MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 6 – Solutions

1. In Midterm exam 1 you might be asked to solve questions similar to the ones here, without the use of 'basis' and 'coordinates', this is the way questions are solved in this HW solutions. In HW7 there will appear a few similar questions, asking for a solution using 'coordinates' while providing all the required justifications, you will be able to learn from the solutions to HW7 how I expect a complete solution using coordinates to look like.

   i. The vector is not in the span. Proof: By the definition of "span" the vector $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ is in this span iff it is a linear combination of the vectors $\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$. That is, it is in the span iff there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \alpha \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} + \beta \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} + \gamma \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}.$$

   We rewrite the linear combination written on the right hand side of the equation:

$$\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 2\alpha - \beta - 2\gamma & \beta + \gamma \\ \alpha + 3\beta + 2\gamma & -\alpha - \gamma \end{pmatrix}.$$

   Such an equality between matrices holds iff the matrices are equal at each entry, that is, iff

$$\begin{cases} 2\alpha - \beta - 2\gamma = 1 \\ \beta + \gamma = 2 \\ \alpha + 3\beta + 2\gamma = 2 \\ -\alpha - \gamma = -1 \end{cases}$$

So, the vector is in the span iff this linear system has a solution. Let us write it in matrix form and bring it to echelon form:

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ -1 & 0 & -1 & -1 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & -2 & 1 \\ -1 & 0 & -1 & -1 \end{array}\right) \xrightarrow{R_3-2R_1, R_4+R_1}$$
$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & -7 & -6 & -3 \\ 0 & 3 & 1 & 1 \end{array}\right) \xrightarrow{R_3+7R_2, R_4-3R_2} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & -2 & -5 \end{array}\right) \xrightarrow{R_4+2R_3}$$
$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 17 \end{array}\right).$$

This echelon form contains a row with a 'lie': $0 = 17$. This implies that the linear system has no solution, which means that the vector is not in the span.

ii. The vector is in the span. Proof: By the definition of "span" the vector $2 + 3x + 2x^2 - x^3$ is in this span iff it is a linear combination of the vectors $1 - x^3, 2 + x + x^2, 3 - x$. That is, it is in the span iff there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$2 + 3x + 2x^2 - x^3 = \alpha(1 - x^3) + \beta(2 + x + x^2) + \gamma(3 - x).$$

We rewrite the linear combination written on the right hand side of the equation:

$$2 + 3x + 2x^2 - x^3 = (\alpha + 2\beta + 3\gamma) + (\beta - \gamma)x + \beta x^2 - \alpha x^3.$$

Such an equality between polynomial holds (for every x) iff the coefficients of the polynomials are equal at each entry, that is, iff

$$\begin{cases} \alpha + 2\beta + 3\gamma = 2 \\ \beta - \gamma = 3 \\ \beta = 2 \\ -\alpha = -1 \end{cases}$$

So the vector is in the span iff this system has a solution. We could write it as a matrix and bring to echelon form, but this seems to be redundant here, as the system is easy to solve: The third equation holds iff $\beta = 2$, inserting this to the second equation we find that we need $\gamma = -1$. For the fourth equation we need $\alpha = 1$. So, the last three equations hold iff $\alpha = 1$, $\beta = 2$ and $\gamma = -1$. We insert these to the first equation to check if the system has a solution: We get $1 + 2 \cdot 2 + 3 \cdot (-1) = 2$, which is a true claim. So, the system does have a solution and therefore the vector is in the span.

iii. The claim is true. Proof— The claim:

$$\text{span}\{\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}\} \subseteq \text{span}\{\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\}$$

holds if and only if both of the vectors $\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$ belong to the span on the right hand side (RHS). Indeed, on one hand, we can apply the following result which we proved in class: Let V be a vector space and W a subspace of V. If $v_1, ..., v_n \in W$ then $\text{span}\{v_1, .., v_n\} \subseteq W$. From this result it follows that if both of the vectors belong to the span on the RHS (which is a subspace, as we proved in class) then their span is a subset of it. On the other hand if one of the vectors does not belong to the span on the RHS then their span cannot be a subset of it, as their span contains, in particular, both of these vectors (which is again a fact we showed in class).
So, by the definition of a span, we need to check whether both of the vectors $\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$ are linear combinations of the vectors $\begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}$ We do this in much the same way as in question 1.
For the vector $\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix}$: It is a linear combination of this three vectors iff there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix} = \alpha \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} + \beta \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} + \gamma \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}.$$

We rewrite the linear combination on the right hand side:

$$\begin{pmatrix} 5 & -2 \\ -5 & -3 \end{pmatrix} = \begin{pmatrix} 2\alpha - \beta - 2\gamma & \beta + \gamma \\ \alpha + 3\beta + 2\gamma & -\alpha - \gamma \end{pmatrix}.$$

Such an equality between matrices holds iff the matrices are equal at each entry, that is, iff

$$\begin{cases} 2\alpha - \beta - 2\gamma = 5 \\ \beta + \gamma = -2 \\ \alpha + 3\beta + 2\gamma = -5 \\ -\alpha - \gamma = -3 \end{cases}$$

So, the vector is in the span iff this linear system has a solution. Let us write it in matrix form and bring it to echelon form:

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & 5 \\ 0 & 1 & 1 & -2 \\ 1 & 3 & 2 & -5 \\ -1 & 0 & -1 & -3 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 3 & 2 & -5 \\ 0 & 1 & 1 & -2 \\ 2 & -1 & -2 & 5 \\ -1 & 0 & -1 & -3 \end{array}\right) \xrightarrow{R_3 - 2R_1, R_4 + R_1}$$
$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & -7 & -6 & 15 \\ 0 & 3 & 1 & -8 \end{array}\right) \xrightarrow{R_3 + 7R_2, R_4 - 3R_2} \left(\begin{array}{ccc|c} 1 & 3 & 2 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -2 & -2 \end{array}\right) \xrightarrow{R_4 + 2R_3}$$
$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This resulted in a row with no "lie" so the linear system has a solution and therefore the vector is a linear combination of the other given three vectors.

For the vector $\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix}$: It is a linear combination of this three vectors iff there exist scalars $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} = \alpha \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} + \beta \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} + \gamma \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}.$$

We rewrite the linear combination on the right hand side:

$$\begin{pmatrix} 1 & 1 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 2\alpha - \beta - 2\gamma & \beta + \gamma \\ \alpha + 3\beta + 2\gamma & -\alpha - \gamma \end{pmatrix}.$$

Such an equality between matrices holds iff the matrices are equal at each entry, that is, iff

$$\begin{cases} 2\alpha - \beta - 2\gamma = 1 \\ \beta + \gamma = 1 \\ \alpha + 3\beta + 2\gamma = 4 \\ -\alpha - \gamma = -1 \end{cases}$$

So, the vector is in the span iff this linear system has a solution. Let us write it in matrix form and bring it to echelon form:

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 3 & 2 & 4 \\ -1 & 0 & -1 & -1 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 2 & -1 & -2 & 1 \\ -1 & 0 & -1 & -1 \end{array}\right) \xrightarrow{R_3-2R_1, R_4+R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & -7 & -6 & -7 \\ 0 & 3 & 1 & 3 \end{array}\right) \xrightarrow{R_3+7R_2, R_4-3R_2} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{array}\right) \xrightarrow{R_4+2R_3}$$

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This resulted in a row with no "lie" so the linear system has a solution and therefore the vector is a linear combination of the other given three vectors.
Since both vectors are linear combinations of the other given three vectors, we conclude that the span on the LHS is a subspace of the span on the RHS.

iv. The claim is true. Proof – Two sets are equal iff each one of them is a subset of the other one. So we need to check two inclusions.

We start with:

$$\operatorname{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\} \subseteq \operatorname{span}\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}.$$

This inclusion will hold iff both of the vectors in the left span belong to the right span, that is, iff:

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \in \operatorname{span}\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}.$$

The justification for this equivalence is exactly as in Q1(iii), so we skip it here. The vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ clearly belongs to the span on the right as every span contains each one of the vectors generating it (we showed this in class). So it remains to check whether the vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ belongs to the span on the right.

We studied in recitation the following algorithm to determine whether a vector $b \in \mathbb{R}^m$ is a linear combination of other vectors $v_1, .., v_n \in \mathbb{R}^m$. This holds iff the equation $(A|b)$ has a solution where A is the matrix who's columns are $v_1, .., v_n$. We use this here:

$$\left(\begin{array}{ccc|c} 3 & 1 & 1 & 1 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3 \leftrightarrow R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 2 & 0 & 2 & 1 \\ 3 & 1 & 1 & 1 \end{array}\right) \xrightarrow{R_3 - 3R_1, R_2 - 2R_1}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 4 & 1 \\ 0 & -2 & 4 & 1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 4 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We reached echelon form, and there is no row with a lie, so the linear system has a solution. This implies that the vector is a linear combination of the other given vectors which in turn implies the inclusion of sets we wanted to prove. So, indeed,

$$\text{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\} \subseteq \text{span}\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}.$$

We now turn to check the opposite inclusion:

$$\text{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\} \supseteq \text{span}\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}.$$

As before, this holds iff the following holds:

$$\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \in \text{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\}$$

As before the vector $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ belongs to the span on the right hand side in a trivial way. It remains to check for the other two vectors. We can again use the algorithm from recitation, applying it to both of these vectors at the same time, with a doubly augmented matrix:

$$\left(\begin{array}{cc|c|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & -1 \end{array}\right) \xrightarrow{R_3 - R_1} \left(\begin{array}{cc|c|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & -1 & -2 & -2 \end{array}\right) \xrightarrow{R_2 + R_1}$$

$$\left(\begin{array}{cc|c|c} 1 & 1 & 3 & 1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We reached echelon form and there is no row with a 'lie', in neither one of the linear systems described by this doubly augmented matrix. So, both of these linear systems have solutions, implying that both vectors are linear combinations of the other two given vectors. This in turn implies that the inclusion we were checking holds, that is, we found that

$$\text{span}\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}\} \supseteq \text{span}\{\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\}.$$

Since both sides of the inclusion hold the sets are indeed equal.

v. The claim is true. Proof – We studied in recitation the following algorithm to determine whether vectors $v_1, .., v_n \in \mathbb{R}^m$ are a spanning set for $\mathbb{R}^m$. This holds iff the equation $(A|b)$ has a solution for every $b \in \mathbb{R}^m$ where A is the matrix who's columns are $v_1, .., v_n$. We use this here:

$$\left(\begin{array}{cc|c} 1 & -1 & b_1 \\ -1 & 2 & b_2 \end{array}\right) \xRightarrow{R_2+R_1} \left(\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 1 & b_2 + b_1 \end{array}\right).$$

We reached echelon form and conclude that for all b_1, b_2 there is no row with a 'lie'. So, the system has a solution for all $b \in \mathbb{R}^2$. This implies that the given vectors were indeed a spanning set for $\mathbb{R}^2$.

vi. The claim is false. Using the same method as in Q1(v) we get:

$$\left(\begin{array}{cc|c} 1 & -1 & b_1 \\ -1 & 1 & b_2 \end{array}\right) \xRightarrow{R_2+R_1} \left(\begin{array}{cc|c} 1 & -1 & b_1 \\ 0 & 0 & b_2 + b_1 \end{array}\right).$$

We reached echelon form and find that if $b_2 + b_1 \neq 0$ then the echelon form has a row with a 'lie' and therefore the linear system has no solution. For example if $b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ then the system has no solution. This, by the algorithm from recitation, implies that $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not a linear combination of $\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. So $\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}\}$ is not a spanning set for $\mathbb{R}^2$ as there is a vector in $\mathbb{R}^2$ which is not a linear combination of them.

vii. The claim is true. Proof – There are two ways to prove this claim. First way: The set $\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\}$ is a spanning set for $\mathbb{R}_3[x]$ iff $\text{span}\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\} = \mathbb{R}_3[x]$. Two sets are equal iff each one of them is included in the other. It is clear that $\text{span}\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\} \subseteq \mathbb{R}_3[x]$ as each one of the vectors generating the span belongs to $\mathbb{R}_3[x]$. So, it remains to check whether $\mathbb{R}_3[x] \subseteq \text{span}\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\}$. We learned in class that $\{1, x, x^2, x^3\}$ is a spanning set for $\mathbb{R}_3[x]$, that is, $\text{span}\{1, x, x^2, x^3\} = \mathbb{R}_3[x]$. It follows that $\{1-x+x^2, x-x^2+x^3, 1+$

$x^2-x^3, x^3\}$ is a spanning set for $\mathbb{R}_3[x]$ iff the following condition holds:

$$\text{span}\{1,x,x^2,x^3\} \subseteq \text{span}\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\}.$$

This happens iff the following condition holds:

$$1,x,x^2,x^3 \in \text{span}\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\}.$$

The justification for this last step is exactly as in Q1(iii) so we skip it here. We now need to check whether each one of the four vectors on the left is a linear combination of the four vectors on the right. But this is rather tedious if we are not allowed to use coordinates, so we will prefer to solve in another way.

Second way: The set $\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\}$ is a spanning set for $\mathbb{R}_3[x]$ if every vector in $\mathbb{R}_3[x]$ is a linear combination of the vectors in this set. That is, if for every $a_0+a_1x+a_2x^2+a_3x^3 \in \mathbb{R}_3[x]$ there exist scalars $\alpha,\beta,\gamma,\delta \in \mathbb{R}$ such that

$$a_0+a_1x+a_2x^2+a_3x^3 = \alpha(1-x+x^2)+\beta(x-x^2+x^3)+\gamma(1+x^2-x^3)+\delta x^3.$$

We rewrite the linear combination on the right hand side.

$$a_0+a_1x+a_2x^2+a_3x^3 = (\alpha+\gamma)+(-\alpha+\beta)x+(\alpha-\beta+\gamma)x^2+(\beta-\gamma+\delta)x^3.$$

Two polynomials are equal (for every x) if their coefficients are equal. So we get the linear system:

$$\begin{cases} \alpha+\gamma = a_0 \\ -\alpha+\beta = a_1 \\ \alpha-\beta+\gamma = a_2 \\ \beta-\gamma+\delta = a_3 \end{cases}$$

It follows that $\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\}$ is a spanning set iff this system has a solution for every $a_0,a_1,a_2,a_3, \in \mathbb{R}$.
We put this system in matrix form and bring it to echelon form.

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a_0 \\ -1 & 1 & 0 & 0 & a_1 \\ 1 & -1 & 1 & 0 & a_2 \\ 0 & 1 & -1 & 1 & a_3 \end{array}\right) \xRightarrow{R_2+R_1, R_3-R_1} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a_0 \\ 0 & 1 & 1 & 0 & a_1+a_0 \\ 0 & -1 & 0 & 0 & a_2-a_0 \\ 0 & 1 & -1 & 1 & a_3 \end{array}\right) \xRightarrow{R_3+R_2, R_4-R_2}$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a_0 \\ 0 & 1 & 1 & 0 & a_1+a_0 \\ 0 & 0 & 1 & 0 & a_2+a_1 \\ 0 & 0 & -2 & 1 & a_3-a_1-a_0 \end{array}\right) \xRightarrow{R_4+2R_3} \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & a_0 \\ 0 & 1 & 1 & 0 & a_1+a_0 \\ 0 & 0 & 1 & 0 & a_2+a_1 \\ 0 & 0 & 0 & 1 & a_3+2a_2+a_1-a_0 \end{array}\right).$$

We reached echelon form and find that it has no row with a 'lie' for all $a_0,a_1,a_2,a_3 \in \mathbb{R}$. This means that the linear system has a solution for all $a_0,a_1,a_2,a_3 \in \mathbb{R}$, which implies that all vectors in $\mathbb{R}_3[x]$ are linear combinations of the vectors $\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\}$. This means that $\{1-x+x^2, x-x^2+x^3, 1+x^2-x^3, x^3\}$ is a spanning set for $\mathbb{R}_3[x]$.

vii. The claim is False. Following the idea of the "second way" to solve such problems from Q1(vi), the set is spanning for $M_2(\mathbb{R})$ iff for every $\begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix} \in M_2(\mathbb{R})$ there exist $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\alpha \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} + \beta \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix} + \gamma \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

We rewrite the linear combination on the right hand side and obtain:

$$\begin{pmatrix} -\alpha - \beta + 3\gamma & \alpha + \gamma \\ \alpha + 2\beta & \alpha + \beta - \gamma \end{pmatrix} = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}.$$

Two matrices are equal iff they are equal entry by entry, so we end up with the following system:

$$\begin{pmatrix} -\alpha - \beta + 3\gamma = b_1 \\ \alpha + \gamma = b_2 \\ \alpha + 2\beta = b_3 \\ \alpha + \beta - \gamma = b_4 \end{pmatrix}$$

So, the given set is spanning for $M_2(\mathbb{R})$ iff this linear system has a solution for all b_1, b_2, b_3, b_4. We write it in matrix form and bring it to echelon form.

$$\left(\begin{array}{ccc|c} -1 & -1 & 3 & b_1 \\ 1 & 0 & 1 & b_2 \\ 1 & 2 & 0 & b_3 \\ 1 & 1 & -1 & b_4 \end{array}\right) \xrightarrow{R_2+R_1, R_3+R_1, R_4+R_1} \left(\begin{array}{ccc|c} -1 & -1 & 3 & b_1 \\ 0 & -1 & 4 & b_2+b_1 \\ 0 & 1 & 3 & b_3+b_1 \\ 0 & 0 & 2 & b_4+b_1 \end{array}\right) \xrightarrow{R_3+R_2}$$

$$\left(\begin{array}{ccc|c} -1 & -1 & 3 & b_1 \\ 0 & -1 & 4 & b_2+b_1 \\ 0 & 0 & 7 & b_3+b_2+2b_1 \\ 0 & 0 & 2 & b_4+b_1 \end{array}\right). \xrightarrow{R_3-3R_4} \left(\begin{array}{ccc|c} -1 & -1 & 3 & b_1 \\ 0 & -1 & 4 & b_2+b_1 \\ 0 & 0 & 1 & b_3+b_2-b_1-3b_4 \\ 0 & 0 & 2 & b_4+b_1 \end{array}\right). \xrightarrow{R_4-2R_3}$$

$$\left(\begin{array}{ccc|c} -1 & -1 & 3 & b_1 \\ 0 & -1 & 4 & b_2+b_1 \\ 0 & 0 & 1 & b_3+b_2-b_1-3b_4 \\ 0 & 0 & 0 & 7b_4-2b_3-2b_2+3b_1 \end{array}\right).$$

We find that if $7b_4 - 2b_3 - 2b_2 + 3b_1 \neq 0$ then the echelon form has a row with a "lie" which implies that in such a case the system has no solution. For example: if $b_1 = 0, b_2 = 0, b_3 = 0, b_4 = 1$ the system has no solution. This means that $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ is not a linear combination of$\left\{\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 0 & -1 \end{pmatrix}\right\}$, which implies that this set is not a spanning set for $M_2(\mathbb{R})$.

Remark. After we will learn dimensions this questions will be solved in one line, like this: The dimension of $M_2(\mathbb{R})$ is 4 and therefore it does not have a spanning system which contains only 3 elements, so the given set is not a spanning set for the space.

2. i. We are looking for a spanning set for the vector space: $W := \{A \in M_n(\mathbb{R}) : A \text{ is diagonal }\}$ (we use the notation W for the space so that it will be easier to refer to it). As a rule, it is always easier to find a spanning set for a vector space if we can parameterize the vectors in this space. In this case the set can be parameterized as follows

$$W = \{\begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} : a_1, a_2, a_3, \ldots, a_n \in \mathbb{R}\}.$$

Once we have a parametrization it is easy to describe any vector from the space as a linear combination of some specific other vectors in the space. In this case:

$$\begin{pmatrix} a_1 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & \cdots & 0 \\ 0 & 0 & a_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix} = a_1 \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + a_2 \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} +$$

$$+ a_3 \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \ldots + a_n \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

Let us denote by E_{jj} the $n \times n$ matrix who is equal to zero in all entries but the jj entry, where it is equal to one. (The jj entry is the entry in the j'th row and j'th column). We found above that every matrix in W is a linear combination of matrices from the set $\{E_{11}, E_{22}, E_{33}, \ldots, E_{nn}\}$. Since each one of the matrices E_{jj} belongs to W (they are all clearly diagonal) it follows that $\{E_{11}, E_{22}, E_{33}, \ldots, E_{nn}\}$ is a spanning set for W.

Remark. In fact, this set is a basis for W, indeed, it is easy to check that $\{E_{11}, E_{22}, E_{33}, \ldots, E_{nn}\}$ is linearly independent. Intuitively, finding a parametrization for the set will result in finding a spanning system, while finding a parametrization that uses the least amount of parameters possible will result in finding a basis. (This is not stated accurately enough to be a "theorem", this is just a discussion about intuition).

ii. We need to find a spanning system for the vector space: $W = \{ \begin{pmatrix} a+b+c \\ a-2b \\ 3a-2c \\ 4c-b \end{pmatrix} : a, b, c \in \mathbb{R}\}$

W is already presented using parametrization, so we do not need to parameterize it ourselves. Once we have a parametrization it is easy to describe any vector from the space as a linear combination of some specific other vectors in the space. In this case:

$$\begin{pmatrix} a+b+c \\ a-2b \\ 3a-2c \\ 4c-b \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix} + b \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix} + c \begin{pmatrix} 1 \\ 0 \\ -2 \\ 4 \end{pmatrix}.$$

Since each one of the vectors $\begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 4 \end{pmatrix}$ belongs to W (Why? How would you show this?), and every vector in W is a linear combination of these vectors, it follows that $\{ \begin{pmatrix} 1 \\ 1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -2 \\ 4 \end{pmatrix} \}$ is a spanning set for W.

iii. We need to find a spanning set for the space $W := \{A \in M_2(\mathbb{R}) : A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}\}$. Our first step will be to parameterize the space:

$$W = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \} =$$

$$\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a+2b \\ c+2d \end{pmatrix} = 0\} = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a+2b = 0 \text{ and } c+2d = 0\} =$$

$$= \{ \begin{pmatrix} -2b & b \\ -2d & d \end{pmatrix} : b, d \in \mathbb{R}\}.$$

Once we have a parametrization it is easy to describe any vector from the space as a linear combination of some specific other vectors in the space. In this case:

$$\begin{pmatrix} -2b & b \\ -2d & d \end{pmatrix} = b \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}$$

Since each one of the vectors $\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}$ belongs to W (Why? How would you show this?), and every vector in W is a linear combination of these two vectors, it follows that $\{\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ -2 & 1 \end{pmatrix}\}$ is a spanning set for W.

iv. We need to find a spanning set for the space $W := \{p(x) \in \mathbb{R}_3[x] : p'(1) = 0\}$. Our first step will be to parameterize the space:

$$W = \{a_0 + a_1x + a_2x^2 + a_3x^3 : a_1 + 2a_2 + 3a_3 = 0\} =$$
$$\{a_0 + (-2a_2 - 3a_3)x + a_2x^2 + a_3x^3 : a_0, a_2, a_3 \in \mathbb{R}\}.$$

Once we have a parametrization it is easy to describe any vector from the space as a linear combination of some specific other vectors in the space. In this case:

$$a_0 + (-2a_2 - 3a_3)x + a_2x^2 + a_3x^3 = a_0 \cdot 1 + a_2(-2x + x^2) + a_3(-3x + x^3)$$

Since each one of the vectors $1, -2x+x^2, -3x+x^3$ belongs to W (Why? How would you show this?), and every vector in W is a linear combination of these three vectors, it follows that $\{1, -2x + x^2, -3x + x^3\}$ is a spanning set for W.

v. **Remark** Due to a missprint I made when writing this question, it turned out to be more annoying then I planned. It is enough if you know to solve this question for, say, $n = 3$ and $n = 4$.

We need to find a spanning set for the space $W = \{p(x) \in \mathbb{R}_n[x] : p(1) = p(-1)\}$. It is easier to divide to two cases and work on each one separately: The case that n is even and the case that n is odd. Let us first assume that n is odd. Our first step will be to parameterize the space:

$$W = \{a_0+a_1x+a_2x^2+a_3x^3 \ldots +a_nx^n : a_0+a_1+a_2+a_3+\ldots+a_n = a_0-a_1+a_2-a_3+\ldots-a_n\} =$$
$$\{a_0 + a_1x + a_2x^2 + a_3x^3 \ldots + a_nx^n : a_1 + a_3 + \ldots + a_n = 0\} =$$
$$= \{a_0 + (-a_3 - a_5 - \ldots - a_n)x + a_2x^2 + a_3x^3 + \cdots + a_nx^n : a_0, a_2, a_3, a_4 \ldots a_n \in \mathbb{R}\}$$

Once we have a parametrization it is easy to describe any vector from the space as a linear combination of some specific other vectors in the space. In this case:

$$a_0 + (-a_3 - a_5 - \ldots - a_n)x + a_2x^2 + a_3x^3 + \cdots + a_nx^n =$$
$$= a_0 \cdot 1 + a_2x^2 + a_3(-x + x^3) + a_4x^4 + a_5(-x + x^5) + \ldots + a_n(-x + x^n)$$

Since each one of the vectors $1, x^2, -x + x^3, x^4, -x + x^5, \ldots, -x + x^n$ belongs to W (Why? How would you show this?), and every vector in W is a linear combination of these three vectors, it follows that $\{1, x^2, -x+x^3, x^4, -x + x^5, \ldots, -x + x^n\}$ is a spanning set for W.

Let us now assume that n is even. Our first step will be to parameterize the space:

$$W = \{a_0+a_1x+a_2x^2+a_3x^3+\dots+a_nx^n : a_0+a_1+a_2+a_3+\dots+a_n = a_0-a_1+a_2-a_3+\dots-a_{n-1}+a_n\} =$$

$$\{a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n : a_1 + a_3 + \dots + a_{n-1} = 0\} =$$

$$\{a_0+(-a_3-a_5-\dots-a_{n-1})x+a_2x^2+a_3x^3+\dots+a_{n-1}x^{n-1}+a_nx^n : a_0, a_2, a_3, a_4 \dots a_{n-1}, a_n \in \mathbb{R}\}$$

Once we have a parametrization it is easy to describe any vector from the space as a linear combination of some specific other vectors in the space. In this case:

$$a_0 + (-a_3 - a_5 - \dots - a_n)x + a_2x^2 + a_3x^3 \dots + a_{n-1}x^{n-1} + a_nx^n =$$

$$= a_0 \cdot 1 + a_2x^2 + a_3(-x + x^3) + a_4x^4 + a_5(-x + x^5) + \dots + a_{n-1}(-x + x^{n-1}) + a_nx^n.$$

Since each one of the vectors $1, x^2, -x+x^3, x^4, -x+x^5, \dots, -x+x^{n-1}, x^n$ belongs to W (Why? How would you show this?), and every vector in W is a linear combination of these three vectors, it follows that $\{1, x^2, -x+x^3, x^4, -x + x^5, \dots, -x + x^{n-1}, x^n\}$ is a spanning set for W.

3. i. By definition, a set is linearly **dependent** if there is a nontrivial linear combination of its elements which is equal to the zero vector. Otherwise, the set is linearly **independent**, that is, it is linearly independent if the only linear combination of its elements which is equal to the zero vector is the trivial one. So, the given set is linearly **dependent** if there are α, β, γ **not all of them zero** such that

$$\alpha \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} + \beta \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} + \gamma \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The set is linearly **independent** if the only solution to the above equation is $\alpha = \beta = \gamma = 0$. We rewrite the linear combination on the left side of the equation:

$$\begin{pmatrix} 2\alpha - \beta - 2\gamma & \beta + \gamma \\ \alpha + 3\beta + 2\gamma & -\alpha - \gamma \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Such an equality between matrices holds iff the matrices are equal at each entry, that is, iff

$$\begin{cases} 2\alpha - \beta - 2\gamma = 0 \\ \beta + \gamma = 0 \\ \alpha + 3\beta + 2\gamma = 0 \\ -\alpha - \gamma = 0 \end{cases}$$

So, the set is linearly **independent** iff this homogeneous linear system has exactly one solution. Let us write it in matrix form:

$$\left(\begin{array}{ccc|c} 2 & -1 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 2 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right)$$

We have brought this matrix to echelon form in Q1(i), this is what we obtained:

$$\left(\begin{array}{ccc|c} 1 & 3 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

There are no free variables in this echelon form, so the corresponding homogenous linear system has exactly one solution (as a homogenous system always has at least one solution) and therefore the set is linearly **independent**.

ii. By an algorithm we studied in class and recitation a set of vectors $v_1, ..., v_m \in \mathbb{R}^m$ is linearly **independent** iff the homogeneous system $(A|0)$ has exactly one solution where A is the matrix who's columns are $v_1, ..., v_m$. We use this here and consider the matrix:

$$\left(\begin{array}{ccc|c} 3 & 1 & 1 & 0 \\ 2 & 0 & 2 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right)$$

We brought this matrix to echelon form in Q1(iv) and obtained:

$$\left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

We see that there is a free variable in this homogenous linear system, so it has an infinite amount of solutions (as a homogenous system always has at least one solution) which implies that the set we are considering is linearly **dependent**.

iii. We consider the equation:

$$\alpha(1 - x^3) + \beta(2 + x + x^2) + \gamma(3 - x) + \delta(1 + x + x^2 + x^3) = 0,$$

where the 0 on the RHS should be thought of as the constant polynomial 0. The given set is linearly **independent** iff this equation has only the trivial solution $\alpha = \beta = \gamma = \delta = 0$ and it is linearly **dependent** if this equation has also a non-trivial solution. The justification for this fact is the same as in Q3(i) so we skip it here. We rewrite the linear combination on the left:

$$(\alpha + 2\beta + 3\gamma + \delta) + (\beta - \gamma + \delta)x + (\beta + \delta)x^2 + (-\alpha + \delta)x^3 = 0.$$

Such an equality between polynomial holds (for every x) iff the coefficients of the polynomials are equal at each entry (recall that the 0 on the RHS should be thought of as the constant polynomial 0), that is, iff

$$\begin{cases} \alpha + 2\beta + 3\gamma + \delta = 0 \\ \beta - \gamma + \delta = 0 \\ \beta + \delta = 0 \\ -\alpha + \delta = 0 \end{cases}$$

So, the set is linearly **independent** iff this homogeneous linear system has exactly one solution. Let us write it in matrix form:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 \end{array}\right) \xrightarrow{R_4+R_1} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 3 & 2 & 0 \end{array}\right) \xrightarrow{R_3-R_2, R_4-2R_2}$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 \end{array}\right) \xrightarrow{R_4-5R_3} \left(\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

There are no free variables in this echelon form, so the corresponding linear system has exactly one solution (as a homogenous system always has at least one solution). It follows that the set we are considering is linearly **independent**.

iv. Here, we use the following trigonometric identity which holds for every $x \in \mathbb{R}$:

$$\sin^2 x + \cos^2(x) = 1.$$

It follows that the following relation holds for every $x \in \mathbb{R}$:

$$1 \cdot f(x) + 1 \cdot g(x) + (-1) \cdot h(x) = 0.$$

This is a nontrivial linear combination of f, g, h which is equal to the constant zero function (which is the zero vector in the vector space of functions). This implies that the set we were given is linearly **dependent**.

4. i. Consider the equation:

$$\alpha(w_1 + w_2 + w_3) + \beta(w_2 + w_3) + \gamma w_3 = 0_V.$$

The set we were given is linearly **independent** iff this equation has only the trivial solution $\alpha = \beta = \gamma = 0$ and it is linearly **dependent** if this equation has also a non-trivial solution. The justification for this fact is the same as in Q3(i) so we skip it here. We rewrite the linear combination on the left:

$$(\alpha + \beta + \gamma)w_1 + (\beta + \gamma)w_2 + \gamma w_3 = 0_V.$$

Since $\{w_1, w_2, w_3\}$ are linearly **independent** this equality holds iff the scalars multiplying w_1, w_2, w_3 are all equal zero. That is, it holds iff:

$$\begin{cases} \alpha+\beta+\gamma=0 \\ \beta+\gamma=0 \\ \gamma=0 \end{cases}$$

So, the set we were given is linearly **independent** iff this homogeneous linear system has exactly one solution. The equation is already in echelon form and we see that there are no free variables, so this system has exactly one solution (as a homogenous system always has at least one solution) and therefore the set $\{w_1+w_2+w_3,_2+w_3, w_3\}$ is linearly **independent**.

ii. Consider the equation:

$$\alpha(w_1+2w_2+w_3)+\beta(w_2+w_3)+\gamma(w_1+w_2)=0_V.$$

The set we were given is linearly **independent** iff this equation has only the trivial solution $\alpha=\beta=\gamma=0$ and it is linearly **dependent** if this equation has also a non-trivial solution. The justification for this fact is the same as in Q3(i) so we skip it here. We rewrite the linear combination on the left:

$$(\alpha+\gamma)w_1+(2\alpha+\beta+\gamma)w_2+(\alpha+\beta)w_3=0_V.$$

Since $\{w_1, w_2, w_3\}$ are linearly **independent** this equality holds iff the scalars multiplying w_1, w_2, w_3 are all equal zero. That is, it holds iff:

$$\begin{cases} \alpha+\gamma=0 \\ 2\alpha+\beta+\gamma=0 \\ \alpha+\beta=0 \end{cases}$$

So, the set we were given is linearly **independent** iff this homogeneous linear system has exactly one solution. We bring it to echelon form:

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{R_3-R_1, R_2-2R_1} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{R_3-R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

We reached echelon form and we see that there is a free variable in the corresponding homogenous linear system. This implies that this system has an infinite amount of solutions (as a homogenous system always has at least one solution), and therefore that the given set is linearly **dependent**

Remark: Another, faster way to solve this question: If you looked at the question carefully enough, before starting to solve it, you might have noticed that the first vector in the set we were given $w_1 + 2w_2 + w_3$ is a sum of the two other vectors, $w_1 + w_2$ and $w_2 + w_3$. This means that $w_1 + 2w_2 + w_3$ is a linear combination of $w_1 + w_2$ and $w_2 + w_3$. We proved in class that a set is linearly **dependent** iff at least one of its elements is a linear combination of the rest of the vectors. It follows that the given set is linearly **dependent**

5. i. The claim is false: For example, take $S = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ and $T = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\}$. Then $S \subset T$, S is linearly independent (why?) and T is linearly dependent (why?).

ii. The claim is true. Proof: Let us denote $S = \{v_1, ..., v_n\}$ (the notation covers all of the cases we should consider since we were given that S is finite), and $T = \{v_1, ..., v_n, w_1, ..., w_k\}$ so that $S \subset T$ as we were given. We need to prove that S is linearly independent so let us consider the equation

$$\alpha_1 v_1 + \ldots + \alpha_n v_n = 0_V.$$

The set is linearly **independent** iff this equation has only the trivial solution $\alpha_1 = \ldots = \alpha_n = 0$. We rewrite the equation above by adding to the LHS k times the zero vector:

$$\alpha_1 v_1 + \ldots + \alpha_n v_n + 0 \cdot w_1 + \ldots + 0 \cdot w_k = 0_V.$$

(note: we know that if the scalar zero is multiplied by a vector then this is equal to the zero vector, we stated this in class). This last equation is a linear combination of the vectors in T which is equal to the zero vector. Since we were given that T is linearly independent it follows that all of the coefficients in this linear combination are equal to zero. This implies that $\alpha_1 = \ldots = \alpha_n = 0$ which in turn implies that S is linearly independent, as was explained above.

iii. The claim is true, in fact, it follows from the true claim in Q5(ii) as $S \cap T \subseteq S$.

iv. The claim is false: For example, take $S = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$ and $T = \{\begin{pmatrix} 2 \\ 0 \end{pmatrix}\}$. Then S and T are both linearly independent (why?) but $S \cup T = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}\}$ is linearly dependent (why?).

v. The claim is true. To prove that two sets are equal we need to show that each one of them is a subset of the other.
We start with $W + U \subseteq \text{span}(S \cup T)$. To show this we show that every vector which belongs to $W + U$ belongs also to $\text{span}(S \cup T)$. Indeed, if $v \in W + U$ then there exist $w \in W$ and $u \in U$ such that $v = w + u$. Since $w \in W = \text{span}S$ then w is a linear combination of the vectors in S. Since $u \in U = \text{span}T$ then u is a linear combination of the vectors in S. This implies that $v = w + u$ is a linear combination of vectors from

S and from T, that is, v is a linear combination of vectors from $S \cup T$, which implies that $v \in \text{span}(S \cup T)$.
Next, we show that $\text{span}(S \cup T) \subseteq W + U$. By a claim we studied in class, it is enough to show that $S \cup T \subseteq W + U$ since this will imply that the span of $S \cup T$ is a subset of the subspace $W + U$ as well (we proved in previous HW that $W + U$ is indeed a subspace). So, let $v \in S$. Since every span contains each one of the vectors generating it (we showed this in class) it follows that $v \in \text{span}S = W$. Since every subspace of a vector space contains the zero vector of the space (we proved this in class), we know that $0_V \in U$. It follows that

$$v = v + 0_V \in W + U.$$

So, $S \subseteq W + U$. In the same way one can prove that $T \subseteq W + U$. Together, these imply that $S \cup T \subseteq W + U$ and therefore that $\text{span}(S \cup T) \subseteq W + U$ (as was explained in the beginning of this paragraph).

6. In this question we will use the following two algorithms which were discussed in class and recitation:
   Let $v_1, ..., v_n \in \mathbb{R}^m$ and A be the $m \times n$ matrix who's columns are $v_1, ..., v_n$.
   - ◇ $\{v_1, ..., v_n\}$ is linearly independent iff the homogenous system $(A|0)$ has exactly one solution.
   - ◇ $\{v_1, ..., v_n\}$ is a spanning system for $\mathbb{R}^m$ iff for every $b \in \mathbb{R}^m$ the system $(A|b)$ has at least one solution.

   i. The claim is true. If $\{v_1, ..., v_n\}$ is linearly independent then the homogenous system $(A|0)$ has exactly one solution. This implies that in the echelon form of $(A|0)$ there are no free variables. So every variable is leading. Since the system has n variables this implies that it has n leading variables. Since every leading variable opens a different row this implies that there are at least n rows. Since the system has m rows this implies that $m \geq n$.

   ii. The claim is false. Consider the set $\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}\}$. It is a set of 2 vectors in $\mathbb{R}^3$ but it is linearly **dependent** (why?).

   iii. This question was solved in HW2 Q4(b) (Check that these are indeed the same questions!). The claim is true, we add a proof here as well. If $\{v_1, ..., v_n\}$ spans $\mathbb{R}^m$ then for every $b \in \mathbb{R}^m$ the system $(A|b)$ has at least one solution. This implies that the echelon form of A does not have a zero row (this was proved twice already in HW's, say in HW2 Q4(a)). Since in echelon form every non-zero row opens with a different variable, this implies that the number of variables is bigger then the number of rows, that is $n \geq m$.

iv. The claim is false. Consider the set $\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \end{pmatrix}\}$. It is a set of 3 vectors in $\mathbb{R}^2$ but it does not span $\mathbb{R}^2$ (why?).

v. The claim is true, this follows by combining the results in Q6(i) and Q6(iii).

vi. The claim is true. Indeed, we proved in class (when we proved equivalent conditions for a matrix to be invertible) that if $n = m$ then: the homogenous system $(A|0)$ has exactly one solution iff for every $b \in \mathbb{R}^m$ the system $(A|b)$ has at least one solution. The result now follows by applying the relations stated above, that is:

  - ◇ $\{v_1, ..., v_n\}$ is linearly independent iff the homogenous system $(A|0)$ has exactly one solution.
  - ◇ $\{v_1, ..., v_n\}$ is a spanning system for $\mathbb{R}^m$ iff for every $b \in \mathbb{R}^m$ the system $(A|b)$ has at least one solution.