MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 5

Note: We will refer to the vector space axioms in the following order:

1. $v + w \in V \qquad \forall v, w \in V$
2. $v + w = w + v \qquad \forall v, w \in V$
3. $v + (w + u) = (v + w) + u \qquad \forall v, w, u \in V$
4. $\exists 0_V \in V$ such that $w + 0_V = w \qquad \forall w \in V$
5. for every $w \in V$ $\exists - w \in V$ such that $w + (-w) = 0_V$.
6. $\alpha v \in V \qquad \forall v \in V \forall \alpha \in \mathbb{R}$
7. $(\alpha\beta)v = \alpha(\beta v) \qquad \forall v \in V \forall \alpha, \beta \in \mathbb{R}$
8. $1 \cdot v \qquad \forall v \in V$
9. $(\alpha + \beta)v = \alpha v + \beta v \qquad \forall v \in V \forall \alpha, \beta \in \mathbb{R}$
10. $\alpha(v + w) = \alpha v + \alpha w \qquad \forall v, w \in V \forall \alpha \in \mathbb{R}$

1. ii. Denote:

$$W = \{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} : x - y + 2z = 0\}.$$

This is a vector space with the prescribed operations. Proof:

Property 1: Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$ be two vectors in W. This implies that $v_1 - v_2 + 2v_3 = 0$ and that $w_1 - w_2 + 2w_3 = 0$. By the definition of the sum,

$$v + w = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \\ v_4 + w_4 \end{pmatrix}.$$

To check whether $v + w$ belongs to W we look at

$$(v_1 + w_1) - (v_2 + w_2) + 2(v_3 + w_3) = v_1 - v_2 + 2v_3 + w_1 - w_2 + 2w_3$$

from the equalities $v_1 - v_2 + 2v_3 = 0$ and $w_1 - w_2 + 2w_3 = 0$ it follows that the last expression is equal to zero so $v + w$ indeed belongs to W.

Property 2: Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$ be two vectors in W.
From the definition of the sum

$$v + w = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \\ v_4 + w_4 \end{pmatrix} =$$

Since the sum is commutative in $\mathbb{R}$ the last expression is equal two

$$\begin{pmatrix} w_1 + v_1 \\ w_2 + v_2 \\ w_3 + v_3 \\ w_4 + v_4 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} + \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = w + v$$

Property 3: Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix}$ and $u = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ be three vectors in W. From the definition of the sum

$$(v + w) + u = \left(\begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} \right) + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 \\ v_2 + w_2 \\ v_3 + w_3 \\ v_4 + w_4 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} =$$

$$= \begin{pmatrix} (v_1 + w_1) + u_1 \\ (v_2 + w_2) + u_2 \\ (v_3 + w_3) + u_3 \\ (v_4 + w_4) + u_4 \end{pmatrix} =$$

Since the sum is associative in $\mathbb{R}$ the last expression is equal two

$$= \begin{pmatrix} v_1 + (w_1 + u_1) \\ v_2 + (w_2 + u_2) \\ v_3 + (w_3 + u_3) \\ v_4 + (w_4 + u_4) \end{pmatrix} =$$

$$= \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} + \begin{pmatrix} w_1 + u_1 \\ w_2 + u_2 \\ w_3 + u_3 \\ w_4 + u_4 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} + \left(\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \right) = v + (w + u)$$

Property 4: The 4-tuple $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ belongs to W as $0 - 0 + 2 \cdot 0 = 0$ and it acts as the zero of the space. Indeed, let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ be in W. By the definition of the sum

$$v + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} v_1 + 0 \\ v_2 + 0 \\ v_3 + 0 \\ v_4 + 0 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} = v$$

Property 5: Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ be in W. This implies that $v_1 - v_2 + 2v_3 = 0$. Then $-v = \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \\ -v_4 \end{pmatrix}$ is also in W since $(-v_1) - (-v_2) + 2(-v_3) = -(v_1 - v_2 + 2v_3) = 0$ and it is the inverse additive of v. Indeed,

$$v + (-v) = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix} + \begin{pmatrix} -v_1 \\ -v_2 \\ -v_3 \\ -v_4 \end{pmatrix} = \begin{pmatrix} v_1 - v_1 \\ v_2 - v_2 \\ v_3 - v_3 \\ v_4 - v_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Property 6: Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$ be in W. This implies that $v_1 - v_2 + 2v_3 = 0$. Let $\alpha \in \mathbb{R}$. By the definition of multiplication by a scalar

$$\alpha v = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \\ \alpha v_3 \\ \alpha v_4 \end{pmatrix}.$$

So, to check if αv is in W we need to look at

$$\alpha v_1 - \alpha v_2 + 2\alpha v_3 = \alpha(v_1 - v_2 + 2v_3).$$

Since $v_1 - v_2 + 2v_3 = 0$ the last expression is also equal 0 and therefore αv is in W.

Properties 7,8,9,10 of this question are proved in the same way, with the same justification, as properties 7,8,9,10 from part (i) . This is much the same as the fact that properties 2,3 of this question are proved in the same way, with the same justification, as properties 2,3 from part (i).

iii. This is not a vector space. Property 8 does not hold. Indeed, consider $v = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ in $\mathbb{R}^2$ then:

$$1 \odot v = 1 \odot \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \neq v.$$

iv. This is not a vector space. For example, condition 2 does not hold. Indeed, consider $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$ then

$$v + w = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1+1 \\ 0+1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

while

$$w + v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+0 \\ 1+0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So, for these specific v and w, $v + w \neq w + v$.

v. This is a vector space. Proof:

Property 1: From the definition of the sum it is clear that the sum of two elements in $\mathbb{R}^2$ is also a 2-tuple and therefore is also in $\mathbb{R}^2$.

Property 2: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be two elements in $\mathbb{R}^2$. Then by the definition of the sum:

$$v \oplus w = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 - 3 \\ v_2 + w_2 - 2 \end{pmatrix}$$

Since the sum in $\mathbb{R}$ is commutative the last expression is equal to

$$\begin{pmatrix} w_1 + v_1 - 3 \\ w_2 + v_2 - 2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \oplus \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = w + v$$

Property 3: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be three elements in $\mathbb{R}^2$. Then by the definition of the sum:

On one hand,

$$(v \oplus w) \oplus u = \left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right) \oplus \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} v_1 + w_1 - 3 \\ v_2 + w_2 - 2 \end{pmatrix} \oplus \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} =$$

$$\begin{pmatrix} (v_1 + w_1 - 3) + u_1 - 3 \\ (v_2 + w_2 - 2) + u_2 - 2 \end{pmatrix} =$$

Since the sum in $\mathbb{R}$ is associative we get

$$(v \oplus w) \oplus u = \begin{pmatrix} v_1 + w_1 + u_1 - 6 \\ v_2 + w_2 + u_2 - 4 \end{pmatrix}$$

On the other hand

$$v \oplus (w \oplus u) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \left(\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \oplus \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} w_1 + u_1 - 3 \\ w_2 + u_2 - 2 \end{pmatrix} =$$

$$\begin{pmatrix} v_1 + (w_1 + u_1 - 3) - 3 \\ v_2 + (w_2 + u_2 - 2) - 2 \end{pmatrix} =$$

Since the sum in $\mathbb{R}$ is associative we get

$$v \oplus (w \oplus u) = \begin{pmatrix} v_1 + w_1 + u_1 - 6 \\ v_2 + w_2 + u_2 - 4 \end{pmatrix}$$

So,

$$v \oplus (w \oplus u) = v \oplus (w \oplus u).$$

Property 4: The 2-tuple $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ belongs to $\mathbb{R}^2$ and acts as the zero of the space. Indeed, let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be in $\mathbb{R}^2$ then,

$$v \oplus \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} v_1 + 3 - 3 \\ v_2 + 2 - 2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v$$

Property 5: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be in $\mathbb{R}^2$ then $-v = \begin{pmatrix} -v_1 + 6 \\ -v_2 + 4 \end{pmatrix}$ belongs to $\mathbb{R}^2$ and acts as the additive inverse of v. Indeed,

$$v \oplus (-v) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} -v_1 + 6 \\ -v_2 + 4 \end{pmatrix} = \begin{pmatrix} v_1 + (-v_1 + 6) - 3 \\ v_2 + (-v_2 + 4) - 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}.$$

Where in property 4 we found that $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ is the zero of the space.

Property 6: From the definition of scalar multiplication it is clear that the multiplication of a 2-tuple with a scalar in this space will again be a 2-tuple and therefore in $\mathbb{R}^2$.

Property 7: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be in $\mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$. Then on one hand,

$$(\alpha\beta) \odot v = (\alpha\beta) \odot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \alpha\beta v_1 - 3\alpha\beta + 3 \\ \alpha\beta v_2 - 2\alpha\beta + 2 \end{pmatrix}.$$

while on the other hand

$$\alpha \odot (\beta \odot v) = \alpha\Big(\beta \odot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\Big) = \alpha \odot \begin{pmatrix} \beta v_1 - 3\beta + 3 \\ \beta v_2 - 2\beta + 2 \end{pmatrix} =$$

$$\begin{pmatrix} \alpha(\beta v_1 - 3\beta + 3) - 3\alpha + 3 \\ \alpha(\beta v_2 - 2\beta + 2) - 3\alpha + 3 \end{pmatrix} = \begin{pmatrix} \alpha\beta v_1 - 3\alpha\beta + 3 \\ \alpha\beta v_2 - 2\alpha\beta + 2 \end{pmatrix}.$$

So we got that,

$$(\alpha\beta) \odot v = \alpha \odot (\beta \odot v).$$

Property 9: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be in $\mathbb{R}^2$ and $\alpha, \beta \in \mathbb{R}$. Then on one hand,

$$(\alpha + \beta) \odot v = (\alpha + \beta) \odot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} =$$

$$= \begin{pmatrix} (\alpha+\beta)v_1 - 3(\alpha+\beta) + 3 \\ (\alpha+\beta)v_2 - 2(\alpha+\beta) + 2 \end{pmatrix} = \begin{pmatrix} \alpha v_1 + \beta v_1 - 3\alpha - 3\beta + 3 \\ \alpha v_2 + \beta v_2 - 2\alpha - 2\beta + 2 \end{pmatrix}.$$

While on the other hand,

$$(\alpha \odot v) \oplus (\beta \odot v) = \begin{pmatrix} \alpha v_1 - 3\alpha + 3 \\ \alpha v_2 - 2\alpha + 2 \end{pmatrix} \oplus \begin{pmatrix} \beta v_1 - 3\beta + 3 \\ \beta v_2 - 2\beta + 2 \end{pmatrix} =$$

$$\begin{pmatrix} (\alpha v_1 - 3\alpha + 3) + (\beta v_1 - 3\beta + 3) - 3 \\ (\alpha v_2 - 2\alpha + 2) + (\beta v_2 - 2\beta + 2) - 2 \end{pmatrix} = \begin{pmatrix} \alpha v_1 + \beta v_1 - 3\alpha - 3\beta + 3 \\ \alpha v_2 + \beta v_2 - 2\alpha - 2\beta + 2 \end{pmatrix}.$$

So, we get that

$$(\alpha + \beta) \odot v = (\alpha \odot v) \oplus (\beta \odot v).$$

Property 9: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be in $\mathbb{R}^2$ and $\alpha \in \mathbb{R}$. Then on one hand,

$$\alpha \odot (v \oplus w) = \alpha \odot (\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}) = \alpha \odot \begin{pmatrix} v_1 + w_1 - 3 \\ v_2 + w_2 - 2 \end{pmatrix} =$$

$$= \begin{pmatrix} \alpha(v_1 + w_1 - 3) - 3\alpha + 3 \\ \alpha(v_2 + w_2 - 2) - 2\alpha + 2 \end{pmatrix} = \begin{pmatrix} \alpha v_1 + \alpha w_1 - 6\alpha + 3 \\ \alpha v_2 + \alpha w_2 - 4\alpha + 4 \end{pmatrix}.$$

While on the other hand,

$$(\alpha \odot v) \oplus (\alpha \odot w) = (\alpha \odot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}) \oplus (\alpha \odot \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}) = \begin{pmatrix} \alpha v_1 - 3\alpha + 3 \\ \alpha v_2 - 2\alpha + 2 \end{pmatrix} \oplus \begin{pmatrix} \alpha w_1 - 3\alpha + 3 \\ \alpha w_2 - 2\alpha + 2 \end{pmatrix} =$$

$$\begin{pmatrix} (\alpha v_1 - 3\alpha + 3) + (\alpha w_1 - 3\alpha + 3) - 3 \\ (\alpha v_2 - 2\alpha + 2) + (\alpha w_2 - 2\alpha + 2) - 2 \end{pmatrix} = \begin{pmatrix} \alpha v_1 + \alpha w_1 - 6\alpha + 3 \\ \alpha v_2 + \alpha w_2 - 4\alpha + 4 \end{pmatrix}.$$

So we get that

$$\alpha \odot (v \oplus w) = (\alpha \odot v) \oplus (\alpha \odot w)$$

Property 8: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be in $\mathbb{R}^2$ then

$$1 \odot v = 1 \odot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 \cdot v_1 - 3 \cdot 1 + 3 \\ 1 \cdot v_2 - 2 \cdot 1 + 2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v$$

vi. This is not a vector space. For example, property 8 does not hold. Indeed, consider $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ which is in $\mathbb{R}^2$ then,

$$1 \odot v = 1 \odot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \cdot 1 \\ 2 \cdot 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \neq v.$$

vii. Denote $W = \{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R} : x_1 > 0, x_2 > 0\}$. This is a vector space with the given operations. Proof:
Property 1: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be in W. This implies that $v_1, v_2, w_1, w_2 > 0$. Then

$$v \oplus w = \begin{pmatrix} v_1 w_1 \\ v_2 w_2 \end{pmatrix}.$$

Since $v_1, v_2, w_1, w_2 > 0$ we get that $v_1 w_1 > 0$ and $v_2 w_2 > 0$. So $v \oplus w$ is in W.
Property 2: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ be in W. Then,

$$v \oplus w = \begin{pmatrix} v_1 w_1 \\ v_2 w_2 \end{pmatrix} =$$

Since the product in $\mathbb{R}$ is commutative the last expression is equal to:

$$\begin{pmatrix} w_1 v_1 \\ w_2 v_2 \end{pmatrix} = w \oplus v.$$

Property 3: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, $w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$ and $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ be in W. Then,

$$(v \oplus w) \oplus u = \begin{pmatrix} v_1 w_1 \\ v_2 w_2 \end{pmatrix} \oplus \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} (v_1 w_1) u_1 \\ (v_2 w_2) u_2 \end{pmatrix}$$

Since the product in $\mathbb{R}$ is associative the last expression is equal to:

$$\begin{pmatrix} v_1 (w_1 u_1) \\ v_2 (w_2 u_2) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} w_1 u_1 \\ w_2 u_2 \end{pmatrix} = v \oplus (w \oplus u)$$

Property 4: The 2-tuple $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ belongs to W as both of its entries are positive. It acts as the zero of the space. Indeed, let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be in W then

$$v \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \cdot 1 \\ v_2 \cdot 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v$$

Property 5: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ be in W. This implies that $v_1, v_2 > 0$ then $-v = \begin{pmatrix} v_1^{-1} \\ v_2^{-1} \end{pmatrix}$ is defined as v_1 and v_2 are both different from zero. Since $v_1, v_2 > 0$ then $v_1^{-1}, v_2^{-1} > 0$ and therefore $-v$ belongs to W. We claim that this is the additive inverse of v. Indeed,

$$v \oplus (-v) = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \oplus \begin{pmatrix} v_1^{-1} \\ v_2^{-1} \end{pmatrix} = \begin{pmatrix} v_1 v_1^{-1} \\ v_2 v_2^{-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

where in property 4 we found that $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the zero of the space.

Property 6: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in W and α in $\mathbb{R}$. This implies that $v_1, v_2 > 0$. Then

$$\alpha \odot v = \begin{pmatrix} v_1^\alpha \\ v_2^\alpha \end{pmatrix}.$$

Since $v_1, v_2 > 0$ we get that $v_1^\alpha > 0$ and $v_2^\alpha > 0$. So $\alpha \odot v$ is in W.

Property 7: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in W and $\alpha, \beta \in \mathbb{R}$. Then on one hand,

$$(\alpha\beta) \odot v = \begin{pmatrix} v_1^{\alpha\beta} \\ v_2^{\alpha\beta} \end{pmatrix}.$$

While on the other hand

$$\alpha \odot (\beta \odot v) = \alpha \odot \begin{pmatrix} v_1^\beta \\ v_2^\beta \end{pmatrix} = \begin{pmatrix} (v_1^\beta)^\alpha \\ (v_2^\beta)^\alpha \end{pmatrix} = \begin{pmatrix} v_1^{\alpha\beta} \\ v_2^{\alpha\beta} \end{pmatrix}.$$

So we get that

$$(\alpha\beta) \odot v = \alpha \odot (\beta \odot v).$$

Property 8: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in W. Then

$$1 \odot v = 1 \odot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} v_1^1 \\ v_2^1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = v$$

Property 9: Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in W and $\alpha, \beta \in \mathbb{R}$. Then on one hand

$$(\alpha + \beta) \odot v = \begin{pmatrix} v_1^{(\alpha+\beta)} \\ v_2^{(\alpha+\beta)} \end{pmatrix} = \begin{pmatrix} v_1^\alpha v_1^\beta \\ v_2^\alpha v_2^\beta \end{pmatrix}.$$

While on the other hand,

$$(\alpha \odot v) \oplus (\beta \odot v) = \begin{pmatrix} v_1^\alpha \\ v_2^\alpha \end{pmatrix} \oplus \begin{pmatrix} v_1^\beta \\ v_2^\beta \end{pmatrix} = \begin{pmatrix} v_1^\alpha v_1^\beta \\ v_2^\alpha v_2^\beta \end{pmatrix}.$$

So we get that

$$(\alpha+\beta)\odot v=(\alpha\odot v)\oplus(\beta\odot v).$$

Property 10: Let $v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ and $v=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ in W and let $\alpha\in\mathbb{R}$. Then on one hand

$$\alpha\odot(v\oplus w)=\alpha\odot\begin{pmatrix} v_1w_1 \\ v_2w_2 \end{pmatrix}=\begin{pmatrix} (v_1w_1)^\alpha \\ (v_2w_2)^\alpha \end{pmatrix}=\begin{pmatrix} v_1^\alpha w_1^\alpha \\ v_2^\alpha w_2^\alpha \end{pmatrix}$$

While on the other hand,

$$(\alpha\odot v)\oplus(\alpha\odot w)=\begin{pmatrix} v_1^\alpha \\ v_2^\alpha \end{pmatrix}\oplus\begin{pmatrix} w_1^\alpha \\ w_2^\alpha \end{pmatrix}=\begin{pmatrix} v_1^\alpha w_1^\alpha \\ v_2^\alpha w_2^\alpha \end{pmatrix}.$$

So we get that

$$\alpha\odot(v\oplus w)=(\alpha\odot v)\oplus(\alpha\odot w).$$

2. i We first use property (8) to get: $2v+v=2v+1\cdot v$ we then use property (9) and find that $2v+v=2v+1\cdot v=(2+1)v=3v$.

   ii. From property (4) we know that $0_V=0_V+0_V$. So,

   $$\alpha\cdot 0_V=\alpha(0_V+0_V).$$

   From property (10) we therefore get

   $$\alpha\cdot 0_V=\alpha\cdot 0_V+\alpha\cdot 0_V.$$

   From property (5) the vector $\alpha\cdot 0_V$ has an additive inverse in V, let us denote it by $-(\alpha\cdot 0_V)$. We now add this vector to both sides of the equation we obtained:

   $$\alpha\cdot 0_V+(-(\alpha\cdot 0_V))=(\alpha\cdot 0_V+\alpha\cdot 0_V)+(-(\alpha\cdot 0_V)).$$

   From property (3):

   $$\alpha\cdot 0_V+(-(\alpha\cdot 0_V))=\alpha\cdot 0_V+\Big(\alpha\cdot 0_V+(-(\alpha\cdot 0_V))\Big).$$

   From the definition of the additive inverse (property (5)) we get:

   $$0_V=\alpha\cdot 0_V+0_V.$$

   Finally, from property (4) we get:

   $$0_V=\alpha\cdot 0_V.$$

   iii. This follows almost immediately from the definition of the additive inverse: To prove that some vector x is the additive inverse of some vector y we need to show that $y+x=0_V$. So, to prove that v is the additive inverse of $-v$ we need to prove that $(-v)+v=0$. But since $(-v)$ is the additive inverse of v we have $v+(-v)=0_V$, applying property (2) we get the result.

iv. From property (2) we have:

$$(u+w)+(v+z)=(w+u)+(v+z).$$

From property (3) the right hand side is equal to:

$$w+(u+(v+z)).$$

This completes the proof. (Note that when we applied property (3) we considered $(v+z)$ as one vector who has a very annoying notation).

3. i. The subset W is not a subspace, as it is not closed to multiplication by scalars: $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ is in W but $-2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -2 \\ -2 \end{pmatrix}$ is not in W.

ii. This is a subspace. By the theorem we studied in class it is enough to prove the following three things:

The set is not empty: putting $x=y=0$ we find that $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$.

The set is closed to summation: indeed, let $A = \begin{pmatrix} a_1 & 2a_1+3a_2 \\ a_2 & a_1-a_2 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & 2b_1+3b_2 \\ b_2 & b_1-b_2 \end{pmatrix}$ be in W, then

$$A+B = \begin{pmatrix} a_1+b_1 & 2a_1+3a_2+2b_1+3b_2 \\ a_2+b_2 & a_1-a_2+b_1-b_2 \end{pmatrix} =$$

$$= \begin{pmatrix} (a_1+b_1) & 2(a_1+b_1)+3(a_2+b_2) \\ (a_2+b_2) & (a_1+b_1)-(a_2+b_2) \end{pmatrix}.$$

So, by putting $x=(a_1+b_1)$ and $y=(a_2+b_2)$ we find that $A+B$ is in W.

The set is closed to multiplication by a scalar: indeed, let $B = \begin{pmatrix} b_1 & 2b_1+3b_2 \\ b_2 & b_1-b_2 \end{pmatrix}$ be in W and $\alpha \in \mathbb{R}$ then

$$\alpha B = \begin{pmatrix} \alpha b_1 & \alpha(2b_1+3b_2) \\ \alpha b_2 & \alpha(b_1-b_2) \end{pmatrix} = \begin{pmatrix} (\alpha b_1) & 2(\alpha b_1)+3(\alpha b_2) \\ (\alpha b_2) & (\alpha b_1)-(\alpha b_2) \end{pmatrix}.$$

So, by putting $x=(\alpha b_1)$ and $y=(\alpha b_2)$ we find that αB is in W.

iii. This is not a subspace. Indeed, we studied in class and in recitation that if V is a vector space and $W \subset V$ is a subspace of V then $0_V \in W$. However, the zero of $\mathbb{R}_4[x]$ is the constant function 0 (i.e., the function which satisfies $p(x)=0$ for every $x \in \mathbb{R}$), and this function does not belong to W (as it's value $p(1)$ equals 0 and not 1). So W is not a subspace.

iv. This is a subspace. By the theorem we studied in class it is enough to prove the following three things:

The set is not empty: indeed, the constant function 0 (i.e., the function which satisfies $p(x) = 0$ for every $x \in \mathbb{R}$) belongs to the set.

The set is closed to summation: indeed, let $p(x)$ and $q(x)$ be in W, this implies that $p(1) = q(1) = 0$. Their sum is the polynomial $(p+q)(x)$ which satisfies at every point $x \in \mathbb{R}$ the relation $(p+q)(x) = p(x)+q(x)$. In particular, inserting $x = 1$ we get $(p+q)(1) = p(1)+q(1) = 0+0 = 0$, so $(p+q)(x)$ belongs to W.

The set is closed to multiplication by scalar: indeed, let $p(x)$ be in W and $\alpha \in \mathbb{R}$, this implies that $p(1) = 0$. Their multiplication is the polynomial $(\alpha p)(x)$ which satisfies at every point $x \in \mathbb{R}$ the relation $(\alpha p)(x) = \alpha p(x)$. In particular, inserting $x = 1$ we get $(\alpha p)(1) = \alpha \cdot p(1) = \alpha \cdot 0 = 0$, so $(\alpha p)(x)$ belongs to W.

Remark. This subspace can be parameterized in the following way:

$$W = \{a_0+a_1x+a_2x^2+a_3x^3+a_4x^4 : a_0+a_1+a_2+a_3+a_4 = 0 \text{ and } a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}\} =$$

$$= \{-(a_1 + a_2 + a_3 + a_4) + a_1x + a_2x^2 + a_3x^3 + a_4x^4 : a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}\}.$$

This parametrization could be used to prove the claim that W is a subspace in much the same way that the parametrization in part (ii) was used. However, proving like that will result in a much longer and more annoying proof, so we chose to present the proof you see above.

v. The subset W is not a subspace as it is not closed to multiplication by scalars: $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is in W but $\sqrt{2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{2} \\ \sqrt{2} \\ \sqrt{2} \end{pmatrix}$ is not in W.

vi. This is a subspace. By the theorem we studied in class it is enough to prove the following three things:

The set is not empty: the matrix $\underline{0} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ satisfies the condition $A\underline{0} = \underline{0}$ and is therefore in W.

The set is closed to summation: Indeed, let $B, C \in W$, this implies that $AB = AC = \underline{0}$. Then, since we stated in class that matrix summation and multiplication are distributive, their sum satisfies: $A(B + C) = AB + AC = \underline{0} + \underline{0} = \underline{0}$. So, $B + C \in W$.

The set is closed to multiplication by scalar: Indeed, let $B \in W$ and $\alpha \in \mathbb{R}$. This implies that $AB = \underline{0}$. Then, since we stated in class that matrix multiplication associates with scalar multiplication: $A(\alpha B) = \alpha(AB) = \alpha \cdot \underline{0} = \underline{0}$. So, $\alpha B \in W$.

viii. This is a subspace. By the theorem we studied in class it is enough to prove the following three things:

The set is not empty: the constant function 0 (i.e., the function which satisfies $f(x) = 0$ for every $x \in \mathbb{R}$) belongs to the set since it is clearly twice differentiable and both its first and second derivatives are also the constant function 0 and therefore trivially satisfy the relation $f''(x) + 3f'(x) - f(x) = 0$ for all x in $\mathbb{R}$.

The set is closed to summation: Indeed, let $f, g \in W$, this implies that both f and g are twice differentiable over $\mathbb{R}$ and that $f''(x) + 3f'(x) - f(x) = g''(x) + 3g'(x) - g(x) = 0$ for all x in $\mathbb{R}$. Their sum $(f + g)(x)$ is the function defined by the relation $(f + g)(x) = f(x) + g(x)$ for all x in $\mathbb{R}$.
We know from calculous courses that the sum of two differential functions is differential and moreover $(f+g)'(x) = f'(x)+g'(x)$ for all $x \in \mathbb{R}$. Similarly, $(f+g)(x)$ is twice differentiable and $(f+g)''(x) = f''(x)+g''(x)$ for all $x \in \mathbb{R}$. So, for all $x \in \mathbb{R}$ we have,

$$(f + g)''(x) + 3(f + g)'(x) - (f + g)(x) =$$
$$= f''(x) + g''(x) + 3(f'(x) + g'(x)) - (f(x) + g(x)) =$$
$$= (f''(x) + 3f'(x) - f(x)) + (g''(x) + 3g'(x) - g(x)) = 0 + 0 = 0.$$

This implies that $(f + g)(x) \in W$.

The set is closed to multiplication by scalar: Indeed, let $f \in W$ and $\alpha \in \mathbb{R}$. This implies that f is twice differentiable over $\mathbb{R}$ and that $f''(x) + 3f'(x) - f(x) = 0$ for all x in $\mathbb{R}$. Their multiplication $(\alpha f)(x)$ is the function defined by the relation $(\alpha f)(x) = \alpha f(x)$ for all x in $\mathbb{R}$.
We know from calculous courses that the multiplication of a differential function by a constant is differential and moreover that $(\alpha f)'(x) = \alpha f'(x)$ for all $x \in \mathbb{R}$. Similarly, $(\alpha f)(x)$ is twice differentiable and $(\alpha f)''(x) = \alpha f''(x)$ for all $x \in \mathbb{R}$. So, for all $x \in \mathbb{R}$ we have,

$$(\alpha f)''(x) + 3(\alpha f)'(x) - (\alpha f)(x) = \alpha f''(x) + 3\alpha f'(x) - \alpha f(x) =$$
$$= \alpha(f''(x) + 3f'(x) - f(x)) = \alpha \cdot 0 = 0.$$

This implies that $(\alpha f)(x) \in W$.

4. i. The claim is true. Proof: to show that $U \cap W$ is also a subspace we will apply the theorem studied in class and show that the following three conditions hold:

The set is nonempty: We proved in class that the zero vector of a space always belongs to its subspaces so we have $0_V \in U$ and $0_V \in W$. It follows that $0_V \in U \cap W$ and therefore that $U \cap W$ is nonempty.

The set is closed to summation: Let $v, w \in U \cap W$ then $v, w \in U$ and $v, w \in W$. Since both U and W are subspaces they are both closed to summation. So, $v+w \in U$ and $v+w \in W$. It follows that $v+w \in U \cap W$.

The set is closed to multiplication by scalar: Let $v \in U \cap W$ and $\alpha \in \mathbb{R}$. Since both U and W are subspaces they are both closed to multiplication by scalar. So, $\alpha v \in U$ and $\alpha v \in W$. It follows that $\alpha v \in U \cap W$.

ii. The claim is false. There are many counterexamples here is one: Take $V = \mathbb{R}^2, U = \{\begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R}\}$, and $W = \{\begin{pmatrix} 0 \\ y \end{pmatrix} : x \in \mathbb{R}\}$.

It is easy to check that both U and W are subspaces of V, we will skip this here. However, $U \cup W$ is not a subspace, as it is not closed to summation. Indeed, the two-tuple $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ belongs to U and the two tuple $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ belongs to W, so they both belong to $U \cup W$, but their sum, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ does not belong to $U \cup W$ as it is neither in U nor in W.

iii. The claim is true. Proof: to show that $U + W$ is also a subspace we will apply the theorem studied in class and show that the following three conditions hold.

The set is nonempty: We proved in class that the zero vector of a space always belongs to its subspaces so we have $0_V \in U$ and $0_V \in W$. It follows that $0_V = 0_V + 0_V \in U + W$ and therefore that $U + W$ is nonempty.

The set is closed to multiplication by scalar: Let $v, w \in U + W$. That means that there exist vectors $v_1, w_1 \in U$ and vectors $v_2, w_2 \in W$ such that $v = v_1 + v_2$ and $w = w_1 + w_2$. This implies that

$$v + w = v_1 + v_2 + w_1 + w_2 = (v_1 + w_1) + (v_2 + w_2).$$

Since both U and W are both subspaces they are both closed to summation. So, $v_1 + w_1 \in U$ and $v_2 + w_2 \in W$. It follows that

$$v + w = (v_1 + w_1) + (v_2 + w_2) \in U + W.$$

The set is closed to summation: Let $v \in U + W$ and $\alpha \in \mathbb{R}$. That means that there exist vectors $v_1 \in U$ and $v_2 \in W$ such that $v = v_1 + v_2$. This implies that

$$\alpha v = \alpha v_1 + \alpha v_2.$$

Since both U and W are both subspaces they are both closed to multiplication by scalar. So, $\alpha v_1 \in U$ and $\alpha v_2 \in W$. It follows that

$$\alpha v = \alpha v_1 + \alpha v_2 \in U + W.$$

5. The goal of this question was to encourage you to think of the geometry of $\mathbb{R}^2$ and $\mathbb{R}^3$ from the point of view of linear algebra. The hope was that you will attempt to solve these questions by first drawing pictures of the appropriate sets.

i. Any union of two lines in $\mathbb{R}^2$ will satisfy this condition. Say the union

$$\{\begin{pmatrix} x \\ 0 \end{pmatrix} : x \in \mathbb{R}\} \cup \{\begin{pmatrix} 0 \\ y \end{pmatrix} : x \in \mathbb{R}\}.$$

We have seen this example also in the solution to Q4(ii) and proved there that it is not closed to summation. It is easy to check that this set is closed to multiplication by scalar, we skip this.

The set $\{\begin{pmatrix} x \\ y \end{pmatrix} : xy \geq 0\}$ also provides such an example, it is recommended to draw a sketch of the set and first convince yourself that this is a good example through the drawing, and then check computationally that it indeed has the required properties (compare to Q3(vii)).

These examples can give you ideas for many other examples.

ii. Any 'half line' will provide such an example, say $\{\begin{pmatrix} x \\ 0 \end{pmatrix} : x > 0\}$. It is easy to check that it satisfies the required properties so we skip this.

Another type of examples are represented by the following example: $\{\begin{pmatrix} x \\ y \end{pmatrix} : x > 0, y > 0\}$.

A third type is represented by the following two examples $\{\begin{pmatrix} x \\ y \end{pmatrix} :$ x, y are both integers $\}$ and $\{\begin{pmatrix} x \\ y \end{pmatrix} : x, y$ are both rational numbers $\}$ (compare this last one to Q3(v)).

These examples can give you ideas for many other examples.

iii. There are many examples for this, "most" of the subsets of $\mathbb{R}^2$ will satisfy these requirements. For example: the set $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\}$ which contains exactly 2 vectors, but does not contain their sum or any multiplication of either of them, by a scalar different from 1.

iv. A reasonable guess after our discussion in class about $\mathbb{R}^2$ will be that $\mathbb{R}^3$ has exactly these types of subspaces:

The trivial subspace $\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\}$ — This is a subspace of dimension 0.

The subspaces which are strait lines going through the origin— These are a subspaces of dimension 1.
The subspaces which are plains going through the origin— These are a subspaces of dimension 2.
The trivial subspace $\mathbb{R}^3$ — This is a subspace of dimension 3.

We will show in class that indeed, these are exactly all of the subspaces in $\mathbb{R}^3$.