

ECE 6560 Final Project - Image Smoothing

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1 Problem Description

Although image processing techniques began to appear around the 1960s, the proliferation of inexpensive digital computing power has greatly widened the application pool. Image processing techniques can now be found in areas such as digital photography, medical imaging, and object detection/tracking.

All these applications can be affected by a very common problem: image noise. Noise can render further processing ineffective, thus there exist a variety of approaches by which we can attempt to smooth and denoise images. Traditional image processing techniques may rely on fixed kernels that can be swept over an image in order to smooth it. However, this type of approach can result in uniform smoothing that blurs the edges of an image.

We can instead leverage PDEs to describe how an image should be updated depending on its local characteristics. More specifically, we will examine Anisotropic Diffusion. This technique can be used to reduce image noise while lessening the blurring that is done to edges.

Linear diffusion (FIX PUT TV DIFFUSION AND PAPER LINK) filtering as described in [1] (update citation!!!) is a simple, yet effective PDE method for image smoothing. The method has a linear energy functional. This means that at each point in the image, the rate of diffusion is linear. Relatively smooth areas that contain some noise will have the noise lessened, while strong edges will only slowly diffuse (as opposed to a quadratic penalty which would yield a great amount of diffusion).

Instead of continuing the diffusion at any gradient, this project will explore what happens if we attempt to have a near zero amount of diffusion for the higher gradients (where we have edges).

<https://www.mia.uni-saarland.de/weickert/Papers/book.pdf>

2 Mathematical Modeling

Before explicitly developing our PDE, we must first understand the behavior that we want our system to have. The Calculus of Variations is an extremely powerful tool that will help us arrive at a PDE that reflects the desired behavior. Examine the following equation

$$E(I) = \int_0^1 \int_0^1 L(I, I_x, I_y, x, y) dx dy$$

$E(I)$ is the energy functional. Similar to the common understanding of a function, it can output a single value. The key difference is that the input to the energy functional, is another entire function that will depend on traditional variables like x, y , etc.

Note that all notation before section (4) is for continuous space: $x, y \in \mathbb{R}$. Thus, the input to the energy functional can be thought of as an image with an infinite amount of pixels: $I(x, y)$.

During class the following example was introduced. If we wanted to measure how noisy a certain image is, we could measure the magnitude of the image's gradient.

$$E(I) = \int_0^1 \int_0^1 \frac{1}{2} (\|\nabla I\|)^2 dx$$

A noisier image will have a larger value for $E(I)$ while a smoother image will have a smaller value for $E(I)$. By beginning with an energy functional and applying the mechanics of the Calculus of Variations, we arrive at the 2-dimensional linear heat equation $I_t = \Delta I$. Although this process will be shown for an energy functional of our choice in section (3), it is important to understand the implications of the chosen energy functional.

3 Derivation of PDE

Now, let's introduce the Euler-Lagrange equation

$$\begin{aligned} L_f - \frac{\partial}{\partial x} L_{f'} &= 0 \text{ (1-D)} \\ L_I - \frac{\partial}{\partial x} L_{I_x} - \frac{\partial}{\partial y} L_{I_y} &= 0 \text{ (2-D)} \end{aligned}$$

We can begin working towards obtaining our PDE by setting up a gradient descent

$$\begin{aligned} I_t &= -\nabla_I E \\ I_t &= -L_I + \frac{\partial}{\partial x} L_{I_x} + \frac{\partial}{\partial y} L_{I_y} \end{aligned}$$

We will now have to compute terms L_I , L_{I_x} , L_{I_y} using the previously obtained energy functional

$$L(I, I_x, I_y, x, y) = \frac{\lambda}{1+e^\alpha}, \text{ where } \alpha = -\frac{1}{\beta}(\|\nabla_I\|)$$

$$\text{We will also include a term } \epsilon \text{ such that } \|\nabla_I\| = \sqrt{I_x^2 + I_y^2 + \epsilon^2}$$

ϵ is a constant used to prevent stability issues and will be examined more closely in section (4).

$$\begin{aligned} L_I &= \frac{\partial}{\partial I}(L) \\ L_I &= 0 \\ L_{I_x} &= \frac{\partial}{\partial I_x}(L) \\ L_{I_x} &= \frac{\lambda}{\beta} \frac{e^\alpha}{(1+e^\alpha)^2} \frac{I_x}{\sqrt{I_x^2 + I_y^2 + \epsilon^2}} \\ L_{I_y} &= \frac{\partial}{\partial I_y}(L) \\ L_{I_y} &= \frac{\lambda}{\beta} \frac{e^\alpha}{(1+e^\alpha)^2} \frac{I_y}{\sqrt{I_x^2 + I_y^2 + \epsilon^2}} \end{aligned}$$

Now that we have obtained our expressions for L_{I_x} and L_{I_y} , we must compute their partial derivatives as shown by the Euler-Lagrange equation. This will be shown for $\frac{\partial}{\partial x} L_{I_x}$. $\frac{\partial}{\partial y} L_{I_y}$ will be obtained by examining the expression of $\frac{\partial}{\partial x} L_{I_x}$.

Let ϕ denote $\frac{e^\alpha}{(1+e^\alpha)^2}$ and let γ denote $\frac{I_x}{\sqrt{I_x^2 + I_y^2 + \epsilon^2}}$. We can begin finding $\frac{\partial}{\partial x} L_{I_x}$ by using the product-rule $\frac{\partial}{\partial x}(\gamma)\phi + \frac{\partial}{\partial x}(\phi)\gamma$. We will start with the left side of the sum. Note that we must include $\frac{\lambda}{\beta}$ in the final expression.

$$\begin{aligned} &\frac{\partial}{\partial x}(\gamma)\phi \\ &\frac{\partial}{\partial x}\left(\frac{I_x}{\sqrt{I_x^2 + I_y^2 + \epsilon^2}}\right)\phi \\ &\left(\frac{I_{xx}}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} + \frac{I_x}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{3}{2}}}(I_x I_{xx} + I_y I_{xy})\right)\phi \end{aligned}$$

We can now examine the right side of $\frac{\partial}{\partial x}(\gamma)\phi + \frac{\partial}{\partial x}(\phi)\gamma$.

$$\begin{aligned} & \frac{\partial}{\partial x}(\phi)\gamma \\ & \frac{\partial}{\partial x}\left(\frac{e^\alpha}{(1+e^\alpha)^2}\right)\gamma \\ & \frac{\partial}{\partial x}((e^\alpha)(1+e^\alpha)^{-2})\gamma \end{aligned}$$

We see that we will need to again perform the product-rule between (e^α) and $(1+e^\alpha)^{-2}$. Taking the partial derivative of (e^α)

$$-\frac{1}{\beta}e^\alpha \frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}(I_x I_{xx} + I_y I_{xy})$$

Taking the partial derivative of $(1+e^\alpha)^{-2}$

$$-2(1+e^\alpha)^{-3}\left(-\frac{1}{\beta}e^\alpha \frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}(I_x I_{xx} + I_y I_{xy})\right)$$

Thus, after factoring common terms, $\frac{\partial}{\partial x}((e^\alpha)(1+e^\alpha)^{-2})$ yields

$$\left[-\frac{1}{\beta}(e^\alpha)\left(\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}\right)(I_x I_{xx} + I_y I_{xy})\right][(1+e^\alpha)^{-2} + (e^\alpha)(-2(1+e^\alpha)^{-3})]$$

We have reached the final expression for $\frac{\partial}{\partial x}L_{I_x}$

$$\begin{aligned} \frac{\partial}{\partial x}L_{I_x} = & \frac{\lambda}{\beta}\left[\left(\frac{I_{xx}}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}\right) - \left(\frac{I_x}{(I_x^2+I_y^2+\epsilon^2)^{\frac{3}{2}}}\right)(I_x I_{xx} + I_y I_{xy})\left(\frac{e^\alpha}{(1+e^\alpha)^2}\right) + \right. \\ & \left. \left(\frac{I_x}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}\right)\left[-\frac{1}{\beta}(e^\alpha)\left(\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}\right)(I_x I_{xx} + I_y I_{xy})\right][(1+e^\alpha)^{-2} + (e^\alpha)(-2(1+e^\alpha)^{-3})]\right] \end{aligned}$$

$\frac{\partial}{\partial y}L_{I_y}$ can be obtained by modifying $\frac{\partial}{\partial x}L_{I_x}$

$$\begin{aligned} \frac{\partial}{\partial y}L_{I_y} = & \frac{\lambda}{\beta}\left[\left(\frac{I_{yy}}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}\right) - \left(\frac{I_y}{(I_x^2+I_y^2+\epsilon^2)^{\frac{3}{2}}}\right)(I_x I_{xy} + I_y I_{yy})\left(\frac{e^\alpha}{(1+e^\alpha)^2}\right) + \right. \\ & \left. \left(\frac{I_y}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}\right)\left[-\frac{1}{\beta}(e^\alpha)\left(\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}\right)(I_x I_{xy} + I_y I_{yy})\right][(1+e^\alpha)^{-2} + (e^\alpha)(-2(1+e^\alpha)^{-3})]\right] \end{aligned}$$

Our final gradient-descent PDE is

$$\begin{aligned}
I_t = & \frac{\lambda}{\beta} \left[\left(\frac{I_{xx}}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) - \left(\frac{I_x}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{3}{2}}} \right) (I_x I_{xx} + I_y I_{xy}) \left(\frac{e^\alpha}{(1+e^\alpha)^2} \right) + \right. \\
& \left. \left(\frac{I_x}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) \left[-\frac{1}{\beta} (e^\alpha) \left(\frac{1}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) (I_x I_{xx} + I_y I_{xy}) \right] [(1+e^\alpha)^{-2} + (e^\alpha)(-2(1+e^\alpha)^{-3})] + \right. \\
& \left. \left(\frac{I_{yy}}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) - \left(\frac{I_y}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{3}{2}}} \right) (I_x I_{xy} + I_y I_{yy}) \left(\frac{e^\alpha}{(1+e^\alpha)^2} \right) + \right. \\
& \left. \left(\frac{I_y}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) \left[-\frac{1}{\beta} (e^\alpha) \left(\frac{1}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) (I_x I_{xy} + I_y I_{yy}) \right] [(1+e^\alpha)^{-2} + (e^\alpha)(-2(1+e^\alpha)^{-3})] \right]
\end{aligned}$$

Where $\alpha = -\frac{1}{\beta} (\|\nabla I\|)$ and $\|\nabla I\| = \sqrt{I_x^2 + I_y^2 + \epsilon^2}$

4 Discretization and Implementation

The PDE that was obtained in the previous section is only applicable for continuous time and space variables. We must discretize the PDE so that it can actually be implemented in software.

4.1 Numerical Differentiation

Before explicitly discretizing the PDE, we must also obtain numeric approximations for its partial derivatives. Beginning with the left side of the PDE, let's approximate I_t . It is important to understand that the PDE is defining how the image will be smoothed over successive iterations. In other words, it specifies how the image is updated as time increases. Thus, it is appropriate to use a forward difference.

The forward difference for a single (space) variable can be obtained using the numeric differentiation method shown in class

$$f(x, t) = f(x, t)$$

$$f(x, t + \Delta t) = f(x, t) + \Delta t f'(x, t) + O(\Delta t^2)$$

After eliminating $f(x, t)$ terms on the right side of the above equations, we obtain our approximation. Note that the approximation includes an error term $O(\Delta t)$.

$$f'(x, t) = \frac{f(x, t + \Delta t) - f(x, t)}{\Delta t}$$

Expanding this process to two (space) dimensions yields

$$I_t(x, y, t) = \frac{I(x, y, t + \Delta t) - I(x, y, t)}{\Delta t}$$

Continuing with the right side of the PDE, approximations are needed for the following partial derivatives: I_x , I_y , I_{xx} , I_{yy} , and I_{xy} . Since these approximations capture how the image is changing with respect to space, it is more appropriate to use a central difference rather than a forward difference.

Suppose we are trying to approximate I_x . A forward difference would introduce some bias because the value of I_x at a certain point is dependent on an adjacent image value in the positive x-direction only. The central difference will balance out the value of I_x since both directions are considered. The intent is to make the approximation more robust to variations in image content.

The central difference for a single (space) variable can be obtained using the numeric differentiation method shown in class

$$f(x, t) = f(x, t)$$

$$f(x + \Delta x, t) = f(x, t) + \Delta x f'(x, t) + O(\Delta x^2)$$

$$f(x - \Delta x, t) = f(x, t) - \Delta x f'(x, t) + O(\Delta x^2)$$

After eliminating $f(x, t)$ terms on the right side of the above equations, we obtain our approximation. Note that the approximation includes an error term $O(\Delta x)$.

$$f'(x, t) = \frac{f(x + \Delta x, t) - f(x - \Delta x, t)}{2\Delta x}$$

Expanding this process to two (space) dimensions yields

$$I_x(x, y, t) = \frac{I(x + \Delta x, y, t) - I(x - \Delta x, y, t)}{2\Delta x}$$

$$I_y(x, y, t) = \frac{I(x, y + \Delta y, t) - I(x, y - \Delta y, t)}{2\Delta y}$$

In order to obtain approximations for I_{xx} and I_{yy} , we can simply add additional terms to the Taylor Series expansion and then repeat the elimination process

$$I_{xx}(x, y, t) = \frac{I(x + \Delta x, y, t) - 2I(x, y, t) + I(x - \Delta x, y, t)}{\Delta x^2}$$

$$I_{yy}(x, y, t) = \frac{I(x, y + \Delta y, t) - 2I(x, y, t) + I(x, y - \Delta y, t)}{\Delta y^2}$$

Finally, we need to obtain an approximation for the mixed partial derivative I_{xy} . This is achieved by taking the central difference approximation for I_x and applying another central difference with respect to y

$$I_x(x, y, t) = \frac{I(x + \Delta x, y, t)}{2\Delta x} - \frac{I(x - \Delta x, y, t)}{2\Delta x}$$

$$I_{xy}(x, y, t) = \frac{\partial}{\partial y} \left(\frac{I(x + \Delta x, y, t)}{2\Delta x} \right) - \frac{\partial}{\partial y} \left(\frac{I(x - \Delta x, y, t)}{2\Delta x} \right)$$

$$I_{xy}(x, y, t) = \frac{1}{2\Delta y} \left(\frac{I(x + \Delta x, y + \Delta y, t)}{2\Delta x} - \frac{I(x + \Delta x, y - \Delta y, t)}{2\Delta x} \right) - \frac{1}{2\Delta y} \left(\frac{I(x - \Delta x, y + \Delta y, t)}{2\Delta x} - \frac{I(x - \Delta x, y - \Delta y, t)}{2\Delta x} \right)$$

$$I_{xy}(x, y, t) = \frac{I(x + \Delta x, y + \Delta y, t) - I(x + \Delta x, y - \Delta y, t) - I(x - \Delta x, y + \Delta y, t) + I(x - \Delta x, y - \Delta y, t)}{4\Delta x \Delta y}$$

4.2 Parameter Selection

Now that we have obtained all the necessary numeric approximations of the partial derivatives, we can move towards defining a scheme that discretizes the PDE. This involves selecting some important parameters.

First, we will examine what is an appropriate time-step Δt . Previously, we defined $I_t(x, y, t) = \frac{I(x, y, t + \Delta t) - I(x, y, t)}{\Delta t}$. In our implementation, Δt will multiply the right side of the PDE. The product will then be added to the current image to obtain the updated image. Thus, we must find a CFL condition so that we can find an appropriate time-step. Recall that during the derivation of the PDE, a term ϵ was included so that $\|\nabla I\| = \sqrt{I_x^2 + I_y^2 + \epsilon^2}$. Suppose the gradient became very small, all terms I_x and I_y in the PDE would go to zero and we would be left with $I_t = \frac{\lambda}{\beta} \left[\frac{I_{xx} + I_{yy}}{(\epsilon^2)^{\frac{1}{2}}} \right]$. Then we simply have the heat equation

$$I_t = \frac{\lambda}{\beta \epsilon} \Delta I$$

We can include our constants with the form of the CFL condition for the two-dimensional linear heat equation as shown in class

$$\frac{\lambda}{\beta \epsilon} \leq \frac{1}{4} \Delta x^2$$

$$\Delta t \leq \frac{1}{4} \frac{\beta \epsilon}{\lambda} \Delta x^2$$

With a bound set for Δt , we can now shift our focus to Δx and Δy . Since a digital image is composed of a finite number of individual pixels, we will compute the different partial derivative approximations by simply taking our current pixel and applying an offset of +1 or -1 to the current pixel's x and y indices. For example, for I_x

$$I_x(x, y, t) = \frac{I(x + \Delta x, y, t)}{2\Delta x} - \frac{I(x - \Delta x, y, t)}{2\Delta x}, \text{ we can compute } I_x \text{ at pixel } (a, b) \text{ as}$$

$$I_x(a, b, t) = \frac{I(a + 1, b, t)}{2(1)} - \frac{I(a - 1, b, t)}{2(1)}$$

Finally, we will iterate through each pixel in the image and compute how the pixel should be updated based on the PDE.

4.3 Implementation Summary

PDE

$$\begin{aligned}
I_t = & \frac{\lambda}{\beta} \left[\left(\frac{I_{xx}}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) - \left(\frac{I_x}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{3}{2}}} \right) (I_x I_{xx} + I_y I_{xy}) \left(\frac{e^\alpha}{(1+e^\alpha)^2} \right) + \right. \\
& \left. \left(\frac{I_x}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) \left[-\frac{1}{\beta} (e^\alpha) \left(\frac{1}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) (I_x I_{xx} + I_y I_{xy}) \right] [(1+e^\alpha)^{-2} + (e^\alpha)(-2(1+e^\alpha)^{-3})] \right] + \\
& \left(\frac{I_{yy}}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) - \left(\frac{I_y}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{3}{2}}} \right) (I_x I_{xy} + I_y I_{yy}) \left(\frac{e^\alpha}{(1+e^\alpha)^2} \right) + \\
& \left(\frac{I_y}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) \left[-\frac{1}{\beta} (e^\alpha) \left(\frac{1}{(I_x^2 + I_y^2 + \epsilon^2)^{\frac{1}{2}}} \right) (I_x I_{xy} + I_y I_{yy}) \right] [(1+e^\alpha)^{-2} + (e^\alpha)(-2(1+e^\alpha)^{-3})]
\end{aligned}$$

Where $\alpha = -\frac{1}{\beta}(\|\nabla I\|)$ and $\|\nabla I\| = \sqrt{I_x^2 + I_y^2 + \epsilon^2}$

Partial Derivatives

$$I_t(x, y, t) = \frac{I(x, y, t + \Delta t) - I(x, y, t)}{\Delta t}$$

$$I_x(x, y, t) = \frac{I(x + \Delta x, y, t) - I(x - \Delta x, y, t)}{2\Delta x}$$

$$I_y(x, y, t) = \frac{I(x, y + \Delta y, t) - I(x, y - \Delta y, t)}{2\Delta y}$$

$$I_{xx}(x, y, t) = \frac{I(x + \Delta x, y, t) - 2I(x, y, t) + I(x - \Delta x, y, t)}{\Delta x^2}$$

$$I_{yy}(x, y, t) = \frac{I(x, y + \Delta y, t) - 2I(x, y, t) + I(x, y - \Delta y, t)}{\Delta y^2}$$

$$I_{xy}(x, y, t) = \frac{I(x + \Delta x, y + \Delta y, t) - I(x + \Delta x, y - \Delta y, t) - I(x - \Delta x, y + \Delta y, t) + I(x - \Delta x, y - \Delta y, t)}{4\Delta x \Delta y}$$

CFL Condition

$$\Delta t \leq \frac{1}{4} \frac{\beta \epsilon}{\lambda} \Delta x^2$$

Parameters

$$\Delta t, \Delta x, \Delta y, \beta, \epsilon, \lambda$$

Format of Implemented PDE

$$I(x, y, t + \Delta t) = I(x, y, t) + \Delta t(I_t)$$

5 Experimental Procedure

6 Experimental Results

7 Summary

References

- [1] Thomas, L. & Ari, R. d. *Biological Feedback* (CRC Press, USA, 1990).