MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 3 – Solutions

1. i. Here are some examples of such equivalence relations:
   a. Example 1: "For $a, b \in S$ we say that $a \sim b$ if b has the same amount of siblings as a does". Short Proof:
      - Reflexivity: For $a \in S$, a has the same amount of siblings as herself/himself, so $a \sim a$.
      - Symmetry: Let $a, b \in S$ be such that $a \sim b$. This means that b has the same amount of siblings as a does, so clearly a has the same amount of siblings as b does, and therefore $b \sim a$.
      - Transitivity: Let $a, b, c \in S$ be such that $a \sim b$, and $b \sim c$. This means that b has the same amount of siblings as a does and c has the same amount of siblings as b does. It follows that c has the same amount of siblings as a does, as they both have the same amount of siblings as b. So, $a \sim c$.
   b. Example 2: "For $a, b \in S$ we say that $a \sim b$ if for some $k \in \mathbb{N}$ there exist k people $c_1, c_2..c_k \in S$, such that a has had a conversation with c_1, c_1 has had a conversation with c_2, c_2 has had a conversation with c_3, and so on up to c_k, who has had a conversation with b.". Short Proof:
      - Reflexivity: For $a \in S$, since a is in the 1564 class, a has certainly had a good thought in their mind, and so, a has had a conversation with themselves. It follows that $a \sim a$.
      - Symmetry: Let $a, b \in S$ be such that $a \sim b$. This means that for some $k \in \mathbb{N}$ there exist k people $c_1, c_2..c_k \in S$, such that a has had a conversation with c_1, c_1 has had a conversation with c_2, c_2 has had a conversation with c_3, and so on up to c_k has had a conversation with b. But then, if we consider the same sequence of people backwards, that is if we consider the people $c_k, ..., c_2, c_1$, then b has had a conversation with c_k, c_k has had a conversation with $c_{(k-1)}$, $c_{(k-1)}$ has had a conversation with $c_{(k-2)}$, and so on up to c_1, who has had a conversation with b. So $b \sim a$.
      - Transitivity: Let $a, b, c \in S$ be such that $a \sim b$, and $b \sim c$. This means that for some $k \in \mathbb{N}$ there exist k people $c_1, c_2..c_k \in S$, such that a has had a conversation with c_1, c_1 has had a conversation with c_2, c_2 has had a conversation with c_3, and so on up to c_k has had a conversation with b, and also for some $m \in \mathbb{N}$ there exist m people $d_1, d_2, .., d_m \in S$, such that b has had a conversation with d_1, d_1 has had a conversation with d_2, and so on up to d_m who has had a conversation with c. So, for the positive integer $k+m+1$ we found a chain of $k+m+1$ people, $c_1, .., c_k, b, d_1, .., d_m$ which

connects between a and c in the required way. It follows that $a \sim c$.

c. Example 3: First, for $x \in S$ let us denote by Age(x) the age of x, rounded up to the closest integer. We consider the relation: "For $a, b \in S$ we say that $a \sim b$ if Age(a)−Age(b) is an even number". Short Proof:
   - Reflexivity: For $a \in S$, Age(a)-Age(a)= 0. Since 0 is an even number, it follows that $a \sim a$.
   - Symmetry: Let $a, b \in S$ be such that $a \sim b$. This means that Age(a)-Age(b) is even. Since

     $$\mathrm{Age}(b) - \mathrm{Age}(a) = -\big(\mathrm{Age}(a) - \mathrm{Age}(b)\big)$$

     , and since the positive inverse of an even number is an even number, it follows that Age(b)-Age(a) is even. So $b \sim a$.
   - Transitivity: Let $a, b, c \in S$ be such that $a \sim b$, and $b \sim c$. This means that Age(a)-Age(b) is even and Age(b)-Age(c) is even. Since

     $$\mathrm{Age}(a) - \mathrm{Age}(c) = \big(\mathrm{Age}(a) - \mathrm{Age}(b)\big) + \big(\mathrm{Age}(b) - \mathrm{Age}(c)\big)$$

     , and since the sum of two even numbers is an even number, it follows that Age(a)-Age(c) is even. So $a \sim c$.

ii. We give two examples, an additional example can be found in the corresponding question on the HW page. Example 1: The relation "For $a, b \in S$ we say that $a \sim b$ if a and b had a conversation" is not an equivalence relation because it is not transitive, it could happen that a and b had a conversation, as well as b and c, but a never talked to c. Example 2: The relation ""For $a, b \in S$ we say that $a \sim b$ if b is taller then a" is not an equivalence relation because it is not reflexive, no person is taller then herself/himself, nor is it symmetric (why?).

iii. Proof that the given relation is an equivalence relation:
   - Reflexivity: For $(x, y) \in \mathbb{R}^2$, we have $x = x$ and so $(x, y) \sim (x, y)$.
   - Symmetry: Let $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ be such that $(x_1, y_1) \sim (x_2, y_2)$. This means that $x_1 = x_2$ and so $x_2 = x_1$. Therefore $(x_2, y_2) \sim (x_1, y_2)$.
   - Transitivity: Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathbb{R}^2$ be such that $(x_1, y_1) \sim (x_2, y_2)$, and $(x_2, y_2) \sim (x_3, y_3)$. This means that $x_1 = x_2$ and $x_2 = x_3$. It follows that $x_1 = x_3$ and therefore that $(x_1, y_1) \sim (x_3, y_3)$.

   An equivalence class of a point $(a, b) \in \mathbb{R}^2$ is given by

   $$[(a, b)] = \{(x, y) \in \mathbb{R}^2 : x = a\}$$

   and so, the equivalence classes are all the lines parallel to the x-axis.

iv. The given relation is indeed not an equivalence relation. For example: $(3, 5) \nsim (3, 5)$, since $3 + 3 \neq 0$, and so the relation is not reflexive.

v. For $A, B \in M_{m\times n}(\mathbb{R})$ we defined that $A \sim B$ if B can be obtained from A by performing a sequence of row operations.

vi. Any matrix obtained from the given matrix by performing row operations will provide a good example. We skip this here.

2. To determine whether the two given matrices are row equivalent, we bring both of them to REF and use the theorem that was proved in class: Two matrices are row equivalent iff they have the same reduced echelon form.

i.

$$\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix} \xRightarrow{R_2-3R_1} \begin{pmatrix} 1 & 3 \\ 0 & -10 \end{pmatrix} \xRightarrow{\frac{-1}{10}R_2}$$
$$\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \xRightarrow{R_1-3R_2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix} \xRightarrow{R_2-4R_1} \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$$

We conclude that the matrices are not row equivalent.

ii.

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix} \xRightarrow{R_2+R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 3 & 3 \end{pmatrix} \xRightarrow{\frac{1}{3}R_2}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \xRightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 3 & -1 \\ 2 & 2 & 5 \end{pmatrix} \xRightarrow{R_1\leftrightarrow R_2} \begin{pmatrix} 2 & 2 & 5 \\ 0 & 3 & -1 \end{pmatrix} \xRightarrow{\frac{1}{2}R_1,\frac{1}{3}R_2}$$
$$\begin{pmatrix} 1 & 1 & 2.5 \\ 0 & 1 & -\frac{1}{3} \end{pmatrix} \xRightarrow{R_1-R_2} \begin{pmatrix} 1 & 0 & 2\frac{5}{6} \\ 0 & 1 & -\frac{1}{3} \end{pmatrix}$$

We conclude that the matrices are not row equivalent.

iii.

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ -2 & -2 & 8 \end{pmatrix} \xRightarrow{R_3+2R_1} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 10 \end{pmatrix} \xRightarrow{\frac{1}{3}R_2,\frac{1}{10}R_3}$$
$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \xRightarrow{R_3-R_2,R_1-R_2} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & -1 & 1 \end{pmatrix} \xrightarrow{R_3 \leftrightarrow R_1} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_2 - R_1}$$
$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix} \xrightarrow{R_3 - R_2} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 + R_2}$$
$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

We conclude that the matrices are not row equivalent.

3. a. This claim was proved in class. We now present exactly the same proof. For the given matrix A one can perform a sequence of row operations, say k operations, to obtain its echelon form. In other words, there exist row operations $R_1, .., R_k$, such that the echelon form of A is equal to $R_k \cdot ... \cdot R_1 A$. Now, let $e \in \mathbb{R}^m$ be the m-tuple who's last entry is 1 and all other of its entries are 0, that is

$$e = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.$$

Let $b \in \mathbb{R}^m$ be the m-tuple given by $b = R_1^{-1} \cdot ... \cdot R_k^{-1} e$ (this is a short way of writing that b is obtained from e by performing the inverse row operations to the ones described above, in opposite order).
We claim that b is the required m-tuple. In other words, we claim that $(A|b)$ has no solution. Indeed, if we perform on $(A|b)$ the same row operations one performs to obtain from A its echelon form. We get

$$(R_k \cdot ... \cdot R_1 A | R_k \cdot ... \cdot R_1 b) = (R_k \cdot ... \cdot R_1 A | R_k \cdot ... \cdot R_1 \cdot R_1^{-1} \cdot ... \cdot R_k^{-1} e) = (R_k \cdot ... \cdot R_1 A | e).$$

This matrix $(R_k \cdot ... \cdot R_1 A | e)$ has on the left side of the line the echelon form of A, which we know has a row of zeros, so on the left side of the line the last row is a zero row. On the right side of the line, the last entry is 1 (this is how we chose e). So, this echelon form of the augmented matrix has a row which is a 'lie', $0 = 1$, this implies that indeed, the system has no solution.

b. Since the matrix has n columns, there are at most n pivots in its echelon form. As the matrix has m rows, and $m > n$ there must be a row with no pivot in the echelon form of A. A row with no pivot is a zero row, so A must have a zero row in its echelon form. The claim now follows from part (a).

c. Since the system $(A|0)$ has infinitely many solutions, the echelon form of A has a free variable. Since A has n columns, which correspond to n variables, there are at most $n-1$ leading variables in the echelon form. Since A has n rows it follows that in the echelon form of A there is a row with no leading variable. A row with no leading variable is a zero row. So the echelon form of A has a zero row. By part (a) of this question it now follows that there exists $b \in \mathbb{R}^n$ such that $(A|b)$ has no solution.

4. Proof:

$a \Rightarrow b$: If $(A|0)$ has exactly one solution then there are no free variables in the corresponding echelon form of A. Since A has n columns, there are n variables, if non of them is free then they are all pivot variables. So in the echelon form of A there are n pivot variables. Since A has n rows, and in each row there is at most one pivot entry, it follows that in **every** row of the echelon form of A there is a pivot entry. A row with a pivot entry is not the zero row. So, in the echelon form of A there is no zero row. For $b \in \mathbb{R}^n$ we consider now the linear system $(A|b)$, if we now perform on $(A|b)$ the same row operations as we performed above on $(A|0)$, we will obtain an echelon form of $(A|b)$ with no row of zeroes to the left of the line. This means that in this echelon form there is no row with a 'lie', which implies that $(A|b)$ has a solution. Since this holds for every $b \in \mathbb{R}^n$, we completed this direction of the proof.

$b \Rightarrow a$: Well, this is exactly what we proved in Question 3 part c. (Check that you understand why these two claims are the same!)

5. There are many counter examples to each one of these statements, we give just one example in each case.

a. Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. Note that A is in echelon form. This echelon form has a free variable (the third one) so $(A|0)$ has infinity many solutions. On the other hand this echelon form has no zero row, so for every $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ the augmented matrix $(A|b)$, which is already in echelon form, has no row with a 'lie'. This means that for every $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ the linear system $(A|b)$ has a solution.

b. Take $A = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then B is obtained from A by reducing the second row from the first row, and therefore A and B are row equivalent. It is easy to check that $(A|b)$ has an infinite amount of solutions while $(B|b)$ has no solution.

c. Take $A = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix}$. Then B is obtained from A by replacing the order of the first and second row and therefore $A \sim B$. On the other hand, any column opperation performed on A will keep its second row a zero row (in the same way that row operations keep a zero column), and so B cannot be obtained from A by performing row operations.

d. Take $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Then $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ are solutions to $(A|b)$, but clearly $u + v = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$ is not.

e. Take $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. Then $u = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is a solution to $(A|b)$, but clearly $5u = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$ is not.