MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Midterm exam — Example 2, Solutions

- ◇ The exam starts when this fact is indicated by the instructor. The exam ends at 2:55. The length of the exam is roughly one hour and 15 minutes.
- ◇ The use of calculators is NOT permitted.
- ◇ The use of written notes is NOT permitted.
- ◇ There are 7 questions with points as indicated (with 100 points in all).
- ◇ Explain yourself clearly and justify all of your claims. If you use a result which was stated in class or recitation then make sure to indicate this fact explicitly. If you use a result that was stated in a homework assignment then you need to add a proof of this result.

Name: ______________________________

Recitation group: _______________

IMPORTANT REMARKS:

1. The only goal of this sample exam is to give you a sense of the structure of the exam. **Please do not assume any similarity between the questions here, and the questions in the exam. Some questions might be similar, others might be completely different. Moreover, please do not assume any similarity between the subjects represented in this example, and the subjects appearing in the exam itself. I strongly recommend to study all parts of the material and to look at all parts of the HWs.**
2. At some point in the exam you are asked to solve a question without the use of 'basis' or 'coordinates'. You can find many examples of such solutions in the solution to HW6. Solutions with Coordinates can be found in the solution set to HW8.
3. Please, feel free to write me with any question that you might have, but do so with enough time in advance, so that I will have enough time to respond.
4. This file contains solutions, if you want to first solve the questions yourself then use the file which contains only the questions.

1. [20 points] Is the following matrix invertible? If so find its inverse.

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$$

Solution. By a theorem studied in class, a matrix is invertible iff its REF is the identity matrix. We check this and look for a possible inverse at the same time, using the algorithm studied in class.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2-2R_1, R_3-3R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & -2 & -3 & 0 & 1 \end{array}\right) \xrightarrow{\frac{-1}{2}R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & 1.5 & 0 & -0.5 \end{array}\right) \xrightarrow{R_2+2R_3, R_1-R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -0.5 & 0 & 0.5 \\ 0 & 1 & 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & 1.5 & 0 & -0.5 \end{array}\right).$$

We conclude that the matrix is invertible and its inverse is equal to:

$$\begin{pmatrix} -0.5 & 0 & 0.5 \\ 1 & 1 & -1 \\ 1.5 & 0 & -0.5 \end{pmatrix}$$

Remarks.

1. A few students made small computational errors when computing the inverse. Several students made a computational error which resulted in them obtaining a matrix with a column of zeroes. I consider this a 'bigger' mistake then just calculational error, as I would expect students to realize that a matrix with a zero column cannot be invertible (why?). If in an exam you obtain a result which you recognize cannot be correct, but cannot find your computational error, then please add some words like: "I know there is a computational error because ... but I cannot find the error".

ii. It is good to end this answer by checking that the matrix you received is indeed the inverse (that is, that $AA^{-1} = I$). It will both allow you to find computational errors and express to me that you understand what an inverse is.

2. [20 points] Consider the vector space:

$$\{p(x) \in \mathbb{R}_3[x] : p(1) = p'(1)\}.$$

Find a spanning system for this space.

(Note: You do not need to prove that this is a vector space. This question should be solved without the use of 'basis' and 'coordinates'.)

Solution. We start by finding a parametrization of the space:

$$\begin{gathered}\{p(x) \in \mathbb{R}_3[x] : p(1) = p'(1)\} = \\ = \{a + bx + cx^2 + dx^3 : a + b + c + d = b + 2c + 3d\} = \\ = \{a + bx + cx^2 + dx^3 : a = c + 2d\} = \\ = \{(c + 2d) + bx + cx^2 + dx^3 : b, c, d \in \mathbb{R}\}.\end{gathered}$$

A general vector in the space has therefore the form

$$(c + 2d) + bx + cx^2 + dx^3 = c(1 + x^2) + bx + d(2 + x^3).$$

We see that all elements in the space are linear combinations of the vectors $1 + x^2, x, 1 + x^3$.

We check that these three vectors belong to the space by checking that they satisfy the condition $p(1) = p'(1)$: Indeed, for the vector $1+x^2$ we have $1 + 1^2 = 2 \cdot 1$, for the vector x we have $1 = 1$ and for the vector $2 + x^3$ we have $2 + 1^3 = 3 \cdot 1^2$. As all of these three vectors belong to the space and all other elements in the space are linear combinations of them, we conclude that $\{1 + x^2, x, 1 + x^3\}$ is a spanning set for the space.

Remarks.

1. This question is very much the same as Q2(iv) from HW5.
2. Some students obtained the correct result but made a "short cut" with the computation that allowed them to get the result but did not provide any explanation for why the result they got is indeed a spanning set. Always assume that in such questions I not only check if you can follow an algorithm, but also check if you understand the definitions of the notions involved.
3. As this question seemed to be difficult to many students I was not very strict with the "no coordinates" restriction. I will be next time.

3. [15 points] Is the following statement correct? Explain your answer.

$$\operatorname{span}\{5-4x-5x^2-3x^3, 1+x+x^2\} \subseteq \operatorname{span}\{1+x+x^2, 3x+4x^2+x^3, 2-x-x^3\}$$

(Note: This question should be solved without the use of 'basis' and 'co-ordinates'.)

Solution. We use the following fact which was proved in class: If V is a vs, $W \subseteq V$ is a subspace and $v_1, ..., v_n \in W$ then $\operatorname{span}\{v_1, ..., v_n\} \subseteq W$. We also use the fact proved in class that the span of a set of vectors is a subspace. Combining these two, we know that if

$$5-4x-5x^2-3x^3, 1+x+x^2 \subseteq \operatorname{span}\{1+x+x^2, 3x+4x^2+x^3, 2-x-x^3\}$$

then the answer to the question is "yes". (While if one of these elements is not in $\operatorname{span}\{1+x+x^2, 3x+4x^2+x^3, 2-x-x^3\}$ then the answer is "no" (why?)).

We check for each vector separately. For $1+x+x^2$, it is easy to see that it belongs to the span on the RHS as

$$1+x+x^2 = 1(1+x+x^2)+0(3x+4x^2+x^3)+0(2-x-x^3).$$

(You could also write that we mentioned in class that a vector always belong to a span of vectors if it is one of these vectors).

For $5-4x-5x^2-3x^3$ we need to check if there exist $a, b, c \in \mathbb{R}$ such that

$$5-4x-5x^2-3x^3 = a(1+x+x^2)+b(3x+4x^2+x^3)+c(2-x-x^3),$$

holds for every x. We rewrite the equation on the RHS to get

$$5-4x-5x^2-3x^3 = (a+2c)+(a+3b-c)x+(a+4b)x^2+(b-c)x^3.$$

Since two polynomials are equal iff their coefficients are equal we obtain the following linear system:

$$\begin{cases} a+2c=5 \\ a+3b-c=-4 \\ a+4b=-5 \\ b-c=-3 \end{cases}$$

We write in matrix form and reduce to echelon form to check if this system indeed has a solution.

$$\begin{pmatrix} 1 & 0 & 2 & 5 \\ 1 & 3 & -1 & -4 \\ 1 & 4 & 0 & -5 \\ 0 & 1 & -1 & -3 \end{pmatrix} \xRightarrow{R_2-R_1, R_3-R_1} \begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 3 & -3 & -9 \\ 0 & 4 & 0 & -10 \\ 0 & 1 & -1 & -3 \end{pmatrix} \xRightarrow{R_2-3R_4, R_3-4R_4}$$

$$\begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 \\ 0 & 1 & -1 & -3 \end{pmatrix} \xRightarrow{R_2 \leftrightarrow R_4} \begin{pmatrix} 1 & 0 & 2 & 5 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The echelon form has no row with a "lie" and therefore the system has a solution. We conclude that $5 - 4x - 5x^2 - 3x^3$ belongs to the span and therefore that the answer to the question is "yes".

Remarks.

1. In this question I was strict with the requirement of "no coordinates".
2. Some students obtained the correct result, but did not mention which theorem from class they are applying (or even that they are applying a theorem). Even in computational questions you should point out what is it that makes your computations "work".

4. [15 points] Let $A \in M_3(\mathbb{R})$. Consider the set
$$W = \{B \in M_3 : AB = BA\},$$
with the 'usual' operations of summation and multiplication by a scalar done entry by entry. Is the set W with these operations a vector space?

Solution. We note that the set we were given is a subset of $M_3(\mathbb{R})$. We proved in class that $M_3(\mathbb{R})$ with the same operations of sum and multiplication by scalar is a vector space. So, it is enough to prove that W is a subspace of $M_3(\mathbb{R})$ since this will in particular imply that W is a vector space (by the definition of a subspace). By a theorem studied in class, it is enough to check the following three things.

- ◇ The subset is not empty: Indeed the zero vector of $M_3(\mathbb{R})$ satisfies $A0 = 0$ and $0A = 0$ which imply that $0A = A0$, so the zero vector satisfies the condition which defines W and therefore $0 \in W$.
- ◇ The subset is closed to summation: Indeed, if $B, C \in W$ then $AB = BA$ and $AC = CA$. This (and the distributivity between summation and multiplication of matrices, which was stated in class,) imply that
$$A(B + C) = AB + AC = BA + CA = (B + C)A.$$
So, $B+C$ satisfies the condition which defines W and therefore $B+C \in W$.
- ◇ The subset is closed to multiplication by scalar: Indeed, if $B \in W$ and $a \in \mathbb{R}$ then $AB = BA$. This (and the relation between multiplication by scalar and multiplication of matrices, which was stated in class,) imply that
$$A(aB) = a(AB) = a(BA) = (aB)A.$$
So, aB satisfies the condition which defines W and therefore $aB \in W$.

We found that W satisfies all three conditions of being a subspace and therefore W is a subspace of $M_3(\mathbb{R})$ and in particular a vector space.

Remarks.

1. This question is very similar to Q3(vi) from HW4.
2. Several students thought that the answer to the question is "no" as it seemed to them that there are no more then 2 or 3 matrices that commute with A. Please note that in general, one can obtain different matrices that commute with A for example: I, 0, A^2, A^3, $A^3 - 2A + I$, and so on.

5. [10 points] Determine whether the following claim is **true or false**. If it is true then prove it, if it is false then show this by providing a counterexample:

Let V be a vector space over $\mathbb{R}$ and let $v_1, v_2, v_3, v_4 \in V$. If $\{v_1, v_2, v_3\}$ is a spanning set for V and $v_4 = 2v_1 + 3v_3$ then $\{v_2, v_3, v_4\}$ is also spanning set for V.

Solution. The claim is true. There were three different proofs given by students in class, here they are:

Proof I. Since $\{v_1, v_2, v_3\}$ is a spanning set for V it is enough to show that span$\{v_2, v_3, v_4\}$ =span$\{v_1, v_2, v_3\}$. We apply the same results from class as in Q3. Since a vector always belongs to a span of vectors if it is one of these vectors (proved in class) we know that v_2 and v_3 belong to span$\{v_1, v_2, v_3\}$. Since $v_4 = 2v_1 + 3v_3 = 2v_1 + 0v_2 + 3v_3$ we know that v_4 is also in span$\{v_1, v_2, v_3\}$. It follows from the theorem mentioned above that

$$\text{span}\{v_2, v_3, v_4\} \subseteq \text{span}\{v_1, v_2, v_3\}.$$

In the same way, since a vector always belongs to a span of vectors if it is one of these vectors (proved in class) we know that v_2 and v_3 belong to span$\{v_2, v_3, v_4\}$. Since $v_1 = \frac{1}{2}v_4 - \frac{3}{2}v_3 = \frac{1}{2}v_4 + 0v_2 + (-\frac{3}{2})v_3$ we know that v_1 is also in span$\{v_2, v_3, v_4\}$. It follows from the theorem mentioned above that

$$\text{span}\{v_1, v_2, v_3\} \subseteq \text{span}\{v_2, v_3, v_4\}$$

. We conclude that span$\{v_2, v_3, v_4\}$ =span$\{v_1, v_2, v_3\} = V$ as requested.

Proof II. Since $\{v_1, v_2, v_3\}$ is a spanning set for V it is enough to show that span$\{v_2, v_3, v_4\}$ =span$\{v_1, v_2, v_3\}$. We use the following lemma from class: Let V be a vector space and $v_1, ..., v_n, w \in V$. Then w is a linear combination of $v_1, ..., v_n$ iff span$\{v_1, ..., v_n\}$ =span$\{v_1, ..., v_n, w\}$. So, since $v_4 = 2v_1 + 3v_3 = 2v_1 + 0v_2 + 3v_3$ we have that v_4 is a linear combination of v_1, v_2, v_3 and therefore from the lemma we get that span$\{v_1, v_2, v_3, v_4\}$ =span$\{v_1, v_2, v_3\}$. Since $v_1 = \frac{1}{2}v_4 - \frac{3}{2}v_3 = \frac{1}{2}v_4 + 0v_2 + (-\frac{3}{2})v_3$ we have that v_1 is a linear combination of v_2, v_3, v_4 and therefore from the lemma we get that span$\{v_1, v_2, v_3, v_4\}$ =span$\{v_2, v_3, v_4\}$. We conclude that

$$\text{span}\{v_1, v_2, v_3\} = \text{span}\{v_2, v_3, v_4\}$$

as requested.

Proof III. We want to show that every $w \in V$ is a linear combination of v_2, v_3, v_4. So, let $w \in V$. Since $\{v_1, v_2, v_3\}$ is a spanning set for V we know that there exist specific $a, b, c \in \mathbb{R}$ such that

$$w = av_1 + bv_2 + cv_3.$$

We substitute $v_1 = \frac{1}{2}v_1 - \frac{3}{2}3v_3$ and get

$$w = a(\frac{1}{2}v_4 - \frac{3}{2}v_3) + bv_2 + cv_3.$$

We rewrite the RHS:

$$w = bv_2 + \frac{a}{2}v_4 + (c - \frac{3a}{2})v_3.$$

We found a linear combination of v_2, v_3, v_4 which is equal to w so $w \in \operatorname{span}\{v_2, v_3, v_4\}$ as requested.

Remarks.

1. Several students went in the direction of proof III, but instead of showing that if w is a linear combination of $\{v_1, v_2, v_3\}$ then it is also a linear combination of $\{v_2, v_3, v_4\}$ they showed the opposite direction (that is, they showed that if w is a linear combination of $\{v_2, v_3, v_4\}$ then it is also a linear combination of $\{v_1, v_2, v_3\}$). Be careful with what you are given, and what you are trying to prove.

6. [10 points] Determine whether the following claim is **true or false**. If it is true then prove it, if it is false then show this by providing a counterexample:

   Let V be a vector space over $\mathbb{R}$. Let $v_1, ...v_n \in V$ be such that none of them is the zero vector. If $\{v_1, v_2, ..., v_n\}$ is linearly **dependent** then **each one** of these vectors is a linear combination of the rest of the vectors.

   Solution. The claim is false. There are many counterexamples, here is one: consider the set $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ in $\mathbb{R}^2$. It is linearly dependent as

$$2\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 2 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0,$$

but $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is not a linear combination of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. (Any linear combination of these two vectors will have a zero in its second entry.)

7. [10 points] Let $A \in M_{m\times n}(R)$. Prove that if the homogeneous linear system $(A|0)$ has exactly one solution then $m \geq n$.

Solution. If $(A|0)$ has exactly one solution then an echelon form of A will have no free variables (we proved in class that if there is a free variable then a homogeneous system will have an infinite amount of solutions). It follows that in an echelon form of A all of the variables are leading variables. Since every row in an echelon form can have at most one leading variable, it follows that the number of rows (m) is bigger or equal to the number of variables (n) and. So, $m \geq n$.