Experiment 3: Frequency Domain Analysis of Linear Circuits using synchronous detection

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Abstract

In this experiment we studied the frequency response of linear circuits, driven by square, triangular and sinusoidal voltage generators. We conducted these experiments to learn about Fourier analysis and the transfer function using synchronous detection. We ran a Lock-In Amplifier into all of our circuits to simultaneously detect the frequency response of low- and high- pass filters and an RLC circuit. The time constant was the primary focus for this part, for the low-pass filter we saw a fairly close relation of the theoretical time constant and our computed time constant. For the high-pass filter we were able to see that our equations held but not quite up to our theoretical calculations. The RLC circuit also has some issues matching up with theory, but the model seemed correct. Finally, while looking at the Fourier components through square and triangle waves we saw a similar thing, good matching of linear relationship but deviation from theoretical values. One of the main sources of error was our measurement method, we very easily could have under- or overestimated our voltages when reading them off. Another potential source of error comes from our analysis, it's possible that our models might have been correct but we implemented them poorly into Origin.

1 Purpose

The purpose of this experiment is to study the frequency response on linear circuits through the transfer function. The transfer function represents the complex voltage read by our Lock-In Amplifier. By looking at the transfer function at a number of different frequencies on a High-Pass filter, Low-Pass filter and RLC circuit we can determine the critical frequencies of each of these systems. We wanted to test the theory that a transfer function could completely describe the frequency response of and LTI system. For the High-Pass filter and RLC circuit we obtain the right general trend but caught some disagreement between theory and fitted models. Our Low-Pass filter behaved much better and the transfer closely described our frequency response. Finally, for our study of Fourier series with our function generator we obtain the correct linear relationships but not quite our predicted values. We concluded that, in fact, the transfer function does allow us to gain some understanding of the frequency response of such systems, but we could not get values for our constraints to follow our predictions. In addition to using our transfer function to study the frequency response of different types of systems we also wanted to study the Fourier analysis of systems being done inside the Lock-In Amplifier.

2 Theory

In this lab what we spend most time looking at the transfer function. We can think of the transfer function as the response to some system being driven at some particular ω. The transfer function fully describes the response of a Linear Time Invariant (LTI) system.

$$x(t) = H(\omega)F(t) \tag{2.0}$$

The transfer equation can also be represented in another way, the ratio of the voltage out to the voltage in.

$$H(\omega) = \frac{V_{\text{out}}}{V_{\text{in}}} \tag{2.1}$$

The transfer function is a complex function that allows us to do Fourier analysis on a system. It can be broken up into two different parts, the real part and the complex. In our experiments we used a Lock-in Amplifier to measure each the real and complex parts of this transfer function in order to measure our RC and LC values. Each of our systems that we studied (Low-Pass Filter, High-Pass filter and RLC circuit) have their own transfer functions.

The transfer function for the Low-Pass Filter is as follows.

$$H(\omega) = \frac{1 - j\omega RC}{1 + (\omega RC)^2} \tag{2.2}$$

This transfer function can be broken up into a real and an imaginary part, as shown below.

$$H_{\text{real}}(\omega) = \frac{1}{1 + (\omega RC)^2} \tag{2.3}$$

$$H_{\text{imag}}(\omega) = \frac{-\, j\omega RC}{1 + (\omega RC)^2} \tag{2.4}$$

For a High-Pass Filter the transfer function is as follows.

$$H(\omega) = \frac{\omega RC}{1 + (\omega RC)^2}(\omega RC + j) \tag{2.5}$$

Again, we can break this down into real and imaginary components.

$$H_{\text{real}}(\omega) = \frac{(\omega RC)^2}{1 + (\omega RC)^2} \tag{2.6}$$

$$H_{\text{imag}}(\omega) = \frac{\omega RC}{1 + (\omega RC)^2} \tag{2.7}$$

Finally, for the RLC circuit our transfer function is defined in the following equation.

$$H(\omega) = \frac{1}{(1 - \omega^2 LC) + j(\omega RC)} \tag{2.8}$$

We can also find the real and imaginary components of this transfer function, though it does require more work than the High- and Low- Pass filters.

$$H_{\text{real}} = \frac{1 - \omega^2 LC}{(1 - \omega^2 LC)^2 + (\omega RC)^2} \tag{2.9}$$

$$H_{\text{imag}} = \frac{\omega RC}{(1 - \omega^2 LC)^2 + (\omega RC)^2} \tag{2.10}$$

These real and imaginary parts of the transfer function are useful for us because they are what our Lock-In Amplifier measures.

The way a Lock-In Amplifier works is some circuit inside of the amplifier works to produce a sinusoidal output at the same frequency as the reference signal, this is called the In-Phase component which will represent the real component. In addition the amplifier also generates a wave that is 90° out of phase with respect to the reference, this is called the Quadrature representing the imaginary component. These two signals represent two vectors whose sum represents the magnitude of the voltage of the source. This sum can be expressed through the norm of the In-phase and Quadrature components.

This is exceedingly useful for us because the In-Phase and Quadrature voltages are outputted by the Lock-In Amplifier to our computers through a program Jordan Sickle, our lab TA, created. Knowing the In-Phase and Quadrature voltages and controlling the voltage of our source we can measure the transfer function of our system. Measuring the input and output voltages we can find the real component using the In-Phase component and we can find the imaginary component using the Quadrature component. Together we can find the magnitude of the transfer function.

$$|H(\omega)| = \sqrt{H_R^2 + H_I^2} \tag{2.12}$$

Plotting this transfer function we can find out the frequency response of the system.

In addition we can look at what angle between our real and imaginary functions using the following.

$$\Theta = \arctan\left(\frac{H_I}{H_R}\right) \tag{2.13}$$

When we are looking at the response of the systems we can identify the so-called critical frequency of a system. This critical frequency corresponds to when the power of the circuit has decreased by a factor of 2, AKA -3dB. For Low- and High- Pass filters this critical frequency is defined by looking at the time constant of the system.

$$\tau = RC \tag{2.14}$$

The critical frequency is one over 2π of this time constant.

$$f_{\text{crit}} = \frac{1}{2\pi\tau} = \frac{1}{2\pi RC} \tag{2.15}$$

Meanwhile, for RLC circuits the critical frequency is measured by the following equation.

$$\omega_0 = 2\pi f_0 = \frac{1}{\sqrt{LC}} \tag{2.16}$$

Additionally for RLC circuits the quality factor can be found with the product of the time constant and this critical frequency.

$$Q = \frac{1}{\omega_0 \tau} \tag{2.17}$$

As we mentioned earlier, the Lock-In Amplifier uses Fourier Analysis. We look at this in depth in Part D. This leads us into Fourier Series. A Fourier series is a way of approximating any waveform on an interval as a sum of sinusoidal waves.

$$g(t) = a_0 + \sum_{n=0}^{\infty} a_n \cos\frac{2n\pi t}{T} + \sum_{m=1}^{\infty} b_m \sin\frac{2m\pi t}{T} \tag{2.18}$$

Where the coefficients a_n and b_m are described by the following processes.

$$a_n = \frac{2}{T}\int_0^T g(t)\cos\frac{2n\pi t}{T}dt \ (n = 0, 1, ...) \qquad (2.18)$$

$$b_m = \frac{2}{T}\int_0^T g(t)\sin\frac{2m\pi t}{T}dt \; (m = 1, 2, ...) \tag{2.19}$$

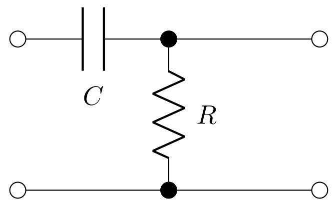


Figure 1: Model High-Pass filter used in Part A

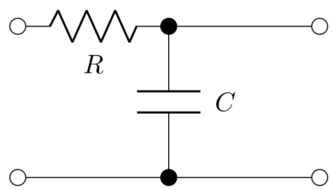


Figure 2: Model Low-Pass filter used in Part B

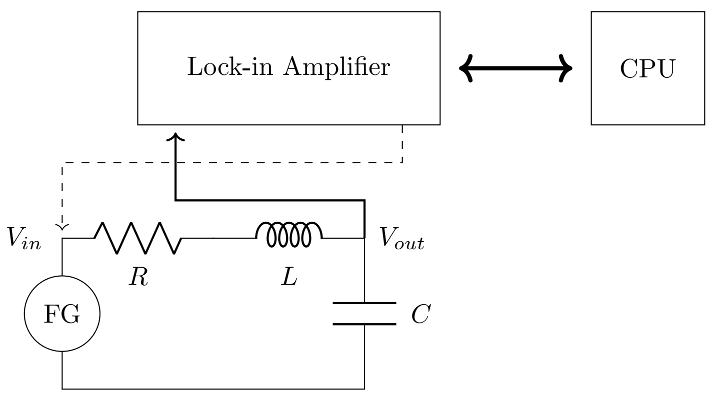


Figure 3: Detailed setup for the RLC circuit portion of the lab where we can see that the V_{in} of this system is just going to be the voltage supplied by the function generator.

3 Experiment

3.1 Equipment

For this experiment we utilize a number of different (this time real) components. The experiment centered around the SR-830 Digital Lock-in Amplifier. In addition our lab TA Jordan Sickle created a software that would allow us to interact digitally with the real SR-830.

For our experiments we used three different set ups. For Part A we used a resistor and a capacitor to create a low-pass filter. For Part B we used the same elements but in a different configuration. For Part C we used an inductor, capacitor and resistor. Finally for part D we used a function generator with a rectangular and triangular wave generator plugged directly into our Lock-In Amplifier. During the duration of the experiment we maintain the voltage source at a 100 mV

During the entire digital measurement taking process we utilized Jordan's executable program and varied the time constant and sensitivity of the Lock-In Amplifier so as to not overload the machine while maximizing our precision at the same time.

Data Analysis Note

All of our data analysis was done with origin in its non-linear curve fitting feature. One note is that while we plotted with respect to frequency, f, our fits are all with respect to angular frequency, ω. This was to make the algorithm work because it wouldn't converge for us while using $2\pi f$. Thus all of the fits for RC and LC are off by some factor of 2π, as described below.

$$RC_f = \frac{RC_\omega}{2\pi},\ LC_f = \frac{LC_\omega}{2\pi}$$

These can both be seen by the simple relation $\omega = 2\pi f$

3.2 Part A: High-Pass

For the first part of this experiment we utilized a High-Pass filter. A High-Pass Filter is one that takes frequencies above a certain threshold amount while attenuating lower frequencies. We can think of it as filtering out low frequencies.

An example of what our High-Pass Filter looks like in a circuit version can be seen in Figure 1.

Table 1: values for High Pass filter

R (kΩ)	C (nF)	$RC,\ \tau$ (s)	f_{crit} (Hz)
2.5 ± 1	52 ± 1	$1.30\times10^{-4} \pm 5.21\times10^{-5}$	1224.3 ± 490.3

Table 2: Data for In-phase and Quadrature voltages across High Pass filter across several frequencies

Frequency (Hz)	In-Phase (mV)	Quad. (mV)	Freq. (Hz)	In-Phase (mV)	Quad. (mV)
100	0.03 ± 0.005	1.591 ± 0.002	5000	40.8 ± 0.05	46.3 ± 0.05
200	0.008 ± 0.008	3.168 ± 0.12	5250	43.27 ± 0.01	46.54 ± 0.01
300	0.01 ± 0.015	4.74 ± 0.255	5500	45.68 ± 0.05	46.68 ± 0.05
400	0.447 ± 0.001	6.299 ± 0.001	5750	48.01 ± 0.05	46.68 ± 0.05
500	-0.01 ± 0.005	7.85 ± 0.69	6000	50.27 ± 0.04	46.6 ± 0.03
515	0.73 ± 0.001	8.08 ± 0.001	6100	51.2 ± 0.02	46.52 ± 0.02
530	0.773 ± 0.001	8.311 ± 0.001	6150	51.61 ± 0.01	46.48 ± 0.02
550	0.829 ± 0.002	8.62 ± 0.002	6200	52.05 ± 0.02	46.45 ± 0.03
600	0.982 ± 0.002	9.384 ± 0.003	6250	52.48 ± 0.02	46.39 ± 0.01
650	1.145 ± 0.005	10.145 ± 0.01	6300	52.9 ± 0.05	46.35 ± 0.05
700	1.324 ± 0.001	10.903 ± 0.004	6310	53.01 ± 0.03	46.35 ± 0.03
725	1.417 ± 0.003	11.281 ± 0.004	6315	53.02 ± 0.02	46.32 ± 0.01
750	1.513 ± 0.002	11.656 ± 0.001	6320	53.07 ± 0.04	46.33 ± 0.03
800	1.715 ± 0.005	12.4 ± 0.001	6340	53.25 ± 0.05	46.3 ± 0.03
900	2.156 ± 0.003	13.899 ± 0.001	6360	53.41 ± 0.03	46.29 ± 0.03
1000	2.64 ± 0.02	15.34 ± 0.01	6380	53.57 ± 0.05	46.27 ± 0.03
1250	4.05 ± 0.05	18.89 ± 0.05	6400	53.75 ± 0.03	46.23 ± 0.02
1500	5.71 ± 0.01	22.24 ± 0.03	6450	54.17 ± 0.03	46.17 ± 0.02
1750	7.6 ± 0.02	25.43 ± 0.02	6500	54.5 ± 0.02	46.12 ± 0.02
2000	9.71 ± 0.01	28.42 ± 0.01	6600	55.37 ± 0.02	45.97 ± 0.01
2250	11.99 ± 0.01	31.12 ± 0.01	6800	56.95 ± 0.01	45.67 ± 0.01
2500	14.4 ± 0.01	33.62 ± 0.02	7000	58.45 ± 0.02	45.33 ± 0.02
2750	16.941 ± 0.003	35.87 ± 0.01	7250	60.32 ± 0.02	44.84 ± 0.02
3000	19.54 ± 0.01	37.88 ± 0.01	7500	62.1 ± 0.02	44.3 ± 0.02
3250	22.22 ± 0.15	39.66 ± 0.15	8000	65.38 ± 0.02	43.09 ± 0.02
3500	24.92 ± 0.01	41.21 ± 0.02	8500	68.39 ± 0.01	41.73 ± 0.01
3750	27.65 ± 0.1	46.55 ± 0.15	9000	71.13 ± 0.01	40.28 ± 0.01
4000	30.34 ± 0.13	43.66 ± 0.1	9500	73.63 ± 0.01	38.76 ± 0.01
4250	30.05 ± 0.06	44.58 ± 0.08	10000	75.91 ± 0.01	37.2 ± 0.01
4500	35.7 ± 0.1	45.3 ± 0.1	15000	90.28 ± 0.01	21.36 ± 0.02
4750	38.29 ± 0.07	45.87 ± 0.05	20000	96.49 ± 0.01	7.42 ± 0.01

During the data taking process of Table 2 we adjusted the spacing between frequencies in order to sample more around our critical frequency to get more resolution around that point.

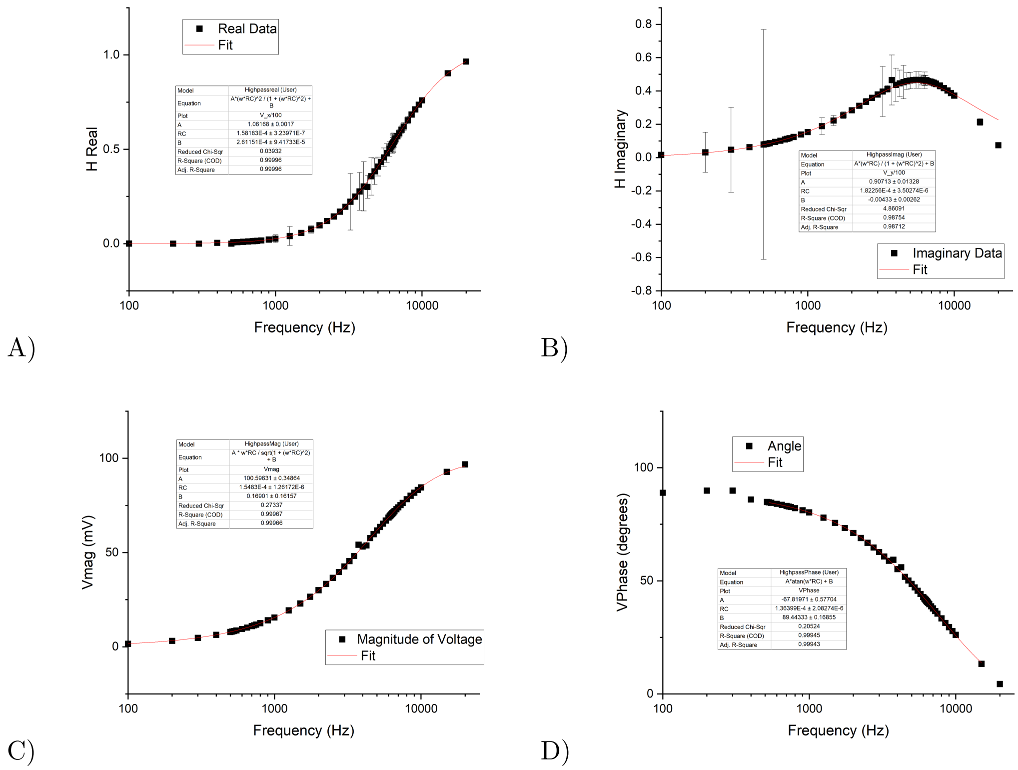


Figure 4: All for High-Pass Filter: A) Real component transfer function found with the In-Phase voltage, B) Imaginary component transfer function found with Quadrature voltage, C) Magnitude of the transfer function, found by norm of real and imaginary and D) Phase of voltage found using Eqn. 2.13

Though it is hard to read on these plots we obtained several values of RC from our Origin fits.

Table 3: Time constants obtained from Origin fits

	RC_ω (s rad)	RC_f* (s)	f_{crit} (Hz)
Real	$1.58{\times}10^{-4} \pm 3.23{\times}10^{-7}$	$2.51{\times}10^{-5} \pm 5.14{\times}10^{-8}$	6340 ± 12.98
Imag	$1.82{\times}10^{-4} \pm 3.50{\times}10^{-6}$	$2.90{\times}10^{-5} \pm 5.56{\times}10^{-7}$	5488 ± 105.2
Mag	$1.54{\times}10^{-4} \pm 1.26{\times}10^{-6}$	$2.40{\times}10^{-5} \pm 2.01{\times}10^{-7}$	6631 ± 55.62
Phase	$1.36{\times}10^{-4} \pm 2.08{\times}10^{-6}$	$2.16{\times}10^{-5} \pm 3.31{\times}10^{-7}$	7368 ± 112.9

*see Data Analysis Note (section 3.1) for additional information

Taking the average of the fits' estimates for RC we obtain $2.49{\times}10^{-5} \pm 2.85{\times}10^{-7}$ s. Based on this we can calculate an approximate percentage error

$$\%\text{ error} = (100)\frac{|2.49\times10^{-5} - 1.30\times10^{-4}|}{1.30\times10^{-4}} = 80.8\%$$

This is an extremely high percent error. Looking at our above average obtained from the fit and our value with its error bars for RC in Table 3 we can see there is not any overlap between our expected and experimental values. We can also see that we have incredibly high margin of error for our resistance. This margin of error means that at least some of the reason that our experimental time constant does not match up to the expected.

Another source of error that we could be seeing here comes from the method by which we measured the voltages. On the program Jordan created the values for In-Phase and Quadrature voltages oscillated around some central number, it is possible that in attempting to get our average we over- or underestimated. This error could compound and produce the result that we see.

One thing that we would like to mention for consideration is that based on the fits we made the critical frequency should be around 5000-7000 Hz, but when we look at the plot we notice that the critical frequency is no where near these values. This suggests that there is something likely wrong with our fit that we are using in Origin.

In the future of this experiment perhaps performing more tests on the resistor in order to more precisely find its resistance as well as taking the average of the readings to get closer to the true average. Both of these would help us get a closer theoretical value to our fits.

3.3 Part B: Low-Pass

For the second part of the experiment we used a Low-Pass Filter, which just does the opposite of what we wanted with the High-Pass filter. We attenuate high frequencies and look only at the lower ones.

The diagram of this can be seen in Figure 2, in the Theory section.

Table 4: values for Low Pass filter

R (kΩ)	C (nF)	τ (s)	f_{crit} (Hz)
1.99 $\pm$ 1	25 $\pm$ 1	$4.97\times10^{-5} \pm 2.50\times10^{-5}$	3199 $\pm$ 1612

For our high pass filter the transfer function is described in Eqn. 2.5, with the real and imaginary parts being Eqns. 2.6 and 2.7, respectively.

Though it is difficult to see in Figure 5 (a couple pages ahead) we obtained several values for the time constant based on the transfer equation equations.

Table 5: Time constants obtained from Origin fits

	RC_ω (s rad)	RC_f* (s)
Real	$2.61\times10^{-4} \pm 1.90\times10^{-6}$	$4.15\times10^{-5} \pm 3.02\times10^{-7}$
Imag	$2.66\times10^{-4} \pm 1.65\times10^{-6}$	$4.23\times10^{-5} \pm 2.63\times10^{-7}$
Mag	$2.39\times10^{-4} \pm 5.63\times10^{-7}$	$3.80\times10^{-5} \pm 8.96\times10^{-8}$
Phase	$1.95\times10^{-4} \pm 1.35\times10^{-6}$	$3.10\times10^{-5} \pm 2.15\times10^{-7}$

*see Data Analysis Note (section 3.1) for additional information

Taking the average of the fits' estimates for RC from Table 5, we obtain $3.82\times10^{-5} \pm 2.17\times10^{-7}$ s. Based on this average we can find an approximate percentage error for our RC value/

$$\%\ \text{error} = (100)\frac{|3.82 \times 10^{-5} - 4.97 \times 10^{-5}|}{4.97 \times 10^{-5}} = 23.1\%$$

Table 6: Observation of in-phase and quadrature voltages at several frequencies across our Low-Pass Filter

Freq. (Hz)	In-Phase (mV)	Quad. (mV)
100	99.58 ± 0.01	-2.84 ± 0.2
200	99.52 ± 0.05	-5.73 ± 0.1
300	98.98 ± 0.02	-8.7 ± 0.1
400	98.5 ± 0.02	-11.4 ± 0.05
500	97.86 ± 0.02	-14.22 ± 0.03
600	97.07 ± 0.05	-16.93 ± 0.02
700	96.17 ± 0.04	-19.59 ± 0.05
800	95.14 ± 0.05	-22.15 ± 0.05
900	93.98 ± 0.05	-24.67 ± 0.03
1000	92.75 ± 0.02	-27.07 ± 0.03
1100	91.4 ± 0.01	-29.4 ± 0.02
1250	89.21 ± 0.01	-32.73 ± 0.02
1500	85.22 ± 0.01	-37.73 ± 0.03
1750	80.86 ± 0.02	-42.13 ± 0.02
2000	76.29 ± 0.03	-45.86 ± 0.01
2250	71.59 ± 0.01	-48.95 ± 0.01
2500	66.86 ± 0.02	-51.45 ± 0.01
2750	62.19 ± 0.01	-53.4 ± 0.01
2900	59.42 ± 0.01	-54.32 ± 0.02
2950	58.52 ± 0.01	-54.59 ± 0.01
3000	57.61 ± 0.02	-54.85 ± 0.01
3050	56.72 ± 0.02	-55.08 ± 0.01
3100	55.81 ± 0.01	-55.31 ± 0.01
3125	55.4 ± 0.02	-55.41 ± 0.01
3150	54.95 ± 0.02	-55.5 ± 0.01
3175	54.51 ± 0.02	-55.61 ± 0.01
3200	54.08 ± 0.02	-55.69 ± 0.01
3225	53.66 ± 0.02	-55.79 ± 0.02
3250	53.21 ± 0.01	-55.86 ± 0.01
3275	52.77 ± 0.02	-55.94 ± 0.01
3300	52.35 ± 0.02	-56.02 ± 0.01
3350	51.51 ± 0.02	-56.16 ± 0.01
3400	50.65 ± 0.02	-56.29 ± 0.01
3450	49.82 ± 0.01	-56.39 ± 0.02
3500	49.01 ± 0.01	-56.5 ± 0.01
3750	44.92 ± 0.02	-56.8 ± 0.01
4000	41.21 ± 0.02	-56.82 ± 0.01
4250	37.64 ± 0.01	-56.61 ± 0.01
4500	34.3 ± 0.01	-56.21 ± 0.01
4750	31.15 ± 0.01	-55.62 ± 0.02
5000	28.26 ± 0.01	-54.92 ± 28.26
5250	25.53 ± 0.01	-54.11 ± 25.53
5500	23.01 ± 0.02	-53.22 ± 23.01
5750	20.67 ± 0.01	-52.26 ± 20.67
6000	18.48 ± 0.01	-51.26 ± 18.48
6500	14.63 ± 0.01	-49.15 ± 14.63
7000	11.3 ± 0.01	-47 ± 11.3
7500	8.44 ± 0.02	-44.83 ± 8.44
8000	5.99 ± 0.01	-42.69 ± 5.99
8500	3.87 ± 0.01	-40.59 ± 3.87
9000	2.07 ± 0.01	-38.6 ± 2.07
10000	0.82 ± 0.01	-34.84 ± 0.82
12000	-4.517 ± 0.005	-28.41 ± -4.517
15000	-7.063 ± 0.005	-20.96 ± -7.063
20000	-7.763 ± 0.005	-12.884 ± -7.763

During the data collection process for Table 6 we varied the frequency so that we could have more data points near the critical frequency for increased resolution around that point. In addition we tweaked the time constant as sensitivity of the Lock-in Amplifier to get as much precision as possible without overloading the machine.

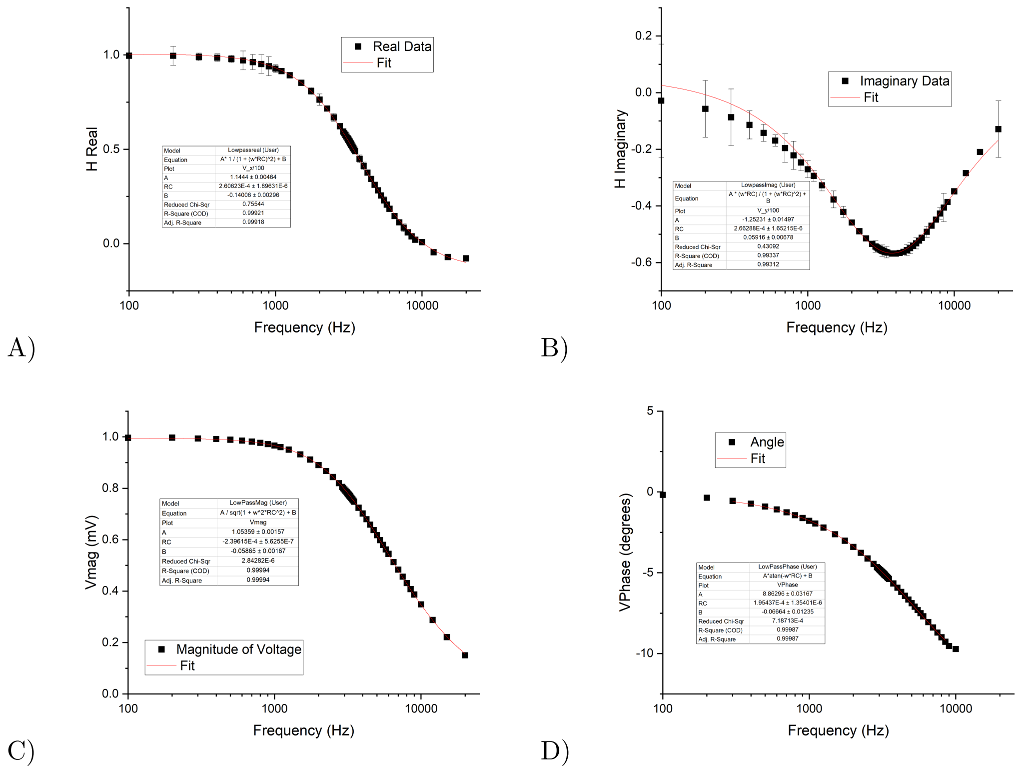


Figure 5: All for Low-Pass filter: A) Real component transfer function found with the In-Phase voltage, B) Imaginary component transfer function found with Quadrature voltage, C) Magnitude of the transfer function, found by norm of real and imaginary and D) Phase of voltage found using Eqn. 2.13

Just looking at our above average obtained from the fit and our value for RC in Table 5 we can see there is significant overlap. The RC values aren't exact, but we can also see that we have incredibly high margin of error for our resistance. This margin of error means that our value for RC could very well match up with the RC value we are getting from the fits, we just don't know the value of R exactly.

Another source of error (that we also mentioned in Part A) that we could be seeing here comes from the method by which we measured the voltages. On the program Jordan created the values for In-Phase and Quadrature voltages oscillated around some central number, it is possible that in attempting to get our average we over- or underestimated. This error could compound and produce the result that we see.

In the future of this experiment perhaps performing more tests on the resistor in order to more precisely find its resistance as well as taking the average of the readings to get closer to the true average. Both of these would help us get a closer theoretical value to our fits.

3.4 Part C: RLC circuit

In this part of the lab we are looking at the transfer function of an RLC circuit in order to determine its frequency dependence.

An example of a detailed RLC circuit set-up for this part can be seen in Figure 3.

Table 7: Values for the RLC circuit

R (Ω)	R_{total}* (Ω)	C (μF)	L (mH)
5.3 ± 0.1	17.3 ± 0.1	0.96 ± 0.02	47 ± 1

*The inductor has internal resistance of 12 Ω

Table 8: Theoretical values of our RLC circuit

τ (s)	LC (s^2/rad^2)	ω_0 (rad/s)	f_c (Hz)	Q
$1.66\times10^{-5} \pm 3.59\times10^{-7}$	$4.51\times10^{-8} \pm 1.34\times10^{-9}$	4707 ± 49	749.2 ± 7.80	41.74 ± 0.626

The transfer function for an RLC circuit is described in Eqn 2.8, the the real and imaginary components being Eqns. 2.9 and 2.10, respectively.

Once again, though it is difficult to see on Figure 6 (a couple pages ahead) we obtained several values for RC and LC based on our Origin fits, they are displayed in Table 9.

Table 9: RC and LC values obtained by Origin fits

	RC_ω (s rad)	RC_f (s)
Real	$5.31\times10^{-5} \pm 3.97\times10^{-6}$	$8.85\times10^{-6} \pm 6.30\times10^{-7}$
Imag	$5.03\times10^{-5} \pm 9.03\times10^{-7}$	$8.01\times10^{-6} \pm 1.43\times10^{-7}$
Mag	$5.27\times10^{-5} \pm 2.80\times10^{-6}$	$8.39\times10^{-6} \pm 4.46\times10^{-7}$
Phase	$4.86\times10^{-5} \pm 2.57\times10^{-6}$	$7.73\times10^{-6} \pm 4.09\times10^{-7}$
	LC_ω (s^2/rad)	LC_f (s^2/rad^2)
Real	$2.27\times10^{-7} \pm 3.97\times10^{-10}$	$3.61\times10^{-8} \pm 6.32\times10^{-11}$
Imag	$2.25\times10^{-7} \pm 2.37\times10^{-10}$	$3.68\times10^{-8} \pm 3.77\times10^{-11}$
Mag	$2.28\times10^{-7} \pm 8.39\times10^{-10}$	$3.62\times10^{-8} \pm 1.34\times10^{-10}$
Phase	$2.26\times10^{-7} \pm 2.18\times10^{-11}$	$3.60\times10^{-8} \pm 3.47\times10^{-12}$

We can take an average of our fits to find our experimental RC and LC values. This gives us $8.25\times10^{-6} \pm 4.07\times10^{-7}$ s for our time constant as well as $3.63\times10^{-8} \pm 5.96\times10^{-11}$ s^2/rad^2 for LC. We can find the approximate percentage error on both of these:

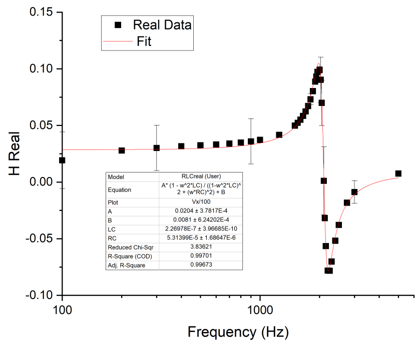
$$\%\ \text{error}_{RC} = (100)\frac{|8.25\times10^{-6} - 1.66\times10^{-5}|}{1.66\times10^{-5}} = 50.3\%$$

$$\%\ \text{error}_{LC} = (100)\frac{|3.63\times10^{-8} - 4.51\times10^{-8}|}{4.51\times10^{-8}} = 19.5\%$$

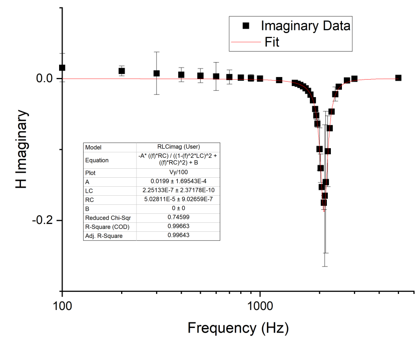
Looking at our time constant we see that there is significant departure from our expected value. Our LC value still has some disagreement with our expected value. A lot of what this is comes from a couple different places, most of which we have mentioned in other parts of our analysis. The first part of this error comes from the method of our measurement of the output voltages, when we were measuring we had to essentially randomly pick values off the screen for what we would imagine a reasonable guess for our average would be. This method isn't perfect and it does explain some of the error that we are receiving here. Another potential source of error was with our lab set-up. As further discussed in the conclusion, we had a lot of troubles getting this lab working and since everything is virtual it is very difficult for us to know what our set-up is actually like. There could have been (and were) a number of incorrect things about our set-up that we would not know.

Table 10: Measurement of in-phase and quadrature voltages across our RLC circuit

Frequency (Hz)	In-Phase (mV)	Quadrature (mV)
100	1.925 $\pm$ 0.025	1.53 $\pm$ 0.02
200	2.784 $\pm$ 0.002	1.078 $\pm$ 0.007
300	3.013 $\pm$ 0.02	0.735 $\pm$ 0.03
400	3.166 $\pm$ 0.001	0.5515 $\pm$ 0.01
500	3.244 $\pm$ 0.002	0.411 $\pm$ 0.01
600	3.324 $\pm$ 0.001	0.305 $\pm$ 0.02
700	3.4 $\pm$ 0.002	0.218 $\pm$ 0.01
800	3.492 $\pm$ 0.002	0.144 $\pm$ 0.001
900	3.596 $\pm$ 0.02	0.074 $\pm$ 0.005
1000	3.726 $\pm$ 0.001	-0.001 $\pm$ 0.002
1250	4.177 $\pm$ 0.001	-0.217 $\pm$ 0.001
1500	4.988 $\pm$ 0.001	-0.607 $\pm$ 0.001
1550	5.233 $\pm$ 0.001	-0.735 $\pm$ 0.002
1600	5.514 $\pm$ 0.001	-0.896 $\pm$ 0.001
1650	5.845 $\pm$ 0.001	-1.099 $\pm$ 0.001
1700	6.242 $\pm$ 0.001	-1.366 $\pm$ 0.004
1750	6.715 $\pm$ 0.002	-1.73 $\pm$ 0.002
1800	7.311 $\pm$ 0.005	-2.26 $\pm$ 0.003
1850	8.028 $\pm$ 0.001	-3.05 $\pm$ 0.002
1900	8.873 $\pm$ 0.002	-4.283 $\pm$ 0.001
1925	9.326 $\pm$ 0.002	-5.192 $\pm$ 0.006
1950	9.731 $\pm$ 0.002	-6.367 $\pm$ 0.005
2000	9.905 $\pm$ 0.003	-9.94 $\pm$ 0.03
2025	9.034 $\pm$ 0.02	-12.632 $\pm$ 0.02
2050	6.994 $\pm$ 0.02	-15.288 $\pm$ 0.003
2100	0.113 $\pm$ 0.03	-17.498 $\pm$ 0.01
2125	-3.177 $\pm$ 0.001	-16.513 $\pm$ 0.1
2150	-5.648 $\pm$ 0.002	-14.602 $\pm$ 0.1
2200	-7.81 $\pm$ 0.002	-10.248 $\pm$ 0.05
2250	-7.831 $\pm$ 0.001	-6.995 $\pm$ 0.001
2300	-7.013 $\pm$ 0.002	-4.653 $\pm$ 0.004
2400	-5.176 $\pm$ 0.002	-2.192 $\pm$ 0.01
2500	-3.782 $\pm$ 0.001	-1.145 $\pm$ 0.004
2750	-1.821 $\pm$ 0.001	-0.253 $\pm$ 0.001
3000	-0.876 $\pm$ 0.01	-0.021 $\pm$ 0.002
5000	0.758 $\pm$ 0.001	0.103 $\pm$ 0.002



A)



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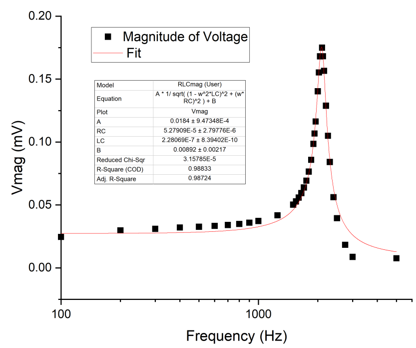

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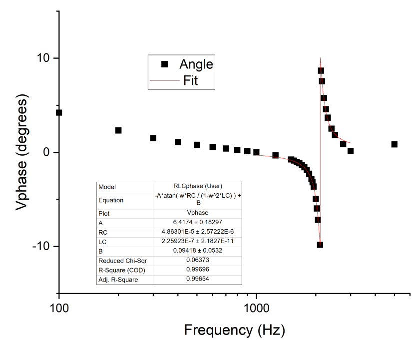


D)

Figure 6: All for RLC circuit: A) Real component transfer function found with the In-Phase voltage, B) Imaginary component transfer function found with Quadrature voltage, C) Magnitude of the transfer function, found by norm of real and imaginary and D) Phase of voltage found using Eqn. 2.13

3.5 Part D: Lock-in amplifier

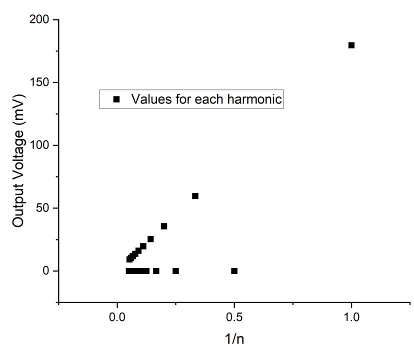
For this part of the experiment we worked with function generators that outputted square and triangular waves. We hooked up the generators to the Lock-In Amplifier in order to study the Fourier analysis being done. As discussed a little bit in the Theory section, we can reconstruct any periodic function as the sum of sines and cosines. This is exactly what we are looking at here. We formulated an expression for all the coefficients of the Fourier series and then compared them to those that we measured from the lock-in amplifier.

Table 11: Square Wave

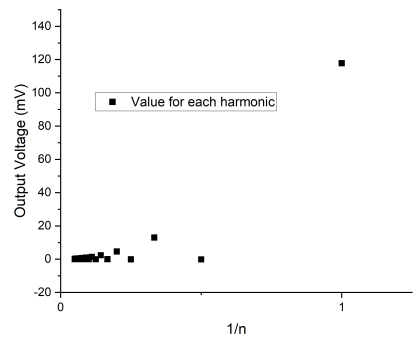
Harmonic	In-Phase (mV)	Quad. (mV)
1	179.6 ± 0.1	-0.585 ± 0.05
2	0 ± 0.005	-0.705 ± 0.01
3	59.65 ± 0.1	-0.43 ± 0.05
4	0 ± 0.02	-0.705 ± 0.02
5	35.66 ± 0.02	-0.41 ± 0.05
6	0 ± 0.015	-0.695 ± 0.02
7	25.42 ± 0.02	-0.304 ± 0.04
8	0 ± 0.05	-0.71 ± 0.04
9	19.73 ± 0.02	-0.33 ± 0.03
10	0 ± 0.05	-0.725 ± 0.02
11	16.13 ± 0.02	-0.269 ± 0.04
12	0 ± 0.03	-0.66 ± 0.02
13	13.63 ± 0.02	-0.304 ± 0.04
14	0 ± 0.02	-0.62 ± 0.03
15	11.8 ± 0.02	-0.32 ± 0.03
16	0 ± 0.03	-0.65 ± 0.01
17	10.4 ± 0.02	-0.29 ± 0.03
18	0 ± 0.03	-0.67 ± 0.02
19	9.278 ± 0.015	-0.28 ± 0.03
20	0 ± 0.03	-0.64 ± 0.03

Table 12: Triangular Wave

Harmonic	In-Phase (mV)	Quad (mV)
1	0.89 ± 0.05	-117.82 ± 0.05
2	-0.17 ± 0.02	-0.14 ± 0.01
3	0.17 ± 0.05	-13.03 ± 0.01
4	-0.02 ± 0.01	-0.06 ± 0.01
5	0.16 ± 0.01	-4.66 ± 0.01
6	-0.06 ± 0.03	0 ± 0.01
7	0.07 ± 0.01	-2.38 ± 0.01
8	-0.02 ± 0.01	-0.02 ± 0.01
9	0.07 ± 0.01	-1.43 ± 0.01
10	-0.04 ± 0.02	0 ± 0.02
11	0.05 ± 0.01	-0.96 ± 0.01
12	-0.02 ± 0.01	0 ± 0.01
13	0.05 ± 0.005	-0.69 ± 0.02
14	0.01 ± 0.005	0 ± -0.02
15	0.04 ± 0.01	-0.52 ± 0.02
16	-0.015 ± 0.005	0 ± 0.005
17	0.04 ± 0.01	-0.4 ± 0.05
18	-0.02 ± 0.005	0 ± 0.005
19	0.03 ± 0.01	-0.32 ± 0.005
20	-0.01 ± 0.01	0 ± 0.005



A)



B)

Figure 7: Plot of one over harmonic number versus Output voltage for A) Square wave and B) Triangular wave

We'll start our analysis of our systems with the square wave. Based on Eqns. 2.18 and 2.19 we can find the coefficients for our Fourier series for our wave. We can derive the fact that $a_n = 0$ for all n since the cosine component is always going to be zero. For b_n we derive the following result:

$$b_n = \begin{cases} \dfrac{4A}{\pi n} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

(where A is the amplitude of the pulse, in our case 100 mV)

Here it is easy to see that all the even terms for the square wave are all going to be zero, which we can see approximately holds for the In-Phase component. Thus the slope of the output voltage to one over the harmonic number should be $\frac{4A}{\pi}$.

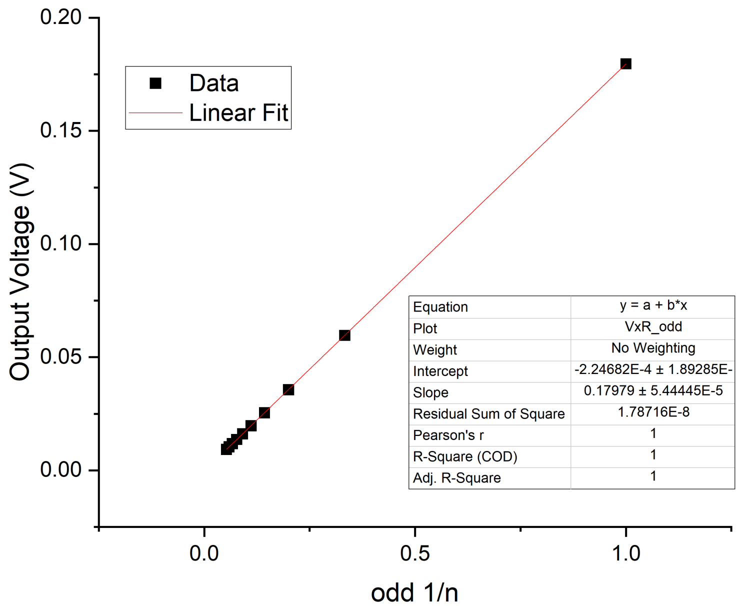


Figure 8: Odd number harmonics with linear fit for square wave

From Figure 8 we obtain the value for the slope of the linear fit to be $0.180 \pm 5.44 \times 10^{-5}$ comparing this to our expected value of 0.127 we can find our percentage error.

$$\% \text{ error} = (100)\frac{|0.180 - 0.127|}{0.127} = 41.7\%$$

For our triangle wave we can again employ Eqns 2.18 and 2.19 to find our coefficients. Similarly to the square wave, one of these terms will go to zero in our Fourier expansion, namely b_n this time. b_n will go to zero in this case because there will be a $\sin(\pi n)$ term that will always evaluate 0. For a_n we derive the following result:

$$a_n = \begin{cases} 4A\frac{1-(-1)^n}{\pi^2 n^2} & \text{for odd } n \\ 0 & \text{for even } n \end{cases}$$

Thus we can see that all the even n terms are going to be zero again, which we can see in our Quadrature output. In addition, we can see that for all the odd terms the (-1)^n will just be -1 and thus the expression becomes $\frac{8A}{\pi^2 n^2}$. From this we can determine the slope of the voltage versus one over harmonic number squared is going to be $\frac{8A}{\pi^2}$.

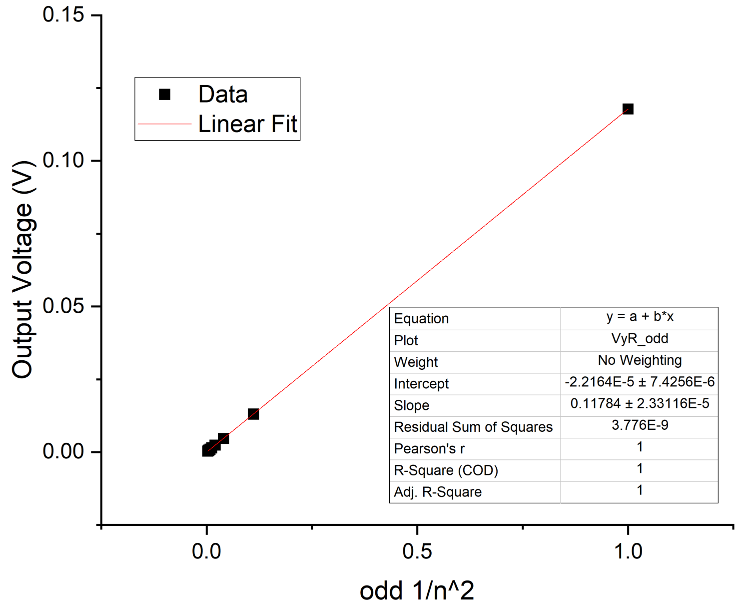


Figure 9: Odd number harmonics with linear fit for triangular wave

From Figure 9 we obtain the value for the slope of the linear fit to be $0.117 \pm 2.33\times10^{-5}$, comparing this to our expected value of 0.081 we can find our percentage error:

$$\%\text{ error} = (100)\frac{|0.117 - 0.081|}{0.081} = 44.4\%$$

For both of these we see a fairly high error. We suspect that most of our error comes from our measurement, since we can clearly see that the linear relationship holds. As with before, when we measured the voltages from the Lock-In Amplifier we were basically randomly selecting values *near* the true mean. In addition it is possible that our amplitude was different than 100 mV, which would change the value of our slope. Our final theory on this could be from there being some resistance in our wire causing some increased voltage, which we would not be able to take into account in our Fourier Series.

4 Conclusion

From all of this we saw that the use of the transfer function described in Eqn. 2.1 was an effective model for predicting the frequency response of LTI systems. For the High-Pass filter we saw pretty large deviation from our expected time constant, but the expected relationship held. Our Low-Pass filter was the best behaved set-up that we examined in this lab. But even the Low-Pass filter didn't quite line up perfectly with our theory due to uncertainties in our circuit's elements. Our RLC circuit had a lot of deviation from our theoretical values, but the LC value seemed a little more well-behaved. Overall for the RLC circuit the model held, we just obtained some weird values. Finally, our study of the

Fourier analysis done by our Lock-In Amplifier yielded a linear relationship, as expected, but again we found some disagreement in our fitted values. In general we saw that there was a pretty strong relation of the theoretical models to the experiment's results; however, when we were constraining our functions with Origin's curve fitting suit we encountered a snag. The theory seemed to match up, then we obtained results that were quite different.

In the future we would like to get back into the lab physically and be able more precisely measure values for our elements and get more hands on to eliminate errors in the set-up of the circuit. As well as becoming more familiar with Origin's curve fitting suite and understanding why our values were so far off.

To say we had troubles with this lab would be an understatement. We encountered a great deal of stress in the retrieval and the analysis of our data. During the lab itself we encountered a snag halfway through Part A that went undetected until half-way through Part C, because of this we had to switch to work with the lab table next to us. This means that for *most* of the data collection we were not any part of taking data, we had to rely on our peers. We have full confidence that our peers did their best to collect data, but sources of potential error in our measurement process could be lost in translation.

Another error that we ran into was that when we measured our voltages we measured 90° out of phase this means that our measured In-Phase voltage was actually our Quadrature and our Quadrature was actually the negative In-phase. We took care of this error relatively easily, but made us wary of any other experimental set-up issues that we could've had otherwise. Since everything is virtual it is very difficult for us to know what our set-up is actually like. There could have been (and were) a number of incorrect things about our set-up that we would not know.