MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 11

Note: Several questions below are not mandatory for submission. These are indicated explicitly.

1. Consider the following linear transformations:

$$S : M_2(\mathbb{R}) \mapsto \mathbb{R}^2 \qquad \text{given by} \qquad SA = A\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$L : \mathbb{R}_3[x] \mapsto \mathbb{R}^2 \qquad \text{given by} \qquad Lp = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}$$

$$\Phi : \mathbb{R}^3 \mapsto \mathbb{R}_3[x] \qquad \text{given by} \qquad \Phi\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b)+(a-2b+c)x+(b-3c)x^2+(a+b+c)x^3$$

$$T_A : \mathbb{R}^2 \mapsto \mathbb{R}^2 \qquad \text{where} \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

Determine which of the following compositions is defined. If it is defined then specify its domain, specify its codomain and find a formula for the composition:

$$\Phi \circ L, L \circ \Phi, T_A^2, S \circ T_A, T_A \circ S,$$
$$\Phi \circ T_A, T_A^2 \circ S, T_A \circ L \circ \Phi.$$

2. [**There is no mandatory submission of this question.**] Complete the following parts of proofs of theorems from class:
   i. If V, W are vector spaces and $T : V \to W$ is a linear transformation then ker$T$ is closed to multiplication by scalar.
   ii. If V, W are vector spaces and $T : V \to W$ is a linear transformation then ImT is closed to multiplication by scalar.
   iii. Corollary of the dimension formula: If V, W are vector spaces such that dim$V$=dim$U$ and $T : V \to W$ is a 1–1 linear transformation then T is onto.
   iv. Corollary of the dimension formula: If V, W are vector spaces such that dim$V$=dim$U$ and $T : V \to W$ is an onto linear transformation then T is 1–1.
   v. If V, W, U are vector spaces and $T : V \to W$, $S : W \to U$ is a linear transformation then $S \circ T$ 'respects' multiplication by scalar.

3. In each of the following you are given a transformation between a space $\mathbb{R}^n$ and a space $\mathbb{R}^m$ for some n and m. Find a matrix A for this transformation such that the transformation is equal to multiplication by this matrix.

i. Let $S : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the composition $S_3 \circ S_2 \circ S_1$ where S_1 is rotation by $\pi/4$ radians counterclockwise, $S_2(x, y) = (x, -y)$, and S_3 is rotation by $\pi/4$ radians clockwise.

ii. The transformation $T_A \circ L \circ \Phi$ where T_A, L, Φ were defined in Q1.

4. **[In this question it is mandatory to submit only odd numbered questions (i,iii,v, and so on).]**

Prove or disprove:

i. There exists $T : \mathbb{R}_2[x] \mapsto M_2(\mathbb{R})$ such that $1 - x + 2x^2$ is in its kernel and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is in its image.

ii. There exists $T : \mathbb{R}^3 \mapsto \mathbb{R}^5$ such that

$$kerT = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z + w = 1 \right\}$$

iii. There exists $T : \mathbb{R}^3 \mapsto \mathbb{R}^5$ such that

$$kerT = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\}$$

iv. For $U = \{p \in \mathbb{R}_3[x] : p(1) = p'(1)\}$ there exists an isomorphism $T : U \mapsto M_2(\mathbb{R})$.

v. For $U = \{p \in \mathbb{R}_3[x] : p(1) = p'(1)\}$ there exists an isomorphism $T : U \mapsto \mathbb{R}_2[x]$.

vi. There exists $T : \mathbb{R}^4 \mapsto \mathbb{R}^6$ such that dimker$T$=2dimImT.

vii. There exists $T : \mathbb{R}^4 \mapsto \mathbb{R}^6$ such that dimker$T$=2dimImT+1.

5. Let V be a vector space over $\mathbb{R}$ such that dim$V$=3 and let $T : V \mapsto V$ be a linear transformation which satisfies $T^3 = 0$ and $T^2 \neq 0$. Prove:

i. There exists $v \in V$ which is not the zero vector such that $Tv = 0$.

ii. Im$T \subseteq$kerT^2 and Im$T^2 \subseteq$ker$T$.

iii. ker$T \subset$kerT^2 and ker$T \neq$kerT^2

iv. dimker$T = 1$.

6. Let V, W be two vector spaces over $\mathbb{R}$. Assume that $T : V \mapsto W$ and $S : W \mapsto V$ are linear transformations which satisfy:

$$S \circ T = id_V$$

Prove or disprove:

i. T is 1–1.

ii. S is 1–1.

iii. $\dim W = \dim V$.

iv. If $\dim W = \dim V$ then S is an isomorphism.

v. If T is not onto then $\dim W \neq \dim V$.

7. **[In this question it is mandatory to submit only odd numbered questions (i,iii,v, and so on).]**

a. Consider the following linear transformations $T, S : \mathbb{R}_3[x] \mapsto \mathbb{R}_3[x]$ given by

$$Tp(x) = p'(x) \qquad \text{and} \qquad Sp(x) = p(x+1)$$

and consider the following bases of $\mathbb{R}^3[x]$:

$$E = \{1, x, x^2, x^3\}$$

and

$$B = \{1, 1+x, (1+x)^2, (1+x)^3\}$$

i. Find $[T]_E^B$, $[T]_B^B$.

ii. $[T]_B^E$, $[T]_E^E$.

iii. Find $[S]_E^E$, $[S]_B^B$.

iv. Find $[T \circ S]_E^E$.

v. $[T \circ S]_B^B$.

b. For each one of the following transformations choose a basis for the domain and the codomain of the transformation. Write the matrix that represents the transformation with respect to these bases, and use it to find a basis for the kernel and image of the transformation.

i. The transformations L from Q1.

ii. The transformations Φ from Q1.

8. Consider the vector space,

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}.$$

Let $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and consider the linear transformation $L : W \mapsto W$ defined by $LA = AH - HA$.

i. Prove that L indeed acts from W to W.

ii. Find an ordered basis for W and denote it B.

iii. Find $[L]_B^B$.

iv. Use $[L]_B^B$ to find a basis for the kernel and image of L.

9. Consider the linear transformation $S : \mathbb{R}_2[x] \mapsto \mathbb{R}^3$ which is defined by

$$Sp = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$$

   i. Is S an isomorphism? Justify your answer.

   ii. If your answer to part (i) was "yes" then find S^{-1}.

10. **[In this question it is mandatory to submit only odd numbered questions (i,iii,v, and so on).]**

   i. Consider the following ordered bases of $\mathbb{R}^3$:

$$B = (\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix})$$

$$C = (\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix})$$

$$E = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$$

   Find the following matrices of transition from basis to basis:

$$[id]_E^B; [id]_E^C; [id]_B^E; [id]_C^E; [id]_C^B; [id]_B^C.$$

   ii. Consider the transformations S and T and the bases B and E from Q7(a). Find the following matrices of transition from basis to basis:

$$[id]_E^B; [id]_B^E.$$

   Check that the formula of transition from basis to basis holds in the following cases:

$$[T]_B = [id]_B^E [T]_E [id]_E^B$$
$$[S]_E = [id]_E^B [S]_B [id]_B^E$$

   iii. Consider the vector space W and the linear transformation L in 8. Here are two bases of W:

$$B = (\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$$

$$C = (\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}).$$

   Find $[id]_B^C$, $[id]_C^B$ and check that the formula of transition from basis to basis holds in the following case:

$$[L]_C = [id]_C^B [L]_B [id]_B^C$$

11. Let V be a vector space such that $\dim V = 3$. Assume that $B = (v_1, v_2, v_3)$ and $C = (w_1, w_2, w_3)$ be bases of V such that

$$[id]_C^B = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 0 & 0 \\ 3 & 5 & 2 \end{pmatrix}$$

a. Are $\{w_1, v_2, v_3\}$ are linearly independent? Justify your answer.

b. Is it true that $w_2 = v_1 + v_3 - v_2$?

c. Find $[w_1]_B$.