Chen

克拉默法则的证明
(The proof of Cramer’s Rule)

1.知识储备：

1.1 行列式的完全展开式及代数余子式

对于 n 阶行列式:

$$\begin{vmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{vmatrix}$$

其完全展开式 **D** 及代数余子式 $\mathbf{A}_{ij}$ 分别为

D = $\sum_{j_1 j_2 \ldots j_n} (-1)^{\tau(j_1 j_2 \ldots j_n)} a_{1j_1} a_{2j_2} \ldots a_{nj_n}$

$\mathbf{A}_{ij} = (-1)^{i+j} \mathbf{M}_{ij}$，其中 $\mathbf{M}_{ij}$ 为原行列式将元素 a_{ij} 所在的第 i 行、第 j 列划去后形成的新的行列式

1.2 线性代数之行列式及其性质

①经转置行列式的值不变，即| $\mathbf{A}^T$ |= | **A** |

②某行有公因数 k，可把 k 提到公因式外，特别地，某行元素全为 0，则其行列式的值为 0

③某行互换行列式变号，特别地，若两行相等，行列式值为 0；两行成比例，行列式值为 0

④某行所有元素都是两个数的和，则可携程两个行列式之和

⑤某行的 k 倍加至另一行，行列式的值不变

2. 证明内容及证明方法：

如果线性方程组

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2, \\ \cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

的系数矩阵$\boldsymbol{A}$的行列式$d = |\boldsymbol{A}| \neq 0$，那么这一个方程组有且仅有一个解

$$x_1 = \frac{d_1}{d}, x_2 = \frac{d_2}{d}, \dots, x_n = \frac{d_n}{d},$$

其中d_i是把矩阵$\boldsymbol{A}$中第i列换成方程组的常数项所成的矩阵的行列式.

2.1 快速证明法：

（1）解的正确性

$x_1 = \frac{d_1}{d}, x_2 = \frac{d_2}{d}, \dots, x_n = \frac{d_n}{d}$ 是解　　即$a_{i1}\frac{d_1}{d} + a_{i2}\frac{d_2}{d} + \cdots + a_{in}\frac{d_n}{d} = b_i, i = 1, \cdots, n$

$$0 = \begin{vmatrix} b_i & a_{i1} & a_{i2} & \cdots & a_{in} \\ b_1 & a_{11} & a_{12} & \cdots & a_{1n} \\ b_2 & a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = b_i d - a_{i1}d_1 - a_{i2}d_2 - \cdots - a_{in}d_n$$

$$\begin{vmatrix} b_1 & a_{11} & a_{13} & \cdots & a_{1n} \\ b_2 & a_{21} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & a_{n1} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = -\begin{vmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & b_n & a_{n3} & \cdots & a_{nn} \end{vmatrix}$$

$$\Rightarrow a_{i1}d_1 + a_{i2}d_2 + \cdots + a_{in}d_n = b_i d$$

$$\Rightarrow a_{i1}\frac{d_1}{d} + a_{i2}\frac{d_2}{d} + \cdots + a_{in}\frac{d_n}{d} = b_i$$

$$i = 1, \cdots, n$$

（2）解的唯一性

若有$\alpha_1,\alpha_2,\cdots,\alpha_n$是方程组的解

$$d\alpha_1 = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \alpha_1 = \begin{vmatrix} a_{11}\alpha_1 & a_{12} & \cdots & a_{1n} \\ a_{21}\alpha_1 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}\alpha_1 & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} a_{11}\alpha_1 + a_{12}\alpha_2 + \cdots + a_{1n}\alpha_n & a_{12} & \cdots & a_{1n} \\ a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1}\alpha_1 + a_{n2}\alpha_2 + \cdots + a_{nn}\alpha_n & a_{n2} & \cdots & a_{nn} \end{vmatrix}$$

$$= \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ b_n & a_{n2} & \cdots & a_{nn} \end{vmatrix} = d_1 \quad \Rightarrow \quad \alpha_1 = \frac{d_1}{d}$$

其它类似可证

2.2 传统证明法：

（1）解的正确性：

$x_1 = \frac{d_1}{d}, x_2 = \frac{d_2}{d}, \dots, x_n = \frac{d_n}{d}$ 是解　　即$a_{i1}\frac{d_1}{d} + a_{i2}\frac{d_2}{d} + \cdots + a_{in}\frac{d_n}{d} = b_i, i = 1,\cdots,n$

$$a_{i1}\frac{d_1}{d} + a_{i2}\frac{d_2}{d} + \cdots + a_{in}\frac{d_n}{d} = \frac{1}{d}(a_{i1}d_1 + a_{i2}d_2 + \cdots + a_{in}d_n) = \frac{1}{d}\sum_{j=1}^{n} a_{ij}d_j$$

$$d_j = \begin{vmatrix} a_{11} & \cdots & a_{1,j-1} & b_1 & a_{1,j+1} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2,j-1} & b_2 & a_{2,j+1} & \cdots & a_{2n} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ a_{n1} & \cdots & a_{n,j-1} & b_n & a_{n,j+1} & \cdots & a_{nn} \end{vmatrix} = b_1A_{1j} + b_2A_{2j} + \cdots + b_nA_{nj} = \sum_{s=1}^{n} b_sA_{sj}$$

$$\frac{1}{d}\sum_{j=1}^{n} a_{ij}d_j = \frac{1}{d}\sum_{j=1}^{n} \underline{a_{ij}}\left(\sum_{\underline{s=1}}^{n} b_sA_{sj}\right) = \frac{1}{d}\sum_{j=1}^{n}\sum_{s=1}^{n} a_{ij}A_{sj}b_s = \frac{1}{d}\sum_{\underline{s=1}}^{n}\sum_{\underline{j=1}}^{n} a_{ij}A_{sj}\underline{b_s}$$

$$= \frac{1}{d}\sum_{\underline{s=1}}^{n}\left(\sum_{j=1}^{n} \underline{a_{ij}A_{sj}}\right)b_s = \frac{1}{d}\left(\sum_{j=1}^{n} \underline{a_{ij}A_{ij}}\right)\underline{b_i} = \frac{1}{d}db_i = b_i \qquad i = 1,\cdots,n$$

（2）解的唯一性:

若有$c_1, c_2, \cdots, c_n$是方程组的解 则$a_{i1}c_1 + a_{i2}c_2 + \cdots + a_{in}c_n = \sum_{j=1}^n a_{ij}c_j = b_i$

$$A_{1k}\sum_{j=1}^n a_{1j}c_j + A_{2k}\sum_{j=1}^n a_{2j}c_j + \cdots + A_{nk}\sum_{j=1}^n a_{nj}c_j = \boxed{\sum_{i=1}^n A_{ik}\sum_{j=1}^n a_{ij}c_j} = \sum_{i=1}^n b_iA_{ik} = d_k$$

$$= \sum_{i=1}^n\sum_{j=1}^n a_{ij}A_{ik}c_j$$

$$= \sum_{j=1}^n\sum_{i=1}^n a_{ij}A_{ik}c_j$$

$$= \sum_{j=1}^n\left(\sum_{i=1}^n a_{ij}A_{ik}\right)c_j$$

$$= \left(\sum_{i=1}^n a_{ik}A_{ik}\right)c_k = dc_k$$

$$c_k = \frac{d_k}{d}$$

$$k = 1, \cdots, n$$

3.证明来源及参考资料：

[1] 《线性代数》同济教材第五版

[2] *Linear Algebra and Its Applications (Subscription), 6th Edition*
David C. Lay, University of Maryland
Judi J. McDonald, Washington State University
Steven R. Lay, Lee University