MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 4, solutions.

1. Consider the following matrices:

$$A=\begin{pmatrix}1&2\\2&-1\end{pmatrix},\quad B=\begin{pmatrix}1&0&3\\2&1&1\end{pmatrix},\quad F=\begin{pmatrix}1&2&3\end{pmatrix}$$

$$C=\begin{pmatrix}1&2\\3&1\\2&-1\end{pmatrix},\quad D=\begin{pmatrix}1&0&3\\1&1&-2\\2&-1&1\end{pmatrix},\quad E=\begin{pmatrix}1\\2\\-1\end{pmatrix}$$

$$\bigstar\quad AB=\begin{pmatrix}1&2\\2&-1\end{pmatrix}\begin{pmatrix}1&0&3\\2&1&1\end{pmatrix}=$$

$$=\begin{pmatrix}1+4&0+2&3+2\\2-2&0-1&6-1\end{pmatrix}=\begin{pmatrix}5&2&5\\0&-1&5\end{pmatrix}$$

$$\bigstar\quad BA=\begin{pmatrix}1&0&3\\2&1&1\end{pmatrix}\begin{pmatrix}1&2\\2&-1\end{pmatrix}$$

This is not defined as the number of columns in B is different then the number of rows in A.

$$\bigstar\quad D^2=\begin{pmatrix}1&0&3\\1&1&-2\\2&-1&1\end{pmatrix}\begin{pmatrix}1&0&3\\1&1&-2\\2&-1&1\end{pmatrix}=$$

$$=\begin{pmatrix}1+0+6&0+0-3&3+0+3\\1+1-4&0+1+2&3-2-2\\2-1+2&0-1-1&6+2+1\end{pmatrix}=\begin{pmatrix}7&-3&6\\-2&3&-1\\3&-2&9\end{pmatrix}$$

$$\bigstar\quad B^2=\begin{pmatrix}1&0&3\\2&1&1\end{pmatrix}\begin{pmatrix}1&0&3\\2&1&1\end{pmatrix}$$

This is not defined as the number of columns in B is different then the number of rows in B.

$$\bigstar\quad DC=\begin{pmatrix}1&0&3\\1&1&-2\\2&-1&1\end{pmatrix}\begin{pmatrix}1&2\\3&1\\2&-1\end{pmatrix}=$$

$$\begin{pmatrix}1+0+6&2+0-3\\1+3-4&2+1+2\\2-3+2&4-1-1\end{pmatrix}=\begin{pmatrix}7&-1\\0&5\\1&2\end{pmatrix}$$

$$\bigstar\quad CB=\begin{pmatrix}1&2\\3&1\\2&-1\end{pmatrix}\begin{pmatrix}1&0&3\\2&1&1\end{pmatrix}=$$

$$= \begin{pmatrix} 1+4 & 0+2 & 3+2 \\ 3+2 & 0+1 & 9+1 \\ 2-2 & 0-1 & 6-1 \end{pmatrix} = \begin{pmatrix} 5 & 2 & 5 \\ 5 & 1 & 10 \\ 0 & -1 & 5 \end{pmatrix}$$

$$\bigstar \quad BC = \begin{pmatrix} 1 & 0 & 3 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1+0+6 & 2+0-3 \\ 2+3+2 & 4+1-1 \end{pmatrix} = \begin{pmatrix} 7 & -1 \\ 7 & 4 \end{pmatrix}$$

$$\bigstar \quad FE = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} = 1+4-3 = 2$$

$$\bigstar \quad EF = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -1 & -2 & -3 \end{pmatrix}$$

$$\bigstar \quad CE = \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

This is not defined as the number of columns in C is different then the number of rows in E.

$$\bigstar \quad EC = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 1 \\ 2 & -1 \end{pmatrix}$$

This is not defined as the number of columns in E is different then the number of rows in C.

2. i. We follow the algorithm studied in class.

◆ For the matrix A:

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2-2R_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & -5 & -2 & 1 \end{array}\right) \xrightarrow{\frac{1}{-5}R_2}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 1 & 2/5 & -1/5 \end{array}\right) \xrightarrow{R_1-2R_2} \left(\begin{array}{cc|cc} 1 & 0 & 1/5 & 2/5 \\ 0 & 1 & 2/5 & -1/5 \end{array}\right)$$

We conclude that A is invertible and that

$$A^{-1} = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix}$$

◆ For the matrix B:

$$\left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ -2 & 6 & 2 & 1 \end{array}\right) \xrightarrow{R_2+2R_1} \left(\begin{array}{cc|cc} 1 & -3 & 1 & 0 \\ 0 & 0 & 4 & 1 \end{array}\right)$$

We conclude that B is not invertible, as it's reduced echelon form is different from I_2.

♦ For the matrix C:

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right) \xRightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right) \xRightarrow{R_2 - 2R_1}$$

$$\left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -2 \end{array}\right) \xRightarrow{-R_2} \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -1 & 2 \end{array}\right) \xRightarrow{R_1 - 2R_2}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 2 & -3 \\ 0 & 1 & -1 & 2 \end{array}\right)$$

We conclude that C is invertible and that

$$C^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$$

♦ For the matrix D:

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \xRightarrow{R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & -1 & 0 & 1 \end{array}\right) \xRightarrow{R_3 + R_2, R_1 - R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 2 & -1 & 1 & 1 \end{array}\right) \xRightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right) \xRightarrow{R_1 + R_3, R_2 - R_3}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & -1/2 & 1/2 & 1/2 \end{array}\right)$$

We conclude that D is invertible and that

$$D^{-1} = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix}$$

♦ For the matrix E:

$$\left(\begin{array}{ccc|ccc} 3 & 0 & 2 & 1 & 0 & 0 \\ 2 & -1 & -2 & 0 & 1 & 0 \\ -1 & 2 & 6 & 0 & 0 & 1 \end{array}\right) \xRightarrow{R_1 \leftrightarrow R_3} \left(\begin{array}{ccc|ccc} -1 & 2 & 6 & 0 & 0 & 1 \\ 2 & -1 & -2 & 0 & 1 & 0 \\ 3 & 0 & 2 & 1 & 0 & 0 \end{array}\right) \xRightarrow{-R_1}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -6 & 0 & 0 & -1 \\ 2 & -1 & -2 & 0 & 1 & 0 \\ 3 & 0 & 2 & 1 & 0 & 0 \end{array}\right) \xRightarrow{R_2 - 2R_1, R_3 - 3R_1} \left(\begin{array}{ccc|ccc} 1 & -2 & -6 & 0 & 0 & -1 \\ 0 & 3 & 10 & 0 & 1 & 2 \\ 0 & 6 & 20 & 1 & 0 & 3 \end{array}\right) \xRightarrow{R_3 - 2R_2}$$

$$\left(\begin{array}{ccc|ccc} 1 & -2 & -6 & 0 & 0 & -1 \\ 0 & 3 & 10 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 & -2 & -1 \end{array}\right)$$

We conclude that E is not invertible, since if a matrix is row equivalent to a matrix with a zero row then it's REF cannot be I_3.

ii. We consider the equation,

$$A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

The matrix A is invertible, so we can multiply **the left of** both of the sides of the equation by A^{-1}. We obtain,

$$\begin{pmatrix} x \\ y \end{pmatrix} = A^{-1}\begin{pmatrix} 1 \\ -2 \end{pmatrix} =$$

$$= \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1/5 - 4/5 \\ 2/5 + 2/5 \end{pmatrix} = \begin{pmatrix} -3/5 \\ 4/5 \end{pmatrix}$$

iii. The matrix B is not invertible, still we can solve the equation as we have learned to do in chapter 1 of the course. The equation we received, when in matrix form, looks like this:

$$\left(B\middle|\begin{pmatrix} 1 \\ -2 \end{pmatrix}\right).$$

We insert B and bring this to echelon form, as usual.

$$\left(\begin{array}{cc|c} 1 & -3 & 1 \\ -2 & 6 & -2 \end{array}\right) \xrightarrow{R_2+2R_1} \left(\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 0 & 0 \end{array}\right)$$

This represents the linear system, which has only one linear equation, $x - 3y = 1$. y is a free variable, so it can be any number, we denote this by $y = t$, $t \in \mathbb{R}$. We insert $y = t$ and get $x = 1 + 3t$. So this system has an infinite amount of solutions, all of the form $\begin{pmatrix} 1+3t \\ t \end{pmatrix}$. We were asked to find one solution, we have many to choose from, for example, we can choose $(16, 5)$, which indeed solves this system, and was obtained by putting $t = 5$ in the parametrization of the set of all solutions.

iv. Since D is invertible, we can multiply **the left of** both of the sides of the equation $DG = E$ by D^{-1} and get

$$G = D^{-1}E = \begin{pmatrix} 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 3 & 0 & 2 \\ 2 & -1 & -2 \\ -1 & 2 & 6 \end{pmatrix} = \begin{pmatrix} 0 & 3/2 & 5 \\ 3 & -3/2 & -3 \\ -1 & 1/2 & 1 \end{pmatrix}$$

3. a. The claim is false, there are many counterexamples, we give one: Take $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ then $A \neq 0$ but it is easy to verify that $A^2 = 0$. Note that to obtain a counterexample, we had to use a matrix A which is not invertible (though, of course, not every non–invertible matrix will provide such a counterexample).

b. The claim is true. Indeed,

$$AB^2 = A(BB) = (AB)B = (BA)B = B(AB) = B(BA) = (BB)A = B^2A.$$

(Make sure that you know to explain each step in this long equality!)

c. The claim is false (it would have been correct if B were invertible, so when we look for a counterexample we should try only non–invertible B's). There are many counterexamples, we give one: Take $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 5 \\ 3 & 7 \end{pmatrix}$. Then $AB = CB = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix}$, but clearly $A \neq C$.

4. di. Recall that we proved in class, that if for a $n \times n$ matrix C there exists a $n \times n$ matrix D so that $CD = I_n$ then C is invertible and $C^{-1} = D$ (that is, if a matrix is invertible–on–the–right then it is invertible). Correspondingly, we compute

$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AI_nA^{-1} = (AI_n)A^{-1} = AA^{-1} = I_n.$$

It therefore follows from the result stated above, that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$. (Check that you can justify each step of the equality above!).

dii. Since AB is invertible, it follows from the definition of invertibility that there exists $D \in M_n(\mathbb{R})$ such that $(AB)D = I_n$. By the associativity property of matrix multiplication we get $A(BD) = I_n$. This implies that A is invertible–on–the–right and therefore, by a thm we proved in class, that it is invertible. Now, since A is invertible A^{-1} exists and is also invertible. Since we are given that AB is also invertible, it follows from part (di) of this question that their multiplication is also invertible. Therefore, $A^{-1}(AB) = (A^{-1}A)B = B$ is also invertible.

ei. The claim is false, there are many counterexamples, we give one: Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then A and B are both non invertible as they have a zero row (why does this imply that they are not invertible?) but $A + B = I$ which is clearly invertible ($I^{-1} = I$).

eii. The claim is false. Take for example $A = I_n$ and $B = -I_n$, then they are both invertible but their sum $A + B = 0$ is not invertible.

diii. The claim is true. Recall that we proved in (dii) that if AB is invertible then A and B are invertible, and therefore, due to (di), so is BA.

g. The claim is true. Recall that we proved in (dii) that if AB is invertible then A and B are invertible. So, it follows that if A^3 is invertible and $A^3 = A^2 \cdot A$ then A is invertible (and so is A^2).

5. We prove using the formula $(AB)_{ij} = \sum_{k=1}^{n}(A)_{ik}(B)_{kj}$ (for two matrices, A with n columns and B with n rows). Several of these claims can easily be

proved without this formula, we use it here so you will have some examples of this type of proof.

i. For $A+B$: We need to show that for every $1 \leq i,j \leq n$ such that $i \neq j$ we have $(A+B)_{ij} = 0$. Indeed, fix such i and j then, $(A+B)_{ij} = (A)_{ij} + (B)_{ij}$. Since A and B are diagonal and $i \neq j$ we know that $(A)_{ij} = (B)_{ij} = 0$. It follows that $(A+B)_{ij} = (A)_{ij} + (B)_{ij} = 0 + 0 = 0$.

   For AB: We need to show that for every $1 \leq i,j \leq n$ such that $i \neq j$ we have $(AB)_{ij} = 0$. Indeed, fix such i and j then,

$$(AB)_{ij} = \sum_{k=1}^{n} (A)_{ik}(B)_{kj} = (A)_{i1}(B)_{1j} + (A)_{i2}(B)_{2j} + ... + (A)_{in}(B)_{nj}.$$

   Since we assume that $i \neq j$ it follows that for every $1 \leq k \leq n$, at least one of the following holds: $i \neq k$ or $k \neq j$. Since A and B are diagonal this means that at least one of the following holds: $(A)_{ik} = 0$ or $(B)_{kj} = 0$. This means that for every $1 \leq k \leq n$ we have: $(A)_{ik}(B)_{kj} = 0$, so all the summands in the sum above are equal zero and therefore $(AB)_{ij} = 0$.

ii. We have

$$tr(AB) = \sum_{i=1}^{n} (AB)_{ii} = \sum_{i=1}^{n} (\sum_{k=1}^{n} (A)_{ik}(B)_{ki}).$$

   Since what that is written here is just a sum of sums, we do not need to write the parenthesis, and we are allowed to change the order of summation. So,

$$tr(AB) = \sum_{i=1}^{n} \sum_{k=1}^{n} (A)_{ik}(B)_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{n} (A)_{ik}(B)_{ki}.$$

   Since multiplication is commutative among real numbers, we get

$$tr(AB) = \sum_{k=1}^{n} \sum_{i=1}^{n} (B)_{ki}(A)_{ik} =$$

$$= \sum_{k=1}^{n} (\sum_{i=1}^{n} (B)_{ki}(A)_{ik}) = \sum_{k=1}^{n} (BA)_{kk} = tr(BA).$$

iii. By definition,

$$\left((AB)^T\right)_{ij} = (AB)_{ji} = \sum_{k=1}^{n} A_{jk}B_{ki} = \sum_{k=1}^{n} A^T_{kj} B^T ik = \sum_{k=1}^{n} B^T ik A^T_{kj} = (B^T A^T)_{ij}.$$

   Since they are equal at each and every entry, we conclude that indeed $(AB)^T = B^T A^T$.