AAE 5626: Orbital Mechanics for Engineers

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Preliminary Orbit Determination

The objective of *orbit determination* is to estimate the orbit of an object based on Earth-bound measurements, e.g. from a radar. This is a practical application in which (possibly multiple) point measurements are used to determine what orbit an object is traveling in. Recall that during our discussion of the implicit solution of the TBP, we learned that $(\mathbf{r}, \mathbf{v})$ information at any *one* point is sufficient to completely determine the orbit using a procedure that we now know very well. So, why exactly are we studying "orbit determination"? The reason is two-fold:

i. It is true that $(\mathbf{r}, \mathbf{v})$ information at one point is enough the completely determine the orbit of an object. Unfortunately in real life, velocity vector measurements are difficult to obtain. It is much easier to simply bounce signals off objects and measure their position vectors. Therefore, the velocity measurement may not be available.

ii. Orbital elements are more *sensitive* to variations in the velocity vector than the radius vector. Which means that a small error in the measured velocity vector amplifies to bigger errors in the computed orbital elements; more than the amplification caused by errors in the measured radius vector. So, if possible we try to avoid using the velocity vector measurements even if they are available. This is a practical consideration.

We will work with the following assumptions while on the current topic:

i. The object is performing Keplerian motion. This is not so obvious/trivial because the *measured* vectors contain the influence of all perturbations e.g. J_2, atmosphere, third-body effects. We however, will used this "corrupted" data in the Keplerian framework assuming the perturbations were *not* present. This is the reason the topic of these notes is "preliminary". More advance methods are needed to fine tune the orbital elements to account for non-Keplerian effects.

ii. In our study, the central object will be the Earth.

Orbit determination is a very common application employed very frequently in the practice of astrodynamics. Every day new objects are sighted in space (e.g. debris, unknown asteroids, meteors etc.) and it is required to know what orbit they are traveling in. These objects carry potential threats to active spacecraft and even our planet if they are on steep incoming orbits. We will consider a couple of methods that are employed to use measured data to determine unknown orbits:

I. **Gibbs' problem.** Determine the orbit form three non-collinear radius vector measurements.

II. **Lambert's problem.** Determine the orbit from two radius vector measurements and the time of flight between the two measurements.

In the Gibbs' method, three radius vector measurements are used such that the time at which the measurements were made is not available. In the Lambert's problem, only two radius vector measurements are used, but the time of flight between these two points is known. The Lambert's problem is of central importance in the design of interplanetary missions. It is yet unsolved (in closed form) and is one of the two holy-grail problems in two-body astrodynamics, the other of course being the Kepler's equation.

Gibbs' Method

In the Gibbs method, the input is three measured radius vectors, $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ obtained at three separate times, t_1, t_2 and t_3 respectively. The times of measurement are not known. However, it is known that $t_1 < t_2 < t_3$, i.e. $\mathbf{r}_1$ was measured first and $\mathbf{r}_3$ last. There is no loss of generality in making this assumption. Fig.(1) shows a possible scenario and Fig.(2) shows the input/output relationship.

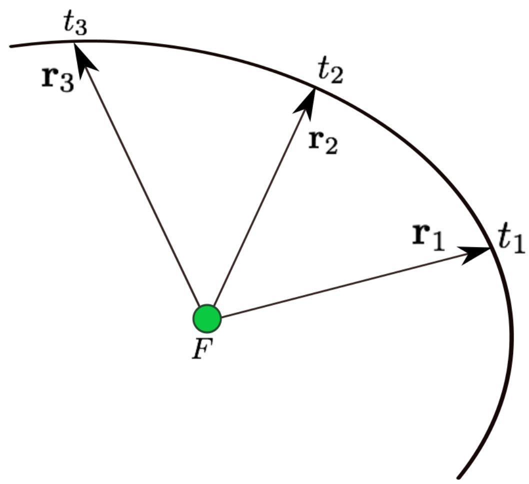

Figure 1: A possible scenario for the Gibbs preliminary orbit determination method.

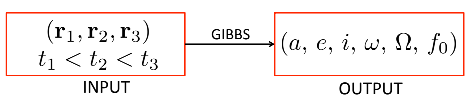

Figure 2: Input/Output: Gibbs preliminary orbit determination.

As mentioned in the previous section, the Gibbs' method assumes that the three are mutually non-collinear. This is a fairly safe bet, but you must still make sure that it is true by taking pair-wise cross products. With the assumption of non-collinearity, the basic idea behind the Gibbs' method is that Keplerian motion is planar. This property is exploited in conjunction with the so-called *hodograph equation* to determine the object's orbit. The hodograph equation is a special relationship between the velocity vector and the radius vector, one that defines the locus of the tip of the velocity vector: see HW solutions for its derivation.

- We will first consider planarity of the two-body motion, by virtue of which all the three vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ lie in the same plane (this is not news!). We can obtain the normal to this plane by taking the cross product between any two of them, e.g.

$$\hat{\mathbf{n}}_{23} \triangleq \frac{\mathbf{r}_2 \times \mathbf{r}_3}{\|\mathbf{r}_2 \times \mathbf{r}_3\|} \tag{1}$$

Clearly,

$$\hat{\mathbf{n}}_{23} \;\|\; \hat{\mathbf{i}}_h. \text{ In fact, } \; \hat{\mathbf{n}}_{23} \;\equiv\; \hat{\mathbf{i}}_h \tag{2}$$

Since the unit angular momentum is also normal to the plane of the orbit. Next, employ an argument similar to the one used in the $F - G$ solution of the TBP: since the vectors $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ are coplanar and non-collinear, any two of them can be used as a set of basis vectors to represent *all radius vectors in the orbital plane.* For example, using $(\mathbf{r}_1, \mathbf{r}_3)$ as a basis pair, we can write, without any simplifying approximation,

$$\mathbf{r}_2 \;=\; a_1\, \mathbf{r}_1 \;+\; a_3\, \mathbf{r}_3 \tag{3}$$

where, a_1 and a_3 are amplitudes that must be determined.

- The other ingredient in the Gibbs method is the so-called *hodograph equation*:

$$\mathbf{v} \;=\; \frac{\mu}{h}\left(e\,\hat{\mathbf{i}}_h \times \hat{\mathbf{i}}_e \;+\; \hat{\mathbf{i}}_h \times \hat{\mathbf{e}}_r \right) \tag{4}$$

The beauty of the above equation is that it gives $\mathbf{v}$ directly as a function of $\mathbf{r}$, **if** $\mathbf{h}$ and $\mathbf{c}$ are known. In the present case, $\mathbf{h}$ and $\mathbf{c}$ are *not known*. However, if we can use the given data $(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ to compute these quantities, Eq.(4) can be used to determine the velocity vectors at all three times, i.e. $(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Then basically you are free to use any one of the three pairs, $(\mathbf{r}_1, \mathbf{v}_1)$, $(\mathbf{r}_2, \mathbf{v}_2)$ or $(\mathbf{r}_2, \mathbf{v}_2)$ to determine the orbital elements.

- Therefore, the problem formulation in the Gibbs' approach reduces to the following schematic:

$$\underbrace{\boxed{(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)}}_{\text{input}} \;\xrightarrow[\text{Gibbs Method}]{}\; \boxed{(\mathbf{h}, \mathbf{c})} \;\xrightarrow[\text{Hodograph}]{}\; \boxed{(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)} \;\xrightarrow[\text{Any } (\mathbf{r}_i, \mathbf{v}_i) \text{ pair}]{}\; \boxed{\text{orbital elements}} \tag{5}$$

- Therefore, we have reduced the problem to using the given information to determine the $\mathbf{h}$ and $\mathbf{c}$ vectors. Let us do this one at a time:

A. **Angular momentum, h.** We begin with the definition of the eccentricity vector:

$$\mathbf{c} \;=\; \mu e \hat{\mathbf{i}}_e \;=\; \mathbf{v} \times \mathbf{h} - \frac{\mu}{r}\mathbf{r} \tag{6}$$

Also recall the following relationship, obtained in our notes on TBP

$$\mathbf{c} \cdot \mathbf{r} \;=\; h^2 \;-\; \mu r \tag{7}$$

Of course, the above relationship holds at all three points t_1, t_2 and t_3 such that $\mathbf{c} \cdot \mathbf{r}_i \;=\; h^2 - \mu r_i$, $i \;=\; 1, 2, 3$. Now, let us take the dot product between $\mathbf{r}_2$ and $\mathbf{c}$, using Eq.(3),

$$\mathbf{c} \cdot \mathbf{r}_2 \;=\; a_1\, \mathbf{r}_1 \cdot \mathbf{c} \;+\; a_3\, \mathbf{r}_3 \cdot \mathbf{c} \tag{8}$$

Now using Eq.(7) in the above, we get

$$(h^2 - \mu r_2) \;=\; a_1\,(h^2 - \mu r_1) \;+\; a_3\,(h^2 - \mu r_3) \tag{9}$$

The above is nice.. but it would be nicer if we could get rid of the coefficients a_1 and a_3. To this end, take the cross product of $\mathbf{r}_2$ with $\mathbf{r}_1$ and $\mathbf{r}_3$ respectively, using Eq.(3):

$$\mathbf{r}_2 \times \mathbf{r}_1 \;=\; (a_1\,\mathbf{r}_1 \;+\; a_3\,\mathbf{r}_3) \times \mathbf{r}_1 \;=\; a_3\,\mathbf{r}_3 \times \mathbf{r}_1 \tag{10}$$

Similarly,

$$\mathbf{r}_2 \times \mathbf{r}_3 \;=\; (a_1\,\mathbf{r}_1 \;+\; a_3\,\mathbf{r}_3) \times \mathbf{r}_3 \;=\; a_1\,\mathbf{r}_1 \times \mathbf{r}_3 \;=\; -a_1\,\mathbf{r}_3 \times \mathbf{r}_1 \tag{11}$$

To rid Eq.(9) of the coefficients, multiply it with $(\mathbf{r}_3 \times \mathbf{r}_1)$

$$(h^2 - \mu r_2)(\mathbf{r}_3 \times \mathbf{r}_1) \;=\; a_1\,(h^2 - \mu r_1)(\mathbf{r}_3 \times \mathbf{r}_1) \;+\; a_3\,(h^2 - \mu r_3)(\mathbf{r}_3 \times \mathbf{r}_1) \tag{12a}$$

$$\underset{\text{Eqs.(10),(11)}}{=} \; -(h^2 - \mu r_1)(\mathbf{r}_2 \times \mathbf{r}_3) \;+\; (h^2 - \mu r_3)(\mathbf{r}_2 \times \mathbf{r}_1) \tag{12b}$$

The coefficients a_1 and a_3 have now vanished, as desired!

Rearrange Eq.(12b) to get,

$$h^2\, \underbrace{\left[\mathbf{r}_1 \times \mathbf{r}_2 \;+\; \mathbf{r}_2 \times \mathbf{r}_3 \;+\; \mathbf{r}_3 \times \mathbf{r}_1\right]}_{\triangleq\, \mathbf{D}} \;=\; \mu\, \underbrace{\left[r_1(\mathbf{r}_2 \times \mathbf{r}_3) \;+\; r_2(\mathbf{r}_3 \times \mathbf{r}_1) \;+\; r_3(\mathbf{r}_1 \times \mathbf{r}_2)\right]}_{\triangleq\, \mathbf{N}} \tag{13}$$

Note that the two newly defined vectors, $\mathbf{D}$ and $\mathbf{N}$ are known because they are functions of only the measured data, namely, $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$. And so we get,

$$h^2\,\mathbf{D} \;=\; \mu\mathbf{N} \tag{14}$$

Or,

$$h \;=\; \sqrt{\frac{\mu N}{D}} \tag{15}$$

where, $N \;=\; \|\mathbf{N}\|$ and $D \;=\; \|\mathbf{D}\|$. All we now need is the unit vector $\hat{\mathbf{i}}_h$ and we are done. Taking a closer look at $\mathbf{D}$, note that it contains three cross products... but, based on our discussion surrounding Eq.(2), we see that each one of these three terms points in the same direction as $\hat{\mathbf{i}}_h$. Thus, the vector $\mathbf{D}$ points in the same direction as $\hat{\mathbf{i}}_h$. Mathematically speaking,

$$\frac{\mathbf{D}}{D} \;=\; \hat{\mathbf{i}}_h \tag{16}$$

Combining Eqs.(15) and (16), we get

$$\boxed{\mathbf{h} \;=\; \sqrt{\frac{\mu N}{D}}\,\frac{\mathbf{D}}{D} \;=\; \sqrt{\frac{\mu N}{D^3}}\,\mathbf{D}} \tag{17}$$

B. **Eccentricity vector, c.** To begin, consider the triple product

$$(\mathbf{r}_i \times \mathbf{r}_j) \times \mathbf{c} \;=\; -\mathbf{c} \times (\mathbf{r}_i \times \mathbf{r}_j) \;=\; (\mathbf{c}\cdot\mathbf{r}_i)\mathbf{r}_j \;-\; (\mathbf{c}\cdot\mathbf{r}_j)\mathbf{r}_i \tag{18}$$

The above equation holds for all $i,\, j \;=\; 1,2,3$ and is nontrivial only if $i \neq j$. Now, using Eq.(7), $(\mathbf{c}\cdot\mathbf{r}_i) \;=\; (h^2 - \mu r_i)$, $i = 1,2,3$. Therefore, we can write out all three equations contained above for various combinations of i and j:

$$(\mathbf{r}_1 \times \mathbf{r}_2) \times \mathbf{c} \;=\; \underbrace{(\mathbf{c}\cdot\mathbf{r}_1)}_{(h^2-\mu r_1)}\;\mathbf{r}_2 \;-\; \underbrace{(\mathbf{c}\cdot\mathbf{r}_2)}_{(h^2-\mu r_2)}\;\mathbf{r}_1 \tag{19a}$$

$$(\mathbf{r}_2 \times \mathbf{r}_3) \times \mathbf{c} \;=\; \underbrace{(\mathbf{c}\cdot\mathbf{r}_2)}_{(h^2-\mu r_2)}\;\mathbf{r}_3 \;-\; \underbrace{(\mathbf{c}\cdot\mathbf{r}_3)}_{(h^2-\mu r_3)}\;\mathbf{r}_2 \tag{19b}$$

$$(\mathbf{r}_3 \times \mathbf{r}_1) \times \mathbf{c} \;=\; \underbrace{(\mathbf{c}\cdot\mathbf{r}_3)}_{(h^2-\mu r_3)}\;\mathbf{r}_1 \;-\; \underbrace{(\mathbf{c}\cdot\mathbf{r}_1)}_{(h^2-\mu r_1)}\;\mathbf{r}_3 \tag{19c}$$

Now, add up all the three equations above to get:

$$\underbrace{\big[\mathbf{r}_1 \times \mathbf{r}_2 \;+\; \mathbf{r}_2 \times \mathbf{r}_3 \;+\; \mathbf{r}_3 \times \mathbf{r}_1\big]}_{\mathbf{D}} \times \mathbf{c} \;=\; h^2\cancel{\big[\mathbf{r}_2 - \mathbf{r}_1 + \mathbf{r}_3 - \mathbf{r}_2 + \mathbf{r}_1 - \mathbf{r}_3\big]} +$$

$$\mu\big[\underbrace{(r_2 - r_3)\mathbf{r}_1 + (r_3 - r_1)\mathbf{r}_2 + (r_1 - r_2)\mathbf{r}_3}_{\triangleq\,\mathbf{S}}\big] \tag{20}$$

Thus we get

$$\mathbf{D} \times \mathbf{c} \;=\; \mu\mathbf{S} \tag{21}$$

where the vector $\mathbf{S}$ is defined in Eq.(20) and is known in terms of the measured information. Well, we have not been successful in completely isolating the vector $\mathbf{c}$. However, it turns out that Eq.(21) is sufficient for our purposes. Recall that the vector $\mathbf{D}$ points along $\hat{\mathbf{i}}_h$. Thus the LHS of Eq.(21) is

$$\mathbf{D} \times \mathbf{c} \;=\; D\mu e\,\hat{\mathbf{i}}_h \times \hat{\mathbf{i}}_e \tag{22}$$

which is all we need for the hodograph (Eq.(4)): note that we are interested only in the term $\hat{\mathbf{i}}_h \times \hat{\mathbf{i}}_e$. Thus using Eqs. (21) and (22) together,

$$e\,\hat{\mathbf{i}}_h \times \hat{\mathbf{i}}_e \;=\; \frac{\mathbf{S}}{D} \tag{23}$$

- Combining Eqs.(23) and (16) in Eq.(4) we get

$$\mathbf{v} \;=\; \frac{\mu}{h}\left(\frac{\mathbf{S}}{D} \;+\; \frac{\mathbf{D}}{D} \times \hat{\mathbf{e}}_r\right) \tag{24}$$

Or,

$$\boxed{\mathbf{v} \;=\; \frac{\mu}{Dh}\left(\mathbf{S} \;+\; \mathbf{D} \times \hat{\mathbf{e}}_r\right)} \tag{25}$$

Everything in the RHS is known. Using the above equation, we can determine $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, corresponding to measured vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ respectively. Then, you can pick any one of the three pairs $(\mathbf{r}_i, \mathbf{v}_i)$ $i = 1, 2, 3$ to determine the desired orbital elements, $(a, e, i, \omega, \Omega, f_0)$.

Lambert's Problem

In the Lambert's problem, the input contains measured radius vectors at two points in the orbit, along with the time of flight between these points, i.e. $\mathbf{r}_1$, $\mathbf{r}_2$ and Δt: see Fig.(3) for an example. Fig.(4) shows the input/output relationship.

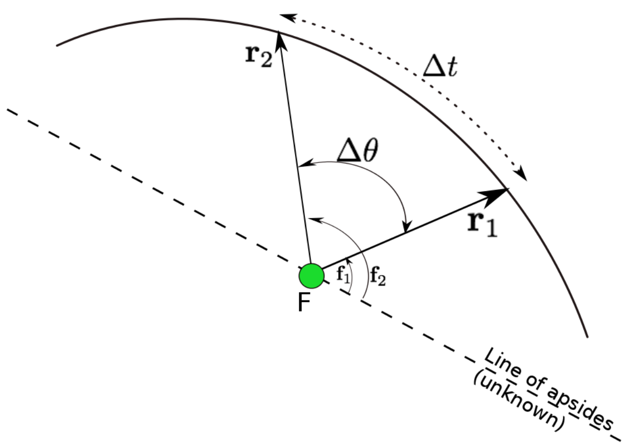

Figure 3: The Lambert's problem.

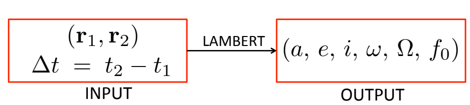

Figure 4: Input/Output: Lambert's problem.

Lambert's problem is one of the holy-grail problems in Astrodynamics. A closed form analytical solution is not available (yet) and therefore it must be solved numerically. It finds application in the design of interplanetary missions, wherein the arrival and departure radius vectors are known, as is the time of flight between them. The objective then is to determine the transfer orbit that takes the spacecraft from its originating planet to the target planet.

- In the Lambert's problem, the first step is to determine the transfer angle (shown as $\Delta\theta$ in Fig.(3)). Note that the true anomalies, f_1 and f_2 of the two points $\mathbf{r}_1$ and $\mathbf{r}_2$ are not known, since the line of apsides is unknown.

- Since we are interested only in the *difference* between the true anomalies, i.e. $\Delta\theta = (f_2 - f_1)$, we can look at the dot product between the two radius vectors,

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = r_1 r_2 \cos\Delta\theta \tag{26}$$

  since $\Delta\theta = \angle(\mathbf{r}_1, \mathbf{r}_2)$. Thus,

$$\Delta\theta = \cos^{-1}\left(\frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) \tag{27}$$

  However, ambiguity remains over the quadrant of the angle $\Delta\theta$ (recall we have encountered this problem several times before).

- To resolve the quadrant of $\Delta\theta$, look at the z-component of $(\mathbf{r}_1 \times \mathbf{r}_2)$:

$$(\mathbf{r}_1 \times \mathbf{r}_2)_z = \hat{\mathbf{i}}_3 \cdot (\mathbf{r}_1 \times \mathbf{r}_2) \tag{28}$$

  Now, we know that (from planarity of the orbit)

$$\mathbf{r}_1 \times \mathbf{r}_2 = r_1 r_2 \sin\Delta\theta \hat{\mathbf{i}}_h \tag{29}$$

  Using Eq.(29) in Eq.(28),

$$(\mathbf{r}_1 \times \mathbf{r}_2)_z = \hat{\mathbf{i}}_3 \cdot (r_1 r_2 \sin\Delta\theta \hat{\mathbf{i}}_h) = (r_1 r_2 \sin\Delta\theta)\hat{\mathbf{i}}_3 \cdot \hat{\mathbf{i}}_h \tag{30}$$

  Now, from orbital geometry, we know that the angle between the angular momentum vector and the inertial z-axis is nothing but the orbital inclination, i.e., $(\hat{\mathbf{i}}_3 \cdot \hat{\mathbf{i}}_h = \cos i)$. Thus,

$$\sin\Delta\theta = \frac{(\mathbf{r}_1 \times \mathbf{r}_2)_z}{r_1 r_2 \cos i} \tag{31}$$

  This is where things stand: (a) the numerator of the above equation can be computed from the measured data: simply take the cross product between the two vectors and extract the z-component; (b) in the denominator, both r_1 and r_2 are known (magnitudes of the measured radius vectors) and they are both positive; (c) of course, the orbit inclination is not known. However, if the orbit is *prograde* (this is the most common case), $i \in [0, 90]$ deg. If the orbit is *retrograde* (almost never happens), $i \in [90, 180]$ deg. Therefore, we can safely assume that we are looking for a prograde orbit, due to which $\cos i > 0$. Thus we get from Eq.(31)

$$\text{sign}(\sin\Delta\theta) = \text{sign}[(\mathbf{r}_1 \times \mathbf{r}_2)_z] \tag{32}$$

  Combining Eqs.(27) and (32) we can place the transfer angle in the correct quadrant.

- *Unless otherwise stated, the orbit is always assumed to be prograde.* Retrograde orbits are extremely rare in nature.

- Having now determined the transfer angle, we arrive at the key result: the so called *Lambert's theorem.*

Lambert's Theorem. *The time of flight (Δt) is independent of the eccentricity of the orbit and depends only on the sum ($r_1 + r_2$); the semi-major axis a and ξ, which represents the length of the chord joining the radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$.*

Proof: beyond the scope of this course. See Battin.

Solution of Lambert's Problem

A truly elegant solution of the Lambert's problem exists using the *universal solution* of the two-body problem. The beauty of this approach is that there is no need to predetermine the eccentricity of the underlying orbit and is, in a sense the ideal application for the universal approach.

- First, begin with the $F-G$ solution paradigm (yet another application of this method!). Recall that here we have two pairs of vectors: $(\mathbf{r}_1, \mathbf{v}_1)$ and $(\mathbf{r}_2, \mathbf{v}_2)$. Of these, $\mathbf{r}_1$ and $\mathbf{r}_2$ are known while $\mathbf{v}_1$ and $\mathbf{v}_2$ are not known. We will use the pair $(\mathbf{r}_1, \mathbf{v}_1)$ as the *basis*:

$$\mathbf{r}_2 = F\,\mathbf{r}_1 + G\,\mathbf{v}_1 \tag{33a}$$

$$\mathbf{v}_2 = \dot{F}\,\mathbf{r}_1 + \dot{G}\,\mathbf{v}_1 \tag{33b}$$

- In Eq.(33a), $\mathbf{v}_1$ is the unknown (along with F and G of course). We have,

$$\mathbf{v}_1 = \frac{1}{G}(\mathbf{r}_2 - F\,\mathbf{r}_1) \tag{34}$$

Substitute Eq.(34) in Eq.(33b),

$$\mathbf{v}_2 = \dot{F}\mathbf{r}_1 + \frac{\dot{G}}{G}(\mathbf{r}_2 - F\,\mathbf{r}_1) \tag{35a}$$

$$= \frac{1}{G}[(G\dot{F} - \dot{G}F)\,\mathbf{r}_1 + \dot{G}\mathbf{r}_2] \tag{35b}$$

But, recall that $(G\dot{F} - F\dot{G}) = 1$ (conservation of angular momentum), whereby

$$\mathbf{v}_2 = \frac{1}{G}(\dot{G}\mathbf{r}_2 - \mathbf{r}_1) \tag{36}$$

Looking at Eqs.(34) and (36), we realize that the problem boils down to using the available data $(\mathbf{r}_1, \mathbf{r}_2, \Delta t)$ to determine F, G, and $\dot{G}$. Because then you will have $\mathbf{v}_1$ and $\mathbf{v}_2$, and you can use any one of the pairs $(\mathbf{r}_1, \mathbf{v}_1)$ or $(\mathbf{r}_2, \mathbf{v}_2)$ to completely determine the orbit.

So, we return to the expressions for the Lagrange coefficients F, G, $\dot{F}$ and $\dot{G}$ in terms of the universal variable χ (these were derived in TBP notes):

$$F(\chi) = U_0(\alpha;\chi) + \left(\alpha - \frac{1}{r_1}\right)U_2(\alpha;\chi) \tag{37a}$$

$$G(\chi) = \frac{r_1}{\sqrt{\mu}}U_1(\alpha;\chi) + \frac{\sigma_1}{\sqrt{\mu}}U_2(\alpha;\chi) \tag{37b}$$

$$\dot{F}(\chi) = -\frac{\sqrt{\mu}}{r_1 r_2}U_1(\alpha;\chi) \tag{37c}$$

$$\dot{G}(\chi) = 1 - \frac{1}{r_2}U_2(\alpha;\chi) \tag{37d}$$

Note that we have replaced "r_0" with r_1 and "r" with r_2, and similarly, "σ_0" with "σ_1". A technicality: The above equations do not involve the elapsed time (Δt) between points $\mathbf{r}_1$ and $\mathbf{r}_2$, which is a known quantity in the present context. So, we would like to sneak it in, hopefully simplifying the expressions above. To do so, recall (also from TBP notes),

$$\sqrt{\mu}\Delta t = r_1 U_1(\alpha;\chi) + \sigma_1 U_2(\alpha;\chi) + U_3(\alpha;\chi) \tag{38}$$

whereby,

$$r_1 U_1(\alpha;\chi) + \sigma_1 U_2(\alpha;\chi) = \sqrt{\mu}\Delta t - U_3(\alpha;\chi) \tag{39}$$

The above equation can be used in Eq.(37b) to get

$$G(\chi) \;=\; \Delta t - \frac{U_3(\alpha;\chi)}{\sqrt{\mu}} \tag{40}$$

This is excellent because now we have replaced one unknown variable (σ_1) with a known quantity (Δt) and simultaneously simplified the expression for the Lagrange coefficient "G". Now in Eqs.(37a), (40), (37c) and (37d), the variables r_1, r_2 and $t_2 - t_1 \;=\; \Delta t$ are known. The variables α and χ are unknown.

- Now, there exists another set of equations for F, G, $\dot{F}$ and $\dot{G}$ in terms of $\Delta\theta$. We did not derive these equations in class. Nevertheless they exist, and interestingly, are universally valid for all types of conic sections:

$$F \;=\; 1 \;-\; \frac{\mu r_2}{h^2}(1 \;-\; \cos\Delta\theta) \tag{41a}$$

$$G \;=\; \frac{r_1 r_2}{h}\sin\Delta\theta \tag{41b}$$

$$\dot{F} \;=\; \frac{\mu}{h}\,\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1-\cos\Delta\theta) \;-\; \frac{1}{r_1} \;-\; \frac{1}{r_2}\right] \tag{41c}$$

$$\dot{G} \;=\; 1 \;-\; \frac{\mu r_1}{h^2}(1-\cos\Delta\theta) \tag{41d}$$

In Eqs. (41), the known quantities are: $(r_1, r_2, \Delta t, \Delta\theta)$ ($\Delta\theta$ was found in Eq.(27)). while the only unknown is h (of course, in addition to F, G, $\dot{F}$ and $\dot{G}$).

- As the first step to solving for the unknowns, the Lagrange coefficients $(F, G, \dot{F}, \dot{G})$ can be eliminated by equating the set [(37a), (40), (37c) and (37d)] with the set [(41)]:

$$U_0(\alpha;\chi) \;+\; \left(\alpha - \frac{1}{r_1}\right)U_2(\alpha;\chi) \;=\; 1 \;-\; \frac{\mu r_2}{h^2}(1 \;-\; \cos\Delta\theta) \tag{42a}$$

$$\Delta t - \frac{U_3(\alpha;\chi)}{\sqrt{\mu}} \;=\; \frac{r_1 r_2}{h}\sin\Delta\theta \tag{42b}$$

$$-\frac{\sqrt{\mu}}{r_1 r_2}U_1(\alpha;\chi) \;=\; \frac{\mu}{h}\,\frac{1-\cos\Delta\theta}{\sin\Delta\theta}\left[\frac{\mu}{h^2}(1-\cos\Delta\theta) \;-\; \frac{1}{r_1} \;-\; \frac{1}{r_2}\right] \tag{42c}$$

$$1 - \frac{1}{r_2}U_2(\alpha;\chi) \;=\; 1 \;-\; \frac{\mu r_1}{h^2}(1-\cos\Delta\theta) \tag{42d}$$

It appears we have four equations involving three unknowns. Thankfully this is not true because there exists one constraint among these equations: the conservation of the angular momentum: by virtue of which $(G\dot{F} - F\dot{G}) = 1$. Thus there are three equations in three unknowns (α, h, χ) and everything is in place.

- To make things easier on ourselves, we let us define two constants:

$$K_1 \;=\; \mu(1-\cos\Delta\theta) \tag{43a}$$

$$K_2 \;=\; \sqrt{\frac{\mu r_1 r_2}{K_1}}\sin\Delta\theta \tag{43b}$$

Let us get rid of h using Eq.(42d) (or equivalently, Eq.(42a)):

$$\frac{U_2(\alpha;\chi)}{r_2} \;=\; \frac{r_1 K_1}{h^2} \tag{44}$$

I.E.,

$$\boxed{h \;=\; \sqrt{\frac{K_1 r_1 r_2}{U_2(\alpha;\chi)}}} \tag{45}$$

- Next, substitute the above expression into Eq.(42b) (G equation) to get

$$\Delta t \;-\; \frac{U_3(\alpha;\chi)}{\sqrt{\mu}} \;=\; \frac{r_1 r_2\sqrt{U_2(\alpha;\chi)}}{\sqrt{K_1 r_1 r_2}}\sin\Delta\theta \tag{46}$$

Or,

$$\sqrt{\mu}\Delta t \;=\; U_3(\alpha;\chi) \;+\; \left[\sqrt{\frac{\mu r_1 r_2}{K_1}}\sin\Delta\theta\right]\sqrt{U_2(\alpha;\chi)} \tag{47}$$

Using Eq.(43b),

$$\boxed{\sqrt{\mu}\Delta t \;=\; U_3(\alpha;\chi) \;+\; K_2\sqrt{U_2(\alpha;\chi)}} \tag{48}$$

- Moving on, consider the $\dot{F}$ equation (Eq.(42c)): using the definition of K_1,

$$\frac{K_1}{h\sin\Delta\theta}\left(\frac{K_1}{h^2}-\frac{1}{r_1}-\frac{1}{r_2}\right) \;=\; -\frac{\sqrt{\mu}}{r_1 r_2}U_1(\alpha;\chi) \tag{49}$$

Substitute for h and $U_1(\alpha;\chi)$:

$$\frac{K_1\,\sqrt{U_2(\alpha;\chi)}}{\sqrt{K_1 r_1 r_2}\,\sin\Delta\theta}\left(\frac{K_1 U_2(\alpha;\chi)}{K_1 r_1 r_2}-\frac{1}{r_1}-\frac{1}{r_2}\right) \;=\; -\frac{\sqrt{\mu}}{r_1 r_2}\,U_1(\alpha;\chi) \tag{50}$$

From now on, let us drop the arguments of the universal functions for brevity, i.e. write $U_i(\alpha;\chi)$ simply as "U_i", $i=0,1,\ldots$. Canceling terms in Eq.(50),

$$\frac{\sqrt{K_1 U_2}}{\sin\Delta\theta}\left(\frac{U_2-(r_1+r_2)}{r_1 r_2}\right) \;=\; -\sqrt{\frac{\mu}{r_1 r_2}}\,U_1 \tag{51}$$

Or,

$$U_2 \;=\; (r_1+r_2) \;-\; \underbrace{\left[\sqrt{\frac{\mu r_2 r_2}{K_1}}\;\sin\Delta\theta\right]}_{K_2\text{ (Eq.(43b))}}\frac{U_1}{\sqrt{U_2}} \tag{52}$$

Thus, finally

$$\boxed{U_2 \;=\; (r_1+r_2) \;-\; K_2\,\frac{U_1}{\sqrt{U_2}}} \tag{53}$$

- Recall now the expanded forms of the universal functions U_n:

$$U_n(\alpha;\chi) \;\doteq\; \chi^n\left(\frac{1}{n!}-\frac{\alpha\chi^2}{(n+2)!}+\frac{(\alpha\chi^2)^2}{(n+4)!}-\frac{(\alpha\chi^2)^3}{(n+6)!}\right) \tag{54}$$

In particular, consider U_2 and U_3:

$$U_2(\alpha;\chi) \;=\; \chi^2\underbrace{\left(\frac{1}{2!}-\frac{\alpha\chi^2}{4!}+\frac{(\alpha\chi^2)^2}{6!}-\frac{(\alpha\chi^2)^3}{8!}+\ldots\right)}_{\text{define as } C(z),\text{ where } z\doteq\alpha\chi^2} \tag{55a}$$

$$U_3(\alpha;\chi) \;=\; \chi^3\underbrace{\left(\frac{1}{3!}-\frac{\alpha\chi^2}{5!}+\frac{(\alpha\chi^2)^2}{7!}-\frac{(\alpha\chi^2)^3}{9!}+\ldots\right)}_{\text{define as } S(z),\text{ where } z\doteq\alpha\chi^2} \tag{55b}$$

(55c)

Note the definition of functions $C(z)$ and $S(z)$ above, with $z=\alpha\chi^2$. They are called *Stumpff functions*.

- Let us now return to Eq.(53) where we left off our solution of the Lambert's problem. Use the above developments for the term $U1/\sqrt{U2}$ to get,

$$\frac{U_1}{\sqrt{U_2}} = \frac{\cancel{\chi}}{\cancel{\sqrt{\chi^2}}} \frac{\left(1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \cdots\right)}{\sqrt{C(z)}} \tag{56}$$

which, you will acknowledge is nice... since the above equation involves a single (although composite) unknown, z. Let us define the function on the RHS as $L(z)$. Upon close examination, you will notice (please do this yourself) that the numerator of the function $L(z)$ can be written as:

$$\left(1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \cdots\right) = 1 - zS(z) \tag{57}$$

Thus Eq.(56) can be cleaned up as

$$L(z) = \frac{U_1}{\sqrt{U_2}} = \frac{(1 - zS(z))}{\sqrt{C(z)}} \tag{58}$$

- Eq.(53) thus becomes,

$$U_2 = (r_1 + r_2) - K_2 L(z) \triangleq Y(z) \tag{59}$$

which is interesting because the right hand side is purely a function of z. Now, using Eq.(55a) and (59) together, we get

$$\chi^2 C(z) = Y(z) \tag{60}$$

Or,

$$\boxed{\chi^2 = \frac{Y(z)}{C(z)}} \tag{61}$$

- To finish, return to Eq.(48). Using Eqs.(55b) and (59), we get

$$\sqrt{\mu}\Delta t = \chi^3 S(z) + K_2\sqrt{Y(z)} \tag{62}$$

Substituting for χ from Eq.(61),

$$\sqrt{\mu}\Delta t = \left(\frac{Y(z)}{C(z)}\right)^{3/2} S(z) + K_2\sqrt{Y(z)} \tag{63}$$

We have finally managed to get one (nonlinear) equation in one unknown (z):

$$\boxed{\left(\frac{Y(z)}{C(z)}\right)^{3/2} S(z) + K_2\sqrt{Y(z)} = \sqrt{\mu}\Delta t} \tag{64}$$

Solve for z.

- The above equation is very interesting. In fact, it is deceptive. Its looks all nice and compact, because it is in analytical form, i.e. contains all terms in their functional form, e.g. $C(z)$ in Eq.(55a). Nevertheless, you can see there is a problem. The analytical forms are not *closed* since all the functions involved above are written as infinite sums (i.e. open sums).

  When you go to a computer to solve the above system, how would you evaluate these functions? Clearly you cannot implement an open sum. You must therefore truncate the summations by including "a sufficiently large" number of terms, perhaps via a while-loop. Essentially, you need to prescribe a tolerance for convergence of the sums and stop including additional terms if their contribution is less than the prescribed tolerance.

For example consider the evaluation of $S(z)$ at $z = 5$. Let us set a tolerance of $\epsilon = 10^{-4}$. We have, using Eq.(55b),

$$S(z = 5) = \frac{1}{3!} - \frac{5}{5!} + \frac{5^2}{7!} - \frac{5^3}{9!} + \cdots \tag{65a}$$

$$= 1.667 \times 10^{-1} - 4.167 \times 10^{-2} + 4.960 \times 10^{-3} - 3.445 \times 10^{-4} - \underbrace{1.566 \times 10^{-5}}_{< \epsilon:\ \textbf{STOP!!}} \tag{65b}$$

$$= 1.296 \times 10^{-1} \tag{65c}$$

All other "open" functions must be evaluated similarly. You probably realize that having a general expression for the k-th term of these functions would be very useful for MATLAB®implementation. We had this same discussion also in TBP notes.

- Let us return to the solution of Eq.(64). Redefine the LHS as $H(z)$ and the RHS as the constant b. We are therefore looking to solve $H(z) = b$ or $H(z) - b = 0$. We already know how to do this via Newton Raphson iterations.

- There is only one unpleasant thing left to be done. Recall that the Newton-Raphson iterations are based on the first derivative of the nonlinear function. In the present problem, this is going to be very nasty considering the form of Eq.(64). The derivative is actually defined piece-wise because the function $H(z)$ is ill-behaved near $z = 0$:

$$\frac{dH}{dz}(z) = \left(\frac{Y(z)}{C(z)}\right)^{3/2} \left[\frac{1}{2z}\left(C(z) - \frac{3}{2}\frac{S(z)}{C(z)}\right) + \frac{3}{4}\frac{S^2(z)}{C(z)}\right] + \frac{K_2}{8}\left[\frac{3S(z)}{C(z)}\sqrt{Y(z)} + K_2\sqrt{\frac{C(z)}{Y(z)}}\right] \tag{66}$$

The above equation holds *only if* $z \neq 0$. If $z = 0$, the following expression must be used:

$$\frac{dH}{dz}(0) = \frac{\sqrt{2}}{40}Y(0)^{3/2} + \frac{K_2}{8}\left[\sqrt{Y(0)} + K_2\sqrt{\frac{1}{2Y(0)}}\right] \tag{67}$$

Again, we need to consider the practical (computational) aspect of Eq.(67). On a computer, you will never encounter an actual zero. So when should you use Eq.(66)? Every computer and/or software has a limit on its accuracy (you can say its their personal definition of a "zero"). In MATLAB®, this limit is called "eps", and its numerical value is 2.2204×10^{-16}. So, when checking for the "zeroness" of z, you can use: *"if $|z| < eps$"*. That's as good as saying *"if $z == 0$"*. In fact, its better because the latter will never happen on a computer.

- We are now ready to set up the algorithm for Lambert's problem:

  Algorithm: Solve Lambert's (sub)-problem for z via Newton Iterations.

  - **Step 1.** Initial guess: $z^{(0)}$.
  - **Step 2.** Set $k = 1$.
    - **Step 2a.** $k - th$ correction:

$$\Delta z^{(k)} = \frac{b - H(z^{(k-1)})}{\left.\frac{dH}{dz}\right|_{z^{(k-1)}}} \tag{68}$$

    - **Step 2b.** $k - th$ approximation:

$$z^{(k)} = z^{(k-1)} + \Delta z^{(k)} \tag{69}$$

– **Step 3.** Stopping criteria:

IF $\left|\Delta z^{(k)}\right| \leq \epsilon$ (threshold on correction magnitude) **OR** $k \geq N$ (limit on number of iterations for robustness),

Return $z^{(k)}$ as final approximation.

ELSE $k = k + 1$; go to Step 2a.

- Once z is known, solve for the semi-major axis as

$$a = \frac{1}{\alpha} = \frac{\chi^2}{z} \underset{\text{Eq.(61)}}{=} \frac{Y(z)}{zC(z)} \tag{70}$$

Also, Eq.(45) can be used to determine the angular momentum.

- As a result of the above analysis, we now know all the terms in the right hand side of *both* equation sets [(37)] and [(41)]. You are free to use either set to obtain the Lagrange parameters $(F, G, \dot{F}, \dot{G})$. Then, use Eqs.(34) and (36) to determine the velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively.

- To finish, use either the pair $(\mathbf{r}_1, \mathbf{v}_1)$ or $(\mathbf{r}_2, \mathbf{v}_2)$ to completely determine the sought orbit.

Example Problems

Gibbs' Method

Three measurements of a *resident space object* were obtained by a radar,

$$\mathbf{r}_{1,\mathcal{I}} = [5887, -3520, -1204]\ km \tag{71a}$$
$$\mathbf{r}_{2,\mathcal{I}} = [5572, -3457, -2376]\ km \tag{71b}$$
$$\mathbf{r}_{3,\mathcal{I}} = [5088, -3289, -3480]\ km \tag{71c}$$

Determine the orbit of this object.

- First determine the magnitudes,

$$r_1 = \|\mathbf{r}_1\| = 6.9640 \times 10^6\ m \tag{72a}$$
$$r_2 = \|\mathbf{r}_2\| = 6.9745 \times 10^6\ m \tag{72b}$$
$$r_3 = \|\mathbf{r}_3\| = 6.9868 \times 10^6\ m \tag{72c}$$

- Thus the unit vectors are,

$$\hat{\mathbf{e}}_{r1} = \frac{\mathbf{r}_1}{r_1} = [0.8454, -0.5055, -0.1729] \tag{73a}$$
$$\hat{\mathbf{e}}_{r2} = \frac{\mathbf{r}_2}{r_2} = [0.7989, -0.4957, -0.3407] \tag{73b}$$
$$\hat{\mathbf{e}}_{r3} = \frac{\mathbf{r}_3}{r_3} = [0.7282, -0.4707, -0.4981] \tag{73c}$$

- Recall the hodograph,

$$\mathbf{v}_i = \frac{\mu}{h}\left(e\hat{\mathbf{i}}_h \times \hat{\mathbf{i}}_e + \hat{\mathbf{i}}_h \times \hat{\mathbf{e}}_{ri}\right) \tag{74}$$

which is equivalent to (following Gibbs),

$$\mathbf{v}_i = \frac{\mu}{Dh}(\mathbf{S} + \mathbf{D} \times \hat{\mathbf{e}}_{ri}) \tag{75}$$

where,

$$\mathbf{S} = (r_2 - r_3)\,\mathbf{r}_1 + (r_3 - r_1)\,\mathbf{r}_2 + (r_1 - r_2)\,\mathbf{r}_3 \tag{76}$$

also,

$$\mathbf{D} = \mathbf{r}_1 \times \mathbf{r}_2 + \mathbf{r}_2 \times \mathbf{r}_3 + \mathbf{r}_3 \times \mathbf{r}_1 \tag{77}$$

and

$$\mathbf{N} = r_1\,(\mathbf{r}_2 \times \mathbf{r}_3) + r_2\,(\mathbf{r}_3 \times \mathbf{r}_1) + r_3\,(\mathbf{r}_1 \times \mathbf{r}_2) \tag{78}$$

Moreover, we know that $h = \sqrt{\mu N/D}$

- Plugging in the numbers, we get

$$\mathbf{S} = [1.2049, -0.9900, -2.8464] \times 10^9\ m^2 \tag{79a}$$
$$\mathbf{D} = [1.2734, 2.1949, -0.2243] \times 10^{11}\ m^2 \tag{79b}$$
$$\mathbf{N} = [0.8956, 1.5438, -0.1578] \times 10^{18}\ m^3 \tag{79c}$$

- Thus we get $h = 5.2949 \times 10^{10}\ m^2/s$. Using all this information in Eq.(75), we get

$$\mathbf{v}_1 = [-1.4208,\ 0.06108, 7.4693] \times 10^3\ m/s \tag{80a}$$
$$\mathbf{v}_2 = [-2.5025,\ 0.72325,\ -7.1313] \times 10^3\ m/s \tag{80b}$$
$$\mathbf{v}_3 = [-3.5070,\ 1.3625,\ -6.5790] \times 10^3\ m/s \tag{80c}$$

- For no good reason, we choose to use the pair $(\mathbf{r}_2, \mathbf{v}_2)$ to determine the orbital elements, leading to the following numbers (look at the TBP notes for details on how to obtain orbital elements from $(\mathbf{r}, \mathbf{v})$ information):

$$a = \frac{-\mu}{2\varepsilon} = 7034.7\ km \tag{81a}$$
$$e = \sqrt{1 - p/a} = 0.0125 \tag{81b}$$
$$i = \cos^{-1}(\hat{\mathbf{i}}_h(3)) = 95.05\,\text{deg} \tag{81c}$$
$$\Omega = \tan^{-1}(\hat{\mathbf{i}}_h(1)/(-\hat{\mathbf{i}}_h(2))) = 149.9\,\text{deg} \tag{81d}$$
$$\omega = \tan^{-1}(\hat{\mathbf{i}}_e(3)/\hat{\mathbf{i}}_y(3)) = 150.5\,\text{deg} \quad \text{and,} \tag{81e}$$
$$f_2 = 47.43\,\text{deg} \quad \text{(careful with quadrant)} \tag{81f}$$

Lambert's Problem

At some point in time, the radius vector of a spacecraft was measured to be

$$\mathbf{r}_{1,\mathcal{I}} = [-0.3730, -1.4581, 0.5976] \times 10^7\ m \tag{82}$$

One hour, 38 minutes and 46 seconds later, its radius vector was found to be

$$\mathbf{r}_{2,\mathcal{I}} = [1.8520, -2.1920, 0.0431] \times 10^7\ m \tag{83}$$

Determine complete details of the orbit of this object.

- The first thing to do is to obtain the angle swept by the object in between the two measurements. Using Eq.(27),

$$\Delta\theta = \cos^{-1}\left(\frac{\mathbf{r}_1 \cdot \mathbf{r}_2}{r_1 r_2}\right) = 0.9948\ rad = 57\,\text{deg} \tag{84}$$

We *assumed* above that $\Delta\theta \in [0, 90]$ deg. Confirm this by looking at the sign of the transfer angle: $\text{sign}(\sin\Delta\theta) = \text{sign}[(\mathbf{r}_1 \times \mathbf{r}_2)_z] = \text{sign}(3.518 \times 10^{14}) = +1$.

- Next, obtain other known data: $r_1 = \|\mathbf{r}_1\| = 1.6193 \times 10^7\ m$, $r_2 = \|\mathbf{r}_2\| = 2.8699 \times 10^7\ m$. Time of flight: $\Delta t = 3600 + 38 * 60 + 46 = 5926\ s$. Also, we have the constants K_1 and K_2:

$$K_1 = \mu(1 - \cos\Delta\theta) = 1.8151 \times 10^{14} \tag{85a}$$

$$K_2 = \sqrt{\frac{\mu r_1 r_2}{K_1}} \sin\Delta\theta = 2.6793 \times 10^7 \tag{85b}$$

- It is time to execute the Newton-Raphson algorithm to approximate $z\ (= \alpha\chi^2)$ using Eq.(64). We will use a universal threshold of $\epsilon = 10^{-15}$ for the following: (a.) to determine when to truncate the evaluation of the *"open"* Stumpff functions, $C(z)$ and $S(z)$; and (b.) to use as the error-threshold for breaking the Newton iterative loop.
  - Initial guess, $z^{(0)} = 0$. Set threshold: $\epsilon = 10^{-15}$.
  - The evolution of iterations is shown in Table.(1). Note that it is almost impossible to do this by hand. The evaluation of various functions involved in Eq.(64) alone is too much work. A code was written to execute the iterations: see Carmen.

Table 1: Newton-Raphson Iteration Evolution for Lambert's Problem

Iteration # (k)	Correction: Δz^k	Approximation: $z^{(k)}$
0	-	0 (initial guess)
1	1.154757539947943	1.154757539947943
2	$6.007098336430585 \times 10^{-2}$	1.214828523312249
3	$2.469760508357477 \times 10^{-6}$	1.214830993072757
4	$-4.406172613806225 \times 10^{-15}$	1.214830993072753
5	$\underbrace{4.89574734867358 \times 10^{-16}}_{< \epsilon:: \textbf{STOP!!}}$	1.214830993072753

  - At the end of the iterations, $z = \alpha\chi^2 = 1.2148$.

- At the above determined value of z, we determine the various functions: $C(z) = 4.5139 \times 10^{-1}$, $S(z) = 1.5683 \times 10^{-1}$, $Y(z) = U_2(\alpha;\chi) = 1.2611 \times 10^7$. Thus, substitution in Eq.(61) gives $a = 23,000\ km$. Finally, using Eq.(45), we get $h = 8.1785 \times 10^{10}\ m^2/s$.

- For no good reason, we use the set [(37)] of equations to determine the Lagrange parameters. Note that you are free to use the set [(41)] as well. We get: $F = 0.22122$, $G = 4.7657 \times 10^3$, $\dot{F} = -1.8381 \times 10^{-4}$ and $\dot{G} = 0.56058$.

- Substitution of the Lagrange parameters into Eqs.(34) and (36) gives:

$$\mathbf{v}_1 = [4.0592, -3.9226, -0.18691]\ km/s \tag{86a}$$

$$\mathbf{v}_2 = [2.9611, 0.48122, -1.2032]\ km/s \tag{86b}$$

- We are almost all done. Again for no good reason, we pick the pair $(\mathbf{r}_1, \mathbf{v}_1)$ to determine the orbit. If you wish, you may use the pair $(\mathbf{r}_2, \mathbf{v}_2)$. The final answers are:

$$a = \frac{-\mu}{2\varepsilon} = 23,000\ km \tag{87a}$$

$$e = \sqrt{1 - p/a} = 0.52 \tag{87b}$$

$$i = \cos^{-1}(\hat{\mathbf{i}}_h(3)) = 25.5\,\deg \tag{87c}$$

$$\Omega = \tan^{-1}(\hat{\mathbf{i}}_h(1)/(-\hat{\mathbf{i}}_h(2))) = 132\,\mathrm{deg} \tag{87d}$$

$$\omega = \tan^{-1}(\hat{\mathbf{i}}_e(3)/\hat{\mathbf{i}}_y(3)) = 35\,\mathrm{deg} \quad \text{and,} \tag{87e}$$

$$f_1 = 86\,\mathrm{deg} \quad \text{(careful with quadrant)} \tag{87f}$$