MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 10 – Solutions

1. i.
$$T : \mathbb{R}^3 \mapsto M_2(\mathbb{R})$$
given by,
$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y & y-2z \\ 3x+z & 0 \end{pmatrix}$$
This is a linear transformation. To show this we need to show that T satisfies both of the conditions in the definition of a linear transformation.

a. Let $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \in \mathbb{R}^3$. We need to show that $T(a+b) = Ta + Tb$. On one hand,
$$T(a+b) = T\Big(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\Big) =$$
$$= T\begin{pmatrix} a_1+b_1 \\ a_2+b_2 \\ a_3+b_3 \end{pmatrix} = \begin{pmatrix} a_1+b_1+a_2+b_2 & a_2+b_2-2(a_3+b_3) \\ 3(a_1+b_1)+a_3+b_3 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} a_1+b_1+a_2+b_2 & a_2+b_2-2a_3-2b_3 \\ 3a_1+3b_1+a_3+b_3 & 0 \end{pmatrix}$$
While on the other hand,
$$Ta + Tb = T\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + T\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} =$$
$$= \begin{pmatrix} a_1+a_2 & a_2-2a_3 \\ 3a_1+a_3 & 0 \end{pmatrix} + \begin{pmatrix} b_1+b_2 & b_2-2b_3 \\ 3b_1+b_3 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} a_1+b_1+a_2+b_2 & a_2+b_2-2a_3-2b_3 \\ 3a_1+3b_1+a_3+b_3 & 0 \end{pmatrix}$$
So indeed, we get that $T(a+b) = Ta + Tb$.

b. Let $a = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \in \mathbb{R}^3$ and $\beta \in \mathbb{R}$. We need to show that $T(\beta a) = \beta Ta$. Indeed,
$$T(\beta a) = T\begin{pmatrix} \beta a_1 \\ \beta a_2 \\ \beta a_3 \end{pmatrix} =$$

$$= \begin{pmatrix} \beta a_1 + \beta a_2 & \beta a_2 - 2\beta a_3 \\ 3\beta a_1 + \beta a_3 & 0 \end{pmatrix} = \beta \begin{pmatrix} a_1 + a_2 & a_2 - 2a_3 \\ 3a_1 + a_3 & 0 \end{pmatrix} =$$

$$= \beta T a$$

ii.

$$T : \mathbb{R}_2[x] \mapsto \mathbb{R}^3$$

given by,

$$Tp = \begin{pmatrix} p(2) \\ p'(2) \\ p''(2) \end{pmatrix}$$

This is a linear transformation. To show this we need to show that T satisfies both of the conditions in the definition of a linear transformation.

a. Let $p, q \in \mathbb{R}_2[x]$. We need to show that $T(p+q) = Tp + Tq$. Indeed, from the rules of derivatives we studied in calculus we get,

$$T(p+q) = \begin{pmatrix} (p+q)(2) \\ (p+q)'(2) \\ (p+q)''(2) \end{pmatrix} = \begin{pmatrix} p(2) + q(2) \\ p'(2) + q'(2) \\ p''(2) + q''(2) \end{pmatrix} =$$

$$= \begin{pmatrix} p(2) \\ p'(2) \\ p''(2) \end{pmatrix} + \begin{pmatrix} q(2) \\ q'(2) \\ q''(2) \end{pmatrix} = Tp + Tq.$$

b. Let $p \in \mathbb{R}_2[x]$ and $\beta \in \mathbb{R}$. We need to show that $T(\beta p) = \beta T p$. Indeed, from the rules of derivatives we studied in calculus we get,

$$T(\beta p) = \begin{pmatrix} (\beta p)(2) \\ (\beta p)'(2) \\ (\beta p)''(2) \end{pmatrix} = \begin{pmatrix} \beta p(2) \\ \beta p'(2) \\ \beta p''(2) \end{pmatrix} =$$

$$= \beta \begin{pmatrix} p(2) \\ p'(2) \\ p''(2) \end{pmatrix} = \beta T p.$$

iii.

$$T : M_2(\mathbb{R}) \mapsto M_2(\mathbb{R})$$

given by,

$$TA = A^2$$

This is not a linear transformation. Consider, for example the matrix I_2 and the scalar 3. Then $TI_2 = I_2^2 = I_2$ while $T(3I_2) = (3I_2)^2 = 9I_2$. So, $T(3I_2) \neq 3TI_2$.

iv.
$$T : \mathbb{R}_2[x] \mapsto \mathbb{R}$$
given by,
$$Tp = \int_0^1 p(x)dx$$
This is a linear transformation. To show this we need to show that T satisfies both of the conditions in the definition of a linear transformation.

a. Let $p, q \in \mathbb{R}_2[x]$. We need to show that $T(p+q) = Tp + Tq$. Indeed, by the rules for integration which we studied in calculus,
$$T(p+q) = \int_0^1 (p+q)(x)dx = \int_0^1 (p(x)+q(x))dx = \int_0^1 p(x)dx + \int_0^1 q(x)dx = Tp+Tq.$$
b. Let $p \in \mathbb{R}_2[x]$ and $\beta \in \mathbb{R}$. We need to show that $T(\beta p) = \beta Tp$. Indeed, by the rules for integration which we studied in calculus,
$$T(\beta p) = \int_0^1 (\beta p)(x)dx = \int_0^1 \beta p(x)dx = \beta \int_0^1 p(x)dx = \beta Tp.$$

v. Fix $B \in M_3(\mathbb{R})$ and consider the function:
$$T : M_3(\mathbb{R}) \mapsto M_3(\mathbb{R})$$
given by,
$$TA = AB$$
This is a linear transformation. To show this we need to show that T satisfies both of the conditions in the definition of a linear transformation.

a. Let $A, C \in M_3(\mathbb{R})$. We need to show that $T(A+C) = TA+TC$. Indeed, by the rules of matrix multiplication which we stated in class,
$$T(A+C) = (A+C)B = AB + CB = TA + TC.$$
b. Let $A \in M_3(\mathbb{R})$. We need to show that $T(\beta A) = \beta TA$. Indeed, by the rules of matrix multiplication which we stated in class,
$$T(\beta A) = (\beta A)B = \beta(AB) = \beta TA.$$

vi.
$$T : M_2(\mathbb{R}) \mapsto M_2(\mathbb{R})$$
given by,
$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c+1 & 2a+3b+2 \\ d-b-8 & 2a \end{pmatrix}$$
This is not a linear transformation. Indeed, we have,
$$T\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -8 & 0 \end{pmatrix}.$$

So, by the 'thumbs–rule' we proved in class T is not linear, as the image of the zero vector is not the zero vector.

2. i. For
$$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 7 & -6 \\ 0 & 5 & -5 \end{pmatrix}$$
consider the operator
$$T_A : \mathbb{R}^3 \mapsto \mathbb{R}^4$$
where the notation T_A was defined in class.

a. We start with the kernel of T_A. We studied in class that the kernel of this transformation is equal to the null space of A. To find a basis for this space we bring the matrix to echelon form.

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 7 & -6 \\ 0 & 5 & -5 \end{pmatrix} \xrightarrow{R_2-3R_1, R_3-R_1} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 5 \\ 0 & 5 & -5 \\ 0 & 5 & -5 \end{pmatrix} \xrightarrow{R_3+R_2, R_4+R_2} \begin{pmatrix} 1 & 2 & -1 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The corresponding linear system is:
$$\begin{cases} x + 2y - z = 0 \\ -5y + 5z = 0 \end{cases}$$

So, z is a free variable and can be any real number. We denote this by $z = t$, $t \in \mathbb{R}$. It follows that $y = t$ and $x = -t$. So the null space of A is

$$\{\begin{pmatrix} -t \\ t \\ t \end{pmatrix} : t \in \mathbb{R}\} = \{t \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} : t \in \mathbb{R}\} = \text{span}\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\}.$$

So $\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\}$ is a spanning set for null A, and since it contains a single vector which is different from zero, this set is also linearly independent. So, $\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\}$ is a basis for null A.

We now turn to the image of T_A. We studied in class that the image of this transformation is equal to the column space of A.

That is, the image of the transformation is equal to

$$\text{span}\left\{\begin{pmatrix}1\\3\\1\\0\end{pmatrix},\begin{pmatrix}2\\1\\7\\5\end{pmatrix},\begin{pmatrix}-1\\2\\-6\\-5\end{pmatrix}\right\}.$$

To find a basis for this space we follow the algorithm described in class, write the vectors as rows of a matrix and bring it to echelon form

$$\begin{pmatrix}1&3&1&0\\2&1&7&5\\-1&2&-6&-5\end{pmatrix}\xrightarrow{R_2-2R_1,R_3+R_1}\begin{pmatrix}1&3&1&0\\0&-5&5&5\\0&5&-5&-5\end{pmatrix}\xrightarrow{R_3+R_2}$$
$$\begin{pmatrix}1&3&1&0\\0&-5&5&5\\0&0&0&0\end{pmatrix}$$

As we studied, a bases for the space is given by the non–zero rows:

$$\left\{\begin{pmatrix}1\\3\\1\\0\end{pmatrix},\begin{pmatrix}0\\-5\\5\\5\end{pmatrix}\right\}$$

b. The kernel has a basis containing 1 element, so $\dim(\ker T_A) = 1$. The image has a basis containing two elements, so $\dim(\mathrm{Im} T_A) = 2$. The domain is $\mathbb{R}^3$ which has dimension 3. Indeed, $3 = 2 + 1$.

c. The transformation is not onto, as the image is not equal to the codomain. Indeed, the codomain is $\mathbb{R}^4$ which has dimension 4 while we have seen that the dimension of the image is 2. By a result from class this means that the image is only a subspace of the codomain, and not equal to it.
Remark: Another way to justify: we proved in class that if the dimension of the domain is smaller then the dimension of the codomain, then the transformation cannot be onto.

d. The transformation is not $1-1$. Indeed, the dimension of the kernel is 1 which implies that the kernel is not equal to the set $\{\underline{0}\}$ (The dimension of $\{\underline{0}\}$ is zero).

ii.

$$S : M_2(\mathbb{R}) \mapsto \mathbb{R}^2$$

given by

$$SA = A\begin{pmatrix}3\\-2\end{pmatrix}$$

a. We start by rewriting the transformation:

$$S\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 3a-2b \\ 3c-2d \end{pmatrix}.$$

We first consider the kernel of S. A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the kernel of S iff $SA = \underline{0}$, that is if

$$\begin{pmatrix} 3a-2b \\ 3c-2d \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

We conclude that

$$\ker(S) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} : 3a-2b=0, 3c-2d=0\right\} =$$

$$= \left\{\begin{pmatrix} \frac{2}{3}b & b \\ \frac{2}{3}d & d \end{pmatrix}\right\}.$$

Every vector in ker$S$ can therefore be written in the following way:

$$\begin{pmatrix} \frac{2}{3}b & b \\ \frac{2}{3}d & d \end{pmatrix} = b\begin{pmatrix} \frac{2}{3} & 1 \\ 0 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}.$$

Both of the matrices on the RHS belong to ker$S$ (check!) and every other matrix in ker$S$ can be written as a linear combination of them. We conclude that the set

$$\left\{\begin{pmatrix} \frac{2}{3} & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ \frac{2}{3} & 1 \end{pmatrix}\right\}$$

is a spanning set for ker$S$. This set contains two vectors, and neither of this vectors is a scalar multiplying the other (which is the only way that two vectors can be linearly dependent). This means that this set is linearly independent and therefore a basis for ker$S$.
We claim that the image of the transformation is equal to $\mathbb{R}^2$ and therefore the set $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ is a basis for ImS. To see that ImS=$\mathbb{R}^2$ we first find the dimension of ImS. By the dimension formula we studied in class we know that

$$\dim(M_2(\mathbb{R})) = \dim(\ker S) + \dim(\mathrm{Im} S).$$

We proved in class that $\dim(M_2(\mathbb{R})) = 4$ and we just showed that $\dim(\ker S) = 2$. We conclude that $\dim(\mathrm{Im} S) = 2$. We proved in class that ImS is a subspace of the codomain, that is, im$S$ is a subspace of $\mathbb{R}^2$. From the result we proved in class regarding the dimension of a subspace, we know that the dimension of ImS is equal to the dimension of $\mathbb{R}^2$ iff these two spaces are equal. Since the dimension of $\mathbb{R}^2$ is 2 we conclude that indeed ImS=$\mathbb{R}^2$.

b. This was done during the solution of part (a).

c. The transformation is onto. We proved during the solution of part (a) that the image of S is equal to the codomain.

d. The transformation is not $1-1$. Indeed, the dimension of the kernel is 2 which implies that the kernel is not the set $\{\underline{0}\}$, (The dimension of $\{\underline{0}\}$ is zero).

iii.

$$L : \mathbb{R}_3[x] \mapsto \mathbb{R}^2$$

given by

$$Lp = \begin{pmatrix} p(2)-p(1) \\ p'(0) \end{pmatrix}$$

a. We start by rewriting the transformation:

$$L(a+bx+cx^2+dx^3) = \begin{pmatrix} b+3c+7d \\ b \end{pmatrix}$$

We first consider the kernel of L. A polynomial p is in the kernel iff it satisfies $Lp = \underline{0}$. We conclude that

$$\ker L = \{a+bx+cx^2+dx^3 : b+3c+7d=0, b=0\} =$$

$$= \{a+cx^2+dx^3 : 3c+7d=0\} = \{a-\frac{7}{3}dx^2+dx^3\}$$

So every vector in the kernel can be written as

$$a-\frac{7}{3}dx^2+dx^3 = a\cdot 1 + d(-\frac{7}{3}x^2+x^3).$$

Both of the vectors in the set $\{1, -\frac{7}{3}x^2+x^3\}$ belongs to the kernel (check!) and every other polynomial in $\ker L$ can be written as a linear combination of them. We conclude that the set $\{1, -\frac{7}{3}x^2+x^3\}$ is a spanning set for $\ker L$. This set contains two vectors, and neither of this vectors is a scalar multiplying the other (which is the only way that two vectors can be linearly dependent). This means that this set is linearly independent and therefore a basis for $\ker L$.

We claim that the image of the transformation is equal to $\mathbb{R}^2$ and therefore the set $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ is a basis for ImL. To see that ImL=$\mathbb{R}^2$ we first find the dimension of ImL. By the dimension formula we studied in class we know that

$$\dim(\mathbb{R}_3[x]) = \dim(\ker L) + \dim(\mathrm{Im} L).$$

We proved in class that $\dim(\mathbb{R}_3[x]) = 4$ and we just showed that $\dim(\ker L) = 2$. We conclude that $\dim(\mathrm{Im} L) = 2$. We proved in class that ImL is a subspace of the codomain, that is, im$L$ is a subspace of $\mathbb{R}^2$. From the result we proved in class regarding the dimension of a subspace, we know that the dimension of ImL is equal to the dimension of $\mathbb{R}^2$ iff these two spaces are equal. Since the dimension of $\mathbb{R}^2$ is 2 we conclude that indeed ImL=$\mathbb{R}^2$.

b. This was done during the solution of part (a).

c. The transformation is onto. We proved during the solution of part (a) that the image of L is equal to the codomain.

d. The transformation is not $1-1$. Indeed, the dimension of the kernel is 2 which implies that the kernel is not the set $\{\underline{0}\}$ (The dimension of $\{\underline{0}\}$ is zero).

iv.

$$\Phi : \mathbb{R}^3 \mapsto \mathbb{R}_3[x]$$

given by

$$\Phi\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b) + (a-2b+c)x + (b-3c)x^2 + (a+b+c)x^3$$

a. A 3–tuple $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ belongs to the kernel of Φ iff its image under this transformation is the constant zero polynomial. Since two polynomials are equal iff their coefficients are equal this means that $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ is in the kernel iff it is a solution to the following homogeneous linear system.

$$\begin{cases} a+b=0 \\ a-2b+c=0 \\ b-3c=0 \\ a+b+c=0 \end{cases}$$

We write this system in a matrix form and bring it to echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{R_2-R_1, R_4-R_1} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_3+3R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -8 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_4 \leftrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right) \xrightarrow{R_4+8R_3} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

All of the variables in the echelon form are leading variables so the only solution to the corresponding homogeneous linear system is the trivial solution. We conclude that $\ker\Phi = \{\underline{0}\}$ and therefore a basis for this kernel is the empty set $\{\}$.
We now turn to study the image of the transformation. A polynomial is in the image iff it can be expressed as $(a+b)+(a-2b+c)x+(b-3c)x^2+(a+b+c)x^3$ with $a,b,c \in \mathbb{R}$. This implies that

$$\mathrm{Im}(\Phi) = \{(a+b)+(a-2b+c)x+(b-3c)x^2+(a+b+c)x^3 : a,b,c \in \mathbb{R}\}$$

Every polynomial in this space can be expressed as,

$$(a+b)+(a-2b+c)x+(b-3c)x^2+(a+b+c)x^3 = a(1+x+x^3)+b(1-2x+x^2+x^3)+c(x-3x^2+x^3)$$

Each element of the set $\{1+x+x^3, 1-2x+x^2+x^3, x-3x^2+x^3\}$ belongs to the space $\mathrm{Im}\Phi$ (check!) and every other element of the space is a linear combination of these three vectors. We conclude that $\{1+x+x^3, 1-2x+x^2+x^3, x-3x^2+x^3\}$ is a spanning set for the space. We claim that $\dim(\mathrm{Im}\phi)=3$ and therefore a set of three vectors which is a spanning set for the space is a basis for the space, so $\{1+x+x^3, 1-2x+x^2+x^3, x-3x^2+x^3\}$ is a basis for the space. To see that $\dim(\mathrm{Im}\phi)=3$ we use the dimension formula,

$$\dim(\mathbb{R}^3) = \dim(\ker\Phi) + \dim(\mathrm{Im}\Phi).$$

We proved in class that $\dim(\mathbb{R}^3) = 3$ and we have just seen that $\ker\Phi = \{\}$ so $\dim(\ker\Phi) = 0$. We conclude that indeed, $\dim(\mathrm{Im}\phi) = 3$ and $\{1+x+x^3, 1-2x+x^2+x^3, x-3x^2+x^3\}$ is a basis for the space.

b. This was done in part (a).

c. The transformation is not onto. We proved in class that if the dimension of the domain is smaller then the dimension of the codomain then the transformation cannot be onto.

d. The transformation is $1-1$. Indeed, we proved in part (a) that $\ker\Phi = \{0\}$.

3 & 4. i. ⋄. The claim is false. Indeed, every zero transformation from a non–zero vector space will provide a counter example. Let us give a different counterexample, just for fun. Consider the linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ which is defined by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2a+b \\ c \end{pmatrix}.$$

The set $\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$ is linearly independent (why?), but the set of their images under the transformation T is $\{\begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ which is linearly dependent (why?).

$\diamond$. The answer doesn't change if we add the condition that T is onto. In fact, in the counterexample we gave above the linear transformation is onto (check!).

$\diamond$. The answer changes if we add the condition that T is $1-1$. To be precise, let us formulate the statement which we claim is **true**:

If $v_1, ..., v_n$ is linearly independent in V and T is a linear $1-1$ transformation then $Tv_1, ..., Tv_n$ is linearly independent in W.

Proof: To show that $Tv_1, ..., Tv_n$ is linearly independent we assume that $a_1, .., a_n \in \mathbb{R}$ are such that

$$a_1Tv_1 + ... + a_nTv_n = 0_W.$$

Our goal is to prove that $a_1 = ... = a_n = 0$ which will imply that the **only** linear combination of $Tv_1, ..., Tv_n$ which is equal to the zero of the space is the trivial linear combination. By definition this will imply that $Tv_1, ..., Tv_n$ are linearly independent.
So, we use the fact that T is linear and get that

$$a_1Tv_1 + ... + a_nTv_n = T(a_1v_1 + ... + a_nv_n).$$

We insert this relation to the equation above and get,

$$T(a_1v_1 + ... + a_nv_n) = 0_W.$$

We proved in class that since T is $1-1$ the only pre–image of 0_W is 0_V. This implies that,

$$a_1v_1 + ... + a_nv_n = 0_V.$$

Since we are given that $v_1, ..., v_n$ are linearly independent, the only linear combination of them which is equal to 0_V is the trivial one. We conclude that $a_1 = ... = a_n = 0$ and therefore $Tv_1, ..., Tv_n$ are linearly independent.

ii. The claim is true. Proof: To show that $v_1, ..., v_n$ is linearly independent we assume that $a_1, .., a_n \in \mathbb{R}$ are such that

$$a_1v_1 + ... + a_nv_n = 0_V.$$

Our goal is to prove that $a_1 = ... = a_n = 0$ which will imply that the **only** linear combination of $v_1, ..., v_n$ which is equal to the zero of the space is the trivial linear combination. By definition this will imply that $v_1, ..., v_n$ are linearly independent.

So, we apply the transformation T to both sides of the equation to get:

$$T(a_1v_1 + ... + a_nv_n) = T(0_V).$$

we use the fact that T is linear and get that

$$T(a_1v_1 + ... + a_nv_n) = a_1Tv_1 + ... + a_nTv_n.$$

We insert this relation to the equation above and get,

$$a_1Tv_1 + ... + a_nTv_n = T(0_V).$$

We insert also the 'thumbs rule' from class: $T(0_V) = 0_W$. We get,

$$a_1Tv_1 + ... + a_nTv_n = 0_W.$$

Since we are given that $Tv_1, ..., Tv_n$ are linearly independent, the only linear combination of them which is equal to 0_W is the trivial one. We conclude that $a_1 = ... = a_n = 0$ and therefore $v_1, ..., v_n$ are linearly independent.

iii. ◇ The claim is false, in fact any transformation which is not onto will provide a counterexample. For instance, consider the transformation $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ which is defined by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

Then the set $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ is a spanning set for $\mathbb{R}^2$, as we proved in class, but the set of their images under T, $\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$ is not a spanning set for $\mathbb{R}^3$ ($\mathbb{R}^3$ is of dimension 3 so by a claim we proved in class, no set of 2 vectors can be a spanning set for this space).

◇ The answer does not change if we add the condition that T is $1-1$. In fact, the linear transformation in the counterexample we constructed above is $1-1$ (check!).

◇ The answer changes if we add the condition that T is onto. To be precise, let us formulate the statement which we claim is **true**:

If $v_1, ..., v_n$ is a spanning set in V and T is an onto linear transformation then $Tv_1, ..., Tv_n$ is a spanning set in W.

Proof: We want to show that every vector in W is a linear combination of $Tv_1, ..., Tv_n$, this will imply that the set indeed spans W. So, let $w \in W$. Since T is onto w has a pre-image, that is, there exists $v \in V$ such that $Tv = w$. Since $v_1, ..., v_n$ is a spanning set in V there exist $a_1, .., a_n \in \mathbb{R}$ such that

$$v = a_1v_1 + ... + a_nv_n.$$

We apply the transformation T on both sides and get,

$$Tv = T(a_1v_1 + ... + a_nv_n).$$

We insert the relation $Tv = w$ on the LHS and use the fact that T is linear on the RHS. We get,

$$w = a_1Tv_1 + ... + a_nTv_n.$$

So w is a linear combination of $Tv_1, ..., Tv_n$. Since this process works for every $w \in W$, we conclude that every vector in W is a linear combination of $Tv_1, ..., Tv_n$ and therefore the set indeed spans W.

iv. ◇ The claim is false. Consider for example the transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ which is defined by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then the set $\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\}$ is not a spanning set for $\mathbb{R}^3$ ($\mathbb{R}^3$ is of dimension 3 so by a claim we proved in class, no set of 2 vectors can be a spanning set for this space). However, the set of its images $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ is a spanning set for $\mathbb{R}^2$, as we proved in class.

◇ The condition we were given, that $Tv_1, ..., Tv_n$ is a spanning set in W, implies that T is onto. Therefore, adding the condition that T is onto is adding a redundant condition, which will not change the answer. To see that if $Tv_1, ..., Tv_n$ is a spanning set in W then T is onto, recall first that we proved in class that ImT is a subspace for W. Next, since $Tv_1, ..., Tv_n \in$ImT, it follows from a result we proved in class that their span is a subspace of ImT. So, if $Tv_1, ..., Tv_n$ is a spanning set in W, it follows that $W \subset$ImT and therefore $W =$ImT. So T is indeed onto.

◇ The answer changes if we add the condition that T is $1-1$. To be precise, let us formulate the statement which we claim is **true**:

If $Tv_1, ..., Tv_n$ is a spanning set in W and T is $1-1$ then $v_1, ..., v_n$ is a spanning set in V.

Proof: We want to show that every vector in V is a linear combination of $v_1, ..., v_n$, this will imply that the set indeed spans V. So, let $v \in V$. Since $Tv_1, ..., Tv_n$ is a spanning set in W there exist $a_1, .., a_n \in \mathbb{R}$ such that

$$Tv = a_1Tv_1 + ... + a_nTv_n.$$

Since T is linear we get,

$$Tv = T(a_1v_1 + ... + a_nv_n).$$

Since T is $1-1$ it follows that

$$v = a_1v_1 + ... + a_nv_n.$$

So v is a linear combination of $v_1, ..., v_n$. Since this process works for every $v \in V$, we conclude that every vector in V is a linear combination of $v_1, ..., v_n$ and therefore the set indeed spans V.

v. The claim is true. To prove that TU is a subspace of W we follow the result from class and show the following three things:

TU is not empty: indeed, since U is a subspace of V then U is not empty. So there exists some $u \in U$. Clearly, $Tu \in TU$ so it follows that TU is not empty.

TU is closed to summation: Let $w_1, w_2 \in TU$. By the definition of TU, there exist $u_1, u_2 \in U$ such that $Tu_1 = w_1$ and $Tu_2 = w_2$. Since we are given that U is a subspace of V, we know that U is closed to addition. It follows that $u_1 + u_2 \in U$ and therefore $T(u_1 + u_2) \in TU$. Since T is linear, we know that $T(u_1 + u_2) = Tu_1 + Tu_2 = w_1 + w_2$. We conclude that $w_1 + w_2 \in TU$ and therefore TU is closed to addition.

TU is closed to multiplication by a scalar: Let $w \in TU$ and $\alpha \in \mathbb{R}$. By the definition of TU, there exist $u \in U$ such that $Tu = w$. Since we are given that U is a subspace of V, we know that U is closed to multiplication by a scalar. It follows that $\alpha u \in U$ and therefore $T(\alpha u) \in TU$. Since T is linear, we know that $T(\alpha u) = \alpha Tu = \alpha w$. We conclude that $\alpha w \in TU$ and therefore TU is closed to multiplication by a scalar.

vi.
- ◇ The claim is false, in fact, the zero transformation allows to construct a simple counterexample. Let us construct another counter example, just for fun. Consider for example the transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^2$ which is defined by

$$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

  The set $U = \{\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}\} \cup \{\begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix}\}$ is not a subspace of $\mathbb{R}^3$ since it is not closed to addition (check!). However, its image satisfies $TU = \mathbb{R}^2$, so TU is a subspace.
- ◇ The answer does not change if we add the information that T is onto. In fact, the counterexample we gave above was of a linear transformation that is onto.

- $\diamond$ The answer changes if we add the condition that T is $1-1$. To be precise, let us formulate the statement which we claim is **true**:

  If U is a subset of V, the set $\{Tu : u \in U\}$ is a subspace of W, and T is a $1-1$ transformation, then U is a subspace of V.

  Proof: To prove that U is a subspace of V we follow the result from class and show the following three things:

  U is not empty: indeed, since TU is a subspace of W then TU is not empty. By the definition of TU this implies that U is not empty.

  U is closed to summation: Let $u_1, u_2 \in U$. This implies that $Tu_1, Tu_2 \in TU$. Since we are given that TU is a subspace of W, we know that TU is closed to addition. It follows that $Tu_1 + Tu_2 \in TU$. Since T is linear, we know that $T(u_1 + u_2) = Tu_1 + Tu_2$. We find that $T(u_1 + u_2) \in TU$. By the definition of TU, this implies that there exists $v \in U$ such that $Tv = T(u_1 + u_2)$. Since T is $1-1$ we know that $v = u_1 + u_2$ and therefore, that $u_1 + u_2 \in U$. So U is closed to summation.

  U is closed to multiplication by a scalar: Let $u \in U$ and $\alpha \in \mathbb{R}$. This implies that $Tu \in TU$. Since we are given that TU is a subspace of W, we know that TU is closed to multiplication by a scalar. It follows that $\alpha Tu \in TU$. Since T is linear, we know that $\alpha Tu = T(\alpha u)$. We find that $T(\alpha u) \in TU$. By the definition of TU, this implies that there exists $v \in U$ such that $Tv = T(\alpha u)$. Since T is $1-1$ we know that $v = \alpha u$ and therefore, that $\alpha u \in U$. So U is closed to multiplication by a scalar.

5. **Remark:** It was not explicitly stated in the formulation of this question that we are considering linear transformations. Still, while writing the answers I will assume that this is the case. Note however, that the answers can be different if one is allowed to use non–linear transformations.

   i. This claim is false. Indeed, we proved in class that dim $M_2(\mathbb{R}) = 4$ while dim$\mathbb{R}^3 = 3$. In addition, we proved in class that if the dimension of the domain is bigger then the dimension of the codomain, then a linear transformation cannot be $1-1$

   ii. The claim is true. Consider for example, $T : \mathbb{R}^3 \mapsto M_2(\mathbb{R})$ which is defined by

   $$T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}.$$

   One can easily check that ker$T = \{\underline{0}\}$ and therefore T is $1-1$.

iii. The claim is true. Consider for example, $T : M_2(\mathbb{R}) \mapsto M_2(\mathbb{R})$ which is defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}.$$

This transformation is not $1-1$ since $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \underline{0}$ and we proved in class that a linear transformation is $1-1$ iff the only vector in the domain who's image is $\underline{0}$ is the zero vector. The transformation is not onto as well. For example, $\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$ is not in the image since all matrices in the image have zero in their last entry.

iv. The claim is false. Indeed, we proved in class that $\dim M_2(\mathbb{R}) = 4$ and $\dim \mathbb{R}_3[x] = 4$. Moreover, we proved in class that if the domain and the codomain have the same dimension and the linear transformation between them is $1-1$ then it is also onto.

v. The claim is true. Consider for example, $T : M_2(\mathbb{R}) \mapsto \mathbb{R}^3$ which is defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The transformation onto, as each $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ has a pre–image, say $\begin{pmatrix} a & b \\ c & 5 \end{pmatrix}$. The transformation is not $1-1$ as the zero 3–tuple has a nontrivial pre–image, say $\begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$ and we proved in class that this happens iff a transformation is not $1-1$.

vi. This claim is false. Indeed, we proved in class that $\dim M_2(\mathbb{R}) = 4$ while $\dim\mathbb{R}^3 = 3$. In addition, we proved in class that if the dimension of the domain is smaller then the dimension of the codomain, then a linear transformation cannot be onto.

vii. The claim is true. Indeed, consider for example $T : M_2(\mathbb{R}) \mapsto \mathbb{R}^3$ which is defined by

$$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}.$$

This transformation is not $1-1$ since $T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \underline{0}$ and we proved in class that a linear transformation is $1-1$ iff the only vector in the domain who's image is $\underline{0}$ is the zero vector. The transformation is not onto as well. For example, $\begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix}$ is not in the image since all matrices in the image have zero in their last entry.