MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 6

1. In each of the following you are given a statement, which may be true or false. Determine whether the statement is correct and show how you reached this conclusion.

   i. $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} \in \text{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$

   ii. $2 + 3x + 2x^2 - x^3 \in \text{span}\{1 - x^3, 2 + x + x^2, 3 - x\}$

   iii. $\text{span}\{\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}\} \subseteq \text{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$

   iv. $\text{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\} = \text{span}\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}$

   v. $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ is a spanning set for $\mathbb{R}^2$.

   vi. $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is a spanning set for $\mathbb{R}^2$.

   vii. $\{1 - x + x^2, x - x^2 + x^3, 1 + x^2 - x^3, x^3\}$ is a spanning set for $\mathbb{R}_3[x]$.

   vii. $\{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix}\}$ is a spanning set for $M_2(\mathbb{R})$.

2. In each of the following you are given a vector space (you do not need to prove that this is indeed a vector space). Find a spanning set for each of these vector spaces.

   i. $\{A \in M_n(\mathbb{R}) : A$ is diagonal $\}$ (The definition of a diagonal matrix was given in HW2).

   ii. $\{\begin{pmatrix} a+b+c \\ a-2b \\ 3a-2c \\ 4c-b \end{pmatrix} : a, b, c \in \mathbb{R}\}$

   iii. $\{A \in M_2(\mathbb{R}) : A\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$

   iv. $\{p(x) \in \mathbb{R}_3[x] : p'(1) = 0\}$

   v. $\{p(x) \in \mathbb{R}_n[x] : p(1) = p(-1)\}$

3. In each of the following you are given a set, determine whether it is linearly independent or linearly dependent, show how you reach your conclusion.
   i. $\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$
   ii. $\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}$
   iii. $\{1 - x^3, 2 + x + x^2, 3 - x, 1 + x + x^2 + x^3\}$
   iv. $\{f(x) = \sin^2 x, g(x) = \cos^2(x), h(x) = 1\}$ (Note that $h(x)$ is the constant function which is equal to 1 for every x).

4. Let V be a vector space and w_1, w_2, w_3 in V be such that $\{w_1, w_2, w_3\}$ is linearly independent. Prove or disprove the following claims.
   i. The set $\{w_1 + w_2 + w_3, w_2 + w_3, w_3\}$ is linearly independent.
   ii. The set $\{w_1 + 2w_2 + w_3, w_2 + w_3, w_1 + w_2\}$ is linearly independent.

5. Let V be a vector space and let $S \subset V$ and $T \subset V$ be two finite subsets of V. Prove or disprove the following claims.
   i. If $S \subset T$ and S is linearly independent then T is linearly independent.
   ii. If $S \subset T$ and T is linearly independent then S is linearly independent.
   iii. If S and T are linearly independent then $S \cap T$ is either empty or linearly independent (Remark: sometimes people consider an empty set to be linearly independent).
   iv. If S and T are linearly independent then $S \cup T$ is linearly independent.
   v. If $W = \text{span}S$ and $U = \text{span}T$ then $W + U = \text{span}(S \cup T)$. (The sum of two subspaces was defined in HW4).

6. Let $v_1, ..., v_n \in \mathbb{R}^m$. Prove or disprove the following claims. (Hint: several of these were given in previous HW's, or in class during our studies of Chapters 1 and 2, but with different formulations. As usual, you are welcome to use whatever was proved in class without repeating the proof.)
   i. If $\{v_1, ..., v_n\}$ is linearly independent then $n \leq m$.
   ii. If $n \leq m$ then $\{v_1, ..., v_n\}$ is linearly independent.
   iii. If $\{v_1, ..., v_n\}$ spans $\mathbb{R}^m$ then $n \geq m$.
   iv. If $n \geq m$ then $\{v_1, ..., v_n\}$ spans $\mathbb{R}^m$
   v. If $\{v_1, ..., v_n\}$ is both linearly independent and spans $\mathbb{R}^m$ then $n = m$.
   vi. If $n = m$ then: $\{v_1, ..., v_n\}$ is linearly independent iff $\{v_1, ..., v_n\}$ spans $\mathbb{R}^m$.