MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 1

1. Solve the following linear systems by presenting them in a matrix form and bringing the matrix to echelon form. If the system has an infinite amount of solutions then present a parametrization of these solutions. In any case please specify how many solutions does the system have.

   i.
$$\begin{cases} 5x+4y-z=0 \\ 2x-4y+z=1 \\ -7x-14y+5z=10 \end{cases}$$

   ii.
$$\begin{cases} 2x+y+2z=2 \\ -x+y-z=2 \\ 3x+2y+z=2 \\ 5x+4y-z=2 \end{cases}$$

   iii.
$$\begin{cases} x_1-x_2+2x_3=0 \\ 2x_1-2x_2+4x_3+x_4+2x_5=4 \\ 3x_1+x_2+6x_3+x_5=-3 \\ x_1+2x_3+2x_4+x_5=4 \end{cases}$$

   iv.
$$\begin{cases} x_1+2x_2+3x_3+4x_4+5x_5=5 \\ x_1+4x_2+6x_3+8x_4+10x_5=10 \\ x_1-x_2+x_3-x_4+x_5=0 \\ -x_1+4x_2+x_3+6x_4+3x_5=5 \end{cases}$$

2. In each of the following, find for which values of $a$ does the linear system have: (i) No solution, (ii) Exactly one solution, (iii) Infinitely many solutions.

   a.
$$\begin{cases} x+y+az=-1 \\ -x+(a-1)y+(2-a)z=a+1 \\ 6x+(5a+6)y+(7a+7)z=a^2 \end{cases}$$

   b
$$\begin{cases} (a+1)x+ay-az=2+a \\ (a+1)x+(a+2)y-(a+2)z=a+4 \\ (a+1)x+ay+(a^2-6)z=a^2-2a+4 \\ (2a+2)x+2ay+(a^2-a-6)z=a^2-a+6 \end{cases}$$

3. Let $A \in M_{m\times n}(\mathbb{R})$ and denote
$$L(A) := \{b \in \mathbb{R}^m : (A|b) \text{ has at least one solution } \}.$$
   Prove
   i. If $b \in L(A)$ and $d \in L(A)$ then $b+d \in L(A)$.
   ii. If $b \in L(A)$ and $t \in \mathbb{R}$ then $t \cdot b \in L(A)$

4. In each of the following parts appears a description of a matrix $A \in M_{m\times n}$. Determine whether or not such a matrix exists (where $m$ and $n$ can be any numbers you want; try to work with small numbers). If you claim that such a matrix exists then provide a concrete example. If you claim that such a matrix does not exist then explain carefully why this is true.
   a. For **every** $b \in \mathbb{R}^n$ there exist infinitely many solutions to the linear system $(A|b)$.
   b. There exists $b_1 \in \mathbb{R}^n$ such that the system $(A|b_1)$ has infinitely many solutions and there exists $b_2 \in \mathbb{R}^n$ such that the system $(A|b_2)$ has no solution.
   c. There exists $b_1 \in \mathbb{R}^n$ such that the system $(A|b_1)$ has exactly one solution, there exists $b_2 \in \mathbb{R}^n$ such that the system $(A|b_2)$ has an infinite amount of solutions and there exists $b_3 \in \mathbb{R}^n$ such that the system $(A|b_3)$ has no solution.
   d. The homogeneous system $(A|0)$ has exactly one solution and the echelon form of $A$ has a row of zeroes.
   e. The echelon form of the matrix has a row of zeroes and for every $b \in \mathbb{R}^n$ there exists exactly one solution to the linear system $(A|b)$.
   f. The 3-tuple $(1, 2, 3)$ is a solution to the homogeneous system $(A|0)$ and the echelon form of $A$ has at least three rows which are not zero rows.
   g. The 3-tuple $(1, 2, 3)$ is a solution to the homogeneous system $(A|0)$ and the echelon form of $A$ has at least two rows which are not zero rows.
   h. The set of solutions of the homogeneous system $(A|0)$ is exactly the set $\{(s+t, s, t) : s, t \in \mathbb{R}\}$.
   i. The set of solutions of the homogeneous system $(A|0)$ is exactly the set $\{(s+t, s-t) : s, t \in \mathbb{R}\}$.