MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 7 solutions - Part 1

1. The dimension is the number of elements in a basis of the space. We count how many elements are in the bases we found for each space and obtain the following dimensions:
   i. 3
   ii. 10
   iii. 6
   iv. 1
   v. 2
   vi. 2
   vii. 2

2. i. We follow the algorithm studied in class and recitation. We write the n-tuples as rows of a matrix and bring this matrix two echelon form.

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & 4 & 1 \\ 1 & 3 & -1 \end{pmatrix} \xrightarrow{R_2-2R_1, R_3-R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 0 & 2 & -3 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 0 \end{pmatrix}$$

The algorithm promises that the rows different from zero give a basis for the space. That is,

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix} \right\}$$

is a basis for the space, and its dimension is 2..

ii. We follow the algorithm studied in recitation and class, write the n-tuples as rows of a matrix and bring it to echelon form

$$\begin{pmatrix} 1 & 1 & 0 & 2 \\ 1 & -1 & 1 & 1 \\ 5 & -1 & 3 & 7 \end{pmatrix} \xrightarrow{R_2-R_1, R_3-5R_1} \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & -6 & 3 & -3 \end{pmatrix} \xrightarrow{R_3-3R_2} \begin{pmatrix} 1 & 1 & 0 & 2 \\ 0 & -2 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The rows different from zero provide a basis for the space. So,

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ -1 \end{pmatrix} \right\}$$

is a basis for the space, and its dimension is 2.

3. i. A vector $b \in \mathbb{R}^m$ is a linear combination of $v_1, ..., v_n$ if and only if there exist $x_1, ..., x_n \in \mathbb{R}$ such that

$$x_1 v_1 + ... + x_n v_n = b.$$

As we have seen in class (and several times in HW's) the left hand side can be written as a multiplication of the matrix

$$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}$$

by the vector $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$. That is, the equation above can be rewritten as

$$Ax = b.$$

So, it follows that b is a linear combination of $v_1, ..., v_n$ if and only if the equation $Ax = b$ has a solution, that is, if and only if the system $(A|b)$ has a solution (which means that $b \in L(A)$).

Remarks. Note that we proved here that $L(A)$ is equal to the column space of A (and, as we will see in upcoming lessons, this space is the same as the space $\mathrm{Im}T_A$ — the image of the linear transformation T_A.). This question was not hard to prove, but the relation we found (that these two spaces are the same space) is very important and very useful.

ii. By part (i) we have

$$L(A) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 4 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 10 \\ -1 \\ 4 \end{pmatrix} \right\}.$$

We follow the algorithm from class and recitation to find a basis for a span of vectors in $\mathbb{R}^n$. We write these vectors as rows of a matrix and bring it to echelon form.

$$\begin{pmatrix} 1 & 3 & 1 & 2 \\ 2 & -1 & 4 & 2 \\ 1 & 10 & -1 & 4 \end{pmatrix} \xrightarrow{R_2-2R_1, R_3-R_1} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -7 & 2 & -2 \\ 0 & 7 & -2 & 2 \end{pmatrix} \xrightarrow{R_3-R_2} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -7 & 2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The rows that are different from zero provide a basis for the space. So the basis we were looking for is,

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -7 \\ 2 \\ -2 \end{pmatrix} \right\}$$

4. i. No. We studied in class that the dimension of $M_{2\times 3}$ is equal to 6, so every basis in this space must have exactly 6 elements, but the set considered here contains only one element (why 1 and not 4?).

ii. We stated in class that $M_2(\mathbb{R})$ is of dimension 4 and the following is an ordered basis for this space:

$$B = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right).$$

We apply the coordinate transformation with respect to the basis B, $[-]_B : M_2(\mathbb{R}) \mapsto \mathbb{R}^4$, to the vectors involved:

$$[\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}]_B = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}$$

$$[\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}]_B = \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}$$

$$[\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}]_B = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}$$

We proved in class that the coordinate transformation preserves all linear structures (it is an isomorphism). So the set we were given is linearly independent iff the following set is linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

There were two different algorithms described in class to check if a set of n-tuples is linearly independent. The first involves writing the n-tuples as columns of a matrix and the second involves writing them as rows of a matrix. We use the second algorithm, write the 4-tuples as rows of a matrix and bring it to echelon form.

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ -1 & 1 & 1 & -1 \\ 1 & 1 & 2 & 2 \end{pmatrix} \xRightarrow{R_2+R_1, R_3-R_1} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix} \xRightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 3 & 1 & 0 \end{pmatrix} \xRightarrow{R_3+3R_2} \begin{pmatrix} 1 & 2 & 0 & 1 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & 7 & 3 \end{pmatrix}$$

The echelon form does not contain any row of zeros. By the algorithm we studied in class this implies that the set of 4-tuples we considered is linearly independent, and therefore that the original set we considered is linearly independent as well.

iii. No. We are working in the same vector space as in Q2(ii), the space $M_2(\mathbb{R})$. We continue working in this space with the same basis B and the same coordinate transformation with respect to this basis B. We first compute the coordinates of the new vector involved with respect to this basis:

$$[\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}]_B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix}$$

We proved in class that the coordinate transformation preserves all linear structures (it is an isomorphism). So the claim we were given is true iff the following claim is true:

$$\begin{pmatrix} 1 \\ 1 \\ 1 \\ -3 \end{pmatrix} \in \text{span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \right\}$$

We follow the algorithm we studied in class to check if an n-tuple is a linear combination of a set of n-tuples: We check if the corresponding linear system has a solution.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & -3 \end{array}\right) \xrightarrow{R_2-2R_1, R_4-R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 3 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -4 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & -1 & -1 \\ 0 & 0 & 1 & -4 \end{array}\right) \xrightarrow{R_3-3R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -7 & -4 \\ 0 & 0 & 1 & -4 \end{array}\right) \xrightarrow{R_4 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & -7 & -4 \end{array}\right) \xrightarrow{R_4+7R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & -32 \end{array}\right)$$

The echelon form contains a row with a 'lie' so the corresponding linear system has no solution and therefore the 4-tuple we considered was not a linear combination of the other 4-tuples. As stated above, since the coordinate transformation preserves all linear structures it follows that the matrix we considered was not a linear combination of the other matrices.

iv. Yes. We showed in Q2(ii) that the set

$$\left\{ \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \right\}$$

is linear independent, and in Q2(iii) we showed that

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}$$

does not belong to the span of this set. We proved in class Lemma 2, which stated that if V is a vector space and $v_1, .., v_n \in V$ are linearly independent then $w \in V$ is a linear combination of $v_1, .., v_n$ iff $\{v_1, .., v_n, w\}$ is linearly independent. Since in our case the vector is not a linear combination of the given linearly independent set, it follows that the set

$$\{ \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \}$$

is linearly independent. This is a set of 4 elements in a vector space of dimension 4 ($M_2(\mathbb{R})$). Since the number of elements is equal to the dimension, the theorem we studied in class assures that if this set is linearly independent then it is also a spanning set. So the set is a basis.

v. Yes. All of the vectors involved in this statement belong to the space $\mathbb{R}_3[x]$. We studied in class that this space is of dimension 4 and that $C = (1, x, x^2, x^3)$ is a basis for this space. We write the coordinates of each one of the vectors involved with respect to the basis C.

$$[2 + x^2 - 2x^3]_C = \begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix}$$

$$[1 - 2x + x^2 - x^3]_C = \begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix}$$

$$[5 + 2x + 2x^2 - 5x^3]_C = \begin{pmatrix} 5 \\ 2 \\ 2 \\ -5 \end{pmatrix}$$

$$[3 + 6x - 3x^3]_C = \begin{pmatrix} 3 \\ 6 \\ 0 \\ -3 \end{pmatrix}$$

We proved in class that the coordinate transformation preserves all linear structures (it is an isomorphism). So the claim we were given is true iff the following claim is true:

$$\begin{pmatrix} 2 \\ 0 \\ 1 \\ -2 \end{pmatrix} \in \text{span}\left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 0 \\ -3 \end{pmatrix} \right\}$$

We follow the same algorithm as in Q2(iii)

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 2 \\ -2 & 2 & 6 & 0 \\ 1 & 2 & 0 & 1 \\ -1 & -5 & -3 & -2 \end{array}\right) \xRightarrow{R_2+2R_1, R_3-R_1, R_4+R_1} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 2 \\ 0 & 12 & 12 & 4 \\ 0 & -3 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xRightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 5 & 3 & 2 \\ 0 & -3 & -3 & -1 \\ 0 & 12 & 12 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \xRightarrow{R_4+4R_3} \left(\begin{array}{ccc|c} 1 & 5 & 3 & 2 \\ 0 & -3 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The echelon form has no row with a lie so the corresponding linear system has a solution and therefore the 4-tuple we considered is a linear combination of the other 4-tuples. As stated above, since the coordinate transformation preserves all linear structures it follows that the polynomial we considered is a linear combination of the other polynomials.

vi. No. We proved in class Lemma 1 which stated that if $v_1, ..., v_n$ are vectors in a vector space V and $w \in V$ then w is a linear combination of $v_1, .., v_n$ iff span$\{v_1, .., v_n\}$=span$\{v_1, .., v_n, w\}$. Sice we showed in Q2(v) that $2 + x^2 - 2x^3$ is a linear combination of $\{1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}$. it follows that

$$\text{span}\{2 + x^2 - 2x^3, 1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\} =$$
$$= \text{span}\{1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}.$$

So the set we were given is a spanning set for the space iff

$$\{1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}.$$

is a spanning set for the space. But this last set has only 3 elements while the dimension of the space $\mathbb{R}_3[x]$ is 4. So this set cannot be a spanning set for the space, due to the result we proved in class which stated that the amount of elements in a spanning set is at least as high as the dimension of the space.

vii. The dimension is 2. Since we have seen in Q2(vi) that

$$\text{span}\{2 + x^2 - 2x^3, 1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\} =$$
$$= \text{span}\{1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\},$$

we conclude that we are looking for the dimension of

$$\mathrm{span}\{1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}.$$

We continue to work in the space $\mathbb{R}_3[x]$ with the basis C as in Q2(v). We have already computed the coordinates of each of the vectors involved with respect to this basis. We proved in class that the coordinate transformation preserves all linear structures (it is an isomorphism). So the dimension of the space we are looking at is equal to the dimension of the following space.

$$\mathrm{span}\left\{\begin{pmatrix} 1 \\ -2 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 2 \\ -5 \end{pmatrix}, \begin{pmatrix} 3 \\ 6 \\ 0 \\ -3 \end{pmatrix}\right\}$$

We follow the algorithm provided in class, write these vectors as rows of a matrix and bring it to echelon form.

$$\begin{pmatrix} 1 & -2 & 1 & -1 \\ 5 & 2 & 2 & -5 \\ 3 & 6 & 0 & -3 \end{pmatrix} \xRightarrow{R_2-5R_1, R_3-3R_1} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 12 & -3 & 0 \\ 0 & 12 & -3 & 0 \end{pmatrix} \xRightarrow{R_3-R_2} \begin{pmatrix} 1 & -2 & 1 & -1 \\ 0 & 12 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

There are two leading variables in the echelon form so the dimension of the row space of the matrix (which is exactly the space we were asked about) is 2.

viii. Yes. We showed in Q2(v) that

$$2 + x^2 - 2x^3 \in \mathrm{span}\{1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}.$$

In addition since a span of a set of vectors always contains each one of these vectors (as was shown in class), we have

$$3 + 6x - 3x^3 \in \mathrm{span}\{1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}.$$

We proved in class that if W is a subspace of a vector space V and $v_1, ..., v_n \in W$ then $\mathrm{span}\{v_1, ..., v_n\} \subseteq W$. It follows that,

$$\mathrm{span}\{2+x^2-2x^3, 3+6x-3x^3\} \subseteq \mathrm{span}\{1-2x+x^2-x^3, 5+2x+2x^2-5x^3, 3+6x-3x^3\}.$$

We proved in class that if V is a vector space and $W \subseteq V$ is a subspace of V then $W = V$ iff $\dim W = \dim V$. Since we showed in Q2(vii) that the dimension of

$$\mathrm{span}\{1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}.$$

is 2, it follows that the spaces will be equal iff the dimension of

$$\mathrm{span}\{2 + x^2 - 2x^3, 3 + 6x - 3x^3\}$$

is also 2. Since this space is spanned by the two vectors $2 + x^2 - 2x^3$ and $3+6x-3x^3$ and since these vectors are linearly independent (two vectors are linearly independent iff neither of them is a scalar multiplying the other) it follows that the dimension of

$$\operatorname{span}\{2+x^2-2x^3, 3+6x-3x^3\}$$

is 2. We conclude that indeed,

$$\operatorname{span}\{2+x^2-2x^3, 3+6x-3x^3\} = \operatorname{span}\{1-2x+x^2-x^3, 5+2x+2x^2-5x^3, 3+6x-3x^3\}.$$

ix. Each vector in the space we are considering has the form:

$$\begin{pmatrix} a-b+c & a+b+4c-d \\ -a+2b-c-2d & -a+b+c+2d \end{pmatrix} =$$

$$= a\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + b\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} + c\begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix} + d\begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}.$$

Each one of the vectors on the RHS belongs to the space we are considering (each one of them is obtained by substituting 1 in one of the variables a, b, c, or d, and zero in the rest of these variables) and every other vector in the space is a linear combination of these vectors so the set

$$\left\{\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}\right\}$$

is a spanning set for the space. We need to find a basis for this space. The space we are considering is a subspace of $M_2(\mathbb{R})$. We stated in class that $M_2(\mathbb{R})$ is of dimension 4 and that the set,

$$D = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}$$

is a basis for it. We apply the coordinate transformation $[-]_D : M_2(\mathbb{R}) \mapsto \mathbb{R}^2$ to the spanning set of the space we are interested in:

$$[\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}]_D = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

$$[\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}]_D = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

$$[\begin{pmatrix} 1 & 4 \\ -1 & 1 \end{pmatrix}]_D = \begin{pmatrix} 1 \\ 4 \\ -1 \\ 1 \end{pmatrix}$$

$$[\begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}]_D = \begin{pmatrix} 0 \\ -1 \\ -2 \\ 2 \end{pmatrix}$$

Since the coordinate transformation maintains all linear structures (it is an isomorphism), if we find a basis to the image of the subspace we are interested in then these will be the coordinates of the basis we are actually looking for. So we look for a basis for the set

$$\text{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -2 \\ 2 \end{pmatrix} \right\}$$

We follow the algorithm we studied in class for finding a basis for a span of n-tuples and write these vectors as rows of a matrix.

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 2 & 1 \\ 1 & 4 & -1 & 1 \\ 0 & -1 & -2 & 2 \end{pmatrix} \xrightarrow{R_2+R_1, R_3-R_1} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 2 \\ 0 & -1 & -2 & 2 \end{pmatrix} \xrightarrow{\text{changing order of rows}}$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -1 & -2 & 2 \\ 0 & 2 & 1 & 0 \\ 0 & 3 & 0 & 2 \end{pmatrix} \xrightarrow{R_3+2R_2, R_4+3R_2} \begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & -6 & 8 \end{pmatrix} \xrightarrow{R_4-2R_3}$$

$$\begin{pmatrix} 1 & 1 & -1 & -1 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & -3 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So the space of 4-tuples is of dimension 3 and a basis for it is given by

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -3 \\ 4 \end{pmatrix} \right\}$$

These 4-tuples give the coordinates of a basis to the space we are actually interested in. We find the corresponding matrices. Denote them A_1, A_2, A_3. Then

$$[A_1]_D = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}$$

This implies that

$$A_1 = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + (-1) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$A_1 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

$$[A_2]_D = \begin{pmatrix} 0 \\ -1 \\ -2 \\ 2 \end{pmatrix}$$

This implies that

$$A_2 = 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + (-1) \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (-2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 2 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$A_2 = \begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}$$

$$[A_3]_D = \begin{pmatrix} 0 \\ 0 \\ -3 \\ 4 \end{pmatrix}$$

This implies that

$$A_3 = 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + (-3) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + 4 \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

So

$$A_3 = \begin{pmatrix} 0 & 0 \\ -3 & 4 \end{pmatrix}$$

Finally, we conclude that the basis we were looking for is

$$\left\{ \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -2 & 2 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -3 & 4 \end{pmatrix} \right\}$$

5. Let $B = (w_1, ..., w_n)$ be the given ordered basis of V. Let $v \in V$, then by definition $[v]_B = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ if

$$v = c_1 w_1 + c_2 w_2 + ... + c_n w_n.$$

Multiplying both sides by α we get

$$\alpha v = \alpha c_1 w_1 + \alpha c_2 w_2 + ... + \alpha c_n w_n.$$

So, by the definition of coordinates,

$$[\alpha v]_B = \begin{pmatrix} \alpha c_1 \\ \alpha c_2 \\ \vdots \\ \alpha c_n \end{pmatrix} = \alpha \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \alpha [v]_B.$$

This completes the proof.

6. a. We first assume that W is a subspace of V and prove that $[W]_B$ is a subspace of $\mathbb{R}^m$. To prove that $[W]_B$ is a subspace we follow the claim we proved in class and show the following three things.
   i. $[W]_B$ is not empty: Indeed, since we are given that W is a subspace then W is not empty, so there exists some $v \in W$. Therefore by the definition of $[W]_B$ we know that $[v]_B \in [W]_B$. So $[W]_B$ is not empty.
   ii. $[W]_B$ is closed to summation: Let $b_1, b_2 \in [W]_B$. By the definition of $[W]_B$ this means that there exists $v_1, v_2 \in W$ such that $[v_1]_B = b_1$ and $[v_2]_B = b_2$. Since we are given that W is a subspace then W is closed to summation so $v_1 + v_2 \in W$. This implies that $[v_1 + v_2]_B \in [W]_B$. Since the coordinate transformation is linear we know that $[v_1 + v_2]_B = [v_1]_B + [v_2]_B$. It follows that $[v_1]_B + [v_2]_B \in [W]_B$. So $[W]_B$ is indeed closed to summation.
   iii. $[W]_B$ is closed to multiplication by a scalar: Let $b \in [W]_B$ and $\alpha \in \mathbb{R}$. By the definition of $[W]_B$ this means that there exists $v \in W$ such that $[v]_B = b$. Since we are given that W is a subspace then W is closed to multiplication by scalar so $\alpha v \in W$. This implies that $[\alpha v]_B \in [W]_B$. Since the coordinate transformation is linear we know that $[\alpha v]_B = \alpha [v]_B$. It follows that $\alpha [v]_B \in [W]_B$. So $[W]_B$ is indeed closed to multiplication by a scalar.

   b. We next assume that $[W]_B$ is a subspace of $\mathbb{R}^m$ and prove that W is a subspace of V . To prove that W is a subspace we follow the claim we proved in class and show the following three things.
   i. W is not empty: Indeed, since we are given that $[W]_B$ is a subspace then $[W]_B$ is not empty, so there exists some $b \in [W]_B$. By the definition of $[W]_B$ this implies that there exists $v \in W$ such that $[v]_B = b$. In particular, this means that W is not empty.
   ii. W is closed to summation: Let $v_1, v_2 \in W$. By the definition of $[W]_B$ this means that $[v_1]_B, [v_2]_B \in [W]_B$. Since we are given that $[W]_B$ is a subspace then $[W]_B$ is closed to summation so $[v_1]_B + [v_2]_B \in [W]_B$. Since the coordinate transformation is linear, this implies that $[v_1 + v_2]_B \in [W]_B$. Since the coordinate transformation is 1–1 this implies that $v_1 + v_2 \in W$. It follows that W is indeed closed to summation.
   iii. W is closed to multiplication by a scalar: Let $v \in W$ and $\alpha \in \mathbb{R}$. By the definition of $[W]_B$ this means that $[v]_B \in [W]_B$. Since we

are given that $[W]_B$ is a subspace then $[W]_B$ is closed to multiplication by scalar so $\alpha[v]_B \in [W]_B$. Since the coordinate transformation is linear, this implies that $[\alpha v]_B \in [W]_B$. Since the coordinate transformation is 1–1 this implies that $\alpha v \in W$. It follows that W is indeed closed to multiplication by scalar.

7. i. The claim is true. Since dim$V$=3 the space V has a basis which consists of three elements, let $\{v_1, v_2, v_3\}$ be such a basis. Let U=span$\{v_1\}$ and W =span$\{v_1, v_2\}$. Since a span of a set of vectors contains each one of these vectors we know that $v_1 \in W$. We studied in class that if a vector space contains a set of vectors then it contains also the span of this set. It follows that $U \subseteq W$. We proved in previous HW's that if $\{v_1, ..., v_n\}$ is a basis for a vector space V and $S \subset \{v_1, ..., v_n\}$ then S is a basis for span$S$. It follows that $\{v_1\}$ is a basis for U and $\{v_1, v_2\}$ is a basis for W. By the definition of the dimension this means that the dimension of U is 1 and the dimension of W is 2.

ii. The claim is true. First, let us recall that a non–trivial subspace of a vector space V is a subspace which is different from V and from $\{0_V\}$. We proved in class that if W is a subspace of V then dim$W$ $\leq$dim$V$ and moreover, $W = V$ iff dim$W$=dim$V$. So, if dim$V$=3 then W is a non–trivial subspace of V iff dim$W \neq 0, 3$.
Further, we are given that W also has a non–trivial subspace U. We claim that this implies that dim$W \neq 1$. Indeed, since U is a subspace of W it follows from the claim we proved in class that dim$U$ $\leq$dim$W$. So, if dim$W = 1$ then dim$U = 0$ or dim$U = 1$. But if dim$U = 0$ then $U = \{0\}$, which is a trivial subspace, while if dim$U = 1$ then dim$U$ =dim$W$, which implies by the claim we proved in class that $U = W$ and therefore, also in this case, U is a trivial subspace. We conclude that dim$W$=2.
Finally, since U is a subspace of W it follows from the claim we proved in class that dim$U$ $\leq$dim$W$. So, if dim$W = 2$ then dim$U = 0$, dim$U = 1$ or dim$U = 2$. Applying exactly the same considerations again we conclude that dim$U = 1$.

iii The claim is false. Here is one counter example:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.$$

Since two vectors are linearly dependent iff one of them is a scalar multiplying the other, it follows that any two vectors from the set $\{v_1, v_2, v_3\}$ is linearly independent (clearly, no vector in this set is a scalar multiplying another vector). However, we have $v_1 + (-1)v_2 + v_3 = 0$ so the set $\{v_1, v_2, v_3\}$ is linearly dependent and therefore is not a basis.

iv. The claim is true. We point out that since a span of a set contains each one of the elements in the set then span$\{v_1, ..., v_n\}$ contains each one of the elements $\{v_1, ..., v_n\}$.

$\rightarrow$ It follows from the definition of a span that $\{v_1, ..., v_n\}$ is a spanning set for span$\{v_1, ..., v_n\}$. We are given that this set is also linearly independent so it is a basis for span$\{v_1, ..., v_n\}$. Since this basis has n elements it follows that the dimension of the space span$\{v_1, ..., v_n\}$ is n.

$\leftarrow$ It follows from the definition of a span that $\{v_1, ..., v_n\}$ is a spanning set for span$\{v_1, ..., v_n\}$. Since this is a set of n elements in a space of dimension n it follows from the claim we proved in class that if this is a spanning set then it is also linearly independent. The claim is proved.

v. The claim is false. We first point out a fact which we proved in previous HW's: If $v_1, ..., v_n, w_1, ..., w_k$ belong to a vector space V then,

$$\text{span}\{v_1, ..., v_n\} + \text{span}\{w_1, ..., w_k\} = \text{span}\{v_1, ..., v_n, w_1, ..., w_k\}$$

We use this to construct our counter example. Let

$$V_1 = \text{span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$$

$$V_2 = \text{span}\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$$

$$V_3 = \text{span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}.$$

The set $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$ is linearly independent (neither of these vectors is a scalar multiplying the other) and $\mathbb{R}^2$ is of dimension 2. The claim we proved in class implies that if a set of two elements is linearly independent in a space of dimension 2, then this set spans the space. It follows that,

$$V_1 + V_3 = \text{span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\} = \mathbb{R}^2.$$

Clearly, we also have,

$$V_1 + V_2 = \text{span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\} = \mathbb{R}^2.$$

So, $V_1 + V_2 = V_1 + V_3$. But $V_2 \neq V_3$ (why?), so the claim is disproved.