MATH-1564, K1, TA: Sam, Instructor: Nitzan, Sigal Shahaf
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1. (a) Consider $v, w \in P_2(\mathbb{R})$ and $a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{R}$.
Property 1:
$v + w = (a_0 + a_1x + a_2x^2) + (b_0 + b_1x + b_2x^2)$
$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$.
Thus, $v + w$ is also contained in P_2.
Property 2:
$v + w = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$
Similarly,
$w + v = (b_0 + a_0) + (b_1 + a_1)x + (b_2 + a_2)x^2$
$= (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$
$= v + w$.
Thus satisfying $v + w = w + v$.
Property 3:
Also consider $u \in P_2(\mathbb{R})$ and $c_0, c_1, c_2 \in \mathbb{R}$.
$(v + w) + u = ((a_0 + b_0) + c_0) + ((a_1 + b_1) + c_1)x + ((a_2 + b_2) + c_2)x^2$
Similarly,
$v + (w + u) = (a_0 + (b_0 + c_0)) + (a_1 + (b_1 + c_1))x + (a_2 + (b_2 + c_2))x^2$
$= ((a_0 + b_0) + c_0) + ((a_1 + b_1) + c_1)x + ((a_2 + b_2) + c_2)x^2$
$= (v + w) + u$
Thus satisfying $(v + w) + u = v + (w + u)$.
Property 4:
Observe the zero polynomial $0 + 0x + 0x^2$ in P_2. Let us denote this zero polynomial as 0_p. Then, we want to show that for all $v \in P_2$, $0_p + v = v$.
$0_p + v = (0 + a_0) + (0 + a_1)x + (0 + a_2)x^2$
$= a_0 + a_1x + a_2x^2$
$= v$
Thus satisfying $0_p + v = v$.
Property 5:
For the given polynomial w, we can see that $-w$, which is $(-b_0) + (-b_1)x + (-b_2)x^2$ is also in P_2. So,
$w + (-w) = (b_0 - b_0) + (b_1 - b_1)x + (b_2 - b_2)x^2$
$= 0 + 0x + 0x^2$
$= 0_p$
Thus satisfying the fact that $w + (-w) = 0_p$.
Property 6:
For $\alpha \in \mathbb{R}$,

$\alpha v = \alpha(a_0 + a_1x + a_2x^2) = (\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2$
Thus, αv is also contained in P_2.
Property 7:
For $\alpha, \beta \in \mathbb{R}$,
$\alpha(\beta v) = \alpha((\beta a_0) + (\beta a_1)x + (\beta a_2)x^2)$
$= (\alpha(\beta a_0)) + (\alpha(\beta a_1))x + (\alpha(\beta a_2))x^2$
$= ((\alpha\beta)a_0) + ((\alpha\beta)a_1)x + ((\alpha\beta)a_2)x^2$
$= (\alpha\beta)v$
Thus satisfying $\alpha(\beta v) = (\alpha\beta)v$.
Property 8:
$1_{\mathbb{R}}v = 1(a_0 + a_1x + a_2x^2)$
$= a_0 + a_1x + a_2x^2 = v$
Thus satisfying $1_{\mathbb{R}}v = v$.
Property 9:
For $\alpha, \beta \in \mathbb{R}$,
$(\alpha+\beta)v = ((\alpha+\beta)a_0) + ((\alpha+\beta)a_1)x + ((\alpha+\beta)a_2)x^2$
$= ((\alpha a_0) + (\beta a_0)) + ((\alpha a_1) + (\beta a_1))x + ((\alpha a_2) + (\beta a_2))x^2$
$= ((\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2) + ((\beta a_0) + (\beta a_1)x + (\beta a_2)x^2)$
$= (\alpha v) + (\beta v)$
Thus satisfying $(\alpha+\beta)v = (\alpha v) + (\beta v)$.
Property 10:
For $\alpha, \beta \in \mathbb{R}$,
$\alpha(v+w) = \alpha((a_0+b_0) + (a_1+b_1)x + (a_2+b_2)x^2)$
$= ((\alpha a_0) + (\alpha b_0)) + ((\alpha a_1) + (\alpha b_1))x + ((\alpha a_2) + (\alpha b_2))x^2$
$= ((\alpha a_0) + (\alpha a_1)x + (\alpha a_2)x^2) + ((\alpha b_0) + (\alpha b_1)x + (\alpha b_2)x^2)$
$= (\alpha v) + (\alpha w)$
Thus satisfying $\alpha(v+w) = (\alpha v) + (\alpha w)$.
Since all 10 properties are satisfied, P_2 is a vector space over the real numbers.

(b) Consider $v, w, u \in V$ where V is the given set, and $v = (x_1, y_1, z_1, w_1), w = (x_2, y_2, z_2, w_2), u = (x_3, y_3, z_3, w_3)$ over the real numbers.
Property 1:
$v + w = (x_1+x_2, y_1+y_2, z_1+z_2, w_1+w_2)$
$= (x_1+x_2) - (y_1+y_2) + 2(z_1+z_2)$
$= (x_1 - y_1 + 2z_1) + (x_2 - y_2 + 2z_2)$
$= 0 + 0 = 0$
Thus, $v + w$ is also contained in set V.
Property 2:
$v + w = (x_1+x_2, y_1+y_2, z_1+z_2, w_1+w_2)$

$= (x_2 + x_1, y_2 + y_1, z_2 + z_1, w_2 + w_1)$
$= w + v$
Thus satisfying $v + w = w + v$.
Property 3:
$v+(w+u) = (x_1+(x_2+x_3), y_1+(y_2+y_3), z_1+(z_2+z_3), w_1+(w_2+w_3))$
Similarly,
$(v+w)+u = ((x_1+x_2)+x_3, (y_1+y_2)+y_3, (z_1+z_2)+z_3, (w_1+w_2)+w_3)$
$= (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3), z_1 + (z_2 + z_3), w_1 + (w_2 + w_3))$
$= v + (w + u)$
Thus satisfying $v + (w + u) = (v + w) + u$.
Property 4:
Observe the zero component $(0, 0, 0, 0)$ which is in the set because $0 - 0 + 2(0) = 0$. Denote this is as 0_V.
$0_v + v = (0 + x_1, 0 + y_1, 0 + z_1, 0 + w_1)$
$= (x_1, y_1, z_1, w_1) = v$
Thus satisfying the fact that there exists a 0_V so that $0_V + v = v$.
Property 5:
For an arbitrary $v \in V$, we can see that $-v$, which is $(-x_1, -y_1, -z_1, -w_1)$, is in the set V since $(-x_1) - (-y_1) + 2(z_1) = -(x_1 - y_1 + 2z_1) = -0 = 0$.
$v + (-v) = (x_1 - x_1, y_1 - y_1, z_1 - z_1, w_1 - w_1)$
$= (0, 0, 0, 0) = 0_V$
Thus satisfying the fact there exists a $-v$ so that $v + (-v) = 0$.
Property 6:
For $a \in \mathbb{R}$,
$av = (ax_1, ay_1, az_1, aw_1)$
$= (ax_1) - (ay_1) + 2(az_1) = a(x_1 - y_1 + 2z_1) = a(0) = 0$
Thus satisfying $av \in V$.
Property 7:
For $a, b \in \mathbb{R}$,
$a(bv) = (a(bx_1), a(by_1), a(bz_1), a(bw_1))$
$= ((ab)x_1, (ab)y_1, (ab)z_1, (ab)w_1) = (ab)v$
Thus satisfying $a(bv) = (ab)v$.
Property 8:
$1_{\mathbb{R}}v = (1x_1, 1y_1, 1z_1, 1w_1) = (x_1, y_1, z_1, w_1) = v$
Thus satisfying $1_{\mathbb{R}}v = v$.
Property 9:
For $a, b \in \mathbb{R}$,
$(a + b)v = ((a + b)x_1, (a + b)y_1, (a + b)z_1, (a + b)w_1)$
$= (ax_1 + bx_1, ay_1 + by_1, az_1 + bz_1, aw_1 + bw_1) = (av) + (bv)$
Thus satisfying $(a + b)v = (av) + (bv)$

Property 10:
For $a \in \mathbb{R}$,
$a(v+w) = (a(x_1+x_2), a(y_1+y_2), a(z_1+z_2), a(w_1+w_2))$
$= (ax_1+ax_2, ay_1+ay_2, az_1+az_2, aw_1+aw_2) = (av)+(aw)$
Thus satisfying $a(v+w) = (av)+(aw)$.
Since all 10 properties are satisfied, V is a vector spasce over the real numbers.

(c) **Property 8:**
According to the set, it has the operations $\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 0 \end{pmatrix}$. However, property 8 states that $1_{\mathbb{R}}v = v$ for $v \in V$. So, for this set, we would get $1 \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1x_1 \\ 0 \end{pmatrix}$. Clearly, $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \neq \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$, so **this set is not a vector space, as it violates the 8th property.**

(d) **Property 9:**
For $a, b \in \mathbb{R}$, according to the operations in the set, $(a+b)v = \begin{pmatrix} (a+b)x_1 \\ (a+b)x_2 \end{pmatrix}$ $= \begin{pmatrix} ax_1+bx_1 \\ ax_2+bx_2 \end{pmatrix}$. By the 9th condition, this should be equivalent to $(av)+(bv)$. According to the operations in the set, $(av)+(bv) = \begin{pmatrix} ax_1 \\ ax_2 \end{pmatrix} \oplus \begin{pmatrix} bx_1 \\ bx_2 \end{pmatrix} = \begin{pmatrix} ax_1+bx_2 \\ ax_2+bx_2 \end{pmatrix}$. Clearly, then, these two are not equal, and **this set is not a vector space, as it violates the 9th property.**

(e) **Property 4:**
Assume that such 0_v existed for this set. Let us say $w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So $w+0_v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0-3 \\ 1+0-2 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \end{pmatrix}$. Clearly, $w+0_v \neq w$, so **this set is not a vector space, as it violates the 4th property.**

(f) **Property 8:**
$1_{\mathbb{R}}v = 1_{\mathbb{R}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$. Given the set operations, this is equal to $\begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$. Therefore, $1_{\mathbb{R}}v \neq v$, and **this set is not a vector space, as it violates the 8th property.**

(g) **Property 5:**
Let us say that w is in this set, and it equals $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Condition 5 states

that there exists an element denoted as $-w$ that is also in the set, which is equivalent to $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$. However, in order for this to be in the set, $x_1, x_2 > 0$, which makes this false. Therefore, for this chosen w, its counterpart $-w$ does not exist in the set, and so **this set is not a vector space as it violates the 5th property.**

2. (a) Let $a = 2, b = 1$. According to property 9 of a vector space, $(a + b)v = av + bv$. We are given the fact that for every $v \in V$ we have $2v + v = 3v$. Clearly, this follows the property since $2v + v = (2 + 1)v = 3v$.

   (b) Using property 6 of vector spaces, we know that for any $a \in \mathbb{R}, v \in V$, $av \in V$. In this scenario, we know that our v is represented by 0_V. Then, we know as a general rule that for any scalar, $a0_V$ will also equal zero. We are guaranteed this because property 6 states that this value will still be contained in V. Since each vector space contains the zero (trivial) vector space, there exists only one 0_V, meaning that any scalar multiplying 0_V will yield itself and not any other form of zero.

   (c) Using property 5 of vector spaces, we are given that for all $v \in V$, there exists a $-v$ so that $v + (-v) = 0$. In this scenario, let us denote v as $-v$, so that would mean $-v + (-(-v)) = 0$. This simplifies to $-v - (-v) = 0$ which is also $-(-v) = v$.

   (d) Using property 3 of vector spaces, we are given that $(u+v)+w = u+(v+w)$. The question states $(u+w)+(v+z) = w+(u+(v+z))$. For simplicity, we denote $(v + z)$ as Y. Then, we have $(u + w) + Y = w + (u + Y)$. It becomes immediately apparent that these two expressions are equal through property 3. Therefore, the expression is valid.

3. (a) **Property 3:** For $a \in \mathbb{R}$, property 3 of a subspace guarantees $aW \in W$. In this scenario, if we choose $a = -1$, and we apply $aW = -1W = \begin{pmatrix} -x_1 \\ -x_2 \\ -x_3 \\ -x_4 \end{pmatrix}$, it is evident that this is no longer in the set W, since all of its components are negative. Thus, **this set is not a subsapce, as it violates the 3rd property.**

   (b) **Property 1:** If $x, y = 0$, the resulting matrix in W is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Therefore, the zero space exists and the set is not empty.
   Property 2: Let $u, v \in W$, then $u + v$

$= \begin{pmatrix} (x_1 + x_2) & (2x_1 + 3y_1) + (2x_2 + 3y_2) \\ (y_1 + y_2) & (x_1 - y_1) + (x_2 - y_2) \end{pmatrix}$
$= \begin{pmatrix} (x_1 + x_2) & 2(x_1 + x_2) + 3(y_1 + y_2) \\ (y_1 + y_2) & (x_1 + x_2) - (y_1 + y_2) \end{pmatrix}$
Here, we know that $x_1 + x_2 \in \mathbb{R}$ and $y_1 + y_2 \in \mathbb{R}$. Therefore, $u + v \in W$.
Property 3: For all $a \in \mathbb{R}$, aW
$= \begin{pmatrix} ax & a(2x+3y) \\ ay & a(x-y) \end{pmatrix} = \begin{pmatrix} ax & 2ax + 3ay \\ ay & ax - ay \end{pmatrix}$.
Here, we know that $ax, ay \in \mathbb{R}$. Therefore, $aw \in W$.
Since all 3 properties are satisfied, W is a subspace.

(c) **Property 2:** Let $u, v \in W$, since all individual polynomials in each element add up to 1 (given by $p(1) = 1$), $u + v = 2$. Clearly, $u + v$ is not in W since its individual polynomials add up to 2. Therefore, **this set is not a subspace, as it violates the 2nd property.**

(d) Take an arbitrary element $v \in W$ and say $v = a + b(1) + c(1)^2 + d(1)^3 = 0$
Property 1: If we set $a, b, c, d = 0$, $v = (0) + (0)(1) + (0)(1)^2 + (0)(1)^3 = 0$, therefore it is an element to the set W, and we have proved that it isn't empty.
Property 2: If we take another element $w \in W$, we know that v, w are polynomials which add up to 0. Therefore, $v + w = 0 + 0 = 0$, and we can conclude that $v + w \in W$.
Property 3: If we take some $a \in \mathbb{R}$, $av = a0 = 0$ since v's polynomials add up to 0. Therefore, we can conclude $av \in W$.
Since all 3 properties are satisfied, W is a subspace.

(e) **Property 3:** For $a \in \mathbb{R}$ and all $w \in W$, let us say $a = \pi$. Then, $aw = \begin{pmatrix} ax_1 \\ ax_2 \\ ax_3 \end{pmatrix}$. However, a rational number multiplied by an irrational number is irrational, which means that $ax_1, ax_2, ax_3 \notin \mathbb{Q}$, which means that $aw \notin W$. Therefore, **this set is not a subsapce, as it violates the 3rd property.**

(f) For all $V, X \in W$:
Property 1: Let us denote the zero matrix as U, then we know it is a part of the set if $AU = 0$. For the ij entry of AU, $AU_{ij} = \sum_{k=1}^{n} a_{ik}u_{kj}$.
Because, every element of u is 0, it follows that each ij entry of $AU = 0$, so therefore we can be certain that the zero matrix is in the set W, and that it is not empty.
Property 2: If $V + X \in W$, then $A(V + X) = 0$. As we already

proved the matrix distributive property, we can simplify this expression to $AV + AX$. Since we know already that $V, X \in W$, $AV, AX = 0$. So, $AV + AX = 0 + 0 = 0$, and therefore $V + X \in W$.

Property 3: For some $b \in \mathbb{R}$, we want to show that $bV \in W$ and subsequently $A(bV) = 0$. For the ij entry of $A(bV)$, $A(bV)_{ij} = \sum_{k=1}^{n} a_{ik}(bv_{kj})$, and since real numbers are multiplicatively communative, this is equivalent to $\sum_{k=1}^{n} b(a_{ik}v_{kj})$. Also, since $V \in W$, it means that $AV = 0$, and each ij element of $AV = 0$, so essentially this evaluates to $\sum_{k=1}^{n} b(0) = 0$. Therefore, we have proved that $bV \in W$ by showing that $A(bV) = 0$.

Since all 3 properties are satisfied, W is a subspace.

(g) Say that $v, w \in W$.

Property 1: Let us choose an arbitrary value for f given as $f(x) = 0$. $f(x)$ is twice differentiable, and $f''(x) + 3f'(x) - f(x) = 0$ for all $x \in \mathbb{R}$. Therefore, the set W is not empty, as $f(x) = 0$ is a part of it.

Property 2: For $x_1, x_2 \in \mathbb{R}$, $v + w$
$= [f''(x_1) + 3f'(x_1) - f(x_1)] + [f''(x_2) + 3f'(x_2) - f(x_2)]$
$= (f''(x_1) + f''(x_2)) + 3(f'(x_1) + f'(x_2)) - (f(x_1) + f(x_2)) = 0 + 0 = 0$.
Therefore, $v + w$ belongs to the set.

Property 3: For some $a \in \mathbb{R}$, av
$= a(f''(x) + 3f'(x) - f(x)) = a(0) = 0$. Therefore, av belongs to the set.

Since all 3 properties are satisfied, W is a subspace.

4. (a) **Property 1:** Since all subspaces have the zero (trivial) space, we are guaranteed that zero space lies in both U and W. In other words, we can say that zero lies in $U \cap W$.

Property 2: Suppose $u, w \in U \cap W$. We then know that u is in U and also in W, whilewv is similarly in both U and W. Therefore, because U is a subspace and u and w are both contained in it, $u + w \in U$, and the same could be said for W. Therefore, it is evident that $u + w$ is in both U and W, and hence $u + w \in U \cap W$.

Property 3: Let $u \in U \cap W$ and $a \in \mathbb{R}$. Since u lies in both U and W, which are subspaces, scalar multiplication is also closed in U and W. Therefore, $ax \in U$ and $ax \in W$. It then can be written as $ax \in U \cap W$.

Since all 3 properties are satisfied, $U \cap W$ is a subspace.

(b) **Property 2:** Let us pick an arbitrary $u \in U$ and $w \in W$. We say that U is part of the y axis subspace, while W is part of the x axis subspace. So, set $u = (1, 0)$ and $w = (0, 1)$. If we perform $u + w$ we get $(1, 1)$ which is clearly not in either U or W. Therefore, $u + w \notin U \cup W$, and **this is not a subspace, as it fails the 2nd property.**

(c) **Property 1:** Since U,W are both subspaces, the zero space is contained in both. Therefore, $0 + 0 = 0 \in U + V$, and there exists at least one solution here.
Property 2: Let $u, w \in U+W$. There exists an $a \in U$ and $b \in W$ which guarantees that $u = a + b$, and there exists a $c \in U$ and $d \in W$ which guarantees that $w = c + d$. Then, we have $u + w = (a + b) + (c + d) = (a + c) + (b + d)$. Since a,c is in the subspace U, $a + c \in U$, and since b,d is in the subspace W, $b + d \in W$. Therefore, it is evident that $u + w \in U + W$.
Property 3: Let $u \in U + W$, there also exists an $a \in U$ and $b \in W$ such that $u = a + b$. Since U,W are subspaces, a scalar $r \in \mathbb{R}$ can be applied, so $ra \in U$ and $rb \in W$. Then $ru = r(a + b) = ra + rb \in U + W$
Since all 3 properties are satisfied, $U + W$ is a subspace.

5. (a) Subspaces of the x-axis and the y-axis. You can scale each of them individually and they will still be in the same axis, i.e. $(cx, 0)$ or $(0, by)$, but you cannot add the two together or they will not be on either axis.

(b) The set $\{(x, y) : x \geq 0, y \geq 0\}$. You can add the ordered pair (x, y) all you want and still get something inside the set, but if you multiply by a negative scalar, it is no longer in the set.

(c) The set $\{(x, y) : x + y = 3\}$. Say that $u = (1, 2)$ and $v = (2, 1)$. If we multiply u by a scalar, $2u = (2, 4)$, which is not included in the set. If we add $u + v$, the result is $(3, 3)$, which is also not in the set. Therefore, this is NOT closed to addition and multiplication.

(d) A surface through the origin, a three dimensional line through the origin, and the point at the origin.