AAE 5626: Orbital Mechanics for Engineers

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Perturbations: Non-Keplerian Motion

Our beloved two body problem is exactly what it sounds like - the motion of two point masses (or equivalently, perfectly symmetrical spheres, both in geometry and mass distribution) under the influence of their gravitational fields. The resulting motion is called *Keplerian* and we know that the governing dynamics is:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} \tag{1}$$

We also know that for the above model, we can define orbital elements, namely a, e, i, ω and Ω. Together, these parameters comprehensively define the shape, size and orientation of the orbit and are *constants over time.*

Reality however, is more complicated. There are always perturbations; due to which the model of Eq.(1) must be modified as follows:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r} + \mathbf{a}_d \tag{2}$$

where, $\mathbf{a}_d$ represents a *disturbing acceleration*, or *perturbing acceleration* which spoils the structure of pure Keplerian motion. The origins of perturbations are many:

i. non-symmetrical nature of the objects involved. In the Earth-spacecraft system, this manifests due to the oblateness of Earth and its asymmetrical mass-distribution.

ii. atmospheric drag

iii. n-body effects (e.g. solar and lunar pull in addition to that of Earth)

iv. solar radiation pressure

v. leaking propellants, etc.

Each of the above perturbative effects is a modeling nightmare. Stated differently, it is <u>very</u> difficult to obtain accurate expressions of the term $\mathbf{a}_d$ corresponding to the above effects. In general (i.e. irrespective of its source), we can write the disturbing acceleration in terms of its components in a local $\hat{\mathbf{e}}_r - \hat{\mathbf{e}}_f$ frame as

$$\mathbf{a}_d = a_r\hat{\mathbf{e}}_r + a_\perp\hat{\mathbf{e}}_f + a_h\hat{\mathbf{i}}_h \tag{3}$$

where, $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_f$ are the local radial and tangential directions respectively and $\hat{\mathbf{i}}_h$ is normal to the $\hat{\mathbf{e}}_r - \hat{\mathbf{e}}_f$ plane. Now, it turns out that under the dynamical model of Eq.(2), we can still define orbital elements, namely a, e, i, ω and Ω, but they are no longer time invariant.

- Let us look at the time variation of the semi-major axis based on the acceleration model of Eq.(2). Recall that from vis-viva,

$$\frac{1}{a} = \frac{2}{r} - \frac{\mathbf{v}\cdot\mathbf{v}}{\mu} \tag{4}$$

where of course, $\mathbf{v}\cdot\mathbf{v} = v^2$. You will soon appreciate the reason behind the "unnecessarily" complicated form of this term in Eq.(4).

Since the semi-major axis is no longer a constant, we can differentiate the above equation without embarrassing ourselves (e.g. ending up with something like $0 = 0$):

$$-\frac{\dot{a}}{a^2} = -\frac{2}{r^2}\dot{r} - \frac{2\mathbf{v}\cdot\mathbf{a}}{\mu} \tag{5}$$

where $\mathbf{a} = \dot{\mathbf{v}}$ is the total acceleration. This is nothing but the RHS of Eq.(2). Recall that in the local $(\hat{\mathbf{e}}_r - \hat{\mathbf{e}}_f)$ frame, $\mathbf{v} = \dot{r}\,\hat{\mathbf{e}}_r + r\dot{f}\,\hat{\mathbf{e}}_f$. Also, combining Eqs.(2) and (3), $\mathbf{a} = (-\mu/r^2 + a_r)\hat{\mathbf{e}}_r + a_\perp\hat{\mathbf{e}}_f + a_h\hat{\mathbf{i}}_h$. Plugging all of this back into Eq.(5),

$$-\frac{\dot{a}}{a^2} = -\frac{2}{r^2}\dot{r} - \frac{2}{\mu}(\dot{r}\,\hat{\mathbf{e}}_r + r\dot{f}\,\hat{\mathbf{e}}_f)\cdot\left(\left(-\frac{\mu}{r^2}+a_r\right)\hat{\mathbf{e}}_r + a_\perp\hat{\mathbf{e}}_f + a_h\hat{\mathbf{i}}_h\right) \tag{6a}$$

$$= \cancel{-\frac{2}{r^2}\dot{r}} - \frac{2}{\mu}\left(\cancel{-\frac{\mu}{r^2}\dot{r}} + a_r\dot{r} + r\dot{f}a_\perp\right) \tag{6b}$$

Or,

$$\dot{a} = \frac{2a^2}{\mu}(\dot{r}\,a_r + r\dot{f}\,a_\perp) \tag{7}$$

Also, recall the following known expressions for $\dot{r}$ and $r\dot{f}$:

$$\dot{f} = \sqrt{\frac{\mu}{p}}(1+e\cos f) \tag{8a}$$

$$\dot{r} = \sqrt{\frac{\mu}{p}}\,e\sin f \tag{8b}$$

Substituting Eqs.(8) in Eq.(7), we have:

$$\dot{a} = \frac{2a^2}{h}\{(1+e\cos f)a_\perp + e\sin f\,a_r\} \tag{9}$$

The above equation is called the *Lagrange planetary equation* for a. Note that the equation holds for general perturbations a_r, $a_\perp$ and a_h. The source could be any one of the five listed on page 1 or something else for that matter. The particular model of perturbation is not important in Eq.(9): what is important is the nature of contribution of each of the components of the perturbation towards the time variation of the semimajor axis. Clearly, a normal (out-of-plane) does not have any effect on the semimajor axis, whereas in-plane radial and tangential perturbations do.

- Just like the semi-major axis, time variations of the other orbital elements can also be obtained following a similar procedure, leading to the following complete set of Lagrange's planetary equations:

$$\dot{a} = \frac{2a^2}{h}[(1+e\cos f)a_\perp + e\sin f\,a_r] \tag{10a}$$

$$\dot{e} = \frac{\sqrt{1-e^2}\sin f}{na}a_r + \frac{\sqrt{1-e^2}}{nea^2}\left[\frac{a^2(1-e^2)}{r} - r\right]a_\perp \tag{10b}$$

$$\dot{i} = \frac{r\cos(\omega+f)}{na^2\sqrt{1-e^2}}a_h \tag{10c}$$

$$\dot{\omega} = \frac{\sqrt{1-e^2}}{nea}\left[-\cos f\,a_r + \frac{2+e\cos f}{1+e\cos f}\sin f\,a_\perp\right] - \frac{r\cot i\sin(\omega+f)}{h}a_h \tag{10d}$$

$$\dot{\Omega} = \frac{r\sin(\omega+f)}{na^2\sqrt{1-e^2}\sin i}a_h \tag{10e}$$

- To complete our (very short) discussion on perturbations, we will look at a particular example of a perturbative effect: namely the Earth's oblateness. We will look the models used for the components a_r, $a_\perp$ and a_h for this perturbation.

Perturbation due to Earth's Oblateness

- It so happens (we did not learn about this) that the acceleration resulting from any conservative force (including gravity) can be written as the negative gradient of a scalar function of the system's position vector, as follows:

$$\mathbf{a} = -\nabla U(\mathbf{r}) \tag{11}$$

The scalar function $U(\mathbf{r})$ turns out to be the *potential energy* of the system, e.g. for the spring-mass system, $U(\mathbf{r}) = U(x) = 1/2kx^2$. For two-body (Keplerian) gravity, we know that $U(\mathbf{r}) = -\mu/r$. Now, if we do not assume that the Earth is spherically symmetrical then the new potential energy function associated with gravity is given as:

$$U(\mathbf{r}) = U(r, \phi) = -\frac{\mu}{r}\left[1 - \sum_{k=2}^{\infty}\left(\frac{R_{eq}}{r}\right)^k J_k P_k(\cos\phi)\right] \tag{12}$$

where,

  - ϕ is the local latitude.
  - r is the radius magnitude of the current location.
  - R_{eq} is the radius of Earth at the equator.
  - $P_k(\cdot)$ is the $k-$th *Legendre polynomial of the first kind* (look up Wikipedia).
  - J_k is the amplitude of the $k-$th zonal harmonic.

The above model of the gravitational potential only accounts for latitudinal variations. If you also wish to account for longitudinal variations (these are much less significant), additional terms must be included in the above expression, involving the Legendre polynomials of the second kind. Latitudinal variations are also called *zonal* variations. This explains the appears of the term "zonal" in the above model. Essentially, a *zonal* model forgoes spherical symmetry, but retains rotational symmetry. In other words, the Earth is considered no longer a sphere but is still assumed symmetric about the equatorial plane, i.e. the southern hemisphere is assumed to be a mirror image of the northern hemisphere.

- There is no easy way of determining the amplitudes of the zonal variations, i.e. the parameters J_k. Based on the tracking data of spacecraft, the following numbers are available:

$$J_2 = 1.08263 \times 10^{-3} \tag{13a}$$
$$J_3 = -2.52 \times 10^{-6} \tag{13b}$$
$$J_4 = -1.61 \times 10^{-6} \tag{13c}$$
$$J_5 = -1.5 \times 10^{-7} \tag{13d}$$
$$J_6 = 5.7 \times 10^{-7} \tag{13e}$$

Clearly, "J_2" is the dominant perturbation by atleast three orders of magnitude and oblateness models typical only retain the perturbative effect due to it and ignore higher order harmonics. Thus, the J_2 gravity potential can be written as:

$$U(r, \phi) = -\frac{\mu}{r}\left[1 - \left(\frac{R_{eq}}{r}\right)^2 J_2 P_2(\cos\phi)\right] \tag{14}$$

- Substitution of the above potential into Eq.(11) gives us, following comparison with Eq.(3):

$$a_r(J_2) = -\frac{3\mu}{2r^2} J_2 \left(\frac{R_{eq}}{r}\right)^2 \left[1 - 3\sin^2 i \sin^2(\omega + f)\right] \tag{15a}$$
$$a_\perp(J_2) = -\frac{3\mu}{2r^2} J_2 \left(\frac{R_{eq}}{r}\right)^2 \sin^2 i \sin[2(\omega + f)] \tag{15b}$$
$$a_h(J_2) = -\frac{3\mu}{2r^2} J_2 \left(\frac{R_{eq}}{r}\right)^2 \sin 2i \sin(\omega + f) \tag{15c}$$

- Substituting the above models in the Langrange's planetary equations (Eqs.(10)) give the instantaneous time variation of the various orbital parameters (only) due to the J_2 oblateness perturbation. The J_2 effect is the most dominant in low earth orbits.

- Recall that perturbations in the two-body problem are generally small (this was the basis of working with Keplerian motion all along this semester). It therefore makes more sense to look at the *time-averaged rate of change of the orbital elements*, as opposed to instantaneous variations in them. For example, the time-averages rate of change of the semi-major axis over a full orbital period is given as

$$\bar{\dot{a}} = \frac{1}{P}\int_0^P \dot{a}\, dt \tag{16}$$

where, $\dot{a}$ is the instantaneous rate of change of a (Eq.(10a)). Similarly time averages can be found for all the other orbital elements as well. Substituting Eqs.(15) into the equation set Eq.(10) and then taking average according to Eq.(16), we get the following *average variation of orbital elements (only) due to the J_2 perturbation:*

$$\bar{\dot{a}} = 0 \tag{17a}$$
$$\bar{\dot{e}} = 0 \tag{17b}$$
$$\bar{\dot{i}} = 0 \tag{17c}$$
$$\bar{\dot{\omega}} = K_2\left(\frac{5}{2}\sin^2 i - 2\right) \tag{17d}$$
$$\bar{\dot{\Omega}} = K_2 \cos i \tag{17e}$$

where,

$$K_2 = -\frac{3}{2}\frac{\sqrt{\mu}\, J_2\, R_{eq}^2}{(1-e^2)^2\, a^{7/2}} \tag{18}$$

- Interestingly, on average over a full period, the semi-major axis, eccentricity and inclination do not change due to Earth's zonal oblateness. Keep in mind - this is an average, i.e. over the span of a period, perturbations *above* the mean are equal to perturbations *below* the mean. However, there is a secular variation in the argument of periapsis and the longitude of nodes due to oblateness.

- It is interesting to look at the effect of J_2 on Ω: Note that for prograde orbits, i.e. $i \in [0, \pi/2]$ (the most common case) $\cos i > 0$. In other words, for prograde orbits, $\bar{\dot{\Omega}} = K_2 \cos i < 0$ since $K_2 < 0$. What does this mean physically? It means that the line of nodes regresses to the west, i.e. the angle Ω reduces over time. The spacecraft rises above the equator more and more to the west as time goes on (see Fig.(1) for an exaggerated depiction. In reality, the average westward regression rate over a single period of a circular orbit of altitude 300 km and inclination 0 deg is 9.82×10^{-5} deg per second. At this rate, the orbit would drift 0.53 deg westward over one full orbit ($P = 1.51$ hours): see Eq.(17e)).

- Similarly, consider the average rate of change of ω: For $i \in [0, 63.4\,\text{deg}]$, $(5/2\sin^2 i - 2) < 0$. For this range of inclinations, $\bar{\dot{\omega}} > 0$. This means that over time, the periapsis advances forward along the orbit, in the direction of motion of the spacecraft.

- Keep in mind, these effects are only due to a single perturbation, namely Earth's oblateness.

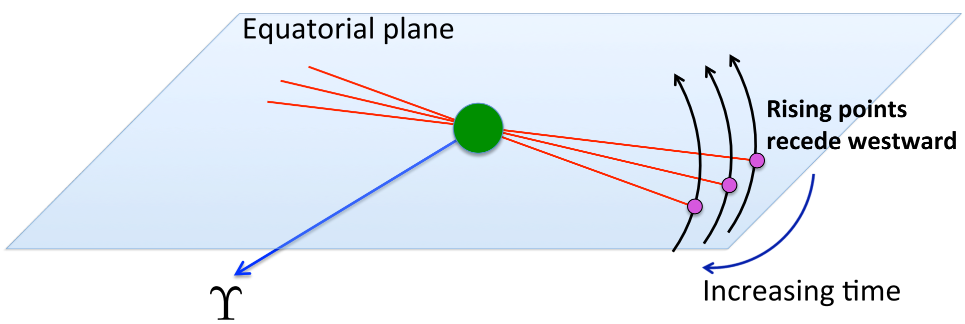

Figure 1: Regression of the line of nodes for prograde orbits.