MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Midterm exam — Example 1, Solutions

- ◇ The exam starts when this fact is indicated by the instructor. The exam ends at 2:55. The length of the exam is roughly one hour and 15 minutes.
- ◇ The use of calculators is NOT permitted.
- ◇ The use of written notes is NOT permitted.
- ◇ There are 7 questions with points as indicated (with 100 points in all).
- ◇ Explain yourself clearly and justify all of your claims. If you use a result which was stated in class or recitation then make sure to indicate this fact explicitly. If you use a result that was stated in a homework assignment then you need to add a proof of this result.

Name: ______________________________

Recitation group: _______________

IMPORTANT REMARKS:

1. The only goal of this sample exam is to give you a sense of the structure of the exam. **Please do not assume any similarity between the questions here, and the questions in the exam. Some questions might be similar, others might be completely different. Moreover, please do not assume any similarity between the subjects represented in this example, and the subjects appearing in the exam itself. I strongly recommend to study all parts of the material and to look at all parts of the HWs.**
2. At some point in the exam you are asked to solve a question without the use of 'basis' or 'coordinates'. You can find many examples of such solutions in the solution to HW6. Solutions with Coordinates can be found in the solution set to HW8.
3. Please, feel free to write me with any question that you might have, but do so with enough time in advance, so that I will have enough time to respond.
4. This file contains solutions, if you want to first solve the questions yourself then use the file which contains only the questions.

1. [20 points] Is the set $\{1-x+x^2, 2+2x-x^2, 1+x^2\}$ linearly independent? Explain your answer.

   (Note: This question should be solved without the use of 'basis' and 'coordinates'.)

   Solution. Yes, the set is linearly independent. Explanation: By definition, a set is linearly **independent** if the only linear combination of its elements which is equal to the zero vector is the trivial one. So, the given set is linearly **independent** if the only solution to the equation:

$$\alpha(1-x+x^2)+\beta(2+2x-x^2)+\gamma(1+x^2)=0$$

is $\alpha=\beta=\gamma=0$. (Where the zero on the RHS should be understood as the constant zero polynomial). We rewrite the linear combination on the LHS:

$$(\alpha+2\beta+\gamma)+(-\alpha+2\beta)x+(\alpha-\beta+\gamma)x^2=0.$$

Such an equality between polynomial holds (for every x) iff the coefficients of the polynomials are equal at each entry (recall that the 0 on the RHS should be thought of as the constant polynomial 0), that is, iff

$$\begin{cases} \alpha+2\beta+\gamma=0 \\ -\alpha+2\beta=0 \\ \alpha-\beta+\gamma=0 \end{cases}$$

So, the set is linearly **independent** iff this homogeneous linear system has exactly one solution. Let us write it in matrix form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ -1 & 2 & 0 & 0 \\ 1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{R_2+R_1, R_3-R_1} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & -3 & 0 & 0 \end{array}\right) \xrightarrow{\frac{-1}{3}R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_2\leftrightarrow R_3} \left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \end{array}\right) \xrightarrow{R_3-4R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right).$$

We reached an echelon form. There are no free variables in this echelon form, so the corresponding linear system has exactly one solution (as a homogenous system always has at least one solution). It follows that the set we are considering is indeed linearly **independent**.

2. [20 points] Consider the vector space:

$$\{A \in M_{2\times 3}(\mathbb{R}) : A \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = 0\}$$

(you do not need to prove that this is a vector space). Find a spanning system for this space.

(Note: This question should be solved without the use of 'basis' and 'coordinates'.)

Solution. We need to find a spanning set for the space

$$W := \{A \in M_{2\times 3}(\mathbb{R}) : A \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}.$$

Our first step will be to parameterize the space:

$$W = \{\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} : \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\} =$$

$$= \{\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} : \begin{pmatrix} a_1 - 2a_2 - a_3 \\ b_1 - 2b_2 - b_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\} =$$

$$= \{\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix} : a_1 - 2a_2 - a_3 = 0 \text{ and } b_1 - 2b_2 - b_3 = 0\} =$$

$$= \{\begin{pmatrix} 2a_2 + a_3 & a_2 & a_3 \\ 2b_2 + b_3 & b_2 & b_3 \end{pmatrix} : a_2, a_3, b_2, b_3 \in \mathbb{R}\}.$$

Once we have a parametrization it is easy to describe any vector from the space as a linear combination of some specific other vectors in the space. In this case:

$$\begin{pmatrix} 2a_2 + a_3 & a_2 & a_3 \\ 2b_2 + b_3 & b_2 & b_3 \end{pmatrix} = a_2 \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + a_3 \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} +$$

$$+ b_2 \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix} + b_3 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

Since each one of the vectors: $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, belongs to W (indeed, they are all 2×3 matrices which, when multiplied by $\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$ give the zero 2-tuple), and every vector in W is a linear combination of these four vectors, it follows that $\{\begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}\}$ is a spanning set for W.

3. [15 points] We say that a function f from $\mathbb{R}$ to $\mathbb{R}$ ($f : \mathbb{R} \to \mathbb{R}$) is even if it satisfies:
$$f(x) = f(-x) \quad \text{for all } x \in \mathbb{R}.$$
Consider the set:
$$W = \{f : f \text{ is an even function from } \mathbb{R} \text{ to } \mathbb{R}\}$$
with the 'usual' operations of sum and multiplication by a scalar performed on functions. Is W with these operations a vector space?

Solution. Yes, this is a vector space. Proof: We stated in class that the set $\{f : \mathbb{R} \to \mathbb{R}\}$ with the 'usual' operations of sum and multiplication by a scalar performed on functions is a vector space. So, to prove that
$$W = \{f : f \text{ is an even function from } \mathbb{R} \text{ to } \mathbb{R}\}$$
with the same operations is a vector space, it is enough to prove that it is a subspace of $\{f : \mathbb{R} \to \mathbb{R}\}$ (as, by definition, every subspace is in particular a vector space). By the theorem we studied in class it is enough to prove the following three things:

The set is not empty: the constant function 0 (i.e., the function which satisfies $f(x) = 0$ for every $x \in \mathbb{R}$) belongs to the set since it trivially satisfies the relation $f(x) = f(-x)$ (both sides are equal zero) for all x in $\mathbb{R}$.

The set is closed to summation: Indeed, let $f, g \in W$, this implies that both f and g are even, that is, $f(x) = f(-x)$ and $g(x) = g(-x)$ for all x in $\mathbb{R}$. Their sum $(f+g)(x)$ is the function defined by the relation $(f+g)(x) = f(x) + g(x)$ for all x in $\mathbb{R}$. So, for all $x \in \mathbb{R}$ we have,
$$(f+g)(x) = f(x) + g(x) = f(-x) + g(-x) = (f+g)(-x).$$
This implies that $(f+g)(x) \in W$.

The set is closed to multiplication by scalar: Indeed, let $f \in W$ and $\alpha \in \mathbb{R}$. This implies that f is even, that is, $f(x) = f(-x)$ for all x in $\mathbb{R}$. Their multiplication $(\alpha f)(x)$ is the function defined by the relation $(\alpha f)(x) = \alpha \cdot f(x)$ for all x in $\mathbb{R}$. So, for all $x \in \mathbb{R}$ we have,
$$(\alpha f)(x) = \alpha \cdot f(x) = \alpha \cdot f(-x) = (\alpha f)(-x).$$
This implies that $(\alpha f)(x) \in W$.

4. [15 points] Consider the set $\mathbb{R}^2$ with the following operation of sum:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \oplus \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} x_1 x_2 \\ y_1 y_2 \end{pmatrix}$$

and the following operation of multiplication by scalar:

$$\alpha \odot \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \alpha x_1 \\ \alpha x_1 \end{pmatrix}.$$

Is $\mathbb{R}^2$ with these operations a vector space?

Solution. No. There are a couple of ways to show this, here is one. By definition, in a vector space the operations of sum and multiplication by scalar must satisfy the distributive relation:

$$\alpha \odot (v \oplus w) = (\alpha \odot v) \oplus (\alpha \odot w).$$

However, in the case described above, if we take for example

$$v = w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \alpha = 2,$$

then:

$$\alpha \odot (v \oplus w) = 2 \odot (\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix}) =$$

$$= 2 \odot \begin{pmatrix} 1 \cdot 1 \\ 0 \cdot 0 \end{pmatrix} = 2 \odot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}.$$

While,

$$(\alpha \odot v) \oplus (\alpha \odot w) = (2 \odot \begin{pmatrix} 1 \\ 0 \end{pmatrix}) \oplus (2 \odot \begin{pmatrix} 1 \\ 0 \end{pmatrix}) =$$

$$= \begin{pmatrix} 2 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \cdot 2 \\ 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

So the two expressions are NOT equal and therefore, as explained above, this is not a vector space.

5. [10 points] Determine whether the following claim is **true or false**. If it is true then prove it, if it is false then show this by providing a counterexample:

   If $\{v_1, v_2, v_3\}$ is a spanning set for V and $\{v_2, v_3, v_4\}$ is also a spanning set for V then $\text{span}\{v_1\} = \text{span}\{v_4\}$.

   Solution. The claim is false. There are many counterexamples, here is one. Consider $V = \mathbb{R}^2$ (one could take also $\mathbb{R}^3$ to construct a counterexample as well as many other spaces, of course). Consider the vectors
$$v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
Then, $\{v_1, v_2, v_3\}$ is a spanning set for $\mathbb{R}^2$ as every $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$ is a linear combination of these three vectors:
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$
and $\{v_2, v_3, v_4\}$ is a spanning set for $\mathbb{R}^2$ as every $\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \in \mathbb{R}^2$ is a linear combination of these three vectors:
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = b_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
But $\text{span}\{v_1\} \neq \text{span}\{v_4\}$ as $v_1 \in \text{span}\{v_1\}$ (indeed, we proved in class that every span contains each one of the vectors generating it) but v_1 does not belong to $\text{span}\{v_4\}$ since $\text{span}\{v_4\} = \{\begin{pmatrix} t \\ -t \end{pmatrix} : t \in \mathbb{R}\}$ and there is no value of t for which $\begin{pmatrix} t \\ -t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

6. [10 points] Determine whether the following claim is **true or false**. If it is true then prove it, if it is false then show this by providing a counterexample:

   Let $n \in \mathbb{N}$ and $A, B \in M_n(\mathbb{R})$. If the linear system $(AB|\underline{0})$ has a solution then the linear system $(A|\underline{0})$ has a solution.

   Solution. The claim is true. Proof: We showed in class that if $C \in M_{m\times n}(\mathbb{R})$ is a matrix and $v \in \mathbb{R}^n$ then v is a solution of $(C|\underline{0})$ if and only if $Cv = \underline{0}$. Applying this we find that if the linear system $(AB|\underline{0})$ has a solution then there exists $v \in \mathbb{R}^n$ (the 'solution') such that $(AB)v = \underline{0}$. Since matrix multiplication is associative, this implies that $A(Bv) = \underline{0}$. It follows (again, from the same fact we showed in class) that Bv is a solution to $(A|\underline{0})$ and therefore that $(A|\underline{0})$ has a solution.

7. [10 points] Let $A \in M_n(R)$ and $b \in \mathbb{R}^n$. Prove that if the linear system $(A|b)$ has exactly one solution then for every $d \in \mathbb{R}^n$ the linear system $(A|d)$ also has exactly one solution.

   Solution. If the linear system $(A|b)$ has exactly one solution then by the theorem we studied in class, which relates the set of solutions of a linear system to the set of solutions of the corresponding homogenous system, we have that the homogenous system $(A|0)$ also has exactly one solution. Since A is a square matrix, it follows from the theorem we studied in class, regarding the equivalent conditions for an invertible matrix, that for every $d \in \mathbb{R}^n$ the system $(A|d)$ has at least one solution. However, since $(A|0)$ has exactly one solution it follows from the same theorem as above, which relates the set of solutions of a linear system to the set of solutions of the corresponding homogenous system, that if $(A|d)$ has at least one solution then it has exactly one solution. (That theorem implies in particular that if $(A|d)$ has at least one solution then the amount of solutions of $(A|d)$ is equal to the amount of solutions of $(A|0)$).