MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Homework 7

1. In each of the following you are given a vector space (you do not need to prove that it is a vector space). Find a basis for this space and **prove** that the system you found is indeed a basis.

   i. $\{\begin{pmatrix} t+2s+r \\ t-s-r \\ t+3r \\ 5r+5s \\ -2t+2s+r \end{pmatrix} : t, r, s \in \mathbb{R}\}$

   ii. $\{A \in M_4(\mathbb{R}) : (A)_{ij} = (A)_{ji} \ \forall 1 \leq i, j \leq 4\}$
   (Remark: This is the collection of all 4×4 **symmetric** matrices).

   iii. $\{A \in M_4(\mathbb{R}) : (A)_{ij} = -(A)_{ji} \ \forall 1 \leq i, j \leq 4\}$
   (Remark: This is the collection of all 4×4 **anti–symmetric** matrices).

   iv. The set of solutions of the homogeneous system
$$\begin{cases} x + y - z = 0 \\ -x + 2y - 5z = 0 \\ 2x + 5y - 8z = 0 \end{cases}$$

   v. $\{p(x) \in \mathbb{R}_2[x] : p(1) = p(2)\}$.

   vi. $\{p(x) \in \mathbb{R}_3[x] : p(1) = 0 \text{ and } p'(1) = 0\}$.

   vii. $\{A \in M_2(\mathbb{R}) : A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0\}$.

2. a Consider the space $\mathbb{R}_2[x]$. Recall that we mentioned in class that $B = (1, x, x^2)$ and $C = (x, x^2, 1)$ are both ordered bases for $\mathbb{R}_2[x]$.

   i. Prove that $D = (1, 1+x, (1+x)^2)$ is also an ordered basis for for $\mathbb{R}_2[x]$.

   ii. Compute $[3 - 2x + x^2]_B$, $[3 - 2x + x^2]_C$ and $[3 - 2x + x^2]_D$.

   iii. Find polynomials $p_1(x), p_2(x), p_3(x) \in \mathbb{R}_2[x]$ which satisfy:
$$[p_1(x)]_B = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, [p_2(x)]_C = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, [p_3(x)]_D = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}.$$

   b. Consider the space $\mathbb{R}^3$.

i. Find all values of k for which $B = \left(\begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} k \\ 7 \\ 1 \end{pmatrix} \right)$ is a basis for $\mathbb{R}^3$.

ii. For $k = 2$ find $\left[\begin{pmatrix} 1 \\ 14 \\ -8 \end{pmatrix} \right]_B$.

iii. For $k = 2$ find $b \in \mathbb{R}^3$ such that $[b]_B = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$.

c. Consider the vector space in Q1(vii) and the basis you found for this space. Give this basis an order and denote this ordered basis B. For each of the following vectors determine whether it belongs to the space and if so find its coordinates with respect to the ordered basis B:

i. $\begin{pmatrix} 6 & -3 \\ -4 & 2 \end{pmatrix}$

ii. $\begin{pmatrix} 0 & 0 \\ 2 & -1 \end{pmatrix}$.

3. Let V be a vector space and let $v_1, ..., v_n \in V$. **Prove** the following claims.

i. If $\{v_1, ..., v_n\}$ is a basis for V then it is a maximal linearly independent set in V.

ii. If $\{v_1, ..., v_n\}$ is a maximal linearly independent set in V then it is a basis in V.

iii. If $v_1, ..., v_n$ is linearly independent then $v_1, ..., v_n$ is a basis for $\text{span}\{v_1, ..., v_n\}$.

iv. If V is a subspace of some vector space W, and $v_1, ..., v_n$ is a basis for V, then for $w \in W$ we have: $\text{span}\{v_1, ..., v_n, w\} = V$ if and only if $\{v_1, .., v_n, w\}$ is linearly dependent.

4. Let $A \in M_n(\mathbb{R})$. Prove that the following are equivalent.

i. A is invertible.

ii. The columns of A form a basis for $\mathbb{R}^n$.

iii. The rows of A form a basis for $\mathbb{R}^n$.

(Hint: You might want to use Q6 from HW6).