MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 8

1. Let V be a vector space over $\mathbb{R}$ and $U, W \subseteq V$ be two subspaces of V.
   i. Prove that there exist a basis B of U and a basis C of W such that $B \cap C$ is a basis for $U \cap W$.
   ii. Is it true that for every basis B of U and every basis C of W the set $B \cap C$ is a basis for $U \cap W$?
   iii. Recall the definition of $U + W$ from previous HW's. Prove the following dimension formula:
   $$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$
   (**Important Remark:** This dimension formula can be used in later HW's and in exams without adding a proof to it.)
   iv. Let $U, W \subset \mathbb{R}_4[x]$ be two subspaces which satisfy $\dim(U) = \dim(W) = 3$ prove that $U \cap W \neq \{0\}$.
   v. Find the dimension of the following space:
   $$\text{span}\{\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}\} \cap \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}\}$$

2. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.
   i. There exists a matrix $A \in M_{4\times 3}$ who's last row is not a linear combination of the rest of it's rows.
   ii. There exists a matrix $A \in M_3(\mathbb{R})$ which satisfies $rank(A) = 2$ and
   $$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0.$$
   iii. There exists a matrix $A \in M_3(\mathbb{R})$ which satisfies $rank(A) = 2$, and
   $$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0 \qquad A \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = 0$$
   .
   iv. There exist two matrices $A, B \in M_3(\mathbb{R})$ such that $A \neq B$, $A \sim B$ (A is row equivalent to B) and the column spaces of A and B are the same.
   v. There exist two matrices $A, B \in M_3(\mathbb{R})$ such that $AB = I$ and
   $$rank(A) + rank(B) = 5.$$

vi. There exists a matrix $A \in M_3(\mathbb{R})$ who's row space is equal to its column space.

3. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.
   a. For any two matrices $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$ we have $\text{rank}(AB) = \text{rank}(A) \cdot \text{rank}(B)$.
   b. If A is a square matrix then its column space is equal to its null space.
   c. The system $(A|b)$ has a solution iff $\text{rank}A = \text{rank}(A|b)$
   d. If $A \in M_{m\times n}(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent.
   e. If $A \in M_n(\mathbb{R})$ is such that the vectors in its rows are linearly independent then the vectors in its columns are also linearly independent

4. Let $A \in M_n(\mathbb{R})$ prove that A is invertible iff $rank(A) = n$.

5. For any two matrices $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$ which satisfy $AB = 0$ prove that $\text{rank}(B) + \text{rank}(A) \leq n$.

6. The following claims are either **true or false**. Determine which case is it for each claim and prove your answer.
   a. For any two $m \times n$ matrices A and B we have $\text{rank}(A+B) = \text{rank}(A) + \text{rank}(B)$.
   b. For any two $m \times n$ matrices A and B we have $\text{rank}(A+B) \leq \text{rank}(A) + \text{rank}(B)$.

6. Let $A \in M_{m\times n}(\mathbb{R})$ and $B \in M_{n\times k}(\mathbb{R})$, consider the matrix AB and prove the following:
   i. The null space of B is a subspace of the null space of AB.
   ii. The column space of AB is a subspace of the column space of A.
   iii. $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$. (Hint: use the previous parts of this question).
   iv. Let $A \in M_3(\mathbb{R})$ be such that $\text{rank}(A)$, $\text{rank}(A^2)$ and $\text{rank}(A^3)$ are three different numbers. Prove that $A^3 = 0$.