- $a, b \in \mathbb{R}$
- $a < b$

Goal: show there is a continuous function $f : [a, b] \to \mathbb{R}$ such that $f(a) < f(b)$.

Theorem: Let $f : [a, b] \to \mathbb{R}$ be continuous. Then $f(a) \leq f(b)$ or $f(a) \geq f(b)$.

Proof: Suppose $f(a) < f(b)$. Let $c = \frac{a+b}{2}$.

If $f(c) \geq f(a)$, then $f(c) > f(b)$.

Let $a_1 = c$, $b_1 = b$.

If $f(c) < f(a)$, then $f(c) < f(b)$.

Let $a_1 = a$, $b_1 = c$.

Repeat this process to get $a_2, b_2, a_3, b_3, \ldots$

Claim: $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

Proof: $a_n \leq a_{n+1} \leq b_{n+1} \leq b_n$.

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$ by the monotone convergence theorem.

Let $c = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n$.

$f(c) \geq f(a)$ and $f(c) \leq f(b)$.

$f(c) \geq f(b)$ and $f(c) \leq f(a)$.

$f(c) = f(a) = f(b)$.

Contradiction.

Exercise: Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $y \in \mathbb{R}$ such that $f(a) < y < f(b)$. Show there is $c \in [a, b]$ such that $f(c) = y$.

Hint: Consider $g : [a, b] \to \mathbb{R}$, $g(x) = f(x) - y$.

Exercise: Let $f : [a, b] \to \mathbb{R}$ be continuous. Let $y \in \mathbb{R}$ such that $f(a) > y > f(b)$. Show there is $c \in [a, b]$ such that $f(c) = y$.

Hint: Consider $g : [a, b] \to \mathbb{R}$, $g(x) = y - f(x)$