ECE 470: Introduction to Robotics

Lecture 13

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Lecture Overview

- Recap: the Big Picture
  - Overview of Robot Dynamics
  - Components of Dynamics Equation
  - Formulation of the Dynamics Equation
- Recall: Previous concept in Kinematics and Statics
  - Motion: Position, Velocity, Acceleration
  - Forces: Generalized Coordinates/Forces, Jacobian, Virtual Work
- Newton-Euler Approach
- Lagrangian Formulation

Robot Mechanics

- **Kinematics**: The science of motion without regards to the forces that cause it
  - Pose of the manipulator in static situations
  - Analyze (linear and angular) motion of bodies (/linkages)
- **Statics**: Bodies in equilibrium and force (/moment) relationship
- **Dynamics**: Concern with the forces (/torque) on bodies that cause motion
  - In ECE 470, we are interested in relating forces (/torque) and motion
  - i.e. Dynamic Equation

Robot Mechanics: Dynamics

- **Dynamics**: Concern with the forces (/torque) on bodies that cause motion

Formulating Dynamic Equations

Recall: Motion (Acceleration)

$$\vec{P}_1 = \vec{P}_{B,ORG} + x\,\hat{i}_B + y\,\hat{j}_B + z\hat{k}_B$$

Differentiating with respecting to time,

$$\vec{V}_1 = \vec{V}_{B,ORG} + \dot{x}\,\hat{i}_B + \dot{y}\,\hat{j}_B + \dot{z}\hat{k}_B + x\dot{\hat{i}}_B + y\dot{\hat{j}}_B + z\dot{\hat{k}}_B$$

$$\vec{V}_1 = \vec{V}_{B,ORG} + \dot{x}\,\hat{i}_B + \dot{y}\,\hat{j}_B + \dot{z}\hat{k}_B + \vec{\omega} \times \vec{P}_{1/B}$$

Differentiate w.r.t. time

$$\dot{\vec{V}}_1 = \dot{\vec{V}}_{B,ORG} + \ddot{x}\,\hat{i}_B + \ddot{y}\,\hat{j}_B + \ddot{z}\hat{k}_B + \vec{\omega} \times \vec{V}_{1/B} + \vec{\omega} \times (\vec{V}_{1/B} + \vec{\omega} \times \vec{P}_{1/B}) + \dot{\vec{\omega}} \times \vec{P}_{1/B}$$

What happen if we are looking at revolute joint?

$$\dot{\vec{V}}_1 = \dot{\vec{V}}_{B,ORG} + \ddot{x}\,\hat{i}_B + \ddot{y}\,\hat{j}_B + \ddot{z}\hat{k}_B + 2\vec{\omega} \times \vec{V}_{1/B} + \vec{\omega} \times \vec{\omega} \times \vec{P}_{1/B} + \dot{\vec{\omega}} \times \vec{P}_{1/B}$$

coriolis acceleration

tangential acceleration

centrifugal acceleration

Recall : Acceleration for “Propagation” from link to link

$$^{0}\omega_{i+1} = {}^{0}\omega_i + {}^{0}_{i+1}R\ \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

Differentiate wrt to time

$$^{0}\dot{\omega}^{0}_{\ i+1} = {}^{0}\dot{\omega}_i + {}^{0}_{i+1}\dot{R}\ \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + {}^{0}_{i+1}R\ \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

$$= {}^{0}\dot{\omega}_i + {}^{0}_{i+1}\dot{R}\ {}^{0}_{i+1}R^T\ {}^{0}_{i+1}R\ \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + {}^{0}_{i+1}R\ \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

$$= {}^{0}\dot{\omega}_i + {}^{0}\Omega_i\ {}^{0}_{i+1}R\ \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + {}^{0}_{i+1}R\ \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

$$= {}^{0}\dot{\omega}_i + {}^{0}\omega_i \times {}^{0}_{i+1}R\ \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + {}^{0}_{i+1}R\ \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}$$

$$\boxed{{}^{i+1}\dot{\omega}^{0}_{\ i+1} = {}^{i+1}_{i}R\ {}^{i}\dot{\omega}_{\ i} + \underbrace{{}^{i+1}_{i}R\ {}^{i}\omega_i \times \dot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\ {}^{i+1}\hat{Z}_{i+1}}_{}}$$

- For prismatic joint,

Since $\dot{\vec{V}}_1 = \dot{\vec{V}}_{B,ORG} + \ddot{x}\,\hat{i}_B + \ddot{y}\,\hat{j}_B + \ddot{z}\hat{k}_B + 2\vec{\omega}\times\vec{V}_{1/B} + \vec{\omega}\times\vec{\omega}\times\vec{P}_{1/B} + \dot{\vec{\omega}}\times\vec{P}_{1/B}$

$$\boxed{{}^{i+1}\dot{v}^{0}_{\ i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{v}^{0}_{\ i} + 0 + 0 + {}^{i}\omega^{0}_{\ i} \times {}^{i}\omega^{0}_{\ i} \times {}^{i}P_{i+1} + {}^{i}\dot{\omega}^{0}_{\ i} \times {}^{i}P_{i+1}\right)}$$

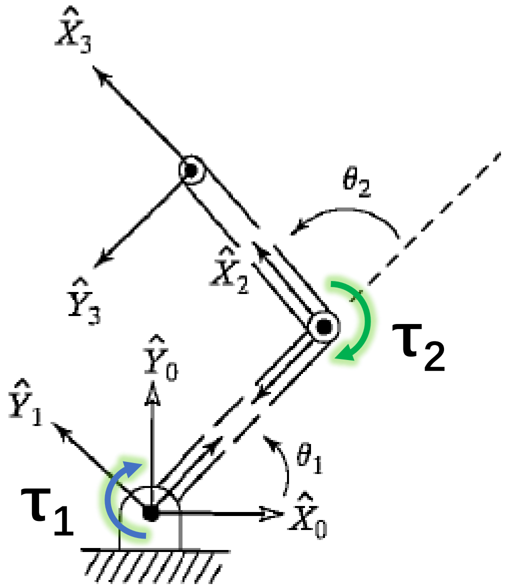
Q5.1 Example of acceleration

Assume ${}^{0}\dot{v}_0 = g\,\hat{Y}_0$

$$ {}^{0}_{1}R = \begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}R = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ {}^{0}v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ {}^{0}\dot{\omega}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ {}^{0}\dot{v}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} $$



Q5.1 Example of acceleration

$$\boxed{{}^{i+1}\omega^{0}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\omega^{0}_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

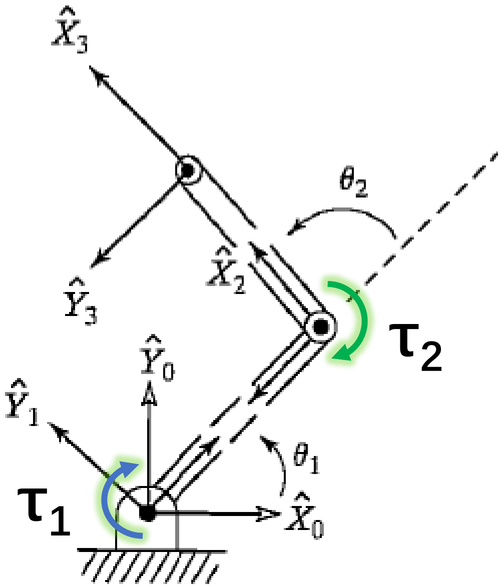
$$\boxed{{}^{i+1}\dot{\omega}^{0}_{i+1} = {}^{i+1}_{i}R\,{}^{i}\dot{\omega}_{i} + {}^{i+1}_{i}R\,{}^{i}\omega_{i} \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}v^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v^{0}_{i} + {}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right)}$$

$$\boxed{{}^{i+1}\dot{v}^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{v}^{0}_{i} + {}^{i}\Omega_{i}\,{}^{i}\Omega_{i}\,{}^{i}P_{i+1} + {}^{i}\dot{\Omega}_{i}\,{}^{i}P_{i+1}\right)}$$

- $i=0$

$${}^{1}\omega_{1} = \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix}, \quad {}^{1}\dot{\omega}_{1} = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 \end{bmatrix}, \quad {}^{1}\dot{v}_{1} = \begin{bmatrix} c_1 & s_1 & 0 \\ -s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix} = \begin{bmatrix} g\,s_1 \\ g\,c_1 \\ 0 \end{bmatrix}$$



Q5.1 Example of acceleration

$$\boxed{{}^{i+1}\omega^{0}_{i+1} = {}^{i+1}_{i}R\;{}^{i}\omega^{0}_{i} + \dot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1}}$$

$$\boxed{{}^{i+1}\dot\omega^{0}_{i+1} = {}^{i+1}_{i}R\;{}^{i}\dot\omega_{i} + {}^{i+1}_{i}R\;{}^{i}\omega_i \times \dot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1} + \ddot\theta_{i+1}\,{}^{i+1}\hat Z_{i+1}}$$

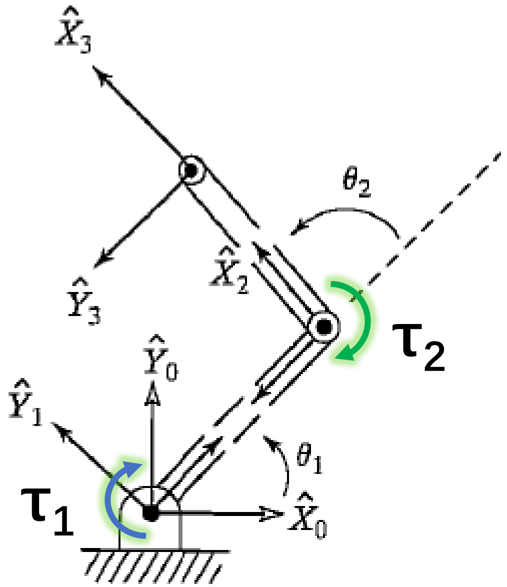
$$\boxed{{}^{i+1}v^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v^{0}_{i} + {}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right)}$$

$$\boxed{{}^{i+1}\dot v^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot v^{0}_{i} + {}^{i}\Omega_i\,{}^{i}\Omega_i\,{}^{i}P_{i+1} + {}^{i}\dot\Omega_i\,{}^{i}P_{i+1}\right)}$$

- i=1

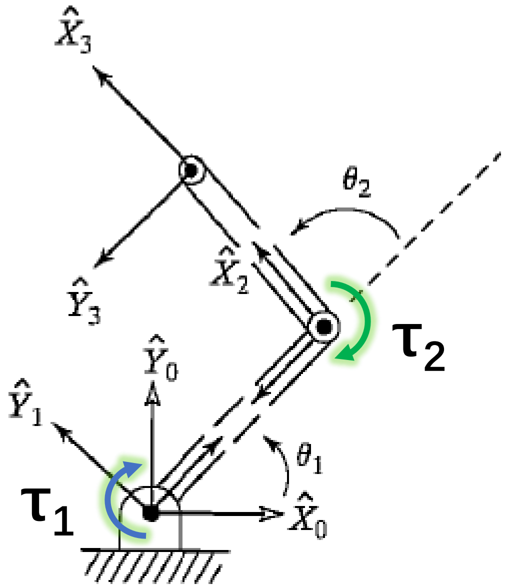
$$ {}^{2}\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \dot\theta_1 + \dot\theta_2 \end{bmatrix},\quad {}^{2}\dot\omega_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot\theta_1 + \ddot\theta_2 \end{bmatrix}, $$

$$ {}^{2}\dot v_2 = \begin{bmatrix} c_2 & s_2 & 0 \\ -s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l_1\dot\theta_1^2 + g\,s_1 \\ l_1\ddot\theta_1 + g\,c_1 \\ 0 \end{bmatrix} = \begin{bmatrix} l_1\ddot\theta_1^2 s_2 - l_1\dot\theta_1^2\,c_2 + g\,s_{12} \\ l_1\ddot\theta_1^2 c_2 + l_1\dot\theta_1^2\,s_2 + g\,c_{12} \\ 0 \end{bmatrix} $$



Q5.1 Example of acceleration

- Given the following 2-link planar manipulator in Q3.4, determine for each link the a) absolute angular acceleration, and b) joint absolute linear acceleration. Express the answer in their own frame.
- $^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
- Assume acceleration due to gravity to be g
  - *i.e.* $^0\dot{v}_0 = g\hat{Y}_0$



$$^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ ^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ ^0\dot{\omega}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ ^0\dot{v}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$$

Recall Generalized Coordinates

Recall Generalized Forces

Recall: Jacobians

- For an *N*-joint robot in 3D space,

Mapping of Velocity Coordinates

$$v_N = [J_1 \quad \ldots J_i \ldots \quad J_N]\begin{bmatrix}\dot{q}_1 \\ \vdots \\ \dot{q}_N\end{bmatrix}$$

Joint velocity is $\dot{\Theta}$ is N x 1,
Jacobian $J(\Theta)$ is 6 x N,
Cartesian velocity is ${}^0v = [\,{}^0\dot{\mathrm{P}} \quad {}^0\dot{\Theta}]^{\mathrm{T}}$ is 6 x 1
Column J_i represents motion contribution of Joint i

Jacobian in Force Domain

6-by-1 Cartesian Force-Moment Vector

6-by-1 torque/force at joints → $\boxed{\tau = J^{\mathrm{T}} F}$

N-by-6 Jacobian Transposed

Newton-Euler Formulation

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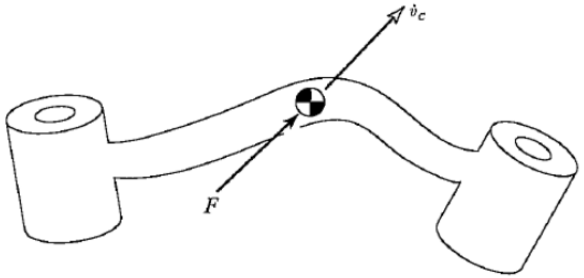
Newton’s Law of Motion

Newton’s 2nd Law

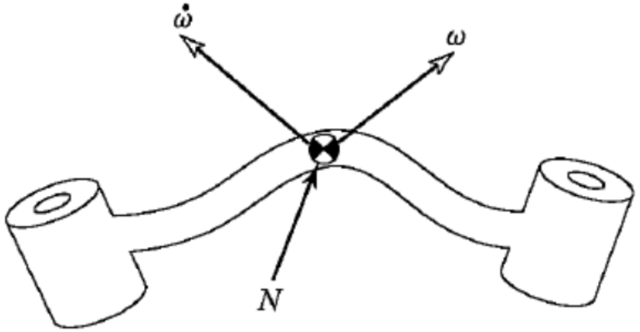
$$\boxed{F_i = m\dot{v}_{C_i}}$$



Euler’s equation (Torque)

$$\boxed{N_i = {}^{C_i}I\,\dot{\omega}_i + \omega_i \times {}^{C_i}I\omega_i}$$

Frame {C} is located at the center of mass



Force and Torque

Notation:

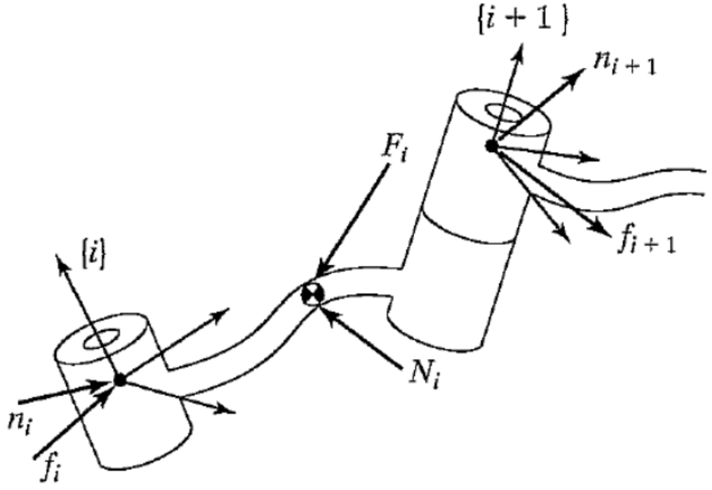
- f_i is the force exerted on link i by link $i-1$
- n_i is the torque exerted on link i by link $i-1$
- F_i is the net force exerted on the CG
- N_i is the net torque exerted on the CG

- Summing forces acting on link *i*,

$$ {}^iF_i = {}^if_i - {}_{i+1}^{\ i}R\ {}^{i+1}f_{i+1} $$

$$ \boxed{{}^if_i = {}_{i+1}^{\ i}R\ {}^{i+1}f_{i+1} + {}^iF_i} $$

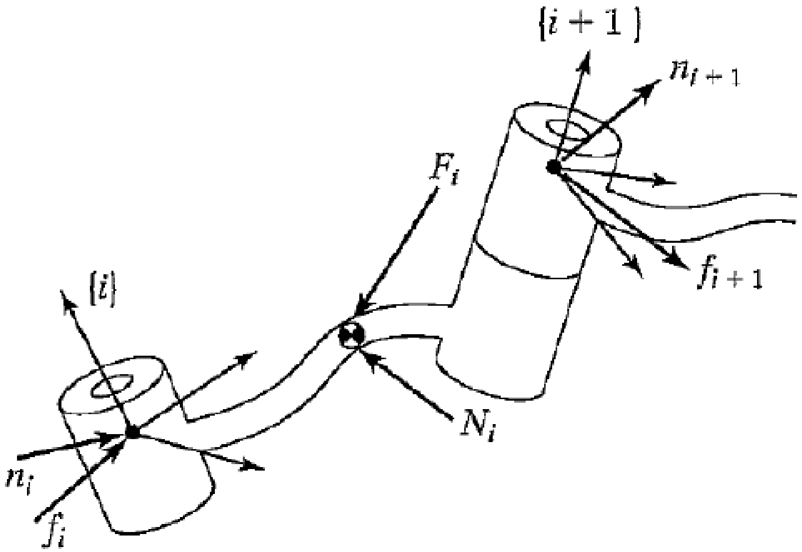
- Summing torques about CM of link *i*,
- $$ {}^iN_i = {}^in_i - {}^in_{i+1} + (-{}^iP_{C_i}) \times {}^if_i - ({}^iP_{i+1} - {}^iP_{C_i}) \times {}^if_{i+1} $$
$$ = {}^in_i - {}_{i+1}^{\ i}R\ {}^{i+1}n_{i+1} - {}^iP_{C_i} \times ({}^if_i - {}_{i+1}^{\ i}R\ {}^{i+1}f_{i+1}) $$
$$ - {}^iP_{i+1} \times {}^if_{i+1} $$



$$ \boxed{{}^in_i = {}^iN_i + {}_{i+1}^{\ i}R\ {}^{i+1}n_{i+1} + {}^iP_{C_i} \times {}^iF_i + {}^iP_{i+1} \times ({}_{i+1}^{\ i}R\ {}^{i+1}f_{i+1})} $$

Force and Torque

- Torque required by motor:
  - $\tau_i = {}^i n_i^T \; {}^i \hat{Z}_i$ (ie dot product of the two vectors)
    - Dot product because the rest are reaction forces
- In the case of prismatic joint, force required by actuator:
  - $F_i = {}^i f_i^T \; {}^i \hat{Z}_i$



Recall in Wk 04: **Jacobian**

For a rotational joint,

$$J_i = \begin{bmatrix} Z_{i-1} \times (P_N - P_{i-1}) \\ Z_{i-1} \end{bmatrix}$$

For a translational joint,

$$J_i = \begin{bmatrix} Z_{i-1} \\ 0 \end{bmatrix}$$

Iterative Newton-Euler Formulation

Outwards

$$\boxed{{}^{i+1}\omega^{0}_{i+1} = {}^{i+1}_{i}R\ {}^{i}\omega^{0}_{i} + \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}\dot{\omega}^{0}_{i+1} = {}^{i+1}_{i}R\dot{\omega} + {}^{i+1}_{i}R\ {}^{i}\omega^{0}_{i} \times \dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}}$$

$$\boxed{{}^{i+1}v^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}v^{0}_{i} + {}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right)}$$

$$\boxed{\begin{aligned}{}^{i+1}\dot{v}^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}^{0}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega^{0}_{i} \times \left({}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}^{0}_{i}\right) \\ +2\,{}^{i+1}\omega^{0}_{i+1} \times \dot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1} + \ddot{d}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}\end{aligned}}$$

$$\boxed{{}^{i+1}\dot{v}^{0}_{i+1} = {}^{i+1}_{i}R\left({}^{i}\dot{\omega}^{0}_{i} \times {}^{i}P_{i+1} + {}^{i}\omega^{0}_{i} \times \left({}^{i}\omega^{0}_{i} \times {}^{i}P_{i+1}\right) + {}^{i}\dot{v}^{0}_{i}\right)}$$

Iterative Newton-Euler Formulation

- Newton and Euler

$$\boxed{{}^{i}\dot{v}^{0}{}_{Ci} = {}^{i}\dot{\omega}^{0}{}_{i} \times {}^{i}P_{Ci} + {}^{i}\omega^{0}{}_{i} \times \left({}^{i}\omega^{0}{}_{i} + {}^{i}P_{Ci} \right) + {}^{i}\dot{v}^{0}{}_{i}}$$

$$\boxed{{}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{Ci+1}}$$

$$\boxed{{}^{i+1}N_{i+1} = {}^{Ci+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{Ci+1}I_{i+1}\,{}^{i+1}\omega_{i+1}}$$

Iterative Newton-Euler Formulation

- Inwards

$$\boxed{{}^{i}f_i = {}^{\;\;i}_{i+1}R \, {}^{i+1}f_{i+1} + {}^{i}F_i}$$

$$\boxed{{}^{i}n_i = {}^{i}N_i + {}^{\;\;i}_{i+1}R \, {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times \left({}^{\;\;i}_{i+1}R \, {}^{i+1}f_{i+1} \right)}$$

$$\boxed{\tau_i = {}^{i}n_i^T \, {}^{i}\hat{Z}_i}$$

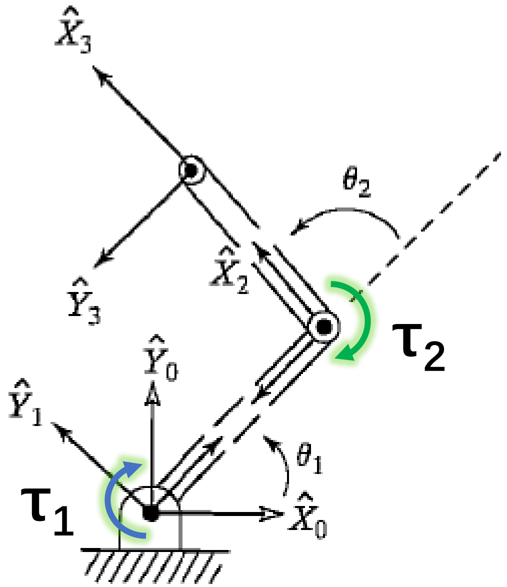
Q5.2 Example of Dynamics

- Given the following 2-link planar manipulator in Q3.4, Given the following two-link planar manipulator, and assuming all the mass exists as a point mass at the distal end of each link, determine the torque required by each motor.

- Previously,
- $^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & l_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $^2_3T = \begin{bmatrix} 1 & 0 & 0 & l_2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- Assume acceleration due to gravity to be g
  - *i.e.* $^0\dot{v}_0 = g\hat{Y}_0$

$$^0\omega_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ ^0v_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ ^0\dot{\omega}_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ ^0\dot{v}_0 = \begin{bmatrix} 0 \\ g \\ 0 \end{bmatrix}$$



Q5.2 Example of Dynamics

$$^{0}_{1}R=\begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{1}_{2}R=\begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$^{0}\omega_0=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad ^{0}v_0=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad ^{0}\dot{\omega}_0=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

$$^{0}\dot{v}_0=\begin{bmatrix}0\\g\\0\end{bmatrix}$$

$$^{i+1}\omega^{0}_{i+1}={}^{i+1}_{i}R\,{}^{i}\omega^{0}_{i}+\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{i+1}\dot{\omega}^{0}_{i+1}={}^{i+1}_{i}R\dot{\omega}+{}^{i+1}_{i}R\,{}^{i}\omega^{0}_{i}\times\dot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}+\ddot{\theta}_{i+1}\,{}^{i+1}\hat{Z}_{i+1}$$

$$^{i+1}v^{0}_{i+1}={}^{i+1}_{i}R\left({}^{i}v^{0}_{i}+{}^{i}\omega^{0}_{i}\times{}^{i}P_{i+1}\right)$$

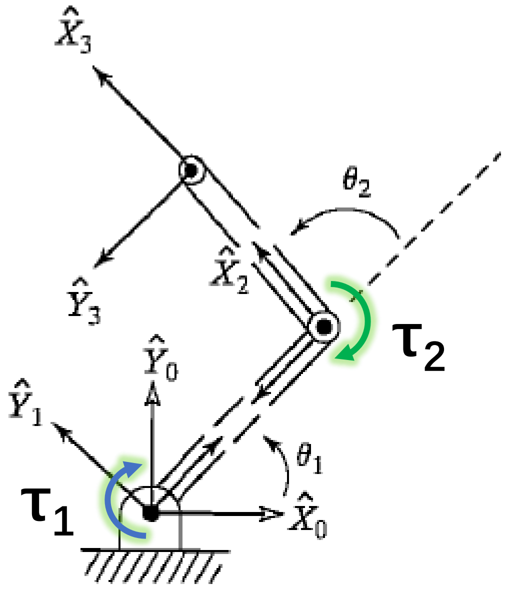
$$^{i+1}\dot{v}^{0}_{i+1}={}^{i+1}_{i}R\left({}^{i}\dot{\omega}^{0}_{i}\times{}^{i}P_{i+1}+{}^{i}\omega^{0}_{i}\times\left({}^{i}\omega^{0}_{i}\times{}^{i}P_{i+1}\right)+{}^{i}\dot{v}^{0}_{i}\right)$$

i=0

$$^{1}\omega_1=\begin{bmatrix}0\\0\\\dot{\theta}_1\end{bmatrix},\quad ^{1}\dot{\omega}_1=\begin{bmatrix}0\\0\\\ddot{\theta}_1\end{bmatrix},\quad ^{1}\dot{v}_1=\begin{bmatrix} c_1 & -s_1 & 0 \\ s_1 & c_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix}0\\g\\0\end{bmatrix}$$

i=1

$$^{2}\omega_2=\begin{bmatrix}0\\0\\\dot{\theta}_1+\dot{\theta}_2\end{bmatrix},\quad ^{2}\dot{\omega}_2=\begin{bmatrix}0\\0\\\ddot{\theta}_1+\ddot{\theta}_2\end{bmatrix},\quad ^{2}\dot{v}_2=\begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix}-l_1\dot{\theta}_1^{\,2}+gs_1\\-l_2\ddot{\theta}_1+gc_1\\0\end{bmatrix}=\begin{bmatrix}l_1\ddot{\theta}_1^{\,2}s_2-l_1\dot{\theta}_1^{\,2}c_2+gs_{12}\\l_1\ddot{\theta}_1^{\,2}c_2-l_1\dot{\theta}_1^{\,2}s_2+gc_{12}\\0\end{bmatrix}$$



Q5.2 Example of Dynamics

$$\boxed{{}^{i}\dot{v}^{0}_{Ci} = {}^{i}\dot{\omega}^{0}_{i} \times {}^{i}P_{Ci} + {}^{i}\omega^{0}_{i} \times \left({}^{i}\omega^{0}_{i} + {}^{i}P_{Ci}\right) + {}^{i}\dot{v}^{0}_{i}}$$

$$\boxed{{}^{i+1}F_{i+1} = m_{i+1}\,{}^{i+1}\dot{v}_{Ci+1}}$$

$$\boxed{{}^{i+1}N_{i+1} = {}^{Ci+1}I_{i+1}\,{}^{i+1}\dot{\omega}_{i+1} + {}^{i+1}\omega_{i+1} \times {}^{Ci+1}I_{i+1}\,{}^{i+1}\omega_{i+1}}$$

For i=0,

$${}^{1}\dot{v}_{C_1} = \begin{bmatrix} gs_1 \\ gc_1 \\ 0 \end{bmatrix} + \begin{bmatrix} -l_1\dot{\theta}_1^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ l_1\ddot{\theta}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -l_1\dot{\theta}_1^2 + gs_1 \\ l_1\ddot{\theta}_1 + gc_1 \\ 0 \end{bmatrix}$$

$${}^{1}F_1 = \begin{bmatrix} -m_1 l_1\dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1\ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix}$$

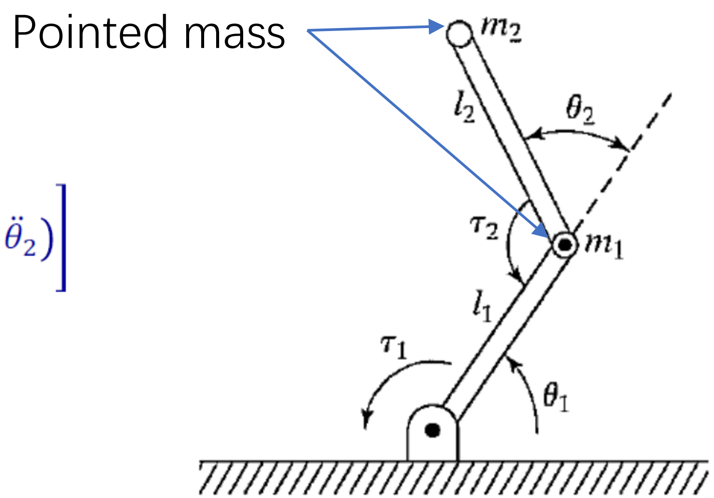
$${}^{1}N_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

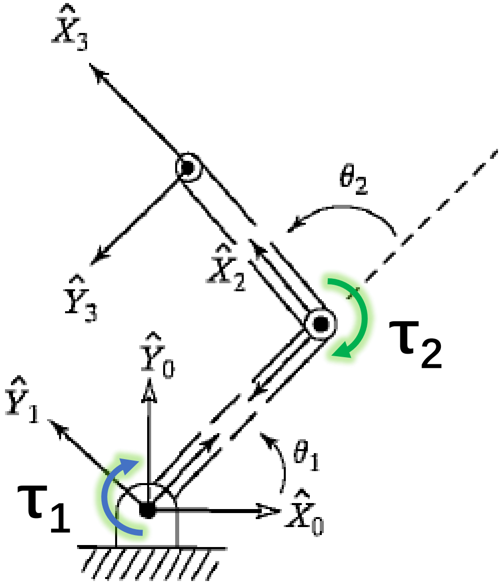
For i=1,

$${}^{2}\dot{v}_{C_2} = \begin{bmatrix} l_1\ddot{\theta}_1^2 s_2 - l_1\dot{\theta}_1^2\, c_2 + g\, s_{12} \\ l_1\ddot{\theta}_1^2 c_2 + l_1\dot{\theta}_1^2\, s_2 + g\, c_{12} \\ 0 \end{bmatrix} + \begin{bmatrix} -l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$${}^{2}F_2 = \begin{bmatrix} m_2 l_1\ddot{\theta}_1 s_2 - m_2 l_1\dot{\theta}_1^2 c_2 + m_2 g s_{12} - m_2 l_2(\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1\ddot{\theta}_1 c_2 + m_2 l_1\dot{\theta}_1^2 s_2 + m_2 g c_{12} + m_2 l_2(\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix}$$

$${}^{2}N_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$





Q5.2 Example of Dynamics

$$\boxed{{}^{i}f_i = {}^{i}_{i+1}R\ {}^{i+1}f_{i+1} + {}^{i}F_i}$$

$$\boxed{{}^{i}n_i = {}^{i}N_i + {}^{i}_{i+1}R\ {}^{i+1}n_{i+1} + {}^{i}P_{C_i} \times {}^{i}F_i + {}^{i}P_{i+1} \times \left({}^{i}_{i+1}R\ {}^{i+1}f_{i+1}\right)}$$

$$\boxed{\tau_i = {}^{i}n_i^T\ {}^{i}\hat{Z}_i}$$

For i =2,

$${}^{2}f_2 = {}^{2}F_2$$

$${}^{2}n_2 = \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 \ddot{\theta}_1 c_2 + m_2 l_1 l_2 \dot{\theta}_1^2 s_2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix}$$

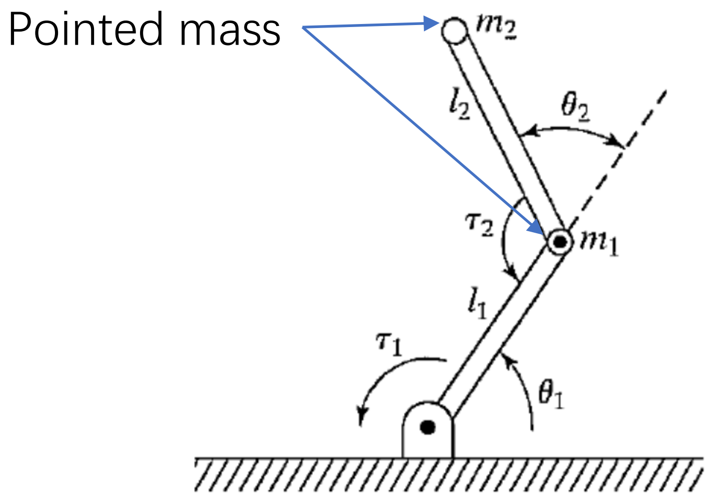
Torque required by motor 2

For i =1,

$${}^{1}f_1 = \begin{bmatrix} c_2 & -s_2 & 0 \\ s_2 & c_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 l_1 s_2 \ddot{\theta}_1 - m_2 l_1 c_2 \dot{\theta}_1^2 + m_2 g s_{12} - m_2 l_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 \\ m_2 l_1 c_2 \ddot{\theta}_1 + m_2 l_1 s_2 \dot{\theta}_1^2 + m_2 g c_{12} + m_2 l_2 (\ddot{\theta}_1 + \ddot{\theta}_2) \\ 0 \end{bmatrix} + \begin{bmatrix} -m_1 l_1 \dot{\theta}_1^2 + m_1 g s_1 \\ m_1 l_1 \ddot{\theta}_1 + m_1 g c_1 \\ 0 \end{bmatrix},$$

$${}^{1}n_1 = \begin{bmatrix} 0 \\ 0 \\ m_2 l_1 l_2 c_2 \ddot{\theta}_1 + m_2 l_1 l_2 s_2 \dot{\theta}_1^2 + m_2 l_2 g c_{12} + m_2 l_2^2 (\ddot{\theta}_1 + \ddot{\theta}_2) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ m_1 l_1^2 \ddot{\theta}_1 + m_1 l_1 g c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ m_2 l_1^2 \ddot{\theta}_1 - m_2 l_1 l_2 s_2 (\dot{\theta}_1 + \dot{\theta}_2)^2 + m_2 l_1 g s_2 s_{12} \\ + m_2 l_1 l_2 c_2 (\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 g c_2 c_{12} \end{bmatrix}.$$

Torque required by motor 1



Dynamic Equation

$$\tau = M(\Theta)\ddot{\Theta} + V(\Theta, \dot{\Theta}) + G(\Theta)$$

$M(\Theta)$ is n x n mass matrix of the manipulator

$V(\Theta, \dot{\Theta})$ is an n x 1 vector of centrifugal and Coriolis terms

$G(\Theta)$ is an n x 1 vector of gravity terms

Dynamic Equation

Using the previous case as the example, $\tau_2 = m_2 l_1 l_2 \ddot{\theta}_1 c_2 + m_2 l_1 l_2 \dot{\theta}_1^2 s_2 + m_2 l_2 g c_{12} + m_2 l_2^2(\ddot{\theta}_1 + \ddot{\theta}_2)$

$$M(\Theta) = \begin{bmatrix} l_2^2 m_2 + 2 l_1 l_2 m_2 c_2 + l_1^2(m_1 + m_2) & l_2^2 m_2 + l_1 l_2 m_2 c_2 \\ l_2^2 m_2 + l_1 l_2 m_2 c_2 & l_2^2 m_2 \end{bmatrix}$$

$$V(\Theta, \dot{\Theta}) = \begin{bmatrix} -m_2 l_1 l_2 s_2 \dot{\theta}_2^2 - 2 m_2 l_1 l_2 s_2 \dot{\theta}_1 \dot{\theta}_2 \\ m_2 l_1 l_2 s_2 \dot{\theta}_1^2 \end{bmatrix}$$

$$G(\Theta) = \begin{bmatrix} m_2 l_2 g c_{12} + (m_1 + m_2) l_1 g c_1 \\ m_2 l_2 g c_{12} \end{bmatrix}$$

Lagrangian Approach

ECE 470: Introduction to Robotics