MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 8

1. Find the dimensions of each one of the spaces in HW7 Question 1.
2. Find a basis to the following spaces and determine the dimension of each of these spaces.
   i. $\text{span}\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}\}$.
   ii. $\text{span}\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 5 \\ -1 \\ 3 \\ 7 \end{pmatrix}\}$.
3. i. Let $v_1, ..., v_n \in \mathbb{R}^m$ and denote by A the matrix who's columns are $v_1, ..., v_n$ that is
   $$A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix}$$
   Denote: $L(A) := \{b \in \mathbb{R}^m : (A|b) \text{ has a solution }\}$. Prove that $L(A) = \text{span}\{v_1, ..., v_n\}$.

   ii. Let
   $$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 10 \\ 1 & 4 & -1 \\ 2 & 2 & 4 \end{pmatrix}.$$
   Find a basis for $L(A)$ and the dimension of $L(A)$.
4. Try to find as '**fast**' a solution as possible to each of these questions, using **dimension considerations**, or any other theorems studied in class. In particular, when an actual computation is needed then use the technique of **coordinates** to solve the questions. Justify all your considerations, including the use of coordinates.
   i. Is the following set a basis for $M_{2\times 3}(\mathbb{R})$?
   $$\{\begin{pmatrix} 1 & 1 & 2 \\ 1 & -3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ 1 & -3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ 1 & -3 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 2 \\ 1 & -3 & 3 \end{pmatrix}\}$$
   ii. Is the following set linearly independent?
   $$\{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\}$$

iii. Is the following statement correct?

$$\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} \in \text{span}\{\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\}$$

iv. Is the following set a basis for $M_2(\mathbb{R})$?

$$\{\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\}$$

v. Is the following statement true?

$$2 + x^2 - 2x^3 \in \text{span}\{1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}$$

vi. Is the following set a spanning set for $\mathbb{R}_3[x]$?

$$\{2 + x^2 - 2x^3, 1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}$$

vii. Find the dimension of the following space:

$$\text{span}\{2 + x^2 - 2x^3, 1 - 2x + x^2 - x^3, 5 + 2x + 2x^2 - 5x^3, 3 + 6x - 3x^3\}$$

viii. Is the following statement correct?

$$\text{span}\{2+x^2-2x^3, 3+6x-3x^3\} = \text{span}\{1-2x+x^2-x^3, 5+2x+2x^2-5x^3, 3+6x-3x^3\}$$

ix. Find a basis for the following space:

$$\{\begin{pmatrix} a - b + c & a + b + 4c - d \\ -a + 2b - c - 2d & -a + b + c + 2d \end{pmatrix} : a, b, c, d \in \mathbb{R}\}$$

5. Let V be a vector space over $\mathbb{R}$ and let B be an ordered basis of V. Prove that if $v \in V$ and $\alpha \in \mathbb{R}$ then $[\alpha v]_B = \alpha[v]_B$.
   (Remark: This was formulated as part of a theorem in class, but the proof was left for HW).
6. Let V be a vector space and $B = (v_1, ..., v_m)$ be an ordered basis of V. Let $W \subseteq V$ be a **subset** of V and denote
   $$[W]_B = \{[w]_B : w \in W\},$$
   so that $[W]_B$ is a **subset** of $\mathbb{R}^m$ ($[W]_B \subseteq \mathbb{R}^m$). Prove that W is a **subspace** of V iff $[W]_B$ is a **subspace** of $\mathbb{R}^m$.
   (Remark: This was formulated as part of a theorem in class, but the proof was left for HW).
7. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.
   i. Let V be a vector space which satisfies $\dim V$=3. Then there exist, a subspace W of V and a subspace U of W (that is, $U \subset W \subset V$) such that $\dim U$=1 and $\dim W$=2.

ii. Let V be a vector space which satisfies dim$V$=3 and let W be a **non–trivial** subspace of V and U be a **non–trivial** subspace of W (that is, $U \subset W \subset V$) then dim$U$=1 and dim$W$=2.

iii Let V be a vector space which satisfies dim$V$=3 and let $v_1, v_2, v_3 \in V$ be such that $\{v_1, v_2\}$ are linearly independent, $\{v_2, v_3\}$ are linearly independent, and $\{v_3, v_1\}$ are linearly independent. Then $\{v_1, v_2, v_3\}$ is a basis for V.

iv. Let V be a vector space and $v_1, ..., v_n \in V$ then: $\{v_1, ..., v_n\}$ is linearly independent iff $\dim\big(\text{span}\{v_1, ..., v_n\}\big) = n$.

v. Let V be a vector space and let $V_1, V_2, V_3 \subset V$ be such that $V_1 + V_2 = V_1 + V_3$ and dimV_2 =dimV_3 then $V_2 = V_3$. (The sum of two subspaces was defined in previous HW's).