ECE 6560 Final Project - Image Smoothing

Alexander Domagala

Spring 2024

1 Problem Description

Although image processing techniques began to appear around the 1960s, the proliferation of inexpensive digital computing power has greatly widened the application pool. Image processing techniques can now be found in areas such as digital photography, medical imaging, and object detection/tracking.

All these applications can be affected by a very common problem: image noise. Noise can render further processing ineffective, thus there exist a variety of approaches by which we can attempt to smooth and denoise images. More traditional image processing techniques may involve fixed convolutional kernels, eg. box blur [2], that indicate how a specific pixel can be updated as a weighted sum of its neighbors. Approaches of this nature will reduce the noise in exchange for the image appearing blurred.

We can instead leverage PDEs to describe how an image should be updated depending on its local characteristics. Approaches that leverage PDEs offer noise reduction while also lessening the blurring that is done to edges.

In this project, we will explore how we can tailor a PDE-based filter to suit a specific image. The goal is that by understanding the characteristics of the image's edges and noise, we will be able to outperform a more generalized PDE-based filter. The filter will be developed using techniques related to Anisotropic Diffusion.

2 Mathematical Modeling

2.1 Introduction

Before explicitly developing our PDE, we must first understand the behavior that we want our system to have. We will leverage the Calculus of Variations which is an extremely powerful tool that will help arrive at a PDE that reflects some desired behavior. Examine the following equation

$$E(I) = \int_0^1 \int_0^1 L(I, I_x, I_y, x, y)dxdy$$

$E(I)$ is the energy functional. Similar to the common understanding of a function, it can output a single value. The key difference is that the input to the energy functional, is another entire function that will depend on traditional variables like x, y, etc.

Note that all notation before section (4) is for continuous space: $x, y \in \mathbb{R}$. Thus, the input to the energy functional can be thought of as an image with an infinite amount of pixels: $I(x, y)$.

During class, the following example was introduced. If we want to measure how noisy a certain image is, we could measure the average value of the image's gradient squared.

$$E(I) = \int_0^1 \int_0^1 \frac{1}{2}(\|\nabla_I\|)^2 dx$$

Noiser images will have larger values for $E(I)$ while smoother images will have a smaller values for $E(I)$. Let's examine how the design of an energy functional impacts image smoothing.

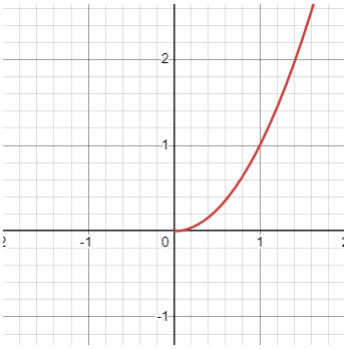
PROBABLY REMOVE
In traditional calculus, if given some function, we can obtain its derivative and find a value that minimizes the function. In the Calculus of Variations, if given some energy functional, we can obtain a PDE (via gradient descent) that sets constraints so that a function can minimize the energy functional.

In the case of the example, minimization yields the Linear Heat Equation

$$I_t = \Delta I$$

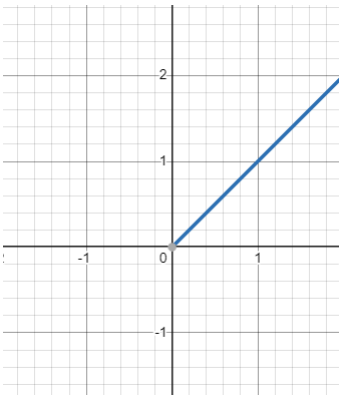
The Linear Heat Equation optimally reduces the average squared value of the gradient in an image. This means that areas of high gradient in the image will be aggresively smoothed. Hence, the edges (which are areas of higher gradient) are not very well preserved with this technique.

Quadratic Penalty: x^2



A major improvement is to instead use a linear penalty. Smoothing that takes place at the edges will be less extreme than the previous case. This means that enough diffusion could take place in areas of low gradient such that the image appears to be denoised. Then, the diffusion can be stopped. Although the edges also experience diffusion, the penalty is much more lenient than the previous case. This method is known as Total Variation Diffusion.

Linear Penalty: x



2.2 Anisotropic Diffusion

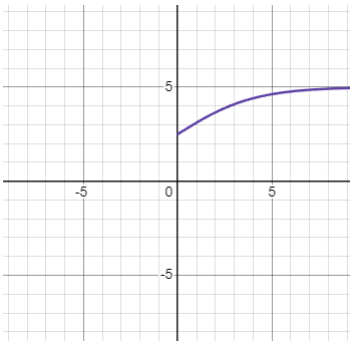
With an understading of energy functionals and how their design will affect image smoothing, we can now propose an energy functional that can be adapted to suit images in which edge preservation is very important.

This generalized approach that uses a penalty that is a function of the magnitude of the gradient is known as Anisotropic or Perona-Malik Diffusion.

$$E(I) = \iint C(\|\nabla_I\|)dxdy$$

The goal of this penalty is to have more smoothing at lower gradients and transition to very strong edge preservation for high gradients. A nearly flat penalty means that almost no diffusion will be occurring for higher gradients. Thus, our filter will only be suited for removing noise that doesn't strongly contrast with the image.

Sigmoidal Penalty: $\frac{\lambda}{1+e^{-x}}$



$\|\nabla_I\|$

3 Derivation of PDE

Now, let's introduce the Euler-Lagrange equation

$$L_f - \frac{\partial}{\partial x} L_{f'} = 0 \text{ (1-D)}$$
$$L_I - \frac{\partial}{\partial x} L_{I_x} - \frac{\partial}{\partial y} L_{I_y} = 0 \text{ (2-D)}$$

We can begin working towards obtaining our PDE by setting up a gradient descent

$$I_t = -\nabla_I E$$
$$I_t = -L_I + \frac{\partial}{\partial x} L_{I_x} + \frac{\partial}{\partial y} L_{I_y}$$

We will now have to compute terms L_I, L_{I_x}, L_{I_y} using the previously obtained energy functional

$L(I, I_x, I_y, x, y) = \frac{\lambda}{1+e^{\alpha}}$, where $\alpha = -\frac{1}{\beta}(\|\nabla_I\|)$
We will also include a term ϵ such that $\|\nabla_I\| = \sqrt{I_x^2 + I_y^2 + \epsilon^2}$
ϵ is a constant used to prevent stability issues and will be examined more closely in section (4).

$L_I = \frac{\partial}{\partial I}(L)$
$L_I = 0$
$L_{I_x} = \frac{\partial}{\partial I_x}(L)$
$L_{I_x} = \frac{\lambda}{\beta} \frac{e^{\alpha}}{(1+e^{\alpha})^2} \frac{I_x}{\sqrt{I_x^2+I_y^2+\epsilon^2}}$
$L_{I_y} = \frac{\partial}{\partial I_y}(L)$
$L_{I_y} = \frac{\lambda}{\beta} \frac{e^{\alpha}}{(1+e^{\alpha})^2} \frac{I_y}{\sqrt{I_x^2+I_y^2+\epsilon^2}}$

Now that we have obtained our expressions for L_{I_x} and L_{I_y}, we must compute their partial derivatives as shown by the Euler-Lagrange equation. This will be shown for $\frac{\partial}{\partial x} L_{I_x}$. $\frac{\partial}{\partial y} L_{I_y}$ will be obtained by examining the expression of $\frac{\partial}{\partial x} L_{I_x}$.

Let ϕ denote $\frac{e^{\alpha}}{(1+e^{\alpha})^2}$ and let γ denote $\frac{I_x}{\sqrt{I_x^2+I_y^2+\epsilon^2}}$. We can begin finding $\frac{\partial}{\partial x} L_{I_x}$ by using the product-rule $\frac{\partial}{\partial x}(\gamma)\phi + \frac{\partial}{\partial x}(\phi)\gamma$. We will start with the left side of the sum. Note that we must include $\frac{\lambda}{\beta}$ in the final expression.

$$\frac{\partial}{\partial x}(\gamma)\phi$$

$$\frac{\partial}{\partial x}\left(\frac{I_x}{\sqrt{I_x^2+I_y^2+\epsilon^2}}\right)\phi$$

$$\left(\frac{I_{xx}}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}} + \frac{I_x}{(I_x^2+I_y^2+\epsilon^2)^{\frac{3}{2}}}(I_x I_{xx} + I_y I_{xy})\right)\phi$$

We can now examine the right side of $\frac{\partial}{\partial x}(\gamma)\phi + \frac{\partial}{\partial x}(\phi)\gamma$.

$$\frac{\partial}{\partial x}(\phi)\gamma$$
$$\frac{\partial}{\partial x}(\frac{e^{\alpha}}{(1+e^{\alpha})^2})\gamma$$
$$\frac{\partial}{\partial x}((e^{\alpha})(1+e^{\alpha})^{-2})\gamma$$

We see that we will need to again perform the product-rule between (e^{α}) and $(1+e^{\alpha})^{-2}$. Taking the partial derivative of (e^{α})

$$-\frac{1}{\beta}e^{\alpha}\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}(I_xI_{xx}+I_yI_{xy})$$

Taking the partial derivative of $(1+e^{\alpha})^{-2}$

$$-2(1+e^{\alpha})^{-3}(-\frac{1}{\beta}e^{\alpha}\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}(I_xI_{xx}+I_yI_{xy}))$$

Thus, after factoring common terms, $\frac{\partial}{\partial x}((e^{\alpha})(1+e^{\alpha})^{-2})$ yields

$$[-\frac{1}{\beta}(e^{\alpha})(\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})(I_xI_{xx}+I_yI_{xy})][(1+e^{\alpha})^{-2}+(e^{\alpha})(-2(1+e^{\alpha})^{-3})]$$

We have reached the final expression for $\frac{\partial}{\partial x}L_{I_x}$

$$\frac{\partial}{\partial x}L_{I_x} =$$

$$\frac{\lambda}{\beta}[(\frac{I_{xx}}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}) - (\frac{I_x}{(I_x^2+I_y^2+\epsilon^2)^{\frac{3}{2}}})(I_xI_{xx}+I_yI_{xy})(\frac{e^{\alpha}}{(1+e^{\alpha})^2})+$$

$$(\frac{I_x}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})[-\frac{1}{\beta}(e^{\alpha})(\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})(I_xI_{xx}+I_yI_{xy})][(1+e^{\alpha})^{-2}+(e^{\alpha})(-2(1+e^{\alpha})^{-3})]]$$

$\frac{\partial}{\partial y}L_{I_y}$ can be obtained by modifying $\frac{\partial}{\partial x}L_{I_x}$

$$\frac{\partial}{\partial y}L_{I_y} =$$

$$\frac{\lambda}{\beta}[(\frac{I_{yy}}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}) - (\frac{I_y}{(I_x^2+I_y^2+\epsilon^2)^{\frac{3}{2}}})(I_xI_{xy}+I_yI_{yy})(\frac{e^{\alpha}}{(1+e^{\alpha})^2})+$$

$$(\frac{I_y}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})[-\frac{1}{\beta}(e^{\alpha})(\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})(I_xI_{xy}+I_yI_{yy})][(1+e^{\alpha})^{-2}+(e^{\alpha})(-2(1+e^{\alpha})^{-3})]]$$

Our final gradient-descent PDE is

$$I_t = \frac{\lambda}{\beta}[(\frac{I_{xx}}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}) - (\frac{I_x}{(I_x^2+I_y^2+\epsilon^2)^{\frac{3}{2}}})(I_xI_{xx}+I_yI_{xy})(\frac{e^\alpha}{(1+e^\alpha)^2})+$$

$$(\frac{I_x}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})[-\frac{1}{\beta}(e^\alpha)(\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})(I_xI_{xx}+I_yI_{xy})][(1+e^\alpha)^{-2}+(e^\alpha)(-2(1+e^\alpha)^{-3})]+$$

$$(\frac{I_{yy}}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}) - (\frac{I_y}{(I_x^2+I_y^2+\epsilon^2)^{\frac{3}{2}}})(I_xI_{xy}+I_yI_{yy})(\frac{e^\alpha}{(1+e^\alpha)^2})+$$

$$(\frac{I_y}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})[-\frac{1}{\beta}(e^\alpha)(\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})(I_xI_{xy}+I_yI_{yy})][(1+e^\alpha)^{-2}+(e^\alpha)(-2(1+e^\alpha)^{-3})]]$$

Where $\alpha = -\frac{1}{\beta}(\|\nabla_I\|)$ and $\|\nabla_I\| = \sqrt{I_x^2+I_y^2+\epsilon^2}$

4 Discretization and Implementation

The PDE that was obtained in the previous section is only applicable for continuous time and space variables. We must discretize the PDE so that it can actually be implemented in software.

4.1 Numerical Differentiation

Before explicitly discretizing the PDE, we must also obtain numeric approximations for its partial derivatives. Beginning with the left side of the PDE, let's approximate I_t. It is important to understand that the PDE is defining how the image will be smoothed over successive iterations. In other words, it specifies how the image is updated as time increases. Thus, it is appropriate to use a forward difference.

The forward difference for a single (space) variable can be obtained using the numeric differentiation method shown in class

$$f(x,t) = f(x,t)$$

$$f(x,t+\Delta t) = f(x,t) + \Delta t f'(x,t) + O(\Delta t^2)$$

After eliminating $f(x,t)$ terms on the right side of the above equations, we obtain our approximation. Note that the approximation includes an error term $O(\Delta t)$.

$$f'(x,t) = \frac{f(x,t+\Delta t) - f(x,t)}{\Delta t}$$

Expanding this process to two (space) dimensions yields

$$I_t(x,y,t) = \frac{I(x,y,t+\Delta t) - I(x,y,t)}{\Delta t}$$

Continuing with the right side of the PDE, approximations are needed for the following partial derivatives: I_x, I_y, I_{xx}, I_{yy}, and I_{xy}. Since these approximations capture how the image is changing with respect to space, it is more appropriate to use a central difference rather than a forward difference.

Suppose we are trying to approximate I_x. A forward difference would introduce some bias because the value of I_x at a certain point is dependent on an adjacent image value in the positive x-direction only. The central difference will balance out the value of I_x since both directions are considered. The intent is to make the approximation more robust to variations in image content.

The central difference for a single (space) variable can be obtained using the numeric differentiation method shown in class

$$f(x,t) = f(x,t)$$

$$f(x+\Delta x,t) = f(x,t) + \Delta x f'(x,t) + O(\Delta x^2)$$

$$f(x-\Delta x,t) = f(x,t) - \Delta x f'(x,t) + O(\Delta x^2)$$

After eliminating $f(x,t)$ terms on the right side of the above equations, we obtain our approximation. Note that the approximation includes an error term $O(\Delta x)$.

$$f'(x,t) = \frac{f(x+\Delta x,t)-f(x-\Delta x,t)}{2\Delta x}$$

Expanding this process to two (space) dimensions yields

$$I_x(x,y,t) = \frac{I(x+\Delta x,y,t)-I(x-\Delta x,y,t)}{2\Delta x}$$

$$I_y(x,y,t) = \frac{I(x,y+\Delta y,t)-I(x,y-\Delta y,t)}{2\Delta y}$$

In order to obtain approximations for I_{xx} and I_{yy}, we can simply add additional terms to the Taylor Series expansion and then repeat the elimination process

$$I_{xx}(x,y,t) = \frac{I(x+\Delta x,y,t)-2I(x,y,t)+I(x-\Delta x,y,t)}{\Delta x^2}$$

$$I_{yy}(x,y,t) = \frac{I(x,y+\Delta y,t)-2I(x,y,t)+I(x,y-\Delta y,t)}{\Delta y^2}$$

Finally, we need to obtain an approximation for the mixed partial derivative I_{xy}. This is achieved by taking the central difference approximation for I_x and applying another central difference with respect to y

$$I_x(x,y,t) = \frac{I(x+\Delta x,y,t)}{2\Delta x} - \frac{I(x-\Delta x,y,t)}{2\Delta x}$$

$$I_{xy}(x,y,t) = \frac{\partial}{\partial y}\left(\frac{I(x+\Delta x,y,t)}{2\Delta x}\right) - \frac{\partial}{\partial y}\left(\frac{I(x-\Delta x,y,t)}{2\Delta x}\right)$$

$$I_{xy}(x,y,t) = \frac{1}{2\Delta y}\left(\frac{I(x+\Delta x,y+\Delta y,t)}{2\Delta x} - \frac{I(x+\Delta x,y-\Delta y,t)}{2\Delta x}\right) - \frac{1}{2\Delta y}\left(\frac{I(x-\Delta x,y+\Delta y,t)}{2\Delta x} - \frac{I(x-\Delta x,y-\Delta y,t)}{2\Delta x}\right)$$

$$I_{xy}(x,y,t) = \frac{I(x+\Delta x,y+\Delta y,t)-I(x+\Delta x,y-\Delta y,t)-I(x-\Delta x,y+\Delta y,t)+I(x-\Delta x,y-\Delta y,t)}{4\Delta x\Delta y}$$

4.2 Parameter Selection

Now that we have obtained all the necessary numeric approximations of the partial derivatives, we can move towards defining a scheme that discretizes the PDE. This involves selecting some important parameters.

First, we will examine what is an appropriate time-step Δt. Previously, we defined $I_t(x, y, t) = \frac{I(x,y,t+\Delta t)-I(x,y,t)}{\Delta t}$. In our implementation, Δt will multiply the right side of the PDE. The product will then be added to the current image to obtain the updated image. Thus, we must find a CFL condition so that we can find an appropriate time-step. Recall that during the derivation of the PDE, a term ϵ was included so that $\|\nabla_I\| = \sqrt{I_x^2 + I_y^2 + \epsilon^2}$. Suppose the gradient became very small, all terms I_x and I_y in the PDE would go to zero and we would be left with $I_t = \frac{\lambda}{\beta}[(\frac{I_{xx}+I_{yy}}{(\epsilon^2)^{\frac{1}{2}}})]$. Then we simply have the heat equation

$$I_t = \frac{\lambda}{\beta\epsilon}\Delta I$$

We can include our constants with the form of the CFL condition for the two-dimensional linear heat equation as shown in class

$$\frac{\lambda}{\beta\epsilon} \leq \frac{1}{4}\Delta x^2$$

$$\Delta t \leq \frac{1}{4}\frac{\beta\epsilon}{\lambda}\Delta x^2$$

With a bound set for Δt, we can now shift our focus to Δx and Δy. Since a digital image is composed of a finite number of individual pixels, we will compute the different partial derivative approximations by simply taking our current pixel and applying an offset of +1 or -1 to the current pixel's x and y indices. For example, for I_x

$I_x(x, y, t) = \frac{I(x+\Delta x,y,t)}{2\Delta x} - \frac{I(x-\Delta x,y,t)}{2\Delta x}$, we can compute I_x at pixel (a, b) as

$I_x(a, b, t) = \frac{I(a+1,b,t)}{2(1)} - \frac{I(a-1,b,t)}{2(1)}$

Finally, we will iterate through each pixel in the image and compute how the pixel should be updated based on the PDE.

4.3 Implementation Summary

PDE

$$I_t = \frac{\lambda}{\beta}[(\frac{I_{xx}}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}) - (\frac{I_x}{(I_x^2+I_y^2+\epsilon^2)^{\frac{3}{2}}})(I_xI_{xx}+I_yI_{xy})(\frac{e^\alpha}{(1+e^\alpha)^2})+$$

$$(\frac{I_x}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})[-\frac{1}{\beta}(e^\alpha)(\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})(I_xI_{xx}+I_yI_{xy})][(1+e^\alpha)^{-2}+(e^\alpha)(-2(1+e^\alpha)^{-3})]+$$

$$(\frac{I_{yy}}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}}) - (\frac{I_y}{(I_x^2+I_y^2+\epsilon^2)^{\frac{3}{2}}})(I_xI_{xy}+I_yI_{yy})(\frac{e^\alpha}{(1+e^\alpha)^2})+$$

$$(\frac{I_y}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})[-\frac{1}{\beta}(e^\alpha)(\frac{1}{(I_x^2+I_y^2+\epsilon^2)^{\frac{1}{2}}})(I_xI_{xy}+I_yI_{yy})][(1+e^\alpha)^{-2}+(e^\alpha)(-2(1+e^\alpha)^{-3})]]$$

Where $\alpha = -\frac{1}{\beta}(\|\nabla_I\|)$ and $\|\nabla_I\| = \sqrt{I_x^2+I_y^2+\epsilon^2}$

Partial Derivatives

$$I_t(x,y,t) = \frac{I(x,y,t+\Delta t)-I(x,y,t)}{\Delta t}$$

$$I_x(x,y,t) = \frac{I(x+\Delta x,y,t)}{2\Delta x} - \frac{I(x-\Delta x,y,t)}{2\Delta x}$$

$$I_y(x,y,t) = \frac{I(x,y+\Delta y,t)-I(x,y-\Delta y,t)}{2\Delta y}$$

$$I_{xx}(x,y,t) = \frac{I(x+\Delta x,y,t)-2I(x,y,t)+I(x-\Delta x,y,t)}{\Delta x^2}$$

$$I_{yy}(x,y,t) = \frac{I(x,y+\Delta y,t)-2I(x,y,t)+I(x,y-\Delta y,t)}{\Delta y^2}$$

$$I_{xy}(x,y,t) = \frac{I(x+\Delta x,y+\Delta y,t)-I(x+\Delta x,y-\Delta y,t)-I(x-\Delta x,y+\Delta y,t)+I(x-\Delta x,y-\Delta y,t)}{4\Delta x\Delta y}$$

CFL Condition

$\Delta t \leq \frac{1}{4}\frac{\beta\epsilon}{\lambda}\Delta x^2$

Parameters

$\Delta t, \Delta x, \Delta y, \beta, \epsilon, \lambda$

Format of Implemented PDE

$I(x, y, t + \Delta t) = I(x, y, t) + \Delta t(I_t)$

5 Experimental Procedure

1. Select image and load into program.
2. Run gradient histogram to understand characteristics of image.
3. Update smoothing parameters.
4. Run smoothing algorithm on image, measure and record MSE during each iteration.
5. After MSE reaches minimum and begins to increase, run 20 iterations past this point.

6 Experimental Results

7 Summary

References

[1] Thomas, L. & Ari, R. d. *Biological Feedback* (CRC Press, USA, 1990).

[2] Thomas, L. & Ari, R. d. *Biological Feedback* (CRC Press, USA, 1990).