Experiment 2: Transients and Oscillations in an RLC Circuit

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Abstract

In this experiment we studied transients of RLC circuits, driven by both square and sinusoidal voltage generators, at a number of different frequencies. A close relationship was found between theory and our experiments. Theoretical results follow a second-order linear differential equation and have three responses: critical, under- and over-damping. We demonstrate that the period squared and capacitance follow a linear relationship described by the inductance and some constants, as expected. In addition, we describe the relationship between the rate of exponential decay and the resistance to determine the extra resistance of the circuit. Furthermore we determined a linear relation for the resistances needed for critical damping as a function of capacitance. Finally we examined the magnitude of impedance and frequency in order to determine the resonant frequency, bandwidth and quality factor. One failure of this experiment comes from our quality and bandwidth not following our theoretical trend, which was due to our experimental setup more than wrong theory.

1 Purpose

The purpose of this experiment was to study the behavior of the transient response of an RLC circuit based on varying the capacitance and resistance of the circuit. In the first part of the experiment we looked at the voltage, as a function of time, across the capacitor. We varied the capacitance and found the period of oscillations in order to better understand the relationship between the two values. In the next part of the experiment we added back in the resistor and evaluated the log decrement as a function of resistance in order to find the extra series resistance due to the other components. Next we wanted to determine the relationship between resistance and critical damping, so we varied the resistance along several capacitances. We varified our theoretical model by plotting the resistance squared by 1/C. Finally we found the voltage across the capacitor as a function of frequency in a slightly modified, sinusoidally-driven circuit. We looked at the impedance as a function of frequency in order to determine the resonant frequency, the bandwidth and quality factor. We compared this to the theoretical model

2 Theory

For an RLC Circuit our 2nd order linear differential equation takes the following form:

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V(t) \tag{2.1}$$

We have our inductance term that acts as inertia, our resistance that dampens the oscillations and our capacitance that acts as our restoring force. V(t) is our driving force function. Inductance has the units of Henry's, Resistance has the units of Ohm's, Capacitance has the units of Farad's and Voltage has the units of Volts.

This equation governs our system and gives it the form of a damped oscillation with a driving function, $V(t)$. Solving the homogeneous differential equation we get the roots

$$s_{1,2} = a \pm b \tag{2.2}$$

$$a = \frac{R}{2L},\ b = \sqrt{\left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right)} \tag{2.3}$$

These RLC transients depend on the sign of b^2, the discriminant. We have three possible cases: over-damped (positive discriminant), critically damped (discriminant vanishes) and under-damped (negative discriminant). These situations are described as follows

If $b^2 > 0$ then the roots are real and the system is said to be over-damped. This means the roots are purely exponential and the response will drop off exponentially If $b^2 = 0$ the roots are also real and the system is said to be critically damped. The same exponential behavior will occur as $b^2 > 0$ but the decay will happen much quicker. If $b^2 < 0$ the roots will be imaginary and the system is said to be under-damped. The imaginary solutions mean that the system will oscillate along a decreasing exponential, eventually decaying all the way to 0. This oscillations occurs at a frequency and period of

$$f = \frac{1}{T} = \left(\frac{1}{2\pi}\right)\sqrt{\left(\frac{1}{LC}\right) - \left(\frac{R}{2L}\right)^2} \tag{2.3}$$

Since the oscillations decay exponentially we can describe this with a quantity known as the log decrement, δ. The log decrement is defined as

$$\delta = \ln\frac{q(t_{max})}{q(t_{max}+T)} = aT \tag{2.4}$$

The aT part of this log decrement equation comes from the exponent of our function and serves as our second definition of log decrement.

Furthermore, we can define the quality factor, Q, in order to describe how long it takes for the signal to decay until is roughly zero. Quality factor can be described by

$$Q = 2\pi\frac{\text{Total stored energy}}{\text{Decrease in energy per period}} = \frac{\pi}{\delta} \tag{2.5}$$

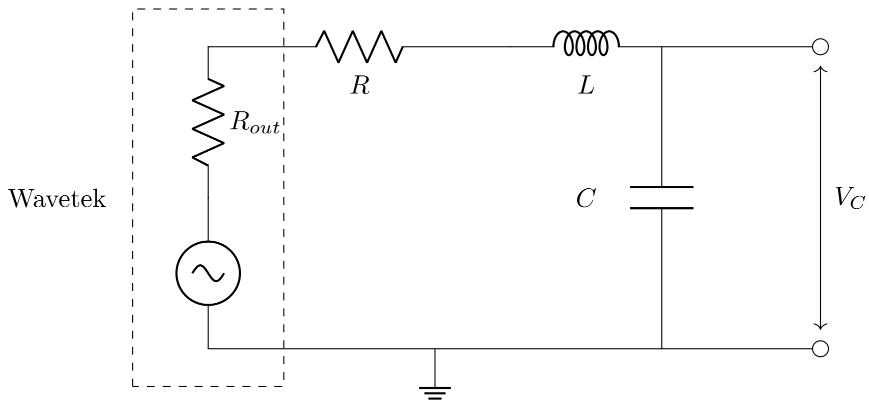

Figure 1: Circuit setup for parts A, B and C

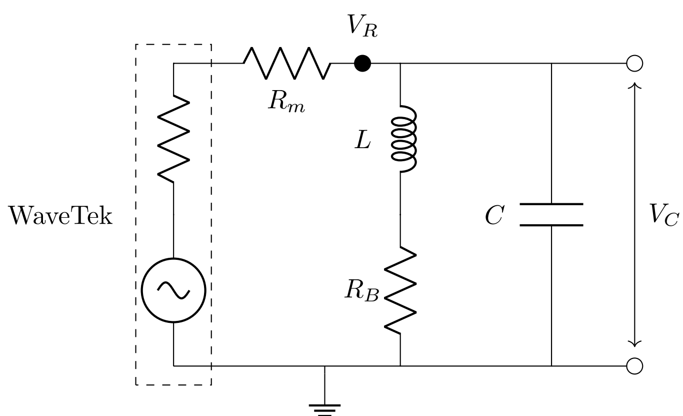

Figure 2: Circuit setup for part D

3 Experiment

3.1 Equipment

For parts A - C, we conducted our simulations through the program Multisim. We used a generic function generator, a thermal noise maker (to better simulate an actual laboratory environment), two resistors, an inductor, a capacitor and a generic oscilloscope.

For part D, we varied our approach and used Multisim Live. Here we utilized a function generator, a thermal noise maker, three resistors, an inductor, a capacitor and two oscilloscopes.

In both of these cases the function generator and one of the resistors were analogous to the "Wavetek" function generator used in actual lab. Throughout the entire lab we left the thermal noise generator at 100kΩ, 27°C and 1MHz as well as keeping the internal resistance of our fake WaveTek at 50Ω

For our data analysis and plot production we used the program Origin 2020 and Python.

3.2 Part A: Determine dependence of frequency on capacitance

For this part of the experiment we removed the first resistor, set our inductor to 70 mH and varied our capacitance. We tested 14 values of capacitance ranging from 0.05 to 10.0 μF in order to learn more about the dependence of the period of oscillations for the transient on these difference values of capacitance

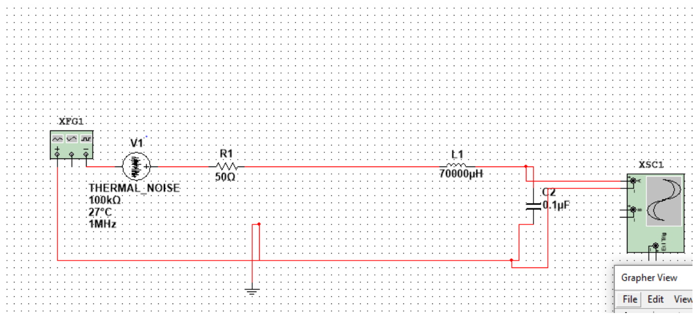

Figure 3: Our multisim setup for part A

The general form of a sine function for any damped oscillation is the following:

$$\sin(\omega(x - x_0)) \tag{3.2.1}$$

However, when Origin computes a non-linear curve fit for our damped sine function it uses this form:

$$\sin\left(\frac{\pi(x - x_0)}{\mathrm{w}}\right) \tag{3.2.2}$$

We can find the relationship between Origin's w and the period T fairly easily now. Knowing that ω is 2π over the period and also that ω is π over w we derive the following relationship:

$$\frac{\mathrm{w}}{2} = \mathrm{T} \tag{3.2.3}$$

Transient response with 0.3μF Capacitor

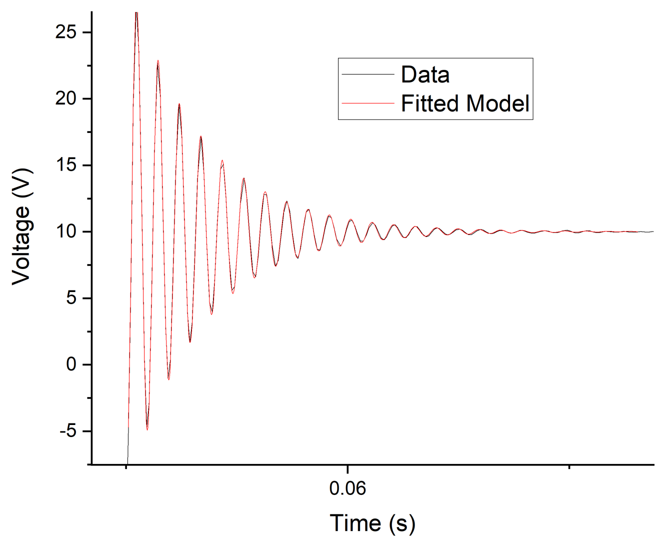

Figure 4: Example of transient response of voltage across the capacitor for part A of the experiment, this particular one is 0.3 μF

Taking our values for w and capacitance, as well as finding our values of T we produce the following table:

Table 1: Period of oscillations and Origin's w as a function of capacitance.

C (μF)	w (ms)	T (ms)
0.05	0.197 $\pm$ 2.59$\times10^{-5}$	0.394 $\pm$ 5.18$\times10^{-5}$
0.1	0.278 $\pm$ 4.55$\times10^{-5}$	0.556 $\pm$ 9.10$\times10^{-5}$
0.2	0.394 $\pm$ 6.89$\times10^{-5}$	0.788 $\pm$ 1.38$\times10^{-4}$
0.3	0.483 $\pm$ 1.02$\times10^{-4}$	0.966 $\pm$ 2.04$\times10^{-4}$
0.4	0.557 $\pm$ 1.47$\times10^{-4}$	1.114 $\pm$ 2.94$\times10^{-4}$
0.5	0.622 $\pm$ 1.51$\times10^{-4}$	1.244 $\pm$ 3.02$\times10^{-4}$
0.6	0.682 $\pm$ 1.87$\times10^{-4}$	1.364 $\pm$ 3.74$\times10^{-4}$
0.7	0.736 $\pm$ 2.66$\times10^{-4}$	1.472 $\pm$ 5.32$\times10^{-4}$
0.8	0.788 $\pm$ 2.43$\times10^{-4}$	1.576 $\pm$ 4.86$\times10^{-4}$
0.9	0.835 $\pm$ 2.22$\times10^{-4}$	1.670 $\pm$ 4.44$\times10^{-4}$
1.0	0.887 $\pm$ 2.61$\times10^{-4}$	1.774 $\pm$ 5.22$\times10^{-4}$
2.0	1.24 $\pm$ 6.65$\times10^{-4}$	2.480 $\pm$ 1.33$\times10^{-3}$
5.0	1.98 $\pm$ 5.45$\times10^{-3}$	3.960 $\pm$ 1.09$\times10^{-2}$
10.0	2.85 $\pm$ 3.40$\times10^{-3}$	5.700 $\pm$ 6.80$\times10^{-3}$

The errors bars in this table are generated by Origin and correspond to the fit being not exactly perfect as our data contains noise.

Looking at Equation 2.3 and making a simplification we can determine what the slope of a graph of $1/T^2$ versus $1/C$ would be.

Our simplification is that

$$\frac{1}{LC} \gg \left(\frac{R}{2L}\right)^2 \tag{3.2.4}$$

Using our values of R and L from above (50 Ω and 70 mH, respectively) and using a sample value of 1μF for C we get the following:

$$1.43 \times 10^7 \gg 1.27 \times 10^5$$

Thus we can simplify Equation 2.3 to

$$\left(\frac{1}{T}\right)^2 \approx \left(\frac{1}{2\pi}\right)^2 \frac{1}{LC} \tag{3.2.5}$$

and find the slope of the graph of 1/T^2 versus 1/C should be

$$m = \frac{1}{4\pi^2 L} \tag{3.2.6}$$

1/T^2 versus 1/C

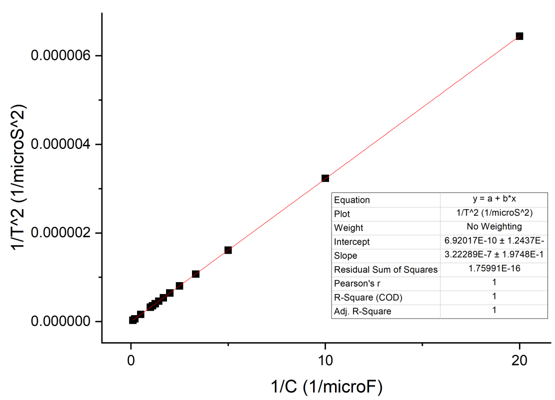

Figure 5: Display of linear relationship of 1/T^2 versus 1/C

By figure 5, the R values for the plot are all 1, therefore we know that the linear relationship is the correct one

We know our theoretical value for the slope should be 0.361 and from the linear regression we obtain $3.22 \times 10^{-7} \pm 1.974 \times 10^{-10}$ (accounting for units this changes to $0.322 \pm 1.974{\times}10^{-4}$). This comes in at around 10% error.

Comparing now T^2 versus C, from equation 3.2.5 we get the following result:

$$T^2 \approx (2\pi)^2 LC \tag{3.2.7}$$

Thus the slope of this graph is $(2\pi)^2 L$

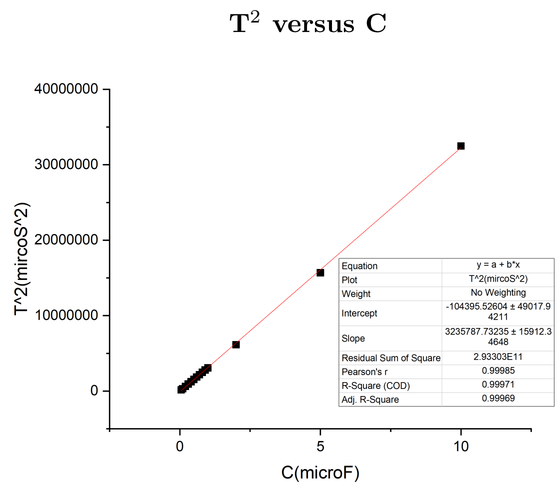

Figure 6: Display of linear relationship of T^2 versus C

Similarly to above our theoretical slope should be 2.76 and from our linear regression we get $3.22 \times 10^6 \pm 1.59 \times 10^4$ (accounting for units this changes to $3.22 \pm 1.59 \times 10^{-2}$). This comes in at around 16% error.

Sources of error in our measurement could come from a number of different places. As previously mentioned, the error on our data points come directly from Origin and the noise in our data producing some non-perfect fits. In addition, to simplify things we truncated our w values at three decimal places and more decimal precision could have produced clearer results.

3.3 Part B: Determine the dependence of log decrements on resistance

For part B we used essentially the same circuit as part A, we just added back in the resistor. The capacitor was set to 1μF, the inductor set to 70 mH, and we varied the resistance on these values: 100, 90, 80, 70, 60, 50, 40, 30, 20, 10 and 0 Ω.

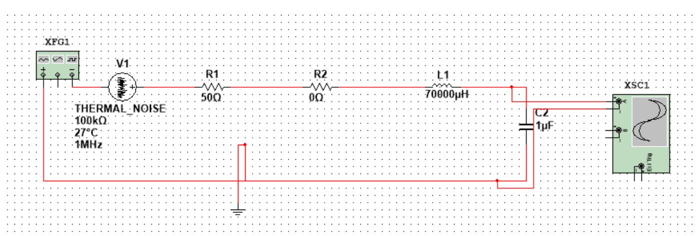

Figure 7: Our multisim setup for part B

During this part of the experiment we are trying to find the relationship between how the rate of damping of the transient depends on the resistance. In order we are measuring the log decrement (Eqn 2.4) of all the plots in order to determine this relationship. According to our theory, the log decrement is aT, or the exponent of our equation. Knowing this we can find the following theoretical relationship:

$$\delta = aT = \left(\frac{R}{2L}\right) T = \left(\frac{R}{2L}\right) \frac{2\pi}{\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}} = \pi \frac{R}{L}\sqrt{LC}\frac{1}{\sqrt{1 - \frac{R^2C}{4L}}} \approx \pi\sqrt{\frac{C}{L}}R \tag{3.2.1}$$

Table 2: Log decrement of first couple peaks on all resistances

Resistance (Ω)	δ_1	δ_2	δ_3	δ_{avg}
0	0.544 ± 0.030	0.514 ± 0.030	0.589 ± 0.030	0.549 ± 0.030
10	0.623 ± 0.030	0.627 ± 0.030	0.600 ± 0.030	0.631 ± 0.030
20	0.757 ± 0.030	0.764 ± 0.030	0.739 ± 0.030	0.754 ± 0.030
30	0.892 ± 0.040	0.815 ± 0.040		0.854 ± 0.040
40	0.968 ± 0.040	1.014 ± 0.040		0.990 ± 0.040
50	1.110 ± 0.040	1.098 ± 0.040		1.104 ± 0.040
60	1.204 ± 0.040	1.211 ± 0.040		1.208 ± 0.040
70	1.310 ± 0.040	1.256 ± 0.040		1.283 ± 0.040
80	1.391 ± 0.040	1.470 ± 0.040		1.431 ± 0.040
90	1.468 ± 0.045			1.468 ± 0.045
100	1.592 ± 0.045			1.592 ± 0.045

Note about Table 2, δ_1 is defined as the log decrement between the first and second peaks, δ_2 is between third and fourth peaks and δ_3 is between fifth and sixth peaks. We took an average of them in order to get the most accurate value. In addition, as the resistance increased the peak voltage decreased and thus it became more difficult to get an accurate reading of the third, fourth, fifth and sixth peaks, therefore we omitted those values. The errors in measurement are our best guesses, as the peaks decreased in magnitude it became more and more difficult to measure the magnitudes accurately

Resistance versus average log decrement

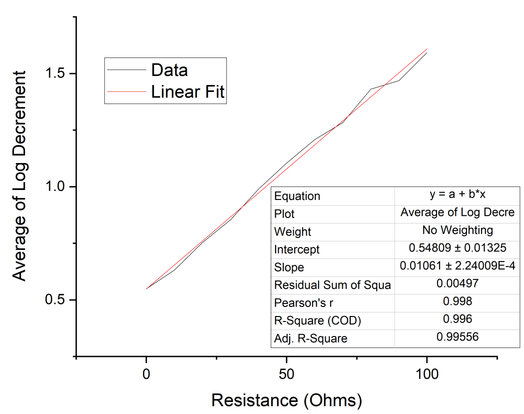

Figure 8: Linear relationship between the resistance and the average of our log decrements, from Table 2

As we can see in Figure 8, we obtain a linear relationship between the resistance and the log decrement, as expected. The slope of this linear relationship is $0.01061 \pm 2.24\times10^{-4}$

and (from Eqn. 3.2.1) the theoretical value for this slope should be 0.0118. This yields an approximate 10% error, which is fairly close. In addition, we find that our resistance intercept is -51.657 $\pm$ 2.29 Ω, which is the extra resistance of the entire circuit, which makes sense with our value of 50 Ω inside the Wavetek.

One of the main sources of this error comes from when we are finding the peaks on Origin, we manually select values we perceive as close to the maximum. Even a little bit of error in the selection of our peak will cause our log decrement to change. One possible fix to this would be using a peak-finding function within Origin to more accurately describe each of the peaks.

3.4 Part C: Determine the value of resistance for critical damping

The experimental setup for this part is the same as in part B, see figure 6. This part of the experiment we wanted to examine the critical damping response. This is where $b^2 = 0$ therefore our governing equation for the part of the experiment was:

$$b^2 = \left(\frac{R}{2L}\right)^2 - \left(\frac{1}{LC}\right) = 0 \tag{3.4.1}$$

There are a lot of variables in this equation that we could have varied, but we wanted to study specifically the effect of varying the capacitance and then finding the so called critical resistance required for critical damping. The theoretical model for this R_{critical} is as follows

$$R_{\text{critical}} = 2\sqrt{\frac{L}{C}} \tag{3.4.2}$$

We ran this experiment for 5 capacitances ranging from 1.0 to 0.01 μF and obtained these results for the critical resistances

Table 3: Capacitances, Theoretical R_{critical} and Experimental R_{critical}

C (μF)	Theoretical R_{critical} (Ω)	Experimental $R_{critical}$ (Ω)
1.0	529.15	530 $\pm$ 1
0.5	748.33	750 $\pm$ 3
0.1	1673.32	1650 $\pm$ 20
0.05	2366.43	2350 $\pm$ 20
0.01	5291.50	5350 $\pm$ 50

The error bars in our experimental R_{critical} come from us guessing approximately where the theoretical R_{critical} should be and setting our resistance near to that.

Example of different types of damping

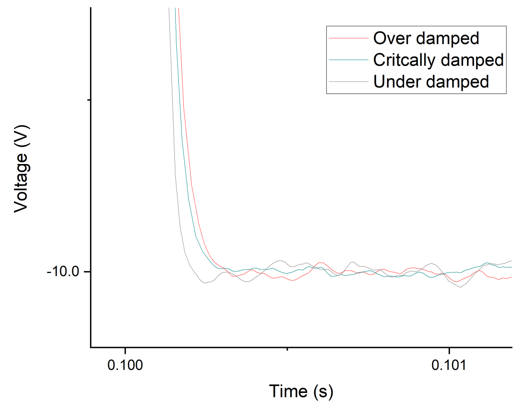

Figure 9: Example data for under, over and critically damped systems

When we plot $R^2_{critical}$ versus 1/C we expect to see a straight line with slope of

$$m = 4L \tag{3.4.3}$$

Which, in our case with our inductance value of 70 mH, should yield a slope of about 0.28

Theoretical R^2_{critical} versus 1/C

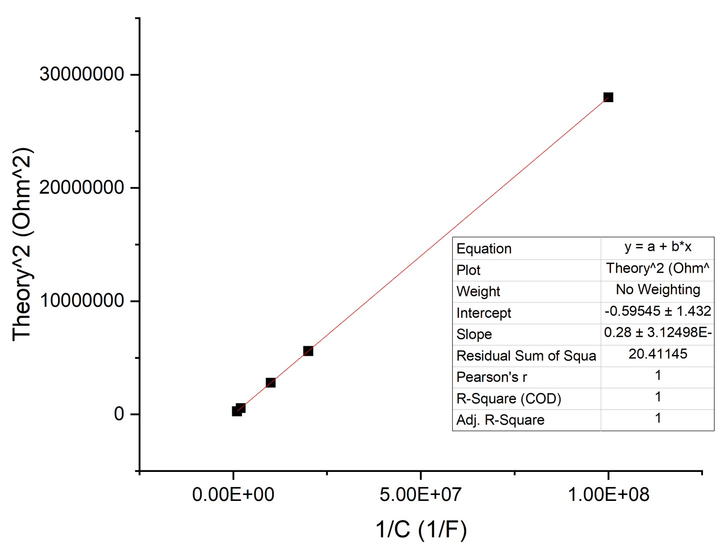

Figure 10: The theoretical plot follows our linear relationship, as expected

Experimental R^2_{critical} versus 1/C

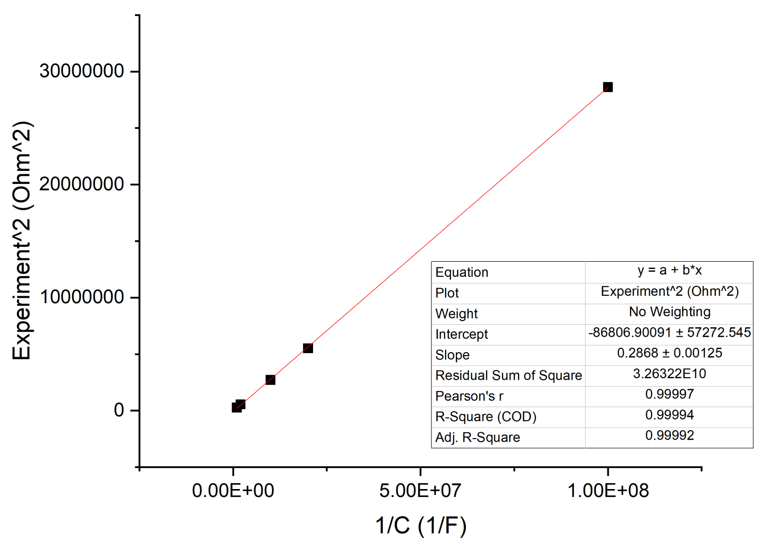

Figure 11: The experimental result also follows this linear trend, with a similar slope as the theory

From Figure 10 and 11 we see that the theoretical slope ($0.28 \pm 3.12\times10^{-8}$) and the experimental slope ($0.2868 \pm 1.25\times10^{-3}$) follow very closely with our trend. The error on the experimental slope has 2% error.

One of the discrepancies that we could have had come from finding our experimental resistance, instead of cranking up the resistor by 1's or 10's we chose to add 100 Ohms most of the time to find the critical resistance. If we were to, instead, change the resistance more finely we could more accurately find the critical resistance.

3.5 Part D: Measure the response of an RLC Circuit to a sinusoidal signal

As we moved into multisim live we changed up the circuit significantly. Now we were reading the voltage across a resistor and the capacitor. We used a the same function generator as before with the same internal resistance. Additionally, we used a 1 kΩ, a 70 mH inductor, 50 Ω resistor and a 1 μF capacitor and two voltage readers, one for the capacitor and the other for the first resistor.

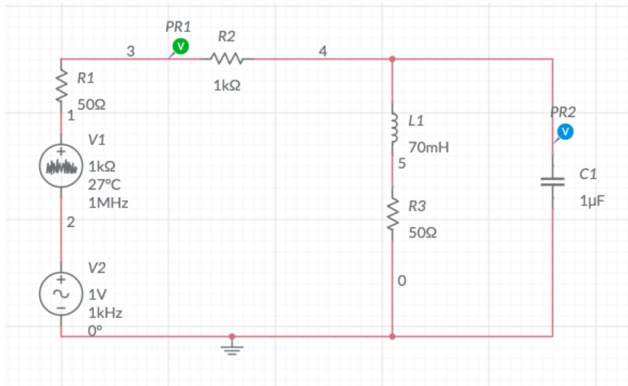

Figure 12: Our multisim live setup for part D

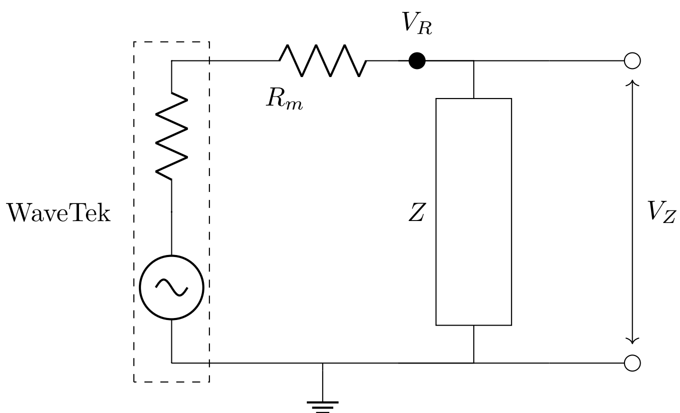

Figure 13: Simplified model setup for part D, where Z is the impedance of the right half of the circuit

Looking at Figure 13 we can see that the circuit, basically, has two components, R_m and Z. V_Z, the reader across the RLC component of the circuit, will read the following:

$$V_Z = V_0 \frac{Z}{R_m + Z} \tag{3.5.1}$$

In general the impedance, Z, is a complex quantity, this is true for our circuit. Since we have a resistor in series with an inductor as well as in parallel with a capacitor we know that the value of the impedance will take the form:

$$Z = \frac{\frac{1}{j\omega C}[j\omega L + R]}{\frac{1}{j\omega C} + j\omega L + R} \tag{3.5.2}$$

where R is is the resistance of the coil and the Wavetek. Equation 3.5.2 can be simplified to the following:

$$Z = \frac{R + j\omega L}{\left(1 - \frac{\omega^2}{\omega_0^2}\right) + j\omega RC} \tag{3.5.3}$$

and the magnitude is:

$$|Z| = \frac{\sqrt{(R)^2 + (\omega L)^2}}{\sqrt{\left(1 - \frac{\omega^2}{\omega_0^2}\right)^2 + (\omega RC)^2}} \tag{3.5.4}$$

where ω_0 is defined as:

$$\omega_0 = 2\pi f_0 = \sqrt{\frac{1}{LC}} \tag{3.5.5}$$

We ran an AC sweep over the range of 1 Hz to 10 MHz with 30 points per decade in order to measure find the resonant frequency of the RLC part of our circuit.

Table 4: Some of our data points for Frequency and Impedance of our circuit

Frequency (Hz)	Voltage across Capacitor (V)
1.00×10^0	$0.045456 \pm 4.55\times10^{-4}$
2.15×10^0	$0.045463 \pm 4.55\times10^{-4}$
4.64×10^0	$0.045495 \pm 4.55\times10^{-4}$
1.00×10^1	$0.045641 \pm 4.56\times10^{-4}$
2.15×10^1	$0.046315 \pm 4.63\times10^{-4}$
4.64×10^1	$0.049352 \pm 4.94\times10^{-4}$
1.00×10^2	$0.062018 \pm 6.20\times10^{-4}$
2.15×10^2	$0.109387 \pm 1.09\times10^{-3}$
4.64×10^2	$0.353698 \pm 3.53\times10^{-3}$
1.00×10^3	$0.225939 \pm 2.26\times10^{-3}$
2.15×10^3	$0.076039 \pm 7.60\times10^{-4}$
4.64×10^3	$0.033195 \pm 3.32\times10^{-4}$
1.00×10^4	$0.015211 \pm 1.52\times10^{-4}$
2.15×10^4	$0.007041 \pm 7.04\times10^{-5}$
4.64×10^4	$0.003266 \pm 3.27\times10^{-5}$
1.00×10^5	$0.001516 \pm 1.51\times10^{-5}$
2.15×10^5	$0.000704 \pm 7.04\times10^{-6}$
4.64×10^5	$0.000327 \pm 3.27\times10^{-6}$
1.00×10^6	$0.000152 \pm 1.52\times10^{-6}$
2.15×10^6	$0.000070 \pm 7.00\times10^{-7}$
4.64×10^6	$0.000033 \pm 3.30\times10^{-7}$
1.00×10^7	$0.000015 \pm 1.50\times10^{-7}$

The error bars here are a standard 1% of each value

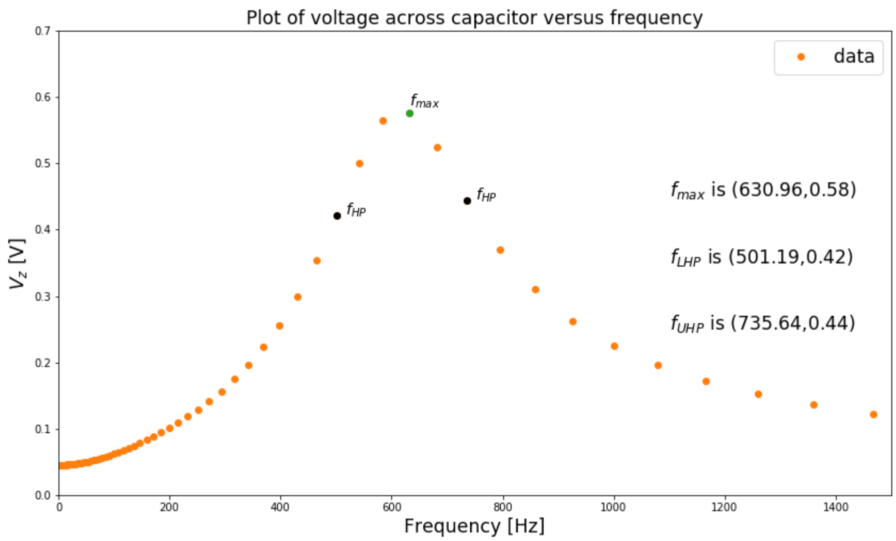

Figure 14: Our data for the voltage across the capacitor

When looking at Figure 14 we can determine a couple of things, primarily the maximum frequency, where we see the big peak. From here we can find the half points of power in order to find the bandwidth. The half points of power are found by multiplying the maximum impedance by $1/\sqrt{2}$. This factor of $\sqrt{2}$ is because power is proportional to voltage squared so in order to half the power we need to find the voltage times $1/\sqrt{2}$. Once we know the f_{max} and the bandwidth (Δf) we can find our quality factor, Q by:

$$Q = \frac{f_{max}}{\Delta f} \tag{3.5.6}$$

Our f_{max} is 630.95 $\pm$ 10 Hz. The half-power points are 501.19 $\pm$ 10 Hz and 735.64 $\pm$ 10 Hz our bandwidth is 243.45 $\pm$ 15 HZ making our Quality factor 2.691 $\pm$ .25.

One of the reasons that we see our error here comes from the fact that we do not have 100% resolution here, we only have a finite amount of data points for the frequencies in our range of interest. Take our half-power points for example, there are very few points along the steep part of this curve, thus getting an accurate assessment of the half points is rather difficult. Another cause for error is due to the nonideality of the wires and elements we are using. Even though we are simulating the circuit, we neglect the resistance of the wires, inductor and capacitor. This accounts could account for difference in quality factor.

4 Conclusions

From all of this we determine that RLC circuits tend to follow rather closely with their theoretical models given by our 2nd order differential equation (Eqn 2.1). We saw how the period of oscillations, squared, depended linearly on the inverse of capacitance of the system. Similarly, we found the linear dependence of the log decrement on the resistance of our circuit. In addition we studied critically damped systems and found the linear relationship between the critical resistance, squared, and the inverse of capacitance. Finally, we studied the frequency response of an RLC circuit and determined the resonant frequency, bandwidth and quality factor as well as comparing those to a theoretical model. Overall, all of our theories held up quite well against our data sets and we found all the expected relationships. In terms of things that we could've done better for this experiment, one of the biggest things was that, especially with the later experiments, we were pressed for time in the lab, therefore we didn't have as much time to accurately dial our equipment more precisely to gain more accurate conclusions. Throughout the lab, due to the software we were using, there was a lot of estimating error because we had to go through and manually select a lot of the data, therefore is we had some graphing suite that could go through and find the peaks for us our results would be more accurate. Finally, some of our error came from not having 100% resolution along the graphs, in the future having a reader with a higher sampling rate could yield much more precise numbers.