MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 11

1. Compute the determinants of the following matrices:

$$\blacklozenge \quad \left|\begin{pmatrix} 2 & 6 & 16 \\ -3 & -6 & 18 \\ 5 & 12 & 35 \end{pmatrix}\right| = \left|\begin{pmatrix} 2 & 6 & 16 \\ -3 & -6 & 18 \\ 3 & 6 & 19 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 2 & 6 & 16 \\ -3 & -6 & 18 \\ 0 & 0 & 37 \end{pmatrix}\right| = 37[-12-(-18)] = 37\cdot 6 = 222$$

$$\blacklozenge \quad \left|\begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 3 & 2 & 0 \end{pmatrix}\right| = \left|\begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 5 \\ 0 & -4 & -9 \end{pmatrix}\right|$$

$$= 1\cdot[-63-(-20)] = -43$$

$$\blacklozenge \quad \left|\begin{pmatrix} 4 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 4 & -3 \end{pmatrix}\right| = 4\left|\begin{pmatrix} 2 & 0 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & -3 \end{pmatrix}\right|$$

$$= 4\left|\begin{pmatrix} 0 & 2 & -1 \\ -1 & 1 & -1 \\ 0 & 2 & -3 \end{pmatrix}\right| = -4(-1)\left|\begin{pmatrix} 2 & -1 \\ 2 & -3 \end{pmatrix}\right| = 4[-6-(-2)] = -16$$

$$\blacklozenge \quad \left|\begin{pmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}\right| = \left|\begin{pmatrix} 4 & -4 & 2 & -3 \\ 1 & 2 & 0 & 5 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 2 & 0 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 4 & -4 & -6 & -3 \\ 1 & 2 & 4 & 5 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 0 & 0 \end{pmatrix}\right| = -1\left|\begin{pmatrix} 4 & -6 & -3 \\ 1 & 4 & 5 \\ 2 & 3 & 4 \end{pmatrix}\right|$$

$$= -1\left|\begin{pmatrix} 0 & -22 & -23 \\ 1 & 4 & 5 \\ 0 & -5 & -2 \end{pmatrix}\right| = -1(-1)\left|\begin{pmatrix} -22 & -23 \\ -5 & -2 \end{pmatrix}\right| = 44-115 = -71$$

2. i. Let $a, b, c \in \mathbb{R}$. Prove that

$$\left|\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}\right| = (c-a)(c-b)(b-a)$$

We use row reduction operations to obtain,

$$\left|\begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix}\right| = \left|\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} 1 & a & a^2 \\ 0 & b-a & (b-a)(b+a) \\ 0 & c-a & (c-a)(c+a) \end{pmatrix}\right| = (b-a)(c-a)\left|\begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{pmatrix}\right|$$

We proceed by developing the determinant with respect to the first column. We get,

$$(b-a)(c-a)\left|\begin{pmatrix} 1 & a & a^2 \\ 0 & 1 & b+a \\ 0 & 1 & c+a \end{pmatrix}\right| = (b-a)(c-a)\cdot 1\cdot[1\cdot(c+a)-1\cdot(b+a)] = (c-a)(b-a)(c-b)$$

ii. Find the values of a for which the following set is a basis for $\mathbb{R}^3$:

$$\left\{\begin{pmatrix} a-1 \\ -3 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ a+5 \\ 6 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ a-4 \end{pmatrix}\right\}$$

We proved in class that an $n \times n$ matrix is invertible iff it's columns form a basis for $\mathbb{R}^n$. So, the considered 3–tuples form a basis iff the following matrix is invertible.

$$A = \begin{pmatrix} a-1 & 3 & -3 \\ -3 & a+5 & -3 \\ -6 & 6 & a-4 \end{pmatrix}$$

We proved in class that a matrix is invertible iff it's determinant is different from zero. So, the values of a for which the given 3–tuples form a basis, are exactly the values of a for which $|A| \neq 0$. We compute the determinant and find these values.

$$\left|\begin{pmatrix} a-1 & 3 & -3 \\ -3 & a+5 & -3 \\ -6 & 6 & a-4 \end{pmatrix}\right| = \left|\begin{pmatrix} a+2 & 3 & -3 \\ a+2 & a+5 & -3 \\ 0 & 6 & a-4 \end{pmatrix}\right|$$

$$= (a+2)\left|\begin{pmatrix} 1 & 3 & -3 \\ 1 & a+5 & -3 \\ 0 & 6 & a-4 \end{pmatrix}\right| = (a+2)\left|\begin{pmatrix} 1 & 0 & -3 \\ 1 & a+2 & -3 \\ 0 & a+2 & a-4 \end{pmatrix}\right|$$

$$= (a+2)^2\left|\begin{pmatrix} 1 & 0 & -3 \\ 1 & 1 & -3 \\ 0 & 1 & a-4 \end{pmatrix}\right| = (a+2)^2\left|\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 1 & a-4 \end{pmatrix}\right| = (a+2)^2(a-4).$$

We conclude that the given 3–tuples are a basis iff $a \neq -2, 4$.

iii. Assume that,

$$\left|\begin{pmatrix} a & x & l \\ b & y & m \\ c & z & n \end{pmatrix}\right| = 2$$

Find:

$$\left|\begin{pmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{pmatrix}\right|$$

We use the fact that the determinant is a multi-linear function on $M_3(\mathbb{R})$. We get,

$$\left|\begin{pmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{pmatrix}\right| = \left|\begin{pmatrix} 2a & 2b & 2c \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{pmatrix}\right| + \left|\begin{pmatrix} 3x & 3y & 3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{pmatrix}\right|$$

$$= 14\left|\begin{pmatrix} a & b & c \\ l+x & m+y & n+z \\ l & m & n \end{pmatrix}\right| + 21\left|\begin{pmatrix} x & y & z \\ l+x & m+y & n+z \\ l & m & n \end{pmatrix}\right|$$

We note that in the matrix

$$\begin{pmatrix} x & y & z \\ l+x & m+y & n+z \\ l & m & n \end{pmatrix}$$

the second row is a linear combination of the first and third rows (it is their sum). So, the rows of this matrix are linearly dependent and therefore, due to a theorem proved in class, the matrix is not invertible which means that it's determinant is zero. This means that,

$$\left|\begin{pmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{pmatrix}\right| = 14\left|\begin{pmatrix} a & b & c \\ l+x & m+y & n+z \\ l & m & n \end{pmatrix}\right|$$

$$= 14\left|\begin{pmatrix} a & b & c \\ x & y & z \\ l & m & n \end{pmatrix}\right| + 14\left|\begin{pmatrix} a & b & c \\ l & m & n \\ l & m & n \end{pmatrix}\right|.$$

Since a matrix with two equal rows has a determinant equal to zero, we conclude that

$$\left|\begin{pmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{pmatrix}\right| = 14\left|\begin{pmatrix} a & b & c \\ x & y & z \\ l & m & n \end{pmatrix}\right| = 14 \times 2 = 28$$

Where in the last step we used the fact that for every square matrix A we have $|A| = |A^t|$.

3. Let $A, B \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Prove or disprove the following claims:

i. $|A+B| = |A| + |B|$
The claim is false. Counterexample: consider the matrices,

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

then $|A| = |B| = 1$ but $|A+B| = |0| = 0$.

ii. $|\lambda A| = \lambda |A|$
The claim is false. Counterexample: consider the matrix,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and the scalar $\lambda = 2$. Then,

$$2I = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

So, $|2I| = 4 \neq 2 = 2|I|$.

iii. $|\lambda A| = \lambda^n |A|$
The claim is true. Proof: we stated in class that if a matrix B is obtained from A by multiplying **one** row of A by a scalar λ then $|B| = \lambda |A|$. In our case, the matrix λA is obtained from A by multiplying **each one of the n rows** of A by the scalar λ. Applying the result from class n times our claim is proved.

iv. If A is anti-symmetric (that is, $A^t = -A$) and n is odd then A is not invertible.
The claim is true. Indeed, we stated in class that $|A| = |A^t|$. If A is anti-symmetric this implies that $|A| = |-A|$. Applying the statement in part (iii) of this question we find that $|A| = (-1)^n |A|$. Since n is odd this means that $|A| = -|A|$, which implies that $2|A| = 0$ and therefore that $|A| = 0$.

v. If A is anti-symmetric (that is, $A^t = -A$) and n is even then A is not invertible.
The claim is false. Counterexample:

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Clearly, A is both anti-symmetric and invertible. (Why is it 'clear' that A is invertible?).

vi. If $AB = 0$ then $|A^2| + |B^2| = |A|^2 + |B|^2 = 0$.
The claim is false as $A = I$ and $B = 0$ provide a counterexample. However, if we add the condition that both A and B cannot be equal zero then the claim becomes true. Indeed, if $AB = 0$ and both of them

are different from zero then they are both not invertible (why?). This means that $|A| = |B| = 0$ and therefore that $|A^2| + |B^2| = 0$.

vii. If $|A+B| = |A|$ then B is the zero matrix.
The claim is false. Counterexample: consider,

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}.$$

and

$$B = \begin{pmatrix} 2 & 3 \\ 0 & 0 \end{pmatrix}.$$

Clearly, B is not the zero matrix. Due to the rules regarding determinant and row reduction we have,

$$|A+B| = \left| \begin{pmatrix} 0+2 & 1+3 \\ 2 & 3 \end{pmatrix} \right| = \left| \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \right| = |A|$$

4. i. Compute the determinant of the following $n \times n$ matrix:

$$\begin{pmatrix} 4 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 4 \end{pmatrix}$$

We add all rows to the last row and get,

$$\left| \begin{pmatrix} 4 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 4 \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} 4 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 3+n & 3+n & 3+n & \dots & 3+n \end{pmatrix} \right|$$

$$= (3+n) \left| \begin{pmatrix} 4 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix} \right|.$$

We remove the last row from all other rows and get,

$$= (3+n)\left|\begin{pmatrix} 4 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}\right|.$$

$$= (3+n)\left|\begin{pmatrix} 3 & 0 & 0 & \dots & 0 \\ 0 & 3 & 0 & \dots & 0 \\ 0 & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{pmatrix}\right| = (3+n)3^{n-1}.$$

Where in the last step we used the fact that the determinant of a lower triangular matrix is equal to the product of the entries on its diagonal.

ii. For the matrix,

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Prove that $|A| = 1 + (-1)^{(n+1)}$. (Note the 1 on the left lowest corner). We compute the determinant by developing it with respect to the first column. We obtain,

$$|A| = \left|\begin{pmatrix} 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}\right| + (-1)^{n+1}\left|\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix}\right| = 1+(-1)^{(n+1)}.$$

Where in the last step we used the fact that the determinant of a (lower or upper) triangular matrix is equal to the product of the entries on its diagonal.

6. i. Following the algorithm described in class, we start by computing the characteristic polynomial of A. We have,

$$P_A(x) = |A - xI| = \left|\begin{pmatrix} 8-x & 3 & -3 \\ -6 & -1-x & 3 \\ 12 & 6 & -4-x \end{pmatrix}\right|$$

$$= (8-x)\left|\begin{pmatrix} -1-x & 3 \\ 6 & -4-x \end{pmatrix}\right| - 3\left|\begin{pmatrix} -6 & 3 \\ 12 & -4-x \end{pmatrix}\right| + (-3)\left|\begin{pmatrix} -6 & -1-x \\ 12 & 6 \end{pmatrix}\right|$$

$$= (8-x)[(-1-x)(-4-x) - 18] - 3[(-6)(-4-x) - 36] - 3[-36 - 12(-1-x)]$$

$$= (8-x)[x^2+5x-14] - 3[6x-12] - 3[12x-24]$$
$$= (8-x)(x+7)(x-2) - 18[x-2] - 36[x-2]$$
$$= (x-2)[(8-x)(x+7) - 18 - 36]$$
$$= -(x-2)[x^2-x-2] = -(x-2)^2(x+1)$$

The eigenvalues of A are equal to the roots of $P_A(x)$, that is, these are the solutions of the equation,

$$-(x-2)^2(x+1) = 0$$

The matrix A has therefore two eigenvalues, $x = 2$ and $x = -1$. The algebraic multiplicity of 2 is the degree of $x - 2$ in this polynomial, that is, it is equal to 2. The algebraic multiplicity of 2 is the degree of $x - (-1) = x + 1$ in this polynomial, that is, it is equal to 1.
The geometric multiplicity of $x = 2$ is the dimension of the eigenspace V_2. This is the null space of the matrix $A - 2I$. We write this matrix explicitly and bring it to echelon form:

$$\begin{pmatrix} 6 & 3 & -3 \\ -6 & -3 & 3 \\ 12 & 6 & -6 \end{pmatrix} \Rightarrow \begin{pmatrix} 6 & 3 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$
$$\begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

To find this null space we solve $2x + y - z = 0$. The solution is

$$\begin{pmatrix} -\frac{1}{2}t + \frac{1}{2}s \\ t \\ s \end{pmatrix} = t \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} = span\{ \begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix} \}.$$

The two vectors spanning this space are clearly linearly independent (why?) and are therefore a basis for this space. So $dimV_2 = 2$, that is, the geometric multiplicity of 2 is equal to 2.
We can find the geometric multiplicity of -1 without much computation. We proved in class that this quantity is at least 1 and at most equal to the algebraic multiplicity, which in this case is also 1. So, the geometric multiplicity of -1 is equal 1. It follows that the matrix is diagonalizable. Indeed, the sum of geometric multiplicities of all different eigenvalues is equal $1 + 2 = 3$. Since this is equal to the order of the matrix (A is a 3×3 matrix) then, by a theorem we proved in class, A is diagonalizable. Another justification: the characteristic polynomial has only linear terms, and the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity, so, by a theorem we proved in class, A is diagonalizable.
To find a diagonal matrix D and an invertible matrix P so that $A = P^{-1}DP$ it will be useful to first find a basis of $\mathbb{R}^3$ which is composed of eigenvectors of A. To this end we find a basis for the eigenspace

V_{-1}, that is, a basis for the null space of $A+I$. We write this matrix explicitly and bring it to echelon form.

$$\begin{pmatrix} 9 & 3 & -3 \\ -6 & 0 & 3 \\ 12 & 6 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 1 & -1 \\ -2 & 0 & 1 \\ 4 & 2 & -1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 3 & 1 & -1 \\ -2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 3 & 1 & -1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigenspace is therefore

$$V_{-1} = \begin{pmatrix} -\frac{3}{2}t \\ -\frac{1}{2}t \\ t \end{pmatrix} = t\begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix} = span\{\begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix}\}.$$

Since the vector spanning this space is a single vector different from zero it is clearly a basis for the space.

It follows from a theorem we stated in class, that if A is diagonizable then the union of the bases of the eigenspaces of A gives a basis for $\mathbb{R}^3$ which is composed of eigenvectors of A. So, the ordered set

$$B = (\begin{pmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{3}{2} \\ -\frac{1}{2} \\ 1 \end{pmatrix})$$

Is an ordered basis for $\mathbb{R}^3$ which is composed of eigenvectors of A.

The matrix,

$$D = [T_A]_B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is a diagonal matrix which is similar to A. (**Note that to obtain this diagonal matrix we did not need to find the basis B, we did so in order to find the matrix P, as we do below**).

Denote by E the standard basis of $\mathbb{R}^3$ with the standard order. It follows from the relation,

$$[T_A]_E = [id]_E^B[T_A]_B[id]_B^E$$

that

$$A = [id]_E^B D[id]_B^E.$$

Since we proved in class that $[id]_E^B$ and $[id]_B^E$ are inverse to one another, it follows that the matrix P we are looking for is $P = [id]_B^E$. To find this matrix we first find the matrix $[id]_E^B$. By definition of $[id]_E^B$, this is just the matrix who's columns are the elements in the ordered basis B (keeping order). That is,

$$[id]_E^B = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 1 & 0 & -\frac{1}{2} \\ 0 & 1 & 1 \end{pmatrix}$$

The matrix P we were asked to find is the inverse of this matrix, $P = ([id]_E^B)^{-1}$. To find it we need only to follow the algorithm from class and invert the matrix we found here. We skip this computation (but clearly would not skip it in an exam).

ii. Following the algorithm described in class, we start by computing the characteristic polynomial of B. We have,

$$p_B(x) = |B - xI| = \left| \begin{pmatrix} -1-x & 2 & 2 \\ 2 & -1-x & 2 \\ 2 & 2 & -1-x \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} -1-x & 2 & 2 \\ 2 & -1-x & 2 \\ 3-x & 3-x & 3-x \end{pmatrix} \right|$$

$$= (3-x)\left| \begin{pmatrix} -1-x & 2 & 2 \\ 2 & -1-x & 2 \\ 1 & 1 & 1 \end{pmatrix} \right|$$

$$= (3-x)\left| \begin{pmatrix} -3-x & 0 & 0 \\ 0 & -3-x & 0 \\ 1 & 1 & 1 \end{pmatrix} \right| = (3-x)(3+x)^2.$$

The eigenvalues of B are equal to the roots of $p_B(x)$, that is, these are the solutions of the equation,

$$-(x+3)^2(x-3) = 0$$

The matrix B has therefore two eigenvalues, $x = 3$ and $x = -3$. The algebraic multiplicity of 3 is the degree of $x - 3$ in this polynomial, that is, it is equal to 1. The algebraic multiplicity of -3 is the degree of $x - (-3) = x + 3$ in this polynomial, that is, it is equal to 2.

The geometric multiplicity of $x = -3$ is the dimension of the eigenspace V_{-3}. This is the null space of the matrix $B + 3I$. We write this matrix explicitly and bring it to echelon form:

$$\begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To find this null space we solve $x+y+z=0$. The solution is

$$\begin{pmatrix} -t-s \\ t \\ s \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = span\{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\}.$$

The two vectors spanning this space are clearly linearly independent (why?) and are therefore a basis for this space. So $dimV_{-3} = 2$, that is, the geometric multiplicity of -3 is equal to 2.
We can find the geometric multiplicity of 3 without much computation. We proved in class that this quantity is at least 1 and at most equal to the algebraic multiplicity, which in this case is also 1. So, the geometric multiplicity of 3 is equal 1. It follows that the matrix is diagonalizable. Indeed, the sum of geometric multiplicities of all different eigenvalues is equal $1+2=3$. Since this is equal to the order of the matrix (B is a 3×3 matrix) then, by a theorem we proved in class, B is diagonalizable. Another justification: the characteristic polynomial has only linear terms, and the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity, so, by a theorem we proved in class, B is diagonalizable.
To find a diagonal matrix D and an invertible matrix P so that $B = P^{-1}DP$ it will be useful to first find a basis of $\mathbb{R}^3$ which is composed of eigenvectors of B. To this end we find a basis for the eigenspace V_3, that is, a basis for the null space of $B-3I$. We write this matrix explicitly and bring it to echelon form.

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix} \Rightarrow \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & -2 & 1 \\ -2 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & -3 & 3 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 1 & -2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigenspace is therefore

$$V_3 = \begin{pmatrix} t \\ t \\ t \end{pmatrix} = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = span\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}.$$

Since the vector spanning this space is a single vector different from zero it is clearly a basis for the space.

It follows from a theorem we stated in class, that if B is diagonizable then the union of the bases of the eigenspaces of B gives a basis for $\mathbb{R}^3$ which is composed of eigenvectors of B. So, the ordered set

$$S = (\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix})$$

Is an ordered basis for $\mathbb{R}^3$ which is composed of eigenvectors of A.
The matrix,

$$D = [T_B]_S = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

is a diagonal matrix which is similar to B. (**Note that to obtain this diagonal matrix we did not need to find the basis S, we did so in order to find the matrix P, as we do below**).
Denote by E the standard basis of $\mathbb{R}^3$ with the standard order. It follows from the relation,

$$[T_B]_E = [id]_E^S [T_A]_S [id]_S^E$$

that

$$B = [id]_E^S D [id]_S^E.$$

Since we proved in class that $[id]_E^S$ and $[id]_S^E$ are inverse to one another, it follows that the matrix P we are looking for is $P = [id]_S^E$. To find this matrix we first find the matrix $[id]_E^S$. By definition of $[id]_E^S$, this is just the matrix who's columns are the elements in the ordered basis S (keeping order). That is,

$$[id]_E^S = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The matrix P we were asked to find is the inverse of this matrix, $P = ([id]_E^S)^{-1}$. To find it we need only to follow the algorithm from class and invert the matrix we found here. We skip this computation (but clearly would not skip it in an exam).

iii. Following the algorithm described in class, we start by computing the characteristic polynomial of C. We have,

$$p_C(x) = |C - xI| = \left| \begin{pmatrix} 4-x & 0 & 3 \\ 0 & 5-x & 0 \\ 3 & 0 & -4-x \end{pmatrix} \right|$$

$$= (5-x) \left| \begin{pmatrix} 4-x & 3 \\ 3 & -4-x \end{pmatrix} \right|$$

$$= (5-x)[(x-4)(x+4) - 9] = (5-x)(x^2 - 25) = -(x+5)(x-5)^2$$

The eigenvalues of C are equal to the roots of $p_C(x)$, that is, these are the solutions of the equation,

$$-(x+5)(x-5)^2 = 0$$

The matrix C has therefore two eigenvalues, $x = 5$ and $x = -5$. The algebraic multiplicity of 5 is the degree of $x - 5$ in this polynomial, that is, it is equal to 2. The algebraic multiplicity of -5 is the degree of $x - (-5) = x + 5$ in this polynomial, that is, it is equal to 1.
The geometric multiplicity of $x = 5$ is the dimension of the eigenspace V_5. This is the null space of the matrix $C - 5I$. We write this matrix explicitly and bring it to echelon form:

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 3 & 0 & -9 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To find this null space we solve $x + y + z = 0$. The solution is

$$\begin{pmatrix} -3s \\ t \\ s \end{pmatrix} = t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = span\{ \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \}.$$

The two vectors spanning this space are clearly linearly independent (why?) and are therefore a basis for this space. So $dim V_5 = 2$, that is, the geometric multiplicity of 5 is equal to 2.
We can find the geometric multiplicity of -5 without much computation. We proved in class that this quantity is at least 1 and at most equal to the algebraic multiplicity, which in this case is also 1. So, the geometric multiplicity of -5 is equal 1. It follows that the matrix is diagonalizable. Indeed, the sum of geometric multiplicities of all different eigenvalues is equal $1 + 2 = 3$. Since this is equal to the order of the matrix (C is a 3×3 matrix) then, by a theorem we proved in class, C is diagonalizable. Another justification: the characteristic polynomial has only linear terms, and the algebraic multiplicity of each eigenvalue is equal to its geometric multiplicity, so, by a theorem we proved in class, C is diagonalizable.
To find a diagonal matrix D and an invertible matrix P so that $C = P^{-1}DP$ it will be useful to first find a basis of $\mathbb{R}^3$ which is composed of eigenvectors of C. To this end we find a basis for the eigenspace V_{-5}, that is, a basis for the null space of $C + 5I$. We write this matrix explicitly and bring it to echelon form.

$$\begin{pmatrix} 9 & 0 & 3 \\ 0 & 10 & 0 \\ 3 & 0 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 0 & 1 \\ 0 & 10 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

The eigenspace is therefore

$$V_{-5} = \begin{pmatrix} -\frac{1}{3}t \\ 0 \\ t \end{pmatrix} = t \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix} = span\{\begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix}\}.$$

Since the vector spanning this space is a single vector different from zero it is clearly a basis for the space.
It follows from a theorem we stated in class, that if C is diagonizable then the union of the bases of the eigenspaces of C gives a basis for $\mathbb{R}^3$ which is composed of eigenvectors of C. So, the ordered set

$$B = (\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{pmatrix})$$

Is an ordered basis for $\mathbb{R}^3$ which is composed of eigenvectors of C.
The matrix,

$$D = [T_C]_B = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & -5 \end{pmatrix}$$

is a diagonal matrix which is similar to C. (**Note that to obtain this diagonal matrix we did not need to find the basis B, we did so in order to find the matrix P, as we do below**).
Denote by E the standard basis of $\mathbb{R}^3$ with the standard order. It follows from the relation,

$$[T_C]_E = [id]_E^B [T_C]_B [id]_B^E$$

that

$$C = [id]_E^B D [id]_B^E.$$

Since we proved in class that $[id]_E^B$ and $[id]_B^E$ are inverse to one another, it follows that the matrix P we are looking for is $P = [id]_B^E$. To find this matrix we first find the matrix $[id]_E^B$. By definition of $[id]_E^B$, this is just the matrix who's columns are the elements in the ordered basis B (keeping order). That is,

$$[id]_E^B = \begin{pmatrix} 0 & -3 & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

The matrix P we were asked to find is the inverse of this matrix, $P = ([id]_E^B)^{-1}$. To find it we need only to follow the algorithm from class and invert the matrix we found here. We skip this computation (but clearly would not skip it in an exam).

iv. Following the algorithm described in class, we start by computing the characteristic polynomial of E. We have,

$$p_E(x) = |E - xI| = \left| \begin{pmatrix} 1-x & 1 & 0 \\ 0 & 2-x & 1 \\ 0 & 0 & -1-x \end{pmatrix} \right|$$

$$= -(x-1)(x-2)(x+1)$$

The eigenvalues of E are equal to the roots of $p_E(x)$, that is, these are the solutions of the equation,

$$-(x-1)(x-2)(x+1) = 0$$

The matrix E has therefore three eigenvalues, $x = 1$, $x = 2$ and $x = -1$. The algebraic multiplicity of 1 is the degree of $x - 1$ in this polynomial, that is, it is equal to 1. The algebraic multiplicity of 2 is the degree of $x - 2$ in this polynomial, that is, it is equal to 1. The algebraic multiplicity of -1 is the degree of $x - (-1) = x + 1$ in this polynomial, that is, it is equal to 1. We can find the geometric multiplicity of 1 without much computation. We proved in class that this quantity is at least 1 and at most equal to the algebraic multiplicity, which in this case is also 1. So, the geometric multiplicity of 1 is equal 1. Similarly, the geometric multiplicities of 2 and -1 are also equal 1. The matrix is diagonalizable as it is a 3×3 matrix with three different eigenvalues (we proved such a statement in class).

To find a diagonal matrix D and an invertible matrix P so that $E = P^{-1}DP$ it will be useful to first find a basis of $\mathbb{R}^3$ which is composed of eigenvectors of E. To this end we find a basis for the eigenspaces V_1, V_2, and V_{-1}. We start by finding a basis for V_1, that is, a basis for the null space of $E - I$. We write this matrix explicitly and bring it to echelon form.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

The eigenspace is therefore

$$V_1 = \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = span\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \}.$$

Next we find a basis for V_2, that is, a basis for the null space of $E - 2I$. We write this matrix explicitly and bring it to echelon form.

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -3 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow$$

The eigenspace is therefore

$$V_2 = \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = span\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \}.$$

Finally, we find a basis for V_{-1}, that is, a basis for the null space of $E + I$. We write this matrix explicitly and bring it to echelon form.

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The eigenspace is therefore

$$V_{-1} = \begin{pmatrix} \frac{1}{6}t \\ -\frac{1}{3}t \\ t \end{pmatrix} = t \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ 1 \end{pmatrix} = span\{ \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ 1 \end{pmatrix} \}.$$

Since the vector spanning this space is a single vector different from zero it is clearly a basis for the space.
It follows from a theorem we stated in class, that if E is diagonizable then the union of the bases of the eigenspaces of E gives a basis for $\mathbb{R}^3$ which is composed of eigenvectors of E. So, the ordered set

$$B = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{6} \\ -\frac{1}{3} \\ 1 \end{pmatrix})$$

Is an ordered basis for $\mathbb{R}^3$ which is composed of eigenvectors of E.
The matrix,

$$D = [T_E]_B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is a diagonal matrix which is similar to E. (**Note that to obtain this diagonal matrix we did not need to find the basis B, we did so in order to find the matrix P, as we do below**).
Denote by E_1 the standard basis of $\mathbb{R}^3$ with the standard order. It follows from the relation,

$$[T_E]_{E_1} = [id]^B_{E_1} [T_E]_B [id]^{E_1}_B$$

that

$$E = [id]^B_{E_1} D [id]^{E_1}_B.$$

Since we proved in class that $[id]^B_{E_1}$ and $[id]^{E_1}_B$ are inverse to one another, it follows that the matrix P we are looking for is $P = [id]^{E_1}_B$. To find this matrix we first find the matrix $[id]^B_{E_1}$. By definition of $[id]^B_{E_1}$, this is just the matrix who's columns are the elements in the ordered basis B (keeping order). That is,

$$[id]_{E_1}^{B} = \begin{pmatrix} 1 & 1 & \frac{1}{6} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix P we were asked to find is the inverse of this matrix, $P = ([id]_{E_1}^{B})^{-1}$. To find it we need only to follow the algorithm from class and invert the matrix we found here. We skip this computation (but clearly would not skip it in an exam).

v. Following the algorithm described in class, we start by computing the characteristic polynomial of F. We have,

$$p_F(x) = |F - xI| = \left| \begin{pmatrix} 2-x & 1 & 0 & 0 \\ 0 & 2-x & 1 & 0 \\ 0 & 0 & 2-x & 0 \\ 0 & 0 & 0 & 3-x \end{pmatrix} \right|$$
$$= (x-2)^3(x-3)$$

The eigenvalues of F are equal to the roots of $p_F(x)$, that is, these are the solutions of the equation,

$$(x-2)^3(x-3) = 0$$

The matrix F has therefore two eigenvalues, $x = 2$ and $x = 3$. The algebraic multiplicity of 2 is the degree of $x-2$ in this polynomial, that is, it is equal to 3. The algebraic multiplicity of 3 is the degree of $x-3$ in this polynomial, that is, it is equal to 1.

The geometric multiplicity of $x = 2$ is the dimension of the eigenspace V_2. This is the null space of the matrix $C - 2I$. We write this matrix explicitly and bring it to echelon form:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The solution is

$$\begin{pmatrix} t \\ 0 \\ 0 \\ 0 \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = span\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \}.$$

The single vector spanning this space is not zero and is therefore a basis for this space. So $dimV_2 = 1$, that is, the geometric multiplicity of 2 is equal to 1.

We now know that F is not diagonalizable, as it has an eigenvalue who's algebraic and geometric multiplicatives are not equal.

We can find the geometric multiplicity of 3 without much computation. We proved in class that this quantity is at least 1 and at most equal to the algebraic multiplicity, which in this case is also 1. So, the geometric multiplicity of 3 is equal 1.

vi. Following the algorithm described in class, we start by computing the characteristic polynomial of G. We have,

$$p_G(x) = |G - xI| = \left| \begin{pmatrix} -x & 1 & 0 & 0 \\ -1 & -x & 0 & 0 \\ 0 & 0 & 2-x & 5 \\ 0 & 0 & 0 & 3-x \end{pmatrix} \right|$$

$$= (3-x)(2-x)(x^2+1)$$

The eigenvalues of F are equal to the roots of $p_G(x)$, that is, these are the solutions of the equation,

$$(x-3)(x-2)(x^2+1) = 0$$

The matrix G has therefore two eigenvalues (over $\mathbb{R}$!), $x = 2$ and $x = 3$. The algebraic multiplicity of 2 is the degree of $x - 2$ in this polynomial, that is, it is equal to 1. The algebraic multiplicity of 3 is the degree of $x - 3$ in this polynomial, that is, it is equal to 1.

We see that G is not diagonalizable (over $\mathbb{R}$!), as it's characteristic polynomial contains an irreducible nonlinear term (over $\mathbb{R}$!).

We can find the geometric multiplicity of 3 without much computation. We proved in class that this quantity is at least 1 and at most equal to the algebraic multiplicity, which in this case is also 1. So, the geometric multiplicity of 3 is equal 1. Similarly, the geometric multiplicity of 2 is equal 1.

7. There are two different approaches to solve this problem (well, parts i–iv of the problem). Both are important so we will discuss both.

i. Find the eigenvalues of E^2. Is E^2 diagonalizable?

First direction for a solution: We will prove the following general claim: Let $A \in M_n(\mathbb{R})$. If $v \in \mathbb{R}^n$ is an eigenvector of A with respect to an eigenvalue λ, then the same v is an eigenvector of A^2 with respect to an eigenvalue λ^2.

Proof of the claim: we have,

$$A^2 v = A(Av) = A(\lambda v) = \lambda(Av) = \lambda^2 v.$$

This proves the claim.

We now use this claim to solve the question: We proved in Q1 that E is diagonalizable. This implies that there exists a basis of $\mathbb{R}^3$ composed of eigenvectors of E. By the claim we just proved, all of the vectors in this basis are also eigenvectors of E^2 and they correspond to the eigenvalues 1 and 4. So, there is a basis of $\mathbb{R}^3$ composed of eigenvectors of E^2 which correspond to the eigenvalues 1 and 4. This implies that

E^2 is diagonalizable and its only eigenvalues are 1 and 4. (How do I know that there are no more eigenvalues?)
Second direction for a solution: We have seen in Q1 that E is similar to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

That is, there exists an invertible matrix $P \in M_3(\mathbb{R})$ such that,

$$E = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} P.$$

This means that,

$$\begin{aligned} E^2 &= P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} PP^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} P \\ &= P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} I \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} P \\ &= P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2 P = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} P. \end{aligned}$$

So, by definition, E^2 is similar to a diagonal matrix and therefore diagonalizable. Also, the entries on the diagonal of this diagonal matrix are 1 and 2. This means that 1 and 2 are eigenvalues of E^2 and the only ones. (How do I know that these are eigenvalues? How do I know they are the only ones?).

ii. Find the eigenvalues of E^{10}. Is E^{10} diagonalizable?
First direction for a solution: We will prove the following general claim: Let $A \in M_n(\mathbb{R})$ and k be a positive integer. If $v \in \mathbb{R}^n$ is an eigenvector of A with respect to an eigenvalue λ, then the same v is an eigenvector of A^k with respect to the eigenvalue λ^k.
Proof of the claim: we have,

$$A^k v = A^{k-1}(Av) = A^{k-1}(\lambda v) = \lambda A^{k-1} v = \lambda A^{k-2}(Av) = \lambda A^{k-2}(\lambda v) = \lambda^2 A^{k-2} v = ... = \lambda^k v$$

This proves the claim. (**Remark:** Here we gave a lazy presentation of the proof. The correct way to present the proof is to use induction.)
We now use this claim to solve the question: We proved in Q1 that E is diagonalizable. This implies that there exists a basis of $\mathbb{R}^3$ composed of eigenvectors of E. By the claim we just proved, all of the vectors in this basis are also eigenvectors of E^{10} and they correspond to the eigenvalues 1 and 1024. So, there is a basis of $\mathbb{R}^3$ composed of eigenvectors of E^{10}

which correspond to the eigenvalues 1 and 1024. This implies that E^{10} is diagonalizable and its only eigenvalues are 1 and 1024. (How do I know that there are no more eigenvalues?)

Second direction for a solution: We have seen in Q1 that E is similar to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

That is, there exists an invertible matrix $P \in M_3(\mathbb{R})$ such that,

$$E = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} P.$$

This means that,

$$E^{10} = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} PP^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} P \times ... \times P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} P$$

$$= P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{10} P = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1024 & 0 \\ 0 & 0 & 1 \end{pmatrix} P.$$

So, by definition, E^{10} is similar to a diagonal matrix and therefore diagonalizable. Also, the entries on the diagonal of this diagonal matrix are 1 and 1024. This means that 1 and 1024 are eigenvalues of E^2 and the only ones. (How do I know that these are eigenvalues? How do I know they are the only ones?).

iii. Find the eigenvalues of $E^3 - 5E^2 + 2E + 3I$. Is $E^3 - 5E^2 + 2E + 3I$ diagonalizable?

First direction for a solution: We will prove the following general claim: Let $A \in M_n(\mathbb{R})$. If $v \in \mathbb{R}^n$ is an eigenvector of A with respect to an eigenvalue λ, then the same v is an eigenvector of $A^3 - 5A^2 + 2A + 3I$ with respect to the eigenvalue $\lambda^3 - 5\lambda^2 + 2\lambda + 3$.

Proof of the claim: we use the general claim we obtained in part (ii) and get,

$$(A^3 - 5A^2 + 2A + 3I)v = A^3v - 5A^2v + 2Av + 3Iv = \lambda^3 v - 5\lambda^2 v + 2\lambda v + 3v = (\lambda^3 - 5\lambda^2 + 2\lambda + 3)v$$

This proves the claim.

We now use this claim to solve the question: We proved in Q1 that E is diagonalizable. This implies that there exists a basis of $\mathbb{R}^3$ composed of eigenvectors of E. By the claim we just proved, all of the vectors in this basis are also eigenvectors of $E^3 - 5E^2 + 2E + 3I$ and they correspond to the eigenvalues $1 - 5 + 2 + 3 = 1$, $8 - 20 + 4 + 3 = -5$ and $-1 - 5 - 2 + 3 = -5$. So, there is a basis of $\mathbb{R}^3$ composed of eigenvectors of $E^3 - 5E^2 + 2E + 3I$ which correspond to the eigenvalues

1 and -5. This implies that $E^3 - 5E^2 + 2E + 3I$ is diagonalizable and its only eigenvalues are 1 and -5. (How do I know that there are no more eigenvalues?)

Remarks:

a. In fact, we can prove a more general claim: Let $A \in M_n(\mathbb{R})$ and $p(x)$ be a polynomial. If $v \in \mathbb{R}^n$ is an eigenvector of A with respect to an eigenvalue λ, then the same v is an eigenvector of $p(A)$ with respect to the eigenvalue $p(\lambda)$.

b. We could solve this part also by the 'second direction of solution'. It is done in exactly the same way as was done in the parts (i) and (ii).

iv. Is E invertible? If so, find the eigenvalues of E^{-1}. Is E^{-1} diagonalizable?
Yes, E is invertible. (There are many different ways to justify this answer, use one of them).
We turn to finding the eigenvalues and determining diagonalizability.
First direction for a solution: We will prove the following general claim: Let $A \in M_n(\mathbb{R})$ be invertible. If $v \in \mathbb{R}^n$ is an eigenvector of A with respect to an eigenvalue λ, then the same v is an eigenvector of A^{-1} with respect to an eigenvalue λ^{-1}.
Proof of the claim: we have,

$$Av = \lambda v.$$

Clearly, $\lambda \neq 0$. Indeed, otherwise there will be a non trivial solution to the homogeneous system $(A|0)$ which will imply, by a thm from class, that A is not invertible, in contradiction to what we were given.
Since $\lambda \neq 0$ we can multiply both sides by λ^{-1}. We get

$$\lambda^{-1} Av = v,$$

or

$$A(\lambda^{-1} v) = v.$$

We multiply both sides by A^{-1} and get,

$$\lambda^{-1} v = A^{-1} v.$$

This proves the claim.
We now use this claim to solve the question: We proved in Q1 that E is diagonalizable. This implies that there exists a basis of $\mathbb{R}^3$ composed of eigenvectors of E. By the claim we just proved, all of the vectors in this basis are also eigenvectors of E^{-1} and they correspond to the eigenvalues $1, -1$ and $\frac{1}{2}$. So, there is a basis of $\mathbb{R}^3$ composed of eigenvectors of E^{-1} which correspond to the eigenvalues 1 -1 and $\frac{1}{2}$. This implies that E^{-1} is diagonalizable and its only eigenvalues are 1, -1 and $\frac{1}{2}$. (How do I know that there are no more eigenvalues?)

Second direction for a solution: We have seen in Q1 that E is similar to the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

That is, there exists an invertible matrix $P \in M_3(\mathbb{R})$ such that,

$$E = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} P.$$

We use the relation $(AB)^{-1} = B^{-1}A^{-1}$ which was proved in recitation and get,

$$\begin{aligned} E^{-1} &= \left(P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} P\right)^{-1} \\ &= P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}^{-1} P \\ &= P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -1 \end{pmatrix} P. \end{aligned}$$

So, by definition, E^{-1} is similar to a diagonal matrix and therefore diagonalizable. Also, the entries on the diagonal of this diagonal matrix are 1, -1 and $\frac{1}{2}$. This means that 1, -1 and $\frac{1}{2}$ are eigenvalues of E^{-1} and the only ones. (How do I know that these are eigenvalues? How do I know they are the only ones?).

v. Compute E^5.

Following the same idea as in the previous parts, we now from Q1 that,

$$E = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} P.$$

Therefore,

$$E^5 = P^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & -1 \end{pmatrix} P.$$

We found P^{-1} explicitly in Q1. We can find P explicitly as well. It remains to compute the multiplication. We skip it here (but would not skip it in an exam).

8. i. $T : M_2(\mathbb{R}) \mapsto M_2(R)$ given by

$$TA = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} A$$

First we choose a basis for $M_2(\mathbb{R})$ and find a matrix representation of T with respect to this basis. We use the following ordered basis (we proved in class that this is indeed a basis for $M_2(\mathbb{R})$):

$$E = \left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

We find the matrix representation of T with respect to this basis,

$$[T]_E == \left([T \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}]_E ; [T \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}]_E ; [T \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}]_E ; [T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}]_E \right)$$

$$= \left([\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}]_E ; [\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}]_E ; [\begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix}]_E ; [\begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix}]_E \right)$$

$$= \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix}.$$

We find the characteristic polynomial of T. We defined it in class to be equal to the characteristic polynomial of $[T]_E$.

$$p_T(x) = |[T]_E - xI| = \left| \begin{pmatrix} 1-x & 0 & 2 & 0 \\ 0 & 1-x & 0 & 2 \\ 2 & 0 & 4-x & 0 \\ 0 & 2 & 0 & 4-x \end{pmatrix} \right| =$$

$$= \left| \begin{pmatrix} 1-x & 0 & 2 & 0 \\ 2 & 0 & 4-x & 0 \\ 0 & 1-x & 0 & 2 \\ 0 & 2 & 0 & 4-x \end{pmatrix} \right|$$

$$= \left| \begin{pmatrix} 1-x & 2 & 0 & 0 \\ 2 & 4-x & 0 & 0 \\ 0 & 0 & 1-x & 2 \\ 0 & 0 & 2 & 4-x \end{pmatrix} \right|$$

$$= [(1-x)(4-x) - 4]^2 = (x^2 - 5x)^2 = x^2(x-5)^2.$$

The eigenvalues of T are equal to the roots of its characteristic polynomial, that is, these are the solutions to the equation

$$x^2(x-5)^2 = 0.$$

T has therefore two eigenvalues, $x = 0$ and $x = 5$. The algebraic multiplicity of both eigenvalues is equal 2. By a thm we proved in class, T is diagonalizable iff the geometric multiplicity of both eigenvalues is also equal to 2. (Indeed, the characteristic polynomial has only linear terms). We start by finding the dimension of V_0, that is by finding the dimension of $null[T]_E$. We bring this matrix to echelon form,

$$\begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 2 & 0 & 4 & 0 \\ 0 & 2 & 0 & 4 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that in the echelon form there are two free variables, so the dimension of the null space of $[T]_E$ (which is the space V_0) is 2 (it was proved in class that the dimension of a null space of a matrix is equal to the amount of free variables in the echelon form).
We now find the dimension of V_5, that is, the dimension of $null([T]_E - 5I)$. We write this matrix explicitly and bring it to echelon form,

$$\begin{pmatrix} -4 & 0 & 2 & 0 \\ 0 & -4 & 0 & 2 \\ 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 0 & -1 & 0 \\ 0 & 2 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We see that in the echelon form there are two free variables, so the dimension of the null space of $[T]_E - 5I$ (which is the space V_5) is 2 (it was proved in class that the dimension 0f a null space of a matrix is equal to the amount of free variables in the echelon form).
We conclude that both eigenvalues have geometric multiplicity 2 (which is equal to their algebraic multiplicity) and therefore the matrix is diagonalizable.

ii. $T : \mathbb{R}_2[x] \mapsto \mathbb{R}_2[x]$ given by $Tp(x) = x(p(x+1) - p(x))$
First we choose a basis for $\mathbb{R}_2[x]$ and find a matrix representation of T with respect to this basis. We use the following ordered basis (we proved in class that this is indeed a basis for $\mathbb{R}_2[x]$): $E = (1, x, x^2)$.
We find the matrix representation of T with respect to this basis,

$$\begin{aligned}[T]_E &== \Big([T1]_E; [Tx]_E; [Tx^2]_E\Big) \\ &= \Big([0]_E; [x]_E; [2x^2 + x]_E\Big) \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.\end{aligned}$$

We find the characteristic polynomial of T. We defined it in class to be equal to the characteristic polynomial of $[T]_E$.

$$p_T(x) = |[T]_E - xI| = \left| \begin{pmatrix} -x & 0 & 0 \\ 0 & 1-x & 1 \\ 0 & 0 & 2-x \end{pmatrix} \right| =$$

$$-x(x-2)(x-1).$$

The eigenvalues of T are equal to the roots of its characteristic polynomial, that is, these are the solutions to the equation

$$-x(x-2)(x-1)=0.$$

T has therefore three eigenvalues, $x=0$, $x=2$ and $x=1$. Since T acts on a space of dimension 3 (we proved in class that $\dim\mathbb{R}_2[x]=3$) and it has three different eigenvalues then T is diagonalizable (this follows from a theorem we proved in class).

iii. Let V be a vector space and $B=(v_1,v_2,v_3)$ a basis for V. Here we consider the linear transformation $T:V\mapsto V$ which satisfies $Tv_1=5v_1$, $Tv_2=v_2+2v_3$ and $Tv_3=2v_2+v_3$.
First we find a matrix representation of T with respect to the ordered basis B.

$$\begin{aligned}[T]_B &== \Big([Tv_1]_B;[Tv_2]_B;[Tv_3]_B\Big)\\ &= \Big([5v_1]_B;[v_2+2v_3]_B;[2v_2+v_3]_B\Big)\\ &= \begin{pmatrix} 5 & 0 & 0\\ 0 & 1 & 2\\ 0 & 2 & 1\end{pmatrix}.\end{aligned}$$

We find the characteristic polynomial of T. We defined it in class to be equal to the characteristic polynomial of $[T]_B$.

$$p_T(x)=|[T]_B-xI|=\left|\begin{pmatrix} 5-x & 0 & 0\\ 0 & 1-x & 2\\ 0 & 2 & 1-x\end{pmatrix}\right|=$$

$$(5-x)(1-2x+x^2-4)=-(x-5)(x^2-2x-3).$$

The eigenvalues of T are equal to the roots of its characteristic polynomial, that is, these are the solutions to the equation

$$-(x-5)(x^2-2x-3)=0.$$

T has therefore three eigenvalues, $x=5$, $x=-1$ and $x=3$. Since T acts on a space of dimension 3 (it has a basis which contains 3 elements) and it has three different eigenvalues then T is diagonalizable (this follows from a theorem we proved in class).

9. Prove or disprove the following claims.

a. If $A\in M_3(\mathbb{R})$ has rows equal to v $2v$ $3v$ for some $v\in\mathbb{R}^3$ and A has a nonzero eigenvalue then A is diagonalizable.
The claim is true. Let us assume that $v\neq 0$ otherwise the claim is obvious (why?).
We first note that A is not invertible (it's rows are linearly dependent and we proved in class that this is equivalent to being non-invertible). This means that 0 is an eigenvalue of A. Indeed, let us formulate this as a general claim: Let $A\in M_n(\mathbb{R})$ then A is not invertible iff 0 is an

eigenvalue of A. Proof of this claim: We proved in class that A is not invertible iff there is a non–trivial solution to the homogenous system $(A|0)$. That is, A is not invertible iff there exists a non-trivial vector $v \in \mathbb{R}^n$ such that $Av = 0$. This is exactly the condition for 0 being an eigenvalue of A.
So, we know that the matrix A has at least two eigenvalues, 0 and some $\lambda \neq 0$. We can find the geometric multiplicity of the eigenvalue 0. This is the dimension of the null space of A. After performing row operations on A we obtain an echelon form who's first row is v and who's second and third rows are zero rows (this is because the second and third rows of A are equal to a scalar multiplying it's first row). This echelon form has two free variables and therefore the dimension of the null space of A is equal 2 (we proved in class that the dimension of the null space of a matrix is equal to the amount of free variables in an echelon form of this matrix). That is, the geometric multiplicity of 0 is equal 2.
So, the matrix A has two eigenvalues, 0 with geometric multiplicity 2 and $\lambda \neq 0$ with geometric multiplicity at least 1 (we stated in class that the geometric multiplicity of an eigenvalue is at leat 1). This means that the sum of the geometric multiplicities of all the different eigenvalues of A is at least $2+1 = 3$. We proved in class that this sum is at most equal to the order of the matrix. Since A is a 3×3 matrix we conclude that the sum of the geometric multiplicities of all the different eigenvalues of A is equal 3. We proved in class that this sum is equal to the order of the matrix (A is a 3×3 matrix) iff the matrix is diagonalizable. We conclude that A is diagonalizable.

b. If $A \in M_4(\mathbb{R})$ has characteristic polynomial $q_A(x) = x^2(x+5)(x+6)$ and

$$\begin{pmatrix} 0 \\ -1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 3 \\ 4 \end{pmatrix} \in nullA$$

then A is diagonalizable.
The claim is true. A has three different eigenvalues, the roots of the characteristic polynomial, $x = 0$, $x = -5$ and $x = -6$. The eigenvalues $x = -5$ and $x = -6$ have algebraic multiplicity 1 and since their geometric multiplicity is at most equal to this number and at least one (proved in class), we conclude that these two eigenvalues have geometric multiplicity equal to 1.
The eigenvalue 0 has two linearly independent vectors in its eigenspace V_0. Indeed, V_0 is equal to the null space of A and we were given that

this space contains the vectors

$$\begin{pmatrix} 0 \\ -1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 3 \\ 4 \end{pmatrix}$$

which are clearly linearly independent (why?). Since the amount of vectors in a linearly independent set is at most equal to the dimension of the space (proved in class) we conclude that the dimension of V_0 is at least 2. That is, the geometric multiplicity of 0 is at least 2. Since the algebraic multiplicity of 0 is equal to 2, and we proved in class that the geometric multiplicity is at most equal to the algebraic multiplicity, we conclude that the geometric multiplicity of 0 is equal to 2.

So, the matrix A has three eigenvalues, -5 and -6 with geometric multiplicities 1 and 0 with geometric multiplicity 2. The sum of the geometric multiplicities of all the different eigenvalues of A is therefore $1 + 1 + 2 = 4$. We proved in class that this sum is equal to the order of the matrix (A is a 4×4 matrix) iff the matrix is diagonalizable. We conclude that A is diagonalizable.

c. Let $A \in M_n(\mathbb{R})$. Then 0 is an eigenvalue of A iff $|A| = 0$.
The claim is true. Proof: We proved in part (a) of this question that 0 is an eigenvalue of A iff A is not invertible. We proved in class that A is not invertible iff $|A| = 0$. The claim follows.

d. Let $A \in M_n(\mathbb{R})$. If 0 is an eigenvalue of A then its geometric multiplicity is equal to $n - rankA$.
The claim is true. The geometric multiplicity of A is the dimension of the eigenspace V_0. The eigenspace V_0 is exactly the null space of A. It follows from the null–rank theorem which was proved in class that the dimension of the null space of A is equal to $n - rankA$. The claim follows.

e. There exists $A \in M_5(\mathbb{R})$ which is diagonalizable and satisfies $rankA = 1$ and $trA = 0$.
The claim is false. Assume that this claim were true, then A would be similar to a diagonal matrix,

$$D = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 \\ 0 & 0 & 0 & 0 & a_5 \end{pmatrix}.$$

Since the rank of two similar matrices is the same (proved in class) this means that $rankD = 1$. So, four out of the five numbers a_1, a_2, a_3, a_4, a_5 must be equal zero. Since the trace of two similar matrices is the same (proved in class) this means that $trD = 0$. Since, four out of the five numbers a_1, a_2, a_3, a_4, a_5 are equal zero, and their sum is zero, then the fifth number must be equal zero as well. This means that D is the zero matrix. But the only matrix which is similar to the zero matrix is the zero matrix (why?), so A must be the zero matrix as well. This contradicts the fact that $rankA = 1$ because the rank of the zero matrix is zero.

f. If $A \in M_n(\mathbb{R})$ is diagonalizable and 2 is the only eigenvalue of A then $A = 2I$.
The claim is true. If A is diagonalizable then it is similar to a diagonal matrix D. The entries on the diagonal of D must be equal to eigenvalues of A (why? make sure that you know how to explain this). Since 2 is the only eigenvalue of A it follows that $D = 2I$. So, A is similar to $2I$. From the definition of similarity, this means that there exists an invertible matrix $P \in M_n(\mathbb{R})$ such that $A = P^{-1}2IP$. This implies that,
$$A = P^{-1}2IP = 2P^{-1}IP = 2P^{-1}P = 2I.$$
The claim is proved.

g. If $A, B \in M_n(\mathbb{R})$ have the same eigenvalues and A is diagonalizable then so is B.
The claim is false. Here is a counterexample: The matrices
$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$
and
$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
have the same eigenvalues, they both have only the eigenvalue 1 (check!). However, I is diagonalizable (indeed, it is already a diagonal matrix) and A is not (check!!).

h. Let $A \in M_n(\mathbb{R})$ and let $q_A(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + x^n$ be the characteristic polynomial of A. Then A is invertible iff $a_0 \neq 0$.
The claim is true. Proof: The number $x = 0$ is a root of $q_A(x)$ iff $a_0 = 0$ (since $q_A(0) = a_0$). So, 0 is an eigenvalue of A iff $a_0 = 0$. We proved in part (a) of this question that 0 is an eigenvalue of A iff A is not invertible. The claim follows.

i. If $A, B \in M_n(\mathbb{R})$ are similar then they have the same characteristic polynomials.
The claim is true. We proved this in class.

j. If $A, B, C \in M_n(\mathbb{R})$ are such that A and B are similar, and such that A and C are similar, then B and C are similar.
The claim is true. If A and B are similar then there exists an invertible matrix $P \in M_n(\mathbb{R})$ so that $A = P^{-1}BP$. If A and C are similar then there exists an invertible matrix $Q \in M_n(\mathbb{R})$ so that $C = Q^{-1}AQ$.It

follows that

$$C = Q^{-1}AQ = Q^{-1}(P^{-1}BP)Q = (Q^{-1}P^{-1})B(PQ).$$

We proved in class/recitation that if P and Q are invertible then PQ is also invertible and $(PQ)^{-1} = Q^{-1}P^{-1}$. Inserting this to the equality we got above, we obtain,

$$C = (PQ)^{-1}B(PQ).$$

By the definition of similar matrices, this proves that B and C are similar.

k. If $A \in M_n(\mathbb{R})$ and $rankA \leq n-1$ then A is similar to a matrix who's left most column is a zero column.
The claim is true. We first note that by the rank-nullity formula $rankA + dimnullA = n$. So, since $rankA \leq n-1$ it follows that $dimnullA \geq 1$, which implies that there exists a non–zero vector $v \in nullA$. Since v is a single vector different from zero it is linearly independent. We proved in class that every linearly independent set in a vector space can be completed into a basis for the space. It follows that there exist $w, ..., w_{(n-1)} \in \mathbb{R}^n$ so that the ordered set $B = (v, w_1, ..., w_{(n-1)})$ is a basis for $\mathbb{R}^n$. We write the linear transformation $T_A : \mathbb{R}^n \mapsto \mathbb{R}^n$ in coordinates with respect to the basis B. We get,

$$[T_A]_B = ([T_Av]_B; [T_Aw_1]_B; ...; [T_Aw_{(n-1)}]_B).$$

By the definition of similar matrices, the matrix $[T_A]_B$ is similar to the matrix A. Since $v \in nullA$, the first column of $[T_A]_B$ is the vector

$$[T_Av]_B = [0]_B = 0$$

The claim follows.

l. If $A \in M_n(\mathbb{R})$ is diagonalizable and $B \in M_n(\mathbb{R})$ is similar to A then B is also diagonalizable.
The claim is true. Indeed, if A is diagonalizable then there exists a diagonal matrix D so that A is similar to D. By part (j) of this question, if A is similar to D and B is similar to A then B is similar to D. This means that B is diagonalizable and proves the claim.

m. **Note a correction in this question** If $A \in M_3(\mathbb{R})$ satisfies: $rank(A - I) = 2$, $|A + I| = 0$ and there exists $v \neq 0$ such that $Av = 3v$ then A is diagonalizable.
The claim is true. Indeed, first note that if $rank(A - I) = 2$ then $A - I$ is not invertible (by a thm from class) and therefore $|A - I| = 0$. This means that $x = 1$ is a root of the characteristic polynomial of A and therefor an eigenvalue of A. Similarly, $x = -1$ is a root of the characteristic polynomial of A and therefor an eigenvalue of A. Finally, the condition that there exists $v \neq 0$ such that $Av = 3v$ means that 3 is also an eigenvalue of A. So, A has three different eigenvalues, $1, -1, 3$.

Since A is a 3×3 matrix, it follows from a thm we proved in class that A is diagonalizable.

n+o. Both these questions were solved during the solution of Q2 in this page.

10. Let V be a vector space of dimension 5. Does there exist a transformation $T : V \mapsto V$ such that $dimImT = 3$ and:

i. T has 5 distinct eigenvalues?
Such T does not exist. To see this we first note that due to the dimension formula $5 = dimkerT + dimImT$, and since $dimImT = 3$, we know that $dimkerT = 2$. This means that 0 has to be an eigenvalue of T with geometric multiplicity 2. If a transformation T like described above existed, then T would have had 4 more eigenvalues. Since the geometric multiplicity of an eigenvalue is at least 1 (discussed in class), it would follow that the sum of the geometric multiplicities of all the different eigenvalues of T is at least $2+1+1+1+1 = 6$. This contradicts the theorem from class which states that this sum is at most equal to the dimension of the space (which is given to be 5).

ii. T has 4 distinct eigenvalues?
Such a linear transformation exists. Indeed, let $\{v_1, v_2, v_3, v_4, v_5\}$ be a basis for V. We claim that there exists a transformation $T : V \mapsto V$ which has the following images to the vectors of the basis B.

$$Tv_1 = 0$$
$$Tv_2 = 0$$
$$Tv_3 = 3v_3$$
$$Tv_4 = 4v_4$$
$$Tv_5 = 5v_5$$

Indeed, such a transformation exists due to the theorem we proved in class which assures the existence (and uniqueness) of a linear transformation with pre-chosen images for the elements of a fixed basis.
We claim that this transformation T satisfies all of the requirements. First, clearly, $0, 3, 4$, and 5 are all eigenvalues of this transformation, so T has 4 distinct eigenvalues. The kernel of T contains two linearly independent vectors (which?) so $dimkerT \geq 2$, due to a thm from class that states that the amount of vectors in a linearly independent set is at most equal to the dimension of the space. The image of T contains three linearly independent vectors (which?) so $dimImT \geq 3$, due to a thm from class that states that the amount of vectors in a linearly independent set is at most equal to the dimension of the space. Since $dimImT + dimkerT = 5$ it follows that $dimkerT = 2$ and $dimImT = 3$. This completes the proof.

iii. T has 4 distinct eigenvalues and T is not diagonalizable?
Such T does not exist. To see this we first note that due to the dimension formula $5 = dimkerT + dimImT$, and since $dimImT = 3$, we know that $dimkerT = 2$. This means that 0 has to be an eigenvalue of

T with geometric multiplicity 2. If a transformation T like described above existed, then T would have had 3 more eigenvalues. Since the geometric multiplicity of an eigenvalue is at least 1 (discussed in class), it would follow that the sum of the geometric multiplicities of all the different eigenvalues of T is at least $2 + 1 + 1 + 1 = 5$. Since we proved in class that this sum is at most equal to the dimension of the space (which is given to be 5), it follows that this sum must be equal 5. But we proved in class that this sum is equal to the dimension of the space iff the transformation is diagonalizable, So T must be diagonalizable, which contradicts the assumption that it is not.