AAE 5626: Orbital Mechanics for Engineers

Mrinal Kumar©

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A Review of Particle Mechanics

1 Vectors and Reference Frames

- Note the following useful "equation:"

$$\text{Mechanics} = \text{Kinematics} + \text{Kinetics}$$

  Kinematics concerns the "geometrical" aspects of motion, be it translational or rotational motion. In simpler terms, it deals with the relationships among the position, velocity and acceleration vectors, *without considering the inertia of the system or the forces acting on it.*

  On the other hand, *kinetics* concerns how the forces and/or moments influence the velocity and hence the position vectors through system inertia. Again, this applies to both translational and rotational motion. Together, kinematics and kinetics constitute the field of *dynamics* or *mechanics*.

  In addition to kinematics and kinetics, there is one more element of mechanics, namely the "constitutive laws" of forces and moments. This is usually the domain of physicists, who look into the composition of the forces/moments acting on the system, e.g. the gravitational force follows an inverse-square law, friction is directly proportional to the normal reaction, etc.

- Kinematics in modern mechanics is studied with the aid of vectors, which is a somewhat new development ($\sim$ 1901), due to Josiah Gibbs.

- What are vectors?
  - High school notion of vectors: anything with a *magnitude* and *direction*.
  - Graduate school notion of vectors: an element of something known as a **vector space**, or, **linear space**. A vector space defined over a "field of scalars" is a non-empty set that allows two operations:
    1. addition of two vectors, and,
    2. multiplication of a vector with a scalar.

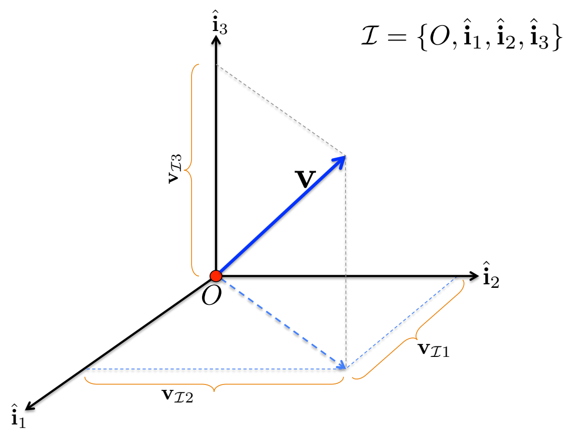
    The above two operations must satisfy the properties of associativity, commutativity and the existence of the so called identity and inverse elements. You must explore deeper into the details!

- As you can see, the "graduate school" definition significantly generalizes the "high school" definition. The latter makes sense only up to three dimensions, while the former is not limited by dimensionality. Moreover, *any* set that satisfies the above conditions qualifies as a set of vectors. Side note: You may be surprised to know that the space of "functions" is also a vector space - despite the fact that it is hard to imagine functions having a "magnitude" and a "direction".

- Getting back to our subject matter, vectors will be used to study mechanics. Some nomenclature: We will use bold lower-case letters to denote vectors, e.g. $\mathbf{v}$. However, when writing by hand (e.g. on the chalkboard), we will denote it using underlines, as follows: $\underline{v}$. (Because it is somewhat difficult to write in bold with chalk!)

- By itself, a vector $\mathbf{v}$ is a fairly abstract entity. Something is needed to express it in. This is where **reference frames** come in. A reference frame (aka coordinate system) may be defined using an *origin*, and three *unit vectors* as follows:
$$\mathcal{I} = \{O, \hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\} \tag{1}$$
In the above equation, the calligraphic $\mathcal{I}$ is the reference frame with origin O and three orthogonal unit vectors, $\hat{\mathbf{i}}_1$, $\hat{\mathbf{i}}_2$ and $\hat{\mathbf{i}}_3$. Note the use of the "hat" to differentiate a unit vector.

- Figure 1 shows the reference frame $\mathcal{I}$ and vector $\mathbf{v}$, so that when expressed in $\mathcal{I}$, $\mathbf{v}$ is given as:
$$\mathbf{v}_{\mathcal{I}} = v_{\mathcal{I}1}\hat{\mathbf{i}}_1 + v_{\mathcal{I}2}\hat{\mathbf{i}}_2 + v_{\mathcal{I}3}\hat{\mathbf{i}}_3 \tag{2}$$
where, the scalars $v_{\mathcal{I}1}$, $v_{\mathcal{I}2}$ and $v_{\mathcal{I}3}$ are called the *components* of $\mathbf{v}$ in frame $\mathcal{I}$ (see Fig. 1). Alternatively, we write: $\mathbf{v}_{\mathcal{I}} = \{v_{\mathcal{I}1}, v_{\mathcal{I}2}, v_{\mathcal{I}3}\}^T$.



$$\mathbf{v}_{\mathcal{I}} = \{v_{\mathcal{I}1}, v_{\mathcal{I}2}, v_{\mathcal{I}3}\}^T$$

Figure 1: Expression of an *abstract* vector $\mathbf{v}$ in the frame of reference $\mathcal{I}$, denoted as $\mathbf{v}_{\mathcal{I}}$

- **A curious chicken and egg problem:** There is a dilemma! This is what we did so far - We started with the vector as an abstract notion ... then we introduced a reference frame in which to express it ... but in order to define the reference frame, we used an origin and three unit ... *vectors* ??!!

- To break this circular thinking, we will constrain the unit vectors to be "orthogonal" to each other and <u>arbitrarily</u> define them *within their own reference frame* in the following manner:
$$\begin{aligned}\hat{\mathbf{i}}_{1,\mathcal{I}} &\doteq \{1,0,0\}^T\\ \hat{\mathbf{i}}_{2,\mathcal{I}} &\doteq \{0,1,0\}^T\\ \hat{\mathbf{i}}_{3,\mathcal{I}} &\doteq \{0,0,1\}^T\end{aligned} \tag{3}$$
Here's why: unit vectors $\hat{\mathbf{i}}_1$, $\hat{\mathbf{i}}_2$ and $\hat{\mathbf{i}}_3$ are, like other vectors, also abstract entities and would thus require another reference frame in which to express them; and so on, ad nauseam. Via the "orthogonality"

property, we ensure that a given unit vector has no component along any of the other unit vectors, and being a *unit* vector, its component along "its own direction" is unity, i.e. 1.
Side note: the above can be extended to N dimensions.

- To avoid tedious notation, $\hat{\mathbf{i}}_1$ will be understood be to $\hat{\mathbf{i}}_{1,\mathcal{I}}$. This abuse of notation will be employed very frequently. Note that the expression $\hat{\mathbf{i}}_{1,\mathcal{E}}$ is also perfectly valid, and stands for "the unit vector $\hat{\mathbf{i}}_1$ of reference frame $\mathcal{I}$ expressed in the reference frame $\mathcal{E}$."

- **Inertial reference frame:** Recall that an inertial frame of reference is one that **does not accelerate**. Clearly, it is a mere idealization because no such frame can actually exist in our universe (everything is revolving around something else!). Yet, the Newton's laws of motion are only valid in an inertial frame. Therefore, we must use an *approximately* inertial RF whose acceleration is significant slower than the acceleration of objects whose motion is being studied.

- In the study of flight mechanics (note no preference given to aircraft here), one encounters the following reference frames:

  1. The **Earth centered inertial RF (ECI)**: $\mathcal{I} = \{O, \hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$. ECI has its origin ($O$) at the Earth's center. Its $x - y$ plane is the same as the Earth's equatorial plane, although sometimes the ecliptic plane (plane of Earth's orbit around the sun) is also used. Its z-axis passes through the North pole and the x-axis ($\hat{\mathbf{i}}_1$) points in the direction of the vernal equinox, which is also known as the first point of Aries. Since the constellation of Aries is represented by the *Ram*, $\hat{\mathbf{i}}_1$ is also written as Υ ($\sim$ horns: Fig.(2(a))). This frame does not rotate with the Earth. While it is assumed to be inertial, you can clearly see that it is actually not because of the motion of the Earth's equatorial plane, and indeed, the center of the Earth. These accelerations are insignificant when studying the motion of spacecraft, and particularly, aircraft around the Earth.

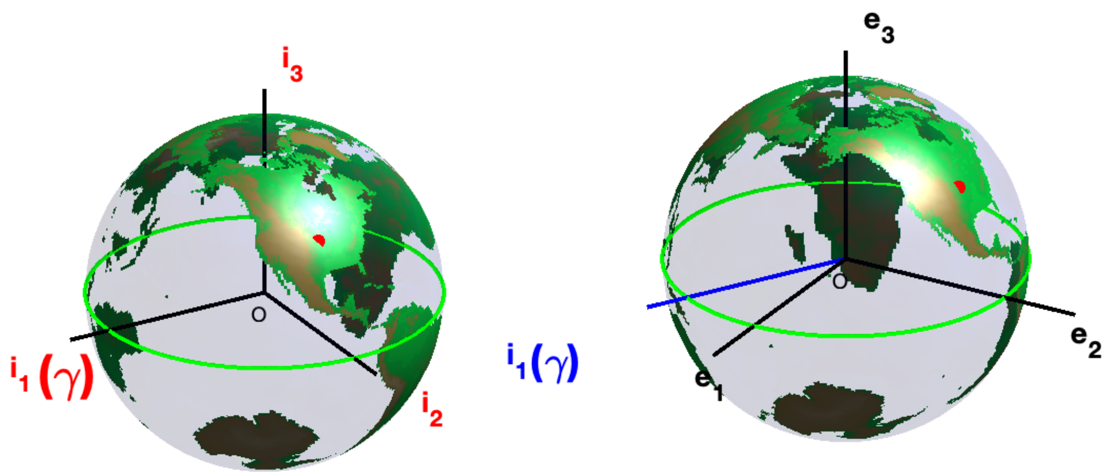
  2. The **Earth centered, Earth fixed RF (ECEF)**: $\mathcal{E} = \{O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$. The origin is at the Earth's center and the $x - y$ plane is the same as the equatorial (or ecliptic) plane (Fig.(2(b))). Being fixed to the Earth, its $x - y$ plane rotates at the same rate as the Earth's rotation. This frame is clearly noninertial, especially when used for analysis of orbital motion. However, when used for aircraft applications, this frame can be safely assumed to be inertial because Earth's rotation rate is insignificant compared to the typical speed of aircraft.

  3. The **Topological RF (TRF)**: $\mathcal{T} = \{S, \hat{\mathbf{t}}_1, \hat{\mathbf{t}}_2, \hat{\mathbf{t}}_3\}$. This noninertial frame is centered at a fixed location on the Earth's surface (typically an observation point). Its x-axis points radially away, normal to the Earth's surface, towards "zenith"; the y-axis points towards the local East and the z-axis points towards the local North (Fig.(3(a))). This frame is noninertial for spacecraft applications, but can be safely considered to be inertial for aircraft motion analysis.

  4. The **Ortibal RF (ORF)**: $\mathcal{O} = \{B, \hat{\mathbf{o}}_1, \hat{\mathbf{o}}_2, \hat{\mathbf{o}}_3\}$. The ORF is centered at the spacecraft in orbit, with the x-axis pointing in the radial direction and the z-axis is perpendicular to the plane of the orbit (Fig.(3(b))). This frame does not rotate with the spacecraft. It is noninertial.

  5. The **Body Fixed RF (BRF)**: $\mathcal{B} = \{B, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$. This RF is centered at the center of gravity of the spacecraft in orbit. In addition, the frame is also fixed to the spacecraft, i.e. rotates with it. As with aircraft motion, the x axis ($\hat{\mathbf{b}}_1$) serves as a structural reference line for the spacecraft. The y axis ($\hat{\mathbf{b}}_2$) is defined outward through the "right" side and the z axis ($\hat{\mathbf{b}}_3$) completes the triad, pointing down through the "bottom" of the spacecraft: see Fig.(4). We will see below that the x, y and z axes are defined in order to describe roll, pitch and yaw motions. These rotations are not as physically appealing as they are for an aircraft (why?). It is clearly noninertial.

Check out the MATLAB®video to get a visual feel for these frames of reference.



(a) ECI　　　　(b) ECEF

Figure 2: The Earth Centered Inertial reference frame (left) and Earth Centered Earth Fixed (ECEF) reference frame. Also shown is Columbus, OH (39.96 N lat/82.99 W long) on the globe.

- Note again that Fig.(1) shows the components of vector $\mathbf{v}$ in frame $\mathcal{I}$. We also know that these components are the projections of $\mathbf{v}$ on the corresponding unit vectors, e.g.

$$v_{\mathcal{I}1} = \mathbf{v}.\hat{\mathbf{i}}_1 \tag{4}$$

Note that both $\mathbf{v}$ and $\hat{\mathbf{i}}_1$ are abstract and we have the freedom to express them in any frame we like. In general, $v_{\mathcal{I}j} = \mathbf{v}.\hat{\mathbf{i}}_j$, $j = 1, 2, 3$.

2 Relationship between reference frames

Consider a vector $\mathbf{v}$ and two reference frames $\mathcal{A} = \{O, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3\}$ and $\mathcal{B} = \{O, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$. Expressed in these frames, vector $\mathbf{v}$ is:

$$\mathbf{v}_\mathcal{A} = \{v_{\mathcal{A}1}, v_{\mathcal{A}2}, v_{\mathcal{A}3}\}^T$$
$$\mathbf{v}_\mathcal{B} = \{v_{\mathcal{B}1}, v_{\mathcal{B}2}, v_{\mathcal{B}3}\}^T$$

Following Eq.(4), $v_{\mathcal{A}1} = \mathbf{v}.\hat{\mathbf{a}}_1$. More specifically,

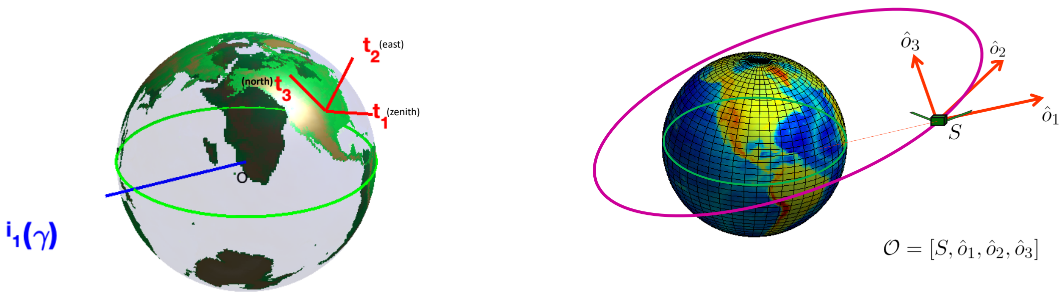
$$v_{\mathcal{A}1} = \mathbf{v}_\mathcal{A}.\hat{\mathbf{a}}_{1\mathcal{A}} = \mathbf{v}_\mathcal{B}.\hat{\mathbf{a}}_{1\mathcal{B}}$$

Now, we know by definition that $\hat{\mathbf{a}}_{1\mathcal{A}} = \{1, 0, 0\}^T$. Also, let $\hat{\mathbf{a}}_{1\mathcal{B}} = \{a_{11\mathcal{B}}, a_{12\mathcal{B}}, a_{13\mathcal{B}}\}^T$. Carrying out the dot products, the above equation becomes:

$$v_{\mathcal{A}1} = v_{\mathcal{B}1}a_{11\mathcal{B}} + v_{\mathcal{B}2}a_{12\mathcal{B}} + v_{\mathcal{B}3}a_{13\mathcal{B}} \tag{5}$$

Similarly,

$$v_{\mathcal{A}2} = v_{\mathcal{B}1}a_{21\mathcal{B}} + v_{\mathcal{B}2}a_{22\mathcal{B}} + v_{\mathcal{B}3}a_{23\mathcal{B}} \tag{6}$$
$$v_{\mathcal{A}3} = v_{\mathcal{B}1}a_{31\mathcal{B}} + v_{\mathcal{B}2}a_{32\mathcal{B}} + v_{\mathcal{B}3}a_{33\mathcal{B}} \tag{7}$$



(a) Topological RF   (b) Orbital RF

Figure 3: The topological reference frame and orbital reference frame are shown. Note that ORF assumes that the spacecraft is a particle entity.

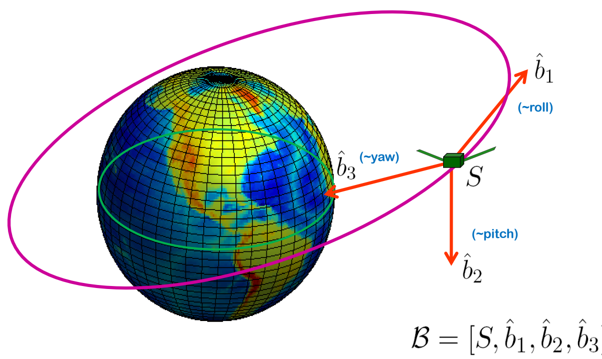


Figure 4: Body reference frame

Eqs.(5)-(7) can be written in matrix form:

$$\begin{Bmatrix} v_{\mathcal{A}1} \\ v_{\mathcal{A}2} \\ v_{\mathcal{A}3} \end{Bmatrix} = \begin{bmatrix} a_{11\mathcal{B}} & a_{12\mathcal{B}} & a_{13\mathcal{B}} \\ a_{21\mathcal{B}} & a_{22\mathcal{B}} & a_{23\mathcal{B}} \\ a_{31\mathcal{B}} & a_{32\mathcal{B}} & a_{33\mathcal{B}} \end{bmatrix} \begin{Bmatrix} v_{\mathcal{B}1} \\ v_{\mathcal{B}2} \\ v_{\mathcal{B}3} \end{Bmatrix} \tag{8}$$

Or,

$$\mathbf{v}_{\mathcal{A}} = \mathbf{R}_{\mathcal{A}\mathcal{B}}\mathbf{v}_{\mathcal{B}} \tag{9}$$

The matrix $\mathbf{R}_{\mathcal{A}\mathcal{B}}$ is called the *Direction Cosine Matrix*, and it tranforms the expression of a vector from the $\mathcal{B}$ frame to the $\mathcal{A}$ frame.

- What is $a_{11\mathcal{B}}$? This is easy to answer: it is the x-component of the unit vector $\hat{\mathbf{a}}_1$ expressed in RF $\mathcal{B}$. Mathematically speaking:

$$a_{11\mathcal{B}} = \hat{\mathbf{a}}_1.\hat{\mathbf{b}}_1 \tag{10}$$

  As usual, we may evaluate the above dot product in any frame we wish. In general,

$$a_{ij\mathcal{B}} = \hat{\mathbf{a}}_i.\hat{\mathbf{b}}_j \tag{11}$$

  With Eq.(11) in mind, the direction cosine matrix can be written as:

$$\mathbf{R}_{\mathcal{A}\mathcal{B}} = \begin{bmatrix} \hat{\mathbf{a}}_1.\hat{\mathbf{b}}_1 & \hat{\mathbf{a}}_1.\hat{\mathbf{b}}_2 & \hat{\mathbf{a}}_1.\hat{\mathbf{b}}_3 \\ \hat{\mathbf{a}}_2.\hat{\mathbf{b}}_1 & \hat{\mathbf{a}}_2.\hat{\mathbf{b}}_2 & \hat{\mathbf{a}}_2.\hat{\mathbf{b}}_3 \\ \hat{\mathbf{a}}_3.\hat{\mathbf{b}}_1 & \hat{\mathbf{a}}_3.\hat{\mathbf{b}}_2 & \hat{\mathbf{a}}_3.\hat{\mathbf{b}}_3 \end{bmatrix} \tag{12}$$

The above equation gives us insight into why the matrix $\mathbf{R}_{\mathcal{AB}}$ is called the "direction cosine matrix". Recall that $\mathbf{v}_1.\mathbf{v}_2 = v_1 v_2 \cos\angle(\mathbf{v}_1, \mathbf{v}_2)$. Also, in the above equation, all vectors are unit vectors, meaning their magnitude equals 1. Therefore, *each element of the above matrix is the cosine of the angle between two unit directions of the reference frames involved.*

- The direction cosine matrix is also known as a rotation matrix, the reason for which we will discover soon. The terms rotation matrix and DCM are used interchangeably.
- DCMs are a type of orthogonal transformation. Generally speaking, an orthogonal transformation is one that preserves the inner product, i.e. a transformation T is orthogonal if $\langle \mathbf{v}, \mathbf{u} \rangle = \langle T\mathbf{v}, T\mathbf{u} \rangle$, where $\mathbf{v}$ and $\mathbf{u}$ are vectors. However, this is beyond the scope of our study. All we need to know is that the DCM has two very nice properties, due to which it preserves the length of vectors and the angle between two vectors. These properties make a DCM an orthogonal transformation.
- Some more technical detail: DCMs are elements of the group of all rotations defined in three dimensional space. This group is called $SO(3)$. In other words, $DCM \in SO(3)$.
- As stated above, DCMs have two very nice properties. These are:

  1. The inverse of a DCM is nothing but its transpose:
  $$\mathbf{R}^{-1} = \mathbf{R}^T \tag{13}$$
  The above equation implies that $\mathbf{R}^T\mathbf{R} = \mathbf{R}\mathbf{R}^T = \mathbf{I}$. This property makes our lives amazingly simple when it comes to taking the inverse of rotation matrices - because the "matrix inverse" operation is simply replaced with the "matrix transpose" operation. We all know the latter is far simpler than the former!

     Consider the following: From Eq.(9), we have: $\mathbf{v}_{\mathcal{B}} = \mathbf{R}^{-1}_{\mathcal{AB}}\mathbf{v}_{\mathcal{A}}$. However, from Eq.(13), $\mathbf{R}^{-1}_{\mathcal{AB}} = \mathbf{R}^T_{\mathcal{AB}}$. Therefore:
  $$\mathbf{v}_{\mathcal{B}} = \mathbf{R}^{-1}_{\mathcal{AB}}\mathbf{v}_{\mathcal{A}} = \mathbf{R}^T_{\mathcal{AB}}\mathbf{v}_{\mathcal{A}} \tag{14}$$
  Also, from our definition of rotation matrices, $\mathbf{v}_{\mathcal{B}} = \mathbf{R}_{\mathcal{BA}}\mathbf{v}_{\mathcal{A}}$. Comparing this with Eq.(14), we have:
  $$\mathbf{R}_{\mathcal{BA}} = \mathbf{R}^{-1}_{\mathcal{AB}} = \mathbf{R}^T_{\mathcal{AB}} \tag{15}$$
  Very useful indeed!

  2. The determinant of a DCM is unity, i.e. $\det(\mathbf{R}) = +1$.

  Due to the above properties, a DCM is a so-called *orthonormal matrix.*

- By virtue of its orthogonality, a DCM preserves the length of vectors and the angle between two vectors.

  Proof:

  1. Preservation of length: Consider the length of a vector $\mathbf{v}$, squared:
  $$\|\mathbf{v}\|^2 = \mathbf{v}.\mathbf{v} = \mathbf{v}^T\mathbf{v}$$
  When computed in RF $\mathcal{A}$, $\|\mathbf{v}_{\mathcal{A}}\|^2 = \mathbf{v}^T_{\mathcal{A}}\mathbf{v}_{\mathcal{A}}$. Similarly, $\|\mathbf{v}_{\mathcal{B}}\|^2 = \mathbf{v}^T_{\mathcal{B}}\mathbf{v}_{\mathcal{B}}$.

     However, $\mathbf{v}_{\mathcal{A}} = \mathbf{R}_{\mathcal{AB}}\mathbf{v}_{\mathcal{B}}$. Therefore we have:
  $$\begin{aligned}\|\mathbf{v}_{\mathcal{A}}\|^2 &= \mathbf{v}^T_{\mathcal{A}}\mathbf{v}_{\mathcal{A}} \\ &= \mathbf{v}^T_{\mathcal{B}}\mathbf{R}^T_{\mathcal{AB}}\mathbf{R}_{\mathcal{AB}}\mathbf{v}_{\mathcal{B}} \\ &= \mathbf{v}^T_{\mathcal{B}}\mathbf{I}\mathbf{v}_{\mathcal{B}} \\ &= \mathbf{v}^T_{\mathcal{B}}\mathbf{v}_{\mathcal{B}} = \|\mathbf{v}_{\mathcal{B}}\|^2\end{aligned}$$

  2. Preservation of angles: This proof is similar to the one above. DIY. Start with the following fact: $\cos\angle(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_1.\mathbf{v}_2$, where $\mathbf{v}_1$ and $\mathbf{v}_2$ are any two vectors.

2.1 Rotation Matrices

- We saw above that a direction cosine matrix is used to relate two reference frames. Each of its nine elements is a dot product between pairs of unit vectors belonging to the reference frames of concern. Unfortunately, this manner of construction of the DCM is neither very convenient, nor physically appealing.
- There exists a very useful result that fixes the above problem: A transformation between any two reference frames in three dimensional space can be characterized using a sequence of three independent so-called *elementary rotations* (aka *canonical rotations*).

  Important: There is an assumption behind the above result: ***Both frames involved must share a common origin:*** See Fig.(5).

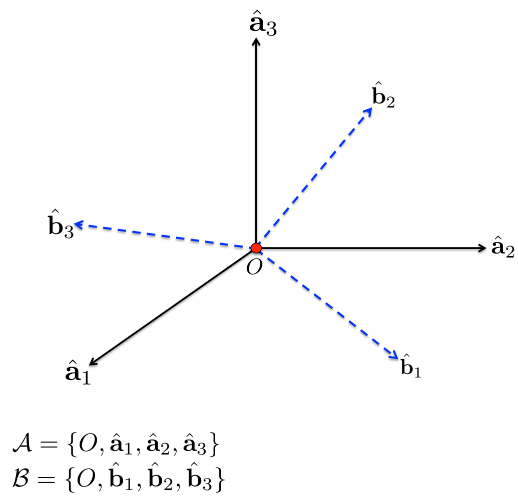


Figure 5: Two reference frames, $\mathcal{A}$ and $\mathcal{B}$ with a common origin, O.

- An important relationship: Let $\mathcal{A} = \{O, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3\}$ and $\mathcal{B} = \{O, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$. Then,

$$\begin{Bmatrix} \hat{\mathbf{a}}_{1,\mathcal{B}} \\ \hat{\mathbf{a}}_{2,\mathcal{B}} \\ \hat{\mathbf{a}}_{3,\mathcal{B}} \end{Bmatrix} = \mathbf{R}_{\mathcal{AB}} \begin{Bmatrix} \hat{\mathbf{b}}_1 \\ \hat{\mathbf{b}}_2 \\ \hat{\mathbf{b}}_3 \end{Bmatrix} \tag{16}$$

  Proof: DIY. Be careful: the individual elements within the curly-brackets on either side of the equation are vectors and not scalars. Eq.(16) will be important in what follows.

2.2 Elementary Rotations

As mentioned above, any general DCM can be constructed using three independent elementary rotations. Since there are three axes in our three dimensional space, we have an elementary rotation defined with respect to each one of these axes:

1. Rotation by angle θ about the z-axis: See Fig.(6). The rotation takes place about the z-axis of frame $\mathcal{A} = \{O, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3\}$, resulting in the new frame $\mathcal{B} = \{O, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$. Clearly, the *axis of rotation*, $\hat{\mathbf{a}}_3$ does not displace, implying that $\hat{\mathbf{b}}_3 = \hat{\mathbf{a}}_3$.

   From Fig.(6), it is easy to see that the new axes, $\hat{\mathbf{b}}_1$, $\hat{\mathbf{b}}_2$ and $\hat{\mathbf{b}}_3$ expressed in frame $\mathcal{A}$ are:

$$\hat{\mathbf{b}}_{1,\mathcal{A}} = \cos\theta\hat{\mathbf{a}}_1 + \sin\theta\hat{\mathbf{a}}_2 \tag{17a}$$

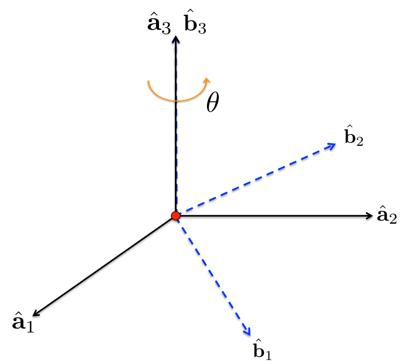


Figure 6: Elementary rotation about the z-axis

$$\hat{\mathbf{b}}_{2,\mathcal{A}} = -\sin\theta\hat{\mathbf{a}}_1 + \cos\theta\hat{\mathbf{a}}_2 \tag{17b}$$

$$\hat{\mathbf{b}}_{3,\mathcal{A}} = \hat{\mathbf{a}}_3 \tag{17c}$$

Or, in matrix form, we have:

$$\begin{Bmatrix} \hat{\mathbf{b}}_{1,\mathcal{A}} \\ \hat{\mathbf{b}}_{2,\mathcal{A}} \\ \hat{\mathbf{b}}_{3,\mathcal{A}} \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{=\mathbf{R}_{\mathcal{BA}} \text{ (compare /w Eq.(16))}} \begin{Bmatrix} \hat{\mathbf{a}}_1 \\ \hat{\mathbf{a}}_2 \\ \hat{\mathbf{a}}_3 \end{Bmatrix} \tag{18}$$

As noted above, the DCM between reference frames $\mathcal{A}$ and $\mathcal{B}$ is nothing but the elementary rotation matrix. This elementary rotation by angle θ about the z-axis is denoted by $\mathbf{R}_3(\theta)$.

It is easy to show that $\mathbf{R}_3(\theta)^{-1} = \mathbf{R}_3(-\theta)$.

2. Rotation by angle θ about the y-axis: See Fig.(7). Upon carrying out similar analysis as for the z elementary rotation, we find that:

$$\mathbf{R}_2(\theta) = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \tag{19}$$

3. Rotation by angle θ about the x-axis: See Fig.(8). The elementary rotation matrix about the x-axis is given as:

$$\mathbf{R}_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \tag{20}$$

2.3 Composition of Elementary Rotations

Elementary rotations shown in the above section are independent of one another and can be sequenced together to compose more complex coordinate transformations.

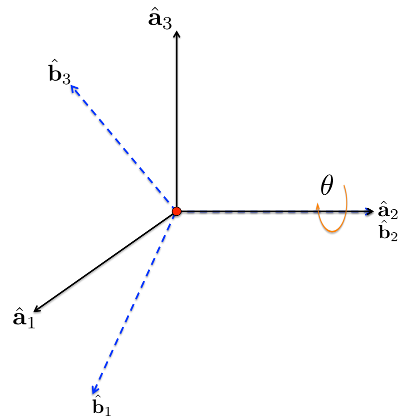


Figure 7: Elementary rotation about the y-axis

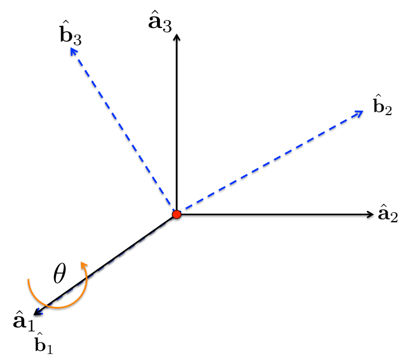


Figure 8: Elementary rotation about the x-axis

1. Composition of two elementary rotations: Consider the following example - Beginning with a RF $\mathcal{A} = \{O, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3\}$, perform an elementary rotation of angle Ω about the z-axis to obtain a new frame $\mathcal{A}' = \{O, \hat{\mathbf{a}}'_1, \hat{\mathbf{a}}'_2, \hat{\mathbf{a}}'_3\}$. Based on the above developments, the DCM between $\mathcal{A}$ and $\mathcal{A}'$ is $\mathbf{R}_{\mathcal{A}'\mathcal{A}} = \mathbf{R}_3(\Omega)$; such that for any vector $\mathbf{v}$:
$$\mathbf{v}_{\mathcal{A}'} = \mathbf{R}_{\mathcal{A}'\mathcal{A}}\mathbf{v}_{\mathcal{A}} = \mathbf{R}_3(\Omega)\mathbf{v}_{\mathcal{A}} \tag{21}$$
Next, perform a rotation of angle i about the x-axis of $\mathcal{A}'$, resulting in a new RF $\mathcal{A}'' = \{O, \hat{\mathbf{a}}''_1, \hat{\mathbf{a}}''_2, \hat{\mathbf{a}}''_3\}$. As above, we get:
$$\mathbf{v}_{\mathcal{A}''} = \mathbf{R}_{\mathcal{A}''\mathcal{A}'}\mathbf{v}_{\mathcal{A}'} = \mathbf{R}_1(i)\mathbf{v}_{\mathcal{A}'} \tag{22}$$
We are interested in the transformation between the original frame ($\mathcal{A}$) and the final frame ($\mathcal{A}''$), i.e. the DCM $\mathbf{R}_{\mathcal{A}''\mathcal{A}}$. This is easy to obtain - note that by definition,
$$\mathbf{v}_{\mathcal{A}''} = \underbrace{\mathbf{R}_{\mathcal{A}''\mathcal{A}}}_{\text{unknown}}\mathbf{v}_{\mathcal{A}} \tag{23}$$
Let us keep this relationship aside for later use. Substituting Eq.(21) in Eq.(22), we get:
$$\mathbf{v}_{\mathcal{A}''} = \mathbf{R}_1(i)\mathbf{R}_3(\Omega)\mathbf{v}_{\mathcal{A}} \tag{24}$$

Compraing the above equation with Eq.(23), we find that $\mathbf{R}_{\mathcal{A}''\mathcal{A}} = \mathbf{R}_1(i)\mathbf{R}_3(\Omega)$. In the same manner, a sequence of rotations about any of two axes can be constructed.

The following is the exhaustive list of two-rotation sequences possible in three dimensional space (the angles have been dropped from the notation for brevity):

$$\begin{array}{ccc} \mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_2\mathbf{R}_3 & \mathbf{R}_3\mathbf{R}_1 \\ \mathbf{R}_2\mathbf{R}_1 & \mathbf{R}_3\mathbf{R}_2 & \mathbf{R}_1\mathbf{R}_3 \end{array}$$

Example continued: consider again the rotation sequence $\mathcal{A} \xrightarrow{\mathbf{R}_3(\Omega)} \mathcal{A}' \xrightarrow{\mathbf{R}_1(i)} \mathcal{A}''$. In other words, an elementary rotation about the z-axis by angle Ω, followed by an elementary rotation about the x-axis by angle i, which leads us to the composite DCM $\mathbf{R}_{\mathcal{A}''\mathcal{A}} = \mathbf{R}_1(i)\mathbf{R}_3(\Omega)$:

$$\mathbf{R}_{\mathcal{A}''\mathcal{A}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos i & \sin i \\ 0 & -\sin i & \cos i \end{bmatrix} \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\sin\Omega & \cos\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\Omega & \sin\Omega & 0 \\ -\cos i\sin\Omega & \cos i\cos\Omega & \sin i \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{bmatrix} \tag{25}$$

2. Composition of three elementary rotations: Following exactly the same procedure as above, we can reconstruct general dimensional rotations using three elementary rotations. Continuing the example from above, consider the following rotation sequence: $\mathcal{A} \xrightarrow{\mathbf{R}_3(\Omega)} \mathcal{A}' \xrightarrow{\mathbf{R}_1(i)} \mathcal{A}'' \xrightarrow{\mathbf{R}_3(\omega)} \mathcal{B}$. We are interested in obtaining the DCM $\mathbf{R}_{\mathcal{B}\mathcal{A}}$, such that $\mathbf{v}_{\mathcal{B}} = \mathbf{R}_{\mathcal{B}\mathcal{A}}\mathbf{v}_{\mathcal{A}}$. Given the sequence of rotations, we have the following developments:

$$\begin{aligned} \mathbf{v}_{\mathcal{B}} &= \mathbf{R}_{\mathcal{B}\mathcal{A}''}\mathbf{v}_{\mathcal{A}''} \\ &= \mathbf{R}_3(\omega)\mathbf{v}_{\mathcal{A}''} \\ &= \mathbf{R}_3(\omega)\underbrace{\mathbf{R}_{\mathcal{A}''\mathcal{A}'}\mathbf{v}_{\mathcal{A}'}}_{=\mathbf{v}_{\mathcal{A}''}} \\ &= \mathbf{R}_3(\omega)\mathbf{R}_1(i)\mathbf{v}_{\mathcal{A}'} \\ &= \mathbf{R}_3(\omega)\mathbf{R}_1(i)\underbrace{\mathbf{R}_{\mathcal{A}'\mathcal{A}}\mathbf{v}_{\mathcal{A}}}_{=\mathbf{v}_{\mathcal{A}'}} \\ &= \mathbf{R}_3(\omega)\mathbf{R}_1(i)\mathbf{R}_3(\Omega)\mathbf{v}_{\mathcal{A}} \end{aligned}$$

Therefore:

$$\mathbf{R}_{\mathcal{B}\mathcal{A}} = \mathbf{R}_3(\omega)\mathbf{R}_1(i)\mathbf{R}_3(\Omega) = \begin{bmatrix} c\omega\, c\Omega - s\omega\, ci\, s\Omega & c\omega\, s\Omega + s\omega\, ci\, c\Omega & s\omega\, si \\ -(s\omega\, c\Omega + c\omega\, ci\, s\Omega) & -s\omega\, s\Omega + c\omega\, ci\, c\Omega & c\omega\, si \\ si\, s\Omega & -\, si\, c\Omega & ci \end{bmatrix} \tag{26}$$

In the above equation c and s stand for cos and sin respectively. The $3-1-3$ rotation sequence shown above will be used in a few weeks to construct the orbital reference frame. Just like there are six transformations possible using a sequence of two rotations, 12 transformations can be constructed using a sequence of three elementary rotations. These can be divided into symmetric and asymmetric sequences as follows:

"Symmetric" transformations:

$$\begin{array}{ccc} \mathbf{R}_1\mathbf{R}_2\mathbf{R}_1 & \mathbf{R}_2\mathbf{R}_1\mathbf{R}_2 & \mathbf{R}_3\mathbf{R}_1\mathbf{R}_3 \\ \mathbf{R}_1\mathbf{R}_3\mathbf{R}_1 & \mathbf{R}_2\mathbf{R}_3\mathbf{R}_2 & \mathbf{R}_3\mathbf{R}_2\mathbf{R}_3 \end{array}$$

"Asymmetric" transformations:

$$\begin{matrix} \mathbf{R}_1\mathbf{R}_2\mathbf{R}_3 & \mathbf{R}_2\mathbf{R}_1\mathbf{R}_3 & \mathbf{R}_3\mathbf{R}_1\mathbf{R}_2 \\ \mathbf{R}_1\mathbf{R}_3\mathbf{R}_2 & \mathbf{R}_2\mathbf{R}_3\mathbf{R}_1 & \mathbf{R}_3\mathbf{R}_2\mathbf{R}_1 \end{matrix}$$

The three angles used to construct the coordinate transformation, e.g. Ω, i and ω in the above $3-1-3$ example are called **Euler's angles.** Another example is the popular $3-2-1$ yaw (ψ) - pitch (θ) - roll (ϕ) Euler sequence used in aircraft dynamics.

Any general coordinate transformation in three dimensions can be constructed using a sequence of at-most three independent elementary Euler rotations. Any of the twelve sequences listed above can be used to achieve this transformation.

Euler angles are special because not only they help us build our DCM in a convenient manner, they also have a very physical feel about them, in the sense that they are used as so called *attitude parameters*.

❣ **Example Transformation of ECI to ORF.** The $3-1-3$ sequence described above is used to build the transformation between the ECI and ORF reference frames. However, to account for the assumption mentioned in the beginning of this document, *we will assume that both the ECI and ORF share the Earth's center as their origin.* In other words: $\mathcal{I} = \{O, \hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ and $\mathcal{O} = \{O, \hat{\mathbf{o}}_1, \hat{\mathbf{o}}_2, \hat{\mathbf{o}}_3\}$, where O is the center of the Earth. Let us visually depict the transformation from $\mathcal{I}$ to $\mathcal{O}$:

1. See Fig.(9): The first elementary Euler rotation by angle Ω about $\hat{\mathbf{i}}_3$ results in the new RF, $\mathcal{I}' = \{O, \hat{\mathbf{i}}'_1, \hat{\mathbf{i}}'_2, \hat{\mathbf{i}}'_3\}$. Based on our foregoing analysis, we have that $\mathbf{v}_{\mathcal{I}'} = \mathbf{R}_3(\Omega)\mathbf{v}_{\mathcal{I}}$, for any vector $\mathbf{v}$.

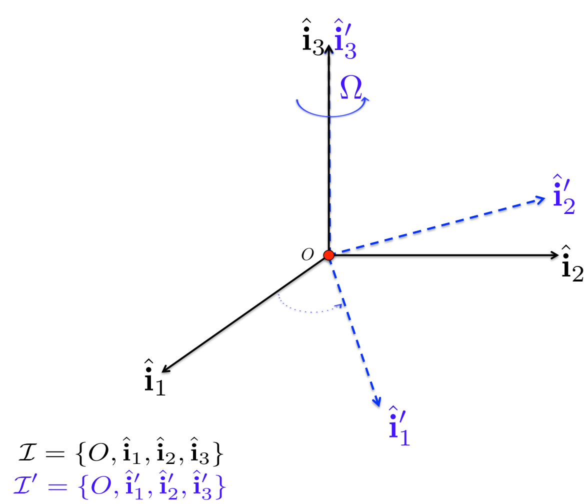


Figure 9: ECI to ORF: Elementary Rotation 1 = $\mathbf{R}_3(\Omega)$

2. See Fig.(10): The second elementary Euler rotation by angle i about $\hat{\mathbf{i}}'_1$ results in the new RF, $\mathcal{I}'' = \{O, \hat{\mathbf{i}}''_1, \hat{\mathbf{i}}''_2, \hat{\mathbf{i}}''_3\}$. We have, $\mathbf{v}_{\mathcal{I}''} = \mathbf{R}_1(i)\mathbf{v}'_{\mathcal{I}}$, for any vector $\mathbf{v}$.

3. See Fig.(11): The third elementary Euler rotation by angle ω about $\hat{\mathbf{i}}''_3$ results in the final (desired) RF, $\mathcal{O} = \{O, \hat{\mathbf{o}}_1, \hat{\mathbf{o}}_2, \hat{\mathbf{o}}_3\}$. Of course, $\mathbf{v}_{\mathcal{O}} = \mathbf{R}_3(\omega)\mathbf{v}_{\mathcal{I}''}, \forall\, \mathbf{v}$.

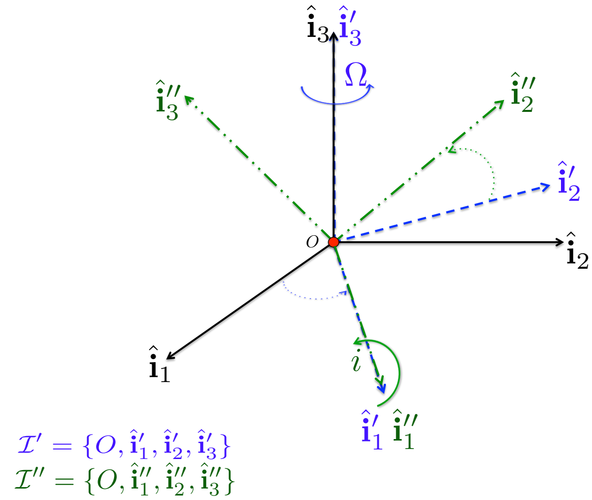


Figure 10: ECI to ORF: Elementary Rotation 2 = $\mathbf{R}_1(\Omega)$

Working backwards:

$$\mathbf{v}_{\mathcal{O}} = \underbrace{\mathbf{R}_3(\omega)}_{\mathbf{R}_{\mathcal{O}\mathcal{I}''}} \mathbf{v}_{\mathcal{I}''} \tag{27a}$$

$$= \underbrace{\mathbf{R}_3(\omega)\mathbf{R}_1(i)}_{=\mathbf{R}_{\mathcal{O}\mathcal{I}'}} \mathbf{v}'_{\mathcal{I}} \tag{27b}$$

$$= \underbrace{\mathbf{R}_3(\omega)\mathbf{R}_1(i)\mathbf{R}_3(\Omega)}_{=\mathbf{R}_{\mathcal{O}\mathcal{I}}} \mathbf{v}_{\mathcal{I}} \tag{27c}$$

- The $\mathbf{R}_{\mathcal{O}\mathcal{I}}$ rotation sequence describes the *orientation of the orbital reference frame with respect to the inertial frame.*
- The $3-1-3$ rotation sequence described above is not the only way to achieve a transformation between ECI and ORF. As pointed out previously, it is in fact one of twelve possible ways that involve Euler angles.
- So, the all important question is: what is the most *primitive relationship* between two concentric reference frames? Or, which relationship between concentric reference frames is *unique*? The answer is: Eq.(12) (or equivalently, Eq.(16)), i.e. the relationship involving direction cosines. All others are simply different and more convenient ways of *characterizing* this relationship. Let us do an example.

❣ **Example** Suppose it is known that the orbital reference frame (ORF) unit vectors expressed in the inertial frame are:

$$\begin{aligned}
\hat{\mathbf{o}}_{1_\mathcal{I}} &= [-0.0800,\ 0.862,\ 0.500]^{\mathrm{T}} \\
\hat{\mathbf{o}}_{2_\mathcal{I}} &= [-0.787,\ -0.362,\ 0.500]^{\mathrm{T}} \\
\hat{\mathbf{o}}_{3_\mathcal{I}} &= [0.612,\ -0.354,\ 0.707]^{\mathrm{T}}
\end{aligned}$$

This is privileged information! Because using Eq.(12) or (16), we directly get:

$$\mathbf{R}_{\mathcal{O}\mathcal{I}} \underset{\text{Eq.(12)}}{=} \begin{bmatrix} \hat{\mathbf{o}}_1.\hat{\mathbf{i}}_1 & \hat{\mathbf{o}}_1.\hat{\mathbf{i}}_2 & \hat{\mathbf{o}}_1.\hat{\mathbf{i}}_3 \\ \hat{\mathbf{o}}_2.\hat{\mathbf{i}}_1 & \hat{\mathbf{o}}_2.\hat{\mathbf{i}}_2 & \hat{\mathbf{o}}_2.\hat{\mathbf{i}}_3 \\ \hat{\mathbf{o}}_3.\hat{\mathbf{i}}_1 & \hat{\mathbf{o}}_3.\hat{\mathbf{i}}_2 & \hat{\mathbf{o}}_3.\hat{\mathbf{i}}_3 \end{bmatrix} = \begin{bmatrix} -0.0800 & 0.862 & 0.500 \\ -0.787 & -0.362 & 0.500 \\ 0.612 & -0.354 & 0.707 \end{bmatrix} \tag{28}$$

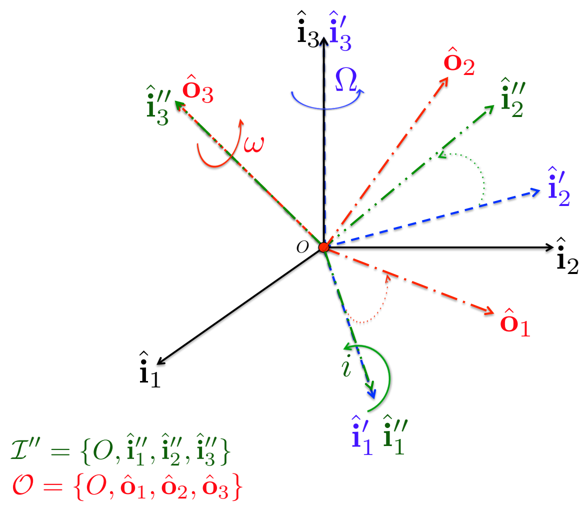


Figure 11: ECI to ORF: Elementary Rotation 3 = $\mathbf{R}_3(\omega)$

The question is, what is the equivalent $3-1-3$ rotation sequence involving angles ω, i and Ω respectively that characterizes the $\mathbf{R}_{\mathcal{OI}}$ matrix given above? To do this, we must fit the rotation matrix of Eq.(28) into the general template given in Eq.(26) by selecting appropriate values for the Euler angles Ω, i and ω, such that:

$$\begin{bmatrix} c\omega\, c\Omega - s\omega\, ci\, s\Omega & c\omega\, s\Omega + s\omega\, ci\, c\Omega & s\omega\, si \\ -(s\omega\, c\Omega + c\omega\, ci\, s\Omega) & -s\omega\, s\Omega + c\omega\, ci\, c\Omega & c\omega\, si \\ si\, s\Omega & -\, si\, c\Omega & ci \end{bmatrix} = \begin{bmatrix} -0.0800 & 0.862 & 0.500 \\ -0.787 & -0.362 & 0.500 \\ 0.612 & -0.354 & 0.707 \end{bmatrix}$$

Clearly, it is easy to obtain the inclination (i) first by comparing the $(3,3)$ elements of the above relationship:

$$\cos i \;=\; 0.707$$

such that $i = \cos^{-1} 0.707 = 45\,\text{deg}$. There is of course some ambiguity here because one could also argue that $\cos^{-1} 0.707 = -45\,\text{deg}$. In effect, this implies a retrograde orbit and we appeal to practical considerations of such an orbit to preclude the possibility.

Moving on, we compare the (1,3) and (2,3) elements respectively to see that

$$\sin\omega \;=\; \frac{0.500}{\sin i} \;=\; 0.707 \quad \text{and,}$$

$$\cos\omega \;=\; \frac{-0.500}{\sin i} \;=\; 0.707$$

Note that it is essential to compute **<u>both</u>** $\cos\omega$ and $\sin\omega$, so that you can place the angle ω in its correct quadrant. It is clear that ω lies in the first quadrant, such that $\omega = 45\,\text{deg}$. Similarly, comparing the $(3,1)$ and $(3,2)$ elements of the above matrices, we get $\sin\Omega = 0.866$ and $\cos\Omega = 0.501$. This indicates that the longitude of ascending node is in the first quadrant, $\Omega = 60\,\text{deg}$.

One can repeat the above procedure for any other rotation sequence (e.g. $1-2-3$) to obtain the corresponding Euler angles that would characterize the given transformation matrix $\mathbf{R}_{\mathcal{OI}}$.

In the ideal two-body problem, the three Euler angles Ω, i and ω are **fixed** after being once ascertained, i.e. they do not change with time. So essentially we are saying that the ORF has a fixed, time-invariant orientation with respect to ECI. If the ORF is built as shown, i.e. with its origin at the Earth's center, without being "attached" to it, it turns out be an inertial frame, just like ECI. (please make sure you are convinced of this)

❢ **Example** An alternate problem is the following: Suppose it is known that the transformation between two reference frames, $\mathcal{O}$ and $\mathcal{I}$ can be characterized via the $3-1-3$ sequence with $\Omega = 135\,\text{deg}$, $i = 10\,\text{deg}$ and $\omega = 80\,\text{deg}$. You are required to determine another Euler sequence, e.g. $3-2-1$, which would also characterize the same matrix.

This problem can be solved as follows:

Step 1. Using the given values of the Euler angles in the $3-1-3$ sequence, construct the $\mathbf{R}_{\mathcal{O}\mathcal{I}}$ matrix, in this case, using Eq.(26).

Step 2. Then, construct a generic $3-2-1$ transformation:

$$\mathcal{I} \xrightarrow{\mathbf{R}_3(\alpha)} \mathcal{I}' \xrightarrow{\mathbf{R}_2(\beta)} \mathcal{I}'' \xrightarrow{\mathbf{R}_1(\gamma)} \mathcal{O} \tag{29}$$

such that, $\mathbf{R}_{\mathcal{O}\mathcal{I}} = \mathbf{R}_1(\gamma)\mathbf{R}_2(\beta)\mathbf{R}_3(\alpha)$. Carry out the matrix multiplications to obtain the expanded generic form.

Step 3. Finally, compare the expanded form with the numerical values of the $\mathbf{R}_{\mathcal{O}\mathcal{I}}$ matrix obtained in Step 1 and solve for the Euler angles α, β and γ of the $3-2-1$ sequence.

3 Rotation rates

- In the above example of transformation from ECI to ORF, the Euler angles were constants, so that the ORF had a time-invariant orientation with respect to the ECI. However, this is true only in the case of the ideal problem of two perfectly spherical bodies. In reality, there are numerous perturbing elements that cause the Euler angles Ω, i and ω to change with time. These include perturbations due to oblateness, atmospheric drag, presence of other bodies, etc. Under such circumstances, the ORF continuously changes orientation with respect to the ECI, *while maintaining the same origin.* We may write ORF as: $\mathcal{O} = \{O, \hat{\mathbf{o}}_1(t), \hat{\mathbf{o}}_2(t), \hat{\mathbf{o}}_3(t)\}$. The instantaneous values of the Euler angles, namely $\Omega(t)$, $i(t)$ and $\omega(t)$ provide the orientation of the ORF with respect to the ECI, and the ORF is no longer an inertial RF.

  Therefore, we are interested in the *angular rate at which ORF rotates with respect to ECI.* This is actually easy to determine.

  Nomenclature: We will write the *angular rate vector* of frame $\mathcal{A}$ with respect to frame $\mathcal{B}$ as $\boldsymbol{\omega}^{\mathcal{A}/\mathcal{B}}$. Important points about this notation:

  1. Distinguish between $\boldsymbol{\omega}$ and ω: the former is a rotation-rate vector written in bold, while the latter is an Euler angle.
  2. Without the superscript, the symbol $\boldsymbol{\omega}$ is meaningless. (why?) $\boldsymbol{\omega}^{\mathcal{A}/\mathcal{B}}$ reads as follows: the angular rate of $\mathcal{A}$ with respect to $\mathcal{B}$, or; ***the angular rate of frame $\mathcal{A}$ as measured by an instrument placed in frame $\mathcal{B}$.*** Clearly, any "rate" or "velocity" like term makes sense only when the point of measurement is specified.
  3. The vector $\boldsymbol{\omega}^{\mathcal{A}/\mathcal{B}}$ by itself is an abstract entity, just like any other vector $\mathbf{v}$. It may be **expressed** in any reference frame of choice. For example, $\boldsymbol{\omega}^{\mathcal{A}/\mathcal{B}}_{\mathcal{C}}$ reads as follows: the angular rate of frame $\mathcal{A}$ as *measured* in frame $\mathcal{B}$ and *expressed* in frame $\mathcal{C}$.

4. A general rule followed in this course: superscripts depict the frame of measurement and subscripts depict the frame of expression. These frames are not required to be the same!

- Recall the basic rule of adding relative velocities: $\mathbf{v}^{\mathcal{A}/\mathcal{B}} = \mathbf{v}^{\mathcal{A}/\mathcal{C}} + \mathbf{v}^{\mathcal{C}/\mathcal{B}}$, where $\mathbf{v}$ stands for velocity.

- Using the rule of adding relative velocities and the sequence of rotations through which the transformation between frames $\mathcal{I}$ and $\mathcal{O}$ was achieved, it is easy to see that:

$$\boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} = \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}''} + \boldsymbol{\omega}^{\mathcal{I}''/\mathcal{I}'} + \boldsymbol{\omega}^{\mathcal{I}'/\mathcal{I}} \tag{30}$$

- The above addition makes perfect sense. Note that all the vectors have been written as abstract entities and not specialized to any particular frame of reference. At the same time, the above equation is not of much practical use because we must write all the vectors in a single frame of reference for the addition to actually take place.

- From Figs.(9)-(11), it is easy to see that:

$$\boldsymbol{\omega}^{\mathcal{I}'/\mathcal{I}} = \dot{\Omega}\ \hat{\mathbf{i}}'_3 \tag{31a}$$
$$\boldsymbol{\omega}^{\mathcal{I}''/\mathcal{I}'} = \dot{i}\ \hat{\mathbf{i}}''_1 \tag{31b}$$
$$\boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}''} = \dot{\omega}\ \hat{\mathbf{o}}_3 \tag{31c}$$

Note that none of the vectors have been expressed in any particular frame of reference (the "abuse of notation" is not in effect).

- Let us now combine Eqs.(30) and (31), and, express all vectors in the ORF:

$$\boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}}_{\mathcal{O}} = \dot{\Omega}\ \hat{\mathbf{i}}'_{3,\mathcal{O}} + \dot{i}\ \hat{\mathbf{i}}''_{1,\mathcal{O}} + \dot{\omega}\ \hat{\mathbf{o}}_{3,\mathcal{O}} \tag{32}$$

- The final step is to obtain the expressions for each of the unit vectors in the ORF:

1. $\hat{\mathbf{i}}'_{3,\mathcal{O}}$:

$$\hat{\mathbf{i}}'_{3,\mathcal{O}} = \mathbf{R}_{\mathcal{O}\mathcal{I}'}\hat{\mathbf{i}}'_{3,\mathcal{I}'} \underset{\text{Eq.(27b)}}{=} \mathbf{R}_3(\omega)\mathbf{R}_1(i)\begin{Bmatrix}0\\0\\1\end{Bmatrix} \tag{33}$$

2. $\hat{\mathbf{i}}''_{1,\mathcal{O}}$:

$$\hat{\mathbf{i}}''_{1,\mathcal{O}} = \mathbf{R}_{\mathcal{O}\mathcal{I}''}\hat{\mathbf{i}}''_{1,\mathcal{I}''} \underset{\text{Eq.(27a)}}{=} \mathbf{R}_3(\omega)\begin{Bmatrix}1\\0\\0\end{Bmatrix} \tag{34}$$

3. $\hat{\mathbf{o}}_{3,\mathcal{O}}$: This of-course is trivial and follows by definition.

Hence, we collect everything together to get:

$$\boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}}_{\mathcal{O}} = \dot{\Omega}\ \mathbf{R}_3(\omega)\mathbf{R}_1(i)\begin{Bmatrix}0\\0\\1\end{Bmatrix} + \dot{i}\ \mathbf{R}_3(\omega)\begin{Bmatrix}1\\0\\0\end{Bmatrix} + \dot{\omega}\ \begin{Bmatrix}0\\0\\1\end{Bmatrix} \tag{35}$$

Carrying out the matrix multiplications, we get:

$$\boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}}_{\mathcal{O}} = \dot{\Omega}\ \begin{Bmatrix}s\omega\, si\\c\omega\, si\\ci\end{Bmatrix} + \dot{i}\ \begin{Bmatrix}c\omega\\-s\omega\\0\end{Bmatrix} + \dot{\omega}\ \begin{Bmatrix}0\\0\\1\end{Bmatrix} \tag{36}$$

(37)

In matrix form,

$$\begin{Bmatrix} \omega_{\mathcal{O},1} \\ \omega_{\mathcal{O},2} \\ \omega_{\mathcal{O},3} \end{Bmatrix} = \begin{bmatrix} s\omega\, si & c\omega & 0 \\ c\omega\, si & -s\omega & 0 \\ ci & 0 & 1 \end{bmatrix} \begin{Bmatrix} \dot{\Omega} \\ \dot{i} \\ \dot{\omega} \end{Bmatrix} \tag{38}$$

where, $\boldsymbol{\omega}_{\mathcal{O}}^{\mathcal{O}/\mathcal{I}} = \{\omega_{\mathcal{O},1}, \omega_{\mathcal{O},2}, \omega_{\mathcal{O},3}\}^T$. Note that the superscripts have been dropped from the components to avoid excessive clutter. Eq.(38) is also written as:

$$\boldsymbol{\omega}_{\mathcal{O}}^{\mathcal{O}/\mathcal{I}} = \mathbf{B}[\boldsymbol{\theta}] \begin{Bmatrix} \dot{\Omega} \\ \dot{i} \\ \dot{\omega} \end{Bmatrix} \tag{39}$$

Important notes:

- The terms $\dot{\Omega}$, $\dot{i}$ and $\dot{\omega}$ are called Euler angle rates.
- The term $\boldsymbol{\theta}$ in the above equation is simply $\boldsymbol{\theta} = [\Omega, i, \omega]$.
- Eq.(39) is an odd equation because (i) the curly bracket on the right hand side is not really a physical vector, and (ii) the matrix $\mathbf{B}[\boldsymbol{\theta}]$ is not a rotation matrix. However, the term on the left hand side **is very much a vector that makes complete physical sense**. It is the angular rate of the ORF measured from the ECI, expressed in the ORF.
- The contents of the $\mathbf{B}$ matrix will be completely different if instead of the $3-1-3$ as in this example, a different rotation sequence is chosen.
- As stated above, matrix $\mathbf{B}$ is *not* a rotation matrix. Indeed, it does not have the nice property of orthogonality, i.e. $\mathbf{B}^{-1} \neq \mathbf{B}^T$. This is made clear upon taking its inverse:

$$\mathbf{B}^{-1} = \frac{1}{\sin i} \begin{bmatrix} s\omega & c\omega & 0 \\ c\omega\, si & -s\omega\, si & 0 \\ s\omega\, ci & -c\omega\, ci & si \end{bmatrix} \tag{40}$$

  The above matrix can be used to find the Euler angle rates if the vector $\boldsymbol{\omega}_{\mathcal{O}}^{\mathcal{O}/\mathcal{I}}$ is given to us (see Eq.(39)).
- Clearly, problems loom large when the second Euler rotation approaches 0 deg or 180 deg. Looking ahead, the Euler angle i will represent the inclination of an orbit. So, watch out as we deal with equatorial (i.e. zero inclination) orbits!

- The above analysis can be repeated for all other rotation sequences.

4 Particle Kinematics

- So far, we have looked at reference frames and the kinematics of RF rotations. We will now consider the motion of particle masses.
- Consider Fig.(12). An inertial frame $\mathcal{I} = \{O, \hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ is shown, along with a particle P.

  Important: all other reference frames used in our analysis will also be centered at O, so that no homogeneous transformations are required.
- The position vector of point P measured in *any* reference frame centered at O is denoted by $\mathbf{r}$. In kinematics, we are interested in finding the velocity and acceleration vectors of P measured in the inertial frame of reference.

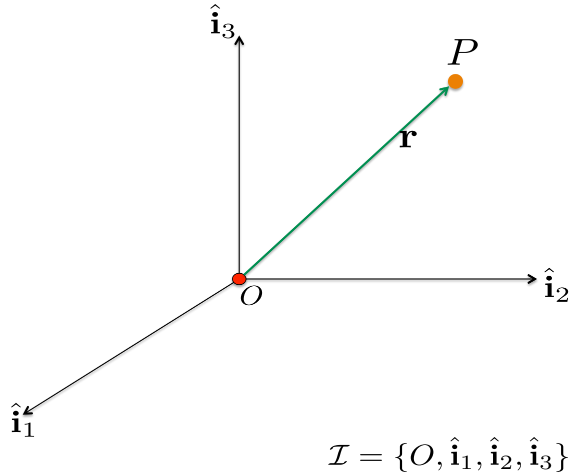


Figure 12: The Geometry of Particle Kinematics

- We will denote the velocity vector of P measured in any reference frame $\mathcal{A}$ as:

$$\mathbf{v}^{P/\mathcal{A}} = \frac{{}^{\mathcal{A}}d\mathbf{r}}{dt} \tag{41}$$

  Recall at the supercript stands for "measured in". The left-superscript on the derivative denotes the frame in which it was measured. As you can see, the vector $\mathbf{v}^{P/\mathcal{A}}$ is abstract in the sense that it has not been expressed in any frame.

- Similarly, we can denote the acceleration vector of P in RF $\mathcal{A}$ as:

$$\mathbf{a}^{P/\mathcal{A}} = \frac{{}^{\mathcal{A}}d\mathbf{v}^{P/\mathcal{A}}}{dt} \tag{42}$$

- If it is well understood (without causing ambiguity) that the object of study is P, we can drop it from the superscripts, leaving us with simply $\mathbf{v}^{\mathcal{A}}$ and $\mathbf{a}^{\mathcal{A}}$.

- As you will recall, the Newton's laws of motion are valid only in an inertial frame of reference. In other words, we must compute $\mathbf{a}^{\mathcal{I}}$.

- To compute $\mathbf{a}^{\mathcal{I}}$, begin with the position vector, *expressed* in the inertial frame:

$$\mathbf{r}_{\mathcal{I}} = r_{\mathcal{I}1}\hat{\mathbf{i}}_1 + r_{\mathcal{I}2}\hat{\mathbf{i}}_2 + r_{\mathcal{I}3}\hat{\mathbf{i}}_3 \tag{43}$$

  As the first step, compute the inertial velocity, expressed also in the inertial frame.

$$\mathbf{v}^{\mathcal{I}}_{\mathcal{I}} = \dot{r}_{\mathcal{I}1}\hat{\mathbf{i}}_1 + \dot{r}_{\mathcal{I}2}\hat{\mathbf{i}}_2 + \dot{r}_{\mathcal{I}3}\hat{\mathbf{i}}_3 \tag{44}$$

  Note that Eq.(44) is obtained easily from Eq.(43) because the unit vectors of the inertial frame are time invariant. In the same manner, the acceleration $\mathbf{a}^{\mathcal{I}}$ can also be obtained by taking another time derivative.

- While the above appears to be pretty straightforward, it is highly inconvenient. This is because the position vector is best expressed in some local reference frame, and not in the inertial reference frame as shown in Eq.(43).

Consider Fig.(13): imagine a local reference frame (we will call it an orbital frame), whose x-axis is in the same direction as the radius vector of particle P. Normal to this axis, we draw the "local horizontal plane", or the "local horizon" as shown. In other words, the x-axis "looks straight up" towards zenith. The y- and z- axes are drawn on the local horizon as shown. To complete its construction, translate the frame so that its origin coincides with O, giving us $\mathcal{O} = \{O, \hat{\mathbf{o}}_1, \hat{\mathbf{o}}_2, \hat{\mathbf{o}}_3\}$.

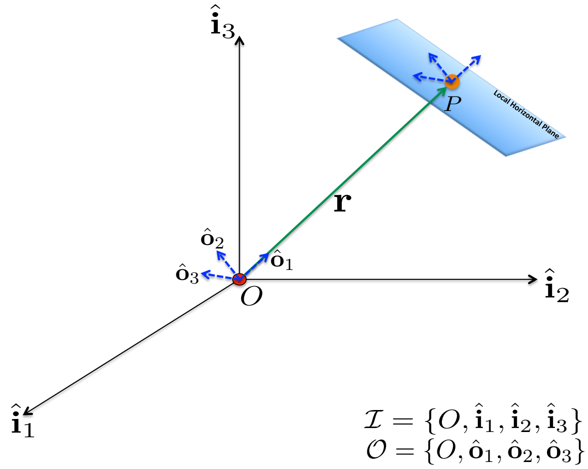


Figure 13: Particle Kinematics: Local Frame

- Clearly, the orbital frame is noninertial because its x-axis, namely $\hat{\mathbf{o}}_1$ points towards the instantaneous locaiton of P and in turn defines a time-varying local horizon. The rotation rate of $\mathcal{O}$ with respect to $\mathcal{I}$ is given by $\boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}}$.

- Now we can express the position vector of P in $\mathcal{O}$:

$$\mathbf{r}_{\mathcal{O}} = r_{\mathcal{O}1}\hat{\mathbf{o}}_1 + r_{\mathcal{O}2}\hat{\mathbf{o}}_2 + r_{\mathcal{O}3}\hat{\mathbf{o}}_3 \tag{45}$$
$$= r\hat{\mathbf{o}}_1 \tag{46}$$

  because, by construction $\hat{\mathbf{o}}_1$ is aligned in the same direction as $\mathbf{r}$, which in turn has zero components along $\hat{\mathbf{o}}_2$ and $\hat{\mathbf{o}}_3$.

- Now comes the fun part. We are interested in computing the velocity vector of P, which can be *measured* in either $\mathcal{I}$, i.e. $\mathbf{v}^{\mathcal{I}}$ or, $\mathcal{O}$, i.e. $\mathbf{v}^{\mathcal{O}}$. The latter is simply convenient, and the former is essential for Newtonian mechanics.

- Following Eq.(41), we have:

$$\mathbf{v}^{\mathcal{O}}_{\mathcal{O}} \underset{\text{Eq.(41)}}{=} \frac{{}^{\mathcal{O}}d\mathbf{r}_{\mathcal{O}}}{dt} \tag{47}$$

$$\underset{\text{Eq.(46)}}{=} \frac{{}^{\mathcal{O}}d}{dt}(r\hat{\mathbf{o}}_1) \tag{48}$$

$$\underset{\text{chain rule}}{=} \dot{r}\hat{\mathbf{o}}_1 + r\underbrace{\cancel{\frac{^{\mathcal{O}}d\hat{\mathbf{o}}_1}{dt}}}_{=0} \tag{49}$$

The second term in Eq.(49) dissapears because the rate of change of unit vector $\hat{\mathbf{o}}_1$ will be measured zero by someone sitting in the $\mathcal{O}$ frame.

- Next, let us compute $\mathbf{v}_{\mathcal{O}}^{\mathcal{I}}$, i.e. the velocity of P as measured in the inertial frame (still expressed in the $\mathcal{O}$ frame).

$$\mathbf{v}_{\mathcal{O}}^{\mathcal{I}} \underset{\text{Eq.(41)}}{=} \frac{^{\mathcal{I}}d\mathbf{r}_{\mathcal{O}}}{dt} \tag{50}$$

$$\underset{\text{Eq.(46)}}{=} \frac{^{\mathcal{I}}d}{dt}(r\hat{\mathbf{o}}_1) \tag{51}$$

$$\underset{\text{chain rule}}{=} \dot{r}\hat{\mathbf{o}}_1 + r\underbrace{\frac{^{\mathcal{I}}d\hat{\mathbf{o}}_1}{dt}}_{\text{what is this?}} \tag{52}$$

Clearly, the second term in Eq.(52) is not zero because the $\mathcal{O}$ frame is rotating with respect to the $\mathcal{I}$ frame. Therefore, someone sitting fixed to the $\mathcal{I}$ frame will measure a non-zero time derivative of the unit vectors of $\mathcal{O}$.

4.1 Transport Theorem

- The derivative of the unit vector in Eq.(52) can be found using a result called the *Transport Theorem*. The proof of the general three dimensional version of the transport theorem is not conclusive. Therefore, we will look at a special case involving rotations in 2D space.
- Consider Fig.(14). Let $\mathcal{E} = \{O, \hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, where $\hat{\mathbf{e}}_3$ is out of the plane of the paper. Let the frame $\mathcal{E}$ be rotating at rate $\boldsymbol{\omega} = \omega\hat{\mathbf{e}_3}$.
- The orientation of unit vectors at time t are $\hat{\mathbf{e}}_1(t)$ and $\hat{\mathbf{e}}_2(t)$. After an infinitesimal time dt, these orientations change to $\hat{\mathbf{e}}_1(t+dt)$ and $\hat{\mathbf{e}}_2(t+dt)$. Consider the **change** in unit vector $\hat{\mathbf{e}}_1$: it has a magnitude of ωdt and direction $\hat{\mathbf{e}}_2$ (see the broken blue line). Of-course, we have assumed that the time-jump $dt \to 0$ so that the angle of rotation $\omega dt \to 0$. Geometry tells us that:

$$\hat{\mathbf{e}}_1(t+dt) = \hat{\mathbf{e}}_1(t) + \omega dt\,\hat{\mathbf{e}}_2(t) \tag{53}$$

Therefore,

$$\underbrace{\hat{\mathbf{e}}_1(t+dt) - \hat{\mathbf{e}}_1(t)}_{=d\hat{\mathbf{e}}_1(t)} = \omega dt\,\hat{\mathbf{e}}_2(t) \tag{54}$$

$$\Rightarrow \frac{d}{dt}\hat{\mathbf{e}}_1(t) = \omega\,\hat{\mathbf{e}}_2(t) \tag{55}$$

- Following the same steps as above for $\hat{\mathbf{e}}_2$:

$$\hat{\mathbf{e}}_2(t+dt) = \hat{\mathbf{e}}_2(t) + \omega dt\,[-\hat{\mathbf{e}}_1(t)] \tag{56}$$

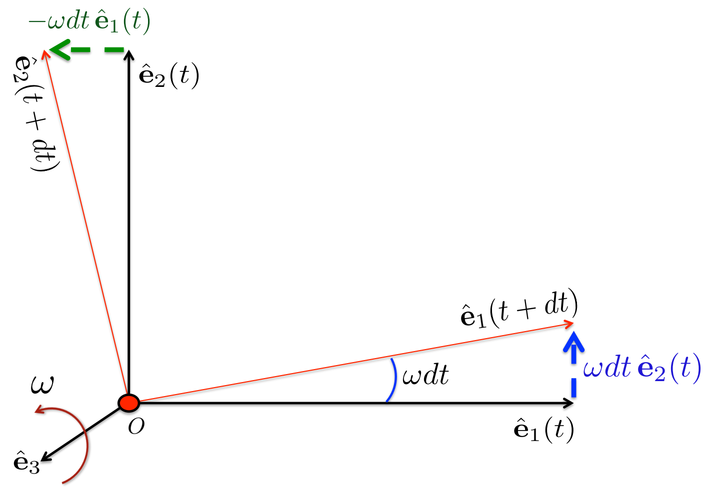


Figure 14: A Simple Version of the Transport Theorem

$$\Rightarrow \underbrace{\hat{\mathbf{e}}_2(t+dt) - \hat{\mathbf{e}}_2(t)}_{=d\hat{\mathbf{e}}_2(t)} = -\omega dt\,\hat{\mathbf{e}}_1(t) \tag{57}$$

$$\Rightarrow \frac{d}{dt}\hat{\mathbf{e}}_2(t) = -\omega\,\hat{\mathbf{e}}_1(t) \tag{58}$$

- So what did we find? In the two dimensional case considered the following is true:

$$\frac{d}{dt}\hat{\mathbf{e}}_1(t) = \omega\,\hat{\mathbf{e}}_2(t) \tag{59}$$

and,

$$\frac{d}{dt}\hat{\mathbf{e}}_2(t) = -\omega\,\hat{\mathbf{e}}_1(t) \tag{60}$$

Interestingly, note that

$$\omega\,\hat{\mathbf{e}}_2(t) = \boldsymbol{\omega} \times \hat{\mathbf{e}}_1(t) \text{ and, } -\omega\,\hat{\mathbf{e}}_1(t) = \boldsymbol{\omega} \times \hat{\mathbf{e}}_2(t) \tag{61}$$

So that, Eqs.(59) and (60) become:

$$\frac{d}{dt}\hat{\mathbf{e}}_1(t) = \boldsymbol{\omega} \times \hat{\mathbf{e}}_1(t) \tag{62a}$$

and,

$$\frac{d}{dt}\hat{\mathbf{e}}_2(t) = \boldsymbol{\omega} \times \hat{\mathbf{e}}_2(t) \tag{62b}$$

- Without providing a proof, we will state a general version of the above found result: Let $\mathcal{A} = \{O, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3\}$ and $\mathcal{B} = \{O, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$ be two reference frames with relative rotation rate $\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}$. Then the following relationships hold:

$$\frac{{}^{\mathcal{A}}d\hat{\mathbf{b}}_1}{dt} = \boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} \times \hat{\mathbf{b}}_1 \tag{63a}$$

$$\frac{{}^{\mathcal{A}}d\hat{\mathbf{b}}_2}{dt} = \boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} \times \hat{\mathbf{b}}_2 \tag{63b}$$

$$\frac{{}^{\mathcal{A}}d\hat{\mathbf{b}}_3}{dt} = \boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} \times \hat{\mathbf{b}}_3 \tag{63c}$$

Note that a leap of faith is required in going from Eqs.(62) to Eqs.(63).

- Nevertheless, we are now ready to return to Eq.(51):

$$\frac{{}^{\mathcal{I}}d}{dt}(r\hat{\mathbf{o}}_1) \underset{\text{chain rule}}{=} \dot{r}\hat{\mathbf{o}}_1 + r\frac{{}^{\mathcal{I}}d\hat{\mathbf{o}}_1}{dt} \tag{64a}$$

$$\underset{\text{Eq. (63a)}}{=} \dot{r}\hat{\mathbf{o}}_1 \; + \; r\,\boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times \hat{\mathbf{o}}_1 \tag{64b}$$

$$= \; \dot{r}\hat{\mathbf{o}}_1 \; + \; \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times (r\hat{\mathbf{o}}_1) \tag{64c}$$

We therefore have

$$\mathbf{v}^{\mathcal{I}}_{\mathcal{O}} \; = \; \frac{{}^{\mathcal{I}}d}{dt}(r\hat{\mathbf{o}}_1) \tag{65a}$$

$$= \; \left[\dot{r}\hat{\mathbf{o}}_1 \; + \; \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}}_{\mathcal{O}} \times (r\hat{\mathbf{o}}_1)\right] \tag{65b}$$

$$= \; \underbrace{[\dot{r}\hat{\mathbf{o}}_1]}_{\mathbf{v}^{\mathcal{O}}_{\mathcal{O}} \;\; !!} + \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}}_{\mathcal{O}} \times \underbrace{[r\hat{\mathbf{o}}_1]}_{\mathbf{r}_{\mathcal{O}}} \tag{65c}$$

Or,

$$\mathbf{v}^{\mathcal{I}}_{\mathcal{O}} \; = \; \mathbf{v}^{\mathcal{O}}_{\mathcal{O}} \; + \; \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}}_{\mathcal{O}} \times \mathbf{r}_{\mathcal{O}} \tag{66}$$

Indeed, there is no need to “express” the above equation in any frame in particular. I.E., we can easily drop the subscripts to get:

$$\mathbf{v}^{\mathcal{I}} \; = \; \mathbf{v}^{\mathcal{O}} \; + \; \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times \mathbf{r} \tag{67}$$

- Or, writing in terms of the “derivative notation”, we have found that:

$$\frac{{}^{\mathcal{I}}d\mathbf{r}}{dt} \; = \; \frac{{}^{\mathcal{O}}d\mathbf{r}}{dt} \; + \; \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times \mathbf{r} \tag{68}$$

- Based on our excursions above, we make a giant leap of faith and state the general result in the form of a theorem:

Theorem 1. *Transport Theorem: Let $\mathcal{A}$ and $\mathcal{B}$ be two reference frames with a common origin and relative rotation rate $\boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}}$. Let $\mathbf{v}$ be any vector. Then, the derivative of $\mathbf{v}$ measured in $\mathcal{B}$ is related to the derivative of $\mathbf{v}$ measured in $\mathcal{A}$ as follows:*

$$\frac{{}^{\mathcal{A}}d\mathbf{v}}{dt} \; = \; \frac{{}^{\mathcal{B}}d\mathbf{v}}{dt} \; + \; \boldsymbol{\omega}^{\mathcal{B}/\mathcal{A}} \times \mathbf{v} \tag{69}$$

Since a conclusive proof does not exist, we will skip it here.

4.2 Capping off Particle Kinematics

- Position vector of P:

$$\mathbf{r}_{\mathcal{O}} = r\hat{\mathbf{o}}_1 \tag{70}$$

- Inertial velocity vector of P:

$$\mathbf{v}^{\mathcal{I}} = \frac{{}^{\mathcal{I}}d\mathbf{r}}{dt} \tag{71a}$$

$$\underset{[69]}{=} \frac{{}^{\mathcal{O}}d\mathbf{r}}{dt} + \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times \mathbf{r} \tag{71b}$$

$$\Rightarrow \mathbf{v}^{\mathcal{I}} \underset{[47]}{=} \mathbf{v}^{\mathcal{O}} + \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times \mathbf{r} \tag{71c}$$

Unlike the radius vector, the velocity vector has not been expressed in any particular frame.

- Inertial acceleration of P:

$$\mathbf{a}^{\mathcal{I}} = \frac{{}^{\mathcal{I}}d\mathbf{v}^{\mathcal{I}}}{dt} \tag{72a}$$

$$\underset{[71c]}{=} \frac{{}^{\mathcal{I}}d}{dt}\left(\mathbf{v}^{\mathcal{O}} + \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times \mathbf{r}\right) \tag{72b}$$

$$\underset{[69]}{=} \left(\frac{{}^{\mathcal{O}}d\mathbf{v}^{\mathcal{O}}}{dt} + \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times \mathbf{v}^{\mathcal{O}}\right) + \left(\frac{{}^{\mathcal{O}}d}{dt}(\boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times \mathbf{r}) + \boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times (\boldsymbol{\omega}^{\mathcal{O}/\mathcal{I}} \times \mathbf{r})\right) \tag{72c}$$

Dropping the superscript $(\mathcal{O}/\mathcal{I})$ for better readability, we get

$$\mathbf{a}^{\mathcal{I}} = \left(\frac{{}^{\mathcal{O}}d\mathbf{v}^{\mathcal{O}}}{dt} + \boldsymbol{\omega} \times \mathbf{v}^{\mathcal{O}}\right) + \left(\frac{{}^{\mathcal{O}}d}{dt}(\boldsymbol{\omega} \times \mathbf{r}) + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})\right) \tag{72d}$$

$$= \underbrace{\frac{{}^{\mathcal{O}}d\mathbf{v}^{\mathcal{O}}}{dt}}_{\doteq\, \mathbf{a}^{\mathcal{O}}} + \boldsymbol{\omega} \times \mathbf{v}^{\mathcal{O}} + \frac{{}^{\mathcal{O}}d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times \underbrace{\frac{{}^{\mathcal{O}}d\mathbf{r}}{dt}}_{=\,\mathbf{v}^{\mathcal{O}}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{72e}$$

$$\Rightarrow \mathbf{a}^{\mathcal{I}} = \mathbf{a}^{\mathcal{O}} + \underbrace{2\boldsymbol{\omega} \times \mathbf{v}^{\mathcal{O}}}_{\sim\ Coriolis} + \frac{{}^{\mathcal{O}}d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \underbrace{\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})}_{\sim Transport\ acc.} \tag{72f}$$

The Coriolis acceleration and transport (aka centrifugal) acceleration terms have been identified in Eq.(72f).

- **Special case - planar motion:** Consider Fig.(15). We have an inertial frame $\mathcal{I} = \{O, \hat{\mathbf{i}}_1, \hat{\mathbf{i}}_2, \hat{\mathbf{i}}_3\}$ where $\hat{\mathbf{i}}_3$ is out of the plane of the paper. A particle P moves arbitrarily on this plane, along a trajectory as shown. Define a local frame $\mathcal{E} = \{O, \hat{\mathbf{e}}_r, \hat{\mathbf{e}}_\theta, \hat{\mathbf{i}}_3\}$. Clearly, $\boldsymbol{\omega}^{\mathcal{E}/\mathcal{I}} = \dot{\theta}\,\hat{\mathbf{i}}_3$.

  - Position of P:

$$\mathbf{r}_{\mathcal{E}} = r\hat{\mathbf{e}}_r \tag{73}$$

  - Velocity of P:

1. measured in RF $\mathcal{E}$:

$$\mathbf{v}_{\mathcal{E}}^{\mathcal{E}} = \dot{r}\hat{\mathbf{e}}_r \tag{74}$$

2. measured in RF $\mathcal{I}$:

$$\mathbf{v}_{\mathcal{E}}^{\mathcal{I}} \underset{[(71c)]}{=} \mathbf{v}_{\mathcal{E}}^{\mathcal{E}} + \boldsymbol{\omega}^{\mathcal{E}/\mathcal{I}} \times \mathbf{r}_{\mathcal{E}} \tag{75a}$$

$$= \dot{r}\,\hat{\mathbf{e}}_r + r\dot{\theta}\,\hat{\mathbf{e}}_\theta \tag{75b}$$

– Inertial acceleration of P:

$$\mathbf{a}^{\mathcal{I}} \underset{[(72f)]}{=} \mathbf{a}^{\mathcal{O}} + 2\boldsymbol{\omega} \times \mathbf{v}^{\mathcal{O}} + \frac{{}^{\mathcal{O}}d\boldsymbol{\omega}}{dt} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) \tag{76a}$$

$$= \ddot{r}\,\hat{\mathbf{e}}_r + 2\omega\dot{r}\,\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_r + \dot{\omega}r\,\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_r + \omega^2 r\,\hat{\mathbf{e}}_3 \times (\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_r) \tag{76b}$$

$$= (\ddot{r} - \omega^2 r)\,\hat{\mathbf{e}}_r + (2\omega\dot{r} + \dot{\omega}r)\,\hat{\mathbf{e}}_\theta \tag{76c}$$

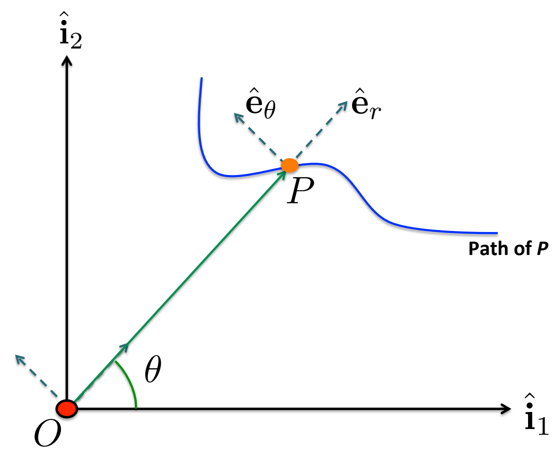


Figure 15: Particle in a Plane

An Enumeration of All Euler Rotation Sequences

Recall that the transformation between two concentric reference frames in three dimensional space can be characterized via a sequence of at-most three independent elementary rotations. The rotation angles are called Euler angles. For each three-rotation sequence, let the first, second and third Euler angle be ψ, θ and ϕ respectively. A total of twelve sequences are possible. The general expressions for the transformation in each case in given below. Denote $\mathbf{\Theta} = [\psi, \theta, \phi]'$. Symbolically,

$$\mathcal{A} \quad \underset{\longrightarrow}{\mathbf{\Theta}} \quad \mathcal{B}$$

Then, the transformations listed below are $\mathbf{R}_{\mathcal{B}\mathcal{A}}$. Also, let $\boldsymbol{\omega}_{\mathcal{B}}^{\mathcal{B}/\mathcal{A}} = \mathbf{B}[\mathbf{\Theta}]\dot{\mathbf{\Theta}}$.

Symmetric Rotation Sequences

1. **1-2-1** Sequence:

$$\mathbf{R}_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} c\theta & s\theta s\psi & -s\theta c\psi \\ s\phi s\theta & -s\phi c\theta s\psi + c\phi c\psi & s\phi c\theta c\psi + c\phi s\psi \\ c\phi s\theta & -c\phi c\theta s\psi - s\phi c\psi & c\phi c\theta c\psi - s\phi s\psi \end{bmatrix} \tag{77a}$$

$$\mathbf{B}[\mathbf{\Theta}] = \begin{bmatrix} c\theta & 0 & 1 \\ s\phi s\theta & c\phi & 0 \\ c\phi s\theta & -s\phi & 0 \end{bmatrix}, \quad \mathbf{B}^{-1}[\mathbf{\Theta}] = \frac{1}{s\theta} \begin{bmatrix} 0 & s\phi & c\phi \\ 0 & c\phi s\theta & -s\phi s\theta \\ s\theta & -s\phi c\theta & -c\phi c\theta \end{bmatrix} \tag{77b}$$

2. **1-3-1** Sequence:

$$\mathbf{R}_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} c\theta & s\theta c\psi & s\theta s\psi \\ -c\phi s\theta & c\phi c\theta c\psi - s\phi s\psi & c\phi c\theta s\psi + s\phi c\psi \\ s\phi s\theta & -s\phi c\theta c\psi - c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi \end{bmatrix} \tag{78a}$$

$$\mathbf{B}[\mathbf{\Theta}] = \begin{bmatrix} c\theta & 0 & 1 \\ -c\phi s\theta & s\phi & 0 \\ s\phi s\theta & c\phi & 0 \end{bmatrix}, \quad \mathbf{B}^{-1}[\mathbf{\Theta}] = \frac{1}{s\theta} \begin{bmatrix} 0 & -c\phi & s\phi \\ 0 & s\phi s\theta & c\phi s\theta \\ s\theta & c\phi c\theta & -s\phi c\theta \end{bmatrix} \tag{78b}$$

3. **2-1-2** Sequence:

$$\mathbf{R}_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta & -s\phi c\theta c\psi - c\phi s\psi \\ s\theta s\psi & c\theta & s\theta c\psi \\ c\phi c\theta s\psi + s\phi c\psi & -c\phi s\theta & c\phi c\theta c\psi - s\phi s\psi \end{bmatrix} \tag{79a}$$

$$\mathbf{B}[\mathbf{\Theta}] = \begin{bmatrix} s\phi s\theta & c\phi & 0 \\ c\theta & 0 & 1 \\ -c\phi s\theta & s\phi & 0 \end{bmatrix}, \quad \mathbf{B}^{-1}[\mathbf{\Theta}] = \frac{1}{s\theta} \begin{bmatrix} s\phi & 0 & -c\phi \\ c\phi s\theta & 0 & s\phi s\theta \\ -s\phi c\theta & s\theta & c\phi c\theta \end{bmatrix} \tag{79b}$$

4. **2-3-2** Sequence:

$$\mathbf{R}_{\mathcal{B}\mathcal{A}} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & c\phi s\theta & -c\phi c\theta s\psi - s\phi c\psi \\ -s\theta c\psi & c\theta & s\theta s\psi \\ s\phi c\theta c\psi + c\phi s\psi & s\phi s\theta & -s\phi c\theta s\psi + c\phi c\psi \end{bmatrix} \tag{80a}$$

$$\mathbf{B}[\boldsymbol{\Theta}] = \begin{bmatrix} c\phi s\theta & -s\phi & 0 \\ c\theta & 0 & 1 \\ s\phi s\theta & c\phi & 0 \end{bmatrix}, \quad \mathbf{B}^{-1}[\boldsymbol{\Theta}] = \frac{1}{s\theta}\begin{bmatrix} c\phi & 0 & s\phi \\ -s\phi s\theta & 0 & c\phi s\theta \\ -c\phi c\theta & s\theta & -s\phi c\theta \end{bmatrix} \tag{80b}$$

5. **3-1-3** Sequence:

$$\mathbf{R}_{\mathcal{BA}} = \begin{bmatrix} -s\phi c\theta s\psi + c\phi c\psi & s\phi c\theta c\psi + c\phi s\psi & s\phi s\theta \\ -c\phi c\theta s\psi - s\phi c\psi & c\phi c\theta c\psi - s\phi s\psi & c\phi s\theta \\ s\theta s\psi & -s\theta c\psi & c\theta \end{bmatrix} \tag{81a}$$

$$\mathbf{B}[\boldsymbol{\Theta}] = \begin{bmatrix} s\phi s\theta & c\phi & 0 \\ c\phi s\theta & -s\phi & 0 \\ c\theta & 0 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1}[\boldsymbol{\Theta}] = \frac{1}{s\theta}\begin{bmatrix} s\phi & c\phi & 0 \\ c\phi s\theta & -s\phi s\theta & 0 \\ -s\phi c\theta & -c\phi c\theta & s\theta \end{bmatrix} \tag{81b}$$

6. **3-2-3** Sequence:

$$\mathbf{R}_{\mathcal{BA}} = \begin{bmatrix} c\phi c\theta c\psi - s\phi s\psi & c\phi c\theta s\psi + s\phi c\psi & -c\phi s\theta \\ -s\phi c\theta c\psi - c\phi s\psi & -s\phi c\theta s\psi + c\phi c\psi & s\phi s\theta \\ s\theta c\psi & s\theta s\psi & c\theta \end{bmatrix} \tag{82a}$$

$$\mathbf{B}[\boldsymbol{\Theta}] = \begin{bmatrix} -c\phi s\theta & s\phi & 0 \\ s\phi s\theta & c\phi & 0 \\ c\theta & 0 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1}[\boldsymbol{\Theta}] = \frac{1}{s\theta}\begin{bmatrix} -c\phi & s\phi & 0 \\ s\phi s\theta & c\phi s\theta & 0 \\ c\phi c\theta & -s\phi c\theta & s\theta \end{bmatrix} \tag{82b}$$

Asymmetric Rotation Sequences

7. **1-2-3** Sequence:

$$\mathbf{R}_{\mathcal{BA}} = \begin{bmatrix} c\phi c\theta & c\phi s\theta s\psi + s\phi c\psi & -c\phi s\theta c\psi + s\phi s\psi \\ -s\phi c\theta & -s\phi s\theta s\psi + c\phi c\psi & s\phi s\theta c\psi + c\phi s\psi \\ s\theta & -c\theta s\psi & c\theta c\psi \end{bmatrix} \tag{83a}$$

$$\mathbf{B}[\boldsymbol{\Theta}] = \begin{bmatrix} c\phi c\theta & s\phi & 0 \\ -s\phi c\theta & c\phi & 0 \\ s\theta & 0 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1}[\boldsymbol{\Theta}] = \frac{1}{c\theta}\begin{bmatrix} c\phi & -s\phi & 0 \\ s\phi c\theta & c\phi c\theta & 0 \\ -c\phi s\theta & s\phi s\theta & c\theta \end{bmatrix} \tag{83b}$$

8. **1-3-2** Sequence:

$$\mathbf{R}_{\mathcal{BA}} = \begin{bmatrix} c\phi c\theta & c\phi s\theta c\psi + s\phi s\psi & c\phi s\theta s\psi - s\phi c\psi \\ -s\theta & c\theta c\psi & c\theta s\psi \\ s\phi c\theta & s\phi s\theta c\psi - c\phi s\psi & s\phi s\theta s\psi + c\phi c\psi \end{bmatrix} \tag{84a}$$

$$\mathbf{B}[\boldsymbol{\Theta}] = \begin{bmatrix} c\phi c\theta & -s\phi & 0 \\ -s\theta & 0 & 1 \\ s\phi c\theta & c\phi & 0 \end{bmatrix}, \quad \mathbf{B}^{-1}[\boldsymbol{\Theta}] = \frac{1}{c\theta}\begin{bmatrix} c\phi & 0 & s\phi \\ -s\phi c\theta & 0 & c\phi c\theta \\ c\phi s\theta & c\theta & s\phi s\theta \end{bmatrix} \tag{84b}$$

9. **2-1-3** Sequence:

$$\mathbf{R}_{\mathcal{BA}} = \begin{bmatrix} s\phi s\theta s\psi + c\phi c\psi & s\phi c\theta & s\phi s\theta c\psi - c\phi s\psi \\ c\phi s\theta s\psi - s\phi c\psi & c\phi c\theta & c\phi s\theta c\psi + s\phi s\psi \\ c\theta s\psi & -s\theta & c\theta c\psi \end{bmatrix} \tag{85a}$$

$$\mathbf{B}[\mathbf{\Theta}] = \begin{bmatrix} s\phi c\theta & c\phi & 0 \\ c\phi c\theta & -s\phi & 0 \\ -s\theta & 0 & 1 \end{bmatrix}, \quad \mathbf{B}^{-1}[\mathbf{\Theta}] = \frac{1}{c\theta}\begin{bmatrix} s\phi & c\phi & 0 \\ c\phi c\theta & -s\phi c\theta & 0 \\ s\phi s\theta & c\phi s\theta & c\theta \end{bmatrix} \tag{85b}$$

10. **2-3-1** Sequence:

$$\mathbf{R}_{\mathcal{BA}} = \begin{bmatrix} c\theta c\psi & s\theta & -c\theta s\psi \\ -c\phi s\theta c\psi + s\phi s\psi & c\phi c\theta & c\phi s\theta s\psi + s\phi c\psi \\ s\phi s\theta c\psi + c\phi s\psi & -s\phi c\theta & -s\phi s\theta s\psi + c\phi c\psi \end{bmatrix} \tag{86a}$$

$$\mathbf{B}[\mathbf{\Theta}] = \begin{bmatrix} s\theta & 0 & 1 \\ c\phi c\theta & s\phi & 0 \\ -s\phi c\theta & c\phi & 0 \end{bmatrix}, \quad \mathbf{B}^{-1}[\mathbf{\Theta}] = \frac{1}{c\theta}\begin{bmatrix} 0 & c\phi & -s\phi \\ 0 & s\phi c\theta & c\phi c\theta \\ c\theta & -c\phi s\theta & s\phi s\theta \end{bmatrix} \tag{86b}$$

11. **3-1-2** Sequence:

$$\mathbf{R}_{\mathcal{BA}} = \begin{bmatrix} -s\phi s\theta s\psi + c\phi c\psi & s\phi s\theta c\psi + c\phi s\psi & -s\phi c\theta \\ -c\theta s\psi & c\theta c\psi & s\theta \\ c\phi s\theta s\psi + s\phi c\psi & -c\phi s\theta c\psi + s\phi s\psi & c\phi c\theta \end{bmatrix} \tag{87a}$$

$$\mathbf{B}[\mathbf{\Theta}] = \begin{bmatrix} -s\phi c\theta & c\phi & 0 \\ s\theta & 0 & 1 \\ c\phi c\theta & s\phi & 0 \end{bmatrix}, \quad \mathbf{B}^{-1}[\mathbf{\Theta}] = \frac{1}{c\theta}\begin{bmatrix} -s\phi & 0 & c\phi \\ c\phi c\theta & 0 & s\phi c\theta \\ s\phi s\theta & c\theta & -c\phi s\theta \end{bmatrix} \tag{87b}$$

12. **3-2-1** Sequence:

$$\mathbf{R}_{\mathcal{BA}} = \begin{bmatrix} c\theta c\psi & c\theta s\psi & -s\theta \\ s\phi s\theta c\psi - c\phi s\psi & s\phi s\theta s\psi + c\phi c\psi & s\phi c\theta \\ c\phi s\theta c\psi + s\phi s\psi & c\phi s\theta s\psi - s\phi c\psi & c\phi c\theta \end{bmatrix} \tag{88a}$$

$$\mathbf{B}[\mathbf{\Theta}] = \begin{bmatrix} -s\theta & 0 & 1 \\ s\phi c\theta & c\phi & 0 \\ c\phi c\theta & -s\phi & 0 \end{bmatrix}, \quad \mathbf{B}^{-1}[\mathbf{\Theta}] = \frac{1}{c\theta}\begin{bmatrix} 0 & s\phi & c\phi \\ 0 & c\phi c\theta & -s\phi c\theta \\ c\theta & s\phi s\theta & c\phi s\theta \end{bmatrix} \tag{88b}$$

Advanced Reading[1]

5 Homogeneous Transformations

- In the above treatment of coordinate transformations, an important assumption was made: that both the reference frames involved share a common origin. This is not true in general. Consider two reference frames with different origins: $\mathcal{A} = \{O_A, \hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3\}$ and $\mathcal{B} = \{O_B, \hat{\mathbf{b}}_1, \hat{\mathbf{b}}_2, \hat{\mathbf{b}}_3\}$ as shown in Fig.(16).

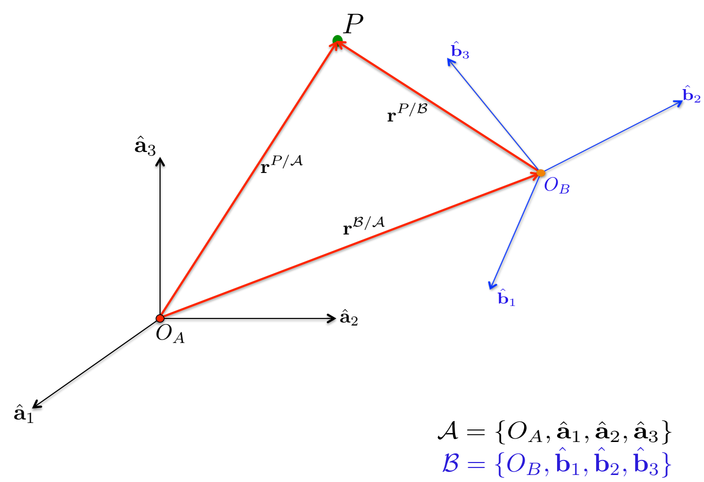


Figure 16: The Geometry of a Homogeneous Transformation

- The location of the origin of frame $\mathcal{B}$, as *measured* in frame $\mathcal{A}$ is denoted by $\mathbf{r}^{\mathcal{B}/\mathcal{A}}$ and it is assumed to be known. Note the use of supercripts rather than subscripts. A point P in space is shown. Its position vector (aka "radius vector") as *measured* in the two reference frames are denoted as $\mathbf{r}^{P/\mathcal{A}}$ and $\mathbf{r}^{P/\mathcal{B}}$. All these three vectors have been shown in Fig.(16), and note that they have not been expressed in any particular reference frame.

- From simple vector geometry, we see that:

$$\mathbf{r}^{P/\mathcal{A}} = \mathbf{r}^{\mathcal{B}/\mathcal{A}} + \mathbf{r}^{P/\mathcal{B}} \tag{89}$$

- Suppose we are given $\mathbf{r}_{\mathcal{B}}^{P/\mathcal{B}}$, i.e., the radius vector of P as measured and expressed in RF $\mathcal{B}$. We are interested in $\mathbf{r}_{\mathcal{A}}^{P/\mathcal{A}}$, i.e. position of P as measured and expressed in $\mathcal{A}$. Therefore, we write out Eq.(89) in RF $\mathcal{A}$:

$$\mathbf{r}_{\mathcal{A}}^{P/\mathcal{A}} = \mathbf{r}_{\mathcal{A}}^{\mathcal{B}/\mathcal{A}} + \mathbf{r}_{\mathcal{A}}^{P/\mathcal{B}} \tag{90}$$

- All we need is $\mathbf{r}_{\mathcal{A}}^{P/\mathcal{B}}$. Imagine a translation of O_B so that it coincides with O_A. We can now use the rotation matrix between frames $\mathcal{A}$ and $\mathcal{B}$, i.e. $\mathbf{R}_{\mathcal{A}\mathcal{B}}$, such that $\mathbf{r}_{\mathcal{A}}^{P/\mathcal{B}} = \mathbf{R}_{\mathcal{A}\mathcal{B}}\, \mathbf{r}_{\mathcal{B}}^{P/\mathcal{B}}$. Using this in Eq.(90):

$$\mathbf{r}_{\mathcal{A}}^{P/\mathcal{A}} = \mathbf{r}_{\mathcal{A}}^{\mathcal{B}/\mathcal{A}} + \mathbf{R}_{\mathcal{A}\mathcal{B}}\, \mathbf{r}_{\mathcal{B}}^{P/\mathcal{B}} \tag{91}$$

[1]Optional

- The above equation can be written in a compact matrix form, which is known as the so-called homogeneous transformation:

$$\begin{Bmatrix} \mathbf{r}_{\mathcal{A}}^{P/\mathcal{A}} \\ 1 \end{Bmatrix} = \underbrace{\begin{bmatrix} \mathbf{R}_{\mathcal{AB}} & \mathbf{r}_{\mathcal{A}}^{\mathcal{B}/\mathcal{A}} \\ \mathbf{0}_{1\times 3} & 1 \end{bmatrix}}_{=\mathfrak{R}_{\mathcal{AB}}} \begin{Bmatrix} \mathbf{r}_{\mathcal{B}}^{P/\mathcal{B}} \\ 1 \end{Bmatrix} \tag{92}$$

  In the above equation, $\mathfrak{R}_{\mathcal{AB}}$ is referred to as the *homogeneous transformation* between frames $\mathcal{A}$ and $\mathcal{B}$.

- Homogeneous transformations can be applied successively in the same way as rotational transformations, i.e., if

$$\begin{Bmatrix} \mathbf{r}_{\mathcal{A}}^{P/\mathcal{A}} \\ 1 \end{Bmatrix} = \mathfrak{R}_{\mathcal{AB}} \begin{Bmatrix} \mathbf{r}_{\mathcal{B}}^{P/\mathcal{B}} \\ 1 \end{Bmatrix},$$

  and,

$$\begin{Bmatrix} \mathbf{r}_{\mathcal{B}}^{P/\mathcal{B}} \\ 1 \end{Bmatrix} = \mathfrak{R}_{\mathcal{BC}} \begin{Bmatrix} \mathbf{r}_{\mathcal{C}}^{P/\mathcal{C}} \\ 1 \end{Bmatrix}$$

  then,

$$\begin{Bmatrix} \mathbf{r}_{\mathcal{A}}^{P/\mathcal{A}} \\ 1 \end{Bmatrix} = \underbrace{\mathfrak{R}_{\mathcal{AB}}\mathfrak{R}_{\mathcal{BC}}}_{=\mathfrak{R}_{\mathcal{AC}}} \begin{Bmatrix} \mathbf{r}_{\mathcal{C}}^{P/\mathcal{C}} \\ 1 \end{Bmatrix} \tag{93}$$

- Homogeneous transformations do not have the nice orthogonality properties.