MATH-1564, K1, TA: Sam, Instructor: Nitzan, Sigal Shahaf
HW7 ; Alexander Guo

1. (a) The basis is: $\left(\begin{pmatrix}1\\1\\1\\0\\-2\end{pmatrix}, \begin{pmatrix}2\\-1\\0\\5\\2\end{pmatrix}, \begin{pmatrix}1\\-1\\3\\5\\1\end{pmatrix}\right)$. To show that something is a basis, we want it to be a spanning set for the given vector space V and linearly independent. To prove the former is true, we want to find the span of our basis which can be denoted as: $\left(a\begin{pmatrix}1\\1\\1\\0\\-2\end{pmatrix} + b\begin{pmatrix}2\\-1\\0\\5\\2\end{pmatrix} + c\begin{pmatrix}1\\-1\\3\\5\\1\end{pmatrix} : a, b, c \in \mathbb{R}\right)$. It is immediately apparent that the span of this equals V, since the given vector space V can also be written as $\left(t\begin{pmatrix}1\\1\\1\\0\\-2\end{pmatrix} + r\begin{pmatrix}2\\-1\\0\\5\\2\end{pmatrix} + s\begin{pmatrix}1\\-1\\3\\5\\1\end{pmatrix} : t, r, s \in \mathbb{R}\right)$. Now, to prove if the basis is linearly independent, we take our basis vectors and combine them into a larger augmented homogeneous matrix: $\left(\begin{array}{ccc|c}1 & 2 & 1 & 0\\1 & -1 & -1 & 0\\1 & 0 & 3 & 0\\0 & 5 & 5 & 0\\-2 & 2 & 1 & 0\end{array}\right)$. We then take RREF which is $\left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right)$. Since there are no free variables and the only solution is the trivial solution, this is indeed linearly independent.

(b) The basis is (in order): $\begin{pmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, \begin{pmatrix}0&0&1&0\\0&0&0&0\\1&0&0&0\\0&0&0&0\end{pmatrix},$

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

To prove this is a basis, let us denote each matrix as A_n where n is its order in the basis set $(1 - 10)$. We first want to show it is a spanning set. By definition, the spanning set is given by $c_1A_1 + c_2A_2 + ... + c_{10}A_{10}$ where $c_1, ..., c_{10} \in \mathbb{R}$. In a single matrix, this is equivalent to $\left(\begin{pmatrix} c_1 & c_2 & c_3 & c_4 \\ c_2 & c_5 & c_6 & c_7 \\ c_3 & c_6 & c_8 & c_9 \\ c_4 & c_7 & c_9 & c_{10} \end{pmatrix} : c_1, ..., c_{10} \in \mathbb{R} \right)$. It is apparent that the span is equal to the original vector space V because it is, by definition, a collection of all symmetric matrices. Therefore, our base is a spanning set. Next, we want to prove that this is linearly independent. To do this, we want to show that the only solution to this system $a_1A_1 + a_2A_2 + ... + a_{10}A_{10} = M_4(0)$ is the trivial solution where $(a_1 - a_{10} = 0)$. To show this, let us attempt to solve the linear system: $\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ a_2 & a_5 & a_6 & a_7 \\ a_3 & a_6 & a_8 & a_9 \\ a_4 & a_7 & a_9 & a_{10} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Therefore, it is clear that each entry of the matrix is equal to 0, and there is only one variable in it, which means that all $a_1 - a_{10} = 0$, making the trivial solution the only solution, and thus this basis is linearly independent.

(c) The basis is (in order): $\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

To prove this is a basis, let us denote each matrix as A_n where n is its order in the basis set $(1 - 6)$. We first want to show it is a spanning set. By definition, the spanning set is given by $c_1A_1 + c_2A_2 + ... + c_6A_6$ where $c_1, ..., c_6 \in \mathbb{R}$. In a single matrix, this is equivalent to

$$\left(\left\{\begin{pmatrix} 0 & c_1 & c_2 & c_3 \\ -c_2 & 0 & c_4 & c_5 \\ -c_2 & -c_4 & 0 & c_6 \\ -c_3 & -c_5 & -c_6 & 0 \end{pmatrix} : c_1, ..., c_6 \in \mathbb{R}\right\}\right).$$ It is apparent that the span is equal to the original vector space V because it is, by its definition a collection of all anti-symmetric matrices. Therefore, our base is a spanning set. Next, we want to prove that this is linearly independent. To do this, we want to show that the only solution to this system $a_1A_1 + a_2A_2 + ... + a_6A_6 = M_4(0)$ is the trivial solution where $(a_1 - a_6 = 0)$. To show this, let us attempt to solve the linear system: $\begin{pmatrix} 0 & a_1 & a_2 & a_3 \\ -a_2 & 0 & a_4 & a_5 \\ -a_2 & -a_4 & 0 & a_6 \\ -a_3 & -a_5 & -a_6 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Therefore, it is clear that each entry of the matrix is equal to 0, and there is only one variable in it, which means that all $a_1 - a_6 = 0$, making the trivial solution the only solution, and thus this basis is linearly independent.

(d) The basis is $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$. We want to show that this spans the solution of the given homogenous equations. First, let us find the solutions, by taking the RREF of the matrix of linear systems given: $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ -1 & 2 & -5 & 0 \\ 2 & 5 & -8 & 0 \end{array}\right) \to$ $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$. Since the third column is free, parameterize $z = t$, then $y = 2t$, and $x = -t$. We can then express the solutions as $t\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ where $t \in \mathbb{R}$. Therefore, the span of our basis is equal to the set of solutions, and it is a spanning set. To show that our basis is linearly independent we want to show that the augmented system only has the trivial solution. $\left(\begin{array}{c|c} -1 & 0 \\ 2 & 0 \\ 1 & 0 \end{array}\right)$. Since there is only one vector, the only way for it to reach zero would be through the trivial solution. Thus, our basis is also linearly independent.

(e) The basis is $(-\frac{1}{3}x^2 + x, 1)$. To prove it is a spanning set, let us find some way to parameterize the given vector space. Since $P(1) = P(2)$, we know that for $a, b, c \in \mathbb{R}$, $a(1)^2 + b(1) + c = a(2)^2 + b(2) + c \to a + b + c = 4a + 2b + c \to a = -\frac{1}{3}b$.

Then, plug this in and we get $P(x) = -\frac{1}{3}bx^2 + bx + c = b(-\frac{1}{3}x^2 + x) + c(1)$. Therefore, the span of the basis equals the vector space, and it is a spanning set. To prove linear independence, we want to show if the linear combination of elements in the basis equals 0. So for some $a_1, a_2 \in \mathbb{R}$, we want to show whether $a_1(-\frac{1}{3}x^2 + x) + a_2(1) = 0$ only yields the trivial solution for a_1, a_2. To simplify this we use the coefficients for each polynomial to make a linear system: $-a_1\frac{1}{3}x^2 = 0x^2, a_1x = 0x, a_2 = 0$. Thus, a_1, a_2 are clearly 0, and the system only has a trivial solution, so it is linearly independent.

(f) The basis is $(-0.5x^3 + x^2, x, 0.5x^3 + 1)$. Let us find the vector space in terms of parameterizations. For $a, b, c \in \mathbb{R}$, $P(1) = P'(1)$ equivalent to $a + b + c + d = 3a + 2b + c$, which means $a = (-0.5b + 0.5d)$. Then, $P(x) = (-0.5b + 0.5d)x^3 + bx^2 + cx + d = b(-0.5x^3 + x^2) + c(x) + d(0.5x^3 + 1)$. Therefore, the span of the basis is equivalent to this vector space, and it is a spanning set. Then, to prove linear independence for the basis, we want to show that the homogenous linear combination of the basis only has the trivial solution. So, for some $a_1, a_2, a_3 \in \mathbb{R}$, $-0.5a_1x^3 + 0.5a_3x^3 = 0x^3, a_1x^2 = 0x^2, a_2x = 0x, a_3 = 0$. It is immediately apparent in the latter three equations that $a_1, a_2, a_3 = 0$, so indeed only the trivial solution exists and the basis is linearly independent.

(g) The basis is $\left(\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}\right)$. The matrix A that satisfies $A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$ can be represented as $\begin{pmatrix} -2b & b \\ -2d & d \end{pmatrix}$. This is the same as $b\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}$. Evidently, the span of the basis equals this, so the span of the basis is a spanning set. We then want to show that for $a_1\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} + a_2\begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, the only solution would be $a_1 = a_2 = 0$. So, we set a linear system for each entry of the matrix: $-2a_1 + 0a_2 = 0, a_1 + 0a_2 = 0, 0a_1 - 2a_2 = 0, 0a_1 + a_2 = 0$. Evidently, $a_1, a_2 = 0$, so the the homogenous linear combination only has the trivial solution, therefore this basis is linearlry independent.

2. (a) i. We first want to prove that D is a spanning set for $\mathbb{R}_2$. Let $S = (1, x, x^2)$ and let $T = (1, 1 + x, (1 + x)^2)$. We then want to show that $S \in Span(T)$.

$$1 = 1(1) + 0(1 + x) + 0(1 + x)^2$$

$$x = -1(1) + 1(1 + x) + 0(1 + x)^2$$

$$x^2 = 1(1) - 2(1 + x) + 1(1 + x)^2$$

Then, because $(1, x, x^2) \in span(T)$, we can use the theorem $S \subseteq Span(T)$, then $Span(S) \subseteq Span(T)$. So, $Span(1, x, x^2) = \mathbb{R}_2(x) \subseteq Span(T)$. Thus, $Span(1, 1 + x, (1 + x)^2) = \mathbb{R}_2(x)$. Then, we want to prove that D is linearly independent, so if $a_1(1) + a_2(1+x) + a_3(1+x)^2 = 0$, only when $a_1, a_2, a_3 = 0$. We simply set up the system of equations: $a_1 + a_2 + a_3 = 0, a_2 x + 2a_3 x = 0x, a_3 x^2 = 0x^2$. It is evident that $a_1, a_2, a_3 = 0$, so the only solution to this system is the trivial one. Therefore, the basis is linearly independent.

ii. $[3 - 2x + x^2]_B = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$[3 - 2x + x^2]_C = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix}$

$[3 - 2x + x^2]_D = \begin{pmatrix} 6 \\ -4 \\ 1 \end{pmatrix}$

iii. $p_1(x) = 1(1) + 3(x) - 1(x^2) = 1 + 3x - x^2$
$p_2(x) = 1(x) + 3(x^2) - (1) = x + 3x^2 - 1$
$p_3(x) = 1(1) + 3(1+x) - (1 + 2x + x^2) = 3 + x - x^2$.

(b) i. When we employ the algorithm for determining the basis, we form the combined matrix and take the echelon form which is $\begin{pmatrix} 1 & 2 & k \\ 0 & -5 & -3k+7 \\ 0 & 0 & -2k+8 \end{pmatrix}$. We know that in order for all vectors in B to be part of the basis, there cannot be any rows of 0 in the resulting matrix. Therefore, when $k = 4$, we end up with a row of zero, so when $k \neq 4$, B is a basis.

ii. When k = 2, we can set up an augmented matrix $\left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 3 & 1 & 7 & 14 \\ -1 & 3 & 1 & -8 \end{array}\right)$. The solution to this which is also $[\begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix}]_B$ is $\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

iii. $\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} - 2\begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -10 \\ 0 \end{pmatrix}$

(c) i. $\left[\begin{pmatrix} 6 & -3 \\ -4 & 2 \end{pmatrix}\right]_B = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$

ii. $\left[\begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix}\right]_B = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

3. (a) If $v_1, ..., v_n$ is a basis, then they are respectively linearly independent. To prove it is

the maximal linearly independent set, we want to show that adding any v results in a linearly dependent set. So, suppose another $u \in V$. However, we know that the basis spans all of V, so the new element u can be written as a linear combination of $v_1, ..., v_n$. Therefore, adding it to the basis would make it linearly dependent. So, the basis is the maximum linearly independent set.

(b) If $v_1, ..., v_n$ is the maximal linearly independent set in V, then adding another component makes it linearly dependent. In other words, this means that every element in V can be expressed as a linear combination of the basis, so when you add that element of V, the basis becomes liniearly dependent. We can then make the conjecture that the basis is indeed a spanning set. We are also given that it is independent, so therefore it is a basis.

(c) Let us fist show that the basis is a spanning set. The basis is $v_1, ..., v_n$, and its span is equal to $Span(v_1, ..., v_n)$ which is exactly equal to the $Span(v_1, ..., v_n)$, so $v_1, ..., v_n$ is a spanning set for $Span(v_1, ..., v_n)$. We are also given that $v_1, ..., v_n$ is linearly independent, so it is a basis.

(d) If $v_1, ..., v_n$ is a basis for V, then that means $Span(v_1, ..., v_n) = V$. Since this is true, the only way for $Span(v_1, ..., v_n, w) = V$ is if $w \in V$. If $w \in V$, it is also true if and only if $w \in Span(v_1, ..., v_n)$, then if and only if w is a linear combination of $v_1, ..., v_n$, which means $v_1, ..., v_n, w$ is linearly dependent. Therefore, $span(v_1, ..., v_n, w) = V$ if and only if $(v_1, ..., v_n, w)$ is linearly dependent.

4. (a) Proving $1 \to 2$: If A is invertible, then its equivalent RREF form is the identity matrix I_n as proved in previous lessons. Since the identity matrix has a pivot in every row and column, it thus spans all of $\mathbb{R}^n$. Additionally, since we know that A and I_n are row equivalent, the spans of their columns should be equal. So, $Span(A) = Span(I_n) = \mathbb{R}^n$. Therefore, we proved that the columns of A span $\mathbb{R}^n$. Similarly, we want to prove that A is linearly independent. Again we find its RREF to be I_n and want to show that

$$a_1 \begin{pmatrix} 1 \\ 0 \\ ... \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 \\ 1 \\ ... \\ 0 \end{pmatrix} + ... + a_n \begin{pmatrix} 0 \\ 0 \\ ... \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \end{pmatrix} \text{ only has the trivial solution.}$$

When we simplify this equation, we get that $a_1, ..., a_n = 0$, which means that the columns in A are also linearly independent. Therefore it is a basis.

(b) Proving $2 \to 3$: Since the columns of A form a basis for $\mathbb{R}^n$, there exists a row equivalent matrix denoted as A' which has at least a pivot in each column, and also spans all of $\mathbb{R}^n$ while being linearly independent. Since A is a square matrix of n dimensions, however, it means that each column must have a pivot. If there is a pivot in each column of a square matrix, its row equivalent RREF form is the identity matrix I_n. Since I_n is row equivalent to A, its columns are also a basis. Furthermore, in an identity matrix, each respective row is the same as each respective column. Therefore, the rows in the Identity matrix are also a basis for $\mathbb{R}^n$ because it is the same as the columns and we are given that the columns are already a basis. Then, since I_n and A are row equivalent, we can reason that the rows in A will also be a basis.

(c) Proving $3 \rightarrow 1$: If the rows of A form basis for $\mathbb{R}^n$, this implies that A is row equivalent to the identity matrix. One of the definitions for invertibility is if its RREF is the identity matrix. Therefore, A is invertible.