KDCR Project 1

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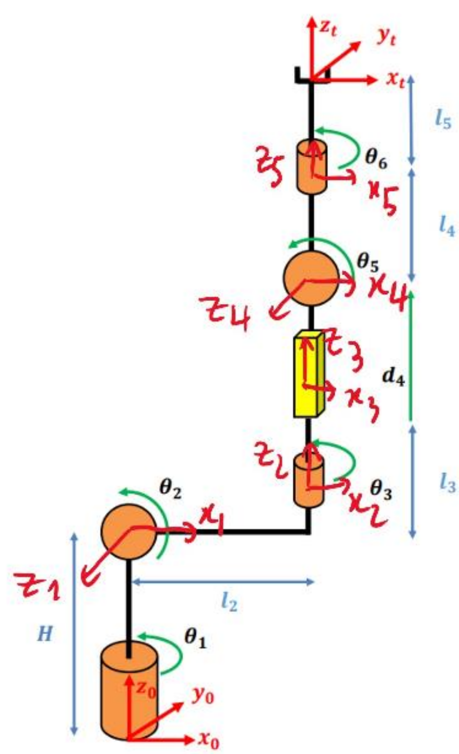
All calculations and programing should be done **parametrically**. Numerical values should only be substituted for creating the figures.
Figure 1 shows a spatial serial manipulator with 6 DOFs - $\theta_1, \theta_2, d_3, \theta_4, \theta_5, \theta_6$

1. Solve the forward kinematics problem for the robot – calculate the homogeneous transformation matrix (4×4) from the tool frame $(\hat{\boldsymbol{x}}_t, \hat{\boldsymbol{y}}_t, \hat{\boldsymbol{z}}_t)$ to the world frame $(\hat{\boldsymbol{x}}_0, \hat{\boldsymbol{y}}_0, \hat{\boldsymbol{z}}_0)$ as a function of the joint variables.
   - The world and tool frames are defined in the figure 1.
   - The revolute joints are drawn in their zero position.
   - The positive direction of the joint angles is as described in the drawing.



Solution:

Firstly, we build the DH diagram and calculate all the homogeneous transformation matrices using Matlab. The results of the calculations are shown below:

$$\text{dh params} = \begin{pmatrix} 0 & \frac{\pi}{2} & \overline{H} & \overline{\theta_1} \\ \overline{l_2} & -\frac{\pi}{2} & 0 & \overline{\theta_2} \\ 0 & 0 & \overline{l_3} & \overline{\theta_3} \\ 0 & \frac{\pi}{2} & \overline{d_4} & 0 \\ 0 & -\frac{\pi}{2} & 0 & \overline{\theta_5} \\ 0 & 0 & \overline{l_4}+\overline{l_5} & \overline{\theta_6} \end{pmatrix}$$

$$A(:,:,1) = \begin{pmatrix} \cos(\theta_1) & 0 & \sin(\theta_1) & 0 \\ \sin(\theta_1) & 0 & -\cos(\theta_1) & 0 \\ 0 & 1 & 0 & H \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A(:,:,2) = \begin{pmatrix} \cos(\theta_2) & 0 & -\sin(\theta_2) & l_2\cos(\theta_2) \\ \sin(\theta_2) & 0 & \cos(\theta_2) & l_2\sin(\theta_2) \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A(:,:,3) = \begin{pmatrix} \cos(\theta_3) & -\sin(\theta_3) & 0 & 0 \\ \sin(\theta_3) & \cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & l_3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A(:,:,4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A(:,:,5) = \begin{pmatrix} \cos(\theta_5) & 0 & -\sin(\theta_5) & 0 \\ \sin(\theta_5) & 0 & \cos(\theta_5) & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad A(:,:,6) = \begin{pmatrix} \cos(\theta_6) & -\sin(\theta_6) & 0 & 0 \\ \sin(\theta_6) & \cos(\theta_6) & 0 & 0 \\ 0 & 0 & 1 & l_4+l_5 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

So the overall homogeneous transformation matrix ${}^0A_t$ is:

$${}^0A_t =$$

$$\begin{pmatrix} \cos(\theta_1)\cos(\theta_2)\cos(\theta_3)\cos(\theta_5)\cos(\theta_6) - \cos(\theta_1)\cos(\theta_2)\sin(\theta_3)\sin(\theta_6) - \cos(\theta_1)\cos(\theta_6)\sin(\theta_2)\sin(\theta_5) - \cos(\theta_5)\cos(\theta_6)\sin(\theta_1)\sin(\theta_3) - \cos(\theta_3)\sin(\theta_1)\sin(\theta_6) & \cos(\theta_1)\sin(\theta_2)\sin(\theta_5)\sin(\theta_6) - \cos(\theta_1)\cos(\theta_2)\cos(\theta_6)\sin(\theta_3) - \cos(\theta_3)\cos(\theta_6)\sin(\theta_1) + \cos(\theta_5)\sin(\theta_1)\sin(\theta_3)\sin(\theta_6) - \cos(\theta_1)\cos(\theta_2)\cos(\theta_3)\cos(\theta_5)\sin(\theta_6) & \sin(\theta_1)\sin(\theta_3)\sin(\theta_5) - \cos(\theta_1)\cos(\theta_5)\sin(\cdots \\ \cos(\theta_1)\cos(\theta_3)\sin(\theta_6) + \cos(\theta_1)\cos(\theta_5)\cos(\theta_6)\sin(\theta_3) - \cos(\theta_2)\sin(\theta_1)\sin(\theta_3)\sin(\theta_6) - \cos(\theta_6)\sin(\theta_1)\sin(\theta_2)\sin(\theta_5) + \cos(\theta_2)\cos(\theta_3)\cos(\theta_5)\cos(\theta_6)\sin(\theta_1) & \cos(\theta_1)\cos(\theta_3)\cos(\theta_6) - \cos(\theta_2)\cos(\theta_6)\sin(\theta_1)\sin(\theta_3) - \cos(\theta_1)\cos(\theta_5)\sin(\theta_3)\sin(\theta_6) + \sin(\theta_1)\sin(\theta_2)\sin(\theta_5)\sin(\theta_6) - \cos(\theta_2)\cos(\theta_3)\cos(\theta_5)\sin(\theta_1)\sin(\theta_6) & -\cos(\theta_5)\sin(\theta_1)\sin(\theta_2) - \cos(\theta_1)\sin(\theta_3)\sin(\cdots \\ \cos(\theta_2)\cos(\theta_6)\sin(\theta_5) - \sin(\theta_2)\sin(\theta_3)\sin(\theta_6) + \cos(\theta_3)\cos(\theta_5)\cos(\theta_6)\sin(\theta_2) & -\cos(\theta_6)\sin(\theta_2)\sin(\theta_3) - \cos(\theta_2)\sin(\theta_5)\sin(\theta_6) - \cos(\theta_3)\cos(\theta_5)\sin(\theta_2)\sin(\theta_6) & \cos(\theta_2)\cos(\theta_5) - \cos(\theta_3\cdots \\ 0 & 0 & 0 \end{pmatrix}$$

2. Solve the inverse kinematics problem – given the full homogeneous transformation matrix representing a possible position and orientation of the robot tool, calculate the values of the joint variables.
   Find all possible solutions and show the multiple solutions in a qualitative drawing.

Solution:

First note that the z axis of the frames 4 5 and t intersects at the same point P which also means that the position of P is only affected by the joint values θ_1,θ_2,d_4. So in order to solve the inverse kinematics we will first solve for θ_1,θ_2,d_4, and then for $\theta_3,\theta_5,\theta_6$

1) Solving for θ_1,θ_2,d_4 :

The position of point P is

$$\mathrm{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} - {}^0R_t \begin{bmatrix} 0 \\ 0 \\ l_4 + l_5 \end{bmatrix}$$

We know that the expression for $P(\theta_1, \theta_2, d_4)$ from ${}^0\mathrm{A}_4$:

$${}^0A_4 = $$
$$= \begin{bmatrix} C_{123} - S_{13} & -C_1S_2 & C_3S_1 + C_{12}S_3 & l_2C_{12} - (d_4 + l_3)\, C_1S_2 \\ C_1S_3 + C_{23}S_1 & -S_{12} & C_2S_{13} - C_{13} & l_2C_2S_1 - (d_4 + l_3)\, S_{12} \\ C_3S_2 & C_2 & S_{23} & H + d_4C_2 + l_3C_2 + l_2S_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Meaning that

$$\begin{bmatrix} P_x \\ P_y \\ P_z \end{bmatrix} = \begin{bmatrix} l_2C_{12} - (d_4 + l_3)\, C_1S_2 \\ l_2C_2S_1 - (d_4 + l_3)\, S_{12} \\ H + d_4C_2 + l_3C_2 + l_2S_2 \end{bmatrix}$$
$$\begin{cases} C_1(l_2C_2 - (d_4 + l_3)\, S_2) = P_x \\ S_1(l_2C_2 - (d_4 + l_3)\, S_2) = P_y \end{cases}$$

So we can calculate the value of θ_1 with

$$\theta_1 = atan2(\pm P_x, \pm P_y)$$

Then we square the three equations:

$$\begin{cases} P_x^2 + P_y^2 = l_2^2C_2^2 + (d_4 + l_3)^2S_2^2 - 2l_2(d_4 + l_3)C_2S_2 \\ P_x^2 - P_y^2 = [l_2^2C_2^2 + (d_4 + l_3)^2S_2^2 - 2l_2(d_4 + l_3)C_2S_2]\, cos(\, 2\theta_1) \\ (P_z - H)^2 = l_2^2S_2^2 + (d_4 + l_3)^2C_2^2 + 2l_2(d_4 + l_3)C_2S_2 \end{cases}$$

Adding them together we can calculate the value of d_4

$$d_4 = \pm\sqrt{P_x^2 + P_y^2 + (P_z - H)^2 - l_2^2} - l_3$$

Finally to calculate the values of θ_2, we can express

$$sin(\, \theta_2) = \frac{P_z - H - d_4C_2 - l_3C_2}{l_2}$$

$$\Rightarrow cos(\, \theta_2) = \frac{\frac{P_x}{\mathrm{C}_1} l_2 - (d_4 + l_3)(H - P_z)}{l_2^2 + (d_4 + l_3)^2}$$

And bring the representation of $cos(\, \theta_2)$ into $sin(\, \theta_2)$ to calculate

$$\theta_2 = atan2(sin(\, \theta_2), cos(\, \theta_2))$$

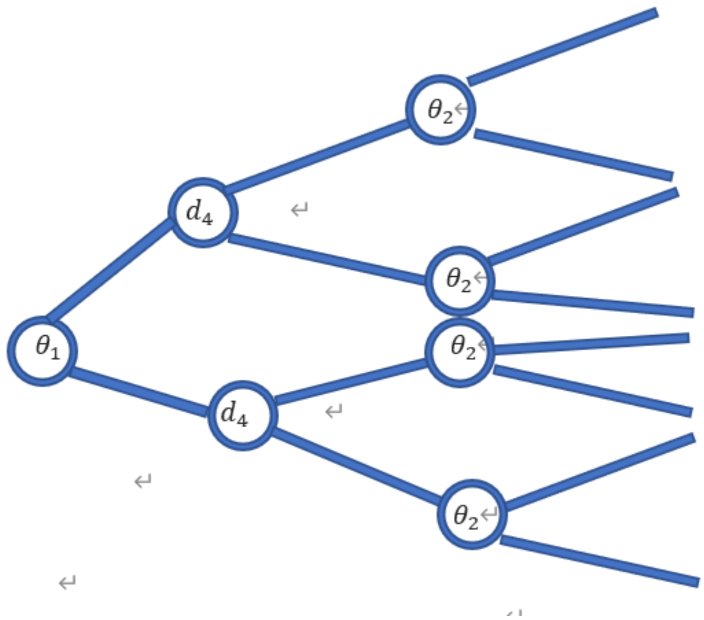
2) Solving for $\theta_3, \theta_5, \theta_6$

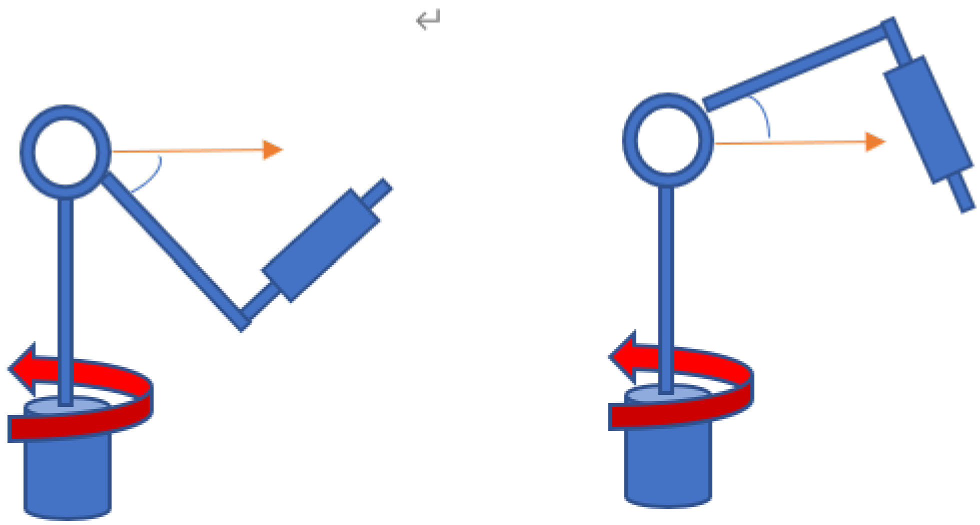
$${}^4A_t = {}^4A_5 * {}^5A_t$$
$$= \begin{bmatrix} * & * & * & -(l_4 + l_5)S_5 \\ * & * & * & (l_4 + l_5)C_5 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

While on the other hand,

$$^{0}P_t = \, ^{0}R_4 * \, ^{4}P_t$$
$$^{4}P_t = (\, ^{0}R_4)' * \, ^{0}P_t$$

Mathematically, we calculated 8 answers just for the case of the three joints θ_1, θ_2, d_4. We briefly sketched some of the solutions as an indication.





3. **Calculate the full Jacobian matrix (6×6) for the robot in the world frame and the tool frame.**

Using the Whitney method to calculate the Jacobian,

U is a unit vector in the direction of the axis of joint, we want to express it in the world frame.

$$\begin{pmatrix} 0 & \sin(\theta_1) & \sigma_2 & \sigma_2 & \cos(\theta_3)\sin(\theta_1)+\cos(\theta_1)\cos(\theta_2)\sin(\theta_3) & \sin(\theta_1)\sin(\theta_3)\sin(\theta_5)-\cos(\theta_1)\cos(\theta_5)\sin(\theta_2)-\cos(\theta_1)\cos(\theta_2)\cos(\theta_3)\sin(\theta_5) \\ 0 & -\cos(\theta_1) & \sigma_1 & \sigma_1 & \cos(\theta_2)\sin(\theta_1)\sin(\theta_3)-\cos(\theta_1)\cos(\theta_3) & -\cos(\theta_5)\sin(\theta_1)\sin(\theta_2)-\cos(\theta_1)\sin(\theta_3)\sin(\theta_5)-\cos(\theta_2)\cos(\theta_3)\sin(\theta_1)\sin(\theta_5) \\ 1 & 0 & \cos(\theta_2) & \cos(\theta_2) & \sin(\theta_2)\sin(\theta_3) & \cos(\theta_2)\cos(\theta_5)-\cos(\theta_3)\sin(\theta_2)\sin(\theta_5) \end{pmatrix}$$

where

$$\sigma_1 = -\sin(\theta_1)\sin(\theta_2)$$

$$\sigma_2 = -\cos(\theta_1)\sin(\theta_2)$$

Then we need to calculate r, r is a vector from any point on the axis of joint i to the

$$\begin{pmatrix} \sigma_5 & \sigma_5 & \sigma_{22}-\sigma_{10}-\sigma_{24}-\sigma_{23}-\sigma_9+\sigma_{21}-\sigma_{20}-\sigma_{19} & \sigma_{22}-\sigma_{24}-\sigma_{23}-\sigma_9+\sigma_{21}-\sigma_{20}-\sigma_{19} & \sigma_8 & \sigma_8 \\ \sigma_6 & \sigma_6 & -\sigma_{11}-\sigma_{12}-\sigma_{18}-\sigma_{17}-\sigma_{16}-\sigma_{15}-\sigma_{14}-\sigma_{13} & -\sigma_{11}-\sigma_{18}-\sigma_{17}-\sigma_{16}-\sigma_{15}-\sigma_{14}-\sigma_{13} & \sigma_7 & \sigma_7 \\ H+d_4\cos(\theta_2)+l_3\cos(\theta_2)+l_2\sin(\theta_2)+\sigma_4+\sigma_3-\sigma_2-\sigma_1 & d_4\cos(\theta_2)+l_3\cos(\theta_2)+l_2\sin(\theta_2)+\sigma_4+\sigma_3-\sigma_2-\sigma_1 & d_4\cos(\theta_2)+l_3\cos(\theta_2)+\sigma_4+\sigma_3-\sigma_2-\sigma_1 & d_4\cos(\theta_2)+\sigma_4+\sigma_3-\sigma_2-\sigma_1 & \sigma_4+\sigma_3-\sigma_2-\sigma_1 & \sigma_4+\sigma_3-\sigma_2-\sigma_1 \end{pmatrix}$$

where

$$\sigma_1 = l_5\cos(\theta_3)\sin(\theta_2)\sin(\theta_5)$$

$$\sigma_2 = l_4\cos(\theta_3)\sin(\theta_2)\sin(\theta_5)$$

$$\sigma_3 = l_5\cos(\theta_2)\cos(\theta_5)$$

$$\sigma_4 = l_4\cos(\theta_2)\cos(\theta_5)$$

$$\sigma_5 = l_2\cos(\theta_1)\cos(\theta_2)-\sigma_9-\sigma_{10}-\sigma_{24}-\sigma_{23}+\sigma_{22}+\sigma_{21}-\sigma_{20}-\sigma_{19}$$

$$\sigma_6 = l_2\cos(\theta_2)\sin(\theta_1)-\sigma_{11}-\sigma_{12}-\sigma_{18}-\sigma_{17}-\sigma_{16}-\sigma_{15}-\sigma_{14}-\sigma_{13}$$

$$\sigma_7 = -\sigma_{18}-\sigma_{17}-\sigma_{16}-\sigma_{15}-\sigma_{14}-\sigma_{13}$$

$$\sigma_8 = \sigma_{22}-\sigma_{23}-\sigma_{24}+\sigma_{21}-\sigma_{20}-\sigma_{19}$$

$$\sigma_9 = d_4\cos(\theta_1)\sin(\theta_2)$$

$$\sigma_{10} = l_3\cos(\theta_1)\sin(\theta_2)$$

$$\sigma_{11} = d_4\sin(\theta_1)\sin(\theta_2)$$

$$\sigma_{12} = l_3\sin(\theta_1)\sin(\theta_2)$$

$$\sigma_{13} = l_5\cos(\theta_2)\cos(\theta_3)\sin(\theta_1)\sin(\theta_5)$$

$$\sigma_{14} = l_4\cos(\theta_2)\cos(\theta_3)\sin(\theta_1)\sin(\theta_5)$$

$$\sigma_{15} = l_5\cos(\theta_1)\sin(\theta_3)\sin(\theta_5)$$

$$\sigma_{16} = l_5\cos(\theta_5)\sin(\theta_1)\sin(\theta_2)$$

$$\sigma_{17} = l_4\cos(\theta_1)\sin(\theta_3)\sin(\theta_5)$$

$$\sigma_{18} = l_4\cos(\theta_5)\sin(\theta_1)\sin(\theta_2)$$

$$\sigma_{19} = l_5\cos(\theta_1)\cos(\theta_2)\cos(\theta_3)\sin(\theta_5)$$

$$\sigma_{20} = l_4\cos(\theta_1)\cos(\theta_2)\cos(\theta_3)\sin(\theta_5)$$

$$\sigma_{21} = l_5\sin(\theta_1)\sin(\theta_3)\sin(\theta_5)$$

$$\sigma_{22} = l_4\sin(\theta_1)\sin(\theta_3)\sin(\theta_5)$$

$$\sigma_{23} = l_5\cos(\theta_1)\cos(\theta_5)\sin(\theta_2)$$

$$\sigma_{24} = l_4\cos(\theta_1)\cos(\theta_5)\sin(\theta_2)$$

end-effector, it is also expressed in the world frame.

With U and r we calculated the Jacobian for the world frame:

$$\begin{pmatrix}
(\sigma_3+\sigma_4)\,(\overline{l_4}+\overline{l_5}) - \cos(\overline{\theta_2})\sin(\overline{\theta_1})\,\overline{l_2} + \sin(\overline{\theta_1})\sin(\overline{\theta_2})\,\overline{d_4} + \sin(\overline{\theta_1})\sin(\overline{\theta_2})\,\overline{l_3} & -\cos(\overline{\theta_1})\,\sigma_9 & \sin(\overline{\theta_5})\,\sigma_2\,(\overline{l_4}+\overline{l_5}) & \sigma_7 & \sin(\overline{\theta_2})\sin(\overline{\theta_3})\,(\sigma_3+\sigma_4)\,(\overline{l_4}+\overline{l_5}) - (\sigma_8-\sigma_5)\,\sigma_{10}\,(\overline{l_4}+\overline{l_5}) & 0 \\
\sigma_1\,(\overline{l_4}+\overline{l_5}) + \cos(\overline{\theta_1})\cos(\overline{\theta_2})\,\overline{l_2} - \cos(\overline{\theta_1})\sin(\overline{\theta_2})\,\overline{d_4} - \cos(\overline{\theta_1})\sin(\overline{\theta_2})\,\overline{l_3} & -\sin(\overline{\theta_1})\,\sigma_9 & -\sin(\overline{\theta_5})\,(\sigma_8-\sigma_5)\,(\overline{l_4}+\overline{l_5}) & \sigma_6 & \sin(\overline{\theta_2})\sin(\overline{\theta_3})\,\sigma_1\,(\overline{l_4}+\overline{l_5}) - \sigma_2\,\sigma_{10}\,(\overline{l_4}+\overline{l_5}) & 0 \\
0 & \cos(\overline{\theta_2})\,\overline{l_2} - \sin(\overline{\theta_2})\,\overline{d_4} - \sin(\overline{\theta_2})\,\overline{l_3} - \cos(\overline{\theta_5})\sin(\overline{\theta_2})\,\overline{l_4} - \cos(\overline{\theta_5})\sin(\overline{\theta_2})\,\overline{l_5} - \cos(\overline{\theta_2})\cos(\overline{\theta_3})\sin(\overline{\theta_5})\,\overline{l_4} - \cos(\overline{\theta_2})\cos(\overline{\theta_3})\sin(\overline{\theta_5})\,\overline{l_5} & \sin(\overline{\theta_2})\sin(\overline{\theta_3})\sin(\overline{\theta_5})\,(\overline{l_4}+\overline{l_5}) & \cos(\overline{\theta_2}) & -\left(\cos(\overline{\theta_2})\sin(\overline{\theta_5}) + \cos(\overline{\theta_3})\cos(\overline{\theta_5})\sin(\overline{\theta_2})\right)(\overline{l_4}+\overline{l_5}) & 0 \\
0 & \sin(\overline{\theta_1}) & \sigma_7 & 0 & \sigma_2 & \sigma_1 \\
0 & -\cos(\overline{\theta_1}) & \sigma_6 & 0 & \sigma_5-\sigma_8 & -\sigma_3-\sigma_4 \\
1 & 0 & \cos(\overline{\theta_2}) & 0 & \sin(\overline{\theta_2})\sin(\overline{\theta_3}) & \sigma_{10}
\end{pmatrix}$$

where

$$\sigma_1 = \sin(\overline{\theta_5})\,\left(\sin(\overline{\theta_1})\sin(\overline{\theta_3}) - \cos(\overline{\theta_1})\cos(\overline{\theta_2})\cos(\overline{\theta_3})\right) - \cos(\overline{\theta_1})\cos(\overline{\theta_5})\sin(\overline{\theta_2})$$

$$\sigma_2 = \cos(\overline{\theta_3})\sin(\overline{\theta_1}) + \cos(\overline{\theta_1})\cos(\overline{\theta_2})\sin(\overline{\theta_3})$$

$$\sigma_3 = \sin(\overline{\theta_5})\,\left(\cos(\overline{\theta_1})\sin(\overline{\theta_3}) + \cos(\overline{\theta_2})\cos(\overline{\theta_3})\sin(\overline{\theta_1})\right)$$

$$\sigma_4 = \cos(\overline{\theta_5})\sin(\overline{\theta_1})\sin(\overline{\theta_2})$$

$$\sigma_5 = \cos(\overline{\theta_2})\sin(\overline{\theta_1})\sin(\overline{\theta_3})$$

$$\sigma_6 = -\sin(\overline{\theta_1})\sin(\overline{\theta_2})$$

$$\sigma_7 = -\cos(\overline{\theta_1})\sin(\overline{\theta_2})$$

$$\sigma_8 = \cos(\overline{\theta_1})\cos(\overline{\theta_3})$$

$$\sigma_9 = \cos(\overline{\theta_2})\,\overline{d_4} + \cos(\overline{\theta_2})\,\overline{l_3} + \sin(\overline{\theta_2})\,\overline{l_2} + \sigma_{10}\,(\overline{l_4}+\overline{l_5})$$

$$\sigma_{10} = \cos(\overline{\theta_2})\cos(\overline{\theta_5}) - \cos(\overline{\theta_3})\sin(\overline{\theta_2})\sin(\overline{\theta_5})$$

$$
\left(\begin{array}{lll}
\sigma_{16}\,\sigma_8-\sigma_{19}\,\sigma_9 & \cos(\theta_1)\,\sigma_{13}\,\sigma_7-\sigma_{17}\,\sigma_1-\sin(\theta_1)\,\sigma_{11}\,\sigma_7 & \sin(\theta_2)\,\sin(\theta_3)\,\sin(\theta_5)\,\left(l_4+\right. \\
\sigma_{18}\,\sigma_9-\sigma_{15}\,\sigma_8 & \sigma_{14}\,\sigma_1-\cos(\overline{\theta_1})\,\sigma_{12}\,\sigma_7+\sin(\overline{\theta_1})\,\sigma_{10}\,\sigma_7 & \sin(\overline{\theta_5})\,\sigma_{25}\,\sigma_{12}\,\left(\overline{l_4}+\overline{l_5}\right)+\sin \\
\sigma_{23}\,\sigma_9-\sigma_{22}\,\sigma_8 & \cos(\overline{\theta_1})\,\sigma_{21}\,\sigma_7-\sigma_3\,\sigma_1+\sin(\overline{\theta_1})\,\sigma_{20}\,\sigma_7 & -\sin(\overline{\theta_5})\,\left(\overline{l_4}+\overline{l_5}\right)\,\left(\cos(\overline{\theta_3})\,\sin(\overline{\theta_1})\,\cos(\theta_1)\,\cos(\theta_5)\,\sin(\theta_2)-\cos(\overline{\theta_1})\,\cos(\overline{\theta_3})\,\cos(\theta_5)\,\sin(\theta_1)\,\sin(\theta_2)-\sin(\overline{\theta_2})\,\sin(\overline{\theta_3})\,\cos(\theta_2)\,\cos(\theta_5)-\cos(\overline{\theta_1})\,\cos(\overline{\theta_3})\,\cos(\theta_1)\,\sin(\theta_3)\,\sin(\theta_5)-\cos(\overline{\theta_3})\,\sin(\overline{\theta_1})\,\sin(\theta_1)\,\sin(\theta_3)\,\sin(\theta_5)+\sin(\overline{\theta_2})\,\sin(\overline{\theta_3})\,\cos(\theta_3)\,\sin(\theta_2)\,\sin(\theta_5)-\cos(\overline{\theta_1})\,\cos(\overline{\theta_3})\,\cos(\theta_2)\,\cos(\theta_3)\,\sin(\theta_1)\,\sin(\theta_5)+\cos(\overline{\theta_3})\,\mathrm{si}
\end{array}\right.
$$

where

$\sigma_1=\sin(\overline{\theta_2})\,\overline{d_4}-\cos(\overline{\theta_2})\,\overline{l_2}+\sin(\overline{\theta_2})\,\overline{l_3}+\cos(\overline{\theta_5})\,\sin(\overline{\theta_2})\,\overline{l_4}+\cos(\overline{\theta_5})\,\sin(\overline{\theta_2})\,\overline{l_5}+\cos(\overline{\theta_2})\,\cos(\overline{\theta_3})\,\sin(\overline{\theta_5})\,\overline{l_4}+\cos(\overline{\theta_2})\,\cos(\overline{\theta_3})\,\sin(\overline{\theta_5})\,\overline{l_5}$

$\sigma_2=\cos(\overline{\theta_2})\,\sin(\overline{\theta_5})+\cos(\overline{\theta_3})\,\cos(\overline{\theta_5})\,\sin(\overline{\theta_2})$

$\sigma_3=\cos(\theta_2)\,\cos(\theta_5)-\cos(\theta_3)\,\sin(\theta_2)\,\sin(\theta_5)$

$\sigma_4=\cos(\theta_2)\,\sin(\theta_5)+\cos(\theta_3)\,\cos(\theta_5)\,\sin(\theta_2)$

$\sigma_5=\sigma_{24}\,\sigma_{26}\,\left(l_4+l_5\right)-\sin(\theta_2)\,\sin(\theta_3)\,\sigma_{28}\,\left(l_4+l_5\right)$

$\sigma_6=\sigma_{25}\,\sigma_{26}\,\left(l_4+l_5\right)-\sin(\theta_2)\,\sin(\theta_3)\,\sigma_{27}\,\left(l_4+l_5\right)$

$\sigma_7=\cos(\theta_2)\,d_4+\cos(\theta_2)\,l_3+\sin(\theta_2)\,l_2+\sigma_{26}\,\left(l_4+l_5\right)$

$\sigma_8=\sigma_{37}\,\left(\overline{l_4}+\overline{l_5}\right)+\cos(\overline{\theta_1})\,\cos(\overline{\theta_2})\,\overline{l_2}-\cos(\overline{\theta_1})\,\sin(\overline{\theta_2})\,\overline{d_4}-\cos(\overline{\theta_1})\,\sin(\overline{\theta_2})\,\overline{l_3}$

$\sigma_9=\sigma_{38}\,\left(\overline{l_4}+\overline{l_5}\right)-\cos(\overline{\theta_2})\,\sin(\overline{\theta_1})\,\overline{l_2}+\sin(\overline{\theta_1})\,\sin(\overline{\theta_2})\,\overline{d_4}+\sin(\overline{\theta_1})\,\sin(\overline{\theta_2})\,\overline{l_3}$

$\sigma_{10}=\sin(\theta_6)\,\sigma_{29}-\cos(\theta_6)\,\sigma_{33}$

$\sigma_{11}=\cos(\theta_6)\,\sigma_{29}+\sin(\theta_6)\,\sigma_{33}$

$\sigma_{12}=\sin(\theta_6)\,\sigma_{30}-\cos(\theta_6)\,\sigma_{36}$

$\sigma_{13}=\cos(\theta_6)\,\sigma_{30}+\sin(\theta_6)\,\sigma_{36}$

$\sigma_{14}=\sigma_{31}+\cos(\theta_2)\,\sin(\theta_5)\,\sin(\theta_6)+\cos(\theta_3)\,\cos(\theta_5)\,\sin(\theta_2)\,\sin(\theta_6)$

$\sigma_{15}=\sin(\theta_6)\,\sigma_{32}-\cos(\theta_6)\,\sigma_{33}$

$\sigma_{16}=\cos(\theta_6)\,\sigma_{32}+\sin(\theta_6)\,\sigma_{33}$

$\sigma_{17}=\cos(\theta_2)\,\cos(\theta_6)\,\sin(\theta_5)-\sigma_{34}+\cos(\theta_3)\,\cos(\theta_5)\,\cos(\theta_6)\,\sin(\theta_2)$

$\sigma_{18}=\sin(\theta_6)\,\sigma_{35}-\cos(\theta_6)\,\sigma_{36}$

$\sigma_{19}=\cos(\theta_6)\,\sigma_{35}+\sin(\theta_6)\,\sigma_{36}$

$\sigma_{20}=\sigma_{37}+\cos(\theta_1)\,\sin(\theta_3)\,\sin(\theta_5)+\cos(\theta_2)\,\cos(\theta_3)\,\sin(\theta_1)\,\sin(\theta_5)$

$\sigma_{21}=\sigma_{38}-\sin(\theta_1)\,\sin(\theta_3)\,\sin(\theta_5)+\cos(\theta_1)\,\cos(\theta_2)\,\cos(\theta_3)\,\sin(\theta_5)$

$\sigma_{22}=\sin(\theta_5)\,\sigma_{39}+\sigma_{37}$

$\sigma_{23}=\sin(\theta_5)\,\sigma_{41}-\sigma_{38}$

$\sigma_{24}=\cos(\overline{\theta_1})\,\cos(\overline{\theta_3})-\cos(\overline{\theta_2})\,\sin(\overline{\theta_1})\,\sin(\overline{\theta_3})$

$\sigma_{25}=\cos(\overline{\theta_3})\,\sin(\overline{\theta_1})+\cos(\overline{\theta_1})\,\cos(\overline{\theta_2})\,\sin(\overline{\theta_3})$

$\sigma_{26}=\cos(\overline{\theta_2})\,\cos(\overline{\theta_5})-\cos(\overline{\theta_3})\,\sin(\overline{\theta_2})\,\sin(\overline{\theta_5})$

$\sigma_{27}=\sin(\overline{\theta_5})\,\left(\sin(\overline{\theta_1})\,\sin(\overline{\theta_3})-\cos(\overline{\theta_1})\,\cos(\overline{\theta_2})\,\cos(\overline{\theta_3})\right)-\cos(\overline{\theta_1})\,\cos(\overline{\theta_5})\,\sin(\overline{\theta_2})$

$\sigma_{28}=\sin(\theta_5)\,\left(\cos(\theta_1)\,\sin(\theta_3)+\cos(\theta_2)\,\cos(\theta_3)\,\sin(\theta_1)\right)+\cos(\theta_5)\,\sin(\theta_1)\,\sin(\theta_2)$

$\sigma_{29}=\cos(\theta_1)\,\cos(\theta_5)\,\sin(\theta_3)-\sigma_{40}+\cos(\theta_2)\,\cos(\theta_3)\,\cos(\theta_5)\,\sin(\theta_1)$

$\sigma_{30}=\sigma_{42}+\cos(\theta_5)\,\sin(\theta_1)\,\sin(\theta_3)-\cos(\theta_1)\,\cos(\theta_2)\,\cos(\theta_3)\,\cos(\theta_5)$

$\sigma_{31}=\cos(\theta_6)\,\sin(\theta_2)\,\sin(\theta_3)$

$\sigma_{32}=\cos(\theta_5)\,\sigma_{39}-\sigma_{40}$

$\sigma_{33}=\cos(\theta_1)\,\cos(\theta_3)-\cos(\theta_2)\,\sin(\theta_1)\,\sin(\theta_3)$

$\sigma_{34}=\sin(\theta_2)\,\sin(\theta_3)\,\sin(\theta_6)$

$\sigma_{35}=\cos(\theta_5)\,\sigma_{41}+\sigma_{42}$

$\sigma_{36}=\cos(\theta_3)\,\sin(\theta_1)+\cos(\theta_1)\,\cos(\theta_2)\,\sin(\theta_3)$

$\sigma_{37}=\cos(\theta_5)\,\sin(\theta_1)\,\sin(\theta_2)$

$\sigma_{38}=\cos(\theta_1)\,\cos(\theta_5)\,\sin(\theta_2)$

$\sigma_{39}=\cos(\theta_1)\,\sin(\theta_3)+\cos(\theta_2)\,\cos(\theta_3)\,\sin(\theta_1)$

$\sigma_{40}=\sin(\theta_1)\,\sin(\theta_2)\,\sin(\theta_5)$

$\sigma_{41}=\sin(\theta_1)\,\sin(\theta_3)-\cos(\theta_1)\,\cos(\theta_2)\,\cos(\theta_3)$

$\sigma_{42}=\cos(\theta_1)\,\sin(\theta_2)\,\sin(\theta_5)$

for the Jacobian in the tool frame, the above is the linear part .

the angular part is:

$$\begin{pmatrix} \sigma_{11} & -\sin\left(\overline{\theta_1}\right)\sigma_{10}-\cos\left(\overline{\theta_1}\right)\sigma_8 & \cos\left(\overline{\theta_2}\right)\sigma_{11}+\cos\left(\overline{\theta_1}\right)\sin\left(\overline{\theta_2}\right)\sigma_{10}-\sin\left(\overline{\theta_1}\right)\sin\left(\overline{\theta_2}\right)\sigma_8 & 0 & \sin\left(\overline{\theta_2}\right)\sin\left(\overline{\theta_3}\right)\sigma_{11}-\sigma_4\sigma_8-\sigma_{10}\sigma_5 & \sigma_{11}\sigma_3-\sigma_8\sigma_1-\sigma_{10}\sigma_2 \\ -\sin(\theta_6)\sigma_{19}-\sigma_{20} & \sin\left(\overline{\theta_1}\right)\sigma_9+\cos\left(\overline{\theta_1}\right)\sigma_7 & \sin\left(\overline{\theta_1}\right)\sin\left(\overline{\theta_2}\right)\sigma_7-\cos\left(\overline{\theta_1}\right)\sin\left(\overline{\theta_2}\right)\sigma_9-\cos\left(\overline{\theta_2}\right)\sigma_{14} & 0 & \sigma_9\sigma_5+\sigma_4\sigma_7-\sin\left(\overline{\theta_2}\right)\sin\left(\overline{\theta_3}\right)\sigma_{14} & \sigma_9\sigma_2+\sigma_7\sigma_1-\sigma_{14}\sigma_3 \\ \sigma_6 & \cos\left(\overline{\theta_1}\right)\sigma_{12}+\sin\left(\overline{\theta_1}\right)\sigma_{13} & \cos\left(\overline{\theta_2}\right)\sigma_6+\sin\left(\overline{\theta_1}\right)\sin\left(\overline{\theta_2}\right)\sigma_{12}-\cos\left(\overline{\theta_1}\right)\sin\left(\overline{\theta_2}\right)\sigma_{13} & 0 & \sigma_{13}\sigma_5+\sigma_{12}\sigma_4+\sin\left(\overline{\theta_2}\right)\sin\left(\overline{\theta_3}\right)\sigma_6 & \sigma_{13}\sigma_2+\sigma_{12}\sigma_1+\sigma_3\sigma_6 \end{pmatrix}$$

where

$$\sigma_1 = \sin\left(\overline{\theta_5}\right)\left(\cos\left(\overline{\theta_1}\right)\sin\left(\overline{\theta_3}\right)+\cos\left(\overline{\theta_2}\right)\cos\left(\overline{\theta_3}\right)\sin\left(\overline{\theta_1}\right)\right)+\cos\left(\overline{\theta_5}\right)\sin\left(\overline{\theta_1}\right)\sin\left(\overline{\theta_2}\right)$$

$$\sigma_2 = \sin\left(\overline{\theta_5}\right)\left(\sin\left(\overline{\theta_1}\right)\sin\left(\overline{\theta_3}\right)-\cos\left(\overline{\theta_1}\right)\cos\left(\overline{\theta_2}\right)\cos\left(\overline{\theta_3}\right)\right)-\cos\left(\overline{\theta_1}\right)\cos\left(\overline{\theta_5}\right)\sin\left(\overline{\theta_2}\right)$$

$$\sigma_3 = \cos\left(\overline{\theta_2}\right)\cos\left(\overline{\theta_5}\right)-\cos\left(\overline{\theta_3}\right)\sin\left(\overline{\theta_2}\right)\sin\left(\overline{\theta_5}\right)$$

$$\sigma_4 = \cos\left(\overline{\theta_1}\right)\cos\left(\overline{\theta_3}\right)-\cos\left(\overline{\theta_2}\right)\sin\left(\overline{\theta_1}\right)\sin\left(\overline{\theta_3}\right)$$

$$\sigma_5 = \cos\left(\overline{\theta_3}\right)\sin\left(\overline{\theta_1}\right)+\cos\left(\overline{\theta_1}\right)\cos\left(\overline{\theta_2}\right)\sin\left(\overline{\theta_3}\right)$$

$$\sigma_6 = \cos(\theta_2)\cos(\theta_5)-\cos(\theta_3)\sin(\theta_2)\sin(\theta_5)$$

$$\sigma_7 = \sin(\theta_6)\sigma_{15}-\cos(\theta_6)\sigma_{16}$$

$$\sigma_8 = \cos(\theta_6)\sigma_{15}+\sin(\theta_6)\sigma_{16}$$

$$\sigma_9 = \sin(\theta_6)\sigma_{17}-\cos(\theta_6)\sigma_{18}$$

$$\sigma_{10} = \cos(\theta_6)\sigma_{17}+\sin(\theta_6)\sigma_{18}$$

$$\sigma_{11} = \cos(\theta_6)\sigma_{19}-\sin(\theta_2)\sin(\theta_3)\sin(\theta_6)$$

$$\sigma_{12} = \sin(\theta_5)\sigma_{21}+\cos(\theta_5)\sin(\theta_1)\sin(\theta_2)$$

$$\sigma_{13} = \sin(\theta_5)\sigma_{22}-\cos(\theta_1)\cos(\theta_5)\sin(\theta_2)$$

$$\sigma_{14} = \sin(\theta_6)\sigma_{19}+\sigma_{20}$$

$$\sigma_{15} = \cos(\theta_5)\sigma_{21}-\sin(\theta_1)\sin(\theta_2)\sin(\theta_5)$$

$$\sigma_{16} = \cos(\theta_1)\cos(\theta_3)-\cos(\theta_2)\sin(\theta_1)\sin(\theta_3)$$

$$\sigma_{17} = \cos(\theta_5)\sigma_{22}+\cos(\theta_1)\sin(\theta_2)\sin(\theta_5)$$

$$\sigma_{18} = \cos(\theta_3)\sin(\theta_1)+\cos(\theta_1)\cos(\theta_2)\sin(\theta_3)$$

$$\sigma_{19} = \cos(\theta_2)\sin(\theta_5)+\cos(\theta_3)\cos(\theta_5)\sin(\theta_2)$$

$$\sigma_{20} = \cos(\theta_6)\sin(\theta_2)\sin(\theta_3)$$

$$\sigma_{21} = \cos(\theta_1)\sin(\theta_3)+\cos(\theta_2)\cos(\theta_3)\sin(\theta_1)$$

$$\sigma_{22} = \sin(\theta_1)\sin(\theta_3)-\cos(\theta_1)\cos(\theta_2)\cos(\theta_3)$$

For the following sections assume $\theta_3, \theta_5, \theta_6 = 0$ and $l_4 = 0, l_5 = 0$.

4. Find all the singular states of the robot with respect to the task of tool position, sketch the states showing the singular direction.

Now that the values for theta3,4,5 and l4,5 are 0, we calculate the new DH table:

$$\begin{pmatrix} 0 & \frac{\pi}{2} & \overline{H} & \overline{\theta_1} \\ \overline{l_2} & -\frac{\pi}{2} & 0 & \overline{\theta_2} \\ 0 & 0 & \overline{d_4}+\overline{l_3} & 0 \end{pmatrix}$$

The new homogeneous transformation matrix

^0A_t =

$$\begin{pmatrix} \cos(\theta_1)\cos(\theta_2) & -\sin(\theta_1) & -\cos(\theta_1)\sin(\theta_2) & -\cos(\theta_1)\,\sigma_1 \\ \cos(\theta_2)\sin(\theta_1) & \cos(\theta_1) & -\sin(\theta_1)\sin(\theta_2) & -\sin(\theta_1)\,\sigma_1 \\ \sin(\theta_2) & 0 & \cos(\theta_2) & H + d_4\cos(\theta_2) + l_3\cos(\theta_2) + l_2\sin(\theta_2) \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where

$$\sigma_1 = d_4\sin(\theta_2) - l_2\cos(\theta_2) + l_3\sin(\theta_2)$$

So the new Jacobian is:

J =

$$\begin{pmatrix} \sigma_3 - \sigma_4 & -\cos(\overline{\theta_1})\,\sigma_2 & -\cos(\overline{\theta_1})\sin(\overline{\theta_2}) \\ \sigma_1 & -\sin(\overline{\theta_1})\,\sigma_2 & -\sin(\overline{\theta_1})\sin(\overline{\theta_2}) \\ 0 & \cos(\overline{\theta_1})\,\sigma_1 + \sin(\overline{\theta_1})\,(\sigma_4 - \sigma_3) & \cos(\overline{\theta_2}) \\ 0 & \sin(\overline{\theta_1}) & 0 \\ 0 & -\cos(\overline{\theta_1}) & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

where

$$\sigma_1 = \cos(\overline{\theta_1})\cos(\overline{\theta_2})\,\overline{l_2} - \cos(\overline{\theta_1})\sin(\overline{\theta_2})\,(\overline{d_4}+\overline{l_3})$$

$$\sigma_2 = \sin(\overline{\theta_2})\,\overline{l_2} + \cos(\overline{\theta_2})\,(\overline{d_4}+\overline{l_3})$$

$$\sigma_3 = \sin(\overline{\theta_1})\sin(\overline{\theta_2})\,(\overline{d_4}+\overline{l_3})$$

$$\sigma_4 = \cos(\overline{\theta_2})\sin(\overline{\theta_1})\,\overline{l_2}$$

And to calculate the singularity, we need the determinant of the linear part of the Jacobian which is equal to:

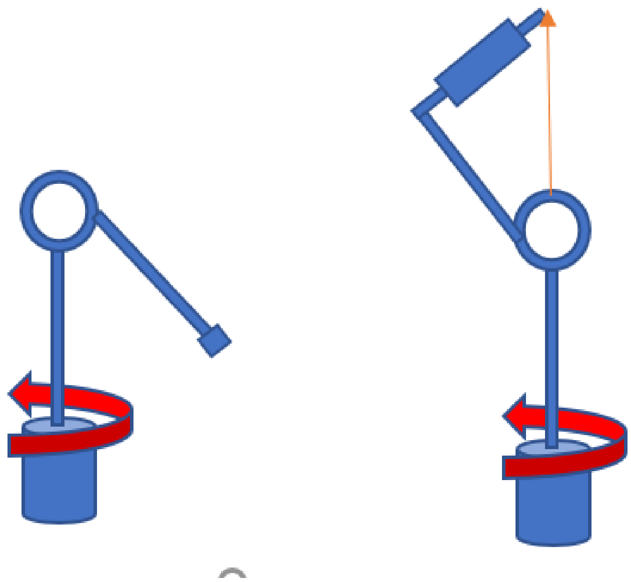
$$\det(\text{J\_linear}) = -(\overline{d_4}+\overline{l_3})\,(\sin(\overline{\theta_2})\,\overline{d_4} - \cos(\overline{\theta_2})\,\overline{l_2} + \sin(\overline{\theta_2})\,\overline{l_3})$$

we can see that singularity occurs when

$$\boldsymbol{d_4 = l_3}$$

Or when

$$\boldsymbol{\tan(\theta_2) = \frac{l_2}{d_4 + l_3}}$$



5. Find the forces and torques in the joints, considering pointed mass $\boldsymbol{M}$ held by the gripper, gravitation on $\hat{z}_0$ direction, links and joints mass are negatable.

Since we only have a mass at the end point, we have the matrix F_e

$$F_e = \begin{bmatrix} 0 \\ 0 \\ -Mg \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Then we use the relation

$$\tau = -J^T * F_e$$

$$\tau = \begin{pmatrix} 0 \\ -\overline{g}\,\overline{m}\,\left(\sin\left(\overline{\theta_2}\right)\overline{d_4} - \cos\left(\overline{\theta_2}\right)\overline{l_2} + \sin\left(\overline{\theta_2}\right)\overline{l_3}\right) \\ \cos\left(\overline{\theta_2}\right)\overline{g}\,\overline{m} \end{pmatrix}$$

6 Trajectory Planning

6.1 Constant Velocity Trajectory

$$\dot{x}(t) = C$$

$$\rightarrow \int_0^T \dot{x}(t)dt = \int_0^T C dt$$

$$x(T) - x(0) = C \cdot [T - 0]$$

$$\rightarrow C = \frac{x(T) - x(0)}{T}$$

$$\rightarrow x(t) = \frac{x(T) - x(0)}{T} \cdot t + x(0)$$

$$\implies [x, \dot{x}, \ddot{x}] = \left[\frac{x(T) - x(0)}{T} \cdot t + x(0), \frac{x(T) - x(0)}{T}, 0\right]$$

6.2 Trapezoidal Velocity Trajectory

There are 3 domains to the piece-wise velocity function:

$$\left[(0 < t < \frac{T}{6}), (\frac{T}{6} < t < \frac{5T}{6}), (\frac{5T}{6} < t < T)\right]$$

Domain 1 ($0 < t < \frac{T}{6}$):

$$\ddot{x}_1 = a$$

$$\dot{x}_1 = at + C_1 \rightarrow \dot{x}_1 = 0 \rightarrow C_1 = 0$$

$$\rightarrow x_1 = \frac{1}{2}at^2 + C_2 \rightarrow x_1(t = 0) = x(0) = C_2$$

$$\rightarrow x_1 = \frac{1}{2}at^2 + x(0)$$

Domain 2 ($\frac{T}{6} < t < \frac{5T}{6}$):

$$\ddot{x}_2 = 0$$

$$\dot{x}_2 = C_1 \rightarrow \dot{x}_2(t = \frac{T}{6}) = a \cdot \frac{T}{6} = C_1$$

$$x_2 = a\frac{T}{6} \cdot t + C_2 \rightarrow x_2(t = \frac{T}{6}) = \frac{1}{2}a\frac{T^2}{36} + x(0) = a\frac{T^2}{72} + C_2 \rightarrow C_2 = x(0) - a\frac{T^2}{72}$$

$$\rightarrow x_2 = a\frac{T}{6} \cdot t - a\frac{T^2}{72} + x(0)$$

Domain 3 ($\frac{5T}{6} < t < T$):

$$\ddot{x}_3 = -a$$

$$\rightarrow \dot{x}_3 = -at + C_1 \rightarrow \dot{x}_3(t = T) = 0 \rightarrow C_1 = a \cdot T$$

$$\rightarrow \dot{x}_3 = -at + a \cdot T$$

$$\rightarrow x_3 = -\frac{1}{2}a \cdot t^2 + aT \cdot t + C_2$$

$$\rightarrow x(t = \frac{5T}{6}) = \frac{1}{8}aT^2 + x(0) = -\frac{1}{2}a \cdot \frac{5T}{6}^2 + aT \cdot \frac{5T}{6} + C_2$$

$$\rightarrow C_2 = x(0) - \frac{13}{36}aT^2$$

$$\rightarrow x_3 = -\frac{1}{2}a \cdot t^2 + aT \cdot t - \frac{13}{36}aT^2 + x(0)$$

Condition on acceleration:

$$x_3(t = T) = \frac{5}{36}aT^2 + x(0) = x(T)$$

$$\implies a = (x(T) - x(0)) \cdot \frac{36}{5T^2}$$

$$\Longrightarrow [x, \dot{x}, \ddot{x}]_{domain1} = \left[\frac{1}{2}at^2 + x(0), at, a\right]$$

$$\Longrightarrow [x, \dot{x}, \ddot{x}]_{domain2} = \left[a\frac{T}{6} \cdot t - a\frac{T^2}{72} + x(0), a \cdot \frac{T}{6}, 0\right]$$

$$\Longrightarrow [x, \dot{x}, \ddot{x}]_{domain3} = \left[-\frac{1}{2}a \cdot t^2 + aT \cdot t - \frac{13}{36}aT^2 + x(0), -at + a \cdot T, -a\right]$$

6.3 Polynomial Velocity

Constraints:

$$x(t=0) = x(0), x(t=T) = x(T)$$

$$\dot{x}(t=0) = 0, \dot{x}(t=T) = 0$$

$$\ddot{x}(t=0) = 0, \ddot{x}(t=T) = 0$$

6 constraints $\Longrightarrow 5_{th}$ order polynomial for the trajectory:

$$x(t) = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0$$

$$\dot{x}(t) = 5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1$$

$$\ddot{x}(t) = 20a_4t^4 + 12a_4t^2 + 6a_3t + 2a_2$$

from plugging in the constraints, we discover $a_0, a_1, a_2 = 0$, and are left with the following system of equations:

$$\begin{bmatrix} 20T^2 & 12T & 6 \\ 5T^2 & 4T & 3 \\ T^5 & T^4 & T^3 \end{bmatrix} \cdot \begin{bmatrix} a_5 \\ a_4 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ x(T) - x(0) \end{bmatrix}$$

Using symbolic MATLAB, we find:

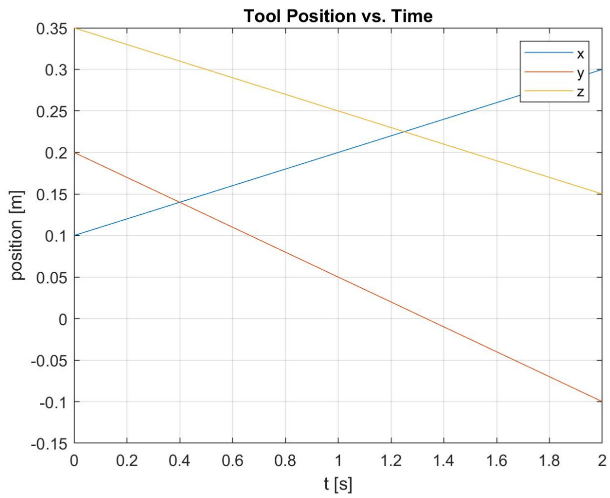
$$\begin{bmatrix} a_5 \\ a_4 \\ a_3 \end{bmatrix} = \begin{bmatrix} 6\frac{x(T)-x(0)}{T^5} \\ -15\frac{x(T)-x(0)}{T^4} \\ 10\frac{x(T)-x(0)}{T^3} \end{bmatrix}$$

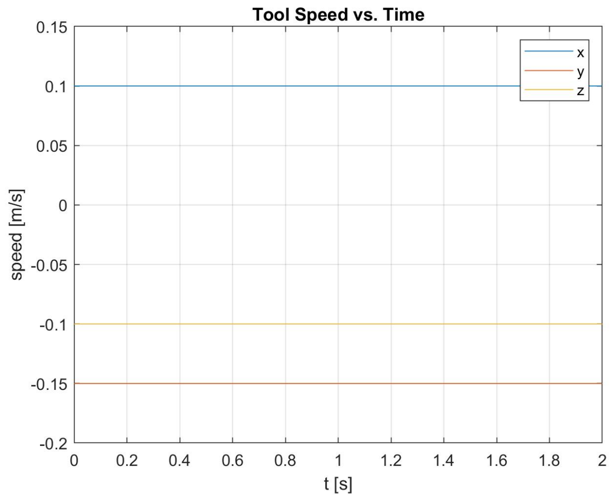
$$\Longrightarrow [x, \dot{x}, \ddot{x}] = \left[a_5t^5 + a_4t^4 + a_3t^3, 5a_5t^4 + 4a_4t^3 + 3a_3t^2, 20a_4t^4 + 12a_4t^2 + 6a_3t\right]$$

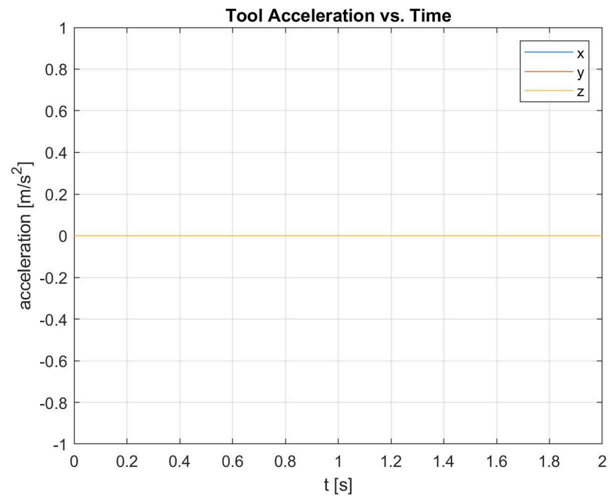
6.4 Tool Trajectory Results

The following graphs display tool position, velocity, and acceleration relative to the world frame.

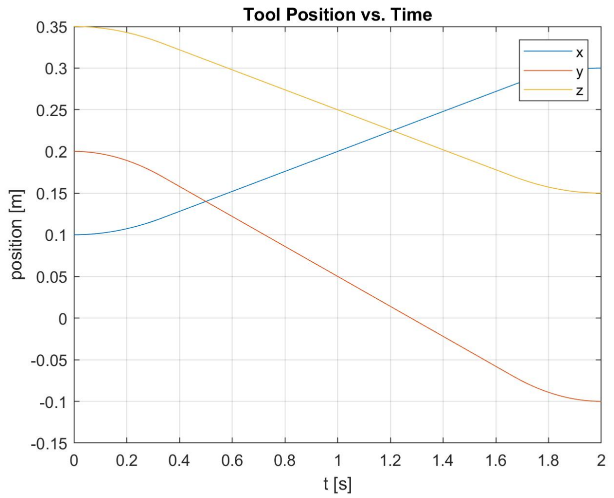
6.4.1 Constant Velocity Profile

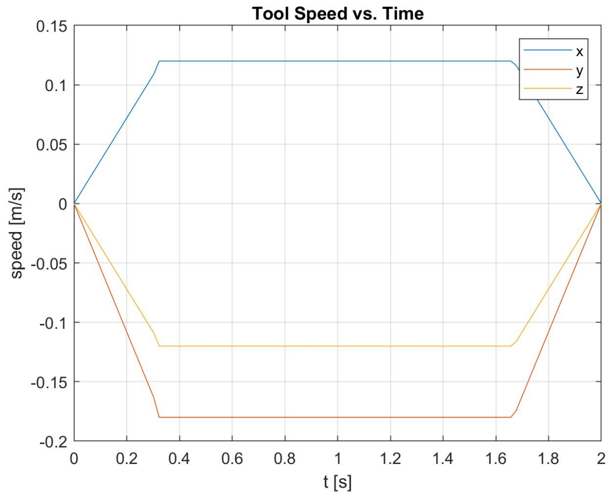


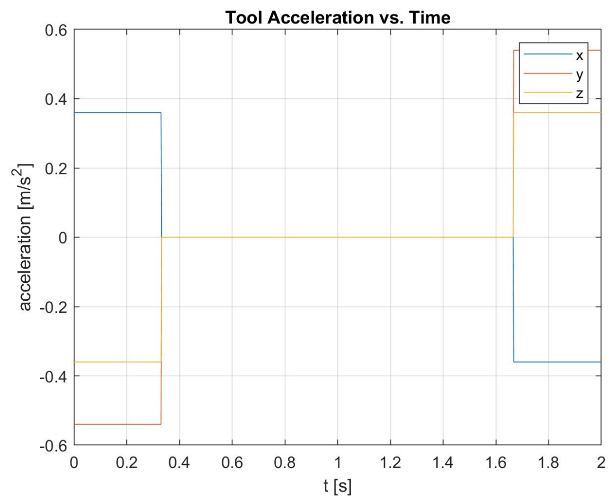




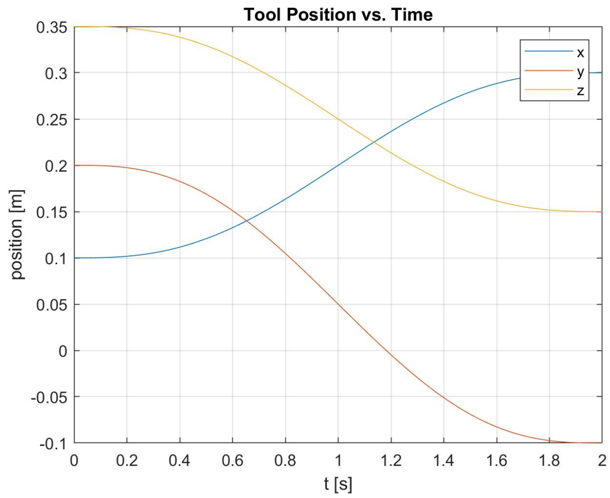
6.4.2 Trapezoidal Velocity Profile

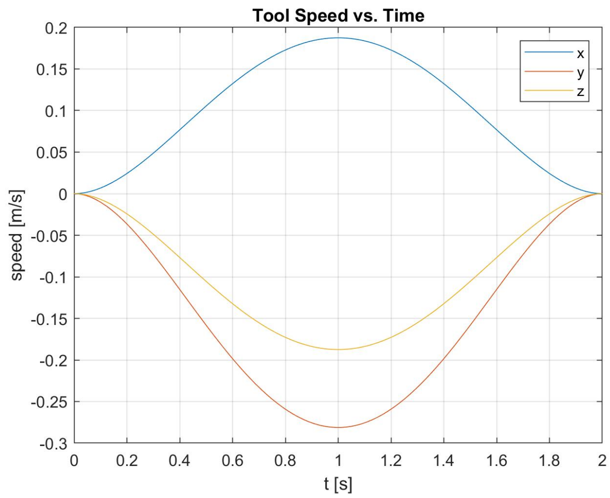


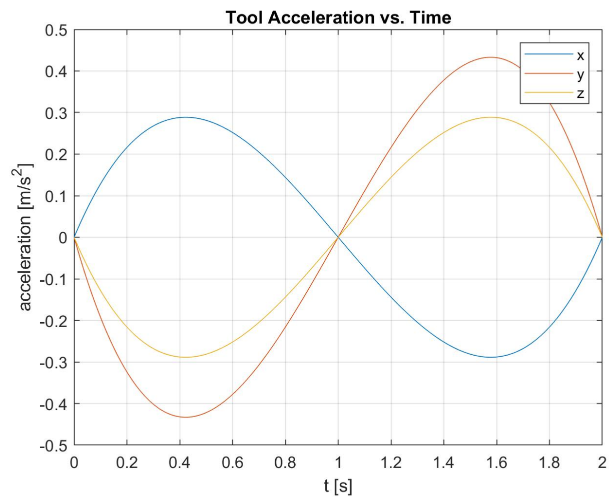




6.4.3 Polynomial Velocity Profile



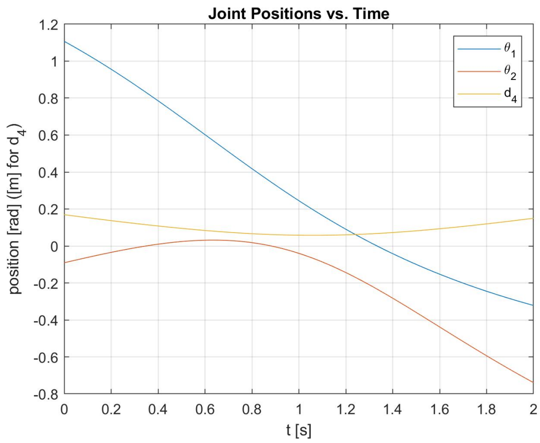


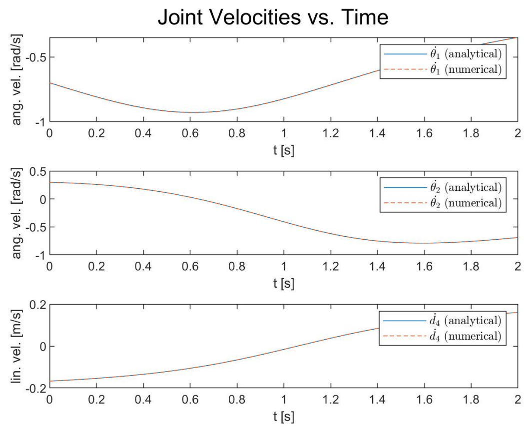


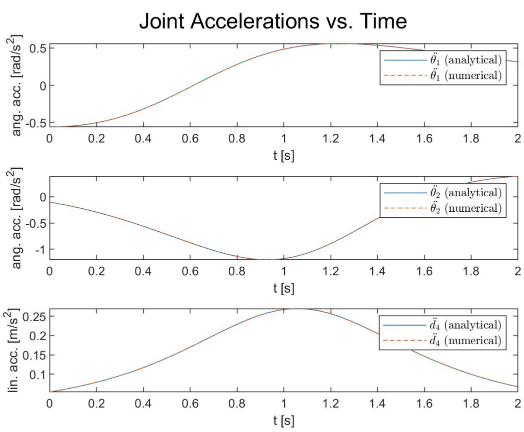
6.5 Joint Trajectory Results

The following graphs of joint position prove that we remain within the mechanical limitations, and therefore this specific trajectory is achievable. Furthermore, we see that the analytical and numerical solutions for joint velocity and acceleration are almost identical, except for a few points of discontinuity which occur in the numerically derived joint velocity and acceleration curves of the Trapezoidal trajectory. This is expected since the gradient function makes use of 2 points next to each other in an array, and therefore if there was a piece-wise jump in the graph, the function will not return an accurate value of the instantaneous derivative.

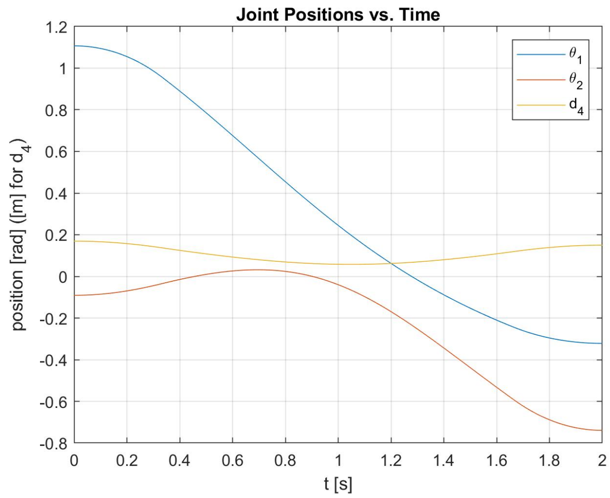
6.5.1 Constant Velocity Profile

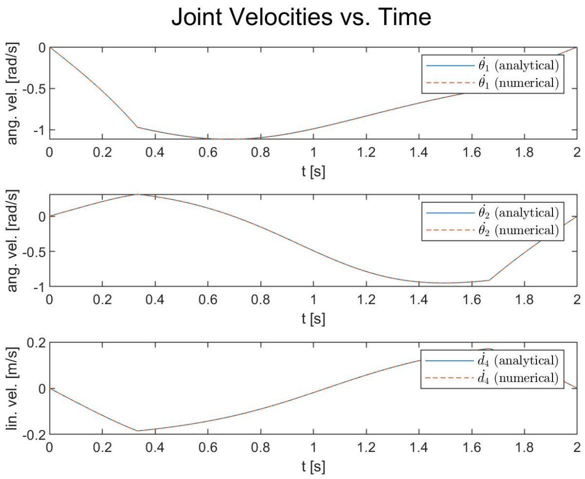


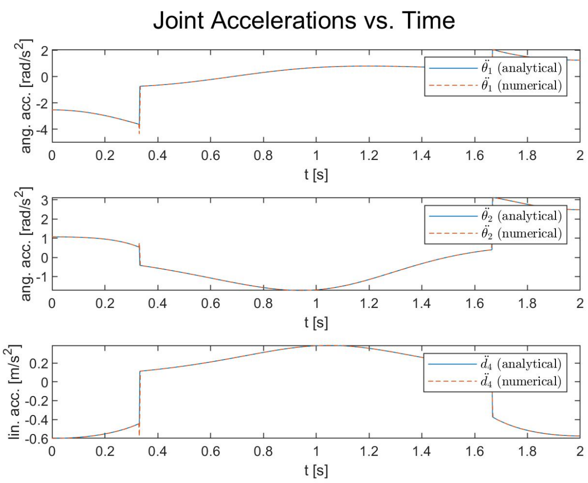




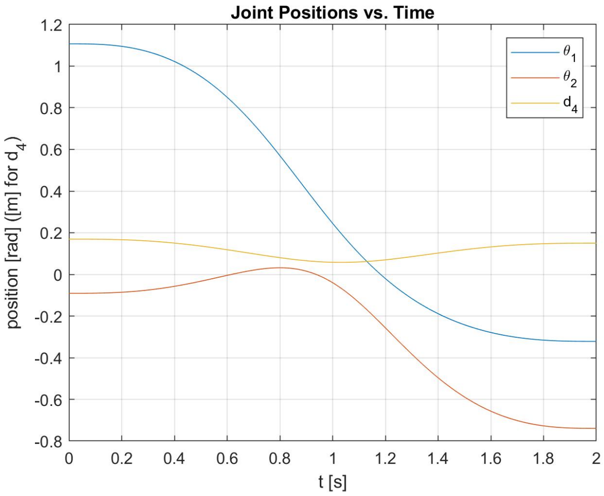
6.5.2 Trapezoidal Velocity Profile

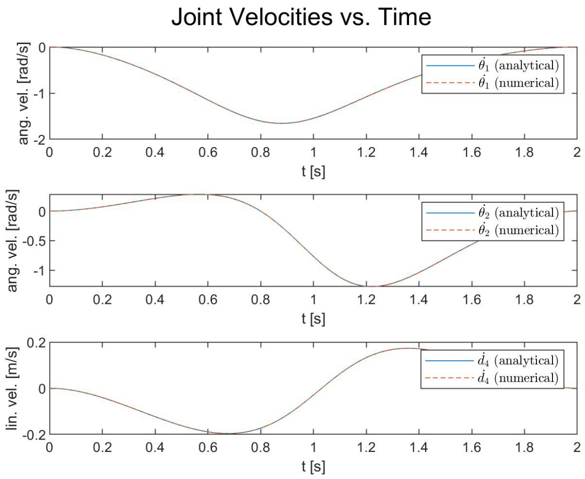


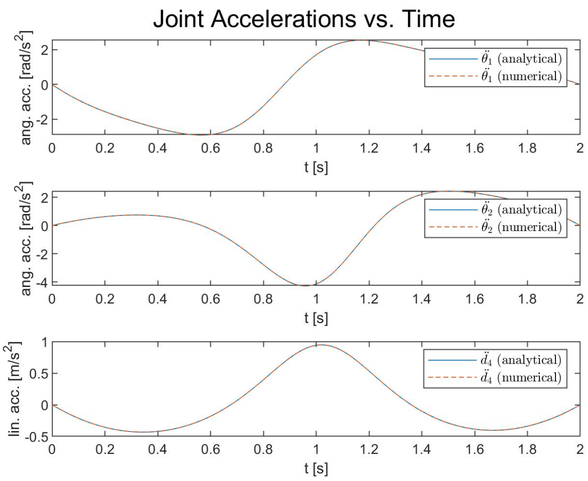




6.5.3 Polynomial Velocity Profile







```matlab
function [x]=forward_kin(q)

%   [x] = forward_kin(q) returns the position of end point given
%       joint values q: [theta1, theta2, d3, theta4, theta5, theta6]
%

 global H l2 l3 l4 l5;
theta1=q(1);    theta2=q(2);    theta3=q(3);
    d4=q(4);    theta5=q(5);    theta6=q(6);

 a = [0 l2 0 0 0 0];
alpha = [sym(pi)/2 -sym(pi)/2 0 sym(pi)/2 -sym(pi)/2 0];
d = [H 0 l3 d4 0 l4+l5];
theta = [theta1 theta2 theta3 0 theta5 theta6];
dhparams = [a' alpha' d' theta']

    for i=1:6
    A(:,:,i)=[
        cos(theta(i)), -sin(theta(i))*cos(alpha(i)), sin(theta(i))*sin(alpha(i)),
a(i)*cos(theta(i));
        sin(theta(i)), cos(theta(i))*cos(alpha(i)), -cos(theta(i))*sin(alpha(i)),
a(i)*sin(theta(i));
        0, sin(alpha(i)), cos(alpha(i)), d(i);
        0 0 0 1
        ];
end
alpha(1)
sum = A(:,:,1);
for i=2:6
    sum = sum*A(:,:,i);
end

x = sum(:,4);
x(4) = [];

end
```

```
function [q]=inverse_kin(xi,elbows, params)
% x - position vector of the tool (3x1)
% elbows – decision values vector for the different solutions, (matrix of 1x3,
first value for joint 2 and second for d4 third for joint1)
% 1 for elbow up, -1 for elbow down.
% T - matrix of 4x4, transformation matrix.
%   joint values q: [theta1, theta2,  d4]

H = params(1);
l2 = params(2);
l3 = params(3);

q = zeros(3,1);

if elbows(3)==1
q(1)=atan2(xi(2),xi(1));
c1=cos(q(1));
s1=sin(q(1));

if elbows(2)==1
    q(3)=sqrt(xi(1)^2+xi(2)^2+(xi(3)-H)^2-l2^2)-l3;

 if elbows(1)==1
     % positive d3 and elbow up for theta 2
    c2=((xi(1)/c1)*l2-(q(3)+l3)*(H-xi(3)))/(l2^2+(q(3)+l3)^2);
    s2=(xi(3)-H-q(3)*c2-l3*c2)/l2;
    q(2)=atan2(s2,c2);
 else
      % positive d3 and elbow down for theta 2
    c2=((xi(1)/c1)*l2-(q(3)+l3)*(H-xi(3)))/(l2^2+(q(3)+l3)^2);
    s2=(xi(3)-H-q(3)*c2-l3*c2)/l2;;
    q(2)=atan2(s2,c2);
 end
end

if elbows(2)==-1
   if elbows(1)==1
     % negative d3 and elbow up for theta 2
    c2=((xi(1)/c1)*l2-(q(3)+l3)*(H-xi(3)))/(l2^2+(q(3)+l3)^2);
    s2=(xi(3)-H-q(3)*c2-l3*c2)/l2;;
    q(2)=atan2(s2,c2);
   else
      % negative d3 and elbow down for theta 2
    c2=((xi(1)/c1)*l2-(q(3)+l3)*(H-xi(3)))/(l2^2+(q(3)+l3)^2);
    s2=(xi(3)-H-q(3)*c2-l3*c2)/l2;;
    q(2)=atan2(s2,c2);
   end
end

end
if elbows(3)==-1
q(1)=atan2(-xi(2),-xi(1));
c1=cos(q(1));
s1=sin(q(1));

if elbows(2)==1
    q(3)=sqrt(xi(1)^2+xi(2)^2+(xi(3)-H)^2-l2^2)-l3;
```

```
 if elbows(1)==1
     % positive d3 and elbow up for theta 2
    c2=((xi(1)/c1)*l2-(q(3)+l3)*(H-xi(3)))/(l2^2+(q(3)+l3)^2);
    s2=(xi(3)-H-q(3)*c2-l3*c2)/l2;
    q(2)=atan2(s2,c2);
 else
      % positive d3 and elbow down for theta 2
    c2=((xi(1)/c1)*l2-(q(3)+l3)*(H-xi(3)))/(l2^2+(q(3)+l3)^2);
    s2=(xi(3)-H-q(3)*c2-l3*c2)/l2;;
    q(2)=atan2(s2,c2);
 end
end

if elbows(2)==-1
   if elbows(1)==1
     % negative d3 and elbow up for theta 2
    c2=((xi(1)/c1)*l2-(q(3)+l3)*(H-xi(3)))/(l2^2+(q(3)+l3)^2);
    s2=(xi(3)-H-q(3)*c2-l3*c2)/l2;;
    q(2)=atan2(s2,c2);
   else
      % negative d3 and elbow down for theta 2
    c2=((xi(1)/c1)*l2-(q(3)+l3)*(H-xi(3)))/(l2^2+(q(3)+l3)^2);
    s2=(xi(3)-H-q(3)*c2-l3*c2)/l2;;
    q(2)=atan2(s2,c2);
   end
end

end
end
```

```
function [J, JL] = jacobian_mat_simplify(qi,params)
% jacobian_mat computes jacobian matrix of the robot given configuration q
%q - 3x1 vector of current joint values
% params=[H,l2,l3]
l2 = params(2);
l3 = params(3);

c1 = cos(qi(1));
s1 = sin(qi(1));
c2 = cos(qi(2));
s2 = sin(qi(2));
d4=qi(3);
sig1=c1*c2*l2-c1*s2*(d4+l3);
sig2=s2*l2+c2*(d4+l3);
sig3=s1*s2*(d4+l3);
sig4=c2*s1*l2;
J =     [sig3-sig4 , -(c1)*sig2 , -c1*s2 ;
        sig1 , -(s1)*sig2 , -s1*s2 ;
        0 , c1*sig1+s1*(sig4-sig3) , c2 ;
        0 , s1 , 0  ;
        0 , -c1 , 0;
        1 , 0 , 0 ];

JL = J(1:3,:);
end
```

```matlab
function [J_dot, JL_dot]=jacobian_mat_dot(qi,q_doti,params)
% J_dot - the jacobian's time derivative.
% q_dot - the joint parameters' time derivative.
% joint values q: [theta1, theta2,  d4]
% params:[H,l2,l3]
l2 = params(2);
l3 = params(3);

c1 = cos(qi(1));
s1 = sin(qi(1));
c2 = cos(qi(2));
s2 = sin(qi(2));
d4 = qi(3);
J_dot = zeros(6,3);
dJdq = zeros(6,3,3);

dJdq(:,:,1)=[c1*s2*(d4+l3)-c1*c2*l2, s1*(s2*l2+c2*(d4+l3)), s1*s2;
            -s1*c2*l2+s1*s2*(d4+l3), -c1*(s2*l2+c2*(d4+l3)), -c1*s2;
            0                     , -s1*(c1*c2*l2-c1*s2*(d4+l3))+c1*(-
s1*c2*l2+s1*s2*(d4+l3))+c1*(c2*s1*l2-s1*s2*(d4+l3))+s1*(c1*c2*l2-c1*s2*(d4+l3))
,0;
            0                     , c1                       ,0;
            0                     , s1                       ,0;
            0                     ,  0                       ,0
    ];
dJdq(:,:,2)=[ s1*c2*(d4+l3)+s2*s1*l2, -c1*(c2*l2-s2*(d4+l3)), -c1*c2;
             -c1*s2*l2-c1*c2*(d4+l3), -s1*(c2*l2-s2*(d4+l3)), -s1*c2;
             0                      ,c1*(-c1*s2*l2-c1*c2*(d4+l3))-
s1*(s1*c2*(d4+l3)+s2*s1*l2), -s2;
             0                      ,0                       ,0;
             0                      ,0                       ,0;
             0                      ,0                       ,0
    ];
dJdq(:,:,3)=[s1*s2, -c1*c2, 0;
             -c1*s2,-s1*c2, 0;
             0,-c1*c1*s2-s1*s1*s2,0;
             0,0,0;
             0,0,0;
             0,0,0
    ];

for i =1:6
    for j=1:3
        for k = 1:3
            J_dot(i,j) = J_dot(i,j) + dJdq(i,j,k)*q_doti(k);
        end
    end
end

JL_dot = J_dot(1:3,:);

end
```

```matlab
function [x] = x_plan(prof,T,n,x0,xf)
%General Description:
    %This function returns a matrix, with each column representing the
    %position vector of the tool in world frame at a particular time t_i

%Parameters:
    %prof: which trajectory profile the user wishes to implement
    %[NOTE: prof accepts 'constant', 'trapezoidal', and 'polynomial']
    %T: time taken to move from x0 to xf
    %n: number of position vectors recorded in T seconds
    %x0: initial position of tool (3x1 vector)
    %xf: final position of tool (3x1 vector)

x = zeros(3,n);
t = linspace(0, T, n);

switch prof

    case 'constant'
        for i=1:n
            x(:,i) = x0 + ((xf-x0)./T).*t(i);
        end

    case 'trapezoidal'
        a = (xf-x0).*(36/(5*T^2));

        for i=1:floor(n/6)
            x(:,i) = ((1/2)*(t(i)^2)).*a + x0;
        end
        for i=floor(n/6)+1:floor(5*n/6)
            x(:,i) = ((T/6)*t(i)-(T^2)/72).*a + x0;
        end
        for i=floor(5*n/6)+1:n
            x(:,i) = (-(1/2)*t(i)^2+T*t(i)-(13/36)*T^2).*a + x0;
        end

    case 'polynomial'
        c5 = (6/T^5).*(xf-x0);
        c4 = (-15/T^4).*(xf-x0);
        c3 = (10/T^3).*(xf-x0);

        for i=1:n
            x(:,i) = c5.*t(i)^5+c4.*t(i)^4+c3.*t(i)^3+x0;
        end
end

end
```

```matlab
function [v] = v_plan(prof,T,n,x0,xf)
%General Description:
    %This function returns a matrix, with each column representing a the
    %velocity vector of the tool in world frame at a particular time t_i

%Parameters:
    %prof: which trajectory profile the user wishes to implement
    %[NOTE: prof accepts 'constant', 'trapezoidal', and 'polynomial']
    %T: time taken to move from x0 to xf
    %n: number of velocity vectors recorded in T seconds
    %x0: initial position of tool (3x1 vector)
    %xf: final position of tool (3x1 vector)

v = zeros(3,n);
t = linspace(0, T, n);

switch prof

    case 'constant'
        for i=1:n
            v(:,i) = (xf-x0)./T;
        end

    case 'trapezoidal'
        a = (xf-x0).*(36/(5*T^2));

        for i=1:floor(n/6)
            v(:,i) = t(i).*a;
        end
        for i=floor(n/6)+1:floor(5*n/6)
            v(:,i) = (T/6).*a;
        end
        for i=floor(5*n/6)+1:n
            v(:,i) = (T-t(i)).*a;
        end

    case 'polynomial'
        c5 = (6/T^5).*(xf-x0);
        c4 = (-15/T^4).*(xf-x0);
        c3 = (10/T^3).*(xf-x0);

        for i=1:n
            v(:,i) = 5.*c5.*t(i)^4+4.*c4.*t(i)^3+3.*c3.*t(i)^2;
        end

end
```

```matlab
function [a] = a_plan(prof,T,n,x0,xf)
%General Description:
    %This function returns a matrix, with each column representing a the
    %acceleration vector of the tool in world frame at a particular time t_i

%Parameters:
    %prof: which trajectory profile the user wishes to implement
    %[NOTE: prof accepts 'constant', 'trapezoidal', and 'polynomial']
    %T: time taken to move from x0 to xf
    %n: number of acceleration vectors recorded in T seconds
    %x0: initial position of tool (3x1 vector)
    %xf: final position of tool (3x1 vector)

a = zeros(3,n);
t = linspace(0, T, n);

switch prof

    case 'constant'
        for i=1:n
            a(:,i) = 0;
        end

    case 'trapezoidal'
        a1 = (xf-x0).*(36/(5*T^2));

        for i=1:floor(n/6)
            a(:,i) = a1;
        end
        for i=floor(n/6)+1:floor(5*n/6)
            a(:,i) = 0;
        end
        for i=floor(5*n/6)+1:n
            a(:,i) = -a1;
        end

    case 'polynomial'
        c5 = (6/T^5).*(xf-x0);
        c4 = (-15/T^4).*(xf-x0);
        c3 = (10/T^3).*(xf-x0);

        for i=1:n
            a(:,i) = 20.*c5.*t(i)^3+12.*c4.*t(i)^2+6.*c3.*t(i);
        end

end
```

```matlab
function [q] = q_plan(x, elbows, params)

% General Description:
%     This function returns a matrix with each column representing the
%     joint position vector at a particular time t_i

% Parameters:
%     x: matrix of preplanned position vectors, forming a trajectory
%     elbows: 1x2 matrix which decides
%     params: [H,l2,l3]

[~,colnum]=size(x);
q = zeros(3,colnum);

for i=1:colnum
    q(:,i) = inverse_kin(x(:,i), elbows, params);
end




function [q_dot] = q_dot_plan(q, v, T, method, params)

% General Description:
%   This function returns a matrix with each column representing a
%   vector of the derivatives of the joint positions with respect to time
%   at a particular time t_i

% Parameters:
% q - matrix of joint position vectors (found previously using q_plan)
% v - matrix of velocity vectors of the tool vector over T seconds
% method - user specifies whether to calculate q_dot numerically, or using
%          analytical definition of x_dot = JL*q_dot
% T-time taken to move from x0 to xf
% params=[H,l2,l3]

[~, colnum] = size(q);
q_dot = zeros(3, colnum);
JL = zeros(3, 3, colnum);
switch method

    case 'numerical'

        for i = 1:3
            q_dot(i,:) = gradient(q(i,:), T/colnum);

        end

    case 'analytical'

        for i=1:colnum
            [~,JL(:,:,i)] = jacobian_mat_simplify(q(:,i), params);
            q_dot(:,i) = inv((JL(:,:,i)))*v(:,i);
        end


end
```

```matlab
function [q_dot2] = q_dot2_plan(q, q_dot, a, T, method, params)

% General Description:
%   This function returns a matrix with each column representing a
%   vector of the derivatives of the joint positions with respect to time
%   at a particular time t_i

% Parameters:
% q - matrix of joint position vectors (found previously using q_plan)
% q_dot - matrix of joint velocity vectors (found previously using q_dot_plan)
% a - matrix of velocity vectors of the tool vector over T seconds
% method - user specifies whether to calculate q_dot numerically, or using
%          analytical definition of x_dot2 -JL_dot*q = JL*q_dot2
% T-time taken to move from x0 to xf
% params=[H,l2,l3]

[~, colnum] = size(q);
q_dot2 = zeros(3, colnum);
J=zeros(6,3,colnum);
JL = zeros(3, 3, colnum);
JL_dot = zeros(3, 3, colnum-1);

switch method

    case 'numerical'

        for i = 1:3
            q_dot2(i,:) = gradient(q_dot(i,:), T/colnum);

        end

    case 'analytical'

        for i=1:colnum
            [~,JL(:,:,i)] = jacobian_mat_simplify(q(:,i), params);
            [~, JL_dot(:,:,i)] = jacobian_mat_dot(q(:,i),q_dot(:,i),params);
            q_dot2(:,i) = inv(JL(:,:,i))*(a(:,i)-JL_dot(:,:,i)*q_dot(:,i));
        end


end
```