ECE 470: Introduction to Robotics
Lecture 02

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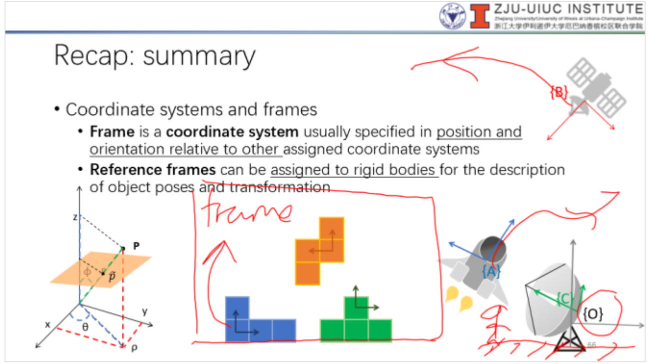
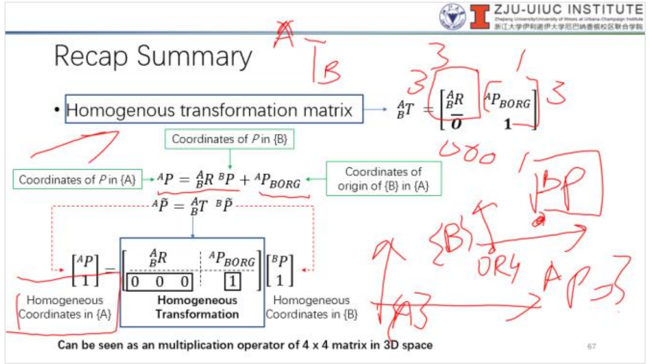
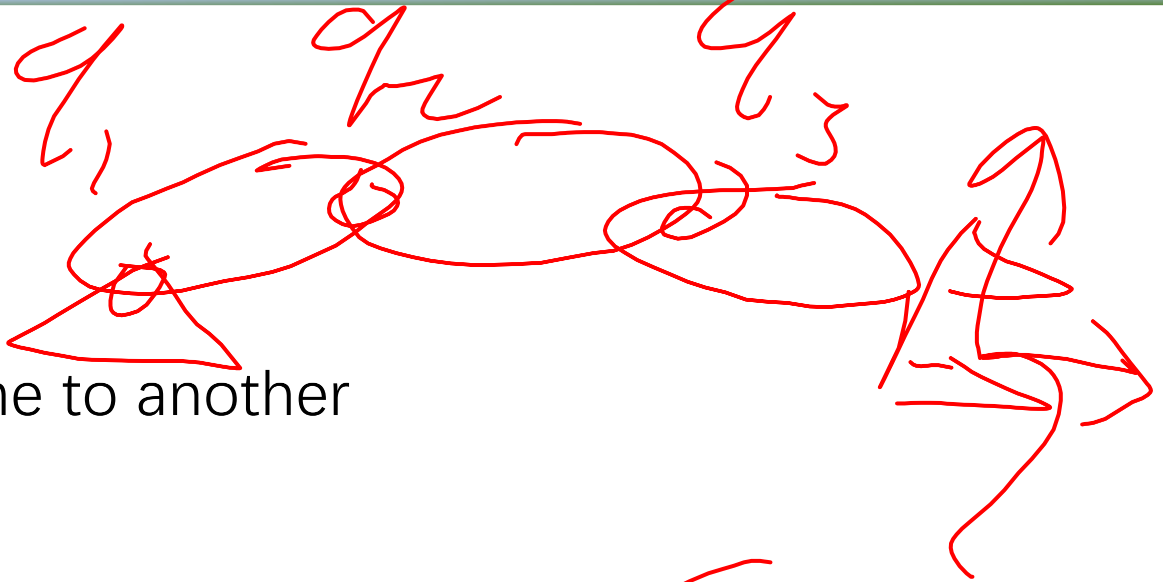
Review on last lecture

- Coordinate System
  - Homogeneous Coordinates
- Spatial representation of Pose/Transformation
  - Reference frame
  - Position/Translation using vector
  - Orientation/Rotation using matrix
  - Pose/Transformation using homogeneous matrix





Overview of this Lecture



- Coordinate transformation
  - Transforming from one reference frame to another
  - Properties of Rotation Matrix
  - Inverse Transformation Matrix
- Introduction to Robot Kinematics
  - Forward kinematics
  - Inverse kinematics

(x, y)

$$(q_1 \quad q_n) \longrightarrow (x, y, \theta)$$

$$(x, y, \theta) \longrightarrow (q_1 \cdots q_n)$$

Recall: Linear Coordinate Transform

trnf

- Transforms a point (*x*, *y*) to (*x*', *y*') such that

$$(x', \quad y') = (ax + by, \quad cx + dy)$$

- A system of linear equations,

$$\begin{aligned} x' &= ax + by \quad \leftarrow 1 \\ y' &= cx + dy \quad \leftarrow 2 \end{aligned}$$

- In matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \begin{pmatrix} x \\ y \end{pmatrix}$$

Coefficient Matrix

Recall: Linear Coordinate Transform

- Vector coordinates as linear combination of basis vectors

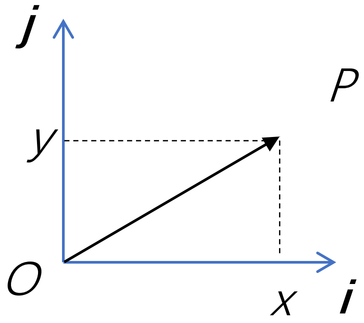
$$\overrightarrow{OP} = x\boldsymbol{i} + y\boldsymbol{j}, \text{ where } \boldsymbol{i} = \begin{pmatrix}1\\0\end{pmatrix}, \text{j}= \begin{pmatrix}0\\1\end{pmatrix}$$

$$\overrightarrow{OP} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$$\begin{array}{cc} \tilde{i} & \tilde{j} \\ \end{array}$$

$${}^{A}P = \begin{pmatrix}\boxed{\begin{matrix}1\\0\end{matrix}} & \boxed{\begin{matrix}0\\1\end{matrix}}\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$$

$${}^{B}P = {}^{B}R_{A}\,{}^{A}P$$



Recall: Linear Coordinate Transform

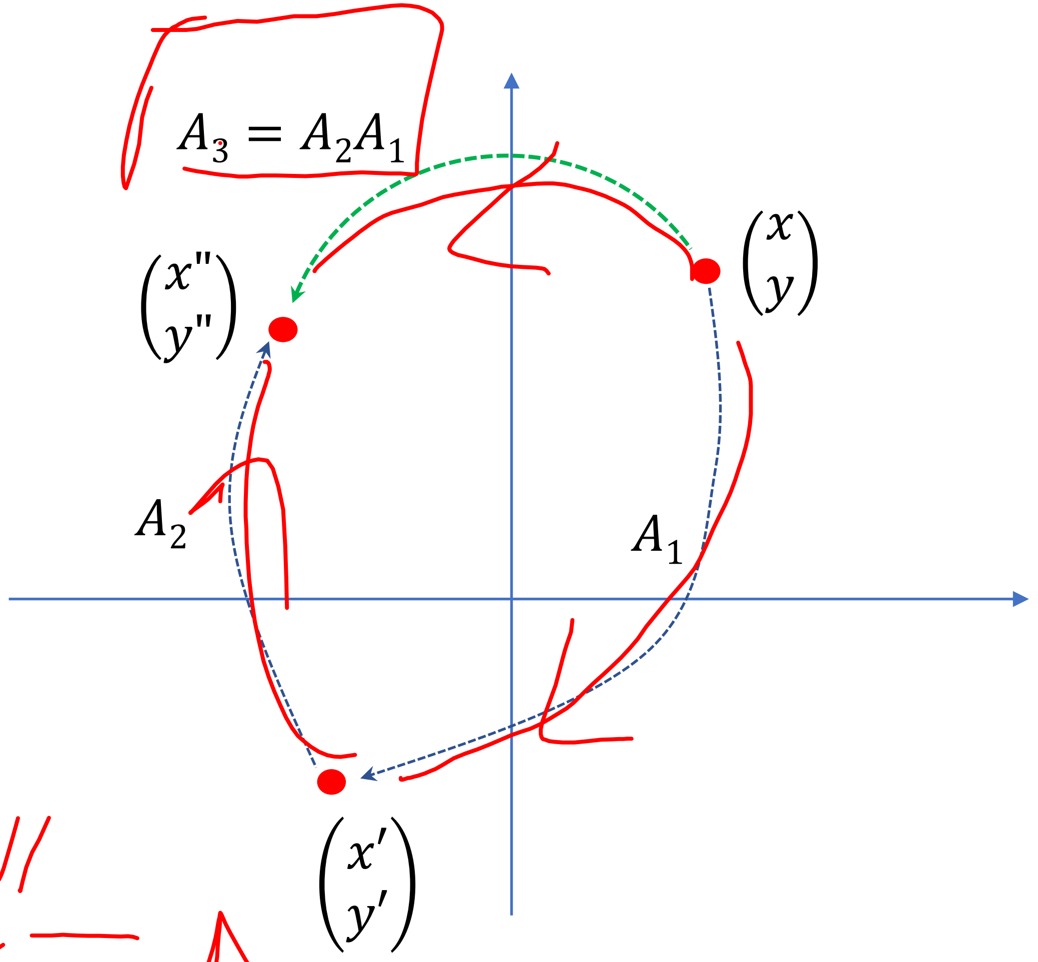
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A_2 = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \quad A_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$A_3 = A_2 A_1 = \begin{pmatrix} ea + fc & eb + fd \\ ga + hc & gb + hd \end{pmatrix}$$



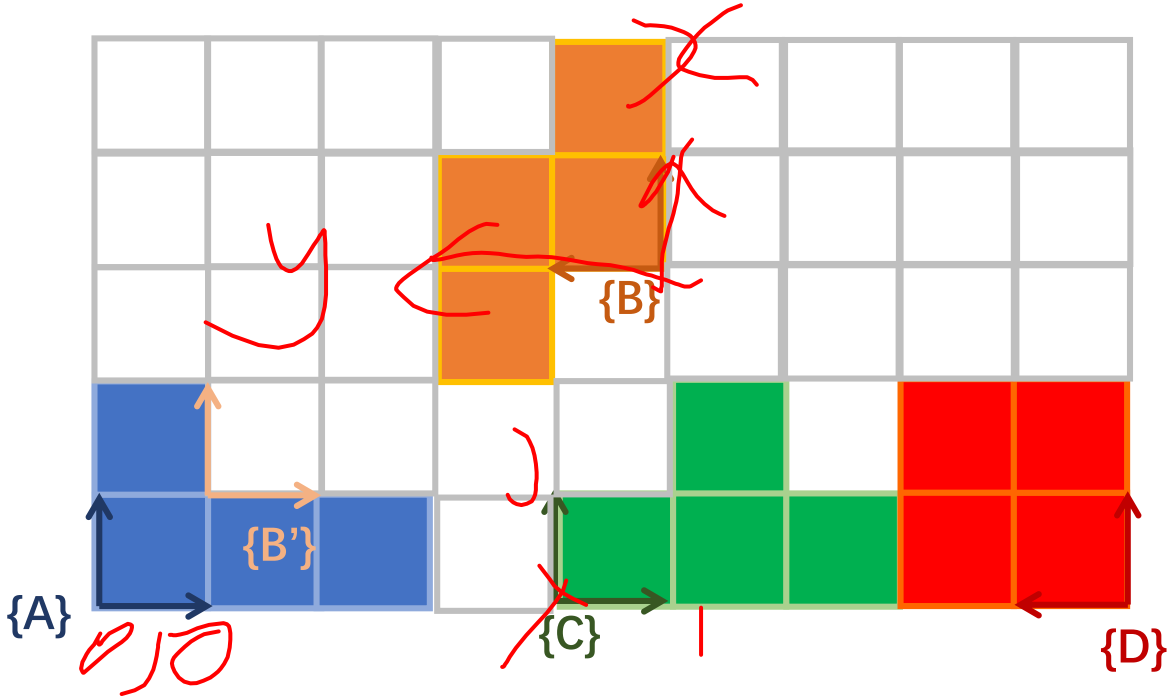
$x'' = A_3 x$

Q1.2: Concept Check

$^A p_C = (4, 0)$

a. Write down the homogeneous transformation $^A T_C$
b. Write down the homogeneous transformation $^C T_B$
c. Find $^A T_B$ using the above two results.

$^C R_B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



$^C p_B = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

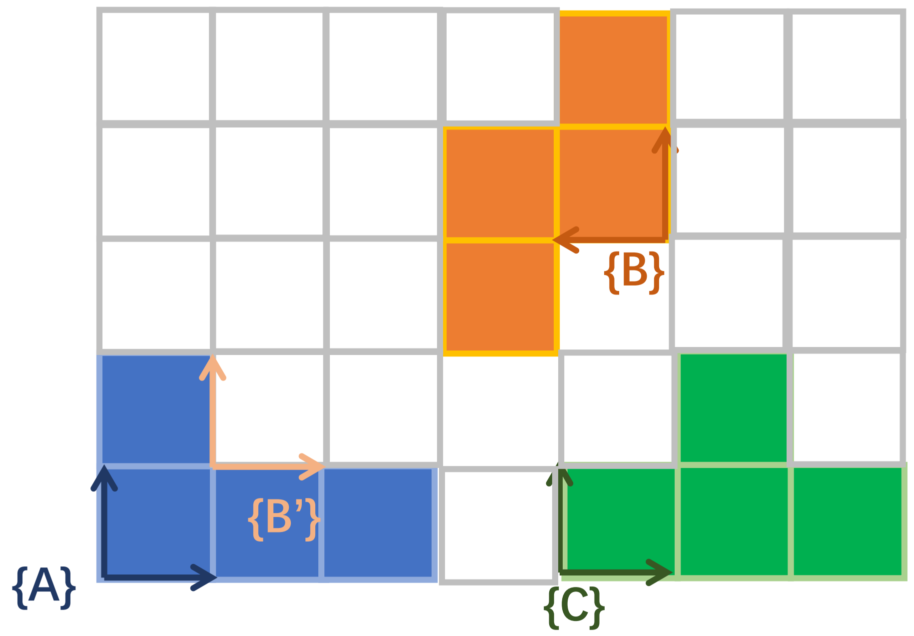
Q1.2: Concept Check

$^{A}T_{B} = {}^{A}T_{C}\,{}^{C}T_{B}$

a. Write down the homogeneous transformation $^{A}T_{C}$

b. Write down the homogeneous transformation $^{C}T_{B}$

c. Find $^{A}T_{B}$ using the above two results.



(a) $^{A}R_{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad {}^{A}P_{C} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} \rightarrow {}^{A}T_{C} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $^{C}R_{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad {}^{C}P_{B} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \rightarrow {}^{C}T_{B} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

(c) $^{A}T_{B} = {}^{A}T_{C}\,{}^{C}T_{B} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

$^{A}R_{B} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \quad {}^{A}P_{B} = \begin{bmatrix} 5 \\ 3 \end{bmatrix} \rightarrow {}^{A}T_{B} = \begin{bmatrix} 0 & -1 & 5 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$

Agrees with Q1.1

Q1.3: Concept Check

a. Is $\mathbf{R_a R_b} \equiv \mathbf{R_b R_a}$? Not true

b. $\mathbf{RR^T}$=? $[I]$

c. What can you conclude from (b)?

$$\boxed{RR^T = I}$$

$$R^T = R^{-1}$$

Q1.3: Concept Check

a. Is $\mathbf{R_a R_b} \equiv \mathbf{R_b R_a}$?

b. $\boxed{\mathbf{RR^T}}$=? $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

c. What can you conclude from (b)?

$R^T \rightarrow$ $\boxed{\text{Transpose}}$

R'

$R R^{-1} = I$

$R^{-1} = R^T$

Properties of Rotation Matrix

- Commutative in 2D space; Not commutative in 3D space
- $\mathbf{R}\mathbf{R}^T=\mathbf{I}$, identity matrix => $\mathbf{R}^T=\mathbf{R}^{-1}$
- Det(**R**)=1
- **R** is normalized: the squares of the elements in any row or column sum to 1
- **R** is orthogonal: the dot product of any pair of rows or any pair of columns is 0
- Rows of **R** represent the coordinates in the original space of unit vectors along the coordinate axes of the rotated space
- Columns of **R** represent the coordinates in the rotated space of unit vectors along the axes of the original space

Q 1.4: Rotation in 3D

{B} is obtained from {A} by rotating θ anti-clockwise about x-axis, which of the following illustrate the correct rotation matrix that maps {B} to {A}?

1. $${}^{A}_{B}R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

2. $${}^{A}_{B}R = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

3. $${}^{A}_{B}R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Standard Rotation in 3D

Rotate θ anti-clockwise about x-axis:

$$ {}^{A}_{B}R_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} $$

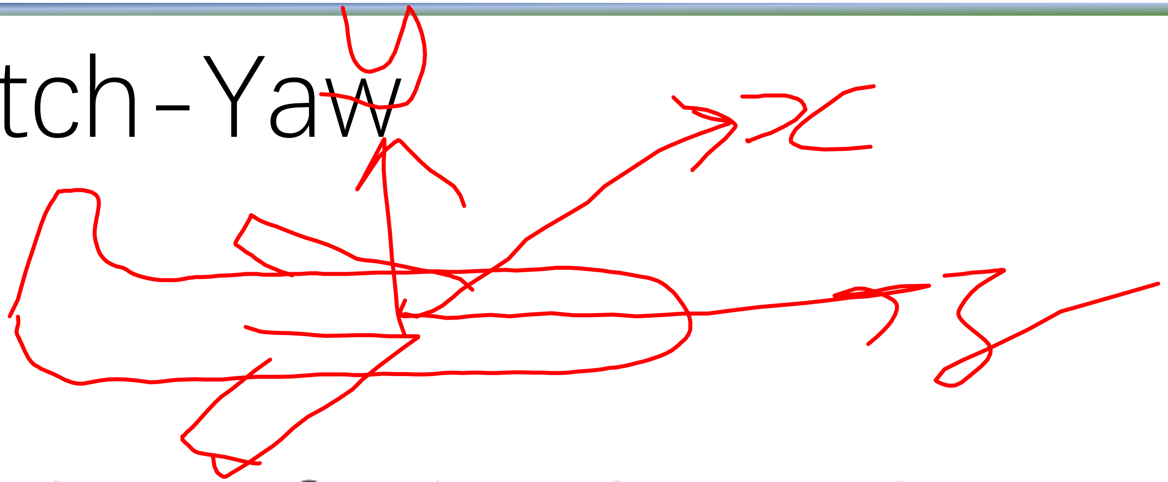
Rotate θ anti-clockwise about y-axis:

$$ {}^{A}_{B}R_Y = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} $$

Rotate θ anti-clockwise about z-axis:

$$ {}^{A}_{B}R_Z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

Rotation in 3D: Roll-Pitch-Yaw



$$
{}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha)
$$

$$
= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{bmatrix}
$$

$$
= \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}
$$

Inverse of Transformation

${}^{A}T_{B} = {}^{B}T_{A} = {}^{A}T_{B}^{-1}$

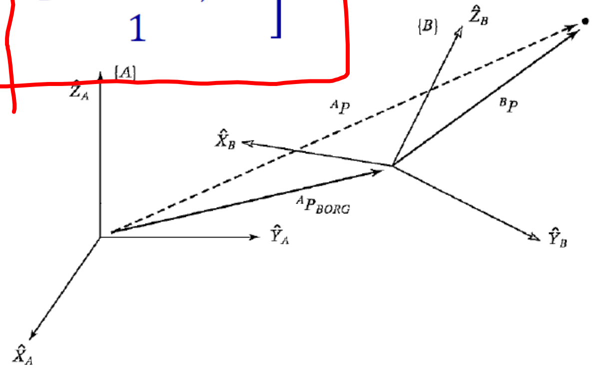
- Inverse of a homogenous transformation, ${}^{A}T_{B}^{-1}$
- ${}^{A}T_{B}^{-1} = {}^{B}T_{A}$
  - Reversing the order of reference

Since $\boxed{{}^{B}_{A}R = {}^{A}_{B}R^{T}\,,}$

$${}^{B}P_{A,ORG} = {}^{B}_{A}R\,.\left(-\,{}^{A}P_{B,ORG}\right) = \boxed{-{}^{A}_{B}R^{T}\,.}\;{}^{A}P_{B,ORG}$$

Hence,

$$\boxed{{}^{B}_{A}T = \begin{bmatrix} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,.\;{}^{A}P_{B,ORG} \\ 0\quad 0\quad 0 & 1 \end{bmatrix}}$$



Inverse of Transformation

- Inverse of a homogenous transformation, ${}^{A}T_{B}^{-1}$
- ${}^{A}T_{B}^{-1} = {}^{B}T_{A}$
  - Reversing the order of reference
- ${}^{B}P = \begin{bmatrix} {}^{A}_{B}R^{\mathrm{T}} & -{}^{A}_{B}R^{\mathrm{T}}\,{}^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} {}^{A}P$
- ${}^{B}T_{A} = \begin{bmatrix} {}^{A}_{B}R^{\mathrm{T}} & -{}^{A}_{B}R^{\mathrm{T}}\,{}^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}$

Recall that ${}^{A}P = {}^{A}_{B}R\,{}^{B}P + {}^{A}P_{BORG}$

$${}^{A}P - {}^{A}P_{BORG} = {}^{A}_{B}R\,{}^{B}P$$

$${}^{A}_{B}R^{-1}({}^{A}P - {}^{A}P_{BORG}) = {}^{B}P$$

$${}^{B}P = {}^{A}_{B}R^{\mathrm{T}}\,({}^{A}P - {}^{A}P_{BORG})$$

$${}^{B}P = {}^{A}_{B}R^{\mathrm{T}}\,{}^{A}P - {}^{A}_{B}R'\,{}^{A}P_{BORG}$$

$${}^{B}P = \begin{bmatrix} {}^{A}_{B}R^{\mathrm{T}} & -{}^{A}_{B}R^{\mathrm{T}}\,{}^{A}P_{BORG} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix} {}^{A}P$$

Robot Kinematics

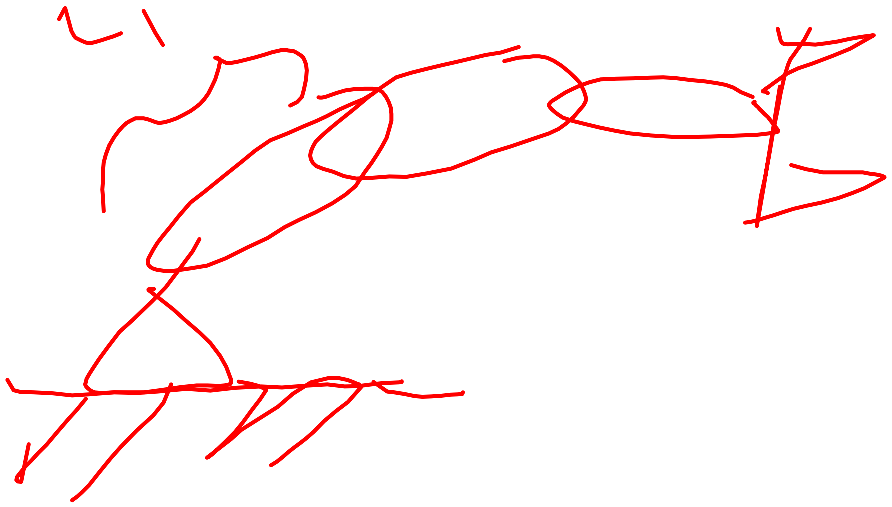
Introduction to Robotics: Fundamentals

Kinematics

- Introduction to Robotic Mechanism
- Frame Assignment for Multi-Body Systems
- Forward Kinematics in Manipulators
- Inverse Kinematics in Manipulators
- Velocity Kinematics
- Jacobian, Velocity and Static Force

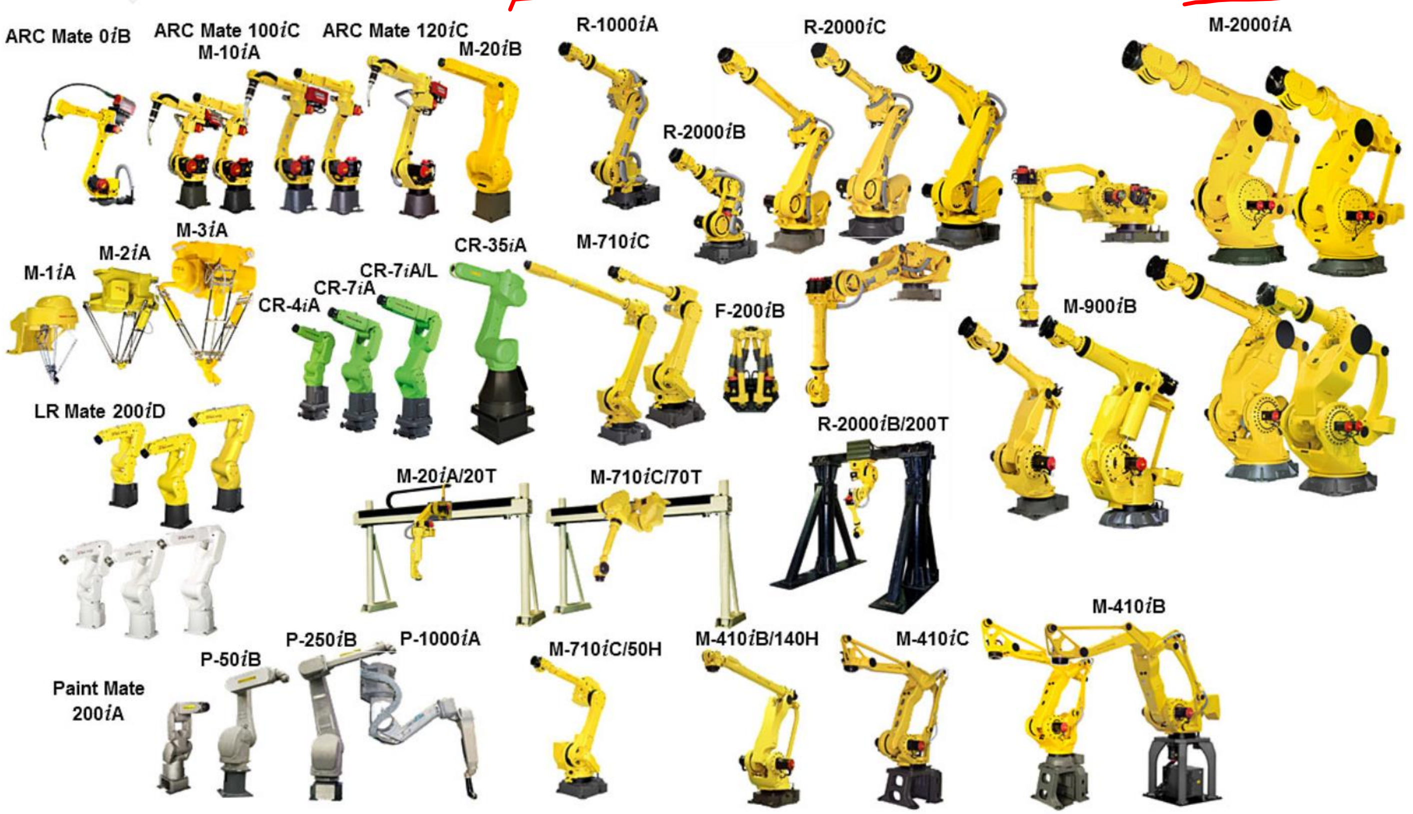
Kinematics

- Kinematics is the science of motion that treats the subject without regard to the forces that cause it.
- Manipulator kinematics
  - Pose of the manipulator linkages in static situations
  - Analyze motion of manipulator (linear and angular velocity of bodies)



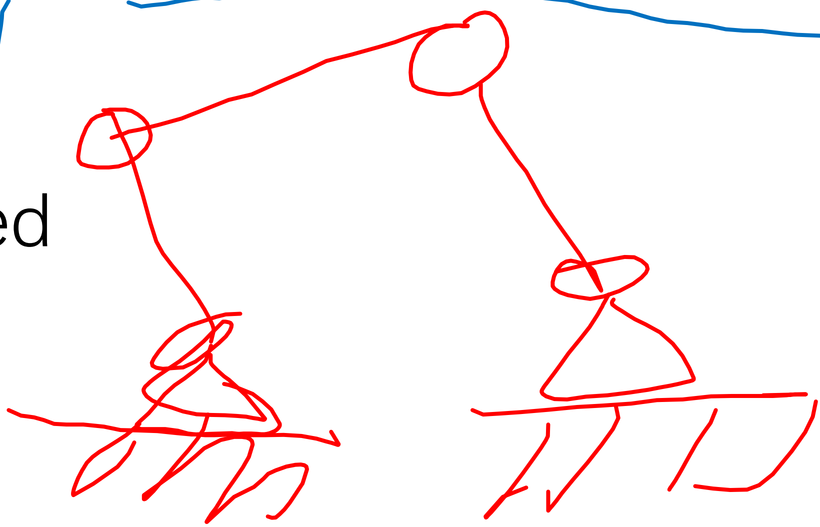
Manipulator Kinematics

- a set of rigid bodies (called **links**) connected (by **joints**) in a chain

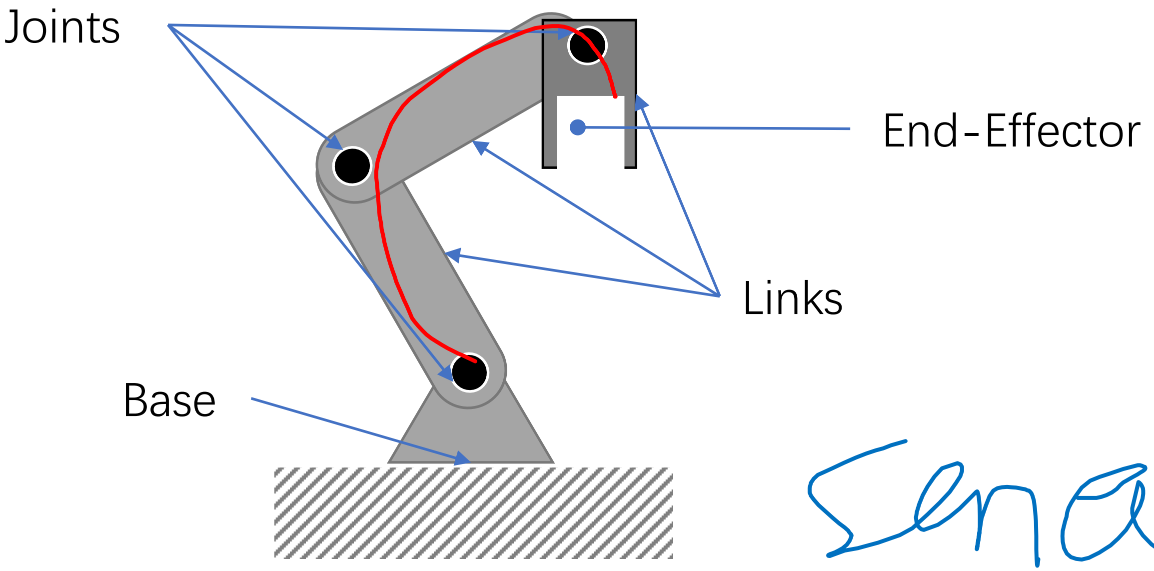


Manipulator Kinematics

- Serial Arm Manipulator will be discussed



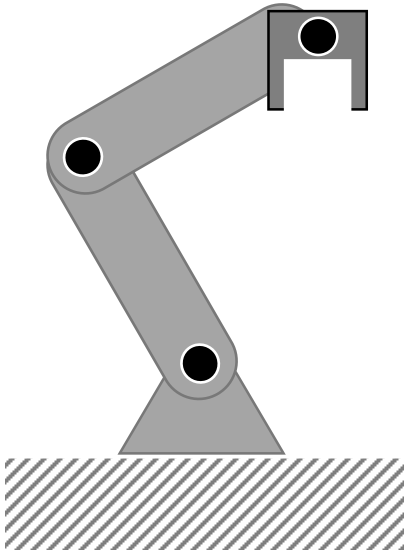
Closed-chain



Serial

Degree-of-Freedom (DOF)

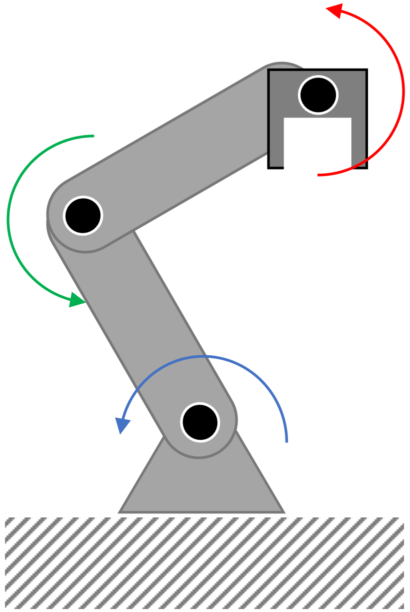
- DOF of a system of bodies
  - number of independent parameters (generalized coordinates) required to fully describe the pose (configuration) of a system of bodies



Q2.1 Concept Check: DOF

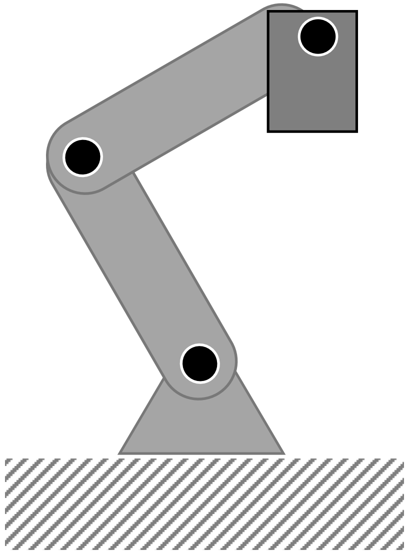
- RRR (Revolute-Revolute-Revolute) serial arm
- What is the number of DOF for the serial arm?

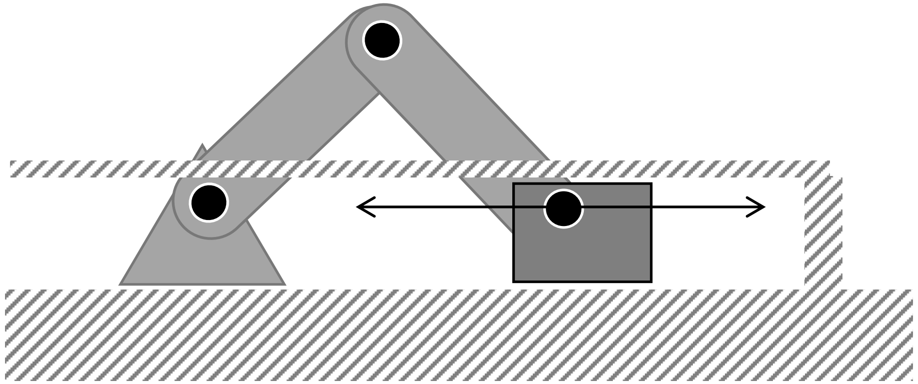
3



Degree-of-Freedom (DOF)

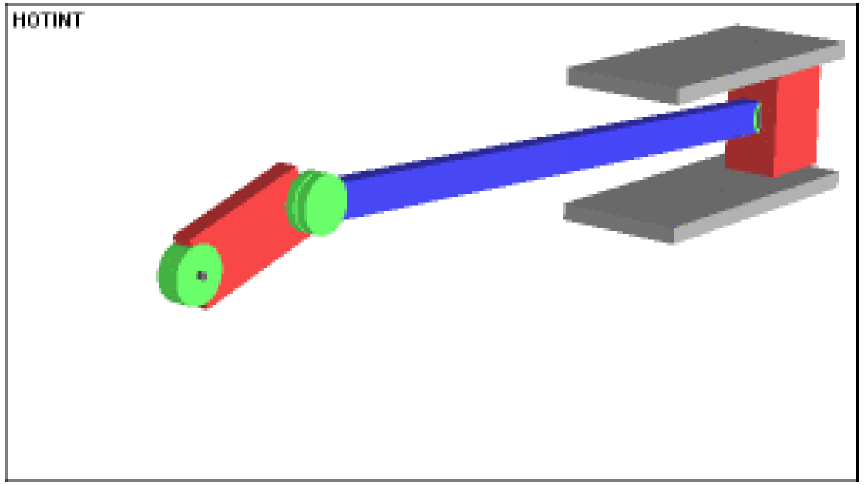
- DOF of a system of bodies
  - number of independent parameters (generalized coordinates) required to fully describe the pose (configuration) of a system of bodies
- May operate in a constrained task space

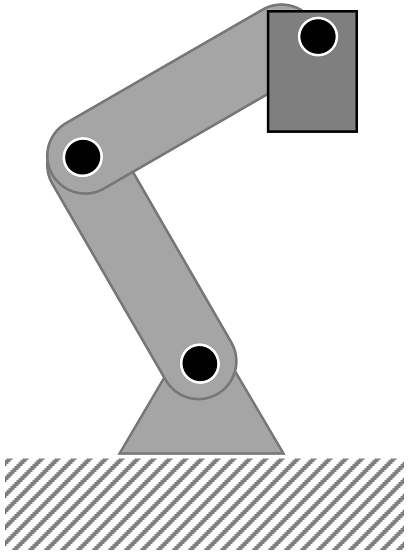


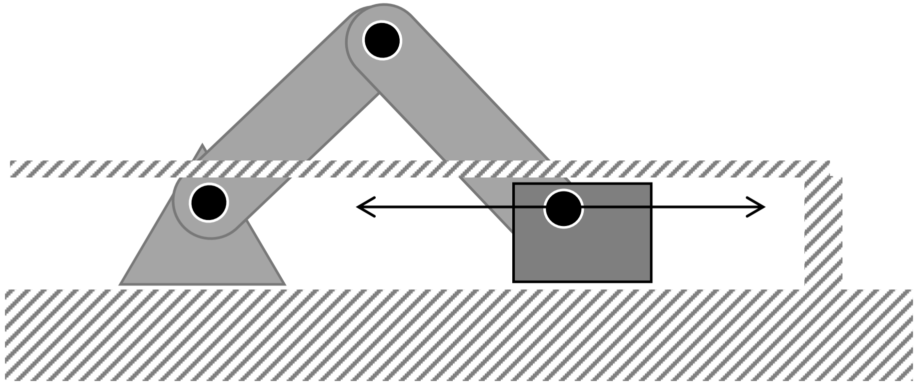


Degree-of-Freedom (DOF)

- May operate in a constrained task space

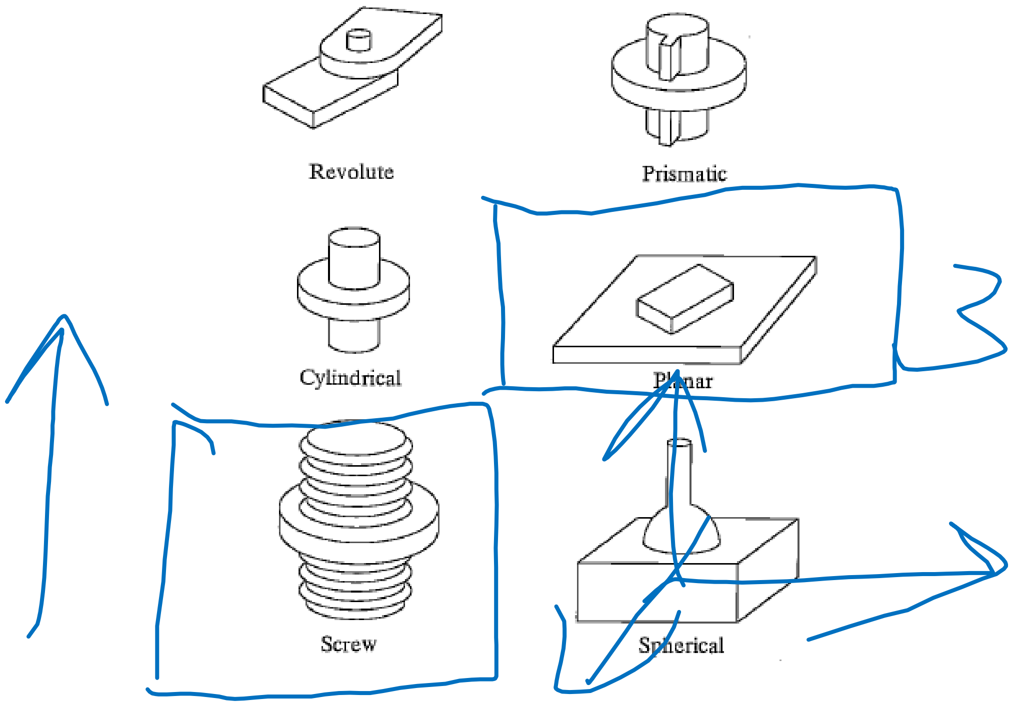






Links and Joints

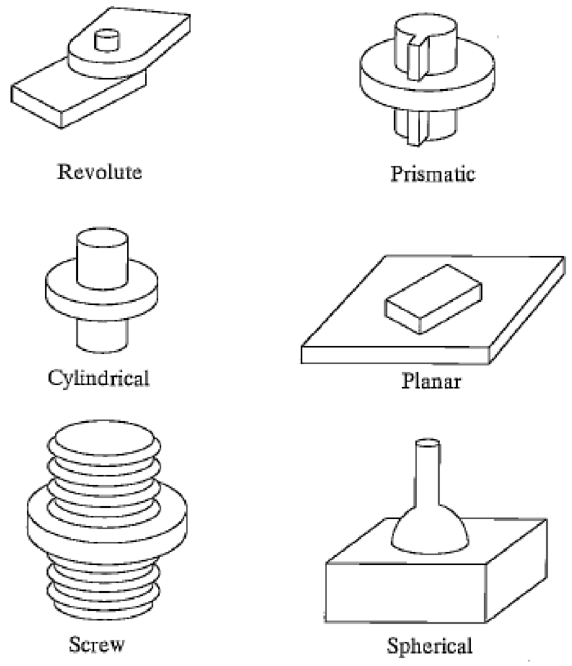
- Links and Joints
  - **Links are rigid bodies** that can move in the DOF provided by the **joints connecting them**



Q2.2: Concept Check

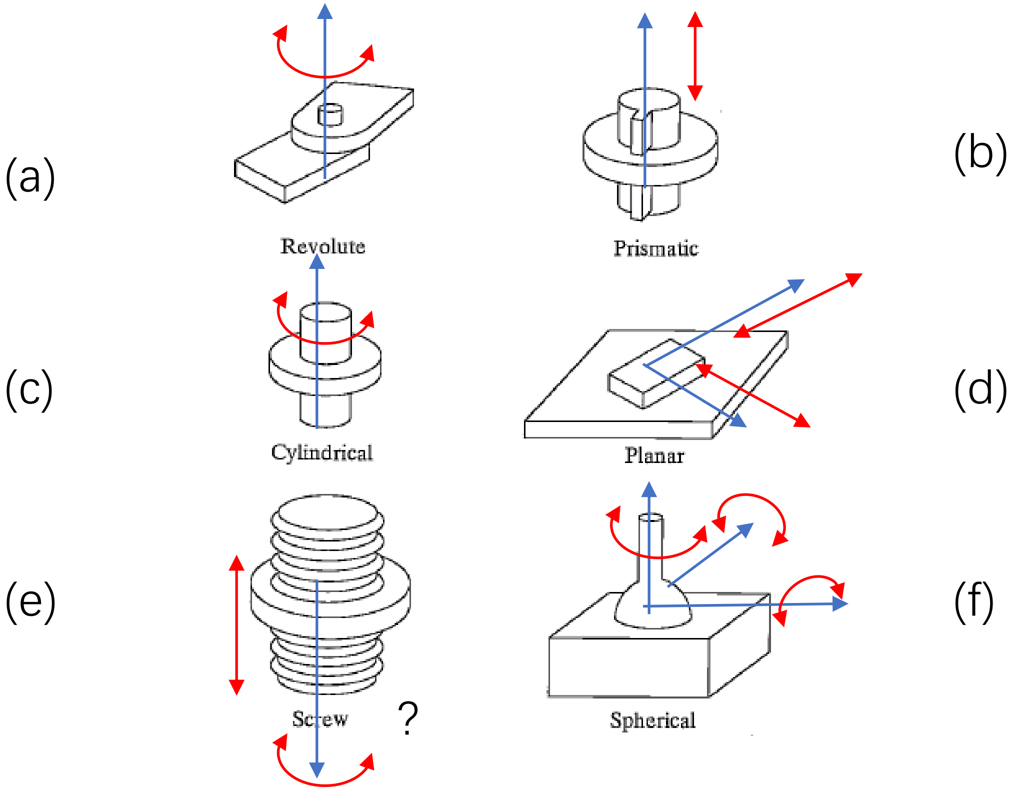
- What is the number of DOF and how do you think they move?

(a) (b) (c) (d) (e) (f)



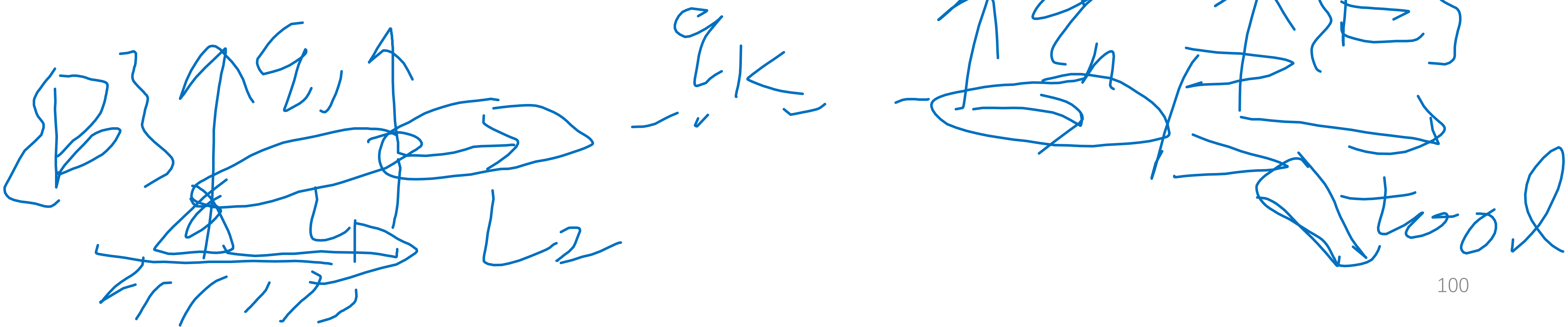
Q2.2: Concept Check

- Links and Joints



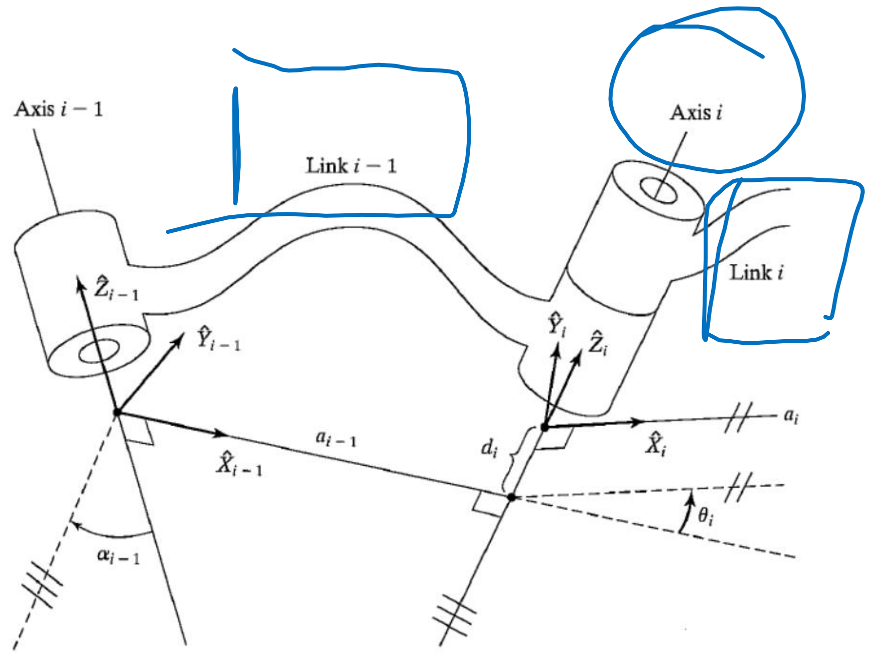
Denavit-Hartenberg (D-H) Convention

- A method to represent the kinematics of a serial arm manipulator
  - For a manipulator with N joints numbered from 1 to N, there are $N + 1$ links, numbered from 0 to N.
  - Joint i connects link $i - 1$ to link i and moves them relative to each other. It follows that link i connects joint i to joint $i + 1$.
  - Link 0 is the base of the robot, typically fixed and link N, the last link of the robot, carries the end-effector or tool.



D-H: Frame Assignment

Frame { *i* } is attached rigidly to link *i*
Frame { *i* } can move relative to Frame { *i* -1} about/along joint *i*



D-H: Notations

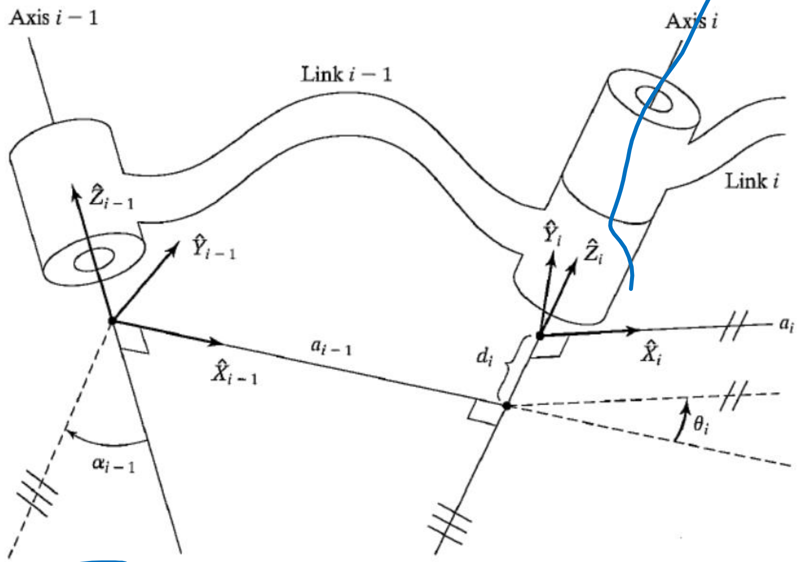
α_{i-1}: Angle from $\hat{Z}_{i-1}$ to $\hat{Z}_i$ measured about $\hat{X}_{i-1}$ (Link Twist)

a_{i-1}: Distance from $\hat{Z}_{i-1}$ to $\hat{Z}_i$ measured along $\hat{X}_{i-1}$ (Link Length)

θ_i: Angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$ (Joint Angle)

d_i: Distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$ (Link offset)

$$^{i-1}_{\ \ i}T = R_x(\alpha_{i-1}) D_x(a_{i-1})\ R_z(\theta_i) D_z(d_i)$$



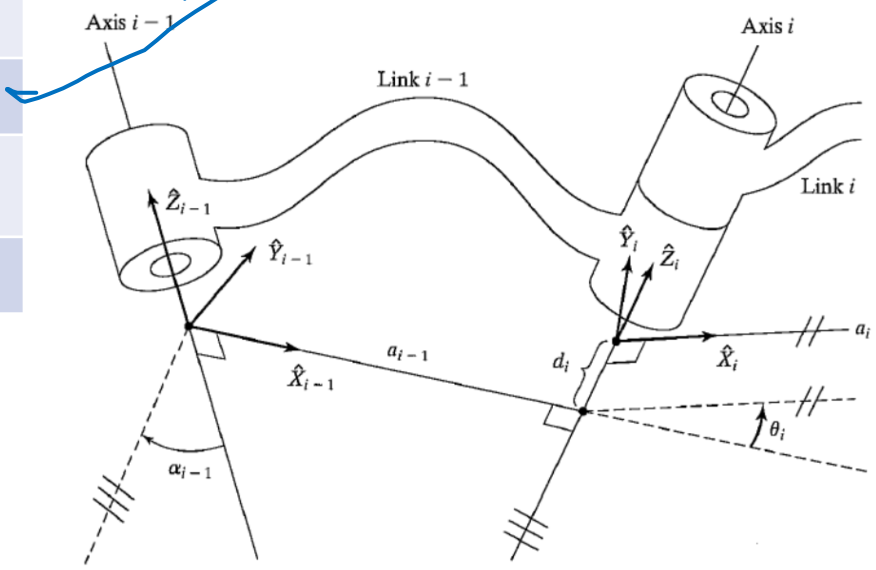
Z_{i-1}

Z_i

D-H Table

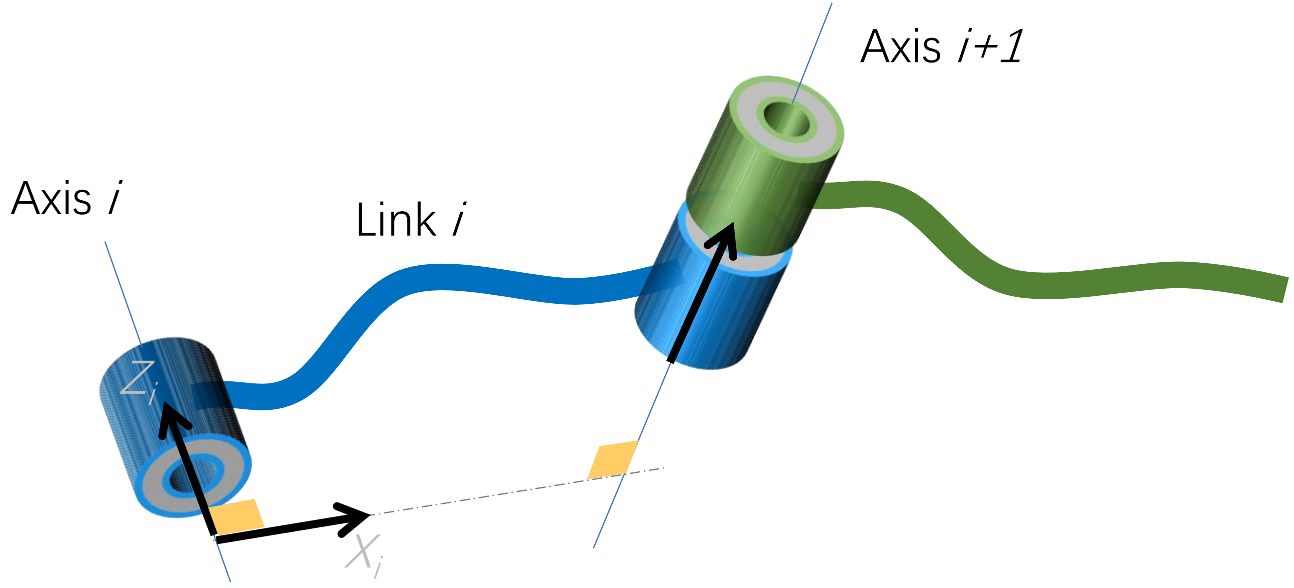
	Link Twist α_{i-1}	Link Length a_{i-1}	Joint Angle θ_i	Link offset d_i
1	α_0	a_0	θ_1	d_1
⋮	⋮	⋮	⋮	⋮
i	α_{i-1}	a_{i-1}	θ_i	d_i
⋮	⋮	⋮	⋮	⋮
N	α_{N-1}	a_{N-1}	θ_N	d_N

$$ {}^{0}_{N}T = {}^{0}_{1}T\,{}^{1}_{2}T \dots\dots {}^{i-1}_{i}T\,{}^{i}_{i+1}T \dots\dots {}^{N-2}_{N-1}T\,{}^{N-1}_{N}T $$



Summary: DH Frame Assignment

1. Identify the joint axes and attach infinite lines along them. For neighboring pair (*i* and *i*+1)
2. Identify the common perpendicular or point of intersection. At the point of intersection, or at the point where the common perpendicular meets i^{th} axis, assign the link-frame origin.
3. Assign the Z_i axis pointing along the i^{th} joint axis.
4. Assign the X_i axis pointing along the direction normal to the two neighboring Z-axes.
5. Assign the Y_i axis to complete a right-hand coordinate system.
6. Assign {0} to match {1}. For {N}, choose an origin location and X direction freely, but generally so as to cause as many linkage parameters as possible to become zero.

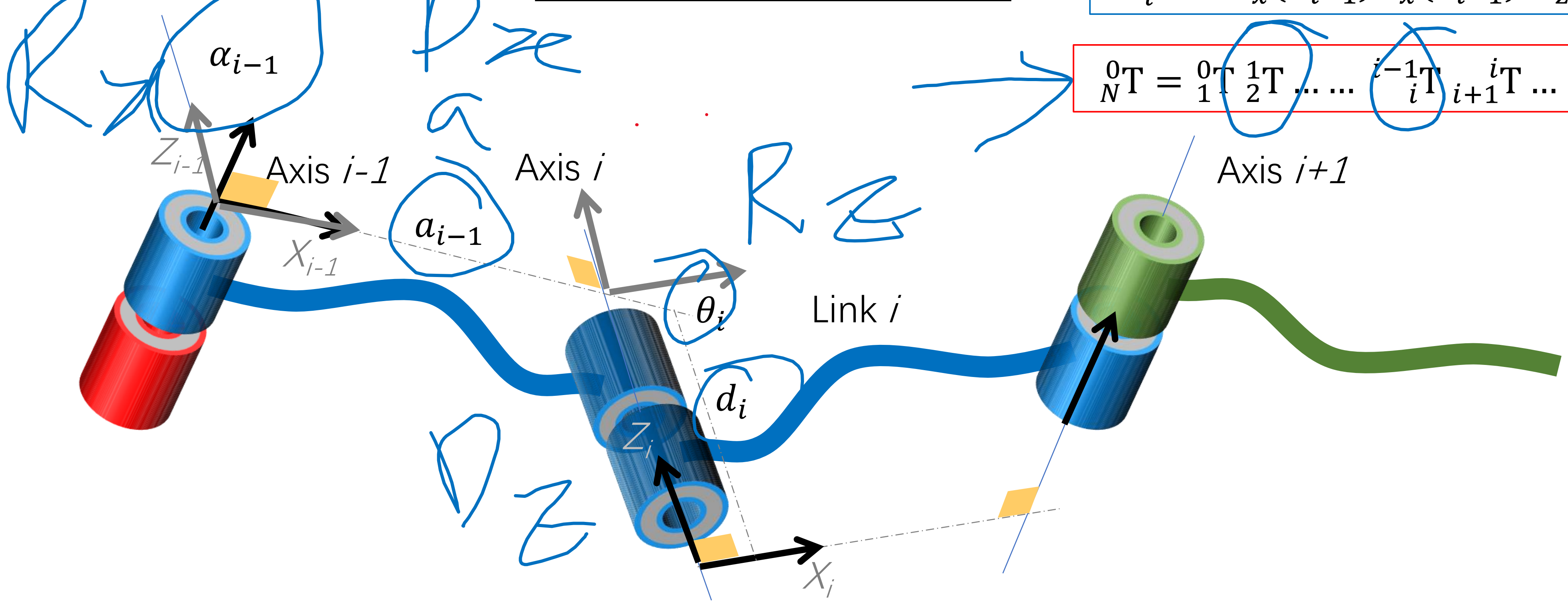


Kinematics Representation in Homogeneous Transformation

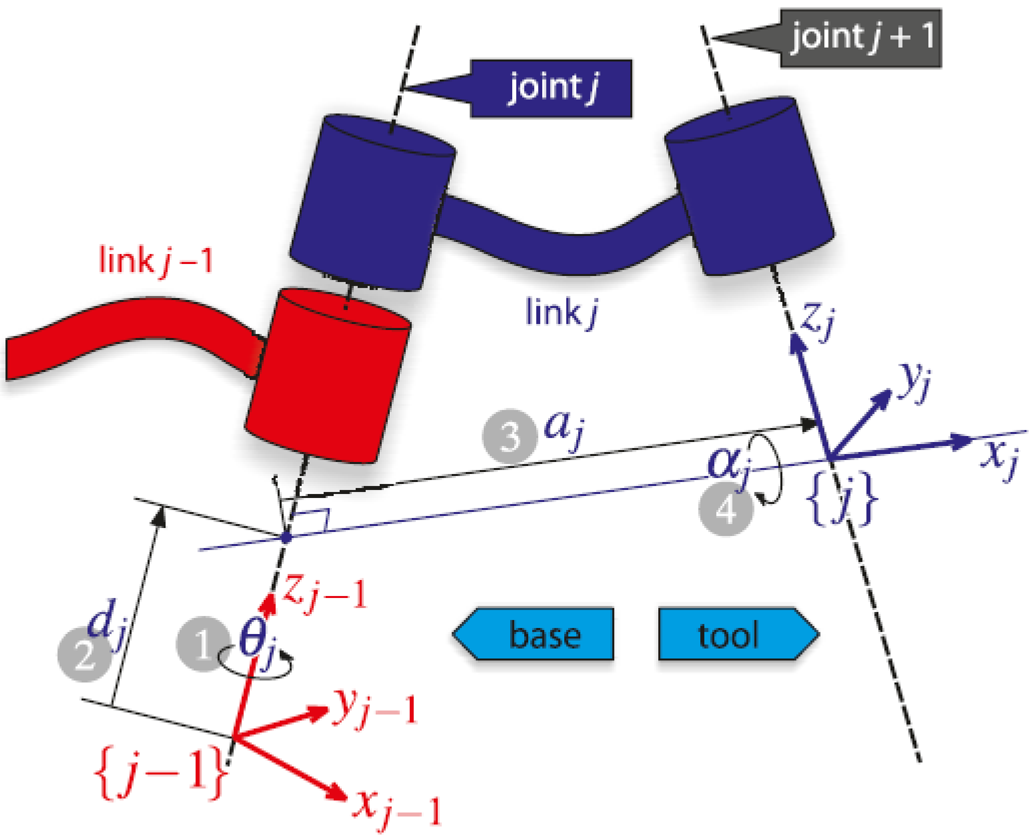
1. Schematic of Serial Arm
2. Establish the DH parameters
3. Tabulate on the DH table
4. Obtain the relevant transformation matrix

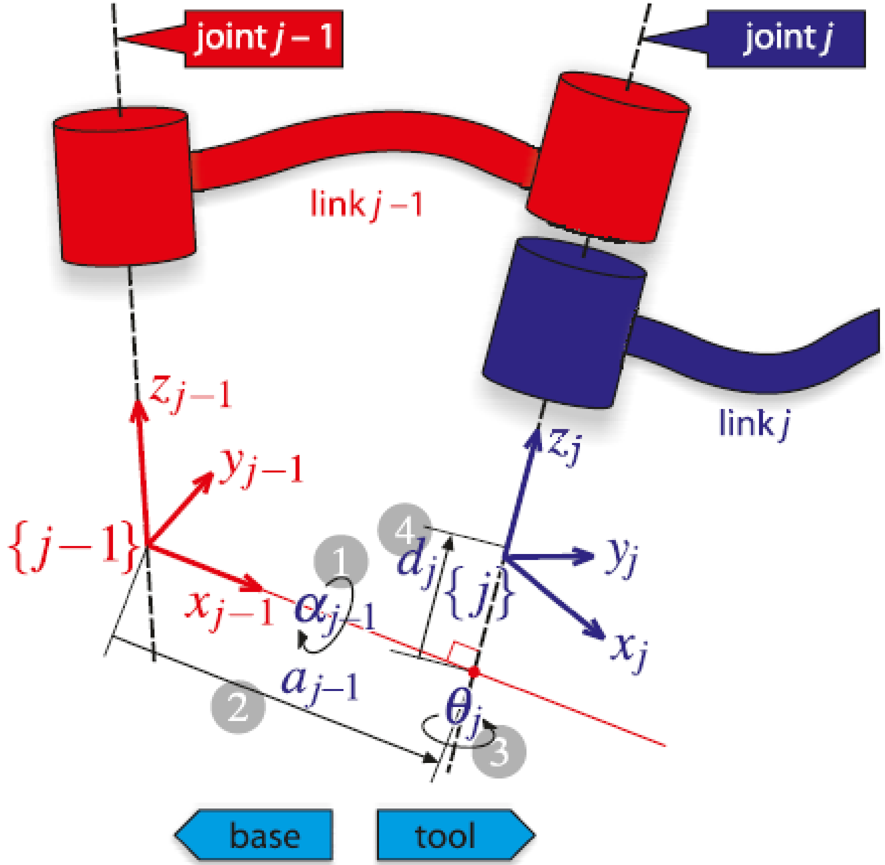
$$ {}^{i-1}_{\ \ i}T = R_x(\alpha_{i-1})D_x(a_{i-1})\ R_z(\theta_i)D_z(d_i) $$

$$ {}^{0}_{N}T = {}^{0}_{1}T\ {}^{1}_{2}T \dots \dots {}^{i-1}_{\ \ i}T\ {}^{\ \ i}_{i+1}T \dots \dots {}^{N-2}_{N-1}T\ {}^{N-1}_{\ \ N}T $$

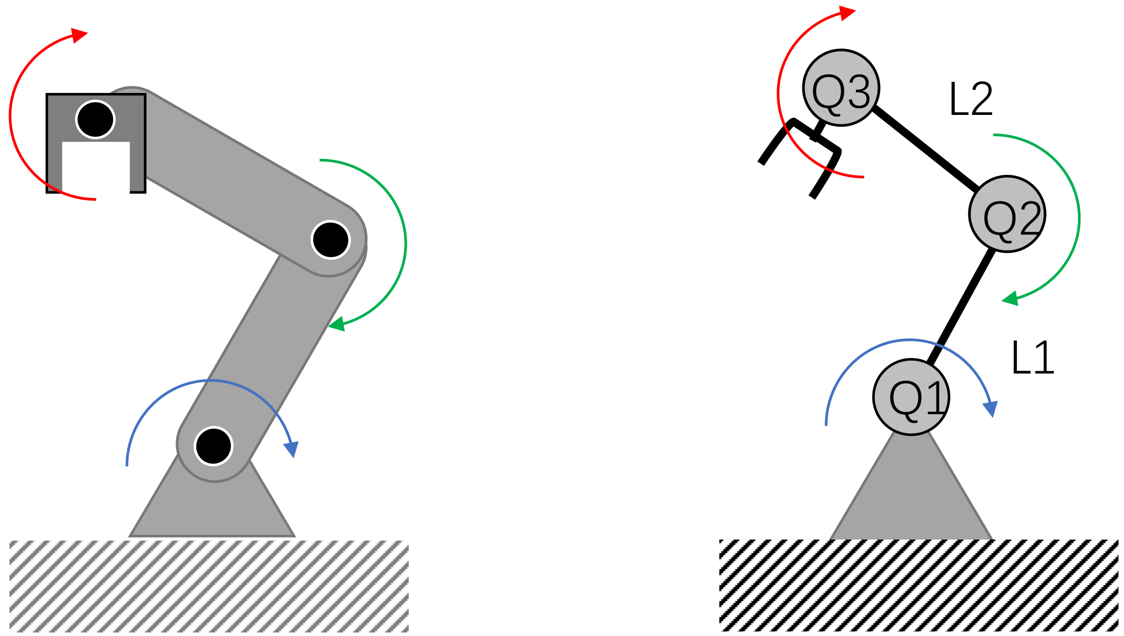


Modified vs. Traditional Convention



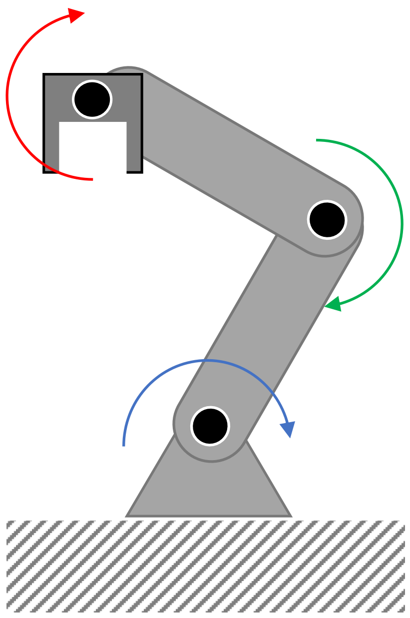


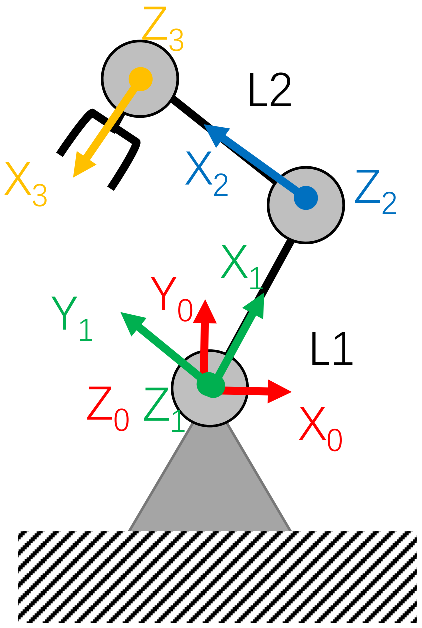
Q2.3: Example on an RRR manipulator



Q2.3: Example on an RRR manipulator

1. Schematic Diagram





2. Frame Assignment

3. DH Parameters & Table

	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	$Q1 = \theta_1$	0
2	0	$L1$	$Q2 = \theta_2$	0
3	0	$L2$	$Q3 = \theta_3$	0

4. Homogenous Transformation

$${}^{i-1}_{\ \ i}\mathrm{T} = R_x(\alpha_{i-1})D_x(\alpha_{i-1})\ R_z(\theta_i)D_z(d_i)$$

$${}^{0}_{3}\mathrm{T} = {}^{0}_{1}\mathrm{T}\ {}^{1}_{2}\mathrm{T}\,{}^{2}_{3}\mathrm{T}$$

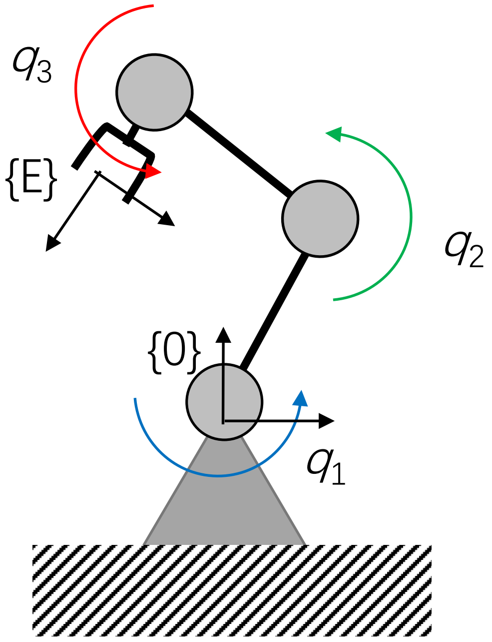
See Textbook Example 3.3, Craig 3rd Ed

Forward Kinematics

Introduction to Robotics: Fundamentals

Forward Kinematics

- **Forward kinematics** is the mapping from joint coordinates, or robot configuration to end-effector pose



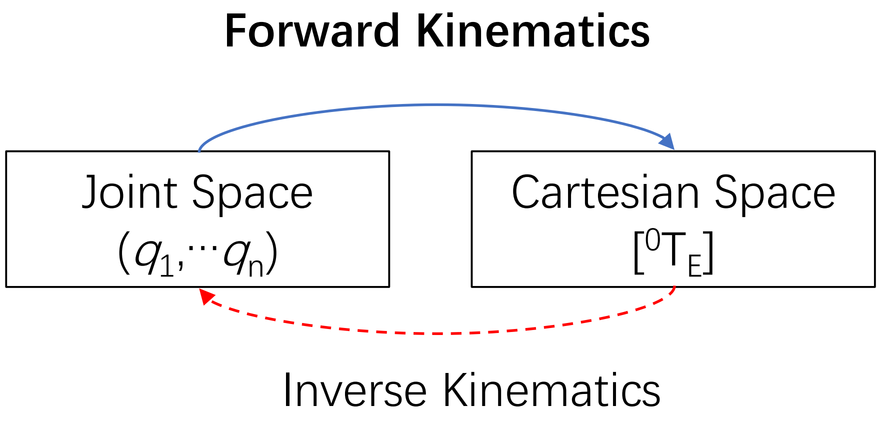
${}^0T_E = F(\mathbf{Q})$,
where $\mathbf{Q} = (q_1, \cdots q_n)$ is the joint coordinate

$$ {}^0_E T = {}^0_1 T \, {}^1_2 T \, {}^2_3 T \, {}^3_E T $$

$$ {}^0_E T = {}^0_1 T(q_1) \cdot {}^1_2 T(q_2) \cdot {}^2_3 T(q_3) \cdot {}^3_E T $$

Mapping between Kinematics Description

- **Forward kinematics** is the mapping from joint coordinates, or robot configuration to end-effector pose



Q2.3: Example on an RRR manipulator

R P

$$
{}^0_1T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

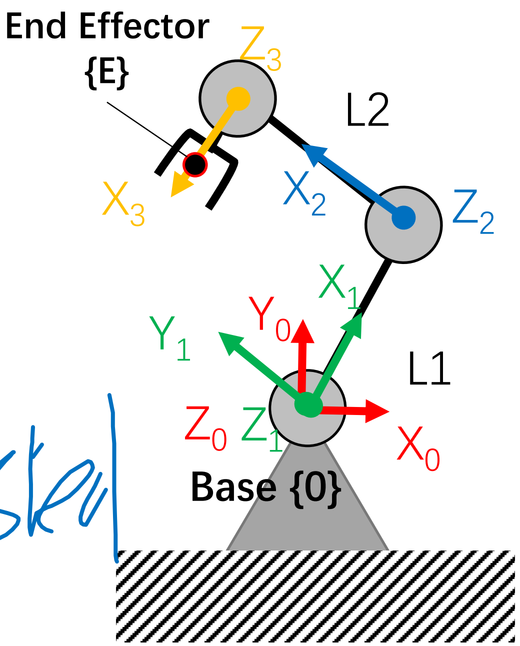
$$
{}^1_2T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & L_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^2_3T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & L_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^3_ET = \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Interested

1. Schematic Diagram



2. Frame Assignment

3. DH Parameters & Table

	α_{i-1}	a_{i-1}	θ_i	d_i
1	0	0	$Q1 = \theta_1$	0
2	0	$L1$	$Q2 = \theta_2$	0
3	0	$L2$	$Q3 = \theta_3$	0

4. Homogenous Transformation

$$
{}^{i-1}_{\ \ i}\mathrm{T} = R_x(\alpha_{i-1})D_x(\alpha_{i-1})\ R_z(\theta_i)D_z(d_i)
$$

$$
{}^0_3\mathrm{T} = {}^0_1\mathrm{T}\,{}^1_2\mathrm{T}\,{}^2_3\mathrm{T}
$$

5. Forward Kinematics

$$
{}^0_E\mathrm{T} = {}^0_1\mathrm{T}(\theta_1)\cdot{}^1_2\mathrm{T}(\theta_2)\cdot{}^2_3\mathrm{T}(\theta_3)\cdot{}^3_E\mathrm{T}
$$

$$
{}^0_E\widetilde{\mathrm{P}} = {}^0_1\mathrm{T}(\theta_1)\cdot{}^1_2\mathrm{T}(\theta_2)\cdot{}^2_3\mathrm{T}(\theta_3)\cdot{}^3_E\widetilde{\mathrm{P}}
$$

Given

Q2.4: Example Forward Kinematics

The serial arm in Q2.3 with the following assigned frames has known link parameters (L1, L2, L3)= (1, 1, 0.5). Find the position of the end effector relative to the base {0} , ${}^{0}P_{E}$ given joint coordinates of (45°, -90°, 45°).

Forward Kinematics

$$ {}^{0}_{E}\mathrm{T} = {}^{0}_{1}\mathrm{T}(\theta_1)\cdot {}^{1}_{2}\mathrm{T}(\theta_2)\cdot {}^{2}_{3}\mathrm{T}(\theta_3)\cdot {}^{3}_{E}\mathrm{T} $$

$$ {}^{0}_{E}\widetilde{\mathrm{P}} = {}^{0}_{1}\mathrm{T}(\theta_1)\cdot {}^{1}_{2}\mathrm{T}(\theta_2)\cdot {}^{2}_{3}\mathrm{T}(\theta_3)\cdot {}^{3}_{E}\widetilde{\mathrm{P}} $$

$$ {}^{0}_{1}T = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 & 0 \\ s\theta_1 & c\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{1}_{2}T = \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 & L_1 \\ s\theta_2 & c\theta_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{2}_{3}T = \begin{bmatrix} c\theta_3 & -s\theta_3 & 0 & L_2 \\ s\theta_3 & c\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{3}_{E}T = \begin{bmatrix} 1 & 0 & 0 & L3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{E}\widetilde{\mathrm{P}} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot {}^{3}_{E}\widetilde{\mathrm{P}} $$

$$ {}^{0}_{E}\widetilde{\mathrm{P}} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot {}^{3}_{E}\widetilde{\mathrm{P}} $$

$$ {}^{0}_{E}\widetilde{\mathrm{P}} = \begin{bmatrix} 1 & 0 & 0 & 2/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5+\sqrt{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} $$

Q2.4: Example Forward Kinematics

The serial arm in Q2.3 with the following assigned frames has known link parameters (L1, L2, L3)= (1, 1, 0.5). Find the position of the end effector relative to the base {0} , ${}^{0}P_{E}$ given joint coordinates of (45°, -90°, 45°).

Forward Kinematics

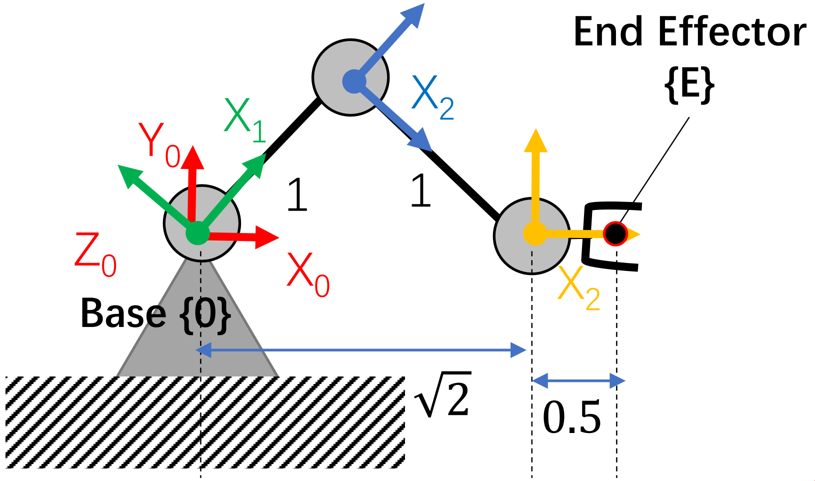
$$ {}^{0}_{E}T = {}^{0}_{1}T(\theta_1)\cdot {}^{1}_{2}T(\theta_2)\cdot {}^{2}_{3}T(\theta_3)\cdot {}^{3}_{E}T $$

$$ {}^{0}_{E}\tilde{P} = {}^{0}_{1}T(\theta_1)\cdot {}^{1}_{2}T(\theta_2)\cdot {}^{2}_{3}T(\theta_3)\cdot {}^{3}_{E}\tilde{P} $$

$$ {}^{0}_{E}\tilde{P} = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot {}^{3}_{E}\tilde{P} $$

$$ {}^{0}_{E}\tilde{P} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 & 1 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot {}^{3}_{E}\tilde{P} $$

$$ {}^{0}_{E}\tilde{P} = \begin{bmatrix} 1 & 0 & 0 & 2/\sqrt{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0.5 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5+\sqrt{2} \\ 0 \\ 0 \\ 1 \end{bmatrix} $$

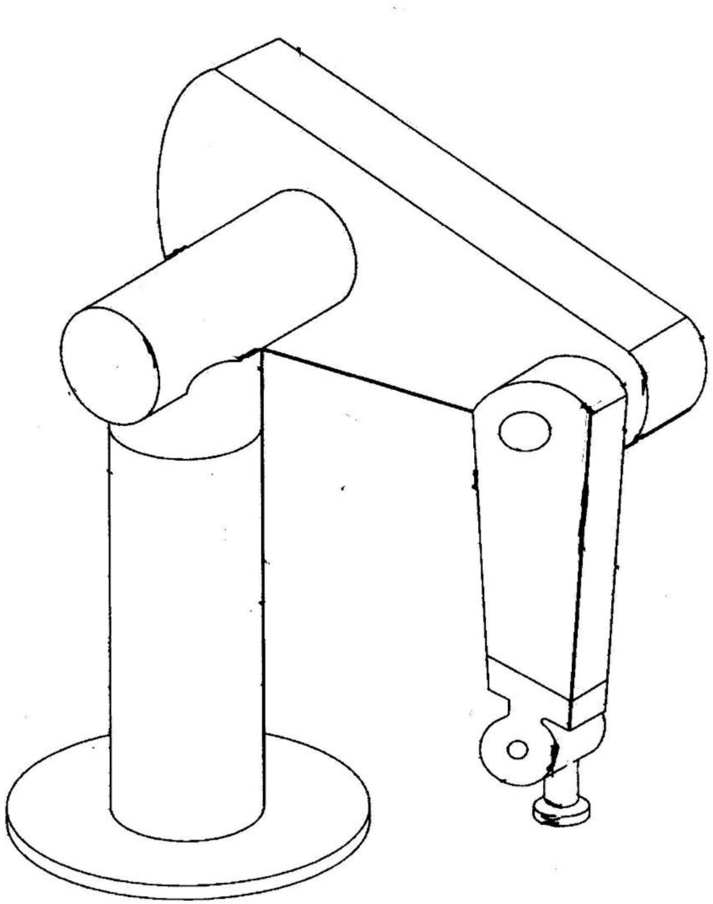


Geometrically

A Notes on Axis Directions

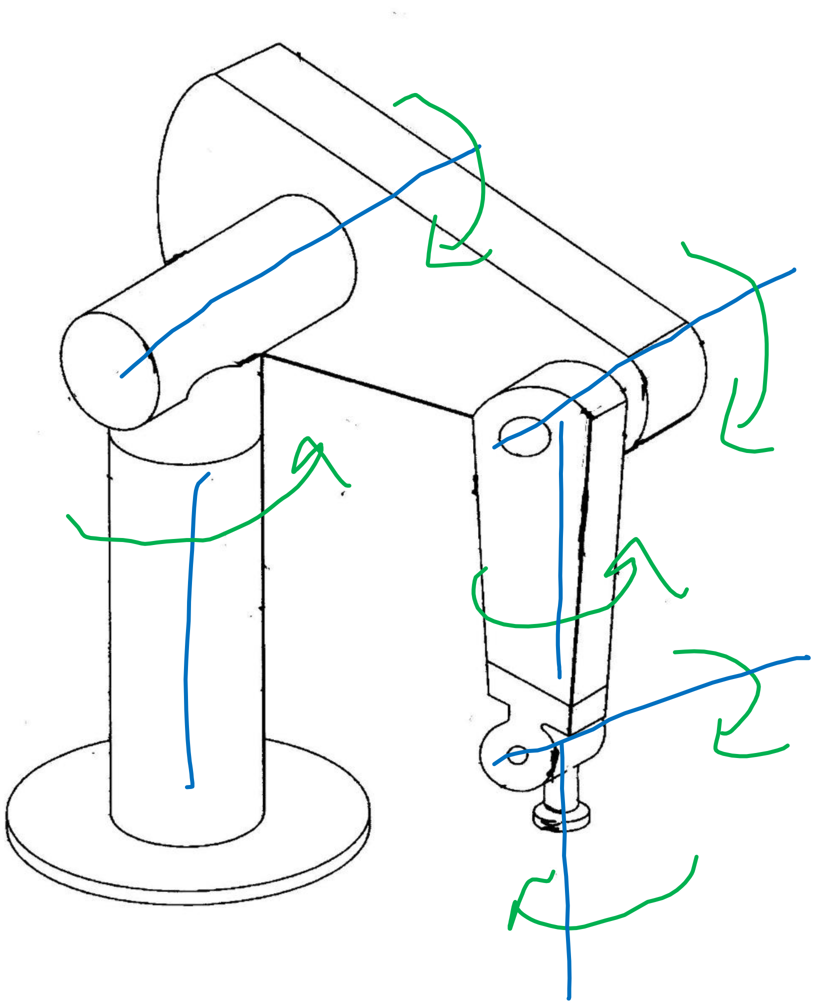
- 2 possible directions for Z-axis along the axis of motion
- 2 possible directions for X-axis perpendicular to skew or intersecting Z-axes (infinite for parallel Z-axes)

Q2.4: Example of Puma 560

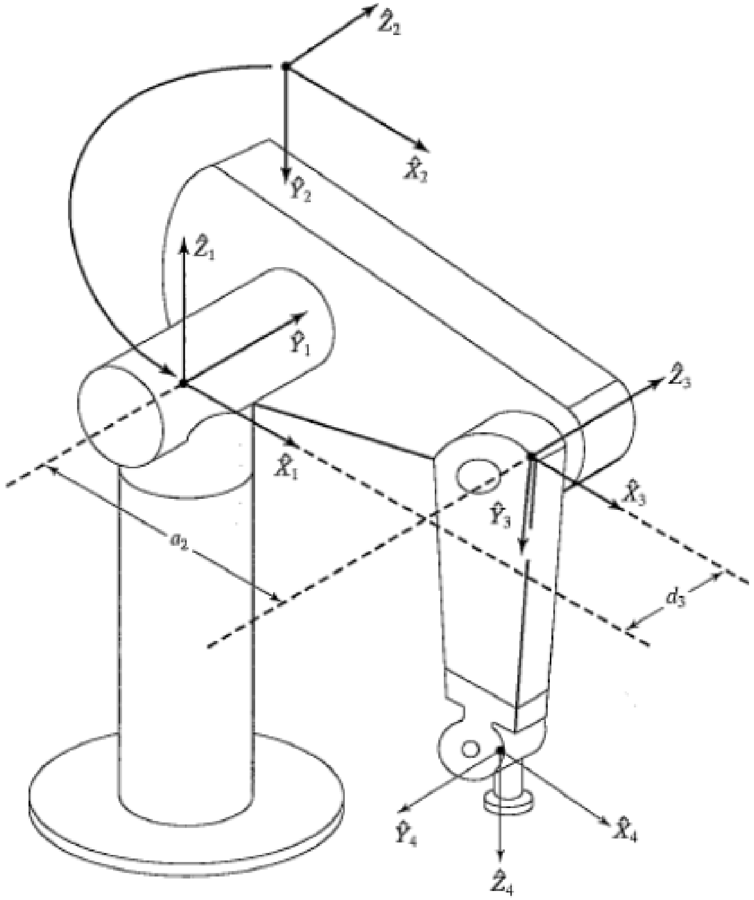


Q2.4: Example of Puma 560

Try it



Q2.4: Example of Puma 560



	α_{i-1}	a_{i-1}	θ_i	d_i
1				
2				
3				
4				
5				
6