MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 5

1. In each of the following you are given a set and two operations: A 'sum', acting between two elements in the set, and a 'multiplication by scalar', acting between one element in the set and a scalar from $\mathbb{R}$. In each case determine whether the set with these two operations gives a vector space over $\mathbb{R}$. If it is a vector space then prove this fact. If it is not a vector space then show this by giving a counterexample. **In this question you are allowed to use only the definition of a vector space, not any other claim given in class.**

   i. The set $P_2(\mathbb{R})$ with the usual operations of summation and multiplication by scalar defined for polynomials.

   ii. The set
   $$\{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x - y + 2z = 0\}$$
   with the usual operations of summation and multiplication by scalar defined for n-tuples.

   iii. The set $\mathbb{R}^2$ with the operations
   $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ 0 \end{pmatrix}$$
   and
   $$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ 0 \end{pmatrix}.$$

   iv. The set $\mathbb{R}^2$ with the operations (note the locations of y_2 in the definition)
   $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_2 \\ x_2 + y_2 \end{pmatrix}$$
   and
   $$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_2 \end{pmatrix}.$$

   v. The set $\mathbb{R}^2$ with the operations
   $$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 - 3 \\ x_2 + y_2 - 2 \end{pmatrix}$$
   and
   $$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 - 3\alpha + 3 \\ \alpha x_2 - 2\alpha + 2 \end{pmatrix}.$$

vi. The set $\mathbb{R}^2$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2\alpha x_1 \\ 2\alpha x_2 \end{pmatrix}.$$

vii. The set $\{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R} : x_1 > 0, x_2 > 0\}$ with the operations

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 y_1 \\ x_2 y_2 \end{pmatrix}$$

and

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \end{pmatrix}.$$

2. Let V be a vector space over $\mathbb{R}$. Prove the following claims. (That is, prove that each one of these claims follows from the definition of a vector space).
   i. For every $v \in V$ we have $2v + v = 3v$.
   ii. For every scalar $\alpha \in \mathbb{R}$ we have $\alpha 0_V = 0_V$.
   iii. The additive inverse of the additive inverse of a vector is equal to the vector, that is, if $v \in V$ then $-(-v) = v$.
   iv. For every $u, v, w, z \in V$ we have $(u + w) + (v + z) = w + (u + (v + z))$.

3. In each of the following you are given a vector space V and a subset W of this space. Determine whether the subset is a subspace. If you claim that the answer is 'yes' then prove this. If you claim that the answer is no then show this by providing a counterexample.
   i. $V = \mathbb{R}^4$ and $W = \{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0\}$.
   ii. $V = M_2(\mathbb{R})$ and $W = \{\begin{pmatrix} x & 2x + 3y \\ y & x - y \end{pmatrix} : x, y \in \mathbb{R}\}$.
   iii. $V = \mathbb{R}_4[x]$ and $W = \{p(x) \in \mathbb{R}_4[x] : p(1) = 1\}$.
   iv. $V = \mathbb{R}_4[x]$ and $W = \{p(x) \in \mathbb{R}_4[x] : p(1) = 0\}$.
   v. $V = \mathbb{R}^3$ and $W = \{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} : x_1 \in \mathbb{Q}, x_2 \in \mathbb{Q}, x_3 \in \mathbb{Q}\}$.

vi. Let $A \in M_{3\times 4}(\mathbb{R})$ be a specific matrix. in this question $V = M_4(\mathbb{R})$ and $W = \{B \in M_4(\mathbb{R}) : AB = 0\}$.

viii. $V = \{f : \mathbb{R} \mapsto \mathbb{R}\}$ and

$$W = \{f \in V : f \text{ is twice diffarentiable and } f''(x) + 3f'(x) - f(x) = 0 \;\; \forall x \in \mathbb{R}\}$$

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4. Let V be a vector space over $\mathbb{R}$ and let $W \subset V$ and $U \subset V$ be two subspaces of V. The following claims are either true or false. Determine whether they are true or false and prove or disprove using a counterexample accordingly.
   a. $U \cap W$ is also a subspace of V.
   b. $U \cup W$ is also a subspace of V.
   c. We define the following subset of V:

   $$U + W := \{u + w : u \in U, w \in W\}.$$

   In this part of the question the claim is: $U + W$ is a subspace of V.

5. i. Give an example of a subset of $\mathbb{R}^2$ that is closed under scalar multiplication but not addition.
   ii. Give an example of a subset of $\mathbb{R}^2$ that is closed under addition but not scalar multiplication.
   iii. Give an example of a subset of $\mathbb{R}^2$ that is closed under neither.
   iv. Identify all of the subspaces of $\mathbb{R}^3$, you do not need to prove your claim, just provide a 'good guess'.