MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 11 – Solutions.

1. Consider the following linear transformations:

$$S : M_2(\mathbb{R}) \mapsto \mathbb{R}^2 \qquad \text{given by} \qquad SA = A\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$L : \mathbb{R}_3[x] \mapsto \mathbb{R}^2 \qquad \text{given by} \qquad Lp = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}$$

$$\Phi : \mathbb{R}^3 \mapsto \mathbb{R}_3[x] \qquad \text{given by} \qquad \Phi\begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b)+(a-2b+c)x+(b-3c)x^2+(a+b+c)x^3$$

$$T_A : \mathbb{R}^2 \mapsto \mathbb{R}^2 \qquad \text{where} \qquad A = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$$

◇ $\Phi \circ L$ is not defined.

◇ $L \circ \Phi : \mathbb{R}^3 \mapsto \mathbb{R}^2$ is defined by

$$(L \circ \Phi)(\begin{pmatrix} a \\ b \\ c \end{pmatrix}) = L\big((a+b) + (a-2b+c)x + (b-3c)x^2 + (a+b+c)x^3\big)$$

$$= \begin{pmatrix} [11a + 9b - 2c] - [3a + b - c] \\ a - 2b + c \end{pmatrix}$$

$$= \begin{pmatrix} 8a + 8b - c \\ a - 2b + c \end{pmatrix}$$

◇ We have seen in class that $T_A^2 = T_{A^2}$. So, to find T_A^2 we only need to find A^2.

$$A^2 = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}.$$

◇ $S \circ T_A$ is not defined.

◇ $T_A \circ S : M_2(\mathbb{R}) \mapsto \mathbb{R}^2$ is defined by

$$\begin{aligned}
(T_A \circ S)\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= T_A\Big(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix}\Big) \\
&= T_A\Big(\begin{pmatrix} 3a - 2b \\ 3c - 2d \end{pmatrix}\Big) \\
&= \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} 3a - 2b \\ 3c - 2d \end{pmatrix} \\
&= \begin{pmatrix} 6a - 4b + 3c - 2d \\ 3a - 2b - 6c + 4d \end{pmatrix}.
\end{aligned}$$

◇ $\Phi \circ T_A$ is not defined.

$\diamond$ Note that T_A^2 has previously been computed. $T_A^2 \circ S : M_2(\mathbb{R}) \mapsto \mathbb{R}^2$ is defined by,

$$(T_A^2 \circ S)B = T_A^2(SB) = T_A^2(B\begin{pmatrix} 3 \\ -2 \end{pmatrix}) = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} B \begin{pmatrix} 3 \\ -2 \end{pmatrix} = 5B\begin{pmatrix} 3 \\ -2 \end{pmatrix}.$$

$\diamond$ Note that $L \circ \Phi$ has previously been computed. $T_A \circ L \circ \Phi : \mathbb{R}^3 \mapsto \mathbb{R}^2$

$$(T_A \circ L \circ \Phi)(\begin{pmatrix} a \\ b \\ c \end{pmatrix})$$

$$= T_A \begin{pmatrix} 8a + 8b - c \\ a - 2b + c \end{pmatrix}$$

$$= T_A \begin{pmatrix} 8 & 8 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

$$= A \begin{pmatrix} 8 & 8 & 1 \\ 1 & -2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

So the transformation is in fact a transformation of the form $T_B : \mathbb{R}^3 \mapsto \mathbb{R}^2$ with

$$B = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 8 & 8 & 1 \\ 1 & -2 & 1 \end{pmatrix} = \begin{pmatrix} 17 & 14 & 3 \\ 6 & 12 & -1 \end{pmatrix}.$$

2. i. Let $v \in$ker$T$ and $\alpha \in \mathbb{R}$. Since $v \in$ker$T$ we know that $Tv = 0$. So, since T is linear, we get

$$T(\alpha v) = \alpha T v = \alpha \cdot 0 = 0.$$

This implies that $\alpha v \in$ker$T$, and so ker$T$ is closed to multiplication by scalar.

ii. Let $w \in$ImT and $\alpha \in \mathbb{R}$. Since $v \in$ker$T$ there exists $v \in V$ so that $Tv = w$. Denote $u = \alpha v$, then, since T is linear,

$$Tu = T(\alpha v) = \alpha T v = \alpha w.$$

This implies that αw has a preimage, so it is in ImT, and therefore ImT is closed to multiplication by scalar.

iii. From the dimension formula we know that dim$V$ =dim(ker$T$)+dim(ImT). Since T is $1-1$ the only preimage of 0 is 0, and therefore ker$T = \{0\}$. This implies that dim(ker$T$)$= 0$, and by inserting to the dimension formula that dim$V$ =dim(ImT). Since we are given that dim$V$=dim$W$ we can conclude that dim$W$ =dim(ImT). Now, we know that ImT is a subspace of W. We proved in class that if a subspace of a vector space has the same dimension as the vector space then they are equal, so we can conclude that W =ImT, and therefore that T is onto.

iv. From the dimension formula we know that $\dim V = \dim(\ker T) + \dim(\mathrm{Im} T)$. Since T is onto we know that $W = \mathrm{Im} T$ and therefore that $\dim W = \dim(\mathrm{Im} T)$. Since we are given that $\dim V = \dim W$ we can conclude that $\dim V = \dim(\mathrm{Im} T)$. By inserting to the dimension formula this implies that $\dim(\ker T) = 0$, and so that $\ker T = \{0\}$. We proved in class that a linear transformation T is$1-1$ if and only if $\ker T = \{0\}$, so the result follows.

v. Let $v \in V$ and $\alpha \in \mathbb{R}$. Then, since both T and S are linear, we have

$$(S \circ T)(\alpha v) = S(T(\alpha v)) = S(\alpha T v) = \alpha S(Tv) = \alpha (S \circ T) v.$$

The result follows.

3. i. Let $S : \mathbb{R}^2 \mapsto \mathbb{R}^2$ be the composition $S_3 \circ S_2 \circ S_1$ where S_1 is rotation by $\pi/4$ radians counterclockwise, $S_2(x, y) = (x, -y)$, and S_3 is rotation by $\pi/4$ radians clockwise.
The required matrix has the form:

$$(S\begin{pmatrix} 1 \\ 0 \end{pmatrix}; S\begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

We compute each one of these vectors

$$S\begin{pmatrix} 1 \\ 0 \end{pmatrix} = S_3(S_2(S_1\begin{pmatrix} 1 \\ 0 \end{pmatrix})) = S_3(S_2(\begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix})) = S_3\begin{pmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

$$S\begin{pmatrix} 0 \\ 1 \end{pmatrix} = S_3(S_2(S_1\begin{pmatrix} 0 \\ 1 \end{pmatrix})) = S_3(S_2(\begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \end{pmatrix})) = S_3\begin{pmatrix} -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

We conclude that the required matrix is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$.

ii. The required matrix was found in Q1 (Last computation there).

4. **Remark** In many parts of this question it was not written explicitly that the transformation considered should be linear. It should have been written, and we will consider only linear transformations in the solutions.

i. There exists $T : \mathbb{R}_2[x] \mapsto M_2(\mathbb{R})$ such that $1 - x + 2x^2$ is in its kernel and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is in its image.
The claim is true. Proof: First, we construct a basis for $\mathbb{R}_2[x]$ that contains the vector $1 - x + 2x^2$. Here is an example of such a basis: $\{1, x, 1 - x + 2x^2\}$. Indeed, it is easy to verify that these vectors are linearly independent (do so!) and we proved in class that a set of 3 linearly independent vectors in a space of dimension 3 is a basis (and we have seen in class that $\mathbb{R}_2[x]$ has dimension 3).
We claim that there exists a linear transformation $T : \mathbb{R}_2[x] \mapsto M_2(\mathbb{R})$ such that the images of these basis elements are as follows:

$$T1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$Tx = 0$$
$$T(1 - x + 2x^2) = 0.$$

Indeed, such a transformation exists due to the theorem we proved in class which assures the existence (and uniqueness) of a linear transformation with pre-chosen images for the elements of a fixed basis. Clearly, this transformation T satisfies both required properties.

ii. There was a misprint in this claim, here is a corrected version: There exists a linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^5$ such that
$$kerT = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 1 \right\}$$
The claim is false. A kernel of a linear transforation is always a subspace of the domain, but the set
$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 1 \right\}$$
is not a subspace of $\mathbb{R}^3$ (Why?).

iii. There exists $T : \mathbb{R}^3 \mapsto \mathbb{R}^5$ such that
$$kerT = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x + y + z = 0 \right\}$$
The claim is true. To prove this we first find a basis for the subspace considered (How do I know that this is a subspace?). We solve the linear equation:
$$\left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x+y+z = 0 \right\} = \begin{pmatrix} -t - s \\ s \\ t \end{pmatrix} = t \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = span\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \}$$
The set
$$\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \}$$
is therefore a spanning set for the subspace. The vectors in it are clearly linearly independent (why?), so this is a basis for the given subspace. We complete the set
$$\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \}$$
into a basis of $\mathbb{R}^3$. We did not learn an algorithm for how to do that, but in this case (as in any case you may encounter in an exam) it is

easy to do. For example, it is easy to verify that the set

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\}$$

is a basis for $\mathbb{R}^3$ (do so!).
We claim that there is a linear transformation $T : \mathbb{R}^3 \mapsto \mathbb{R}^5$ which satisfies:

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Indeed, such a transformation exists due to the theorem we proved in class which assures the existence (and uniqueness) of a linear transformation with pre-chosen images for the elements of a fixed basis.
We claim that this linear transformation satisfies the required condition. To show this, we first note that clearly,

$$\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\} \subset kerT$$

It follows from a claim we proved in class that

$$span\{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}\} \subset KerT.$$

We need to show that in fact these two spaces are equal. We proved in class that if $W \subset V$ are two subspaces then $dimW \leq dimV$ and the spaces are equal iff their dimensions are equal. The dimension of the space on the LHS is 2 (why?). It follows that the dimension of the space on the RHS is at least 2 and that our claim will be proved if we show that in fact this dimension is equal to 2. Since we know that

$dimkerT \geq 2$, it remains to show that $dimkerT \leq 2$. We do this via the dimension formula (null-rank formula).
Due to our construction of T there is a non-zero vector in the image of T (which?), so $dimImT \geq 1$. Since the domain of T is $\mathbb{R}^3$ the dimension of the domain is 3. The dimension formula now gives $3 = dimkerT + dimImT \geq dimkerT + 1$. We conclude that indeed, $dimkerT \leq 2$. Our proof is complete.

iii. For $U = \{p \in \mathbb{R}_3[x] : p(1) = p'(1)\}$ there exists an isomorphism $T : U \mapsto M_2(\mathbb{R})$.
The claim is false. We proved in class that an isomorphism can exist between two spaces only if they have the same dimension. Here the dimension of $M_2(\mathbb{R})$ is 4, as was proved in class. While the dimension of U is strictly less then 4. To see that the dimension of U is less then 4 note that $U \subseteq \mathbb{R}_3[x]$. Also note that this is a strict inclusion, that is, $U \neq \mathbb{R}_3[x]$. Indeed the function x^2, for example, belongs to $\mathbb{R}_3[x]$ but not to U (check!). We proved in class that if $W \subset V$ are two subspaces then $dimW \leq dimV$ and the spaces are equal iff their dimensions are equal. Since the dimension of $\mathbb{R}_3[x]$ is 4 it follows that $dimU < 4$. The proof is complete.

iv. For $U = \{p \in \mathbb{R}_3[x] : p(1) = p'(1)\}$ there exists an isomorphism $T : U \mapsto \mathbb{R}_2[x]$.
The claim is true. We first find a basis for U. We start this by finding a parametrization of U.

$$U = \{p \in \mathbb{R}_3[x] : p(1) = p'(1)\} = \{a+bx+cx^2+dx^3 : a = c+2d\} = \{(c+2d)+bx+cx^2+dx^3\}$$

So every vector in U can be written as

$$(c + 2d) + bx + cx^2 + dx^3 = c(1 + x^2) + bx + d(2 + x^3).$$

The polynomials on the RHS all belong to U (check!) and every other vector in U is a linear combination of them so we conclude that $\{1 + x^2, x, 2 + x^3\}$ is a spanning system for U. It is easy to check that the vectors in this set are linearly independent (do so!) and therefore $\{1 + x^2, x, 2 + x^3\}$ is a basis for U.
We claim that there exists a linear transformation $T : U \mapsto \mathbb{R}_2[x]$ which satisfies

$$T(1 + x^2) = 1$$

$$Tx = x$$

and

$$T(2 + x^3) = x^2$$

Indeed, such a transformation exists due to the theorem we proved in class which assures the existence (and uniqueness) of a linear transformation with pre-chosen images for the elements of a fixed basis.

We want to show that T is an isomorphism. We first note that $ImT \subseteq \mathbb{R}_2[x]$ (the image is always a subset of the codomain) and that by our construction:

$$1, x, x^2 \in ImT.$$

It follows from a claim we proved in class that

$$\mathbb{R}_2[x] = span\{1, x, x^2\} \subseteq ImT.$$

We conclude that $ImT = \mathbb{R}_2[x]$ and therefore T is onto.
Note, that the basis we found for U contains 3 elements, so $dimU = 3$. Also, we have seen in class that $dim\mathbb{R}_2[x] = 3$. So, the domain and the codomain of T have the same dimension. We proved in class that if a linear transformation between two spaces with the same dimension is onto, then this transformation is an isomorphism. We have shown that the transformation T which we constructed satisfies all this, so it is an isomorphism. Our proof is complete.

v. There exists $T : \mathbb{R}^4 \mapsto \mathbb{R}^6$ such that dimker$T$=2dimImT.
The claim is false. Assume that it were true and such T existed. From the dimension formula it will follow that

$$4 = dimImT + dimkerT = 3dimImT.$$

Since a dimension of a vector space is always a positive integer, we arrive at a contradiction.

vi. There exists $T : \mathbb{R}^4 \mapsto \mathbb{R}^6$ such that dimker$T$=2dimImT+1.
The claim is true. Let $\{e_1, e_2, e_3, e_4\}$ be a basis for $\mathbb{R}^4$ (we proved in class that the dimension of this space is 4). We claim that there exists a linear transformation T such that $Te_1 = Te_2 = Te_3 = 0$ while

$$Te_4 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Indeed, such a transformation exists due to the theorem we proved in class which assures the existence (and uniqueness) of a linear transformation with pre-chosen images for the elements of a fixed basis.
We claim that $dimImT = 1$ and $dimkerT = 3$, and therefore the requirements of the question are fulfilled. Indeed, the image contains at least one vector different from zero (which?) so $dimImT \geq 1$, and the kernel contains at least 3 linearly independent vectors (which?) so $dimkerT \geq 3$ (because we proved in class that the amount of elements in a linearly independent sequence is at most the dimension of the space). On the other hand, the dimension formula implies that $4 = dimkerT + dimImT$. Combining these conditions together we find that indeed, $dimImT = 1$ and $dimkerT = 3$.

5. Let V be a vector space over $\mathbb{R}$ such that dim$V$=3 and let $T : V \mapsto V$ be a linear transformation which satisfies $T^3 = 0$ and $T^2 \neq 0$. Prove:
   i. There exists $v \in V$ which is not the zero vector such that $Tv = 0$.
   Proof: We proved in class/recitation that the composition of two isomorphisms is an isomorphism. Therefore, since T^3 is clearly not an isomorphism (why?) Then T is also not an isomorphism. Since the dimensions of the domain and the codomain of T are equal (they are the same space) it follows from a result we proved in class that T is not $1-1$. We showed in class that a transformation is $1-1$ iff the only pre-image of 0 is 0. Since T is not $1-1$ it follows that there exists $v \neq 0 \in V$ such that $Tv = 0$.
   ii. ImT $\subseteq$kerT^2 and ImT^2 $\subseteq$ker$T$.
   Proof: Let $w \in ImT$. This means that there exists $v \in V$ such that $Tv = w$. Then, since $T^3 = 0$ we have:
   $$T^2 w = T^2(Tv) = T^3 v = 0.$$
   This completes the proof of the first claim.
   For the second claim, let $w \in ImT^2$. This means that there exists $v \in V$ such that $T^2 v = w$. Then, since $T^3 = 0$ we have:
   $$Tw = T(T^2 v) = T^3 v = 0.$$
   This completes the proof of the second claim.
   iii. ker$T$ $\subset$kerT^2 and ker$T$ $\neq$kerT^2
   Proof: Let $v \in kerT$. This means that $Tv = 0$. Then,
   $$T^2 v = T(Tv) = T0 = 0.$$
   This completes the proof of the first claim.
   For the second claim, assume that $kerT = kerT^2$. Combining this assumption with what we proved in (ii) we will get ImT $\subseteq$ker$T$. This means that for every $v \in V$
   $$T^2 v = T(Tv) = 0.$$
   Since we were given that $T^2 \neq 0$ we arrive at a contradiction.
   iv. dimker$T = 1$.
   Proof: From part (i) we know that $dimkerT \geq 1$. From the condition $T^2 \neq 0$ we know that $dimkerT^2 \leq 2$ (why?). We proved in class that if $W \subset V$ are two subspaces then $dimW \leq dimV$ and the spaces are equal iff their dimensions are equal. It follows from part (iii) that $dimkerT < dimkerT^2$ and the inequality is strict (it cannot be equality). Combining all these facts we get
   $$1 \leq dimkerT < dimkerT^2 \leq 2.$$
   Since the dimension of a space is always an integer, it follows that $dimkerT = 1$ and $dimkerT^2 = 2$. The proof is complete.

6. Let V, W be two vector spaces over $\mathbb{R}$. Assume that $T : V \mapsto W$ and $S : W \mapsto V$ are linear transformations which satisfy:

$$S \circ T = id_V$$

Prove or disprove:

i. T is 1–1.
The claim is true. Proof: Let $u, v \in V$ be such that

$$Tv = Tu$$

applying S on both sides we get

$$S(Tv) = S(Tu).$$

We rewrite this,

$$(S \circ T)v = (S \circ T)u.$$

Since $(S \circ T)v = id_V$ we conclude that $u = v$. This proves that T is $1-1$

ii. S is 1–1.
The claim is false, we give a counterexample. Consider the linear transformations, $T : \mathbb{R}^2 \mapsto \mathbb{R}^3$ given by

$$T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}$$

and the linear transformation $S : \mathbb{R}^3 \mapsto \mathbb{R}^2$ given by

$$S\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

It is easy to check that $S \circ T = id_{\mathbb{R}^2}$ (do so!) and that S is not $1-1$ (do this as well!).

iii. dim$W$ =dim$V$
The claim is false, the counterexample we constructed in part (ii) works for this case as well.

iv. If dim$W$ =dim$V$ then S is an isomorphism.
The claim is true. Proof: We first show that S is onto. To show this we need to show that every $v \in V$ has a pre-image under S. So, let $v \in V$ and denote $w = Tv$, then

$$Sw = S(Tv) = (S \circ T)v = id_V v = v.$$

This shows that S is indeed onto.
Assume that dim$W$ =dim$V$. We proved in class that in this case if $S : W \mapsto V$ is onto then it is also $1-1$ and therefore an isomorphism. We are done.

v. If T is not onto then $\dim W \neq \dim V$.
The claim is true. Proof: We proved in (i) that T is $1-1$. By a result from class it follows that if $\dim W = \dim V$ then T is onto (and therefore an isomorphism). We get that indeed, if T is not onto then $\dim W \neq \dim V$.

7. a. Consider the following linear transformations $T, S : \mathbb{R}_3[x] \mapsto \mathbb{R}_3[x]$ given by

$$Tp(x) = p'(x) \qquad \text{and} \qquad Sp(x) = p(x+1)$$

and consider the following bases of $\mathbb{R}^3[x]$:

$$E = \{1, x, x^2, x^3\}$$

and

$$B = \{1, 1+x, (1+x)^2, (1+x)^3\}$$

i. Computing the coordinates of a vector with respect to the natural (standard) basis is always simple. Therefore the first two matrices are easy to find.

$$[T]_E^E = ([T1]_E; [Tx]_E; [Tx^2]_E; [Tx^3]_E) = ([0]_E; [1]_E; [2x]_E; [3x^2]_E) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$[T]_E^B = ([T1]_E; [T(1+x)]_E; [T(1+x)^2]_E; [T(1+x)^3]_E) = ([0]_E; [1]_E; [2+2x]_E; [3+6x+3x^2]_E) =$$

$$= \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We turn to the third matrix:

$$[T]_B^E = ([T1]_B; [Tx]_B; [Tx^2]_B; [Tx^3]_B) = ([0]_B; [1]_B; [2x]_B; [3x^2]_B)$$

We compute each one of the columns. The coordinates of the zero vector are always the zero $n - tuple$

$$[0]_B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The coordinates $[1]_B$ are clear, as the constant function 1 is one of the elements in the basis B.

$$[1]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

To find $[2x]_B$ we need to find the coefficients in the linear combinations.

$$2x = a + b(1+x) + c(1+x)^2 + d(1+x)^3.$$

Rewriting the RHS we get

$$2x = (a+b+c+d) + (b+2c+3d)x + (c+3d)x^2 + dx^3$$

Since two polynomials are equal iff all of their coefficients are equal we obtain a linear system. To solve it we write the corresponding augmented matrix and bring it to echelon form. We skip this process here (but would not skip it in an exam). The solution is: $a = -2, b = 2, c = 0, d = 0$. So,

$$[2x]_B = \begin{pmatrix} -2 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

To find $[3x^2]_B$ we need to find the coefficients in the linear combinations.

$$3x^2 = a + b(1+x) + c(1+x)^2 + d(1+x)^3.$$

Rewriting the RHS we get

$$3x^2 = (a+b+c+d) + (b+2c+3d)x + (c+3d)x^2 + dx^3$$

Since two polynomials are equal iff all of their coefficients are equal we obtain a linear system. To solve it we write the corresponding augmented matrix and bring it to echelon form. We skip this process here (but would not skip it in an exam). The solution is: $a = 3, b = -6, c = 3, d = 0$. So,

$$[3x^2]_B = \begin{pmatrix} 3 \\ -6 \\ 3 \\ 0 \end{pmatrix}$$

Finally we get,

$$[T]_B^E = \begin{pmatrix} 0 & 1 & -2 & 3 \\ 0 & 0 & 2 & -6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

To find the fourth matrix:

$$[T]_B^B = ([T1]_B; [T(1+x)]_B; [T(1+x)^2]_B; [T(1+x)^3]_B) = ([0]_B; [1]_B; [2(1+x)]_B; [3(1+x)^2]_B)$$

We compute each one of the columns. The coordinates of the zero vector are always the zero $n - tuple$

$$[0]_B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The coordinates $[1]_B$ are clear, as the constant function 1 is one of the elements in the basis B.

$$[1]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The coordinates $[2(1+x)]_B$ are clear, as the function $1+x$ is one of the elements in the basis B.

$$[2(1+x)]_B = \begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$$

The coordinates $[3(1+x)^2]_B$ are clear, as the function $(1+x)^2$ is one of the elements in the basis B.

$$[3(1+x)^2]_B = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 0 \end{pmatrix}$$

Finally we get,

$$[T]_B^B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

ii. Computing the coordinates of a vector with respect to the natural (standard) basis is always simple. Therefore the first matrix is easy to find.

$$[S]_E^E = ([S1]_E; [Sx]_E; [Sx^2]_E; [Sx^3]_E) = ([1]_E; [x+1]_E; [(x+1)^2]_E; [(x+1)^3]_E) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We turn to the second matrix,

$$[S]_B^B = ([S1]_B; [S(x+1)]_B; [S(x+1)^2]_B; [S(x+1)^3]_B) = ([1]_B; [x+2]_B; [(x+2)^2]_B; [(x+2)^3]_B)$$

We compute each one of the columns. We could follow the same procedure as in part (i) to compute the coordinates, but here a somewhat faster approach is possible. First, as usual, The coordinates $[1]_B$ are clear, as the constant function 1 is an element of the basis,

$$[1]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Next, we note that $x + 2 = 1 + (x + 1)$ so,

$$[x+2]_B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

Similarly, we note that $(x+2)^2 = (1+(x+1))^2 = 1+2(x+1)+(x+1)^2$ so,

$$[(x+2)^2]_B = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}.$$

Again, we note that $(x+2)^3 = (1+(x+1))^3 = 1+3(x+1)+3(x+1)^2+(x+1)^3$ so,

$$[(x+2)^3]_B = \begin{pmatrix} 1 \\ 3 \\ 3 \\ 1 \end{pmatrix}.$$

We conclude that

$$[S]_B^B = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

.

Remark. We found that $[S]_B^B = [S]_E^E$ and $[T]_B^B = [T]_E^E$. This is a lucky coincidence due to the specific transformation and bases we were given. In general, similar matrices are not necessarily equal.

iii. By the theorem we proved in class,

$$[T \circ S]_E^E = [T]_E^E [S]_E^E =$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$[T \circ S]_B^B = [T]_B^B [S]_B^B =$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

b. i.

$$L : \mathbb{R}_3[x] \mapsto \mathbb{R}^2 \qquad \text{given by} \qquad Lp = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}$$

.

We use the following ordered bases. For $\mathbb{R}_3[x]$ we use the ordered basis $B = (1, x, x^2, x^3)$ and for $\mathbb{R}^2$ the ordered basis $E = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$, we proved in class that these are indeed bases for the corresponding spaces. We write the matrix which represents L with respect to these bases:

$$[L]_E^B = ([L1]_E; [Lx]_E; [Lx^2]_E; [Lx^3]_E) = \begin{pmatrix} 0 & 1 & 3 & 7 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We proved in class that the transformation L and the transformation of multiplication by the matrix $[L]_E^B$ have the same linear structure. To find a basis for the kernel of L we first find a basis for the kernel $[L]_E^B$. The kernel of $[L]_E^B$ is equal to it's null space, that is, this is the set of solutions of the corresponding homogenous system. We bring the matrix to echelon form:

$$\begin{pmatrix} 0 & 1 & 3 & 7 \\ 0 & 1 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & 7 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 7 \end{pmatrix}$$

We end up with the equations $y = 0$ and $3z + 7w = 0$. So, x and w are free variables, represented by setting $x = t$, $w = s$ with $t, s \in \mathbb{R}$, and the null space is given by

$$null([L]_E^B) = \begin{pmatrix} t \\ 0 \\ -\frac{7}{3}s \\ s \end{pmatrix} = t \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + s \begin{pmatrix} 0 \\ 0 \\ -\frac{7}{3} \\ 1 \end{pmatrix}$$

So the set $\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -\frac{7}{3} \\ 1 \end{pmatrix}\}$ is a spanning set for $null([L]_E^B)$.
We proved in class that when we follows this algorithm to find a spanning set for the null space of a matrix, we end up with linearly independent vectors. So this set is a basis for $null([L]_E^B)$.
To find a basis for the kernel of L we apply the inverse of the coordinate transformation $[-]_B : \mathbb{R}_3[x] \mapsto \mathbb{R}^4$. That is, we find the two polynomials $p_1, p_2 \in \mathbb{R}_3[x]$ which satisfy:

$$[p_1]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and

$$[p_2]_B = \begin{pmatrix} 0 \\ 0 \\ -\frac{7}{3} \\ 1 \end{pmatrix}$$

These equalities mean that

$$p_1 = 1 \cdot 1 + 0 \cdot x + 0 \cdot x^2 + 0 \cdot x^3 = 1$$

and

$$p_2 = 0 \cdot 1 + 0 \cdot x + (-\frac{7}{3}) \cdot x^2 + 1 \cdot x^3 = -\frac{7}{3}x^2 + \cdot x^3$$

So, $\{1, -\frac{7}{3}x^2 + \cdot x^3\}$ is a basis for $kerL$.
Next, we find a basis for the image of L. We first find a basis for the image of $[L]_E^B$. The image of $[L]_E^B$ is equal to it's column space, that is, this is the set spanned by the columns of the matrix. It follows that, $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \end{pmatrix}\}$ is a spanning set for this space. To find a basis for a space spanned by a collection of vectors, we usually follow an algorithm from class. Here we can be lazy, and simply say that since the left most vector in this set is clearly a linear combination of the other vectors (check!) then by the lemma we proved in class removing him will not change the span. So, $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}\}$ is a spanning set for this space. This spanning set contains two linearly independent vectors (neither of them is a scalar multiplying the other) and is therefore a basis for the column space of $[L]_E^B$. To find a basis for the image of L, we need to find two vectors $v_1, v_2 \in \mathbb{R}^2$ such that $[v_1]_E = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

and $[v_2]_E = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. The natural basis E is very friendly, we get $v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$. We conclude that $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}\}$ is a basis for the image of L.

Remark. Since We proved in class that the transformation L and the transformation of multiplication by the matrix $[L]_E^B$ have the same linear structure, some questions about L can be solved by working with $[L]_E^B$ and forgetting completely about L. For example, questions like: "Find the dimension of the kernel" "Find the dimension of the image". If we were asked to solve these questions then we could just solve them for $[L]_E^B$ (by counting the amount of free variables or pivots in a corresponding echelon form) and the answer will be exactly the same for the transformation L. In the question we were given here, however, we were asked to find not the dimension but rather a basis for the image and the kernel. In this case, we could not 'forget' about the transformation L and we needed to translate back our solutions from n–tuples into elements in the original domain and codomain.

ii.

$$\Phi : \mathbb{R}^3 \mapsto \mathbb{R}_3[x] \qquad \text{given by} \qquad \Phi \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b)+(a-2b+c)x+(b-3c)x^2+(a+b+c)x^3$$

For the space $\mathbb{R}^3$ we use the ordered basis:

$$E = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}).$$

For the space $\mathbb{R}_3[x]$ we use the ordered basis $B = (1, x, x^2, x^3)$. (We proved in class that both are bases for the corresponding spaces).

$$[\Phi]_B^E = ([\Phi \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}]_B; [\Phi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}]_B; [\Phi \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}]_B) =$$

$$= ([1 + x + x^3]_B; [1 - 2x + x^2 + x^3]_B; [x - 3x^2 + x^3]_B) = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$$

We proved in class that the transformation Φ and the transformation of multiplication by the matrix $[\Phi]_E^B$ have the same linear structure. So, to find a basis for the kernel of Φ we can first find a basis for the kernel of $[\Phi]^E_B$, that is, find a basis for the null space of $[\Phi]^E_B$. We bring the matrix to echelon form,

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & -3 & 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

The corresponding linear system has only the trivial solution (no free variables) and therefore the null space of $[\Phi]^B_E$ is $\{0\}$ and a basis for it is the empty set. It follows that a basis for the kernel of Φ is also the empty set.
To find a basis for the image of Φ we can first find a basis for the image of $[\Phi]^B_E$, that is, a basis for the column space of $[\Phi]^B_E$:

$$span\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ 1 \end{pmatrix}\}$$

We note that after bringing this matrix $[\Phi]^B_E$ to echelon form we obtained three leading variables so the rank of this matrix is 3. That is, the dimension of the column space is 3. We proved in class that a spanning set of 3 vectors in a space of dimension 3 is a basis. So

$$\{\begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -3 \\ 1 \end{pmatrix}\}$$

is a basis for the column space of $[\Phi]^B_E$. To obtain a basis for the image of Φ we need to find three vectors $p_1, p_2, p_3 \in \mathbb{R}_3[x]$ such that

$$[p_1]_B = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$[p_2]_B = \begin{pmatrix} 1 \\ -2 \\ 1 \\ 1 \end{pmatrix}$$

and

$$[p_3]_B = \begin{pmatrix} 0 \\ 1 \\ -3 \\ 1 \end{pmatrix}$$

We find the corresponding vectors:

$$p_1 = 1 + x + x^3$$

$$p_2 = 1 - 2x + x^2 + x^3$$

and

$$p_3 = x - 3x^2 + x^3$$

We conclude that $\{1 + x + x^3, 1 - 2x + x^2 + x^3, x - 3x^2 + x^3\}$ is a basis for the image of Φ.

8. Consider the vector space,

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\}.$$

Let $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and consider the linear transformation $L : W \mapsto W$ defined by $LA = AH - HA$.

i. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in W$$

. Then

$$\begin{aligned} LA = L\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} \\ &= \begin{pmatrix} b & a \\ d & c \end{pmatrix} - \begin{pmatrix} c & d \\ a & b \end{pmatrix} \\ &= \begin{pmatrix} b - c & a - d \\ d - a & c - b \end{pmatrix} \end{aligned}$$

We see that the sum of the first and fourth entries of LA is zero: $(b - c) + (c - b) = 0$. This is the condition that defines the subspace W, so $LA \in W$. This proves that indeed, L acts from W to W.

ii. To find an ordered basis for W we first present a parametrization of W:

$$W = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a + d = 0 \right\} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\}.$$

Once we have a parametrization it is easy to find a spanning system. All of the vectors in W have the following form:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

All of the matrices on the RHS belong to W (check!) and we showed that every vector in W is a linear combination of these three matrices, so the set

$$\{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$$

is a spanning set for the space W. One can easily check that this set is linearly independent (do so!) and therefore is a basis for the space. We give this basis an order and denote it B.

$$B = (\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}).$$

iii.

$$[L]_B^B = ([L\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}]_B; [L\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}]_B; [L\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}]_B)$$

$$= ([\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}]_B; [\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}]_B; [\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}]_B) = \begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix}.$$

iv. We proved in class that the transformation L and the transformation of multiplication by the matrix $[L]_B^B$ have the same linear structure. So, to find a basis for the kernel of L we can first find a basis for the kernel of $[L]_B^B$, that is, find a basis for the null space of $[L]_B^B$. We bring the matrix to echelon form,

$$\begin{pmatrix} 0 & 1 & -1 \\ 2 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & -1 \\ -2 & 0 & 0 \end{pmatrix} \Rightarrow$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -2 & 0 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The null space of the matrix is the set of solutions of the corresponding homogeneous system:

$$null([L]_B^B) = \begin{pmatrix} 0 \\ t \\ t \end{pmatrix} = t\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

So, $\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\}$ is a spanning set for the space. Since it contains a single vector that is different from zero it is also a linearly independent set and therefore a basis for the null space of $[L]_B^B$. To find a basis for the kernel of L we need to find a vector $A \in W$ such that

$$[A]_B = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

We find this vector

$$A = 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 1 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We conclude that $\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$ is a basis for the kernel of L.

To find a basis for the image of L we can first find a basis for the image of $[L]_B^B$, that is, find a basis for the column space of $[L]_B^B$:

$$span\{\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\}$$

The left most vector in this set is clearly a linear combination of the other vectors (check!), so by a lemma proved in class, removing it will not change the span. The column space of $[L]_B^B$ is therefore equal to

$$span\{\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$$

The two vectors that are spanning this space are linearly independent (neither of them is a scalar multiplying the other) and therefore the set

$$\{\begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\}$$

is a basis for the column space of $[L]_B^B$. To find a basis for the image of L we need to find two vectors $A_1, A_2 \in W$ such that

$$[A_1]_B = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$$

$$[A_2]_B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We find these vectors:

$$A_1 = 0 \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} - 2 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$A_2 = 1 \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

We conclude that

$$\{\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\}$$

is a basis for the image of L.

9. Consider the linear transformation $S : \mathbb{R}_2[x] \mapsto \mathbb{R}^3$ which is defined by

$$Sp = \begin{pmatrix} p(0) \\ p(1) \\ p(2) \end{pmatrix}$$

i. We use the following ordered bases. For $\mathbb{R}_2[x]$ the basis $B = (1, x, x^2)$ and for $\mathbb{R}^3$ the basis

$$E = (\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}).$$

We write the matrix representing S with respect to these bases:

$$[S]_E^B = ([S1]_E; [Sx]_E; [Sx^2]_E) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

We proved in class that the transformation S and the transformation of multiplication by the matrix $[S]_E^B$ have the same linear structure. So the transformation S is invertible (which is the same as being an isomorphism) iff the matrix $[S]_E^B$ is invertible. There are many ways to check if $[S]_E^B$ is invertible. Keeping part (ii) of this question in mind, we will check if $[S]_E^B$ is invertible by finding its REF, if it is indeed invertible this will allow us to find also the inverse of $[S]_E^B$.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1\frac{1}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right)$$

We see that the REF of $[S]_E^B$ is I, we proved in class that this means that $[S]_E^B$ is invertible. As explained above, this means that S is invertible (which is the same as being an isomorphism).

ii. We proved in class that

$$[S^{-1}]_B^E = ([S]_E^B)^{-1}.$$

We found $([S]_E^B)^{-1}$ in part (i). Using this we get:

$$[S^{-1}]_B^E = \begin{pmatrix} 1 & 0 & 0 \\ -1\frac{1}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}.$$

We express the columns of $[S^{-1}]_B^E$ explicitly:

$$[S^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}]_B = \begin{pmatrix} 1 \\ -1\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

$$[S^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}]_B = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}.$$

and

$$[S^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}]_B = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

We use this to find the value of S^{-1} on the elements of the basis E. We get,

$$S^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 - 1\frac{1}{2}x + \frac{1}{2}x^2$$

$$S^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = 1 + 2x - x^2.$$

and

$$S^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -\frac{1}{2}x + \frac{1}{2}x^2.$$

Finally, we can compute S^{-1}. Indeed, since this is a linear transformation we get,

$$S^{-1}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = aS^{-1}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + bS^{-1}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + cS^{-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= a(1 - 1\frac{1}{2}x + \frac{1}{2}x^2) + b(1 + 2x - x^2) + c(-\frac{1}{2}x + \frac{1}{2}x^2) =$$

$$(a+b) + (-1\frac{1}{2}a + 2b - \frac{1}{2}c)x + (\frac{1}{2}a - b + \frac{1}{2}c)x^2$$

Remark. The value of S^{-1} could be computed directly, without first finding the value on E.

10. This question regards representation of a linear transformation using a matrix and the matrix of transition from basis to basis.

i. Consider the following ordered bases of $\mathbb{R}^3$:

$$B = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$C = \left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right)$$

$$E = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

Find the following matrices of transition from basis to basis:

$$[id]_E^B = \left(\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right]_E ; \left[\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right]_E ; \left[\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right]_E \right) = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$[id]_E^C = \left(\left[\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right]_E ; \left[\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right]_E ; \left[\begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right]_E \right) = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

To find all other required matrices, we use the identities:

$$[id]_B^E = ([id]_E^B)^{-1}$$
$$[id]_C^E = ([id]_E^C)^{-1}$$
$$[id]_C^B = [id]_C^E [id]_E^B$$
$$[id]_B^C = ([id]_C^B)^{-1}$$

We skip the computations of inverting a matrix and multiplying two matrices (though we would not skip it in an exam).

ii. Consider the transformations S and T and the bases B and E from Q3(a) in HW10. Find the following matrices of transition from basis to basis:

$$[id]_E^B = ([1]_E; [1+x]_E; [(1+x)^2]_E; [(1+x)^3) = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$[id]_B^E = ([id]_E^B)^{-1}.$$

We skip the computation of an inverse here, as well as the computations of matrix multiplications, which are the only thing required to check the identities,

$$[T]_B = [id]_B^E [T]_E [id]_E^B$$
$$[S]_E = [id]_E^B [S]_B [id]_B^E$$

Clearly, we would not skip these computations in an exam.

iii. Consider the vector space W and the linear transformation L in Q3(c) in HW10. Here are two bases of W:

$$B = (\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})$$

$$C = (\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}).$$

$$[id]_C^B = ([\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}]_B; [\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}]_B; [\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}]_B) = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$[id]_B^C = ([id]_C^B)^{-1}$$

We skip the computation of an inverse here, as well as the computations of matrix multiplications. Clearly, we would not skip these computations in an exam.

11. Let V be a vector space such that $\dim V = 3$. Assume that $B = (v_1, v_2, v_3)$ and $C = (w_1, w_2, w_3)$ are bases of V such that

$$[id]_C^B = \begin{pmatrix} 2 & 0 & -2 \\ 1 & 0 & 0 \\ 3 & 5 & 2 \end{pmatrix}$$

Recall that,

$$[id]_C^B = ([v_1]_C; [v_2]_C; [v_3]_C).$$

Therefore,

$$[v_1]_C = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix},$$

$$[v_2]_C = \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix}$$

and

$$[v_3]_C = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}.$$

These equalities mean that,

$$v_1 = 2w_1 + w_2 + 3w_3$$

$$v_2 = 5w_3$$

and

$$v_3 = -2w_1 + 2w_3.$$

a. Are $\{w_1, v_2, v_3\}$ linearly independent? Justify your answer.
We found that,

$$\{w_1, v_2, v_3\} = \{w_1, 5w_3, -2w_1 + 2w_3\}.$$

Since,

$$-2w_1 + 2w_3 = -2 \cdot w_1 + \frac{2}{5} \cdot 5w_3,$$

we see that one of the vectors in this set is a linear combination of the other two vectors. So the set is linearly dependent.

b. Is it true that $w_2 = v_1 + v_3 - v_2$?
We found that,

$$v_1 + v_3 - v_2 = (2w_1 + w_2 + 3w_3) + (-2w_1 + 2w_3) - (5w_3) = w_2.$$

So, yes, the claim is true.

c. Find $[w_1]_B$.
We know that,

$$[id]_B^C = ([id]_C^B)^{-1}.$$

We compute this matrix.

$$\left(\begin{array}{ccc|ccc} 2 & 0 & -2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & -2 & 1 & 0 & 0 \\ 3 & 5 & 2 & 0 & 0 & 1 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 & -2 & 0 \\ 0 & 5 & 2 & 0 & -3 & 1 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 2 & 0 & -3 & 1 \\ 0 & 0 & -2 & 1 & -2 & 0 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 & -5 & 1 \\ 0 & 0 & -2 & 1 & -2 & 0 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1/5 & -1 & 1/5 \\ 0 & 0 & 1 & -1/2 & 1 & 0 \end{array}\right).$$

We found that,

$$[id]_B^C = \begin{pmatrix} 0 & 1 & 0 \\ 1/5 & -1 & 1/5 \\ -1/2 & 1 & 0 \end{pmatrix}.$$

The first column of the matrix $[id]_B^C$ is equal to $[w_1]_B$. We conclude that,

$$[w_1]_B = \begin{pmatrix} 0 \\ 1/5 \\ -1/2 \end{pmatrix}.$$