SOLUTIONS

Technical University of Denmark

46750 - Optimization in Modern Energy Systems
Exercise 5

1. **Electricity market properties**

We consider a power system with 3 generators and 2 flexible consumers. The fixed marginal production costs c_g^G (DKK/MWh) and maximum generation capacity $\overline{P}_g$ (MWh) of all generators $g \in \mathcal{G} = \{W_1, G_1, G_2\}$, the fixed marginal utility cost c_d^D (DKK/MWh), minimum $\underline{L}^d$ and maximum load capacity $\overline{L}^d$ of all flexible consumers $d \in \mathcal{D} = \{D_1, D_2\}$ (in MWh) are summarized in the table below.

	marginal cost (DKKMWh)	**Minimum capacity (MWh)**	**Maximum capacity (MWh)**
$\boldsymbol{W_1}$	0	0	80
$\boldsymbol{G_1}$	30	0	80
$\boldsymbol{G_2}$	35	0	80
$\boldsymbol{D_1}$	40	0	100
$\boldsymbol{D_2}$	20	0	50

The market operator wants to find the optimal dispatch of the generators and flexible consumers in order to maximize the social welfare of the system.

(a) Assuming that all market participants are price-takers, formulate this market-clearing as an optimization problem.

Solution:
We define the decision variables $\boldsymbol{p_g}$ as the electricity production of generators $g \in \mathcal{G}$. The economic dispatch problem can be modelled as:

$$\max_{\boldsymbol{p_g}, \boldsymbol{l_d}} \quad \sum_{d\in\mathcal{L}} c_d^D \boldsymbol{l_d} - \sum_{g\in\mathcal{G}} c_g^G \boldsymbol{p_g} \tag{1a}$$

$$\text{s.t.} \quad \sum_{g\in\mathcal{G}} \boldsymbol{p_g} = \sum_{d\in\mathcal{L}} \boldsymbol{l_d} \qquad : \boldsymbol{\lambda} \tag{1b}$$

$$0 \leq \boldsymbol{p_g} \leq \overline{P}_g \qquad : \underline{\boldsymbol{\mu}}_{\boldsymbol{g}}, \overline{\boldsymbol{\mu}}_{\boldsymbol{g}} \qquad \forall g \in \mathcal{G} \tag{1c}$$

$$\underline{L}_d \leq \boldsymbol{l_d} \leq \overline{L}_d \qquad : \underline{\boldsymbol{\sigma}}_{\boldsymbol{d}}, \overline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \qquad \forall d \in \mathcal{L} \tag{1d}$$

where the objective of this optimization problem is to maximize the social welfare (1a), subject to the balance equation (1b), lower-bounds and upper-bounds on the generators' production (1c) and the flexible consumers' loads (1d). The dual variables ($\boldsymbol{\lambda} \in \mathbb{R}$, $\underline{\boldsymbol{\mu}}_{\boldsymbol{g}} \in \mathbb{R}^+$ and $\overline{\boldsymbol{\mu}}_{\boldsymbol{g}} \in \mathbb{R}^+$ for all $g \in \mathcal{G}$, $\underline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \in \mathbb{R}^+$ and $\overline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \in \mathbb{R}^+$ for all $d \in \mathcal{L}$ are defined in front of the constraints they are associated with.

The optimal solutions of this optimization problem are:

$$p^*_{W_1} = 80; p^*_{G_1} = 20; p^*_{G_2} = 0 \text{ (MWh)}$$
$$l^*_{D_1} = 100; l^*_{D_2} = 0 \text{ (MWh)}$$
$$\lambda^* = 20 \text{ (DKK/MWh)}$$

(b) Assuming that all market participants are price-takers, formulate the equilibrium among them.

Solution:

The equilibrium problem representing the simultaneous decisions of all price-taker producers and consumers on this market includes the i) optimal offering problem of all producers; ii) optimal bidding problem of all consumers; and iii) linking constraint (balance equation). These are formulated below.

Optimal offering problem of producers

Each price-taker producer $g \in \mathcal{G}$ aims at finding its optimal production $\boldsymbol{p_g}$ (self-dispatch), based on the forecast market-clearing price $\boldsymbol{\lambda}$. Its optimal offering problem can be formulated as follows:

$$\max_{\boldsymbol{p_g}} \left(\boldsymbol{\lambda} - c^G_g\right) \boldsymbol{p_g} \tag{2a}$$
$$\text{s.t.} \quad 0 \le \boldsymbol{p_g} \le \overline{P}_g \quad : \underline{\boldsymbol{\mu}}_{\boldsymbol{g}}, \overline{\boldsymbol{\mu}}_{\boldsymbol{g}} \tag{2b}$$

where the objective function (2a) represents the profit of the price-taker producer, where the market-clearing price $\boldsymbol{\lambda}$ is considered as a fixed input parameter, subject to constraint (2b) on its minimum and maximum production.

Optimal bidding problem of consumers

Each price-taker consumer $d \in \mathcal{L}$ aims at finding its optimal consumption $\boldsymbol{l_d}$ (self-dispatch), based on the forecast market-clearing price $\boldsymbol{\lambda}$. Its optimal bidding problem can be formulated as follows:

$$\max_{\boldsymbol{l_d}} \left(c^D_d - \boldsymbol{\lambda}\right) \boldsymbol{l_d} \tag{3a}$$
$$\text{s.t.} \quad \underline{L}_d \le \boldsymbol{l_d} \le \overline{L}_d \quad : \underline{\boldsymbol{\sigma}}_{\boldsymbol{d}}, \overline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \tag{3b}$$

where the objective function (3a) represents the surplus of the price-taker consumer, where the market-clearing price $\boldsymbol{\lambda}$ is considered as a fixed input parameter, subject to constraints (3b) on its minimum and maximum consumption.

Linking constraint (balance equation)

The balance equation links the optimal decisions of all these price-taker market participants, such that:

$$\sum_{g \in \mathcal{G}} \boldsymbol{p_g} = \sum_{d \in \mathcal{L}} \boldsymbol{l_d} \quad : \boldsymbol{\lambda}, \tag{4}$$

where the market-clearing price $\boldsymbol{\lambda}$ is the dual variable associate with this constraint.

Note: although the market-clearing price $\boldsymbol{\lambda}$ is considered as a fixed input parameter in each price-taker's optimization problem, it is defined as the dual variable associated wit the linking constraint (4).

(c) Show that the KKT conditions of the market-clearing problem formulated in Question a) and the equilibrium formulated in Question b) are identical. What does this mean for the properties of this market design, *under the assumption of perfect competition*?

Solution:

KKT conditions of market-clearing as an optimization problem:

We can reformulate the market-clearing optimization problem in (1) as a minimization problem and express its Lagrangian function as:

$$\begin{aligned}\mathcal{L}(\boldsymbol{p_g}, \boldsymbol{l_d}, \boldsymbol{\underline{\mu}_g}, \boldsymbol{\overline{\mu}_g}, \boldsymbol{\underline{\sigma}_d}, \boldsymbol{\overline{\sigma}_d}) = &\sum_{g \in \mathcal{G}} c_g^G \boldsymbol{p_g} - \sum_{d \in \mathcal{L}} c_d^D \boldsymbol{l_d} + \boldsymbol{\lambda} \left[\sum_{d \in \mathcal{L}} \boldsymbol{l_d} - \sum_{g \in \mathcal{G}} \boldsymbol{p_g} \right] \\ &+ \sum_{g \in \mathcal{G}} \left[\boldsymbol{\overline{\mu}_g} \left(\boldsymbol{p_g} - \overline{P}_g \right) - \boldsymbol{\underline{\mu}_g} \boldsymbol{p_g} \right] \\ &+ \sum_{d \in \mathcal{L}} \left[\boldsymbol{\overline{\sigma}_d} \left(\boldsymbol{l_d} - \overline{L}_d \right) + \boldsymbol{\underline{\sigma}_d} \left(\underline{L}_d - \boldsymbol{l_d} \right) \right]\end{aligned} \tag{5}$$

Since the market-clearing optimization problem is convex and satisfies the weak Slaters' condition, strong duality holds. Therefore, its KKT conditions are sufficient and necessary optimality conditions. These KKT conditions include the 1st order optimality conditions:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{p_g}} = c_g^G - \boldsymbol{\lambda} + \boldsymbol{\overline{\mu}_g} - \boldsymbol{\underline{\mu}_g} = 0 \quad \forall g \in \mathcal{G}, \tag{6}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{l_d}} = -c_d^D + \boldsymbol{\lambda} + \boldsymbol{\overline{\sigma}_d} - \boldsymbol{\underline{\sigma}_d} = 0 \quad \forall d \in \mathcal{L}, \tag{7}$$

complementarity conditions:

$$\boldsymbol{\underline{\mu}_g} \cdot (-\boldsymbol{p_g}) = 0 \quad \forall g \in \mathcal{G}, \tag{8}$$

$$\boldsymbol{\overline{\mu}_g} \cdot \left(\boldsymbol{p_g} - \overline{P}_g \right) = 0 \quad \forall g \in \mathcal{G} \tag{9}$$

$$\boldsymbol{\underline{\sigma}_d} \cdot \left(\underline{L}_d - \boldsymbol{l_d} \right) = 0 \quad \forall d \in \mathcal{L}, \tag{10}$$

$$\boldsymbol{\overline{\sigma}_d} \cdot \left(\boldsymbol{l_d} - \overline{L}_d \right) = 0 \quad \forall d \in \mathcal{L} \tag{11}$$

the primal feasibility conditions:

$$\sum_{g \in \mathcal{G}} \boldsymbol{p_g} = \sum_{d \in \mathcal{L}} \boldsymbol{l_d} \tag{12}$$

$$0 \le \boldsymbol{p_g} \le \overline{P}_g \quad \forall g \in \mathcal{G} \tag{13}$$

$$\underline{L}_d \le \boldsymbol{l_d} \le \overline{L}_d \quad \forall d \in \mathcal{L} \tag{14}$$

and the dual feasibility conditions:

$$\boldsymbol{\underline{\mu}_g} \ge 0; \ \boldsymbol{\overline{\mu}_g} \ge 0 \quad \forall g \in \mathcal{G} \tag{15}$$

$$\boldsymbol{\underline{\sigma}_g} \ge 0; \ \boldsymbol{\overline{\sigma}_g} \ge 0 \quad \forall d \in \mathcal{L} \tag{16}$$

$$\boldsymbol{\lambda} \text{ free.} \tag{17}$$

These KKT conditions are necessary and sufficient.

KKT conditions of market-clearing as an equilibrium problem:

The KKT conditions of the equilibrium problem include the i) KKT conditions of the optimal offering problem of all producers; ii) KKT conditions of he optimal bidding problem of all consumers; and iii) the linking constraint (balance equation). These are detailed below:

KKT conditions of optimal offering problem of producers

For each producer $g \in \mathcal{G}$, the KKT conditions of the optimal offering problem are:

$$\frac{\partial \mathcal{L}(\boldsymbol{p_g}, \underline{\boldsymbol{\mu}}_{\boldsymbol{g}}, \overline{\boldsymbol{\mu}}_{\boldsymbol{g}})}{\partial \boldsymbol{p_g}} = -\boldsymbol{\lambda} + c_g^G - \underline{\boldsymbol{\mu}}_{\boldsymbol{g}} + \overline{\boldsymbol{\mu}}_{\boldsymbol{g}} = 0 \tag{18a}$$

$$\underline{\boldsymbol{\mu}}_{\boldsymbol{g}} \cdot (-\boldsymbol{p_g}) = 0 \tag{18b}$$

$$\overline{\boldsymbol{\mu}}_{\boldsymbol{g}} \cdot (\boldsymbol{p_g} - \overline{P}_g) = 0 \tag{18c}$$

$$0 \leq \boldsymbol{p_g} \leq \overline{P}_g \tag{18d}$$

$$\underline{\boldsymbol{\mu}}_{\boldsymbol{g}} \geq 0;\ \overline{\boldsymbol{\mu}}_{\boldsymbol{g}} \geq 0 \tag{18e}$$

where (18a) represents the 1st order condition, (18b) - (18c) the complementarity slackness conditions, (18d) the primal constraints, and (18e) the dual constraints. These KKT conditions are necessary and sufficient.

KKT conditions of optimal bidding problem of consumers

For each comsumer $d \in \mathcal{L}$, the KKT conditions of the optimal bidding problem are:

$$\frac{\partial \mathcal{L}(\boldsymbol{l_d}, \underline{\boldsymbol{\sigma}}_{\boldsymbol{d}}, \overline{\boldsymbol{\sigma}}_{\boldsymbol{d}})}{\partial \boldsymbol{l_d}} = \boldsymbol{\lambda} - c_d^D - \underline{\boldsymbol{\sigma}}_{\boldsymbol{d}} + \overline{\boldsymbol{\sigma}}_{\boldsymbol{d}} = 0 \tag{19a}$$

$$\underline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \cdot (\underline{L}_d - \boldsymbol{l_d}) = 0 \tag{19b}$$

$$\overline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \cdot (\boldsymbol{l_d} - \overline{L}_d) = 0 \tag{19c}$$

$$\underline{L}_d \leq \boldsymbol{l_d} \leq \overline{L}_d \tag{19d}$$

$$\underline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \geq 0;\ \overline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \geq 0 \tag{19e}$$

where (19a) represents the 1st order condition, (19b) - (19c) the complementarity slackness conditions, (19d) the primal constraints, and (19e) the dual constraints. These KKT conditions are necessary and sufficient.

Linking constraint (balance equation)

The balance equation links the optimal decisions of all these price-taker market participants, such that:

$$\sum_{g \in \mathcal{G}} \boldsymbol{p_g} = \sum_{d \in \mathcal{L}} \boldsymbol{l_d} \tag{20a}$$

$$\boldsymbol{\lambda} \text{ free} \tag{20b}$$

where the market-clearing price $\boldsymbol{\lambda}$ is the dual variable associate with this constraint.

The KKT conditions of the market-clearing equilibrium problem are identical to those of the market-clearing optimization problem.

(d) Using the formulation of the market-clearing problem and its KKT conditions, show that this market design ensures the following desirable properties:

i. *budget balance* for the market operator, *(Tip: use the balance equation of the* *market-clearing problem.)*

Solution:
In this market, each consumer d pays $\boldsymbol{\lambda l_d}$, and each producer g receives the payment $\boldsymbol{\lambda p_g}$. The balance equality ensures that:
$\sum_{g=1...G} \boldsymbol{p_g} = \sum_{d=1...D} \boldsymbol{l_d}$.
Therefore, by multiplying each side of this equality by $\boldsymbol{\lambda}$, we obtain:
$\sum_{g=1...G} \boldsymbol{\lambda p_g} = \sum_{d=1...D} \boldsymbol{\lambda l_d}$.
This shows that the sum of payments made by consumers is equal to the sum of payments received by producers and that the market is *budget balanced.*
Note: if network constraints are accounted for in this market-clearing problem, and if congestion occurs between two market areas, then the prices in each area differ from one another. As a result, budget balance does not always hold. However, the market remains revenue adequate, i.e. the sum of payments by consumers exceeds the sum of payments received by producers. This difference is called the congestion rent, and can be used by the system operator to invest in new transmission lines.

ii. *cost recovery* for all market participants. *(Tip: use the 1st order and complementarity conditions of the market-clearing problem.)*

Solution:
This property can be intuitively inferred by observing that the dispatch and market-clearing price are the intersection of supply and demand curves. Therefore, only producers whose marginal costs c_g^G are lower (and consumers whose marginal utilities c_d^D are higher) than the market-clearing price are dispatched. For each producer g, we can use the 1st order condition w.r.t. $\boldsymbol{p_g}$, which ensures that $\frac{\partial \mathcal{L}}{\partial \boldsymbol{p_g}} = c_g^G - \boldsymbol{\lambda} + \boldsymbol{\overline{\mu}_g} - \boldsymbol{\underline{\mu}_g} = 0$. By multiplying this equality by $\boldsymbol{p_g}$, we obtain, the profit of producer g:

$$\left(\boldsymbol{\lambda} - c_g^G\right) \boldsymbol{p_g} = \boldsymbol{\overline{\mu}_g p_g} - \boldsymbol{\underline{\mu}_g p_g} \tag{21}$$

As a result, we can consider 3 cases:

1. $\boldsymbol{p_g} = 0$: In this case, the profit of the generator is equal to zero $c_g^G \boldsymbol{p_g} = 0$.

2. $0 < \boldsymbol{p_g} < \overline{P}_g$: In this case, since both the lower- and upper-bounds on its production are non-binding, then the complementarity constraints ensure that $\boldsymbol{\underline{\mu}_g} = 0$ and $\boldsymbol{\overline{\mu}_g} = 0$. As a result, $\left(\boldsymbol{\lambda} - c_g^G\right) \boldsymbol{p_g} = 0$. The marginal producer does not make any profit as it sets the market-clearing price $\boldsymbol{\lambda} = c_g^G$.

3. $\boldsymbol{p_g} = \overline{P}_g$: In this case, since only the lower-bound on its production is non-binding, then the complementarity constraints ensure that $\boldsymbol{\underline{\mu}_g} = 0$ and $\boldsymbol{\overline{\mu}_g} \geq 0$. As a result, $\left(\boldsymbol{\lambda} - c_g^G\right) \boldsymbol{p_g} = \boldsymbol{\overline{\mu}_g p_g} \geq 0$.

In all three cases we observe that the profit $\left(\boldsymbol{\lambda} - c_g^G\right) \boldsymbol{p_g} \geq 0$, ensuring cost recovery for each generator g. By analogy, we can prove cost recovery for each flexible consumer.
Note: If commitment decisions and minimum production are accounted for in the market-clearing, then cost recovery does not hold, requiring the use of

extra remuneration sources. Indeed, since strong duality does not hold for the Unit Commitment (UC) problem (non convex), the market operator must use a 2-step process for pricing electricity. First, they solve the non-convex UC problem and find the optimal commitment variables. Then, they fix these binary commitment variables to this optimal value, and resolve the problem, which is now a simple economic dispatch (ED) problem, and use the dual variable associated with the balance equation as marginal electricity price. However, in this case, a generator may be committed and dispatched because it maximizes the social welfare, although the marginal electricity price is below their marginal production cost. In this case, they will not recover their costs. In these cases, the market operator offers uplift payments to these generators to ensure that they recover their costs.

2. **Optimal offering strategy of a price-maker**

(a) Assuming that producer G_1 acts as a price-maker and is strategic in quantity, and all other market participants are price-takers, formulate its optimal offering strategy as a bilevel optimization problem.

Solution:
The objective of the strategic producer G_1 is to find the optimal quantity $\boldsymbol{p_{G_1}^{DA}}$ to offer in the day-ahead market, in order to maximize its own profit. This can be formulated as a bilevel optimization problem, where the lower-level problem represents the market-clearing problem with this quantity offer:

$$\max_{\boldsymbol{p_g},\boldsymbol{l_d}} \quad \sum_{d\in\mathcal{L}} c_d^D \boldsymbol{l_d} - \sum_{g\in\mathcal{G}} c_g^G \boldsymbol{p_g} \tag{22a}$$

$$\text{s.t.} \quad \sum_{g\in\mathcal{G}} \boldsymbol{p_g} = \sum_{d\in\mathcal{L}} \boldsymbol{l_d} \qquad : \boldsymbol{\lambda} \tag{22b}$$

$$0 \leq \boldsymbol{p_g} \leq \overline{P}_g \qquad : \underline{\boldsymbol{\mu}}_{\boldsymbol{g}}, \overline{\boldsymbol{\mu}}_{\boldsymbol{g}} \qquad \forall g \in \mathcal{G} \setminus \{G_1\} \tag{22c}$$

$$0 \leq \boldsymbol{p_{G_1}} \leq \boldsymbol{p_{G_1}^{DA}} \qquad : \underline{\boldsymbol{\mu}}_{\boldsymbol{G_1}}, \overline{\boldsymbol{\mu}}_{\boldsymbol{G_1}} \tag{22d}$$

$$\underline{L}_d \leq \boldsymbol{l_d} \leq \overline{L}_d \qquad : \underline{\boldsymbol{\sigma}}_{\boldsymbol{d}}, \overline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \qquad \forall d \in \mathcal{L} \tag{22e}$$

where the objective of this optimization problem is to maximize the social welfare (22a), subject to the balance equation (22b), lower-bounds and upper-bounds on the price-taker generators' production (22c), the strategic producer's production (22d), and the flexible consumers' loads (22e). We observe that the optimal quantity offer $\boldsymbol{p_{G_1}^{DA}}$ of the strategic producer impact this market-clearing problem through constraint (22d). However, $\boldsymbol{p_{G_1}^{DA}}$ is considered as a fixed parameter in the market-clearing problem. The dual variables ($\boldsymbol{\lambda} \in \mathbb{R}$, $\underline{\boldsymbol{\mu}}_{\boldsymbol{g}} \in \mathbb{R}^+$ and $\overline{\boldsymbol{\mu}}_{\boldsymbol{g}} \in \mathbb{R}^+$ for all $g \in \mathcal{G}$, $\underline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \in \mathbb{R}^+$ and $\overline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \in \mathbb{R}^+$ for all $d \in \mathcal{L}$ are defined in front of the constraints they are associated with.

And the upper-level problem, representing the strategic offering problem of the strategic producer G_1 can be formulated as follows:

$$\max_{\boldsymbol{p^{DA}_{G_1}}, \boldsymbol{p_{G_1}}, \boldsymbol{\lambda}} \quad \left(\boldsymbol{\lambda} - c^G_{G_1}\right) \boldsymbol{p_{G_1}} \tag{23a}$$

$$\text{s.t.} \quad 0 \le \boldsymbol{p^{DA}_{G_1}} \le \overline{P}_{G_1} \tag{23b}$$

$$\{\boldsymbol{p_{G_1}}, \boldsymbol{\lambda}\} = \text{ primal and dual solutions of Problem (22)} \tag{23c}$$

where the objective function (33a) represents the profit of the strategic producer G_1, subject to constraint (33b) limiting its quantity offer to its maximum production capacity, and constraint (23c) setting the market-clearing outcomes, i.e. the producer's dispatch and market-clearing price, as the optimal solutions to the lower-level problem (22). We observe that these market-clearing outcomes directly impact the objective function of the upper-level problem.

(b) Reformulate this bilevel problem as a single-level optimization problem, so-called Mathematical Problem with Equilibrium Constraints (MPEC), by replacing the lower-level optimization problem by its (necessary and sufficient) KKT conditions.

Solution:
We can reformulate the bilevel problem by including the KKT conditions of the market clearing (lower-level) problem as constraints in the offering strategy (upper-level) problem. Therefore, we start by formulating the KKT conditions of the market clearing problem.
The Lagrangian function can be expressed as:

$$\begin{aligned} \mathcal{L} = &- \left(\sum_{d \in \mathcal{L}} c^D_d \boldsymbol{l_d} - \sum_{g \in \mathcal{G}} c^G_g \boldsymbol{p_g} \right) + \sum_{g \in \mathcal{G}} \left[\boldsymbol{\overline{\mu}_g} \left(\boldsymbol{p_g} - \overline{P}_g \right) - \boldsymbol{\underline{\mu}_g} \boldsymbol{p_g} \right] \\ &+ \sum_{d \in \mathcal{L}} \left[\boldsymbol{\overline{\sigma}_d} \left(\boldsymbol{l_d} - \overline{L}_d \right) - \boldsymbol{\underline{\sigma}_d} \boldsymbol{l_d} \right] + \boldsymbol{\lambda} \left[\sum_{d \in \mathcal{L}} \boldsymbol{l_d} - \sum_{g \in \mathcal{G}} \boldsymbol{p_g} \right] \end{aligned} \tag{24}$$

Since the market clearing optimization problem is convex and has a strictly feasible point, strong duality holds. Therefore, the KKT conditions are sufficient and necessary optimality conditions.
The KKT conditions include the 1st order optimality conditions:

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{p_g}} = c^G_g + \boldsymbol{\overline{\mu}_g} - \boldsymbol{\underline{\mu}_g} - \boldsymbol{\lambda} = 0 \quad \forall g \in \mathcal{G}, \tag{25}$$

$$\frac{\partial \mathcal{L}}{\partial \boldsymbol{l_d}} = -c^D_d + \boldsymbol{\overline{\sigma}_d} - \boldsymbol{\underline{\sigma}_d} + \boldsymbol{\lambda} = 0 \quad \forall d \in \mathcal{L}, \tag{26}$$

as well as the primal and dual feasibility, and complementarity conditions, which are summarized in this compact formulation:

$$\sum_{g\in\mathcal{G}} \boldsymbol{p_g} = \sum_{d\in\mathcal{L}} \boldsymbol{l_d}, \tag{27}$$

$$0 \leq \underline{\boldsymbol{\mu}}_{\boldsymbol{g}} \perp -\boldsymbol{p_g} \leq 0 \quad \forall g \in \mathcal{G}, \tag{28}$$

$$0 \leq \underline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \perp -\boldsymbol{l_d} \leq 0 \quad \forall d \in \mathcal{L}, \tag{29}$$

$$0 \leq \overline{\boldsymbol{\mu}}_{\boldsymbol{g}} \perp (\boldsymbol{p_g} - \overline{P}_g) \leq 0 \quad \forall g \in \mathcal{G} \setminus \{G_1\}, \tag{30}$$

$$0 \leq \overline{\boldsymbol{\mu}}_{\boldsymbol{G_1}} \perp (\boldsymbol{p_{G_1}} - \boldsymbol{p^{DA}_{G_1}}) \leq 0, \tag{31}$$

$$0 \leq \overline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \perp (\boldsymbol{l_d} - \overline{L}_d) \leq 0 \quad \forall d \in \mathcal{L}. \tag{32}$$

Now, the bilevel problem can be formulated as a single-level optimization problem by adding the KKT conditions of the lower-level problem as constraints in the upper-level problem:

$$\max_{\boldsymbol{p^{DA}_{G_1}}, \boldsymbol{p}, \boldsymbol{l}, \boldsymbol{\lambda}, \underline{\boldsymbol{\mu}}, \overline{\boldsymbol{\mu}}, \underline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\sigma}}} \left(\boldsymbol{\lambda} - c^G_{G_1}\right) \boldsymbol{p_{G_1}} \tag{33a}$$

$$\text{s.t.} \qquad 0 \leq \boldsymbol{p^{DA}_{G_1}} \leq \overline{P}_{G_1} \tag{33b}$$

$$\text{constraints (25)-(32).} \tag{33c}$$

(c) Formulate the strong-duality condition for the lower-level problem. Then, linearize the bilinear terms in the upper-level objective function using the strong duality and the KKT conditions of the lower-level problem. *Please, check the book "Complementarity Modelling in Energy Markets" for guidance.*

Solution: We can recast the market-clearing problem in the lower-level as the following minimization problem, such that:

$$\min_{\boldsymbol{p_g}, \boldsymbol{l_d}} \quad -\sum_{d\in\mathcal{L}} c^D_d \boldsymbol{l_d} + \sum_{g\in\mathcal{G}} c^G_g \boldsymbol{p_g} \tag{34a}$$

$$\text{s.t.} \quad \sum_{g\in\mathcal{G}} \boldsymbol{p_g} - \sum_{d\in\mathcal{L}} \boldsymbol{l_d} = 0 \qquad : \boldsymbol{\lambda} \tag{34b}$$

$$\boldsymbol{p_g} \geq 0 \qquad : \underline{\boldsymbol{\mu}}_{\boldsymbol{g}} \qquad \forall g \in \mathcal{G} \tag{34c}$$

$$-\boldsymbol{p_g} \geq -\overline{P}_g \qquad : \overline{\boldsymbol{\mu}}_{\boldsymbol{g}} \qquad \forall g \in \mathcal{G} \setminus \{G_1\} \tag{34d}$$

$$-\boldsymbol{p_{G_1}} \geq -\boldsymbol{p^{DA}_{G_1}} \qquad : \overline{\boldsymbol{\mu}}_{\boldsymbol{G_1}} \tag{34e}$$

$$\boldsymbol{l_d} \geq \underline{L}_d \qquad : \underline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \qquad \forall d \in \mathcal{L} \tag{34f}$$

$$-\boldsymbol{l_d} \geq -\overline{L}_d \qquad : \overline{\boldsymbol{\sigma}}_{\boldsymbol{d}} \qquad \forall d \in \mathcal{L} \tag{34g}$$

The strong duality theorem for this optimization problem that, at optimality the objective, of the primal and dual problems are equal, i.e.

$$-\sum_{d\in\mathcal{L}} c^D_d \boldsymbol{l_d} + \sum_{g\in\mathcal{G}} c^G_g \boldsymbol{p_g} = -\sum_{g\in\mathcal{G}\setminus\{G_1\}} \overline{P}_g \overline{\boldsymbol{\mu}}_{\boldsymbol{g}} - \boldsymbol{p^{DA}_{G_1}} \overline{\boldsymbol{\mu}}_{\boldsymbol{G_1}} + \sum_{d\in\mathcal{L}} \left(\underline{L}_d \underline{\boldsymbol{\sigma}}_{\boldsymbol{d}} - \overline{L}_d \overline{\boldsymbol{\sigma}}_{\boldsymbol{d}}\right), \tag{35}$$

which can be reformulated as:

$$c^G_{G_1} \boldsymbol{p_{G_1}} + \boldsymbol{p^{DA}_{G_1}} \overline{\boldsymbol{\mu}}_{\boldsymbol{G_1}} = -\sum_{g\in\mathcal{G}\setminus\{G_1\}} \left(c^G_g \boldsymbol{p_g} + \overline{P}_g \overline{\boldsymbol{\mu}}_{\boldsymbol{g}}\right) + \sum_{d\in\mathcal{L}} \left(c^D_d \boldsymbol{l_d} + \underline{L}_d \underline{\boldsymbol{\sigma}}_{\boldsymbol{d}} - \overline{L}_d \overline{\boldsymbol{\sigma}}_{\boldsymbol{d}}\right). \tag{36}$$

Based on the complementarity condition (31), $\boldsymbol{p_{G_1}^{DA}}\boldsymbol{\overline{\mu}_{G_1}} = \boldsymbol{p_{G_1}}\boldsymbol{\overline{\mu}_{G_1}}$. Hence, (36) can be reformulated as:

$$\left(c_{G_1}^G + \boldsymbol{\overline{\mu}_{G_1}}\right)\boldsymbol{p_{G_1}} = -\sum_{g\in\mathcal{G}\setminus\{G_1\}}\left(c_g^G \boldsymbol{p_g} + \overline{P}_g\boldsymbol{\overline{\mu}_g}\right) + \sum_{d\in\mathcal{L}}\left(c_d^D \boldsymbol{l_d} + \underline{L}_d\boldsymbol{\underline{\sigma}_d} - \overline{L}_d\boldsymbol{\overline{\sigma}_d}\right). \tag{37}$$

In addition, the 1st order condition (**??**) for producer G_1, enforces that $c_{G_1}^G + \boldsymbol{\overline{\mu}_{G_1}} = \boldsymbol{\underline{\mu}_{G_1}} + \boldsymbol{\lambda}$. As a result, (37) can be reformulated as:

$$\left(\boldsymbol{\underline{\mu}_{G_1}} + \boldsymbol{\lambda}\right)\boldsymbol{p_{G_1}} = -\sum_{g\in\mathcal{G}\setminus\{G_1\}}\left(c_g^G \boldsymbol{p_g} + \overline{P}_g\boldsymbol{\overline{\mu}_g}\right) + \sum_{d\in\mathcal{L}}\left(c_d^D \boldsymbol{l_d} + \underline{L}_d\boldsymbol{\underline{\sigma}_d} - \overline{L}_d\boldsymbol{\overline{\sigma}_d}\right). \tag{38}$$

Finally, based on the complementarity condition (28) for producer G_1, $\boldsymbol{\underline{\mu}_{G_1}}\boldsymbol{p_{G_1}} = 0$. Hence, (38) gives us the following exact linearization of the bilinear term in the upper-level objective function (33a) such that:

$$\boldsymbol{\lambda}\boldsymbol{p_{G_1}} = -\sum_{g\in\mathcal{G}\setminus\{G_1\}}\left(c_g^G \boldsymbol{p_g} + \overline{P}_g\boldsymbol{\overline{\mu}_g}\right) + \sum_{d\in\mathcal{L}}\left(c_d^D \boldsymbol{l_d} + \underline{L}_d\boldsymbol{\underline{\sigma}_d} - \overline{L}_d\boldsymbol{\overline{\sigma}_d}\right). \tag{39}$$

(d) Solve the resulting optimization problem using a programming language of your choice. You can either reformulate the complementarity conditions of the lower-level problems using the Fortuny-Amat linearization or introduce SOS1 variables. *Please, check the Python tutorial on complementarity modelling for guidance.*

Solution: We can now reformulate the complementarity conditions (27)-(32) using the Fortuny-Amat linearization (or introducing SOS1 variables). The resulting optimization problem is a Mixed Integer Linear Program (MILP) which can be solved using Gurobi. Please see the Python code provided.

(e) Compare the social welfare in the market when all market-participants are price-takers and when one market participant is strategic. What can you deduce with respect to the efficiency and incentive compatibility of this market design?

Solution:
The results of the market-clearing problem with the strategic offer of producer G_1 are:

$$\begin{aligned}
&p_{W_1}^* = 80; p_{G_1}^* = 20; p_{G_2}^* = 0 \text{ (MWh)}\\
&l_{D_1}^* = 100; l_{D_2}^* = 0 \text{ (MWh)}\\
&\lambda^* = 35 \text{ (DKK/MWh)}\\
&p_{G_1}^{DA*} = 20
\end{aligned}$$

We observe that the social welfare of the system is unchanged. However, the surplus of the consumers is decreased, while the profit of the producers is increased.

In addition, we observe that by changing its offering strategy producer G_1, instead of offering its *true* preferences, has managed to increase its own profit. Hence, the marker is *not incentive compatible.*

Since other market participants may be able to exercise market power to strategically increase their profits or surplus, this market outcome does not constitute an equilibrium (i.e. market outcome for which all market participant are satisfied and from which they do not which to deviate). Hence, this market outcome is *not efficient*.