Complex Numbers Example Solution

In lecture this week, we went over the following problem:

Let c_1 and c_2 be two complex numbers. Show that the magnitude of $c_1 \cdot c_2$ equals the product of the magnitudes, i.e.

$$|c_1 \cdot c_2| = |c_1| \cdot |c_2|$$

The solution presented in lecture was hand-written and difficult to read. I'm typing it out just to provide more clarity.

$$\text{Take } c_1 = a_1 + ib_1 \text{ and } c_2 = a_2 + ib_2$$

$$|c_1 \cdot c_2|$$

Recall how to multiply complex numbers:

$$c_1 \cdot c_2 = (a_1 + ib_1)(a_2 + ib_2)$$

$$c_1 \cdot c_2 = a_1a_2 + ib_1a_2 + ib_2a_1 + ib_1ib_2$$

Recall that $i^2 = \sqrt{-1}$

$$c_1 \cdot c_2 = (a_1a_2 - b_1b_2) + i(b_1a_2 + b_2a_1)$$

Taking the magnitude

$$|c_1 \cdot c_2| = \sqrt{(a_1a_2 - b_1b_2)^2 + (b_1a_2 + b_2a_1)^2}$$

$$|c_1| \cdot |c_2|$$

Recall the definition of the magnitude:

$$|c_1| = \sqrt{a_1^2 + b_1^2}\,,\ |c_2| = \sqrt{a_2^2 + b_2^2}$$

$$|c_1| \cdot |c_2| = \sqrt{a_1^2 + b_1^2}\sqrt{a_2^2 + b_2^2}$$

Let's now square both sides

$$(|c_1| \cdot |c_2|)^2 = (a_1^2 + b_1^2)(a_2^2 + b_2^2)$$

$$(|c_1| \cdot |c_2|)^2 = a_1^2a_2^2 + a_2^2b_1^2 + a_1^2b_2^2 + b_1^2b_2^2$$

$$|c_1| \cdot |c_2| = \sqrt{a_1^2a_2^2 + a_2^2b_1^2 + a_1^2b_2^2 + b_1^2b_2^2}$$

Now, we have *fairly* similar results. What we need to do from here is F.O.I.L out the left-hand, bottom equation.

$$|c_1 \cdot c_2| = \sqrt{(a_1a_2 - b_1b_2)^2 + (b_1a_2 + b_2a_1)^2}$$

$$|c_1 \cdot c_2| = \sqrt{a_1^2a_2^2 - 2a_1a_2b_1b_2 + b_1^2b_2^2 + b_1^2a_2^2 + 2a_1a_2b_1b_2 + b_2^2a_1^2}$$

$$|c_1 \cdot c_2| = \sqrt{a_1^2a_2^2 + a_2^2b_1^2 + a_1^2b_2^2 + b_1^2b_2^2}$$

Which, from the right-hand, bottom equation, gives us the desired result:

$$\boxed{|c_1 \cdot c_2| = |c_1| \cdot |c_2|}$$