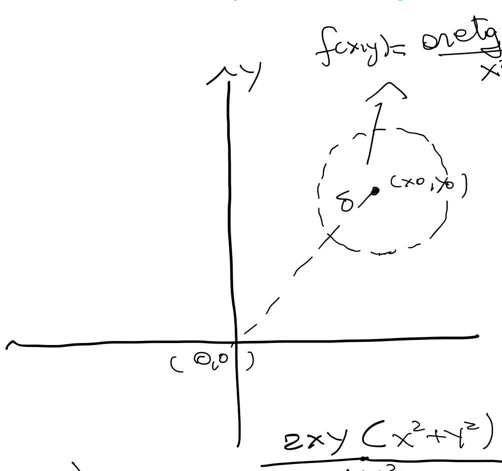
martedì 22 dicembre 2020 08:23

Data la funzione

$$f(x,y)=\begin{cases}\dfrac{\operatorname{arctg} x^2y}{x^2+y^2} & \text{se } (x,y)\neq(0,0)\\[2mm] 0 & \text{se } (x,y)=(0,0)\end{cases}$$

se $(x_0,y_0)\neq(0,0) \Rightarrow \exists\, \delta>0$: $I_\delta(x_0,y_0)\subseteq \mathbb{R}^2\setminus\{(0,0)\}$

$f(x,y)=\dfrac{\operatorname{arctg} x^2y}{x^2+y^2} \quad \forall (x,y)\in I_\delta(x_0,y_0)$



$$\begin{cases} f_x(x,y)=\dfrac{\dfrac{2xy(x^2+y^2)}{1+x^4y^2}-2x\operatorname{arctg}(x^2y)}{(x^2+y^2)^2} \\ f_x \text{ è continua in } I_\delta(x_0,y_0) \end{cases}$$

Analogamente $f_y$ è continua in $I_\delta(x_0,y_0)$ $\Rightarrow$ $f$ è diff. in $I_\delta(x_0,y_0)$

$\Downarrow$

$f$ è diff. in $(x_0,y_0)$

- $f$ è dotata di derivate parziali prime continue in $\mathbb{R}^2\setminus\{(0,0)\}$ $\Rightarrow$ è differenziabile in $\mathbb{R}^2\setminus\{(0,0)\}$
- Studiamo la continuità e la diff. in $(0,0)$.

Continuità

$$\lim_{(x,y)\to(0,0)} f(x,y)=f(0,0) \Leftrightarrow \lim_{(x,y)\to(0,0)} \frac{\operatorname{arctg} x^2y}{x^2+y^2}=0 \;?$$

$E_0=\{(x,y)\in\mathbb{R}^2:\ x=0\ \ y\neq 0\}$

$$f_{|E_0}(x,y)=0 \xrightarrow{(x,y)\to(0,0)} 0$$

$m\in\mathbb{R}$ $\quad m\neq 0$

$E_m=\{(x,y)\in\mathbb{R}^2:\ y=mx,\ \ x\neq0\}$

$$f_{|E_m}(x,y)=\frac{\operatorname{arctg}(mx^3)}{x^2+m^2x^2}=\frac{\operatorname{arctg}(mx^3)}{x^2(1+m^2)}=$$

$$=\underbrace{\frac{\operatorname{arctg}(mx^3)}{mx^3}}_{\downarrow\ 1}\cdot\underbrace{\frac{m\,x^{\cancel{3}}}{\cancel{x^2}(1+m^2)}}_{\downarrow\ 0}\to 0$$

$$\boxed{\lim_{t\to0}\frac{\operatorname{arctg} t}{t}=1}$$

$\forall (x,y)\in\mathbb{R}^2:\quad xy\neq0$

$$\frac{\operatorname{arctg}(x^2y)}{x^2+y^2}=\underbrace{\frac{\operatorname{arctg}(x^2y)}{x^2y}}_{\downarrow\ 1}\ \boxed{\frac{x^2y}{x^2+y^2}}$$

$$\lim_{(x,y)\to(0,0)}\frac{x^2y}{x^2+y^2}=0 \text{ perché}$$

$\forall (x,y)\neq(0,0)$ vale:

$$0\le\left|\frac{x^2y}{x^2+y^2}\right|=\frac{|x|\,|xy|}{x^2+y^2}\le\frac{|x|\ \cancel{(x^2+y^2)}}{\cancel{x^2+y^2}}=|x| \xrightarrow{(x,y)\to(0,0)} 0$$

oppure

$$0\le\left|\frac{x^2y}{x^2+y^2}\right|=\underbrace{\boxed{\frac{x^2}{x^2+y^2}}}_{\le 1}|y|\le|y|$$

- $|x|\le\sqrt{x^2+y^2}$ $\quad |xy|=|x|\cdot|y|\le\left(\sqrt{x^2+y^2}\right)^2$
  $|y|\le\sqrt{x^2+y^2}$

Conclusione: $f$ è continua in $(0,0)$

- DIFFERENZIABILITÀ

Per verificare la diff. in $(0,0)$ uso la definizione (anzi è un fatto equiv.)

$$f \text{ diff. in } (0,0) \Longleftrightarrow \begin{cases}\exists\, f_x(0,0),\ f_y(0,0)\\ \lim\limits_{(h,k)\to(0,0)}\dfrac{f(h,k)-f(0,0)-df(0,0)}{\sqrt{h^2+k^2}}=0\end{cases}$$

$\exists\, f_x(0,0)$ ?

$\Updownarrow$

$\exists\,[Df(x,0)]_{x=0}$ ?

$$f(x,y)=\begin{cases}\dfrac{\operatorname{arctg} x^2y}{x^2+y^2} & \text{se } (x,y)\neq(0,0)\\[2mm] 0 & \text{se } (x,y)=(0,0)\end{cases}$$

$$f(x,0)=\begin{cases}0 & \text{se } x\neq0\\ 0 & \text{se } x=0\end{cases}$$

cioè

$f(x,0)=0 \quad \forall x\in\mathbb{R}$

$\Downarrow$

$\exists\,[Df(x,0)]_{x=0}=0.$

$\Updownarrow$

$\boxed{f_x(0,0)=0}$

- $\exists\, f_y(0,0)$ ?

$\Updownarrow$

$\exists\,[Df(0,y)]_{y=0}$ ?

ma

$f(0,y)=0 \quad \forall y\in\mathbb{R}$

$\Downarrow$

$\exists\,[Df(0,y)]_{y=0}=0$

$\exists\, f_y(0,0)=0.$

Conseguenze:

$$df(0,0)\overset{\text{def}}{=}f_x(0,0)h+f_y(0,0)k=0 \quad \forall (h,k)\in\mathbb{R}^2$$

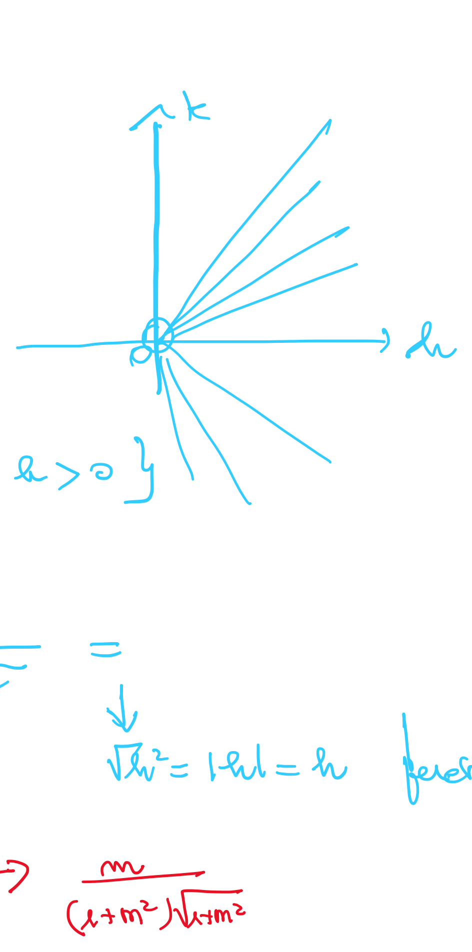
$f$ diff. in $(0,0)$

$\Updownarrow$

$$\lim_{(h,k)\to(0,0)}\frac{\dfrac{\operatorname{arctg} h^2k}{h^2+k^2}}{\sqrt{h^2+k^2}}=0$$

$\Updownarrow$

$$\lim_{(h,k)\to(0,0)}\underbrace{\boxed{\frac{\operatorname{arctg}(h^2k)}{(h^2+k^2)\sqrt{h^2+k^2}}}}_{g(h,k)}=0$$



Sia $m\neq0$

$E_m=\{(h,k)\in\mathbb{R}^2:\ k=mh,\ h>0\}$

$$g_{|E_m}(h,k)=\frac{\operatorname{arctg}(mh^3)}{(1+m^2)h^2\sqrt{(1+m^2)h^2}}=$$

$\sqrt{h^2}=|h|=h$ perché $h>0$

$$=m\frac{\operatorname{arctg}(mh^3)}{(1+m^2)\sqrt{1+m^2}\,(mh^3)}\longrightarrow\frac{m}{(1+m^2)\sqrt{1+m^2}}$$

$m\neq0$

il $\lim\limits_{(h,k)\to(0,0)} g_{|E_m}(h,k)$ DIPENDE da $m$

$\Downarrow$

$$\nexists \lim_{(h,k)\to(0,0)} g(h,k)$$

$\Downarrow$

$f$ non è diff. in $(0,0)$