MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Final exam

- The exam starts when this fact is indicated by the instructor. The exam ends at 5:40. The length of the exam is roughly two hours and 50 minutes.
- The use of calculators is NOT permitted.
- The use of written notes is NOT permitted.
- There are 9 questions with points as indicated (with 100 points in all).
- Explain yourself clearly and justify all of your claims. If you use a result which was stated in class, recitation or homework then make sure to indicate this fact explicitly.

Name: ________________________________

Recitation group: ________________

1. [10 points] Find a basis for the following space. Justify your computations

$$\{x^3 - 2x^2 + x + 1,\ 2x^3 + 3x - 2,\ x^3 - 2x^2 + x - 3,\ x^3 - 6x^2 - 3\}.$$

2. Let $A \in M_4(\mathbb{R})$ be a matrix which satisfies

$$A := \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & a & b \\ 0 & 1 & c & d \end{pmatrix}.$$

Assume that $\dim(\mathrm{null}(A)) = \mathrm{rank}(A)$.

i. [5 points] Find the value of each one of the parameters a, b, c, d.

ii. [5 points] Find a basis for the **row** space of A and a basis for the **column** space of A.

3. Let $T : M_2(\mathbb{R}) \to M_2(\mathbb{R})$ be defined by

$$TA = \begin{pmatrix} 2 & -1 \\ 4 & -2 \end{pmatrix},$$

and consider the following basis of $M_2(\mathbb{R})$:

$$B = (\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}).$$

(You do not need to prove that T is a linear transformation or that B is a basis for the space.)

i. [5 points] Find $[T]_B$.

ii. [5 points] **Use the matrix** $[\mathbf{T}]_{\mathbf{B}}$ to find $dim(ImT)$. Justify your answer.

iii. [5 points] **Use the matrix** $[\mathbf{T}]_{\mathbf{B}}$ to find a **basis** for $kerT$. Justify your answer.

4. [10 points] Consider the matrix:

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -6 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Is A diagonalizable? If so, find a diagonal matrix D which is similar to A. Justify your answers.

5. [10 points] Let V be a vector space of dimension 3 over $\mathbb{R}$ and let $B = (v_1, v_2, v_3)$ and $C = (w_1, w_2, w_3)$ be two ordered bases of V. Assume that

$$[id]_B^C = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 0 & 0 \end{pmatrix}$$

a. [5 points] Is the set $\{w_2, w_3, v_1, v_2\}$ a spanning system for V? Justify your answer.

b. [5 points] Find $[v_1]_C$, $[v_2]_C$ and $[v_3]_C$.

6. [10 points] **Prove** the following claim: If $A \in M_n(R)$ is such that $|A| = 0$, then A is similar to a matrix who's left most column is the zero column (that is, the column with all of its entries being equal to zero).

7. [10 points] Let V be a vector space and let $v_1, v_2, v_3, w, u \in V$ be vectors in V. Assume that $u \notin$span$\{v_1, v_2, v_3\}$ and $u \in$span$\{v_1, v_2, v_3, w\}$. **Prove** each one of the following claims.
   i. [5 points] $w \notin$span$\{v_1, v_2, v_3\}$.
   ii. [5 points] $w \in$span$\{v_1, v_2, v_3, u\}$

8. Let V be a vector space and $T : V \to V$ be a linear transformation which satisfies $T^2 = T$. **Prove** the following claims:
   i. [5 points] If a vector $v \in V$ belongs both to the kernel of T and to the image of T then $v = 0$.
   ii. [5points] If T is 1–1 then $T = id_V$.
   iii. [5points] If λ is an eigenvalue of T then $\lambda = 1$ or $\lambda = 0$.

9. **Prove or disprove** the following claims: Let V be a vector space **of dimension 4** over $\mathbb{R}$.
   i. [5 points] If $T : V \to V$ is a linear transformation which satisfies $\dim(\mathrm{Im}T^2) = 3$ then $\dim(\mathrm{Im}T) = 3$.
   ii. [5 points] If $T : V \to V$ is a linear transformation which satisfies $\dim(\mathrm{Im}T) = 3$ then $\dim(\mathrm{Im}T^2) = 3$.