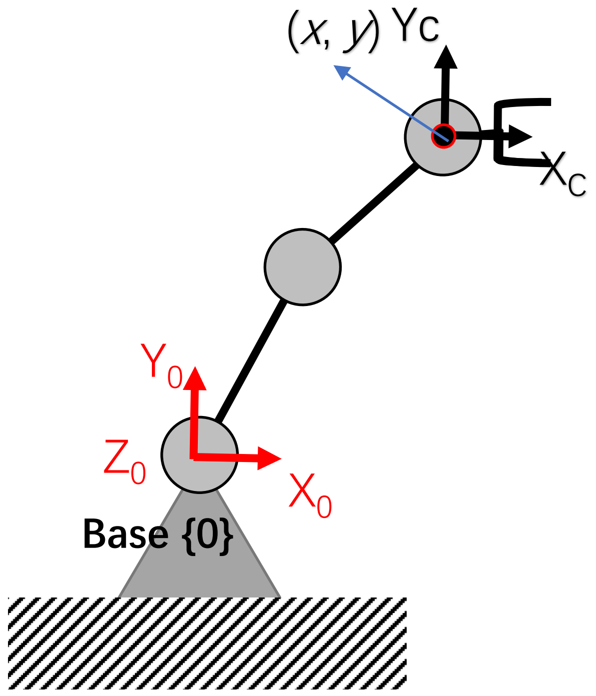
Q3.1: RRR example for Inverse Kinematics

Inverse Kinematics

a) Solve $\theta_1\,\theta_2\,\theta_3$, such that ${}^0_E\mathrm{T} = {}^0_A\mathrm{T}$

Generally

$$\begin{bmatrix} c_{123} & -s_{123} & 0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0 & l_1s_1 + l_2s_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos(\varphi) & -\sin(\varphi) & 0 & x \\ \sin(\varphi) & \cos(\varphi) & 0 & y \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Three unknowns:

Four nonlinear equations:

$$\cos(\varphi) = c_{123} \qquad (1)$$
$$\sin(\varphi) = s_{123} \qquad (2)$$
$$x = l_1c_1 + l_2c_{12} \qquad (3)$$
$$y = l_1s_1 + l_2s_{12} \qquad (4)$$

See textbook Section 4.4

Q3.1: RRR example for Inverse Kinematics

Inverse Kinematics

Algebraic Approach

Three unknowns:
Four nonlinear equations:

$\cos(\varphi) = c_{123}$ (1)

$\sin(\varphi) = s_{123}$ (2)

$x = l_1 c_1 + l_2 c_{12}$ (3)

$y = l_1 s_1 + l_2 s_{12}$ (4)

$C_{123} = \cos(\theta_1 + \theta_2 + \theta_3)$

(3) & (4): $x^2 + y^2 = l_1^2 + l_2^2 + 2 l_1 l_2 c_2$

$^* c_{12} = c_1 c_2 - s_1 s_2$
$s_{12} = s_1 c_2 + c_1 s_2$

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$

- Right hand side must be between -1 and 1, else out of workspace
- θ_2 is solved

(3): $x = l_1 c_1 + l_2 c_{12} = l_1 c_1 + l_2 c_1 c_2 - l_2 s_1 s_2$

$(l_1 + l_2 c_2) c_1 - l_2 s_2 s_1 = R \sin(\theta_1 + \gamma)$ $\quad a \sin\vartheta \pm b \cos\vartheta \equiv R \sin(\vartheta \pm \alpha)$

Where $R = \sqrt{(l_1 + l_2 c_2)^2 + l_2^2 s_2^2}$ and $\gamma = -\tan^{-1} \frac{(l_1 + l_2 c_2)}{l_2 s_2}$

- θ_1 is solved

From (1) or (2), $\theta_1 + \theta_2 + \theta_3 = \phi$

- θ_3 is solved

See textbook Section 4.4

Q3.1: RRR example for Inverse Kinematics

Inverse Kinematics

Geometrical Approach?

See textbook Section 4.4

Velocity Kinematics and Static Forces

Introduction to Robotics: Fundamentals

Schedule Check

- **Lecture**

O. Overview
- Science & Engineering in Robotics

I. Spatial Representation & Transformation
- Coordinate Systems; Pose Representations; Homogeneous Transformations

II. Kinematics
- Multi-body frame assignment; D-H Convention; Joint-space; Work-space; Forward/Inverse Kinematics

Lecture 4

III. Velocity Kinematics and Static Forces
- Translational/Rotational Velocity; Joint torque; Generalized Force Coordinates; Jacobian; Singularity

Fundamentals

Week 1-4

Revision/ Quiz on Week 5

IV. Dynamics
- Lagrangian Formulation; Newton-Euler Equations of Motion

V. Planning
- Joint-based Motion Planning; Cartesian-based Path Planning

VI. Control
- Independent Joint/Feedforward/Inverse Dynamics Controls; Controller Architectures

Essentials

Week 6-9

Revision/ Quiz on Week 10

VII. Robot Vision (and Perception)
- Image Formation; Image Processing; Visual Tracking & Pose Estimation; Vision-based Control & Image-guided robotics

Applied

Week 11-14

Reading Wk/ Exam on Week 15-16

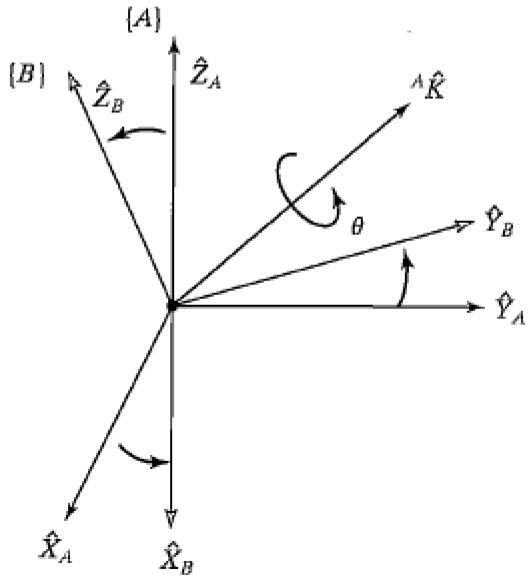
Revisit Orientation/Rotation

- Representation:

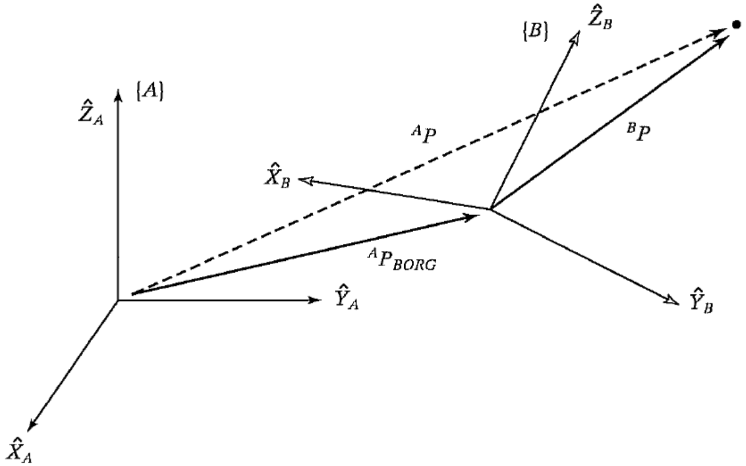
1. Euler Angle
2. Rotation Matrix
3. Rotation Vector
4. Unit Quaternion

Rotation

- Any rotation can be expressed as:
  1. Rotation of angle θ
  2. About some rotation axis $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$



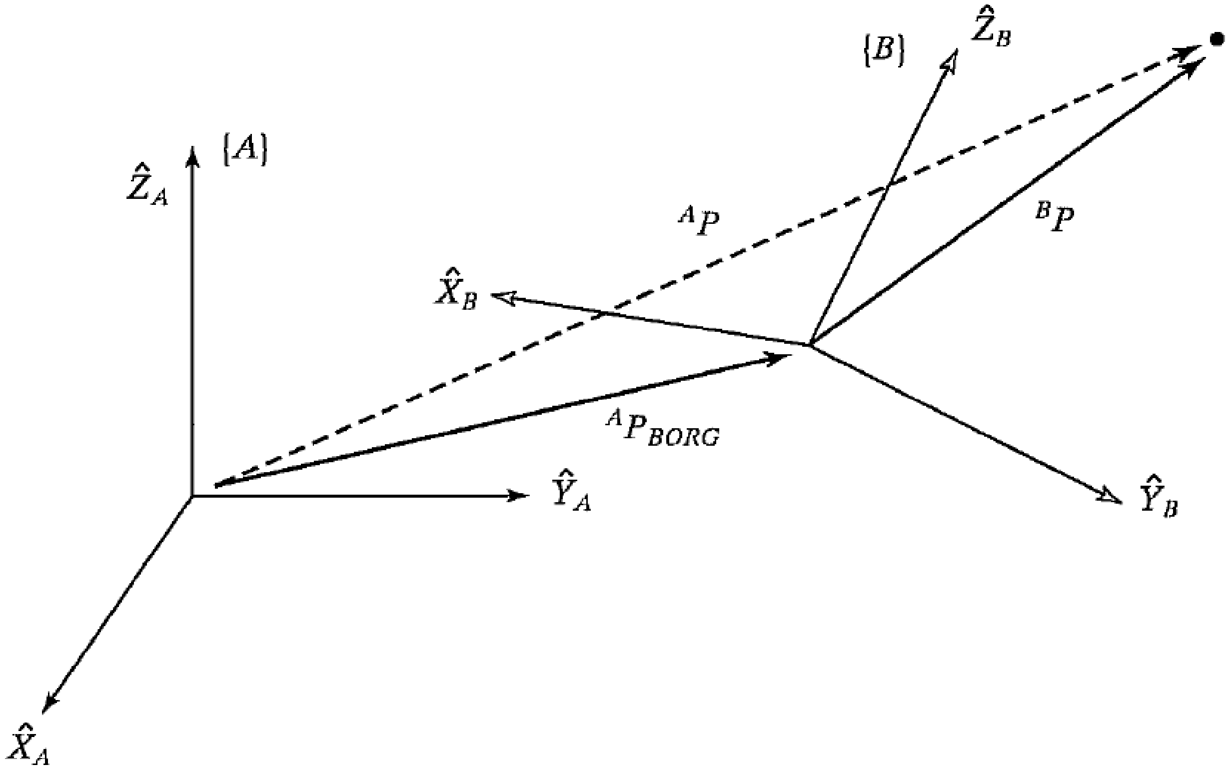
Review: Position



$$ {}^AP = {}^AP_{B,ORG} + {}^A_BR \, . \, {}^BP $$

Concept Check

- It is given that the point is stationary in frame {B}. Can we conclude that the velocity of the point is zero in frame {A}?



Velocity

- 2 Methods to differentiate position to obtain velocity
  1)Cross product using vector form
  2)Skew Matrix

Vector Form

- $\vec{P}_1 = \vec{P}_{B,ORG} + \vec{P}_{1/B}$ (1)

Where $\vec{P}_{1/B}$ is the relative position of point (1) with respect to the B frame origin and,

- $\vec{P}_{1/B} = x\,\hat{i}_B + y\,\hat{j}_B + z\hat{k}_B$ (2)

Hence,

- $\vec{V}_1 = \dot{\vec{P}}_{B,ORG} + \dot{\vec{P}}_{1/B}$

$$= \vec{V}_{B,ORG} + \dot{x}\,\hat{i}_B + \dot{y}\,\hat{j}_B + \dot{z}\hat{k}_B + x\dot{\hat{i}}_B + y\dot{\hat{j}}_B + z\dot{\hat{k}}_B \qquad (3)$$

Vector Form

- $\dot{\widehat{i_B}} = \vec{\omega} \times \widehat{i}_B \qquad \dot{\widehat{j_B}} = \vec{\omega} \times \widehat{j}_B \qquad \dot{\widehat{k_B}} = \vec{\omega} \times \widehat{k}_B$

Hence,

- $$\begin{aligned} \vec{V}_1 &= \vec{V}_{B,ORG} + \dot{x}\,\widehat{i}_B + \dot{y}\,\widehat{j}_B + \dot{z}\hat{k}_B + x\dot{\widehat{i_B}} + y\dot{\widehat{j_B}} + z\dot{\widehat{k_B}} \\ &= \vec{V}_{B,ORG} + \dot{x}\,\widehat{i}_B + \dot{y}\,\widehat{j}_B + \dot{z}\hat{k}_B \\ &\quad + x(\vec{\omega} \times \widehat{i}_B) + y(\vec{\omega} \times \widehat{j}_B) + z(\vec{\omega} \times \widehat{k}_B) \\ &= \vec{V}_{B,ORG} + \dot{x}\,\widehat{i}_B + \dot{y}\,\widehat{j}_B + \dot{z}\hat{k}_B \\ &\quad + \vec{\omega} \times \left(x\,\widehat{i}_B + y\,\widehat{j}_B + z\hat{k}_B\right) \\ &= \vec{V}_{B,ORG} + \dot{x}\,\widehat{i}_B + \dot{y}\,\widehat{j}_B + \dot{z}\hat{k}_B + \vec{\omega} \times \vec{P}_{1/B} \end{aligned}$$

Velocity (using rotation matrix)

$$ {}^{A}P_1 = {}^{A}P_{B,ORG} + {}^{A}_{B}R\ {}^{B}P_{1/B} $$

- Differentiating with time,

$$ {}^{A}V_1 = {}^{A}V_{B,ORG} + {}^{A}_{B}R\ {}^{B}V_{1/B} + {}^{A}_{B}\dot{R}\ {}^{B}P_{1/B} $$
$$ = {}^{A}V_{B,ORG} + {}^{A}_{B}R\ {}^{B}V_{1/B} + {}^{A}_{B}\dot{R}\ {}^{A}_{B}R^{T}\ {}^{A}P_{1/B} $$

  - ${}^{A}V_1$: velocity of point 1 in frame {A}
  - ${}^{A}V_{B,ORG}$: velocity of B,ORG in frame {A}
  - ${}^{B}V_{1/B}$: velocity of point 1 wrt to B in frame {B}

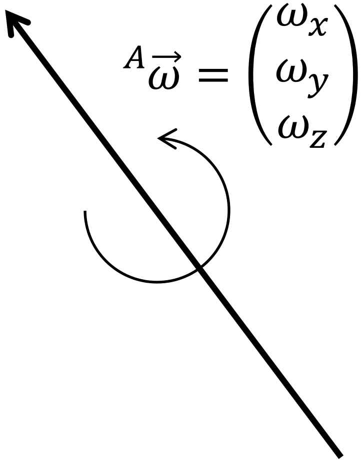
- Question: What is ${}^{A}_{B}\dot{R}\ {}^{A}_{B}R^{T}$?

Using vector form to explain

$$\vec{P}_1 = \vec{P}_{B,ORG} + x\,\hat{i}_B + y\,\hat{j}_B + z\hat{k}_B$$

- Differentiating with respecting to time,

$$\vec{V}_1 = \vec{V}_{B,ORG} + \dot{x}\,\hat{i}_B + \dot{y}\,\hat{j}_B + \dot{z}\hat{k}_B + \underbrace{x\dot{\hat{i}}_B + y\dot{\hat{j}}_B + z\dot{\hat{k}}_B}_{\vec{\omega} \times (x\,\hat{i}_B + y\,\hat{j}_B + z\hat{k}_B)}$$



$${}^{A}\vec{\omega} = \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Skew-symmetric matrices

Velocity (Skew Matrix)

$$^{A}P_1 = {}^{A}P_{B,ORG} + {}^{A}_{B}R \; {}^{B}P_{1/B}$$

- Differentiating with time,

$$^{A}V_1 = {}^{A}V_{B,ORG} + {}^{A}_{B}R \; {}^{B}V_{1/B} + {}^{A}_{B}\dot{R} \; {}^{B}P_{1/B}$$
$$= {}^{A}V_{B,ORG} + {}^{A}_{B}R \; {}^{B}V_{1/B} + {}^{A}_{B}\dot{R} \; {}^{A}_{B}R^{T} \; {}^{A}P_{1/B}$$

  - ${}^{A}V_1$: velocity of point 1 in frame {A}
  - ${}^{A}V_{B,ORG}$: velocity of *B,ORG* in frame {A}
  - ${}^{B}V_{1/B}$: velocity of point 1 wrt to B in frame {B}
  - ${}^{A}_{B}\dot{R} \; {}^{A}_{B}R^{T} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$

Velocity (Skew Matrix)

$$^{A}P_1 = {}^{A}P_{B,ORG} + {}^{A}_{B}R\ {}^{B}P_{1/B}$$

- Differentiating with time,

$$^{A}V_1 = {}^{A}V_{B,ORG} + {}^{A}_{B}R\ {}^{B}V_{1/B} + {}^{A}_{B}\dot{R}\ {}^{B}P_{1/B}$$
$$= {}^{A}V_{B,ORG} + {}^{A}_{B}R\ {}^{B}V_{1/B} + {}^{A}_{B}\dot{R}\ {}^{A}_{B}R^{T}\ {}^{A}P_{1/B}$$

  - ${}^{A}V_1$: velocity of point 1 in frame {A}
  - ${}^{A}V_{B,ORG}$: velocity of B,ORG in frame {A}
  - ${}^{B}V_{1/B}$: velocity of point 1 wrt to B in frame {B}
  - ${}^{A}_{B}\dot{R}\ {}^{A}_{B}R^{T} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}$

Q 3.3 Example of Velocity

A point P_1 is stationary in frame {B}.

If position of the point is ${}^{A}P_{1/B} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and the origin of frame {B} is stationary in frame {A}. Determine ${}^{A}V_1$ if frame {B} is rotating at 2 rad/s about $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ in frame {A} .

Since $\begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}$ is a unit vector, $\vec{\omega} = \begin{pmatrix} 2/\sqrt{3} \\ 2/\sqrt{3} \\ 2/\sqrt{3} \end{pmatrix}$

Q 3.3 Example of Velocity

$$^{A}V_1 = {}^{A}V_{B,ORG} + {}^{A}_{B}R\ {}^{B}V_1 + {}^{A}_{B}\dot{R}\ {}^{A}_{B}R^T\ {}^{A}P_{1/B}$$

$$= 0 + 0 + \begin{pmatrix} 0 & -2/\sqrt{3} & 2/\sqrt{3} \\ 2/\sqrt{3} & 0 & -2/\sqrt{3} \\ -2/\sqrt{3} & 2/\sqrt{3} & 0 \end{pmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{3} \\ 0 \\ 2/\sqrt{3} \end{bmatrix}\ [\mathrm{m/s}]$$

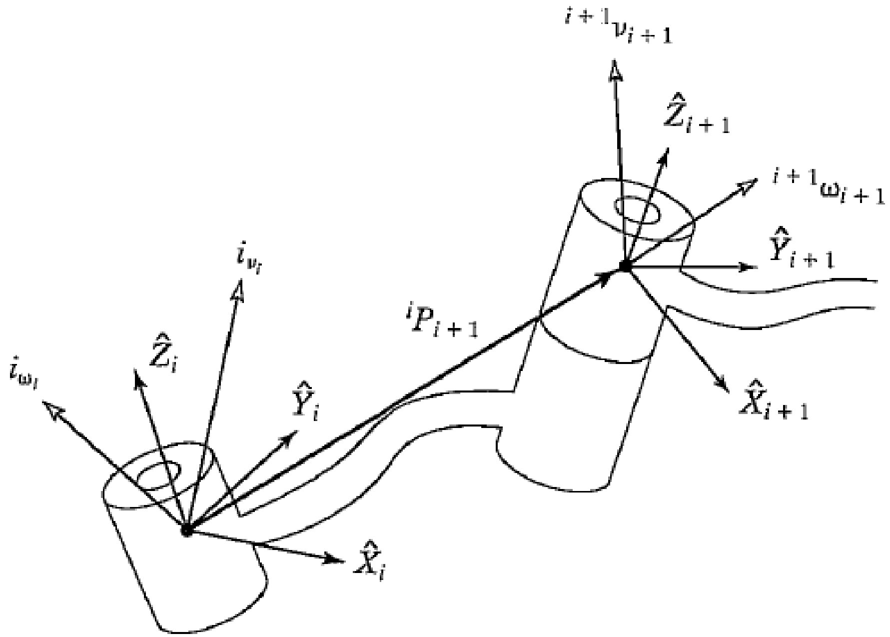
Velocity “Propagation” from link to link

- Rotational velocities can be added when both ω vectors are written with respect to the same frame

$$ {}^{i}\omega^{0}_{i+1} = {}^{i}\omega^{0}_{i} + {}^{i}_{i+1}R\,\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} $$

$$ \boxed{{}^{i+1}\omega^{0}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega^{0}_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}} $$

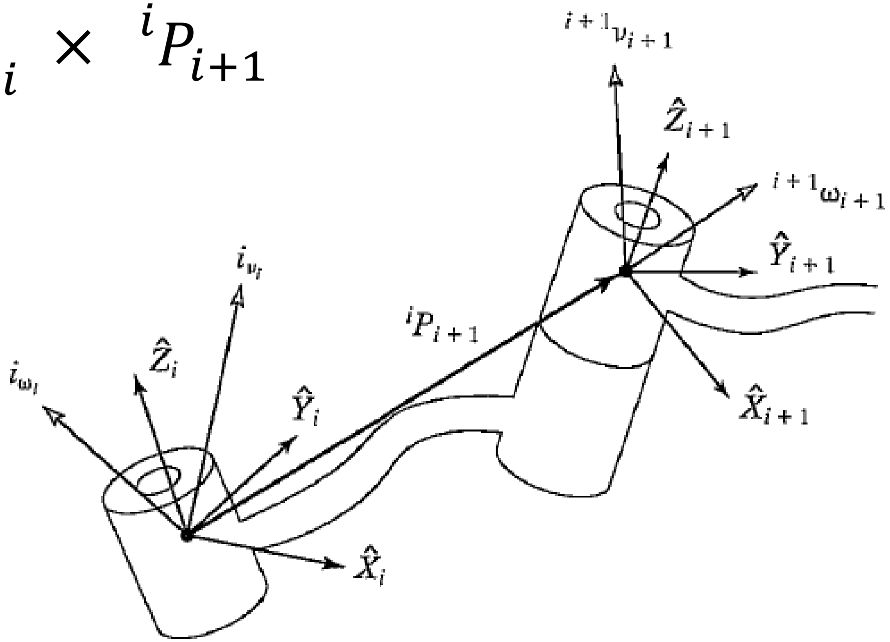
$*\hat{Z}$ in the direction of joint
Notation: In consideration of robot links, frame {0} is used as the reference frame. Meaning to say, ${}^{i+1}\omega_{i+1}$ is the absolute angular velocity of {i+1} expressed in frame {i+1}



Linear Velocity in Vector Form

- General form: $\vec{V}_1 = \vec{V}_{B,ORG} + \dot{x}\,\hat{i}_B + \dot{y}\,\hat{j}_B + \dot{z}\hat{k}_B + \vec{\omega} \times \vec{P}_{1/B}$
- For a fixed length serial manipulator:

$${}^{i}v_{i+1} = {}^{i}v_i + 0 + {}^{i}\omega_i \times {}^{i}P_{i+1}$$



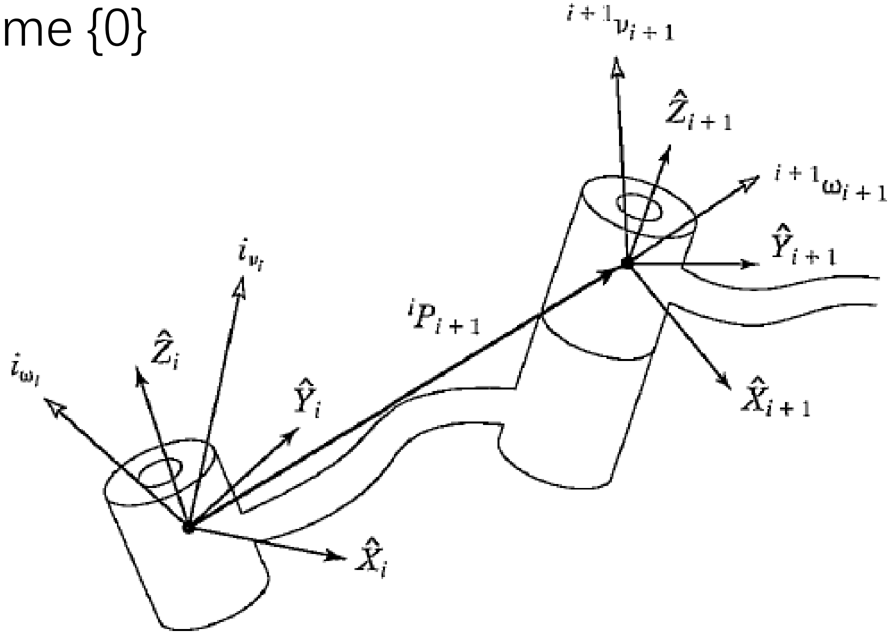
Velocity "Propagation" from link to link

- Linear velocities

$$^{i}v^{0}_{i+1} = {}^{i}v^{0}_{i} + {}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}$$

Differentiate w.r.t to frame {0}

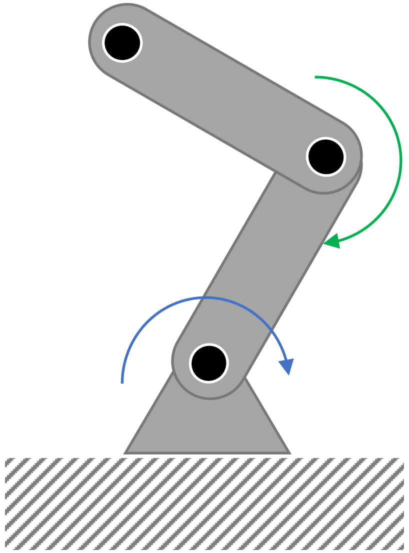
$$\boxed{^{i+1}v^{0}_{i+1} = {}^{i+1}_{i}R \left({}^{i}v^{0}_{i} + {}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1} \right)}$$

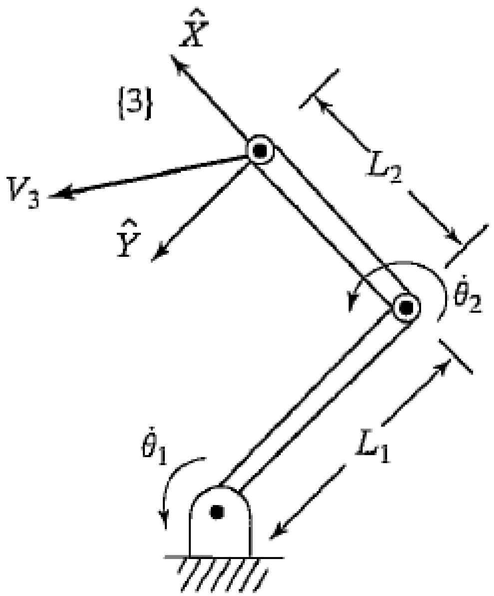


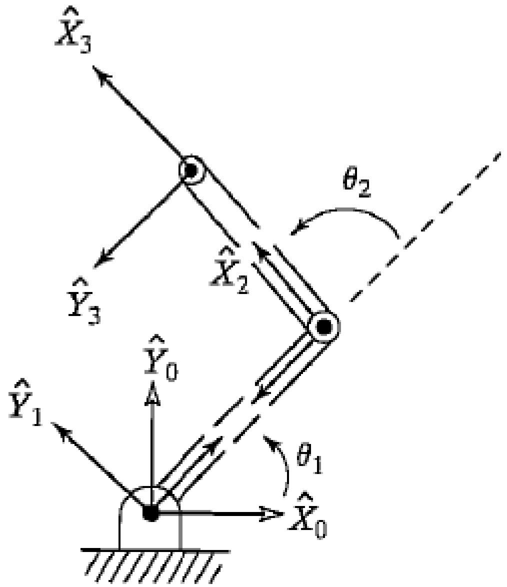
$*\hat{Z}$ in the direction of joint
Notation: In consideration of robot links, frame {0} is used as the reference frame. Meaning to say, $^{i+1}v_{i+1}$ is the absolute velocity of {i+1} origin expressed in frame {i+1}

Q 3.4 Example of Velocity Kinematics

- Given the below two-link manipulator, calculate the absolute velocity of the tip of the arm as a function of joint rates. Give the answer in frame {3} and frame {0}.







Q 3.4 Example of Velocity Kinematics

- $${}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

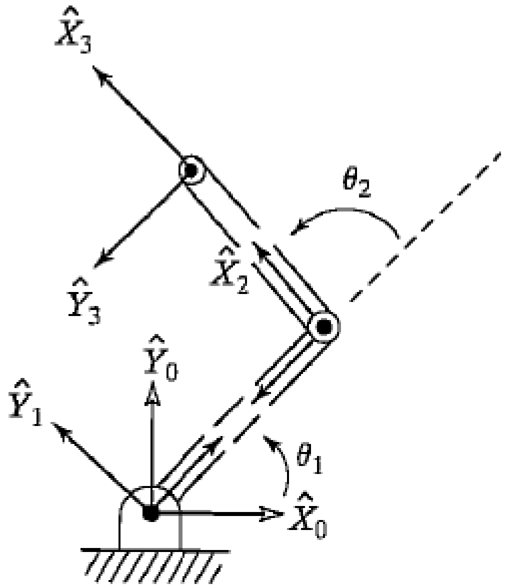
$${}^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

{3} and {2} are rigidly attached

$$\boxed{{}^{i+1}\omega^{0}{}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega^{0}{}_{i} + \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}v^{0}{}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v^{0}{}_{i} + {}^{i}\omega^{0}{}_{i} \times {}^{i}P_{i+1}\right)}$$

- $${}^1\omega^0{}_1 = {}^1_0R\ {}^0\omega^0{}_0 + \dot{\theta}_1\ {}^1\hat{Z}_1 = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}$$
- $${}^1v^0{}_1 = {}^1_0R\left({}^0v^0{}_0 + {}^0\omega^0{}_0 \times {}^0P_1\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

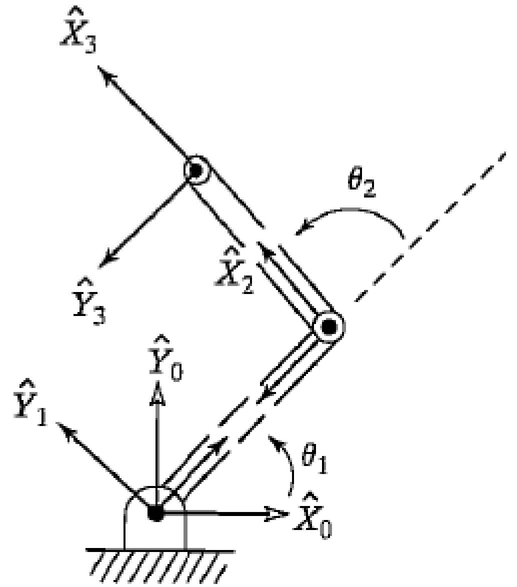


Q 3.4 Example of Velocity Kinematics

$$\boxed{{}^{i+1}\omega^{0}{}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega^{0}{}_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}v^{0}{}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v^{0}{}_{i} + {}^{i}\omega^{0}{}_{i} \times {}^{i}P_{i+1}\right)}$$

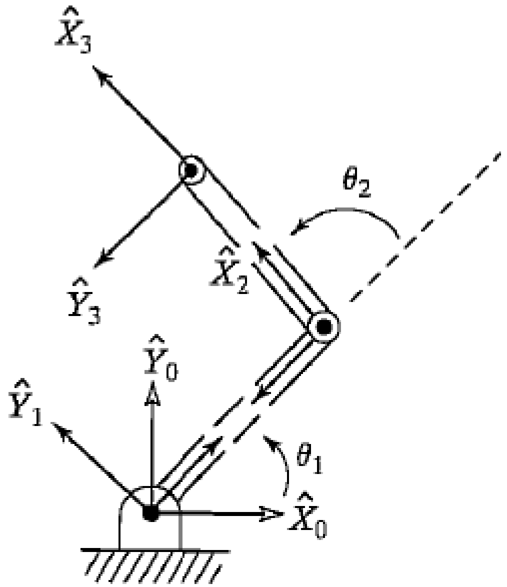
- $${}^{2}\omega^{0}{}_{2} = {}^{2}_{1}R\ {}^{1}\omega^{0}{}_{1} + \dot{\theta}_{2}\,{}^{2}\hat{Z}_{2} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix}$$
- $${}^{2}v^{0}{}_{2} = {}^{2}_{1}R\left({}^{1}v^{0}{}_{1} + {}^{1}\omega^{0}{}_{1} \times {}^{1}P_{2}\right) = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ l_1\dot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 \\ 0 \end{bmatrix}$$
- $${}^{3}\omega^{0}{}_{3} = {}^{2}\omega^{0}{}_{2}$$
- $${}^{3}v^{0}{}_{3} = {}^{3}_{2}R\left({}^{2}v^{0}{}_{2} + {}^{2}\omega^{0}{}_{2} \times {}^{2}P_{3}\right) = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$



Q 3.4 Example of Velocity Kinematics

- ${}^0_3R = {}^0_1R\,{}^1_2R\,{}^2_3R = \begin{bmatrix} c_{12} & -s_{12} & 0 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$
- Hence,

$${}^0v^0{}_3 = {}^0_3R\ {}^3v^0{}_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix}$$



Jacobians

- Jacobian is a multidimensional form of the derivative

$$y_1 = f_1(x_1, x_2, \dots, x_6)$$
$$\vdots$$
$$y_6 = f_6(x_1, x_2, \dots, x_6)$$

- Using vector notation to write these equations:

$$Y = F(X)$$

- Using chain rule,

$$\delta y_1 = \frac{\partial f_1}{\partial x_1}\delta x_1 + \frac{\partial f_1}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_1}{\partial x_6}\delta x_6$$
$$\vdots$$
$$\delta y_6 = \frac{\partial f_6}{\partial x_1}\delta x_1 + \frac{\partial f_6}{\partial x_2}\delta x_2 + \cdots + \frac{\partial f_6}{\partial x_6}\delta x_6$$

In vector notation: $\delta Y = \frac{\partial F}{\partial X}\delta X$

Jacobians

$$\delta Y = \frac{\partial F}{\partial X} \delta X$$

- 6 x 6 matrix of partial derivatives is what we call the Jacobian, J

$$\delta Y = J(X)\, \delta X$$

- Dividing both sides by differential time element,

$$\dot{Y} = J(X)\dot{X}$$

- Jacobians are time-varying linear transformations

Jacobians

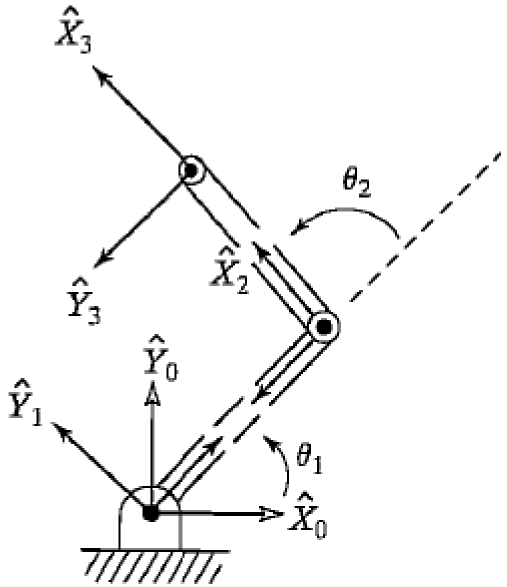
- In general, seen as the mapping of velocities in X to Y
$$\dot{Y} = J(X)\dot{X}$$
- In robotics, used to relate joint velocities to cartesian velocities
$$^{0}v = ^{0}J(\Theta)\,\dot{\Theta}$$
- In 3D space, a six-joint robot,
  - Jacobian $J(\Theta)$ is 6 x 6,
  - Joint velocity is $\dot{\Theta}$ is 6 x 1,
  - Cartesian velocity is $^{0}v = [\,^{0}\dot{\mathrm{P}} \quad ^{0}\dot{\Theta}]^{\mathrm{T}}$ is 6 x 1

Jacobians

- For an *N*-joint robot in 3D space,
  - Joint velocity is $\dot{\Theta}$ is *N* x 1,
  - Jacobian $J(\Theta)$ is 6 x *N*,
  - Cartesian velocity is ${}^{0}v = [\ {}^{0}\dot{\mathrm{P}} \quad {}^{0}\dot{\Theta}]^{\mathrm{T}}$ is 6 x 1
    - Linear velocity stacked with rotational velocity
  - Cartesian velocity is $v_N = [J_1 \quad \ldots J_i \ldots \quad J_N] \begin{bmatrix} \dot{q}_1 \\ \vdots \\ \dot{q}_N \end{bmatrix}$
    - Column J_i represents motion contribution of Joint i

Q3.5 Example on Jacobian

- Using the previous question Q3.4, obtain the 2 x 2 Jacobian that relates joint rates to end-effector velocity in both frame {3} and frame {0}.



Q3.5 Example on Jacobian

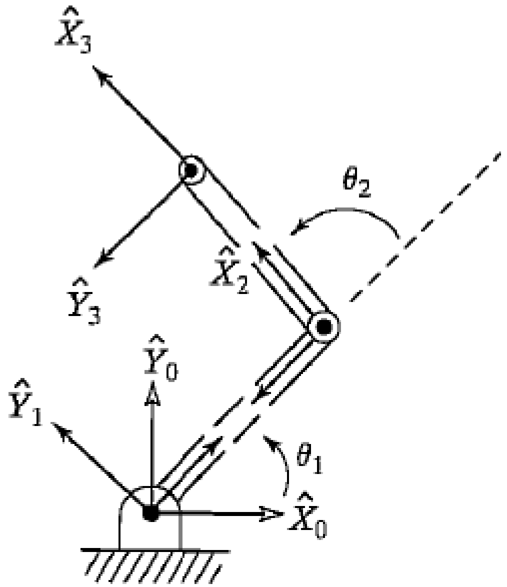
- Using the previous case problem, obtain the 2 x 2 Jacobian that relates joint rates to end-effector velocity in both frame {3} and frame {0}.

$$^{3}v_3 = \begin{bmatrix} l_1 s_2 \dot{\theta}_1 \\ l_1 c_2 \dot{\theta}_1 + l_2(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$^{0}v_3 = \begin{bmatrix} -l_1 s_1 \dot{\theta}_1 - l_2 s_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ l_1 c_1 \dot{\theta}_1 + l_2 c_{12}(\dot{\theta}_1 + \dot{\theta}_2) \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$$^{3}J(\Theta) = \begin{bmatrix} l_1 s_2 & 0 \\ l_1 c_2 + l_2 & l_2 \end{bmatrix}$$

$$^{0}J(\Theta) = \begin{bmatrix} -l_1 s_1 - l_2 s_{12} & -l_2 s_{12} \\ l_1 c_1 + l_2 c_{12} & l_2 c_{12} \end{bmatrix}$$



Changing Jacobian's frame of reference

- Given a Jacobian in frame {B},

$$\begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix} = {}^{B}J(\Theta)\,\dot{\Theta}$$

- The velocity in {B} is described relative to {A} by the transformation

$$\begin{bmatrix} {}^{A}v \\ {}^{A}\omega \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} \begin{bmatrix} {}^{B}v \\ {}^{B}\omega \end{bmatrix} = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} {}^{B}J(\Theta)\,\dot{\Theta}$$

- Hence,

$${}^{A}J(\Theta) = \begin{bmatrix} {}^{A}_{B}R & 0 \\ 0 & {}^{A}_{B}R \end{bmatrix} {}^{B}J(\Theta)$$

Singularities

$$v = J(\Theta)\,\dot{\Theta}$$
$$J^{-1}(\Theta)v = \dot{\Theta}$$

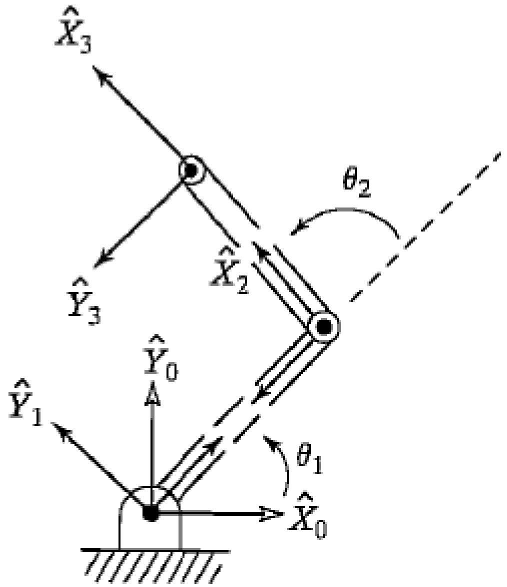
- This is important when a certain velocity vector of the end-effector is desired

- But what happens when Jacobian becomes singular (ie no inverse)?
  - Workspace-boundary singularities
  - Workspace-interior singularities
  - Inverse Jacobian blows up when at singular point

Q3.6 Example on Singularity

Continuing from the case problem, inverse of the Jacobian can be written as:

$$^{0}J^{-1}(\Theta) = \frac{1}{l_1 l_2 s_2}\begin{bmatrix} l_2 c_{12} & l_2 s_{12} \\ -l_1 c_1 - l_2 c_{12} & -l_1 s_1 - l_2 s_{12} \end{bmatrix}$$



For a desired velocity of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ m/s,

$$\dot{\theta}_1 = \frac{c_{12}}{l_1 s_2}$$

$$\dot{\theta}_2 = -\frac{\dot{c}_1}{l_2 s_2} - \frac{c_{12}}{l_1 s_2}$$

As arm stretches out towards $\theta_2 = 0$, both joint rates go to infinity

Jacobian: Static Forces

Introduction to Robotics: Fundamentals