MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
HW2; Solutions

Remark: In several places in this file the term 'iff' is used as a short hand for 'if and only if'.

1. i. We start by writing the linear system in an augmented matrix and continue by performing row operations to obtain an echelon form matrix.

$$\left(\begin{array}{ccc|c} 5 & 4 & -1 & 0 \\ 2 & -4 & 1 & 1 \\ -7 & -14 & 5 & 10 \end{array}\right) \xRightarrow{R_1-2R_2} \left(\begin{array}{ccc|c} 1 & 12 & -3 & -2 \\ 2 & -4 & 1 & 1 \\ -7 & -14 & 5 & 10 \end{array}\right) \xRightarrow[R_3+7R_1]{R_2-2R_1}$$
$$\left(\begin{array}{ccc|c} 1 & 12 & -3 & -2 \\ 0 & -28 & 7 & 5 \\ 0 & 70 & -16 & -4 \end{array}\right) \xRightarrow{\frac{1}{2}R_3} \left(\begin{array}{ccc|c} 1 & 12 & -3 & -2 \\ 0 & -28 & 7 & 5 \\ 0 & 35 & -8 & -2 \end{array}\right) \xRightarrow{R_3+R_2}$$
$$\left(\begin{array}{ccc|c} 1 & 12 & -3 & -2 \\ 0 & -28 & 7 & 5 \\ 0 & 7 & -1 & 3 \end{array}\right) \xRightarrow{R_3\leftrightarrow R_2} \left(\begin{array}{ccc|c} 1 & 12 & -3 & -2 \\ 0 & 7 & -1 & 3 \\ 0 & -28 & 7 & 5 \end{array}\right) \xRightarrow{R_3+4R_2}$$
$$\left(\begin{array}{ccc|c} 1 & 12 & -3 & -2 \\ 0 & 7 & -1 & 3 \\ 0 & 0 & 3 & 17 \end{array}\right)$$

From here we obtain the linear system:

$$\begin{cases} x+12y-3z=-2 \\ 7y-z=3 \\ 3z=17 \end{cases}$$

This system has only one solution, as there are no free variables, $\begin{pmatrix} \frac{1}{7} \\ 1\frac{5}{21} \\ 5\frac{2}{3} \end{pmatrix}$.

ii. We solve in the same way.

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 2 \\ -1 & 1 & -1 & 2 \\ 3 & 2 & 1 & 2 \\ 5 & 4 & -1 & 2 \end{array}\right) \xRightarrow{R_1\leftrightarrow R_2} \left(\begin{array}{ccc|c} -1 & 1 & -1 & 2 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 1 & 2 \\ 5 & 4 & -1 & 2 \end{array}\right) \xRightarrow{-R_1}$$
$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 1 & 2 \\ 5 & 4 & -1 & 2 \end{array}\right) \xRightarrow[R_4-5R_1]{R_2-2R_1, R_3-3R_1} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 3 & 0 & 6 \\ 0 & 5 & -2 & 8 \\ 0 & 9 & -6 & 12 \end{array}\right) \xRightarrow{\frac{1}{3}R_2, \frac{1}{3}R_4}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 5 & -2 & 8 \\ 0 & 3 & -2 & 4 \end{array}\right) \xrightarrow[R_4-3R_2]{R_3-5R_2} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & -2 & -2 \end{array}\right) \xrightarrow{R_4-R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\frac{1}{-2}R_3} \left(\begin{array}{ccc|c} 1 & -1 & 1 & -2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

From here we obtain the linear system:

$$\begin{cases} x - y + z = -2 \\ y = 2 \\ z = 1 \end{cases}$$

This system has only one solution, as there are no free variables, $\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$.

iii. We solve in the same way.

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 2 & -2 & 4 & 1 & 2 & 4 \\ 3 & 1 & 6 & 0 & 1 & -3 \\ 1 & 0 & 2 & 2 & 1 & 4 \end{array}\right) \xrightarrow[R_4-R_1]{R_2-2R_1, R_3-3R_1} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 4 & 0 & 0 & 1 & -3 \\ 0 & 1 & 0 & 2 & 1 & 4 \end{array}\right) \xrightarrow{R_4 \leftrightarrow R_2}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 4 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right) \xrightarrow{R_3-4R_2} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & -8 & -3 & -19 \\ 0 & 0 & 0 & 1 & 2 & 4 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_4}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & -8 & -3 & -19 \end{array}\right) \xrightarrow{R_4+8R_3} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 13 & 13 \end{array}\right) \xrightarrow{\frac{1}{13}R_4}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 4 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{array}\right)$$

From here we obtain the linear system:

$$\begin{cases} x_1 - x_2 + 2x_3 = 0 \\ x_2 + 2x_4 + x_5 = 4 \\ x_4 + 2x_5 = 4 \\ x_5 = 1 \end{cases}$$

We see that x_1, x_2, x_4, x_5 are all leading (pivot) variables and x_3 is a free variable. So x_3 can be any number and we denote this by $x_3 = t$ where

$t \in \mathbb{R}$. We get the set of solutions,

$$\left\{\begin{pmatrix} -1-2t \\ -1 \\ t \\ 2 \\ 1 \end{pmatrix} : t \in \mathbb{R}\right\} = \left\{\begin{pmatrix} -1 \\ -1 \\ 0 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} : t \in \mathbb{R}\right\}.$$

iv. We solve in the same way.

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 5 \\ 1 & 4 & 6 & 8 & 10 & 10 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ -1 & 4 & 1 & 6 & 3 & 5 \end{array}\right) \xrightarrow[R_4+R_1]{R_2-R_1, R_3-R_1} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 5 \\ 0 & 2 & 3 & 4 & 5 & 5 \\ 0 & -3 & -2 & -5 & -4 & -5 \\ 0 & 6 & 4 & 10 & 8 & 10 \end{array}\right) \xrightarrow[\frac{1}{2}R_4]{R_2+R_3}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 5 \\ 0 & -1 & 1 & -1 & 1 & 0 \\ 0 & -3 & -2 & -5 & -4 & -5 \\ 0 & 3 & 2 & 5 & 4 & 5 \end{array}\right) \xrightarrow[-R_2]{R_4+R_3} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 5 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & -3 & -2 & -5 & -4 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_3+3R_2}$$

$$\left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 5 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & -5 & -2 & -7 & -5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{-R_3} \left(\begin{array}{ccccc|c} 1 & 2 & 3 & 4 & 5 & 5 \\ 0 & 1 & -1 & 1 & -1 & 0 \\ 0 & 0 & 5 & 2 & 7 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

From here we obtain the linear system:

$$\begin{cases} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = 5 \\ x_2 - x_3 + x_4 - x_5 = 0 \\ 5x_3 + 2x_4 + 7x_5 = 5 \\ 0 = 0 \end{cases}$$

We see that x_1, x_2, x_3 are all leading (pivot) variables while x_4, x_5 are free variable. So x_4 can be any number and we denote this by $x_4 = t$ where $t \in \mathbb{R}$ and x_5 can also be any number which we denote by $x_5 = s$ where $s \in \mathbb{R}$. We get the set of solutions,

$$\left\{\begin{pmatrix} 0 \\ 1 - 1\frac{2}{5}t - \frac{2}{5}s \\ 1 - \frac{2}{5}t - 1\frac{2}{5}s \\ t \\ s \end{pmatrix} : t, s \in \mathbb{R}\right\} = \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -1\frac{2}{5} \\ -\frac{2}{5} \\ 1 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ -\frac{2}{5} \\ -1\frac{2}{5} \\ 0 \\ 1 \end{pmatrix} : t, s \in \mathbb{R}\right\}.$$

Let us write this set in a nicer way. Since t and s can be any two real numbers, we can substitute $5t$ instead of t and $5s$ instead of s, the set will remain the same (Why? make sure you understand this last step!).

We get that the set of solutions is

$$\{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ -7 \\ -2 \\ 5 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ -2 \\ -7 \\ 0 \\ 5 \end{pmatrix} : t, s \in \mathbb{R}\}.$$

2. a. We start as in question 1, but we proceed carefully, and make sure that we never divide or multiply a row by a scalar that could be zero .

$$\left(\begin{array}{ccc|c} 1 & 1 & a & -1 \\ -1 & a-1 & 2-a & a+1 \\ 6 & 5a+6 & 7a+7 & a^2 \end{array}\right) \xrightarrow[R_3-6R_1]{R_2+R_1} \left(\begin{array}{ccc|c} 1 & 1 & a & -1 \\ 0 & a & 2 & a \\ 0 & 5a & a+7 & a^2+6 \end{array}\right) \xrightarrow{R_3-5R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & a & -1 \\ 0 & a & 2 & a \\ 0 & 0 & a-3 & a^2-5a+6 \end{array}\right) \xrightarrow{a^2-5a+6=(a-2)(a-3)} \left(\begin{array}{ccc|c} 1 & 1 & a & -1 \\ 0 & a & 2 & a \\ 0 & 0 & a-3 & (a-2)(a-3) \end{array}\right)$$

If $a \neq 0$ the matrix is now in echelon form, if $a = 0$ the matrix is not yet in echelon form and we need to continue performing row reductions. So, we divide our discussion into these two cases and proceed with each one separately.

Case 1: $a \neq 0$. In this case the matrix is now in echelon form. When in echelon form we are able to immediately identify how many solutions the corresponding linear system has.

(i) The system has no solution iff there exists a row with zeroes on the left of the line and a number different from zero on the right of the line. In our case this is a possibility only in the third row, which will be zero on the left of the line only when $a = 3$, but in that case the row is zero also on the right of the line. We conclude that no value of a from 'Case 1' gives no solution.

(ii) The system has exactly one solution if we know that it has a solution (so, if there is no row with a 'lie'), and there are no free variables. Here we see that if a belongs to 'Case 1', (that is $a \neq 0$), and $a \neq 3$ then there are no free variables and the system has exactly one solution.

(iii) The system has an infinite amount of solutions if we know that it has a solution (so, if there is no row with a 'lie'), and there is at least one free variables. Here we see that this happens if a is in 'Case 1' and $a = 3$.

Case 2: $a = 0$. In this case the matrix is not yet in echelon form. Let us write the matrix we obtained, inserting $a = 0$, and proceed to obtain an echelon form matrix.

$$\left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & -3 & 6 \end{array}\right) \xrightarrow{R_3+\frac{3}{2}R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{array}\right)$$

We arrive at a row with a 'lie' (formally: 'false'): $0 = 6$, this means that in this case the system has no solution.
We are now ready to sum things up and provide the final answer to the question:

(i) The system has no solution iff $a = 0$.
(ii) The system has exactly one solution iff $a \neq 0, 3$
(iii) The system has an infinite amount of solutions iff $a = 3$.

b. We start as in part a.

$$\left(\begin{array}{ccc|c} a+1 & a & -a & 2+a \\ a+1 & a+2 & -a-2 & a+4 \\ a+1 & a & a^2-6 & a^2-2a+4 \\ 2a+2 & 2a & a^2-a-6 & a^2-a+6 \end{array}\right) \xrightarrow[R_4-2R_1]{R_2-R_1, R_3-R_1}$$

$$\left(\begin{array}{ccc|c} a+1 & a & -a & 2+a \\ 0 & 2 & -2 & 2 \\ 0 & 0 & a^2+a-6 & a^2-3a+2 \\ 0 & 0 & a^2+a-6 & a^2-3a+2 \end{array}\right) \xrightarrow{R_4-R_3}$$

$$\left(\begin{array}{ccc|c} a+1 & a & -a & 2+a \\ 0 & 2 & -2 & 2 \\ 0 & 0 & a^2+a-6 & a^2-3a+2 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{a^2+a-6=(a-2)(a+3), a^2-3a+2=(a-2)(a-1)}$$

$$\left(\begin{array}{ccc|c} a+1 & a & -a & 2+a \\ 0 & 2 & -2 & 2 \\ 0 & 0 & (a-2)(a+3) & (a-2)(a-1) \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The matrix is now in echelon form for all $a \neq -1$. If $a = -1$ then the matrix is not yet in echelon form. As in the previous question, we divide to two cases.

Case 1: $a \neq -1$**.** In this case the matrix is now in echelon form. When in echelon form we are able to immediately identify how many solutions the corresponding linear system has.

(i) The system has no solution iff there exists a row with zeroes on the left of the line and a number different from zero on the right of the line. In our case this is a possibility only in the third row, which will be zero on the left of the line only when $a = -3$ or $a = 2$. In the case that $a = 2$ the row is zero also on the right of the line. So we conclude that in 'case 1' the system has no solution iff $a = -3$.

(ii) The system has exactly one solution if we know that it has a solution (so, if there is no row with a 'lie'), and there are no free variables. Here we see that in 'case 1' this happens iff $a \neq -3, 2$.

(iii) The system has an infinite amount of solutions if we know that it has a solution (so, if there is no row with a 'lie'), and there is at least one free variables. Here we see that in 'case 1' this happens iff $a = 2$.

Case 2: $a = -1$**.** In this case the matrix is not yet in echelon form. Let us write the matrix we obtained, inserting $a = -1$, and proceed to obtain an echelon form matrix.

$$\left(\begin{array}{ccc|c} 0 & -1 & 1 & 1 \\ 0 & 2 & -2 & 2 \\ 0 & 0 & -6 & 6 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2+2R_1, \frac{1}{6}R_3} \left(\begin{array}{ccc|c} 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{ccc|c} 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We arrive at a row with a 'lie' (formally: 'false'): $0 = 4$, this means that in this case the system has no solution.

We are now ready to sum things up and provide the final answer to the question:

(i) The system has no solution iff $a = -1, -3$.

(ii) The system has exactly one solution iff $a \neq -3, 2, -1$

(iii) The system has an infinite amount of solutions iff $a = 2$.

3 i. Proof: Let us use the notations

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, d = \begin{pmatrix} d_1 \\ \vdots \\ d_m \end{pmatrix}, A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Since $b \in L(A)$ there exists $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ which is a solution to $(A|b)$, that is,

$$\begin{cases} a_{11}v_1 + \cdots + a_{1n}v_n = b_1 \\ \vdots \\ a_{m1}v_1 + \cdots a_{mn}v_n = b_n \end{cases} \tag{1}$$

In the same way, since $d \in L(A)$, there exists $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$ which is a solution to the equation $(A|d)$, that is,

$$\begin{cases} a_{11}w_1 + \cdots + a_{1n}w_n = d_1 \\ \vdots \\ a_{m1}w_1 + \cdots a_{mn}w_n = d_n \end{cases} \tag{2}$$

We claim that $v+w$ is a solution to $(A|b+d)$ and therefore $b+d \in L(A)$. To prove this claim we need to show that $v+w = (v_1+w_1, \ldots, v_n+w_n)$

solves the system:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = b_1 + d_1 \\ \vdots \\ a_{m1}x_1 + \cdots a_{mn}x_n = b_n + d_n \end{cases} \tag{3}$$

Let $1 \leq j \leq m$ we show that $v + w = (v_1 + w_1, \ldots, v_n + w_n)$ is a solution to the $j'th$ equation, indeed:

$$a_{j1}(v_1 + w_1) + \cdots + a_{jn}(v_n + w_n) =$$

$$= \big(a_{j1}v_1 + \cdots + a_{jn}v_n\big) + \big(a_{j1}w_1 + \cdots + a_{jn}w_n\big) = b_j + d_j,$$

where the last equality was given to us in (1) and (2). Since this holds for all $1 \leq j \leq m$ we get that $v + w$ solves each one of the equations in (3) and therefore is a solution to the linear system $(A|b + d)$. Since $(A|b + d)$ has a solution we conclude that $b + d \in L(A)$.

Remark: Once we studied the notion of multiplying an n-tuple by a matrix, and the properties of such a multiplication, the proof can be written in a much shorter way. We repeat the proof now using this terminology: Since $b \in L(A)$ there exists $v \in \mathbb{R}^n$ which is a solution to $(A|b)$, that is, $Av = b$. In the same way, since $d \in L(A)$, there exists $w \in \mathbb{R}^n$ which is a solution to the equation $(A|d)$, that is, $Aw = d$. We claim that $v + w$ is a solution to $(A|b + d)$ and therefore $b + d \in L(A)$. Indeed:

$$A(v + w) = Av + Aw = b + d$$

We get that $v + w$ is a solution to the linear system $(A|b + d)$. Since $(A|b + d)$ has a solution we conclude that $b + d \in L(A)$.

ii. Proof: Let us use the notations

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}, A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}.$$

Since $b \in L(A)$ there exists $v = (v_1, \ldots, v_n) \in \mathbb{R}^n$ which is a solution to $(A|b)$, that is,

$$\begin{cases} a_{11}v_1 + \cdots + a_{1n}v_n = b_1 \\ \vdots \\ a_{m1}v_1 + \cdots a_{mn}v_n = b_n \end{cases} \tag{4}$$

We claim that tv is a solution to $(A|tb)$ and therefore $tb \in L(A)$. To prove this claim we need to show that $tv = (tv_1, \ldots, tv_n)$ solves the system:

$$\begin{cases} a_{11}x_1 + \cdots + a_{1n}x_n = tb_1 \\ \vdots \\ a_{m1}x_1 + \cdots a_{mn}x_n = tb_n \end{cases} \tag{5}$$

Let $1 \leq j \leq m$ we show that $tv = (tv_1, \ldots, tv_n)$ is a solution to the $j'th$ equation, indeed:

$$a_{j1}(tv_1) + \cdots + a_{jn}(tv_n) = t\big(a_{j1}v_1 + \cdots + a_{jn}v_n\big) = tb_j,$$

where the last equality was given to us in (4). Since this holds for all $1 \leq j \leq m$ we get that tv solves each one of the equations in (5) and therefore is a solution to the linear system $(A|tb)$. Since $(A|tb)$ has a solution we conclude that $tb \in L(A)$.

Remark: The same type of remark holds for this part as to the previous part. Reprove the claim in this part using multiplication of n-tuple by a matrix, just as we did in the previous part.

4. a. Such a matrix A exists, in fact there are many such matrices. As an example of such a matrix we can take, say,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Indeed, for every $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$ the system $(A|b)$ has an infinite amount of solutions: the augmented matrix is already in echelon form and we can see that, for all $b \in \mathbb{R}^2$, on one hand there is no row with a 'lie', so the system has a solution, and on the other hand the third variable is free, so there is an infinite amount of solutions.

b. Such a matrix A exists, in fact there are many such matrices. As an example of such a matrix we can take, say,

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}.$$

Indeed, for $b_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ the system $(A|b)$ has an infinite amount of solutions while for $b_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ the system has no solution.

c. Such a matrix A does not exist. There are several different ways to prove this, here we demonstrate one way: We will assume that such a matrix exists and show that this leads to a contradiction. This will imply that such a matrix cannot exist.
So, assume that such a matrix A exists. On the one hand, since there exists b_1 such that $(A|b_1)$ has exactly one solution, it follows from the theorem we proved in class, regarding the relation between the set of solutions of a linear system and the set of solutions of the associated homogeneous system, that $(A|0)$ has exactly one solution. On the other hand, since there exists b_2 such that $(A|b_2)$ has an infinite amount of solutions, it follows from the same theorem that $(A|0)$ has an infinite

amount of solutions. So, on one hand $(A|0)$ has exactly one solution, and on the other hand it has an infinite amount of solutions. We reached a contradiction. Our claim is proved, no such matrix A exists.

d. Such a matrix A exists, in fact there are many such matrices. As an example of such a matrix we can take, say,

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

The echelon form of A is

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix},$$

which has a row of zeroes. On the other hand the echelon form of A has no free variables, so $(A|0)$ has exactly one solution, as requested.

e. No such A exists. We prove this by proving the following claim: If the echelon form of A has a row of zeroes then there exists $b \in R^n$ such that $(A|b)$ has no solution. Proof for the claim: Let R_1, R_2,...R_k be the sequence of row operations one needs to perform on A to obtain echelon form (in this order). Recall that we pointed out in class that every row operation has an inverse operation (so that performing on some matrix B a row operation, and then the inverse of this operation, will end up with the same matrix B that we started with). Denote the inverse of the row operations R_1, R_2,...R_k by $R_1^{-1}, R_2^{-1}, ..., R_k^{-1}$ respectively.
Let

$$e = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

be the n-tuple who's all but last entries are zero, and who's last entry is one. We perform on e the inverse of the row operations R_1, R_2,...R_k in opposite order. That is, we perform on e the operation R_k^{-1} then the operation R_{k-1}^{-1} and continue in this way until we perform the operation R_2^{-1} and finally R_1^{-1}. We obtain by this some n-tuple and denote it b. We notice that if we perform on b the operations R_1, R_2,...R_k, in that order, we will return to the n-tuple e.
Recall that we pointed out in class that row operations change each column separately, independently of all other columns. So, if we consider now the augmented matrix $(A|b)$ and perform on it the operations R_1, R_2,...R_k, in that order, we obtain a matrix in echelon form with a last row that is a 'lie': on the left of the line the row is all zeroes (as this was the condition on A) and on the right of the line we have a 1 (as on

the right of the line we now have the n-tuple e). This means that $(A|b)$ has no solution. With this our claim is proved.

Remark: It is much simpler to write a proof for this claim once one learned matrix multiplication and elementary matrices. The same question appears also in HW2 (though in a somewhat different formulation). In the solution to HW2 I will demonstrate how one can use matrix multiplication to express this proof more easily.

f. Such a matrix A does not exist. We will prove this by assuming that such a matrix exists and showing that this leads to a contradiction.
So, assume that such A exists. Note that since the 3-tuple $(1, 2, 3)$ is a solution to $(A|0)$, we know that in $(A|0)$ there are exactly 3 variables. Since the echelon form of A has 3 rows different from zero, and in each such row there has to be a different leading variable, we conclude that all of the variables are leading, and that no variable is free. This means that $(A|0)$ has exactly one solution. However, the 0 n-tuple is always a solution to a homogeneous system so $(0, 0, 0)$ is a solution. So, on one hand we concluded that $(A|0)$ has only one solution, while on the other hand we have that both $(1, 2, 3)$ and $(0, 0, 0)$ are solutions to $(A|0)$. We reached a contradiction and therefore the claim is proved.

g. Such a matrix A exists, in fact there are many such matrices. To find such a matrix we need to find 3-tuples (a, b, c) such that
$$a + 2b + 3c = 0,$$
each such 3 tuple can be a row in our matrix. We solve this equation, it has an infinite amount of solutions, we choose two non–zero solutions so that in one of them $a \neq 0$ and in the other $a = 0$. Say, $(1, 1, -1)$ and $(0, 3, -2)$. The corresponding matrix is
$$A = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \end{pmatrix}.$$
We chose the solutions so that A will already be in echelon form, and it indeed has two rows different from zero. Clearly, $(1, 2, 3)$ solves $(A|0)$, we chose the rows of A to satisfy this.

h. Such a matrix A exists, in fact there are many such matrices. To find such a matrix we note that the corresponding linear system has three variables, say x, y and z. The solution satisfies $x = s + t$, $y = s$ and $z = t$. So, (inserting y and z instead off s and t respectively) we find that the solutions satisfy $x = y + z$ or $x - y - z = 0$. It seems that the linear system, which contains only one equation: $x - y - z = 0$, is the linear system we were looking for. To make sure we solve this system and check that indeed its set of solutions is **exactly** the set $(s + t, s, t)$. This is simple to do here (we skip the computation), we find that indeed this was the required linear system. The corresponding matrix A is a 3×1 matrix $A = (1, -1, -1)$.

Remark: It is important in such exercises to check that we indeed have the required system. This is because the considerations we made to obtain it only proved that the set we were given is a subset of the set of solutions, these considerations did not prove that the two sets are equal, as was required. Hence, we needed to do the 'checking'.

i. This question is tricky. A TA I had in one of the previous times I taught this course thought of this question. The point is to notice that the set we were given satisfies

$$\{(s+t, s-t) : s, t \in \mathbb{R}\} = \mathbb{R}^2.$$

In other words, this set contains **all** of the 2-tuples. Let us prove this fact: let $(a, b) \in \mathbb{R}^2$, we wont to show that there exist $s, t \in \mathbb{R}$ so that $s + t = a$ and $s - t = b$. To check that this linear system has a solution (for all $(a, b) \in \mathbb{R}^2$), we write the corresponding augmented matrix:

$$\left(\begin{array}{cc|c} 1 & 1 & a \\ 1 & -1 & b \end{array}\right) \xRightarrow{R2-R_1} \left(\begin{array}{cc|c} 1 & 1 & a \\ 0 & -2 & b-a \end{array}\right).$$

We find that indeed, the echelon form has no row with a 'lie', which implies that there exists a solution to the system, for all $(a, b) \in \mathbb{R}^2$. We conclude that we were asked to find a linear system, with two variables, for which the set of solutions is **all** of $\mathbb{R}^2$. There are matrices with this property, here are two examples:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We note that the only matrices satisfying this property are matrices which have two columns, and all entries equal to zero.