Exam questions

NLA course

2016

Online rendering on github or nbviewer might be incorrect in some places. Therefore, please read lectures locally on your laptop.

List of questions

1. Vector norms. Forward and backward stability. Disks in different norms. First norm and compressed sensing.
2. Matrix-matrix multiplication. Computer memory hierarchy and blocking. Standard libraries for linear algebra. Strassen algorithm (no need to remember exact formulas) and its complexity (with derivation).
3. Matrix norms. Example of a norm that is not a matrix norm (does not satisfy submultiplicative property). Scalar product. Cauchy-Schwarz-Bunyakovsky inequality.
4. Unitary matrices and their properties. Examples: Fourier matrix, permutation matrix, Householder reflections, Givens rotations.
5. Matrix rank. Skeleton decomposition (with proof). Low-rank matrix storage and matrix-vector product. Instability of matrix rank.
6. Singular value decomposition (SVD). Proof of its existence. Null space and image of A and A^* in terms of singular vectors. Interpretation of SVD as a separation of variables.
7. Eckart-Young theorem (proof for the spectral norm). Applications of SVD: latent semantic analysis, collaborative filtering in recommender systems, dense matrix compression.
8. LU-decomposition. Criteria of its existence and uniqueness (both with proof). Connection with Gaussian elimination. Complexity of decomposing and solving linear system. The concept of pivoting.
9. Cholesky factorization (with proof of its existence).
10. Neumann series. Bounds for $\|(I-A)^{-1}\|$ and $\dfrac{\|(A+\Delta A)^{-1}-A^{-1}\|}{\|A^{-1}\|}$ (both with proofs). Estimate of a perturbation of a solution in terms of condition number (with proof). Linear systems with "consistent" and "inconsistent" right-hand sides and behavior of solution error for large condition numbers.
11. Least squares (LS) problem for over- and underdetermined linear systems. Methods of solving LS via QR, pseudoinverse, SVD, padding into a bigger system.
12. Eigenvalue problem. Eigendecomposition. Criteria of eigendecomposition existence via algebraic and geometric multiplicities (no proof). Example of a nondiagonalizable matrix. Computing eigenvalues using characteristic equation. Is it a good idea? Gershgorin circles theorem (with proof).
13. Power method and its convergence (with bound derivation). Application in PageRank.
14. Schur decomposition (with proof). Normal matrices and their diagonalizability. Properties of eigenvalues of Hermitian, unitary and skew-Hermitian matrices. Variational concept for eigenvalues: Rayleigh quotient.
15. QR decomposition. Proof of its existence. Connection with the Gram-Schmidt algorithm. QR via Cholesky decomposition.
16. QR via Householder reflections and Givens rotations (both with derivation). Comparison of these algorithms.

17. QR algorithm. Convergence theorem (no proof). Accelerating convergence with shifts. Reducing complexity of QR algorithm from $\mathcal{O}(n^4)$ down to $\mathcal{O}(n^3)$.

18. Divide and conquer algorithm for symmetric eigenvalue problems (with derivation).

19. Jacobi method for eigenvalue problem. Its convergence (with proof).

20. Sparse matrix arithmectics: COO, LIL, CSR, CSC formats. Their comparison. Blocking for increasing efficiency of sparse formats.

21. Sparse LU decomposition, connection with graphs. Dependence of fill-in on ordering of graph nodes (with example). Nested dissection and spectral bisection algorithms.

22. Richardson iteration. Optimal choice of parameter. Convergence estimate.

23. Chebyshev iteration. Error bounds via Chebyshev polynomials.

24. Solving linear system as an optimization problem: minimization of residual and energy functional. Approximation of a solution of a linear system by a subspace, Galerkin projection. Krylov subspace and ill-posedness of natural Krylov basis.

25. Ill-posedness of natural Krylov basis. Orthogonal basis in Krylov subspace. Arnoldi relation and its derivation. Lanczos process.

26. Direct Lanczos method for solving linear systems. Conjugate gradient method.

27. Convergence theory of the conjugate gradient method.

28. MINRES method. GMRES method and its connection to Anderson acceleration. Disadvantages of GMRES for nonsymmetric systems. Idea of BiCG and BiCGStab.

29. Preconditioning concept. Right and left preconditioners. Jacobi, Gauss-Seidel and SOR preconditioners.

30. Incomplete LU for preconditioning. ILUt, ILU(k), second-order ILU (ILU2).

31. Inverse and Rayleigh quotient iterations. Speed of convergence. Convergence behavior of inexact inverse iteration.

32. Block power method.

33. Ritz approximation: Ritz values and vectors and their properties. Rayleigh-Ritz method.

34. Arnoldi relation (with derivation). Lanczos and Arnoldi methods for solving partial eigenvalue problem. Their advantages and disadvantages.

35. Convergence bound for $\lambda_{\max}$ in Lanczos method (with derivation).

36. Preconditioned eigensolvers: PINVIT and LOBPCG methods. Finding parameters using Rayleigh-Ritz method.

37. Jacobi-Davidson method: derivation of the Jacobi equation. Connection to the RQ iteration. Subspace acceleration.

38. Fast Fourier Transform (FFT). Cooley-Tukey algorithm (with derivation).

39. Continuous and discrete convolution. Eigendecomposition for circulant matrices (with proof).

40. Product of Toeplitz matrix by vector via FFT. BTTB matrix-by-vector product via 2D FFT.

41. The concept of displacement rank. Displacement rank of Toeplitz matrices.

42. Matrix functions. Matrix exponential and its applications. Problem with evaluating matrix exponential and how to avoid it. Schur-Parlett algorithm. Matrix functions via Pade approximation.

43. Sylvester and Lyapunov equations, their applications. Solving Sylvester equation using Bartels-Stewart method.