MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 10

1. In each of the following you are given two vector spaces and a function between them. Determine whether the function is a linear transformation or not. Prove your claim.

   i.
   $$T : \mathbb{R}^3 \mapsto M_2(\mathbb{R})$$
   given by,
   $$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x+y & y-2z \\ 3x+z & 0 \end{pmatrix}$$

   ii.
   $$T : \mathbb{R}_2[x] \mapsto \mathbb{R}^3$$
   given by,
   $$Tp = \begin{pmatrix} p(2) \\ p'(2) \\ p''(2) \end{pmatrix}$$

   iii.
   $$T : M_2(\mathbb{R}) \mapsto M_2(\mathbb{R})$$
   given by,
   $$TA = A^2$$

   iv.
   $$T : \mathbb{R}_2[x] \mapsto \mathbb{R}$$
   given by,
   $$Tp = \int_0^1 p(x)dx$$

   v. Fix $B \in M_3(\mathbb{R})$ and consider the function:
   $$T : M_3(\mathbb{R}) \mapsto M_3(\mathbb{R})$$
   given by,
   $$TA = AB$$

   vi.
   $$T : M_2(\mathbb{R}) \mapsto M_2(\mathbb{R})$$
   given by,
   $$T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a-c+1 & 2a+3b+2 \\ d-b-8 & 2a \end{pmatrix}$$

2. In each of the following you are given a linear transformation (you don't need to prove that it is a linear transformation). Follow the following directions for each such transformation:
   a. Find a basis for the kernel and the image of this transformation.
   b. Find the dimension of the kernel and the image of this transformation. (Remark: This question will continue in the next HW, you may want to keep a copy of your solution to this part of the question).
   c. Determine whether the transformation is onto. Explain your answer.
   d. Determine whether the transformation is $1-1$. Explain your answer.
   i. For
   $$A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 1 & 2 \\ 1 & 7 & -6 \\ 0 & 5 & -5 \end{pmatrix}$$
   consider the operator
   $$T_A : \mathbb{R}^3 \mapsto \mathbb{R}^4$$
   where the notation T_A was defined in class.
   ii.
   $$S : M_2(\mathbb{R}) \mapsto \mathbb{R}^2$$
   given by
   $$SA = A \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$
   iii.
   $$L : \mathbb{R}_3[x] \mapsto \mathbb{R}^2$$
   given by
   $$Lp = \begin{pmatrix} p(2) - p(1) \\ p'(0) \end{pmatrix}$$
   iv.
   $$\Phi : \mathbb{R}^3 \mapsto \mathbb{R}_3[x]$$
   given by
   $$\Phi \begin{pmatrix} a \\ b \\ c \end{pmatrix} = (a+b) + (a-2b+c)x + (b-3c)x^2 + (a+b+c+d)x^3$$

3. Let V, W be vector spaces over $\mathbb{R}$ and $T : V \mapsto W$ be a linear operator. Let $v_1, ..., v_n \in V$. For each of the following claims determine whether it is **true** or **false**. Prove or disprove your claim accordingly.
   i. If $v_1, ..., v_n$ is linearly independent in V then $Tv_1, ..., Tv_n$ is linearly independent in W.
   ii. If $Tv_1, ..., Tv_n$ is linearly independent in W then $v_1, ..., v_n$ is linearly independent in V.

iii. If $v_1, ..., v_n$ is a spanning set in V then $Tv_1, ..., Tv_n$ is a spanning set in W.

iv. If $Tv_1, ..., Tv_n$ is a spanning set in W then $v_1, ..., v_n$ is a spanning set in V.

v. If U is a subspace of V then the set $\{Tu : u \in U\}$ is a subspace of W.

v. If U is a subset of V and the set $\{Tu : u \in U\}$ is a subspace of W then U is a subspace of V.

4. Consider the claims which you determined are **false** in Q5. Which of them will become **true** if you add one of the following conditions? Prove your answers.
   i. If you add the condition that T is 1-1 (but not necessarily onto).
   ii. If you add the condition that T is onto (but not necessarily 1-1).
5. Prove or disprove the following claims:
   (**Hint:** You may want to use your answers to Q4 while solving this question. Alternatively, material which we will study in class next week may help.)
   i. There exists $T : M_2(\mathbb{R}) \mapsto \mathbb{R}^3$ which is 1-1.
   ii. There exists $T : \mathbb{R}^3 \mapsto M_2(\mathbb{R})$ which is 1-1.
   iii. There exists $T : M_2(\mathbb{R}) \mapsto M_2(\mathbb{R})$ which is neither 1-1 nor onto.
   iv. There exists $T : M_2(\mathbb{R}) \mapsto \mathbb{R}_3[x]$ which is 1-1 but not onto.
   v. There exists $T : M_2(\mathbb{R}) \mapsto \mathbb{R}^3$ which is onto but not 1-1.
   vi. There exists $T : \mathbb{R}^3 \mapsto M_2(\mathbb{R})$ which is onto but not 1-1.
   vii. There exists $T : M_2(\mathbb{R}) \mapsto \mathbb{R}^3$ which neither onto nor 1-1.