MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 4

1. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix}, \quad F = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}, \quad D = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & -2 \\ 2 & -1 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Compute the following expressions if defined: AB, BA, D^2, B^2, DC, CB, BC, FE, EF, CE, EC.

2. Consider the following matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 3 & 0 & 2 \\ 2 & -1 & -2 \\ -1 & 2 & 6 \end{pmatrix}.$$

   i. For each one of these matrices determine whether it is invertible or not, and compute its inverse when relevant.

   ii. Find $x, y \in \mathbb{R}$ which solve the equation
$$A \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

   iii. Find $x, y \in \mathbb{R}$ which solve the equation
$$B \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

   iv. Find a matrix $G \in M_3(\mathbb{R})$ which solves the equation
$$DG = E.$$

3. In each of the following there is a claim, which might be **true or false**. If the claim is true then prove it, and if it is false then provide a counterexample. (For counter examples you may choose any n you wish, but if you want to prove a claim then you should prove it for all possible n's).

   a. If $A \in M_n(\mathbb{R})$ satisfies $A^2 = 0$ then $A = 0$. (Here 0 is the zero matrix).

   b. If $A, B \in M_n(\mathbb{R})$ are such that $AB = BA$ then $AB^2 = B^2A$.

   c. Let $A, B, C \in M_n(\mathbb{R})$. If $AB = CB$ then $A = C$.

4. In each of the following there is a claim, which might be **true or false**. If the claim is true then prove it, and if it is false then provide a counterexample. (For counter examples you may choose any n you wish, but if you want to prove a claim then you should prove it for all possible n's).

   d. If $A, B \in M_n(\mathbb{R})$ are both invertible then AB is also invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

   d. If $A, B \in M_n(\mathbb{R})$ are such that AB is invertible then A and B are also invertible.

   e. If $A, B \in M_n(\mathbb{R})$ are such that $A + B$ is invertible then A and B are both invertible.

   e. If $A, B \in M_n(\mathbb{R})$ are both invertible then $A + B$ is also invertible.

   d. If $A, B \in M_n(\mathbb{R})$ are such that AB is invertible then BA is also invertible.

   g. If $A \in M_n(\mathbb{R})$ is such that A^3 is invertible, then A is invertible.

5. Prove the following claims.

   i A square matrix $A \in M_n(\mathbb{R})$ is called **'diagonal'** if all of its entries that are not on the main diagonal are equal zero, that is, A is diagonal if $(A)_{ij} = 0$ for all $i \neq j$. Here is an example of a diagonal matrix:

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

   Prove that if $A, B \in M_n(\mathbb{R})$ are both diagonal then both $A+B$ and AB are diagonal as well.

   ii. For a square matrix $A \in M_n(R)$ the **'trace'** of A, denoted $tr(A)$, is the sum of all of its entries on the main diagonal, that is $tr(A) = \sum_{i=1}^{n}(A)_{ii}$. Here is an example of a trace computation:

$$tr\begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix} = 3 + 2 + (-1) = 4.$$

   For $A, B \in M_n(\mathbb{R})$ prove that $tr(AB) = tr(BA)$.

   iii. For a square matrix $A \in M_n(R)$ the **'transposed'** of A, denoted A^T, is the matrix obtained by turning each row of A into a column by order, that is $(A^T)_{i,j} = (A)_{j,i}$. Here is an example of a transposed computation:

$$\begin{pmatrix} 3 & 5 & 7 \\ 0 & 2 & 4 \\ 8 & 1 & -1 \end{pmatrix}^T = \begin{pmatrix} 3 & 0 & 8 \\ 5 & 2 & 1 \\ 7 & 4 & -1 \end{pmatrix}.$$

   For $A, B \in M_n(\mathbb{R})$ prove that $(AB)^T = B^T A^T$.