MATH-1564-K1,K2,K3 –Linear Algebra with Abstract Vector Spaces
Final exam

- The exam starts when this fact is indicated by the instructor. The exam ends at 5:40. The length of the exam is roughly two hours and 50 minutes.
- The use of calculators is NOT permitted.
- The use of written notes is NOT permitted.
- There are 9 questions with points as indicated (with 100 points in all).
- Explain yourself clearly and justify all of your claims. If you use a result which was stated in class, recitation or homework then make sure to indicate this fact explicitly.

Name: ________________________________

Recitation group: ________________

1. [10 points] Consider the following linear system:

$$\begin{cases} x + 2y + az = 1 \\ x + ay + z = 1 \\ -x + (10 - 6a)y + a^2 z = a - 4 \end{cases}$$

Find the values of a for which the following statements hold. Justify your answers.

i. The system has exactly one solution.
ii. The system has no solution.
iii. The system has an infinite amount of solutions.

2. Consider the vector space
$$W = \{p \in \mathbb{R}_3[x] : p(1) = 0\},$$
and consider the linear transformation
$$T : W \mapsto W$$
which is defined by
$$Tp(x) = p'(x) - p'(1).$$
Consider also the following ordered basis of W,
$$B = (x - 1, x^2 - 1, x^3 - 1).$$
(You do not need to prove that W is a vector space, nor that T is a linear transformation or that B is a basis for the space.)
   i. [5 points] Find $[T]_B$.
   ii. [5 points] Use the matrix $[T]_B$ to find $dim(ImT)$. Justify your answer.
   iii. [5 points] Use the matrix $[T]_B$ to find a basis for $kerT$. Justify your answer.

3. [10 points] Consider the matrix:

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 4 & 0 \\ 1 & 0 & 3 \end{pmatrix}$$

Is A diagonalizable? If so, find a diagonal matrix D which is similar to A. Justify your answers.

4. [10 points] Let V be a vector space of dimension 3 over $\mathbb{R}$ and let $B = (v_1, v_2, v_3)$ be an ordered basis of V. Let $T : V \mapsto V$ be a linear transformation that satisfies the following relation.

$$[T]_B = \begin{pmatrix} 1 & 0 & 2 \\ 1 & 3 & 2 \\ 1 & 0 & 2 \end{pmatrix}$$

   a. [5 points] Is the set $\{Tv_1, Tv_2, v_3\}$ a basis for V? Justify your answer.
   b. [5 points] For each one of the following vectors determine whether it is an eigenvector of T or not. Justify your answer.
      i. v_2.
      ii. $2v_2 - v_1$.

5. [10 points] Consider the vector space V the basis $B = (v_1, v_2, v_3)$ and the linear transformation $T : V \mapsto V$ given in Question 3. Consider also the ordered set $C = (v_1, v_1 + v_2, v_1 + v_2 + v_3)$. Prove that C is also a basis for V and find $[T]_C$.

6. [10 points] **Prove or disprove** the following claim: Let $A \in M_n(R)$ and denote

$$b = \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}.$$

If the linear system $(A|b)$ has exactly one solution then the linear system $(A^t|b)$ has exactly one solution.

7. [10 points] Let V be a vector space **of dimension** 4 over $\mathbb{R}$ and W be a subspace of V **of dimension** 3. **Prove or disprove** each one of the following claims.
   i. [5 points] If $\{w_1, w_2, w_3\}$ is a basis of W and $v \notin W$ then $\{w_1, w_2, w_3, v\}$ is a basis of V.
   ii. [5 points] If $\{w_1, w_2, w_3, v\}$ is a basis of V and $v \notin W$ then $\{w_1, w_2, w_3\}$ is a basis of W.

8. Let V be a vector space **of dimension** 3 over $\mathbb{R}$. Let $T : V \mapsto V$ and $S : V \mapsto V$ be a linear transformations which satisfy $S \circ T = 0$. **Prove** the following claims:
   i. [5 points] $ImT \subseteq kerS$.
   ii. [5points] If $dim(kerT) = dim(ImS)$ then $ImT = kerS$.
   iii. [5points] If $T \neq 0$ then 0 is an eigenvalue of S.

9. Let V be a vector space **of dimension** 3 over $\mathbb{R}$ and W be a vector space **of dimension** 4 over $\mathbb{R}$. Prove or disprove each one of the following claims:
   i. [5 points] There exist linear transformations $T : V \mapsto W$ and $S : W \mapsto V$ such that $S \circ T = id_V$.
   ii. [5 points] There exist linear transformations $T : V \mapsto W$ and $S : W \mapsto V$ such that $T \circ S = id_W$.