MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 12

1. Compute the determinants of the following matrices:

$$\begin{pmatrix} 2 & 6 & 16 \\ -3 & -6 & 18 \\ 5 & 12 & 35 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ -1 & 5 & 2 \\ 3 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 4 & 0 & 1 \\ -2 & 2 & -1 \\ 0 & 4 & -3 \end{pmatrix}, \quad \begin{pmatrix} 4 & -4 & 2 & 1 \\ 1 & 2 & 0 & 3 \\ 2 & 0 & 3 & 4 \\ 0 & -1 & 2 & 1 \end{pmatrix}$$

2. i. Let $a, b, c \in \mathbb{R}$. Prove that

$$\left| \begin{pmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{pmatrix} \right| = (c-a)(c-b)(b-a)$$

ii. Find the values of a for which the following set is a basis for $\mathbb{R}^3$:

$$\left\{ \begin{pmatrix} a-1 \\ -3 \\ -6 \end{pmatrix}, \begin{pmatrix} 3 \\ a+5 \\ 6 \end{pmatrix}, \begin{pmatrix} -3 \\ -3 \\ a-4 \end{pmatrix} \right\}$$

iii. Assume that,

$$\left| \begin{pmatrix} a & x & l \\ b & y & m \\ c & z & n \end{pmatrix} \right| = 2$$

Find:

$$\left| \begin{pmatrix} 2a+3x & 2b+3y & 2c+3z \\ l+x & m+y & n+z \\ 7l & 7m & 7n \end{pmatrix} \right|$$

3. Let $A, B \in M_n(\mathbb{R})$ and $\lambda \in \mathbb{R}$. Prove or disprove the following claims:
   i. $|A+B| = |A| + |B|$
   ii. $|\lambda A| = \lambda |A|$
   iii. $|\lambda A| = \lambda^n |A|$
   iv. If A is anti-symmetric (that is, $A^t = -A$) and n is odd then A is not invertible.
   v. If A is anti-symmetric (that is, $A^t = -A$) and n is even then A is not invertible.
   vi. If $AB = 0$ then $|A^2| + |B^2| = 0$.
   vii. If $|A+B| = |A|$ then B is the zero matrix.

4. i. Compute the determinant of the following $n \times n$ matrix:

$$\begin{pmatrix} 4 & 1 & 1 & \dots & 1 \\ 1 & 4 & 1 & \dots & 1 \\ 1 & 1 & 4 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 4 \end{pmatrix}$$

ii. For the matrix,

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}$$

Prove that $|A| = 1 + (-1)^{(n+1)}$. (Note the 1 on the left lowest corner).

6. Consider the following matrices.

$$A = \begin{pmatrix} 8 & 3 & -3 \\ -6 & -1 & 3 \\ 12 & 6 & -4 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \qquad c = \begin{pmatrix} 4 & 0 & 3 \\ 0 & 5 & 0 \\ 3 & 0 & -4 \end{pmatrix}$$

$$E = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{pmatrix} \qquad F = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \qquad G = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

For each of these matrices solve the following:

i. Find all of the eigenvalues of the matrix, determine the geometric multiplicity and the algebraic multiplicity of each eigenvalue.

ii. Determine if the matrix is diagonalizable. If it is then find a diagonal matrix D and an invertible matrix P so that the matrix is equal to $P^{-1}DP$.

7. Consider the matrix E from Q1

i. Find the eigenvalues of E^2. Is E^2 diagonalizable?

ii. Find the eigenvalues of E^{10}. Is E^{10} diagonalizable?

ii. Find the eigenvalues of $E^3 - 5E^2 + 2E + 3I$. Is $E^3 - 5E^2 + 2E + 3I$ diagonalizable?

iii. Is E invertible? If so, find the eigenvalues of E^{-1}. Is E^{-1} diagonalizable?

iv. Compute E^5.

8. In each of the following you are given a linear transformation. Determine whether it is diagonalizable.

i. $T : M_2(\mathbb{R}) \mapsto M_2(R)$ given by

$$TA = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} A$$

ii. $T : \mathbb{R}_2[x] \mapsto \mathbb{R}_2[x]$ given by $Tp(x) = x(p(x+1) - p(x))$

iii. Let V be a vector space and $B = (v_1, v_2, v_3)$ a basis for V. Here we consider the linear transformation $T : V \mapsto V$ which satisfies $Tv_1 = 5v_1$, $Tv_2 = v_2 + 2v_3$ and $Tv_3 = 2v_2 + v_3$.

9. Prove or disprove the following claims.

a. If $A \in M_3(\mathbb{R})$ has rows equal to v $2v$ $3v$ for some $v \in \mathbb{R}^3$ and A has a nonzero eigenvalue then A is diagonalizable.

b. If $A \in M_4(\mathbb{R})$ has characteristic polynomial $q_A(x) = x^2(x+5)(x+6)$ and

$$\begin{pmatrix} 0 \\ -1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 3 \\ 4 \end{pmatrix} \in null(A)$$

then A is diagonalizable.

c. Let $A \in M_n(\mathbb{R})$. Then 0 is an eigenvalue of A iff $|A| = 0$.

d. Let $A \in M_n(\mathbb{R})$. If 0 is an eigenvalue of A then its geometric multiplicity is equal to $n - rankA$.

e. There exists $A \in M_5(\mathbb{R})$ which is diagonalizable and satisfies $rankA = 1$ and $trA = 0$.

f. If $A \in M_n(\mathbb{R})$ is diagonalizable and 2 is the only eigenvalue of A then $A = 2I$.

g. If $A, B \in M_n(\mathbb{R})$ have the same eigenvalues and A is diagonalizable then so is B.

h. Let $A \in M_n(\mathbb{R})$ and let $q_A(x) = a_0 + a_1x + a_2x^2 + ... + a_{n-1}x^{n-1} + x^n$ be the characteristic polynomial of A. Then A is invertible iff $a_0 \neq 0$.

i. If $A, B \in M_n(\mathbb{R})$ are similar then they have the same characteristic polynomials.

j. If $A, B, C \in M_n(\mathbb{R})$ are such that A and B are similar, and such that A and C are similar, then B and C are similar.

k. If $A \in M_n(\mathbb{R})$ and $rankA \leq n - 1$ then A is similar to a matrix who's left most column is a zero column.

l. If $A \in M_n(\mathbb{R})$ is diagonalizable and $B \in M_n(\mathbb{R})$ is similar to A then B is also diagonalizable.

m. If $A \in M_3(\mathbb{R})$ satisfies: $rank(A - I) = 2$, $|A + I| = 0$ and there exists v such that $Av = 3v$ then A is diagonalizable.

n. If $A \in M_n(\mathbb{R})$ has eigenvalue λ and corresponding eigenvector v then for every positive integer k the matrix A^k has eigenvalue λ^k and corresponding eigenvector v. What about negative integers?

o. If $A \in M_n(\mathbb{R})$ is diagonalizable then A^2 is diagonalizable.

10. Let V be a vector space of dimension 5. Does there exist a transformation $T : V \mapsto V$ such that $dim Im T = 3$ and:

i. T has 5 distinct eigenvalues?

ii. T has 4 distinct eigenvalues?

iii. T has 4 distinct eigenvalues and T is not diagonalizable?