MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Midterm exam 1

- ◇ The exam starts when this fact is indicated by the instructor. The exam ends at 2:45. The length of the exam is roughly one hour and 15 minutes.
- ◇ The use of calculators is NOT permitted.
- ◇ The use of written notes is NOT permitted.
- ◇ There are 7 questions with points as indicated (with 100 points in all).
- ◇ Explain yourself clearly and justify all of your claims. If you use a result which was stated in class or recitation then make sure to indicate this fact explicitly. If you use a result that was stated in a homework assignment then you need to add a proof of this result.

Name: ________________________________

Recitation group: ________________

IMPORTANT REMARKS:

1. The only goal of this sample exam is to give you a sense of the structure of the exam. **Please do not assume any similarity between the questions here, and the questions in the exam. Some questions might be similar, others might be completely different. Moreover, please do not assume any similarity between the subjects represented in this example, and the subjects appearing in the exam itself. I strongly recommend to study all parts of the material and to look at all parts of the HWs.**
2. At some point in the exam you are asked to solve a question without the use of 'basis' or 'coordinates'. You can find many examples of such solutions in the solution to HW6. Solutions with Coordinates can be found in the solution set to HW8.
3. Please, feel free to write me with any question that you might have, but do so with enough time in advance, so that I will have enough time to respond.
4. This file contains solutions, if you want to first solve the questions yourself then use the file which contains only the questions.

1. [20 points] For which values of a does the following linear system have:
   (i) Exactly one solution.
   (ii) No solutions.
   (iii) An infinite amount of solutions.
   (Do not forget to explain your answer).

$$\begin{cases} x + y + (a+1)z = 1 \\ x + (a+1)y + z = 4 \\ (a+1)x + y + z = 2a+1 \end{cases}$$

Solution. We start by writing the coefficients of this linear system in an augmented matrix and row reducing it to echelon form.

$$\left(\begin{array}{ccc|c} 1 & 1 & a+1 & 1 \\ 1 & a+1 & 1 & 4 \\ a+1 & 1 & 1 & 2a+1 \end{array}\right) \xrightarrow{R_2-R_1, R_3-(a+1)R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & a+1 & 1 \\ 0 & a & -a & 3 \\ 0 & -a & 1-(a+1)^2 & a \end{array}\right) \xrightarrow{R_3+R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & a+1 & 1 \\ 0 & a & -a & 3 \\ 0 & 0 & 1-(a+1)^2-a & a+3 \end{array}\right) \Rightarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & a+1 & 1 \\ 0 & a & -a & 3 \\ 0 & 0 & -a(a+3) & a+3 \end{array}\right)$$

With this we obtain an echelon form matrix for all values of a. (We should be extra careful in checking that this is indeed the case. For example, due to the second row, we should check that the matrix is in echelon form also for the value $a = 0$, but a quick substitution of $a = 0$ shows that we obtained an echelon form matrix also in this case). We are now ready to answer the required questions:

(i) Exactly one solution: This happens when $a \neq 0, -3$. In these cases the echelon form has a pivot in every row and a pivot in every column so, in particular, the there are no free variables and no rows with a 'lie'. These are precisely the conditions given in class for a system to have exactly one solution.

(ii) No solutions: This happens when $a = 0$. In this case the second row becomes $0 = 3$ and the third row also becomes $0 = 3$. So both these rows contain a "lie", making the system inconsistent and therefore having no solution. As discussed in class, a row with a "lie" (namely, zero equal to a nonzero number) in the echelon form of a matrix is the **only** case where a system is inconsistent. Since $a = 0$ is the only case where we obtain a row with a "lie", it is the only value for which there are no solutions.

(iii) An infinite amount of solutions: This happens when $a = -3$. In this case the last equation becomes all zero, leaving z as a free variable, and the first two equations do not contain a "lie". As discussed in class, there is an infinite amount of solutions if and only if there is no row with a "lie" and there is a free variable. Since $a = -3$ is the only value for which both of these conditions hold, this is the only value for which there are an infinite amount of solutions.

2. [20 points] Let $B = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$, and consider the vector space:

$$\{A \in M_{3\times 2}(\mathbb{R}) : AB = 0\}.$$

Find a spanning system for this space.

(Note: You do not need to prove that this is a vector space. This question should be solved without the use of 'basis' and 'coordinates'.)

Solution. We start by finding a parametrization of the vector space:

$$\{A \in M_{3\times 2}(\mathbb{R}) : AB = 0\}$$
$$= \{\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} : \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 0\}$$
$$= \{\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} : \begin{cases} a_1 + 2a_2 = 0 \\ 2a_1 + 4a_2 = 0 \end{cases} \begin{cases} b_1 + 2b_2 = 0 \\ 2b_1 + 4b_2 = 0 \end{cases} \begin{cases} c_1 + 2c_2 = 0 \\ 2c_1 + 4c_2 = 0 \end{cases} \}$$
$$= \{\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix} : a_1 + 2a_2 = 0, b_1 + 2b_2 = 0, c_1 + 2c_2 = 0\}$$
$$= \{\begin{pmatrix} -2a_2 & a_2 \\ -2b_2 & b_2 \\ -2c_2 & c_2 \end{pmatrix} : a_2, b_2, c_2 \in \mathbb{R}\}.$$

We rewrite the last expression loosing the subindex 2, as we do not need it any more. We denote the space by W:

$$W := \{\begin{pmatrix} -2a & a \\ -2b & b \\ -2c & c \end{pmatrix} : a_2, b_2, c_2 \in \mathbb{R}\}.$$

Now, we see that every matrix in the space is of the form:

$$\begin{pmatrix} -2a & a \\ -2b & b \\ -2c & c \end{pmatrix} = a\begin{pmatrix} -2 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ -2 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -2 & 1 \end{pmatrix},$$

and therefore, every matrix in the space is a linear combination of the set

$$E = \{\begin{pmatrix} -2 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ -2 & 1 \end{pmatrix}\}.$$

Since each one of the three vectors in the set E belongs to the space W (as they all satisfy the conditions of the space), it follows that E is a spanning set for W.

Remark. Most students solved this problem correctly, but with **no** explanations. In particular, they did not prove that the system they found indeed spans the space, nor did they prove that it belongs to the space. I credited these solutions with most of the points, but note that I will be more strict in next exams.

3. [15 points] Is the following statement correct? Explain your answer.

$$\operatorname{span}\{2-x-x^3, 2+x+x^3\} \subseteq \operatorname{span}\{1+x+x^2, 3x+4x^2+x^3, 2-x-x^3\}$$

(Note: This question should be solved without the use of 'basis' and 'co-ordinates'.)

Solution. The statement is not correct. To show this we need to show that there is a vector in $\operatorname{span}\{2-x-x^3, 2+x+x^3\}$ which does not belong to $\operatorname{span}\{1+x+x^2, 3x+4x^2+x^3, 2-x-x^3\}$. We will show that $2+x+x^3$ is such a vector. Indeed, we have $2+x+x^3 \in \operatorname{span}\{2-x-x^3, 2+x+x^3\}$ since a span of a collection of vectors always contains each one of these vectors (as was shown in class). On the other hand, we will show that $2+x+x^3$ is not in $\operatorname{span}\{1+x+x^2, 3x+4x^2+x^3, 2-x-x^3\}$.

From the definition of a span, we know that $2+x+x^3 \in \operatorname{span}\{1+x+x^2, 3x+4x^2+x^3, 2-x-x^3\}$ if and only if there exist three scalars $a, b, c \in \mathbb{R}$ such that

$$2+x+x^3 = a(1+x+x^2) + b(3x+4x^2+x^3) + c(2-x-x^3).$$

Rewriting the RHS we get

$$2+x+x^3 = (a+2c) + (a+3b-c)x + (a+4b)x^2 + (b-c)x^3.$$

Comparing the coefficients of the polynomials on both sides, we find that $2+x+x^3 \in \operatorname{span}\{1+x+x^2, 3x+4x^2+x^3, 2-x-x^3\}$ if and only if the following linear system has a solution.

$$\begin{cases} a+2c=2 \\ a+3b-c=1 \\ a+4b=0 \\ b-c=1 \end{cases}$$

Putting this linear system in an augmented matrix, and row reducing it to echelon form, we find

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 1 & 3 & -1 & 1 \\ 1 & 4 & 0 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow{R_2-R_1, R_3-R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 3 & -3 & -1 \\ 0 & 4 & -2 & -2 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow{\frac{1}{3}R_2, \frac{1}{2}R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & -1/3 \\ 0 & 2 & -1 & -1 \\ 0 & 1 & -1 & 1 \end{array}\right) \xrightarrow{R_4-R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & 4/3 \end{array}\right)$$

There is no need to continue row reducing further – the last row now contains a "lie": $0 = -4/3$, and therefore the system has no solution. It follows that indeed, $2 + x + x^3$ is not in $\text{span}\{1 + x + x^2, 3x + 4x^2 + x^3, 2 - x - x^3\}$.

Remarks.

i. Most students solved this problem correctly, but the explanation they used was in the opposite direction: Instead of explaining why if $2+x+x^3$ is not in $\text{span}\{1 + x + x^2, 3x + 4x^2 + x^3, 2 - x - x^3\}$ the statement is not correct, they explained why if $2 + x + x^3$ would have been in $\text{span}\{1 + x + x^2, 3x + 4x^2 + x^3, 2 - x - x^3\}$, the statement would have been correct. This, of course, is not satisfactory explanation.

ii. There were several students who solved this problem with the use of coordinates, but with no explanation: They did not explain which ordered basis they used, they did not indicate the coordinate transformation, nor did they state the relevant theorem from class. Usually, I will not credit such an answer with more then half the points, so be careful.

4. [15 points]
(i) Consider the set

$$W = \{B \in M_2(\mathbb{R}) : B \text{ is invertible } \} \cup \{0\}$$

with the 'usual' operations of summation and multiplication by a scalar done entry by entry. Is the set W with these operations a vector space?
(ii) Let $A, B \in M_3(\mathbb{R})$ be such that AB is invertible. Prove that both A and B are invertible. (You may use any theorem stated in class without proof, but if you use a result stated in recitation for this question, you should add a proof to it.)

Solution.
(i) No, W with these operations is not a vector space. This is due to the fact that this set is not closed to summation: Indeed, the matrices

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \text{and} \qquad J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

are both invertible, in fact, they are each equal to their own inverse: $I \cdot I = I$, $J \cdot J = J$, and therefore we have $I, J \in W$. However, their sum,

$$I + J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

is not invertible. To see this note that the matrix $I + J$ is already in reduced echelon form, and is not equal to I. We proved in class that a matrix is invertible iff its reduced echelon form is equal I. This implies that $I + J$ is not invertible and our claim is proved.

(ii) Since AB is invertible there exists a matrix $C \in M_3(\mathbb{R})$ such that $(AB)C = I$ and $C(AB) = I$. Since $(AB)C = I$ we get, from associativity, that $A(BC) = I$. Denoting $D = BC$ we find that $AD = I$, which means that A 'has an inverse from the right'. We proved in class that if A 'has an inverse from the right' then it is invertible. This completes the proof for A. Similarly, Since $C(AB) = I$ we get, from associativity, that $(CA)B = I$. Denoting $E = CA$ we find that $EB = I$, which means that B 'has an inverse from the left'. We proved in class that if B 'has an inverse from the left' then it is invertible. This completes the proof.

Remarks.
i. Most students who solved part (i) of this problem correctly did so with **no** explanations. In the most part, I ignored the missing explanations for this problem, and gave these students full credit. However, please

note that such missing explanations will not be ignored in the next exam.

ii. All of the students who solved part (*ii*) ignored the fact that 'has an inverse from the right' and 'invertible' are the same for square matrices **due to a theorem.** I ignored this lack when giving credit to this question, but expect you to know that when a result from class is used you must state that explicitly in your solution.

5. [10 points] Determine whether the following claim is **true or false**. If it is true then prove it, if it is false then show this by providing a counterexample:

Let V be a vector space over $\mathbb{R}$ and let $v_1, v_2, v_3 \in V$. If $\{v_1, v_2, v_3\}$ is a spanning system for V then $\{v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3\}$ is also a spanning system for V.

Solution. This claim is true. First, we show that each one of the vectors v_1, v_2, v_3 is a linear combination of the vectors $\{v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3\}$. Indeed, we have

$$\begin{aligned}
v_1 &= \frac{1}{2}(v_1 - v_2) + \frac{1}{2}(v_1 + v_2) + 0 \cdot (v_1 + v_2 + v_3), \\
v_2 &= \big(-\frac{1}{2}\big)(v_1 - v_2) + \frac{1}{2}(v_1 + v_2) + 0(v_1 + v_2 + v_3), \\
v_3 &= 0 \cdot (v_1 - v_2) + (-1)(v_1 + v_2) + 1(v_1 + v_2 + v_3).
\end{aligned}$$

Now, since v_1, v_2, v_3 are all linear combinations of the vectors $\{v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3\}$, we know by the definition of a span that

$$v_1, v_2, v_3 \in \operatorname{span}\{v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3\}.$$

From a result we proved in class, it follows that

$$\operatorname{span}\{v_1, v_2, v_3\} \subseteq \operatorname{span}\{v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3\}.$$

Since $\{v_1, v_2, v_3\}$ is a spanning set for V we conclude that

$$V \subseteq \operatorname{span}\{v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3\}.$$

On the other hand, since v_1, v_2, v_3 belong to V and V is a vector space, that is, closed to summation and multiplication by scalar, we know that $v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3$ all belong to V. Therefore, applying the same result from class again, we have

$$\operatorname{span}\{v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3\} \subseteq V.$$

Since both directions of the inclusion hold we have,

$$\operatorname{span}\{v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3\} = V.$$

So $v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3$ all belong to V and their span is equal to V. This implies that $v_1 - v_2, v_1 + v_2, v_1 + v_2 + v_3$ is a spanning set for V and our proof is complete.

6. [10 points] Determine whether the following claim is **true or false**. If it is true then prove it, if it is false then show this by providing a counterexample:

Let V be a vector space over $\mathbb{R}$ and let v_1, v_2, v_3, v_4 be a spanning system for V. If v_1, v_2, v_3 is not a spanning system for V then v_2, v_3, v_4 is also not a spanning system for V .

Solution. The claim is false. Indeed, consider the following counterexample: Let $V = \mathbb{R}^3$ and

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad v_4 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We see that $\{v_1, v_2, v_3, v_4\}$ is a spanning system for $V = \mathbb{R}^3$ since every $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ can be written as a linear combination of these vectors:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = 0 \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Similarly, $\{v_2, v_3, v_4\}$ is a spanning system for $V = \mathbb{R}^3$ since every $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$ can be written as a linear combination of these vectors:

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

However, $\{v_1, v_2, v_3\}$ is **not** a spanning system for $V = \mathbb{R}^3$. To see this note that any vector in the span of these vectors has a 0 entry in its last coordinate,

$$a \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a+b \\ a+c \\ 0 \end{pmatrix},$$

and therefore $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is not in the span of $\{v_2, v_3, v_4\}$.

Remark. Most students solved this problem correctly, but with **no** explanations. I ignored the missing explanations for this problem, and gave these students full credit. However, please note that such missing explanations will not be ignored in the next exam.

7. [10 points] Let $A \in M_{m\times n}(R)$. Prove that if $m > n$ then there exists $b \in \mathbb{R}^m$ such that the linear system $(A|b)$ has no solution.

Solution. We proved in class that every matrix can be brought to reduced echelon form by applying row operations. Moreover, we have seen that this sequence of row operations can be represented by a sequence of multiplications by elementary matrices. So let $E_1, ..., E_k \in M_{m\times m}(\mathbb{R})$ be elementary matrices so that $B = E_k \cdot ... \cdot E_2 \cdot E_1 \cdot A$ is of reduced echelon form.

Now, consider the reduced echelon form matrix B. Since every column in B contains at most one pivot, there are at most n pivots in B. Since there are m rows in B, and we have $m > n$, we can conclude that B contains at least one row with no pivot. This means that B contains at least one zero row, which will be the bottom row in B.

Let $e \in \mathbb{R}^m$ be the vector satisfying

$$e = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

and denote $b = (E_k \cdot ... \cdot E_2 \cdot E_1)^{-1}e$. Then, we claim that $(A|b)$ has no solution. Indeed, applying the same row operations as above to $(A|b)$ we get the matrix $(B|e)$:

$$E_k \cdot ... \cdot E_2 \cdot E_1(A|b) = (E_k \cdot ... \cdot E_2 \cdot E_1 A | E_k \cdot ... \cdot E_2 \cdot E_1 b) = (B|e).$$

But the bottom row in $(B|e)$ is a "lie": it is of the form $0 = 1$ as the bottom row of B is zero while the bottom coordinate of e is 1. This implies that indeed, $(A|b)$ has no solution.

Remark. The use of elementary matrices in the presentation of this proof is not necessary, just convenient. One could present this proof in other ways. However, the idea that the vector b should be obtained from e by applying the inverse of the row operations applied on A cannot be skipped. There were many students who skipped this part, and in fact chose $b = e$. This counted as only half a solution to this problem.