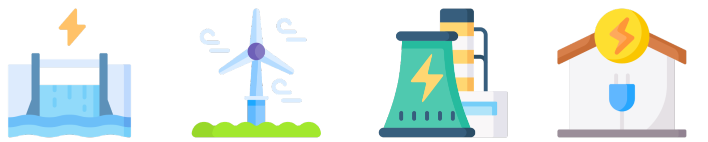
SOLUTIONS

Technical University
of Denmark

46750 - Optimization in Modern Energy Systems
Exercise 2

1. **Economic dispatch**

We consider a power system with 3 generators (G_1, G_2, G_3) and 1 inflexible load (D_1). The production costs (in DKK/MWh), generation capacity and demand (in MWh) are summarized in Figure 1. The system operator wants to dispatch the generators in order to cover this load at the lowest possible cost.



	G1	G2	G3	D1
Cost	$c_1^G = 70$	$c_2^G = 0$	$c_3^G = 150$	–
Capacity	$Q_1^G = 150$	$Q_2^G = 150$	$Q_3^G = 150$	$Q_1^D = 200$

Figure 1: Economic dispatch parameters

(a) Formulate this economic dispatch problem as an optimization problem. Specify the number of variables and constraints of this optimization problem. What do the dual variables associated with each constraint of the economic dispatch represent?

Solution:

We define the non-negative decision variables p_1^G, p_2^G, p_3^G as the electricity production of generators $G1$, $G2$, and $G3$, respectively. The economic dispatch can be modelled as:

$$\min_{\boldsymbol{p^G}} \quad z = \sum_{i=1}^{3} c_i^G \boldsymbol{p_i^G} \tag{1a}$$

$$\text{s.t.} \quad \sum_{i=1}^{3} \boldsymbol{p_i^G} = P_1^D \quad : \boldsymbol{\lambda} \tag{1b}$$

$$\boldsymbol{p_i^G} \leq \overline{P}_i^G \quad : \boldsymbol{\mu_i} \quad \forall i = 1, ..., 3 \tag{1c}$$

$$\boldsymbol{p_i^G} \geq 0 \quad \forall i = 1, ..., 3, \tag{1d}$$

with $\boldsymbol{p^G} = \begin{bmatrix} \boldsymbol{p_1^G} & \boldsymbol{p_2^G} & \boldsymbol{p_3^G} \end{bmatrix}^\top$. The objective of this optimization problem is to minimize the electricity production cost (1a), subject to the balance equation (1b), and upper bounds on the generators' production (1c). The dual variables ($\boldsymbol{\lambda}$, $\boldsymbol{\mu_i}$)

are defined in front of the constraints they are associated with. This optimization problem has **3 variables** and **4 constraints**.

The dual variables associated with each constraint represent the sensitivity of the objective function with respect to the right-hand side of this constraint. In particular, the dual variable associated with the balance equation (1b) represents the cost of the marginal electricity production cost, i.e., if the demand (right hand side of (1b)) increased by a small amount δ, the optimal production cost would increase by $\lambda^*.\delta$. The dual variables associated with the inequality constraints (1c) must take the value 0 when these inequalities are non-binding, and can take a positive value when these inequalities are binding. Indeed, when an inequality is not binding, a small change in its right-hand side would not result in any change in the optimal objective value.

(b) Solve this optimization problem using Python. Provide the values of the optimal primal variables, objective value, and dual variables associated with each constraint. What do you observe w.r.t. to the values (zero or non-zero) taken by the dual variables at optimality and the constraints they are associated with?

Solution: The optimal solutions of this optimization problem are:

$$\begin{aligned} z^* &= 3500 \\ p^{G^*} &= \begin{bmatrix} 50 & 150 & 0 \end{bmatrix}^\top \\ \lambda^* &= 70 \\ \mu^* &= \begin{bmatrix} 0 & 70 & 0 \end{bmatrix}^\top \end{aligned}$$

We observe that these dual variables μ_1^* and μ_3^* take the value 0 while the constraints they are associated with are non-binding. Since these dual variables represent the sensitivities of the objective with respect to the right-hand side of these constraints, when these constraints are non-binding, a small change in their right-hand-side would have no impact on the optimal solutions and objective. Similarly, the dual variable μ_2^* associated with the constraint on the upper bound of generator G_2, which is binding, takes a positive value. This value represents the sensitivity of the objective function with respect to the right-hand-side of this constraint. If the maximum capacity of generator G_2 increased by 1MW, its production at optimality would also increase by 1MWh, and the production of generator G_1 would decrease by the same amount. As a result, the optimal objective value would change by $+c_2^G - c_1^G = -70 = -\mu_2$DKK.

(c) Formulate the dual of the economic dispatch problem. What do the dual variables associated with each constraint of the dual problem represent?

Solution:

The dual of the economic dispatch in (1) can be formulated as:

$$\max_{\{\boldsymbol{\lambda},\boldsymbol{\mu}\}} \quad \tilde{z} = P^D_1 \boldsymbol{\lambda} - \sum_{i=1}^{3} \overline{P}^G_i \boldsymbol{\mu_i} \tag{2a}$$

$$\text{s.t.} \quad \boldsymbol{\lambda} - \boldsymbol{\mu_i} \le c^G_i \quad : \boldsymbol{p^G_i} \quad \forall i = 1, ..., 3 \tag{2b}$$

$$\boldsymbol{\lambda} \text{ free} \tag{2c}$$

$$\boldsymbol{\mu_i} \ge 0 \quad \forall i = 1, ..., 3, \tag{2d}$$

with $\boldsymbol{\mu} = \begin{bmatrix}\boldsymbol{\mu_1} & \boldsymbol{\mu_2} & \boldsymbol{\mu_3}\end{bmatrix}^\top$. This dual problem has **4 variables** (same as the number of constraints in the primal problem) and **3 constraints** (same as the number of variables in the primal problem).

The dual variables associated with each constraint (2b) are denoted in front of the constraints. As the dual of the dual optimization problem (2) is the primal optimization problem (1). The dual variables associated with each constraint (2b) for $i = 1, ..., 3$ represent the electricity production $\boldsymbol{p^G_i}$ of each generator.

Further details: This formulation can be obtained directly using the SOB mnemonic system, or by First reformulating the economic dispatch in (1) in a standard form, such that:

$$\min_{\boldsymbol{p^G}} z = \sum_{i=1}^{3} c^G_i \boldsymbol{p^G_i} \tag{3a}$$

$$\text{s.t.} \quad \sum_{i=1}^{3} \boldsymbol{p^G_i} \ge P^D_1 \quad : \boldsymbol{\lambda^+} \tag{3b}$$

$$\text{quad} \sum_{i=1}^{3} -\boldsymbol{p^G_i} \ge -P^D_1 \quad : \boldsymbol{\lambda^-} \tag{3c}$$

$$-\boldsymbol{p^G_i} \ge -\overline{P}^G_i \quad : \boldsymbol{\mu_i}, \ \forall i = 1, ..., 3 \tag{3d}$$

$$\boldsymbol{p^G_i} \ge 0, \ \forall i = 1, ..., 3, \tag{3e}$$

which is of the form

$$\min_{\boldsymbol{p^G}} z = c^\top \boldsymbol{p^G} \tag{4a}$$

$$\text{s.t.} \quad A.\boldsymbol{p^G} \ge b \tag{4b}$$

$$\boldsymbol{p^G} \ge 0, \tag{4c}$$

with

$$b = \begin{bmatrix} P^D_1 & -P^D_1 & -\overline{P}^G 1 & -\overline{P}^G 2 & -\overline{P}^G 3 \end{bmatrix}^\top$$

$$c = \begin{bmatrix} c^G_1 & c^G_2 & c^G_3 \end{bmatrix}^\top$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

We recall that the dual of this optimization problem (4) is:

$$\max_{\boldsymbol{y}} \ \tilde{z} = b^\top \boldsymbol{y} \tag{5a}$$
$$\text{s.t.} \quad A^\top . \boldsymbol{y} \leq c \tag{5b}$$
$$\boldsymbol{y} \geq 0, \tag{5c}$$

with $\boldsymbol{y} = \begin{bmatrix} \boldsymbol{\lambda}^+ & \boldsymbol{\lambda}^- & \boldsymbol{\mu_1} & \boldsymbol{\mu_2} & \boldsymbol{\mu_3} \end{bmatrix}^\top$. Therefore, the dual of the optimization problem (7) is

$$\max_{\{\boldsymbol{\lambda}^+, \boldsymbol{\lambda}^-, \boldsymbol{\mu}\}} \ \tilde{z} = P_1^D(\boldsymbol{\lambda}^+ - \boldsymbol{\lambda}^-) - \sum_{i=1}^{3} \overline{P}_i^G \boldsymbol{\mu_i} \tag{6a}$$
$$\text{s.t.} \quad (\boldsymbol{\lambda}^+ - \boldsymbol{\lambda}^-) - \boldsymbol{\mu_i} \leq c_i^G : \boldsymbol{p_i^G}, \ \forall i = 1, ..., 3 \tag{6b}$$
$$\boldsymbol{\lambda}^+ \geq 0, \boldsymbol{\lambda}^- \geq 0 \ \forall i = 1, ..., 3, \tag{6c}$$
$$\boldsymbol{\mu_i} \geq 0, \ \forall i = 1, ..., 3. \tag{6d}$$

By introducing the auxiliary free variable $\boldsymbol{\lambda} = (\boldsymbol{\lambda}^+ - \boldsymbol{\lambda}^-)$, (6) can be recast as (2).

(d) Solve this optimization problem using Python. Provide the optimal values of its primal variables and objective value, as well as the values of the dual variables associated with each constraint.

Solution: The optimal solutions of this optimization problem are:

$$\tilde{z}^* = 3500$$
$$\lambda^* = 70$$
$$\mu^* = \begin{bmatrix} 0 & 70 & 0 \end{bmatrix}^\top$$
$$p^{G^*} = \begin{bmatrix} 50 & 150 & 0 \end{bmatrix}^\top$$

(e) Formulate the strong-duality theorem for the economic dispatch, and provide its primal-dual formulation.

Solution:
The strong duality theorem for the primal (1) and dual (2) problems states that, at optimality, $z^* = \tilde{z}^*$, and the primal-dual formulation of the economic dispatch

(1) is:

$$\sum_{i=1}^{3} c_i^G p_i^{G^*} = P_1^D \lambda^* - \sum_{i=1}^{3} \overline{P}_i^G \mu_i^* \tag{7a}$$

$$\sum_{i=1}^{3} \boldsymbol{p_i^G} = P_1^D \tag{7b}$$

$$\boldsymbol{p_i^G} \leq \overline{P}_i^G, \ \forall i = 1, ..., 3 \tag{7c}$$

$$\boldsymbol{p_i^G} \geq 0, \ \forall i = 1, ..., 3, \tag{7d}$$

$$\boldsymbol{\lambda} - \boldsymbol{\mu_i} \leq c_i^G, \ \forall i = 1, ..., 3 \tag{7e}$$

$$\boldsymbol{\lambda} \text{ free} \tag{7f}$$

$$\boldsymbol{\mu_i} \geq 0, \ \forall i = 1, ..., 3, \tag{7g}$$

where (7a) represents the strong duality condition, (7b)-(7d) represent primal feasibility, and (7e)-(7g) represent dual feasibility.