MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 9 – solutions

1. **Remark.** Note that in the solutions below B will denote a basis for W and C will denote a basis for U (in contrast to the notations in the formulation of the question).

   i. Consider the subspace $U \cap W$ (note that we proved in a previous HW that this is indeed a subspace). Let $D = \{v_1, ..., v_m\}$ be a basis for $U \cap W$, note that this implies that $\dim U \cap W = m$.
   Since $D \subset U \cap W$ we have $D \subset W$. Moreover, since D is linearly independent in $U \cap W$ it is linearly independent in W (the definition of linear independence involves only the set considered and the scalars we work with, nothing in the definition changes if we consider this set as a subset of a bigger vector space). So, D belongs to W and is linearly independent there. We proved in class that every linearly independent set in a vector space can be completed into a basis there. It follows that there exist $w_1, .., w_k \in W$ such that $B = \{v_1, ..., v_m, w_1, .., w_k\}$ is a basis for W. In exactly the same way we conclude that there exist $u_1, .., u_n \in U$ such that $C = \{v_1, ..., v_m, u_1, .., u_n\}$ is a basis for U.
   We claim that $D = B \cap C$. Since both B and C contain D it is clear that $D \subseteq B \cap C$. Now, every element in $B \cap C$ belongs both to W and to U, so $B \cap C \subseteq U \cap W$. Moreover, $B \cap C$ is linearly independent in $U \cap W$ (why?). From the claim we studied in class it now follows that the amount of elements in $B \cap C$ cannot be bigger then the dimension of $U \cap W$, that is, it cannot be bigger then m. Since $D \subseteq B \cap C$ and D has exactly m elements we conclude that indeed $D = B \cap C$.
   So, finally, we found a basis B for W and a basis C for U such that $D = B \cap C$ is a basis for $U \cap W$, as requested.

   ii. No. Consider for example the following two subspaces of $\mathbb{R}^3$,

$$W = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$$

$$U = \left\{ \begin{pmatrix} 0 \\ x \\ y \end{pmatrix} : x, y \in \mathbb{R} \right\}.$$

   Consider further the following to sets,

$$B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$C = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

Then B is a basis for W (check!) and C is a basis for V (check!). Moreover, $B \cap C$ is empty so it cannot be a basis for any vector space that contains a vector different from zero. It is clear that $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ is both in W and in U so $W \cap U$ contains a vector different from zero. Our counterexample is complete.

iii. To prove the formula:

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

we continue with the same notations that we used in Q6(i). That is, we assume that $D = \{v_1, ..., v_m\}$ is a basis for $U \cap W$ and therefore $\dim U \cap W = m$, that $B = \{v_1, ..., v_m, w_1, ..., w_k\}$ is a basis for W and therefore $\dim W = m + k$, and that $C = \{v_1, ..., v_m, u_1, .., u_n\}$ is a basis for U and therefore $\dim U = m + n$. To complete the proof we will show that $E = \{v_1, ..., v_m, w_1, ..., w_k, u_1, .., u_n\}$ is a basis for $U + W$ and therefore $\dim(U + W) = m + k + n$. If we show this then the dimension formula will be proved (check!).
We start by recalling a fact which we proved in previous HW's: If $v_1, ..., v_n, w_1, ..., w_k$ belong to a vector space V then,

$$\text{span}\{v_1, ..., v_n\} + \text{span}\{w_1, ..., w_k\} = \text{span}\{v_1, ..., v_n, w_1, ..., w_k\}.$$

In our case this implies that

$$U + W = \text{span}B + \text{span}C = \text{span}(B \cup C).$$

It follows that, $E = B \cup C$ is a spanning set for $U + W$. To prove that E is a basis for $U + W$ it remains to show that E is linearly independent. So, we consider a linear combination of these elements which is equal to the zero of the space: Let $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_k, \gamma_1, ..., \gamma_n \in \mathbb{R}$ be such that,

$$\alpha_1 v_1 + ... + \alpha_m v_m + \beta_1 w_1 + ... + \beta_k w_k + \gamma_1 u_1 + ... + \gamma_n u_n = 0.$$

This implies that

$$\alpha_1 v_1 + ... + \alpha_m v_m + \beta_1 w_1 + ... + \beta_k w_k = -(\gamma_1 u_1 + ... + \gamma_n u_n).$$

Let us denote,

$$x := \alpha_1 v_1 + ... + \alpha_m v_m + \beta_1 w_1 + ... + \beta_k w_k = -(\gamma_1 u_1 + ... + \gamma_n u_n).$$

Then on one side, since

$$x = \alpha_1 v_1 + ... + \alpha_m v_m + \beta_1 w_1 + ... + \beta_k w_k,$$

then x is a linear combination of the elements in B, the basis of W, so $x \in W$. While on the other side, since

$$x = -(\gamma_1 u_1 + ... + \gamma_n u_n),$$

then x is a linear combination of the elements in C, the basis of U, so $x \in U$. We conclude that $x \in U \cap W$. Since D is a basis for $U \cap W$ we can write x as a linear combination of the elements in D: There exist $\delta_1, ..., \delta_m$ such that,

$$x = \delta_1 v_1 + ... + \delta_m v_m.$$

Using two of the different expressions for x, we get

$$\alpha_1 v_1 + ... + \alpha_m v_m + \beta_1 w_1 + ... + \beta_k w_k = \delta_1 v_1 + ... + \delta_m v_m.$$

This implies that,

$$(\alpha_1 - \delta_1)v_1 + ... + (\alpha_m - \delta_m)v_m + \beta_1 w_1 + ... + \beta_k w_k = 0.$$

This is a linear combination of the elements in B which is equal to 0. Since B is a basis for W it is in particular linearly independent. So the coefficients in this linear combination must be equal zero. In particular this implies that $\beta_1 = .. = \beta_k = 0$. We insert this into the equation we started with, that is into

$$\alpha_1 v_1 + ... + \alpha_m v_m + \beta_1 w_1 + ... + \beta_k w_k + \gamma_1 u_1 + ... + \gamma_n u_n = 0.$$

and get

$$\alpha_1 v_1 + ... + \alpha_m v_m + \gamma_1 u_1 + ... + \gamma_n u_n = 0.$$

This is a linear combination of the elements of C which is equal to zero. Since C is a basis for U it is in particular linearly independent. So the coefficients in this linear combination must be equal zero. So, $\alpha_1 = .. = \alpha_m = \gamma_1 = ... = \gamma_n = 0$.
Finally, we found that $\alpha_1 = .. = \alpha_m = \beta_1 = ... = \beta_k = \gamma_1 = ... = \gamma_n = 0$, so the elements in E are indeed linearly independent and the dimension formula is proved.

iv. Since $U, W \subset \mathbb{R}_4[x]$ it follows that $U + W \subseteq \mathbb{R}_4[x]$ (why?). From the claim we proved in class regarding the dimension of subspaces it follows that $\dim(U + W) \leq \dim \mathbb{R}_4[x] = 5$. We insert this information, as well as the conditions $\dim(U) = \dim(W) = 3$ which we were given, into the dimension formula we proved in Q6(iii). We get:

$$3 + 3 - \dim(U \cap W) \leq 5.$$

This implies that

$$1 \leq \dim(U \cap W).$$

Since the dimension of $U \cap W$ is bigger or equal to 1 it must contain a vector which is different from zero (a vector space which contains only the zero vector has dimension zero). Our claim is proved.

v. Denote:

$$W := \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}\}$$

$$U := \text{span}\{\begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}\}$$

It may be a rather annoying procedure to find $W \cap U$ or a basis for it. However, we were not asked to do neither. Since we only need to find the dimension of this space, we can use the dimension formula which we proved in Q6(iii). For this goal we need to find the dimensions of W, U and $W + U$.

We start with W, it is spanned by two vectors which are linearly independent (neither of them is a scalar multiplying the other, which is the only way that two vectors can be linearly dependent). So these two vectors are a basis for W and therefore $\dim W = 2$.

To find the dimension of U we follow the algorithm described in class and recitation, write the 4-tuples as rows of a matrix and find the rank of this matrix.

$$\begin{pmatrix} 0 & 1 & 2 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & 2 & 4 & 1 \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 4 & 1 \\ 2 & 1 & 2 & 3 \\ 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_2 - 2R_1}$$

$$\begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & -3 & -6 & 1 \\ 0 & 1 & 2 & 1 \end{pmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -3 & -6 & 1 \end{pmatrix} \xrightarrow{R_3 + 3R_1}$$

$$\begin{pmatrix} 1 & 2 & 4 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

So, $\dim U = 3$. To find the dimension of $W + U$ we first recall the fact which we proved in previous HW's: If $v_1, ..., v_n, w_1, ..., w_k$ belong to a vector space V then,

$$\text{span}\{v_1, ..., v_n\} + \text{span}\{w_1, ..., w_k\} = \text{span}\{v_1, ..., v_n, w_1, ..., w_k\}.$$

In our case this implies that

$$U + W = \text{span}\{\begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 2 \\ -3 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 4 \\ 1 \end{pmatrix}\}.$$

To find the dimension of $U + W$ we now follow the algorithm described in class and recitation, write the 4-tuples as rows of a matrix and find the rank of this matrix.

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ -1 & 2 & -3 & 1 \\ 0 & 1 & 2 & 1 \\ 2 & 1 & 2 & 3 \\ 1 & 2 & 4 & 1 \end{pmatrix} \xRightarrow{R_2+R_1, R_4-2R_1, R_5-R_1} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \xRightarrow{R_2 \leftrightarrow R_3}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & -1 & 2 \\ 0 & -1 & -2 & 1 \\ 0 & 1 & 2 & 0 \end{pmatrix} \xRightarrow{R_3-3R_2, R_4+R_2, R_5-R_2} \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -7 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{pmatrix} \xRightarrow{R_5-\frac{1}{2}R_4}$$

$$\begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -7 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

So, dim $U + W$=4.
Now, inserting all the information we found into the dimension formula we find that that

$$2 + 3 - \dim(W \cap U) = 4.$$

We conclude that dim$U \cap W$=1.

2. i. The claim is true. (While we know that the rows of the matrix are 3–tuples, and that there are 4 rows, so by the claim we proved in class the rows of the matrix are linearly dependent, still it may be that the fourth row is linearly independent of the rest). Here is an example of such a matrix:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The last row is not a linear combination of the rest of the rows, as any linear combination of the first three rows has zero at it's last entry.

ii. The claim is true, here is an example of such a matrix:

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

This matrix is in echelon form. The rank of a matrix in echelon form is equal to the amount of leading variables in it. It follows that in our case rank$A = 2$. It is easy to check that indeed:

$$A\begin{pmatrix}1\\1\\1\end{pmatrix} = 0.$$

iii. The claim is false. Let us assume that such a matrix exists and obtain a contradiction. This will show that such a matrix cannot exist.
So, let us assume that there exists $A \in M_3(\mathbb{R})$ such that $\text{rank}(A) = 2$, and

$$A\begin{pmatrix}1\\1\\1\end{pmatrix} = 0 \qquad\qquad A\begin{pmatrix}1\\-1\\1\end{pmatrix} = 0.$$

This last condition can be written as follows:

$$\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix} \in \text{null}(A).$$

The vectors $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}$ are linearly independent, (indeed, two vectors are linearly independent iff neither of them is a scalar multiplying the other). So null(A) is a vector space that contains a linearly independent set with two elements. We proved in class that the number of elements in a linearly independent set is never bigger then the dimension of the space. This implies that $\dim(\text{null}(A)) \geq 2$. We proved in class that if $A \in M_{m\times n}(\mathbb{R})$ then $n = \dim(\text{null}(A)) + \text{rank}(A)$. In our case this implies that $3 = \dim(\text{null}(A)) + \text{rank}(A) \geq 2 + \text{rank}(A)$, so $\text{rank}(A) \leq 1$. This contradicts the condition $\text{rank}(A) = 2$.

iv. The claim is true. Indeed, any two different invertible matrices A and B will satisfy these requirements. In such a case, A and B are row equivalent since both of them are row equivalent to I (by the invertible matrix theorem both of these matrices have a REF which is equal to I). Further, we proved in previous HW's that a matrix $A \in M_3(\mathbb{R})$ is invertible iff it's columns are a basis for $\mathbb{R}^3$. So both of these matrices have a column space which is equal to $\mathbb{R}^3$, and therefore these column spaces are the same space. Here is a specific example:

$$A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} \qquad B = \begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}.$$

v. The claim is false. We will assume that such matrices exist and will arrive at a contradiction. This will show that such matrices do not exist.

So, assume there exist two matrices $A, B \in M_3(\mathbb{R})$ such that $AB = I$ and

$$rank(A) + rank(B) = 5.$$

It follows from the invertible matrix theorem we proved in class that if $AB = I$ then A and B are both invertible (in fact, it follows that $A^{-1} = B$ and $B^{-1} = A$). We proved in a previous HW that if a matrix $A \in M_3(\mathbb{R})$ is invertible then its columns span the hole space $\mathbb{R}^3$. So the dimension of the column space of A is 3. In other words, rank(A)=3. In the same way we get that rank(B)=3. So rank(A)+rank(B) $= 3+3 = 6$, which contradicts the condition that the sum of these ranks is 5.

vi. The claim is true, there are many examples. If $A \in M_3(\mathbb{R})$ is an invertible matrix then it satisfies the requirements as both it's column space and it's row space are equal to $\mathbb{R}^3$. However, let us give an example of a matrix which is not invertible:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

Clearly, A is not invertible (it is easy to see that it's rows are linearly dependent) and it's row space is equal to it's column space (as the vectors it's rows and the vectors in it's columns are the same).

3. a. The claim is false. Here is one out of many counterexample:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then,

$$AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So, rank(A)=rank(B) $= 1$ but rank(AB) $= 0 \neq 1 \cdot 1$.

b. This claim is not only false, it is ridiculous. Consider for example $A = I_n$ (or any other invertible matrix) then the null space of A is $\{0\}$ while the column space of A is $\mathbb{R}^n$.

c. The claim is true. Recall that we proved in class that the rank of a matrix is equal to the amount of leading variables in an echelon form of the matrix. So the condition rank$A$=rank($A|b$) is equivalent to the condition that in an echelon form of A and in an echelon form of $(A|b)$ there is the same amount of leading variables. This is the same as saying that in an echelon form of $(A|b)$ there is no leading variable in the last column, which is the same as saying that in an echelon form of $(A|b)$ there is no row with a 'lie'. We proved in class that this happens iff the system $(A|b)$ has a solution.

d. The claim is false. Here is one out of many counterexamples:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}.$$

The vectors in the rows of A are linearly independent,(two vectors are linearly dependent iff one of them is a scalar multiplying the other, which is clearly not the case here). On the other hand, the vectors in the columns of A are linearly dependent – it follows from a result we proved in class that every set of 3 vectors in $\mathbb{R}^2$ is linearly dependent, since the dimension of $\mathbb{R}^2$ is 2.

e. The claim is true. If $A \in M_n(\mathbb{R})$ is such that the vectors in its rows are linearly independent then these are n linearly independent vectors in a space of dimension n ($\mathbb{R}^n$), so by the theorem we proved in class these vectors are a basis for $\mathbb{R}^n$. We proved in a previous HW that in a square matrix this happens iff the vectors in the columns of the matrix are also a basis for $\mathbb{R}^n$ (and these things happen iff the matrix is invertible). Since the vectors in the columns are a basis they are in particular linearly independent

4. $\rightarrow$ We proved in a previous HW that if $A \in M_n(\mathbb{R})$ is invertible then its rows are a basis for $\mathbb{R}^n$. So the space spanned by the rows is all of the space $\mathbb{R}^n$, which is of dimension n. So, $\text{rank}(A) = n$.

   $\leftarrow$ If $rank(A) = n$ then the rows of A span a subspace of $\mathbb{R}^n$ that has dimension n. The space $\mathbb{R}^n$ itself has dimension n. By a result from class a subspace of $\mathbb{R}^n$ has dimension n iff this subspace is $\mathbb{R}^n$ itself. We conclude that the rows of A are a spanning set for $\mathbb{R}^n$. Since the rows of A form a set of n vectors in a space of dimension n, and these vectors form a spanning set for the space, then these vectors form a basis for the space. we proved in a previous HW that if the rows of a square matrix form a basis then this matrix is invertible. we conclude that A is invertible.

5. We claim that if $AB = 0$ then the column space of B is a subspace of the null space of A. To show this we need to show that if $b \in \mathbb{R}^n$ is in the column space of B then $Ab = 0$. So, let b be in the column space of B. This means that b is a linear combination of the columns of B, which implies (as we have seen in class) that there exists $v \in \mathbb{R}^k$ such that $Bv = b$. So, since $AB = 0$ we get,
$$Ab = A(Bv) = (AB)v = 0v = 0.$$
We conclude that indeed the column space of B is a subspace of the null space of A. From the claim we proved in class regarding dimensions of subspaces it follows that dim(column space of B) $\leq$dim(null(A)). We recall that by the definition of rank, the rank of B is equal to the dimension of the column space of B. So, we found that rank$B$ $\leq$dim(null(A)). We now insert this relation into the following theorem which we proved in class
$$n = \text{rank}(A) + \dim(\text{null}(A)) \geq$$
$$\text{rank}(A) + \text{rank}(B).$$
The claim is proved.

6. a. The claim is false. Here is one out of many counterexample:

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then,

$$A + B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So, rank(A)=rank(B) $= 1$ but rank$(A + B) = 0 \neq 1 + 1$.

b. The claim is true. Let $v_1, ..., v_n$ be the m–tuples in the columns of A and $w_1, .., w_n$ be the m–tuples in the columns of B. Then the m–tuples in the columns of $A+B$ are $v_1 + w_1, ..., v_n + w_n$. The rank of a matrix is by definition equal to the dimension of the column space of the matrix. So, the claim that we are trying to prove can be reformulated as follows: if $v_1, ..., v_n, w_1, ..., w_n \in \mathbb{R}^m$ then

$$\dim(\text{span}\{v_1 + w_1, ..., v_n + w_n\}) \leq \dim(\text{span}\{v_1, ..., v_n\}) + \dim(\text{span}\{w_1, ..., w_n\}).$$

We will show that on one hand,

$$\dim(\text{span}\{v_1 + w_1, ..., v_n + w_n\}) \leq \dim(\text{span}\{v_1, ..., v_n, w_1, ..., w_n\}).$$

While on the other hand,

$$\dim(\text{span}\{v_1, ..., v_n, w_1, ..., w_n\}) \leq \dim(\text{span}\{v_1, ..., v_n\}) + \dim(\text{span}\{w_1, ..., w_n\}).$$

The claim that we need to prove follows from these two inequalities.

i. Here we show that:

$$\dim(\text{span}\{v_1 + w_1, ..., v_n + w_n\}) \leq \dim(\text{span}\{v_1, ..., v_n, w_1, ..., w_n\}).$$

First, we note that for every $1 \leq l \leq n$ the vector $v_l + w_l$ is a linear combination of $\{v_1, ..., v_n, w_1, ..., w_n\}$ (by putting 1 as the coefficient of v_l and w_l and zero as the coefficient of all other vectors in this set). So,

$$v_1 + w_1, ..., v_n + w_n \in \text{span}\{v_1, ..., v_n, w_1, ..., w_n\}.$$

We proved in class that if a set of vectors belongs to a subspace then their span also belongs to this subspace so,

$$\text{span}\{v_1 + w_1, ..., v_n + w_n\} \subseteq \text{span}\{v_1, ..., v_n, w_1, ..., w_n\}.$$

By the result regarding dimensions of subspaces we proved in class, we now conclude that indeed,

$$\dim(\text{span}\{v_1 + w_1, ..., v_n + w_n\}) \leq \dim(\text{span}\{v_1, ..., v_n, w_1, ..., w_n\}).$$

ii. Here we show that:

$$\dim(\text{span}\{v_1, ..., v_n, w_1, ..., w_n\}) \leq \dim(\text{span}\{v_1, ..., v_n\}) + \dim(\text{span}\{w_1, ..., w_n\}).$$

Denote:

$$U = \text{span}\{v_1, ..., v_n\},$$
$$W = \text{span}\{w_1, ..., w_n\}.$$

Then, we have proved in previous HW's that

$$U + W = \text{span}\{v_1, ..., v_n, w_1, ..., w_n\}.$$

We proved in the previous HW the following dimension formula

$$\dim(U + W) = \dim(U) + \dim(W) - \dim(U \cap W).$$

In particular, since a dimension of a space is always non–negative it follows from this formula that,

$$\dim(U + W) \leq \dim(U) + \dim(W).$$

Inserting to this inequality the definitions of $U, W, U + W$ from above, we find that indeed,

$$\dim(\text{span}\{v_1, ..., v_n, w_1, ..., w_n\}) \leq \dim(\text{span}\{v_1, ..., v_n\}) + \dim(\text{span}\{w_1, ..., w_n\}).$$

Our claim is finally proved.

7. i. Indeed, let v be in the null space of B. This means that $Bv = 0$. So,

$$(AB)v = A(Bv) = A0 = 0.$$

We found that $(AB)v = 0$ so v is in the null space of AB. The claim is proved.

ii. There was a misprint in this question, it should read as follows: The column space of AB is a subspace of the column space of A. Proof: We have seen in class that a vector $d \in \mathbb{R}^m$ is in the column space of a matrix $C \in M_{m\times n}$ iff there exists $v \in \mathbb{R}^n$ so that $Cv = d$. So, let $b \in \mathbb{R}^m$ be in the column space of AB. This therefore implies that there exists $v \in \mathbb{R}^k$ such that $(AB)v = b$. Denote $w = Bv \in \mathbb{R}^n$ then

$$Aw = A(Bv) = (AB)v = b.$$

This, by the claim written above, implies that b belongs to the column space of A. So indeed, the column space of AB is a subspace of the column space of A.

iii. We need to prove that both of the inequalities, $\text{rank}(AB) \leq \text{rank}(A)$ and $\text{rank}(AB) \leq \text{rank}(B)$, hold.

   i. Here we prove that $\text{rank}(AB) \leq \text{rank}(A)$:
   We showed in Q6(ii) that the column space of AB is a subspace of the column space of A. From the claim we proved in class regarding the dimension of subspaces, it follows that dim(column space of AB) $\leq$dim(column space of A). Since the dimension of the column space of a matrix is equal to the rank of the matrix (this is just the definition of a rank) the result follows.

   ii. Here we prove that $\text{rank}(AB) \leq \text{rank}(B)$:
   We showed in Q6(i) that the null space of B is a subspace of the null space of AB. From the claim we proved in class regarding the dimension of subspaces, it follows that the dim(null$B$) $\leq$dim(nullAB).

Next, from the theorem we proved in class regarding the relation between the dimension of the null space of a matrix and its rank, we find that

$$k = \dim(\text{null}B) + \text{rank}(B) \quad \text{and} \quad k = \dim(\text{null}AB) + \text{rank}(AB)$$

(why are they both equal to k?). So, in particular,

$$\dim(\text{null}B) + \text{rank}(B) = \dim(\text{null}AB) + \text{rank}(AB)$$

Inserting to this inequality the inequality we found above:

$$\dim(\text{null}B) \leq \dim(\text{null}AB),$$

we get that indeed $\text{rank}(AB) \leq \text{rank}(B)$, which is what we wanted to show.

iv. We first note that under the conditions of the question, the matrix A cannot be invertible. Indeed, if it were invertible, then A^2 was invertible as well (we proved in recitation that the multiplication of two invertible matrices is invertible). But if both A and A^2 were invertible then by Q3 in this HW they would both have the same rank, which contradicts the conditions we were given. So, indeed, A is not invertible.
Since A is not invertible Q3 from this HW implies that $\text{rank}A \leq 2$. So, Q6(iii) of this HW implies that

$$0 \leq \text{rank}(A^3) \leq \text{rank}(A^2) \leq \text{rank}(A) \leq 2$$

Since $\text{rank}(A)$, $\text{rank}(A^2)$ and $\text{rank}(A^3)$ are three different integer numbers we conclude that $\text{rank}(A) = 2$, $\text{rank}(A^2) = 1$ and $\text{rank}(A^3) = 0$. The only vector space that has dimension 0 is the vector space that contains only the zero vector. It follows that the only matrix that has rank 0 is the zero matrix (since the rank of a matrix is the dimension of the column space). We conclude that indeed A^3 is the zero matrix.