5.1 Response of a First Order System to Step and Square Wave Inputs

In [0]:

```
!pip install -q pyomo
```

First-Order Differential Equation with Constant Input

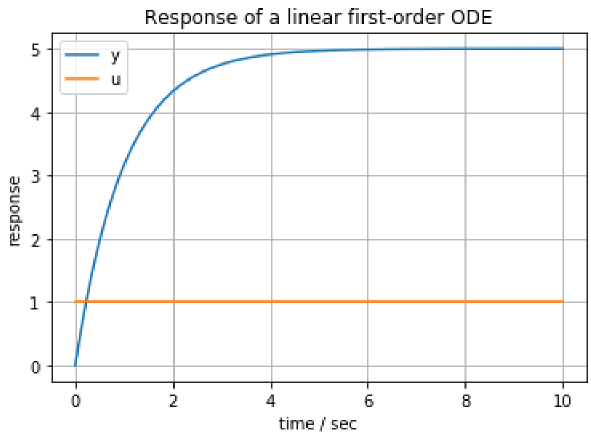
The following cell simulates the response of a first-order linear model in the form

$$\tau\frac{dy}{dt} + y = Ku(t)$$

where τ and K are model parameters, and $u(t) = 1$ is an external process input.

In [2]:

```
% matplotlib inline
from pyomo.environ import *
from pyomo.dae import *
import matplotlib.pyplot as plt

tfinal = 10
tau = 1
K = 5

# define u(t)
u = lambda t: 1

# create a model object
model = ConcreteModel()

# define the independent variable
model.t = ContinuousSet(bounds=(0, tfinal))

# define the dependent variables
model.y = Var(model.t)
model.dydt = DerivativeVar(model.y)

# fix the initial value of y
model.y[0].fix(0)

# define the differential equation as a constraint
model.ode = Constraint(model.t, rule=lambda model, t: tau*model.dydt[t] + model.y[t] ==
K*u(t))

# simulation using scipy integrators
tsim, profiles = Simulator(model, package='scipy').simulate(numpoints=1000)

fig, ax = plt.subplots(1, 1)
ax.plot(tsim, profiles, label='y')
ax.plot(tsim, [u(t) for t in tsim], label='u')
ax.set_xlabel('time / sec')
ax.set_ylabel('response')
ax.set_title('Response of a linear first-order ODE')
ax.legend()
ax.grid(True)
```

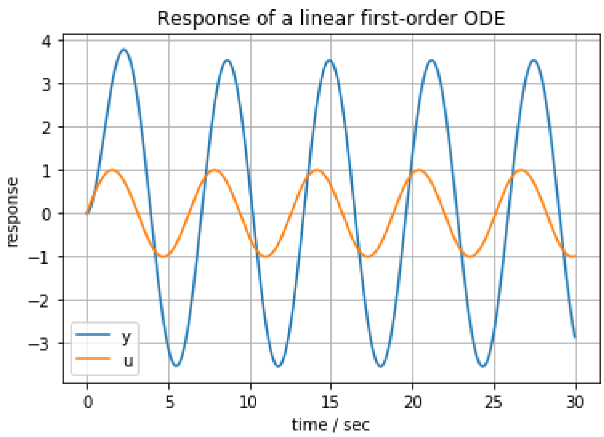


Encapsulating into a Function

In following cells we would like to explore the response of a first order system to changes in parameters and input functions. To facilitate this study, the next cell encapsulates the simulation into a function that can be called with different parameter values and input functions.

In [3]:

```
% matplotlib inline
from pyomo.environ import *
from pyomo.dae import *
import matplotlib.pyplot as plt

def first_order(K=1, tau=1, tfinal=1, u=lambda t: 1):
    model = ConcreteModel()
    model.t = ContinuousSet(bounds=(0, tfinal))
    model.y = Var(model.t)
    model.dydt = DerivativeVar(model.y)
    model.y[0].fix(0)
    model.ode = Constraint(model.t, rule=lambda model, t:
                       tau*model.dydt[t] + model.y[t] == K*u(t))

    tsim, profiles = Simulator(model, package='scipy').simulate(numpoints=1000)

    fig, ax = plt.subplots(1, 1)
    ax.plot(tsim, profiles, label='y')
    ax.plot(tsim, [u(t) for t in tsim], label='u')
    ax.set_xlabel('time / sec')
    ax.set_ylabel('response')
    ax.set_title('Response of a linear first-order ODE')
    ax.legend()
    ax.grid(True)

first_order(5, 1, 30, sin)
```



Analytical Approximation to a Step Input

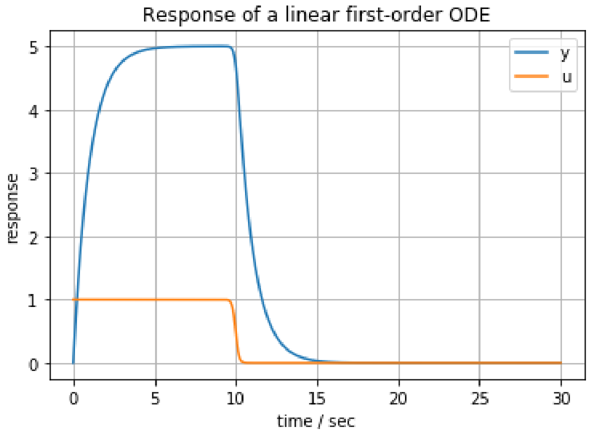
The response to a step change is a common test giving insight into the dynamics of a given system. An infinitely differentiable approximation to a step change is given by the *Butterworth function* $b_n(t)$

$$b_n(t) = \frac{1}{1 + (\frac{t}{c})^n}$$

where n is the order of a approximation, and c is value of t where the step change occurs.

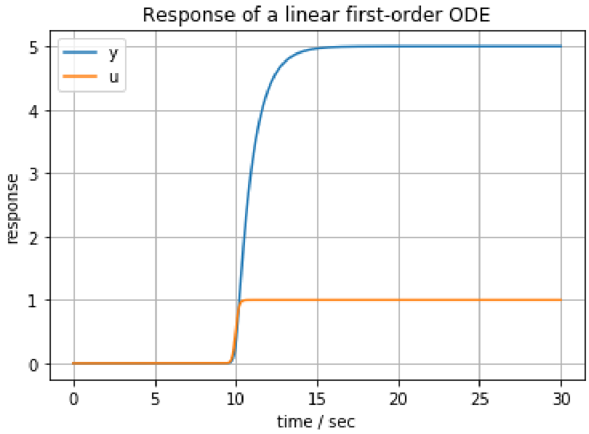
In [4]:

```
u = lambda t: 1/(1 + (t/10)**100)

first_order(5, 1, 30, u)
```



In [5]:

```
u = lambda t: 1 - 1/(1 + (t/10)**100)

first_order(5, 1, 30, u)
```



Analytical Approximation to a Square Wave Input

An analytical approximation to a square wave with frequency f is given by

$$\frac{4}{\pi}\sum_{k=1,3,5,\ldots}^{N}\frac{sin(k\pi/N)}{k\pi/N}\frac{sin(2\pi ft)}{k}$$

where the first term is the *Lanczos* sigma factor designed to suppress the Gibb's phenomenon associated with Fourier series approximations.

In [6]:

```
from math import pi
def square(t, f=1, N=31):
    return (4/pi)*sum((N*sin(k*pi/N)/k/pi)*sin(2*k*f*pi*t)/k for k in range(1, N+1,2))

u = lambda t: square(t, 0.1)

first_order(5, 1, 30, u)
```