MATH-1564, K1, TA: Sam, Instructor: Nitzan, Sigal Shahaf
HW4 ; Alexander Guo

1. (a) $AB = \begin{pmatrix} 5 & 2 & 5 \\ 0 & -1 & 5 \end{pmatrix}$

   (b) $BA =$ Undefined, inner dims don't match. $(2 \times \underline{3})(\underline{2} \times 2)$

   (c) $D^2 = \begin{pmatrix} 7 & -3 & 6 \\ -2 & 3 & -1 \\ 3 & -2 & 9 \end{pmatrix}$

   (d) $B^2 =$ Undefined, inner dims don't match. $(2 \times \underline{3})(\underline{2} \times 3)$

   (e) $DC = \begin{pmatrix} 7 & -1 \\ 0 & 5 \\ 1 & 2 \end{pmatrix}$

   (f) $CB = \begin{pmatrix} 5 & 2 & 5 \\ 5 & 1 & 10 \\ 0 & -1 & 5 \end{pmatrix}$

   (g) $BC = \begin{pmatrix} 7 & -1 \\ 7 & 4 \end{pmatrix}$

   (h) $FE = \begin{pmatrix} 2 \end{pmatrix}$

   (i) $EF = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{pmatrix}$

   (j) $CE =$ Undefined, inner dims don't match. $(3 \times \underline{2})(\underline{3} \times 1)$

   (k) $EC =$ Undefined, inner dims don't match. $(3 \times \underline{1})(\underline{3} \times 2)$

2. (a) i. A is invertible. $A^{-1} = \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & -0.2 \end{pmatrix}$

   ii. B is not invertible. Its REF is $\begin{pmatrix} 1 & -3 \\ 0 & 0 \end{pmatrix}$

   iii. C is invertible. $C^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

   iv. D is invertible. $D^{-1} = \begin{pmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \end{pmatrix}$

v. E is not invertible. Its REF is $\begin{pmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & \frac{10}{3} \\ 0 & 0 & 0 \end{pmatrix}$

(b) $A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \to A^{-1}A\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.2 & 0.4 \\ 0.4 & -0.2 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} \to \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -0.6 \\ 0.8 \end{pmatrix}$

(c) $B\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \to \begin{pmatrix} 1 & -3 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \to \begin{pmatrix} x-3y \\ -2x+6y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} \xrightarrow{R_2+2R_1} \begin{pmatrix} x-3y \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ So: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

(d) $DG = E \to D^{-1}DG = D^{-1}E \to G = \begin{pmatrix} 0.5 & -0.5 & 0.5 \\ 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \end{pmatrix}\begin{pmatrix} 3 & 0 & 2 \\ 2 & -1 & -2 \\ -1 & 2 & 6 \end{pmatrix} \to G = \begin{pmatrix} 0 & 1.5 & 5 \\ 3 & -1.5 & -3 \\ -1 & 0.5 & 1 \end{pmatrix}$

3. (a) **False**. If $A \in M_n(\mathbb{R})$, fix $n = 2$, let us say that $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. $A^2 = 0$, but $A \neq 0$, thus the statement is false.

(b) **True**. Let $A, B \in M_n(\mathbb{R})$. Rewrite $AB^2 = B^2A$ as $A(BB) = (BB)A$. Apply associative property on matrices: $(AB)B = B(BA)$. Since we are given that $AB = BA$, substitute in equation: $(BA)B = B(AB)$. Apply associative property again and get $B(AB) = B(AB)$. We therefore conclude that the terms are equal and that $AB^2 = B^2A$ as long as $AB = BA$.

(c) **False**. If $A, B, C \in M_n(\mathbb{R})$, fix $n = 2$, let us set $A = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $C = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}$. $AB = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}$, and $CB = \begin{pmatrix} 0 & 0 \\ 2 & 2 \end{pmatrix}$. However, $A \neq C$, so clearly the statement is false.

4. (a) **True**. If A,B are invertible square matrices, then we need to prove that $B^{-1}A^{-1}$ is the inverse of AB. Let us denote $B^{-1}A^{-1}$ as C. We want to show that $(AB)C = C(AB) = I$ since it is the definition of inverse matrices. Thus, if we plug in for C: $(AB)(B^{-1}A^{-1})$, by the associative property for matrices, $A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I$. Indeed, AB is invertible, and its inverse is equal to $B^{-1}A^{-1}$. This also works when we plug C in to the left side: $(B^{-1}A^{-1})(AB) = B^{-1}IB = I$

(b) **True**. If AB is invertible, then there exists a C so that $C(AB) = I$. Because matrices are multiplicatively associative, then we get $(CA)B = I$. We claim that B is invertable if $XB = I$. Denote CA as X. Therefore, B is indeed invertible. On the other hand, if AB is invertible there exists a C so that $(AB)C = I$. Given the multiplicative association property of matrices, we get $A(BC)$. We claim that A is invertible if $AX = I$. Denote CA as X. Therefore, A is indeed invertible.

(c) **False**. If $A, B \in M_n(\mathbb{R})$, fix $n = 2$, let us say that $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 \\ 3 & 4 \end{pmatrix}$. $A + B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, and its inverse is $\begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$, so it is indeed invertible. However, A has no inverse, so the statement is false.

(d) **False**. If $A, B \in M_n(\mathbb{R})$, fix $n = 2$, let us say that $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} 0 & -1 \\ -3 & -4 \end{pmatrix}$. $A^{-1} = \begin{pmatrix} -2 & 1 \\ 1.5 & -0.5 \end{pmatrix}$, $B^{-1} = \begin{pmatrix} \frac{4}{3} & -\frac{1}{3} \\ -1 & 0 \end{pmatrix}$ so A,B are both invertible. $A + B = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, but the resulting matrix has no inverse. Therefore, the statement is false.

(e) **True**. If AB is invertible, then from (b), we know that A and B are both invertible. Then, from question (a), we know that the multiplication of two invertible matrices are also invertible. Hence, BA, is invertible, proving our claim to be true.

(f) **True**. If A^3 is invertible, then there exists a B so that $A^3B = I$. We can rearrange this expression to $(AAA)B = I$. With the associative property of matrix products, we have $A(AAB) = I$. We claim that A is invertible if there exists some X so that $AX = I$. Indeed, denote AAB as X, so $AX = I$, therefore A is invertible and the statement is true.

5. (a) Let $A, B \in M_n(\mathbb{R})$, then set $A_{ij} = 0$ for all $i \neq j$. So, $A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & a_{nn} \end{pmatrix}$ and $B = \begin{pmatrix} b_{11} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & b_{nn} \end{pmatrix}$. By regular matrix addition, each respective coordinate is added $a_{ij} + b_{ij}$, and $A + B = \begin{pmatrix} a_{11} + b_{11} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & a_{nn} + b_{nn} \end{pmatrix}$.

Therefore, we conclude that $A+B$ is diagonal. Similarly, by regular matrix multiplication, $AB = \begin{pmatrix} a_{11}b_{11} & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & a_{nn}b_{nn} \end{pmatrix}$, so we also conclude that AB is diagonal.

(b) Let $A, B, C, D \in M_n(\mathbb{R})$. Denote AB as C, then in sum notation for matrix multiplication, $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$, where $1 \leq i \leq n, 1 \leq j \leq n$. Since we are only interested in the trace of matrix C, which is given to us as $tr(C) = \sum_{i=1}^{n}(c)_{ii}$, we substitute in our multiplication equation and get $tr(C) = \sum_{i=1}^{n}\sum_{k=1}^{n} a_{ik}b_{ki}$. On the other hand, denote BA as D, then in sum notation for matrix multiplication, $d_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj}$, where $1 \leq i \leq n, 1 \leq j \leq n$. Since we are only interested in the trace of matrix D, which is given to us as $tr(D) = \sum_{i=1}^{n}(d)_{ii}$, we substitute in our multiplication equation and get $tr(D) = \sum_{i=1}^{n}\sum_{k=1}^{n} b_{ik}a_{ki}$. Then, we reverse the sigmoids, and get $tr(D) = \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ki}b_{ik}$. Because i and k take on the same values, they are interchangable, and thus tr(C) and tr(D) are equivalent. Therefore, $tr(AB) = tr(BA)$.

(c) The ij entry of AB can be written as $AB_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$. If we transpose a matrix, we switch the rows with the columns, so
$(AB_{ij})^T = AB_{ji} = \sum_{k=1}^{n} a_{jk}b_{ki}$.
On the other hand,
$(B^T A^T)_{ij} = \sum_{k=1}^{n}(B^T)_{ik}(A^T)_{kj} = \sum_{k=1}^{n} b_{ki}a_{jk}$.
Since real numbers are multiplicatively communitive, these two outcomes are the same and the matrices are therefore equal. Thus, $(AB)^T = B^T A^T$.