5.3 Transient Heat Conduction in Various Geometries

Rescaling

We'll assume the thermal conductivity k is a constant, and define thermal diffusivity in the conventional way

$$\alpha = \frac{k}{\rho C_p}$$

We will further assume symmetry with respect to all spatial coordinates except r where r extends from $-R$ to $+R$. The boundary conditions are

$$\begin{aligned} T(t,R) &= T_\infty & \forall t > 0 \\ \nabla T(t,0) &= 0 & \forall t \geq 0 \end{aligned}$$

where we have assumed symmetry with respect to r and uniform initial conditions $T(0,r) = T_0$ for all $0 \leq r \leq R$. Following standard scaling procedures, we introduce the dimensionless variables

$$\begin{aligned} T' &= \frac{T - T_0}{T_\infty - T_0} \\ r' &= \frac{r}{R} \\ t' &= t\frac{\alpha}{R^2} \end{aligned}$$

Dimensionless Model

Under these conditions the problem reduces to

$$\frac{\partial T'}{\partial t'} = \nabla^2 T'$$

with auxiliary conditions

$$\begin{aligned} T'(0,r') &= 0 & \forall 0 \leq r' \leq 1 \\ T'(t',1) &= 1 & \forall t' > 0 \\ \nabla T'(t',0) &= 0 & \forall t' \geq 0 \end{aligned}$$

which we can specialize to specific geometries.

Preliminary Code

In [ ]:

```
!pip install -q pyomo
!wget -N -q "https://ampl.com/dl/open/ipopt/ipopt-linux64.zip"
!unzip -o -q ipopt-linux64

ipopt_executable='/content/ipopt'
```

In [ ]:

```
import numpy as np
import matplotlib.pyplot as plt
from mpl_toolkits.mplot3d.axes3d import Axes3D

def model_plot(m):
    r = sorted(m.r)
    t = sorted(m.t)

    rgrid = np.zeros((len(t), len(r)))
    tgrid = np.zeros((len(t), len(r)))
    Tgrid = np.zeros((len(t), len(r)))

    for i in range(0, len(t)):
        for j in range(0, len(r)):
            rgrid[i,j] = r[j]
            tgrid[i,j] = t[i]
            Tgrid[i,j] = m.T[t[i], r[j]].value

    fig = plt.figure(figsize=(10,6))
    ax = fig.add_subplot(1, 1, 1, projection='3d')
    ax.set_xlabel('Distance r')
    ax.set_ylabel('Time t')
    ax.set_zlabel('Temperature T')
    p = ax.plot_wireframe(rgrid, tgrid, Tgrid)
```

Planar Coordinates

Suppressing the prime notation, for a slab geometry the model specializes to

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2}$$

with auxiliary conditions

$$\begin{aligned} T(0,r) &= 0 && \forall 0 \leq r \leq 1 \\ T(t,1) &= 1 && \forall t > 0 \\ \frac{\partial T}{\partial r}(t,0) &= 0 && \forall t \geq 0 \end{aligned}$$

In [11]:

```
!ls -a /content
```

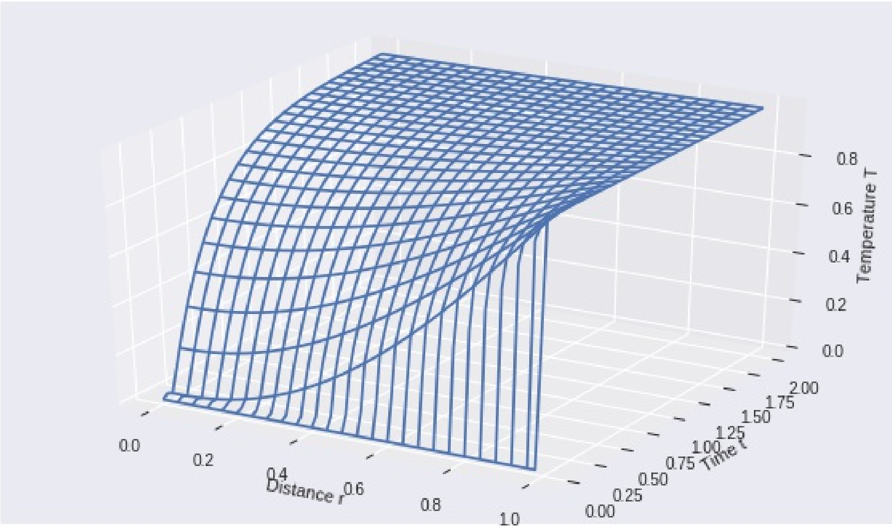
```
.  ..  coin-license.txt  .config  ipopt  ipopt-linux64.zip  sample_data
```

In [12]:

```
from pyomo.environ import *
from pyomo.dae import *

m = ConcreteModel()

m.r = ContinuousSet(bounds=(0,1))
m.t = ContinuousSet(bounds=(0,2))

m.T = Var(m.t, m.r)

m.dTdt   = DerivativeVar(m.T, wrt=m.t)
m.dTdr   = DerivativeVar(m.T, wrt=m.r)
m.d2Tdr2 = DerivativeVar(m.T, wrt=(m.r, m.r))

@m.Constraint(m.t, m.r)
def pde(m, t, r):
    if t == 0:
        return Constraint.Skip
    if r == 0 or r == 1:
        return Constraint.Skip
    return m.dTdt[t,r] == m.d2Tdr2[t,r]

m.obj = Objective(expr=1)

m.ic  = Constraint(m.r, rule=lambda m, r:    m.T[0,r] == 0 if r > 0 and r < 1 else
Constraint.Skip)
m.bc1 = Constraint(m.t, rule=lambda m, t:    m.T[t,1] == 1)
m.bc2 = Constraint(m.t, rule=lambda m, t: m.dTdr[t,0] == 0)

TransformationFactory('dae.finite_difference').apply_to(m, nfe=50, scheme='FORWARD', wr
t=m.r)
TransformationFactory('dae.finite_difference').apply_to(m, nfe=50, scheme='FORWARD', wr
t=m.t)
SolverFactory('ipopt', executable=ipopt_executable).solve(m, tee=True).write()
model_plot(m)
```

```
Ipopt 3.12.8:

******************************************************************************
This program contains Ipopt, a library for large-scale nonlinear optimization.
 Ipopt is released as open source code under the Eclipse Public License (EPL).
         For more information visit http://projects.coin-or.org/Ipopt
******************************************************************************

This is Ipopt version 3.12.8, running with linear solver mumps.
NOTE: Other linear solvers might be more efficient (see Ipopt documentation).

Number of nonzeros in equality constraint Jacobian...:    30347
Number of nonzeros in inequality constraint Jacobian.:        0
Number of nonzeros in Lagrangian Hessian.............:        0

Total number of variables............................:    10299
                     variables with only lower bounds:        0
                variables with lower and upper bounds:        0
                     variables with only upper bounds:        0
Total number of equality constraints.................:    10200
Total number of inequality constraints...............:        0
        inequality constraints with only lower bounds:        0
   inequality constraints with lower and upper bounds:        0
        inequality constraints with only upper bounds:        0

iter    objective    inf_pr   inf_du lg(mu)  ||d||  lg(rg) alpha_du alpha_pr  ls
   0  1.0000000e+00 1.00e+00 0.00e+00  -1.0 0.00e+00    -  0.00e+00 0.00e+00   0
   1  1.0000000e+00 1.50e-12 2.50e-01  -1.7 2.50e+03  -4.0 1.00e+00 1.00e+00h  1
   2  1.0000000e+00 1.82e-12 1.53e-10  -1.7 8.16e-10  -4.5 1.00e+00 1.00e+00h  1

Number of Iterations....: 2
```

```
Number of Iterations....: 2

                                   (scaled)                 (unscaled)
Objective...............:   1.0000000000000000e+00    1.0000000000000000e+00
Dual infeasibility......:   1.5268200213081659e-10    1.5268200213081659e-10
Constraint violation....:   3.6364522504328498e-14    1.8182261252164267e-12
Complementarity.........:   0.0000000000000000e+00    0.0000000000000000e+00
Overall NLP error.......:   1.5268200213081659e-10    1.5268200213081659e-10


Number of objective function evaluations             = 3
Number of objective gradient evaluations             = 3
Number of equality constraint evaluations            = 3
Number of inequality constraint evaluations          = 0
Number of equality constraint Jacobian evaluations   = 3
Number of inequality constraint Jacobian evaluations = 0
Number of Lagrangian Hessian evaluations             = 2
Total CPU secs in IPOPT (w/o function evaluations)   =      0.207
Total CPU secs in NLP function evaluations           =      0.003

EXIT: Optimal Solution Found.
# ==========================================================
# = Solver Results                                         =
# ==========================================================
# ----------------------------------------------------------
#   Problem Information
# ----------------------------------------------------------
Problem:
- Lower bound: -inf
  Upper bound: inf
  Number of objectives: 1
  Number of constraints: 10200
  Number of variables: 10299
  Sense: unknown
# ----------------------------------------------------------
#   Solver Information
# ----------------------------------------------------------
Solver:
- Status: ok
  Message: Ipopt 3.12.8\x3a Optimal Solution Found
  Termination condition: optimal
  Id: 0
  Error rc: 0
  Time: 0.30750346183776855
# ----------------------------------------------------------
#   Solution Information
# ----------------------------------------------------------
Solution:
- number of solutions: 0
  number of solutions displayed: 0
```



Cylindrical Coordinates

Suppressing the prime notation, for a cylindrical geometry the model specializes to

$$\frac{\partial T}{\partial t} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

Expanding,

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial t^2} + \frac{1}{r}\frac{\partial T}{\partial r}$$

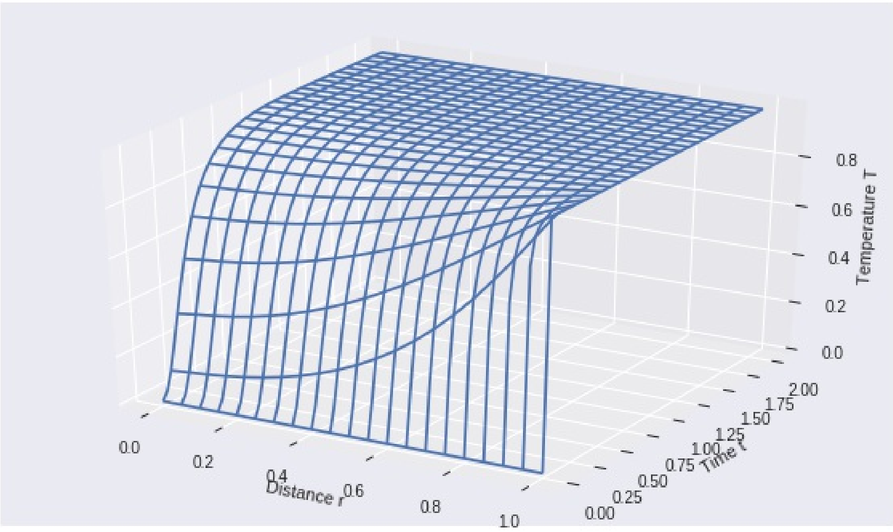
with auxiliary conditions

$$\begin{aligned} T(0,r) &= 0 && \forall 0 \le r \le 1 \\ T(t,1) &= 1 && \forall t > 0 \\ \frac{\partial T}{\partial r}(t,0) &= 0 && \forall t \ge 0 \end{aligned}$$

In [13]:

```
from pyomo.environ import *
from pyomo.dae import *

m = ConcreteModel()

m.r = ContinuousSet(bounds=(0,1))
m.t = ContinuousSet(bounds=(0,2))

m.T = Var(m.t, m.r)

m.dTdt   = DerivativeVar(m.T, wrt=m.t)
m.dTdr   = DerivativeVar(m.T, wrt=m.r)
m.d2Tdr2 = DerivativeVar(m.T, wrt=(m.r, m.r))

m.pde = Constraint(m.t, m.r, rule=lambda m, t, r: m.dTdt[t,r] == m.d2Tdr2[t,r] + (1/r)*m.
dTdr[t,r]
        if r > 0 and r < 1 and t > 0 else Constraint.Skip)

m.ic  = Constraint(m.r, rule=lambda m, r:    m.T[0,r] == 0)
m.bc1 = Constraint(m.t, rule=lambda m, t:    m.T[t,1] == 1 if t > 0 else Constraint.Skip
)
m.bc2 = Constraint(m.t, rule=lambda m, t: m.dTdr[t,0] == 0)

TransformationFactory('dae.finite_difference').apply_to(m, nfe=20, wrt=m.r, scheme='CEN
TRAL')
TransformationFactory('dae.finite_difference').apply_to(m, nfe=50, wrt=m.t, scheme='BAC
KWARD')
SolverFactory('ipopt', executable=ipopt_executable).solve(m).write()

model_plot(m)
```

```
# ==========================================================
# = Solver Results                                         =
# ==========================================================
# ----------------------------------------------------------
#   Problem Information
# ----------------------------------------------------------
Problem:
- Lower bound: -inf
  Upper bound: inf
  Number of objectives: 1
  Number of constraints: 4060
  Number of variables: 4110
  Sense: unknown
# ----------------------------------------------------------
#   Solver Information
# ----------------------------------------------------------
Solver:
- Status: ok
  Message: Ipopt 3.12.8\x3a Optimal Solution Found
  Termination condition: optimal
  Id: 0
  Error rc: 0
  Time: 0.16563725471496582
# ----------------------------------------------------------
#   Solution Information
# ----------------------------------------------------------
Solution:
- number of solutions: 0
  number of solutions displayed: 0
```



Spherical Coordinates

Suppressing the prime notation, for a cylindrical geometry the model specializes to

$$\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)$$

Expanding,

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial t^2} + \frac{2}{r}\frac{\partial T}{\partial r}$$

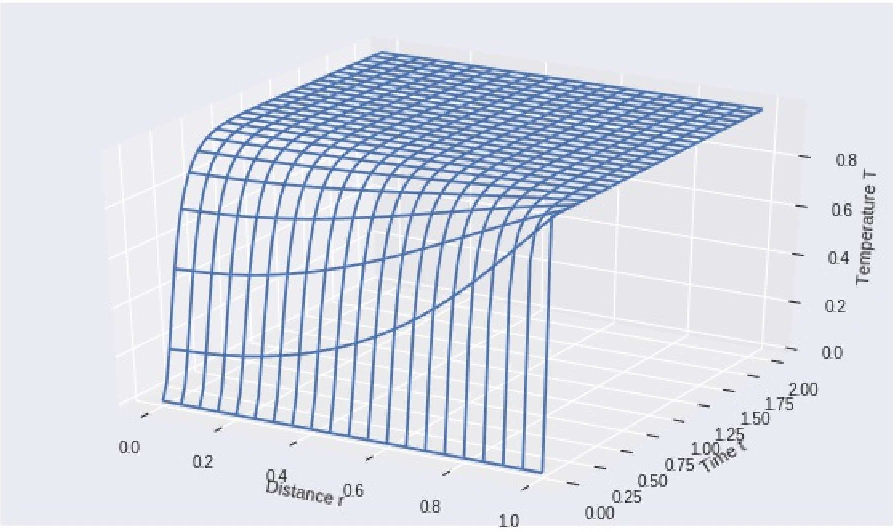
with auxiliary conditions

$$\begin{aligned} T(0,r) &= 0 & \forall 0 \leq r \leq 1 \\ T(t,1) &= 1 & \forall t > 0 \\ \frac{\partial T}{\partial r}(t,0) &= 0 & \forall t \geq 0 \end{aligned}$$

In [14]:

```
from pyomo.environ import *
from pyomo.dae import *

m = ConcreteModel()

m.r = ContinuousSet(bounds=(0,1))
m.t = ContinuousSet(bounds=(0,2))

m.T = Var(m.t, m.r)

m.dTdt   = DerivativeVar(m.T, wrt=m.t)
m.dTdr   = DerivativeVar(m.T, wrt=m.r)
m.d2Tdr2 = DerivativeVar(m.T, wrt=(m.r, m.r))

m.pde = Constraint(m.t, m.r, rule=lambda m, t, r: m.dTdt[t,r] == m.d2Tdr2[t,r] + (2/r)*m.
dTdr[t,r]
        if r > 0 and r < 1 and t > 0 else Constraint.Skip)

m.ic  = Constraint(m.r, rule=lambda m, r:    m.T[0,r] == 0)
m.bc1 = Constraint(m.t, rule=lambda m, t:    m.T[t,1] == 1 if t > 0 else Constraint.Skip
)
m.bc2 = Constraint(m.t, rule=lambda m, t: m.dTdr[t,0] == 0)

TransformationFactory('dae.finite_difference').apply_to(m, nfe=20, wrt=m.r, scheme='CEN
TRAL')
TransformationFactory('dae.finite_difference').apply_to(m, nfe=50, wrt=m.t, scheme='BAC
KWARD')
SolverFactory('ipopt', executable=ipopt_executable).solve(m).write()

model_plot(m)
```

```
# ==========================================================
# = Solver Results                                         =
# ==========================================================
# ----------------------------------------------------------
#   Problem Information
# ----------------------------------------------------------
Problem:
- Lower bound: -inf
  Upper bound: inf
  Number of objectives: 1
  Number of constraints: 4060
  Number of variables: 4110
  Sense: unknown
# ----------------------------------------------------------
#   Solver Information
# ----------------------------------------------------------
Solver:
- Status: ok
  Message: Ipopt 3.12.8\x3a Optimal Solution Found
  Termination condition: optimal
  Id: 0
  Error rc: 0
  Time: 0.15887713432312012
# ----------------------------------------------------------
#   Solution Information
# ----------------------------------------------------------
Solution:
- number of solutions: 0
  number of solutions displayed: 0
```




```
In [ ]:
```