MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 6 – Solutions

1. i. We start by finding a spanning set for the space. Every element in the space can be written as

$$\begin{pmatrix} t+2s+r \\ t-s-r \\ t+3r \\ 5r+5s \\ -2t+2s+r \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix} + s\begin{pmatrix} 2 \\ -1 \\ 0 \\ 5 \\ 2 \end{pmatrix} + r\begin{pmatrix} 1 \\ -1 \\ 3 \\ 5 \\ 1 \end{pmatrix}$$

So, every vector in the space is a linear combination of the vectors on the RHS. These three vectors belong to the space: Indeed they are obtained by substituting in the rule that defines the space the values $t = 1, s = 0, r = 0$ for the first vector, $t = 0, s = 1, r = 0$ for the second vector, and $t = 0, s = 0, r = 1$ for the third vector. So, all three vectors belong to the space and every other vector in the space is a linear combination of them. We conclude that these three vectors are a spanning set for the space. Let us check if they are also a linear independent set. If they are then being both a spanning set and a linear independent set then they are a basis for the space. To check if they are linearly independent we follow the algorithm we studied in class. We write the vectors as columns of a matrix:

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & -1 \\ 1 & 0 & 3 \\ 0 & 5 & 5 \\ 2 & 2 & 1 \end{pmatrix} \xrightarrow{R_2-R_1, R_3-R_1, R_5-2R_1} \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 0 & -2 & 2 \\ 0 & 5 & 5 \\ 0 & -2 & -1 \end{pmatrix} \xrightarrow{\frac{1}{-2}R_3}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -2 \\ 0 & 1 & -1 \\ 0 & 5 & 5 \\ 0 & -2 & -1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & -3 & -2 \\ 0 & 5 & 5 \\ 0 & -2 & -1 \end{pmatrix} \xrightarrow{R_3+3R_2, R_4-5R_2, R_5+2R_2}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 10 \\ 0 & 0 & -3 \end{pmatrix} \xrightarrow{R_4+2R_3, R_5-\frac{3}{5}R_3} \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There are no free variables, so the corresponding linear system has only the trivial solution. We conclude that the only linear combination of our three vectors, that is equal to the zero vector, is the trivial linear combination. So the vectors are linearly independent. It follows that:

$$\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ 0 \\ 5 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \\ 5 \\ 1 \end{pmatrix} \right\}$$

is a basis for the space.

Remark. We have two different algorithms to find if a set in $\mathbb{R}^n$ is linearly independent: We either write them as columns of a matrix and check that there are no free variables (which implies that the only linear combination of the vectors which is equal to zero is the trivial one), or we write them as rows of a matrix and check whether the corresponding echelon form has no row of zeroes. We used the first algorithm here however, in fact, the second algorithm is much more suited to this question. This is because if the system turned out to be linearly dependent, the second algorithm would have immediately provided us with a basis to its span.

ii. We start by writing a parametrization of the space.

$$\{A \in M_4(\mathbb{R}) : (A)_{ij} = (A)_{ji} \ \forall 1 \le i, j \le 4\} = \left\{ \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} \right\}.$$

Every matrix in this space can be written as:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} = a_{11} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_{22} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_{33} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$+ a_{44} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + a_{12} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_{13} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$+ a_{14} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + a_{23} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_{24} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} +$$

$$+ a_{34} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

All of the matrices on the RHS belong to the space as they are all symmetric (check!). Since every other vector in the space is a linear combination of these matrices, we conclude that they are a spanning set for the space. To check if this set of matrices is also linearly independent, we need to check if there is a non-trivial linear combination of them that is equal to the zero matrix. So we look at:

$$a_{11}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_{22}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_{33}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$
$$+a_{44}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + a_{12}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_{13}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$
$$+a_{14}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} + a_{23}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_{24}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} +$$
$$+a_{34}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We rewrite the LHS:

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since two matrices are equal iff they are equal at every entry, we conclude that all of the coefficients in the linear combination are equal zero, and therefore that the equation above has only the trivial solution. So our set of matrices is linearly independent. Since we found that the set

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \right.$$
$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \right\}$$

is both a spanning set for the space and linearly independent, we conclude that it is a basis for the space.

Remark. It may happen that after you follow the outline described here (that is, you are given, or you find, a parametrization for the space, and then you find a spanning system to the space) you will discover that the spanning system you found is linearly **dependent**. To find a basis in such a case, the best way is to at that point (after you found a spanning set, and are now looking for a basis) "translate" the question into a question in $\mathbb{R}^n$ using coordinates and then solve this last part in $\mathbb{R}^n$. (And off course, not to forget to "translate back" your solution, so that you can present a basis to the space you were actually asked about). When this HW was given, we did not yet discuss coordinates with all details. Therefore, in all the questions in this HW, the spanning set you find turns out to be linearly independent. But **be aware that this is not always the case**.

iii. We start by writing a parametrization of the space. We note that on the main diagonal of the matrix we have $(A)_{jj} = -(A)_{jj}$ which implies that $2(A)_{jj} = 0$ and therefore $(A)_{jj} = 0$. So the parametrization looks like this:

$$\{A \in M_4(\mathbb{R}) : (A)_{ij} = -(A)_{ji}\ \forall 1 \le i, j \le 4\} = \{ \begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} \}.$$

Every matrix in this space can be written as:

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = a_{12}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_{13}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$+a_{14}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + a_{23}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$+a_{24}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + a_{34}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

All of the matrices on the RHS belong to the space as they are all antisymmetric (check!). Since every other vector in the space is a linear combination of these matrices, we conclude that they are a spanning set for the space. To check if this set of matrices is also linearly independent, we need to check if there is a non-trivial linear combination of them that is equal to the zero matrix. So we look at:

$$a_{12}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + a_{13}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$+a_{14}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} + a_{23}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} +$$

$$+a_{24}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} + a_{34}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We rewrite the LHS:

$$\begin{pmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Since two matrices are equal iff they are equal at every entry, we conclude that all of the coefficients in the linear combination are equal zero, and therefore that the equation above has only the trivial solution. So our set of matrices is linearly independent. Since we found that the set

$$\left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \right.$$

$$\left. \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \right\}$$

is both a spanning set for the space and linearly independent, we conclude that it is a basis for the space.

iv. To find a parametrization of this set, we need to solve the linear system. We write it in matrix form, and bring it to echelon form.

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 2 & -5 & 0 \\ 2 & 5 & -8 & 0 \end{pmatrix} \xRightarrow{R_2+R_1, R_3-2R_1} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 3 & -6 & 0 \end{pmatrix} \xRightarrow{R_3-R_2}$$

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \xRightarrow{\frac{1}{3}R_2} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We end up with the system

$$\begin{cases} x + y - z = 0 \\ \quad y - 2z = 0 \end{cases}$$

The variable z is free, and therefore can be any number. We denote this by $z = t$ where $t \in \mathbb{R}$. Next, we find from the second equation that $y = 2t$ and then from the first equation that $x = -2t + t = -t$. With this, we have a parametrization to the set of solutions:

$$\left\{ \begin{pmatrix} -t \\ 2t \\ t \end{pmatrix} : t \in \mathbb{R} \right\}$$

A general element in the set can therefore be written as:

$$\begin{pmatrix} -t \\ 2t \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$$

The vector $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ belongs to the space as it solves the given linear system (check!) and every other element in the space is a linear combination of it, so it is a spanning set for the space. The set with only this single element is also linearly independent, since if

$$t\begin{pmatrix}-1\\2\\1\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$$

then t must be zero. (In fact, any set containing one element, who is different from zero, is linearly independent). Since the set

$$\left\{\begin{pmatrix}-1\\2\\1\end{pmatrix}\right\}$$

is both a spanning set for the space, and linearly independent, we conclude that it is a basis for the space.

Remark. In fact, we discussed in class the fact that if you follow this algorithm to find a spanning set for a solution of a linear system, then the spanning set you will find will always be linearly independent and therefore a basis. So, instead of checking for linear independence you can state this remark from class. But be careful, the remark holds **only** for questions that have exactly this form (or that at some point reach this form).

v. As usual, we start by finding a parametrization to the space,

$$\{p(x)\in\mathbb{R}_2[x]:p(1)=p(2)\}=\{a+bx+cx^2:a+b+c=a+2b+4c\}=$$
$$=\{a+bx+cx^2:b=-3c\}=\{a-3cx+cx^2:a,c\in\mathbb{R}\}.$$

Next, we look for a spanning set. Every element in the space can be written as:

$$a-3cx+cx^2=a\cdot 1+c(-3x+x^2).$$

The polynomials 1 and $-3x+x^2$ belong to the space as they both satisfy the rule $p(1)=p(2)$ (check!). Since every other element in the space is a linear combination of these elements we conclude that $\{1,-3x+x^2\}$ is a spanning set for the space. Next, we check if it is also linearly independent. We look at a linear combination of these elements which is equal to the zero of the space:

$$a\cdot 1+c(-3x+x^2)=0$$

We rewrite the RHS,

$$a-3cx+cx^2=0.$$

Two polynomials are equal iff all of their coefficient are equal (note that we think of the zero on the LHS as the constant zero polynomial). It follows that $a=c=0$. This implies that the two polynomials are linearly independent. Since the set $\{1,-3x+x^2\}$ is both linearly independent and a spanning set for the space, we conclude that this is a basis for the space.

vi. We start by finding a parametrization for the space:

$$\{p(x) \in \mathbb{R}_3[x] : p(1) = 0 \text{ and } p'(1) = 0\} =$$

$$= \{a + bx + cx^2 + dx^3 : a + b + c + d = 0 \text{ and } b + 2c + 3d = 0\} =$$

$$= \{a + bx + cx^2 + dx^3 : a + b + c + d = 0 \text{ and } b = -2c - 3d\} =$$

$$= \{a + (-2c - 3d)x + cx^2 + dx^3 : a + (-2c - 3d) + c + d = 0\} =$$

$$= \{a + (-2c - 3d)x + cx^2 + dx^3 : a = c + 2d\} =$$

$$= \{(c + 2d) + (-2c - 3d)x + cx^2 + dx^3 : c, d \in \mathbb{R}\}$$

It follows that every vector in the space has the form:

$$(c + 2d) + (-2c - 3d)x + cx^2 + dx^3 = c(1 - 2x + x^2) + d(2 - 3x + x^3).$$

Since both $1-2x+x^2$ and $2-3x+x^3$ are in the given space (check that they both satisfy the conditions $p(1) = 0$ and $p'(1) = 0$!), and every other vector in the space is a linear combination of these two vectors, we conclude that $\{1 - 2x + x^2, 2 - 3x + x^3\}$ is a spanning set for the space. We now need to check if the set is also linearly independent. We look at:

$$c(1 + 2x + x^2) + d(2 - 3x + x^3) = 0.$$

We rewrite the RHS,

$$(c + 2d) + (-2c - 3d)x + cx^2 + dx^3 = 0$$

Two polynomials are equal iff all of their coefficient are equal (note that we think of the zero on the RHS as the constant zero polynomial). It follows that $d = c = 0$. This implies that the two polynomials are linearly independent. Since the set $\{1 - 2x + x^2, 2 - 3x + x^3\}$ is both linearly independent and a spanning set for the space, we conclude that this is a basis for the space.

Remark. Having two conditions on the polynomials in the set, makes it rather annoying to find a parametrization for the space as was done here. If we use coordinates to translate the question into a question about n-tuples then it becomes a much friendlier question (in fact, the question becomes just as the form that was given in the part Q1(iv)), which is an easy question to solve. Since we did not yet use coordinates in a rigorous form when this HW was given, I am avoiding their use in this solution page. Solutions with coordinates will appear in the solution to HW7.

vii. We find a parametrization for the set.

$$\{A \in M_2(\mathbb{R}) : A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0\} =$$

$$= \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0\} =$$

$$= \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : \begin{pmatrix} a+2b \\ c+2d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\} =$$

$$= \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : a = -2b \text{ and } c = -2d\} =$$

$$= \{\begin{pmatrix} -2b & b \\ -2d & d \end{pmatrix} \in M_2(\mathbb{R}) : b, d \in \mathbb{R}\}.$$

We see that every vector in the space can be written as:

$$\begin{pmatrix} -2b & b \\ -2d & d \end{pmatrix} = b \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}.$$

Since both of the matrices on the RHS are in the space (check!) and every other vector in the space is a linear combination of these two matrices we conclude that $\{\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}\}$ is a spanning set for the space. We need to check that the set is linearly independent. Let us do it a bit different this time: We know that a set is linearly dependent iff there exist in it a vector that is a linear combination of all other vectors. However, in our set there are only two vectors, and neither of them is a linear combination of the other (Why?). We conclude that the set is linearly independent. Since it is both a spanning set for the space and linearly independent, the set is a basis for the space.

2. a Consider the space $\mathbb{R}_2[x]$. Recall that we mentioned in class that $B = (1, x, x^2)$ and $C = (x, x^2, 1)$ are both ordered bases for $\mathbb{R}_2[x]$.

i. We need to prove that the (ordered) set $D = (1, 1+x, (1+x)^2)$ is both linearly independent and a spanning set for the space. We start by checking linear independence:

$$a \cdot 1 + b(1+x) + c(1+x)^2 = 0$$

We rewrite the LHS,

$$(a+b+c) + (b+2c)x + cx^2 = 0$$

Since two polynomials are equal iff all of their coefficients are equal we get $a+b+c = 0$, $b+2c =$, and $c = 0$. It follows that $a = b = c = 0$. So the set is linearly independent.

We now show that it is a spanning set. First we note that each one of the vectors $1, x, x^2$ is a linear combination of the vectors in $D = (1, 1 + x, (1 + x)^2)$. Indeed, it is easy to see that:

$$1 = 1 \cdot 1 + 0 \cdot (1 + x) + 0 \cdot (1 + x)^2$$

$$x = -1 \cdot +1 \cdot (1 + x) + 0 \cdot (1 + x)^2$$

while for the third vector we solve:

$$x^2 = a \cdot 1 + b(1 + x) + c(1 + x)^2.$$

We rewrite the RHS;

$$x^2 = (a + b + c) + (b + 2c)x + cx^2.$$

Since two polynomials are equal iff all of their coefficients are equal we get the equations $c = 1$, $b + 2c = 0$, $a + b + c = 0$ which give $c = 1, b = -2, a = 1$. In particular this implies that the equation has a solution, so indeed x^2 is a linear combination of $1, 1 + x, (1 + x)^2$. From the claim we studied in class, if $v_1, ..., v_n$ belong to some subspace W, then $\text{span}\{v_1, ..., v_n\} \subset W$. So

$$\text{span}\{1, x, x^2\} \subset \text{span}\{1, 1 + x, (1 + x)^2\}.$$

Since $1, x, x^2$ is a spanning set for $\mathbb{R}_2[x]$ we conclude that,

$$\mathbb{R}_2[x] \subset \text{span}\{1, 1 + x, (1 + x)^2\}.$$

On the other hand, all of the vectors in $\{1, 1+x, (1+x)^2\}$ belong to $\mathbb{R}_2[x]$ so (by the same result as above)

$$\text{span}\{1, 1 + x, (1 + x)^2\} \subset \mathbb{R}_2[x].$$

Therefore,

$$\text{span}\{1, 1 + x, (1 + x)^2\} = \mathbb{R}_2[x].$$

This implies that $\{1, 1 + x, (1 + x)^2\}$ is a spanning set for $\mathbb{R}_2[x]$. Since it is also linearly independent, we conclude that the system is a basis for the space.

Remark. Once we have dimension considerations in our disposal (starting HW7), the question becomes much easier: We know that the dimension of $\mathbb{R}_2[x]$ is 3 (proved in class). The theorem we studied in class, implies that if a VS has dimension 3 then any set of 3 vectors which is linearly independent is also a spanning set. So in this question it is enough to prove that the set is linearly independent and then use this theorem to conclude that the set is a basis.

ii. We use the following ordered bases: $B = (1, x, x^2)$, $C = (x, x^2, 1)$ and $D = (1, 1 + x, (1 + x)^2)$.
To compute $[3 - 2x + x^2]_B$ we write $3 - 2x + x^2$ as a linear combination of the elements in B (keeping order).

$$3 - 2x + x^2 = 3 \cdot 1 + (-2)x + 1 \cdot x^2$$

So,

$$[3 - 2x + x^2]_B = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}.$$

To compute $[3 - 2x + x^2]_C$ we write $3 - 2x + x^2$ as a linear combination of the elements in C (keeping order).

$$3 - 2x + x^2 = (-2)x + 1 \cdot x^2 + 3 \cdot 1$$

So,

$$[3 - 2x + x^2]_B = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}.$$

To compute $[3 - 2x + x^2]_D$ we write $3 - 2x + x^2$ as a linear combination of the elements in D (keeping order). This is not as strait forward as the previous two cases, so we express this linear combination with a, b, c as coefficients of the linear combination and solve the required equations to find them.

$$3 - 2x + x^2 = a \cdot 1 + b(1 + x) + c(1 + x)^2.$$

We rewrite the RHS to get,

$$3 - 2x + x^2 = (a + b + c) + (b + 2c)x + cx^2.$$

We obtain the equations $c = 1$, $b + 2c = -2$, and $a + b + c = 3$. It follows that $c = 1$, $b = -4$, and $a = 6$. So,

$$[3 - 2x + x^2]_D = \begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix}.$$

iii. The fact that, $[p_1(x)]_B = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ implies that

$$p_1(x) = 1 \cdot 1 + 3 \cdot x + (-1)x^2 = 1 + 3x - x^2.$$

The fact that $[p_2(x)]_C = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ implies that

$$p_2(x) = 1 \cdot x + 3 \cdot x^2 + (-1) \cdot 1 = -1 + x + 3x^2.$$

The fact that $[p_3(x)]_D = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ implies that

$$p_3(x) = 1 \cdot 1 + 3 \cdot (1+x) + (-1)(1+x)^2 = 3 + x - x^2.$$

b. Consider the space $\mathbb{R}^3$.

i. We have showed in the previous HW (HW6 Q6) that in R^n a set of n vectors is linearly independent iff it is a spanning set (we have also proven a more general version of this claim in class). Since we are working in $\mathbb{R}^3$ and we have 3 vectors, it is enough to check for the values of k that make this set linearly independent. By the algorithm studied in recitation and class, we write these vectors as columns of a matrix and check for the values of k that guaranty that the corresponding homogeneous linear system has only one solution.

$$\left(\begin{array}{ccc|c} 1 & 2 & k & 0 \\ 3 & 1 & 7 & 0 \\ -1 & 3 & 1 & 0 \end{array}\right) \xrightarrow{R_2-3R_1, R_3+R_1} \left(\begin{array}{ccc|c} 1 & 2 & k & 0 \\ 0 & -5 & 7-3k & 0 \\ 0 & 5 & 1+k & 0 \end{array}\right) \xrightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 2 & k & 0 \\ 0 & -5 & 7-3k & 0 \\ 0 & 0 & 8-2k & 0 \end{array}\right)$$

The homogeneous system has exactly one solution iff there are no free variables in the echelon form (we cannot get a "lie" in a homogeneous system). This happens iff $8-2k \neq 0$ that is, iff $k \neq 4$.

ii. We insert $k=2$ and get the ordered basis $B = (\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix})$.

To find $[\begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}]_B$ we need to write this vector as a linear combination of the vectors in the basis (keeping order). So we need to find a, b, c which satisfy:

$$a \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + c \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}$$

We rewrite the LHS using observations which were made in class regarding the relation between linear combinations of n-tuples and multiplication of a matrix by a vector:

$$\begin{pmatrix} 1 & 2 & 2 \\ 3 & 1 & 7 \\ -1 & 3 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}$$

We write this linear system in matrix form and bring it to echelon form

$$\left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 3 & 1 & 7 & 14 \\ -1 & 3 & 1 & -8 \end{array}\right) \xRightarrow{R_2-3R_1,R_3+R_1} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -5 & 1 & 11 \\ 0 & 5 & 3 & -7 \end{array}\right) \xRightarrow{R_3+R_2} \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & -5 & 1 & 11 \\ 0 & 0 & 4 & 4 \end{array}\right)$$

It follows that $c = 1$ and therefore $b = -2$ and $a = 3$.
We conclude that

$$[\begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}]_B = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

iii. We insert $k = 2$ and get the ordered basis $B = (\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix})$.

If $[b]_B = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$ then

$$b = 1 \cdot \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} + (-2) \begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -10 \\ 0 \end{pmatrix}$$

3. i. To show that $\{v_1, ..., v_n\}$ is a maximal linearly independent set for V we need to show that it is a linearly independent set (which it clearly is, at it is a basis) and that if any vectors is added to the set then the resulting set is no longer linearly independent. So let $w \in V$ and consider the set $\{v_1, ..., v_n, w\}$. Since $\{v_1, .., v_n\}$ is a basis, it is in particular a spanning set for V and therefore w is a linear combination of $\{v_1, .., v_n\}$. From a result we proved in class, we know that if one of the vectors in a set is a linear combination of the rest of the vectors, then the set is linearly dependent. Since w is a linear combination of $\{v_1, .., v_n\}$ it follows that $\{v_1, ..., v_n, w\}$ is linearly dependent. Our claim is proved.

   ii. Since we are given that $\{v_1, ..., v_n\}$ is linearly independent, in order to prove that it is a basis we need to show that it is a spanning system for V. Assume for a contradiction that it is not. This means that there exists $w \in V$, such that $w \notin$span$\{v_1, ..., v_n\}$. By a claim from class ('Lemma 2') we conclude that $\{v_1, ..., v_n, w\}$ is linearly independent, which contradicts the fact that $\{v_1, ..., v_n\}$ is a maximal linearly independent set.

   iii. To show that $\{v_1, ..., v_n\}$ is a basis for span $\{v_1, ..., v_n\}$ we need to show that it is a spanning set for the space (as it clearly is, since the space we are considering is its span) and linearly independent, which we are given that it is. So, this question was too simple.

iv. We know that $\operatorname{span}\{v_1, ..., v_n\} = V$ so $\operatorname{span}\{v_1, ..., v_n, w\} = V$ iff

$$\operatorname{span}\{v_1, ..., v_n\} = \operatorname{span}\{v_1, ..., v_n, w\}.$$

By Lemma 1 from class this happens iff w is a linear combination of $\{v_1, ..., v_n\}$. Since $\{v_1, ..., v_n\}$ is a basis and therefore linearly independent we can apply Lemma 2 from class. In this case, w is a linear combination of $\{v_1, ..., v_n\}$ iff $\{v_1, .., v_n, w\}$ is linearly dependent. The claim is proved.

4. $i \Rightarrow ii$ By the theorem regarding invertible matrices we have seen in class, if the matrix is invertible then for every $b \in \mathbb{R}^n$ the linear system $(A|b)$ has at least one solution. By the algorithm to determine whether an n-tuple is in the span of other n-tuples, which was learned in recitation, the columns of this matrix span $\mathbb{R}^n$. Since there are exactly n columns we can apply Q6 from the previous HW (or the more general theorem which will be proved in class): A spanning system of n elements in $\mathbb{R}^n$ (or any space of dimension n) is a basis.

$ii \Rightarrow i$ If the columns are a basis then in particular they are a spanning set for $\mathbb{R}^n$. It follows from the algorithm studied in recitation that if $b \in \mathbb{R}^n$ then $(A|b)$ has a solution. By the theorem we proved in class regarding invertible matrices, since for every $b \in \mathbb{R}^n$ the system $(A|b)$ has a solution the matrix A is invertible.

$i \Rightarrow iii$ If A is invertible, then by the theorem regarding invertible matrices we studied in class, it follows that the REF of A is I. So A is row equivalent to I. We proved in recitation that if two matrices are row equivalent then their rows span the same space. It follows that the rows of A span all of $\mathbb{R}^n$. So the rows of A are a spanning set for $\mathbb{R}^n$. Since there are exactly n rows we can apply Q6 from the previous HW (or the more general theorem which will be proved in class): A spanning system of n elements in $\mathbb{R}^n$ (or any space of dimension n) is a basis.

$iii \Rightarrow i$ Let us consider the REF of A and denote it J. we want to prove that $J = I$ as then, by the theorem we proved in class regarding invertible matrices, it will follow that A is invertible. We proved in class that if two matrices are row equivalent then they span the same space. So the rows of J are a spanning set for $\mathbb{R}^n$.

Since there are exactly n rows we can apply Q6 from the previous HW (or the more general theorem which will be proved in class): A spanning system in $\mathbb{R}^n$ (or any space of dimension n) has at least n elements. So in J there are at least n rows different from zero (why does this follow?). Since there are n rows in J we conclude that non of them is the zero row. Since every row which is different from zero has a leading variable, it follows that there are n leading variables in J. Since J has exactly n columns we conclude that all of the variables are leading and there are no free variables. A REF matrix that all of its variables are leading, and that has no zero rows is I. We are done.

Remark. There are several different ways to prove each of the implications written above. If you wrote a different proof you may very well have written a correct proof. If you are not sure then check with us.