MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 3

1. We discussed in class the concept of equivalence relations and then studied a very specific equivalence relation defined over the set $M_{m\times n}(\mathbb{R})$. The goal of this question is to review these concepts.

   i. Denote by S the set of all students in our class. Give examples of **three different** relations that one can define over the set S which are equivalence relations. Explain why you claim that they are equivalence relations. For example, you can check that the following relation is an equivalence relation: "For $a, b \in S$ we say that $a \sim b$ if b has the same amount of siblings as a does".

   ii. Denote by S the set of all students in our class. Give examples of **three different** relations that one can define over the set S which are **not (necessarily)** equivalence relations. Explain why you claim that they are **not (necessarily)** equivalence relations. For example: the relation defined by: "For $a, b \in S$ we say that $a \sim b$ if b enjoys the company of a", is not (necessarily) an equivalence relation as (unfortunately) it may happen that this relation is not symmetric (that is, b might enjoy the company of a while a does not enjoy the company of b so much). It also may happen that this relation is not transitive (that is, it can happen that c enjoys the company of b, and b enjoys that of a, however c does not enjoy being around a at all). In fact, this relation might even not be reflexive (it can happen that a person extremely suffers from the company of himself...).

   iii. The relation defined over $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 = x_2$ is an equivalence relation. Prove this fact. Can you describe the equivalence classes obtained by this equivalence relation?

   iv. The relation defined over $\mathbb{R}^2$ by $(x_1, y_1) \sim (x_2, y_2)$ if $x_1 + x_2 = 0$ is **not** an equivalence relation. Prove this fact.

   v. In class we defined a very specific relation over $M_{m\times n}(\mathbb{R})$ and proved that this specific relation is an equivalence relation. Complete this sentence: "For $A, B \in M_{m\times n}(\mathbb{R})$ we defined that $A \sim B$ if ...". Recall that in this case, we say that A and B are row equivalent.

   vi. For the specific equivalence relation defined in (v), give examples of 3 different matrices which are in the equivalence class of the matrix

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ -2 & -2 & 8 \end{pmatrix}.$$

   (That is, give examples of matrices which are row equivalent to this matrix).

2. In each of the following parts determine whether the two given matrices are row equivalent. (We discussed how such a question can be solved in class. Examples will be given in the next recitation.)

   i. $\begin{pmatrix} 1 & 3 \\ 3 & -1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$

   ii. $\begin{pmatrix} 1 & 1 & 1 \\ -1 & 2 & 2 \end{pmatrix}$ and $\begin{pmatrix} 0 & 3 & -1 \\ 2 & 2 & 5 \end{pmatrix}$

   iii. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ -2 & -2 & 8 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 1 & -1 & 1 \end{pmatrix}$

3. Prove the following statements.
   a. Let $A \in M_{m\times n}(\mathbb{R})$. If an echelon form of A has a row of zeroes then there exists $b \in \mathbb{R}^m$ such that $(A|b)$ has no solution.

   b. If $m > n$ and $A \in M_{m\times n}(\mathbb{R})$ then there exists $b \in \mathbb{R}^m$ such that $(A|b)$ has no solution.

   c. If $A \in M_n(\mathbb{R})$ is a square $n\times n$ matrix such that the homogenous system $(A|0)$ has infinity many solutions, then there exists $b \in \mathbb{R}^n$ such that $(A|b)$ has no solution.

4. Let $A \in M_n(\mathbb{R})$ be a square $n \times n$ matrix. Prove that the following two statements are equivalent, that is, prove that $a \Rightarrow b$ and $b \Rightarrow a$ (one can also say that "a is true if and only if b is true").

   a. The homogeneous system $(A|0)$ has exactly one solution.

   b. For every $b \in \mathbb{R}^n$ the linear system $(A|b)$ has a solution.

5. The following statements are **false**. Prove that they are **false** by providing a counterexample in each case (you may choose the numbers m and n to be whatever is convenient, try to work with small numbers).
   a. If $A \in M_{m\times n}(\mathbb{R})$ is a matrix such that the homogenous system $(A|0)$ has infinity many solutions, then there exists $b \in \mathbb{R}^n$ such that $(A|b)$ has no solution. (Note that by question (4), we know that this statement is in fact true if A is a square matrix).

   b. Let $A, B \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If A and B are row equivalent then $(A|b)$ and $(B|b)$ have the same amount of solutions.

   c. If $A, B \in M_{m\times n}(\mathbb{R})$ are row equivalent then B can be obtained from A by performing **column** operations (that is, by performing a sequence of operations of the form 'swapping two columns', 'multiplying a column by a scalar different from zero', 'adding to a column another column multiplied by a scalar').

d. Let $A \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If $u, v \in \mathrm{Sol}(A|b)$ then $u + v \in \mathrm{Sol}(A|b)$.

e. Let $A \in M_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$. If $u \in \mathrm{Sol}(A|b)$ and $t \in \mathbb{R}$ then $tu \in \mathrm{Sol}(A|b)$.