



Center for Academic Resources in Engineering (CARE) Peer Exam Review Session

Phys 213 — University Physics: Thermal Physics

Midterm Worksheet Solutions

The problems in this review are designed to help prepare you for your upcoming exam. Questions pertain to material covered in the course and are intended to reflect the topics likely to appear in the exam. Keep in mind that this worksheet was created by CARE tutors, and while it is thorough, it is not comprehensive. In addition to exam review sessions, CARE also hosts regularly scheduled tutoring hours.

Tutors are available to answer questions, review problems, and help you feel prepared for your exam during these times:

Session 1: April 2, 4-6pm Alex and Karan Session 2: None

Can't make it to a session? Here's our schedule by course:

<https://care.engineering.illinois.edu/tutoring-resources/tutoring-schedule-by-course/>

Solutions will be available on our website after the last review session that we host, as well as posted in the zoom chat 30 minutes prior to the end of the session

Step-by-step login for exam review session:

1. Log into [Queue @ Illinois](#)
2. Click “New Question”
3. Add your NetID and Name
4. Press “Add to Queue”
5. Join the zoom link in the staff message

Please do not log into the zoom call without adding yourself to the queue

Good luck with your exam!

1. Consider 5 coins, each initially starting on heads.

(a) What is the entropy, S , of this system in its current configuration?

The first thing to note here is that this system is in a 5H 0T *macrostate* (in this scenario, macrostates are the number of heads up (or tails) in a given configuration). There's only one way to attain 5 heads — all of them are heads! So there is one *microstate* corresponding to this macrostate. We can now use the statistical definition of entropy, $S = k_b \ln(\Omega)$, with the microstates in this configuration, Ω , equalling one. Thus,

$$S = k_b \ln(1) = 0 \quad (1)$$

Which is a reasonable answer because this system is highly uniform.

(b) List all the macrostates available to this system.

The macrostates of this system are all the combinations of heads (H) and tails (T) available to the 5 coins.

5H 0T
4H 1T
3H 2T
2H 3T
1H 4T
0H 5T

(c) Identify the most probable macrostates. Hint: there are two.

The most probable macrostates have the highest number of associated microstates.

3H 2T
2H 3T

(d) How many microstates would lead to the macrostates identified above?

To calculate the number of microstates in either of these macrostates, we must use the combination formula, ${}_nC_r$, where n is the total number of items in the set (5), and r is the number of items we're choosing (hence n choose r).

$${}_5C_3 = {}_5C_2 = 10 \text{ microstates}$$

(e) Calculate the change in entropy, ΔS , if the system changed to either of its most probable macrostates.

The change in entropy is found by taking the difference of the final state entropy, S_{final} , and initial state entropy, $S_{initial}$.

$$\Delta S = S_{final} - S_{initial} = k_b \ln(10) - k_b \ln(1) = \boxed{k_b \ln(10)}$$

Note that the entropy increased, as we would expect by the second law of thermodynamics.

2. Consider two containers A and B filled with the same amount of the same ideal gas. Heat is added to each one but A is held at constant volume while B is held at constant pressure. Both are heated through a temperature increase of 10 K.
- (a) Which of the following is true of their added heats?
- A) $Q_A > Q_B$
 - B) $Q_A = Q_B$
 - C) $\boxed{Q_A < Q_B}$

Since container A is held at constant volume, the gas cannot expand. On the contrary, container B is held at constant pressure, so it does expand. Therefore, more heat must be added to container B to reach the same temperature because some of the added heat goes into the work done by the gas expansion. This is equivalent to saying that the environment does negative work on the gas.

- (b) Which of the following is true of their change in internal energy?
- A) $\Delta U_A > \Delta U_B$
 - B) $\boxed{\Delta U_A = \Delta U_B}$
 - C) $\Delta U_A < \Delta U_B$

The internal energy of a gas only depends on the number of particles, the degrees of freedom per particle, and the temperature, $U = \frac{D}{2} N k_b T$, where D is the degrees of freedom. Each container has the same amount of the same gas and undergo the same temperature change. Therefore, their internal energies must be equal.

3. (This is a continuation of problem 2). Assuming the situation described above is carried out, and both containers increase their temperatures by 10 Kelvin, how much more heat must be added to container B?

From (2b), we know that after the heating process, the gases have equal internal energies. However, from (2a) we know more energy was added to container B as a result of expansion. Using the first law of thermodynamics, $dU = dQ - pdV$, we can say

$$\begin{aligned} Q_B - Q_A &= (U_B - U_A) + p(\Delta V_B - \Delta V_A) \\ &= p\Delta V_B = W_{byB} \end{aligned}$$

Where this equals exactly the additional work done by gas B during expansion.

4. What is the difference between heat capacity, specific heat capacity, and molar heat capacity?

Physically, these three quantities are measuring the same thing: an object's tendency to increase in temperature as heat is added to it. However, the difference between these three is the units that they are expressed in.

Heat Capacity

$$C \equiv \frac{dQ}{dT}$$

Specific Heat

$$c_{sp} \equiv \frac{C}{m}$$

Molar Heat

$$c_{mol} \equiv \frac{C}{n}$$

Where m is the mass and n is the number of moles of the substance.

Therefore, the units on each of these quantities is

$$\text{Heat Capacity: } \frac{\text{J}}{\text{K}}$$

$$\text{Specific Heat: } \frac{\text{J}}{\text{kg} \cdot \text{K}}$$

$$\text{Molar Heat: } \frac{\text{J}}{\text{mol} \cdot \text{K}}$$

5. Substance A has a heat capacity of 3 J/K while substance B has a heat capacity of 5 J/K. Starting from the same temperature, which one cools at a faster rate.

As established above, the units for heat capacity are J/K. Interpreting this, a heat capacity tells us the amount of energy required to raise or lower an object's temperature by 1 Kelvin. So you may be tricked into thinking at first that a *higher* heat capacity means a greater tendency to heat up, but that's not true. A *lower* heat capacity means less energy is required for temperature changes (heating and cooling), so objects with lower heat capacities will change temperature at faster rates.

Substance A

6. True or False: For an ideal gas, the molar heat capacity at constant pressure will always be greater than the molar heat capacity at constant volume.

From the definition of molar heat capacity at constant pressure

$$c_{p,mol} = \frac{1}{n} \frac{dU}{dT} + \frac{p}{n} \frac{dV}{dT}$$

and the definition of molar heat capacity at constant volume

$$c_{v,mol} = \frac{1}{n} \frac{dU}{dT}$$

we can relate the two by substitution, ending up with

$$c_{p,mol} = c_{v,mol} + \frac{p}{n} \frac{dV}{dT}$$

So we already see that $c_{v,mol}$ is less than $c_{p,mol}$ as long as $\frac{p}{n} \frac{dV}{dT} > 0$. Now let's confirm that, for an ideal gas, $\frac{p}{n} \frac{dV}{dT}$ is greater than zero.

Starting from the ideal gas law $pV = Nk_bT$, we can differentiate V with respect to T to get $p \frac{dV}{dT} = Nk_b$. Substituting this into the equation relating the two molar heat capacities and simplifying, we get

$$c_{p,mol} = c_{v,mol} + \frac{p}{n} \frac{dV}{dT}$$

$$c_{p,mol} = c_{v,mol} + \frac{N}{n} k_b$$

$$c_{p,mol} = c_{v,mol} + N_A k_b$$

$$c_{p,mol} = c_{v,mol} + R$$

Note that $\frac{N}{n}$ is Avogadro's Number, N_A , and R is the universal gas constant. Since $R > 0$, we have confirmed that $c_{p,mol} > c_{v,mol}$ for an ideal gas.

True

7. Explain why the heat capacity at constant volume of an ideal solid is twice that of the same amount of a monatomic ideal gas.

At constant volume, heat capacity is the derivative of internal energy

$$C_v = \frac{dU}{dT}$$

Where $U = \frac{D}{2}Nk_bT$ where D is the number of degrees of freedom.

For a ideal solid, there are six degrees of freedom; 3 translational and 3 vibrational ($D = 6$). But for a monatomic ideal gas, there are only 3 degrees of freedom; 3 translational ($D = 3$). When the derivative of internal energy is taken to obtain heat capacity, these constants remain.

8. A sealed container with a mass of 2.7 kg is filled with 4 moles of helium gas. Initially, the helium gas is at a temperature of 140°C and the container is at 38°C. The helium-container system is thermally isolated.

Note that the specific heat of the material making the container is 386 J/(kg·K) and the molar specific heat of helium is 12.5 J/(mol·K).

Find the equilibrium temperature of the system in Celsius and Kelvin.

The helium-container system is isolated from any heat conducting environment. So, by conservation of energy, we know that $Q_{He} = -Q_C$. One thing to see is that the heat capacities in the question have different units — one is specific heat, the other is molar heat. Using the fact that $Q = C\Delta T$ where C is regular heat capacity, we can do the following

$$\begin{aligned} Q_{He} &= -Q_C \\ nc_{mol}(T_f - T_{He,i}) &= -mc_{sp}(T_f - T_{C,i}) \end{aligned}$$

where n is the moles of Helium, m is the mass of the container, and T_f is the equilibrium temperature (which is the same for both the Helium and the container by the definition of thermal equilibrium). Solving for T_f

$$T_f = \frac{nc_{mol}T_{He,i} + mc_{sp}T_{C,i}}{nc_{mol} + mc_{sp}}$$

And when the values given above are put into this equation, we get

$$T_f = 315.8 \text{ K} = 42.7^\circ\text{C}$$

Remember to convert the temperatures to Kelvin in the equations, then back to Celsius for the final answer.

9. A 10 kg block of aluminum initially at 90°C is placed in contact with a 12 kg block of copper initially at 10°C. The molar heat capacities of aluminum and copper are 24.2 J/(K-mol) and 24.5 J/(K-mol), respectively. Molar masses are on the equation sheet.
- (a) When the two blocks come into thermal equilibrium, what temperature will they reach?
- A) 75.4°C
 B) 62.9°C
 C) 49.8°C
 D) 33.5°C
 E) 24.5°C

It is assumed in these problems that when two blocks come into contact, the only energy (heat) flow is between them, and not the surrounding environment. That is, $Q_{Al} = -Q_{Cu}$ (remember that negative sign).

We know from the definition of heat capacity at constant volume that

$$dQ = C_v dT = \frac{m}{m_{mol}} c_{mol} dT$$

where Q is the heat exchanged, C_v is the heat capacity at constant volume, m is the mass, m_{mol} is the molar mass, c_{mol} is the molar heat capacity, and T is the temperature. This is written in terms of moles because the problem gives us the relevant quantities in these units. We also know that the final temperature of the two blocks will be equal after the heat is exchanged because they will be in thermal equilibrium, T_f .

Using this information, we can do the following

$$\begin{aligned} Q_{Al} &= -Q_{Cu} \\ C_{v,Al}(T_f - T_{i,Al}) &= -C_{v,Cu}(T_f - T_{i,Cu}) \end{aligned}$$

Notice that regular heat capacity ($C_v = \frac{m}{m_{mol}} c_{mol}$) is being used here; this makes the resulting expression for T_f , the final temperature of both blocks, look much friendlier.

$$T_f = \frac{C_{v,Al}T_{i,Al} + C_{v,Cu}T_{i,Cu}}{C_{v,Al} + C_{v,Cu}}$$

And just for reference here's what the equation looks like if you don't use heat capacity, instead opting for all the given quantities in the question.

$$T_f = \frac{\frac{m_{Al}}{m_{mol,Al}} c_{mol,Al} T_{i,Al} + \frac{m_{Cu}}{m_{mol,Cu}} c_{mol,Cu} T_{i,Cu}}{\frac{m_{Al}}{m_{mol,Al}} c_{mol,Al} + \frac{m_{Cu}}{m_{mol,Cu}} c_{mol,Cu}}$$

When you use either of these with the given values, you end up with the boxed answer choice. **Remember to convert the temperatures to Kelvin in the equations, then back to Celsius for the final answer.**

(b) How does the entropy of the aluminum change during this process?

- A) 0
- B) +695 J/K
- C) −695 J/K
- D) $+3.211 \times 10^3$ J/K
- E) -3.211×10^3 J/K

From the fundamental relation for entropy, we know that

$$\Delta S = \int_{U_i}^{U_f} \frac{1}{T} dU = \int_{T_i}^{T_f} \frac{C_v}{T} dT$$

Where we use the fact that $dU = C_v dT$ at constant volume, which is applicable here because the aluminum block does not change volume during the heat exchange.

We can now substitute the given values in part (a) into the second integral (remember to convert the temperatures to Kelvin!)

$$\Delta S = \int_{90+273}^{62.9+273} \frac{8963}{T} dT = 8963 \ln \left(\frac{335.9}{363} \right) \approx -695.4 \text{ J/K}$$

Don't get confused about this. The negative entropy here is totally valid *as long as* the entropy increase in the copper block exceeds the magnitude of this entropy change. You can calculate the copper block's entropy to verify that the sum is greater than zero.

10. Consider a sealed container with a volume of 1 m^3 filled with 10^{23} helium atoms and 2×10^{23} molecules of nitrogen. Recall that helium is a monatomic gas and nitrogen is a diatomic gas. Initially the temperature of the gas mixture is 1000 K.

(a) Find the pressure inside the container (Answer in Pa).

- A) 8980
- B) 8290
- C) 4140
- D) 2250

This container holds a mixture of two gases, so the ideal gas law has to be used on each separately, and then sum of the pressures will be the total pressure from the gas mixture.

Helium	Nitrogen
$p_{He} = \frac{N_{He} k_b T}{V}$	$p_N = \frac{N_N k_b T}{V}$

Given these relations, the total pressure is the sum of each partial pressure. Note that the volume and temperature is the same for these two gases.

$$p = p_{He} + p_N = \frac{(N_{He} + N_N) k_b T}{V}$$

(b) Find the ratio of the total molecular rotational energy to the total translational energy (molecular and atomic) inside the container. Assume that equipartition applies.

- A) 0.333
- B) 0.444
- C) 0.666
- D) 0.777

First recall that the formula for the internal energy of a gas as a result of D degrees of a freedom is $U = \frac{D}{2} N k_b T$.

The total translational energy is the sum of the energy from the He atoms and the N atoms, both of which have 3 translational degrees of freedom per particle. Let T be the total translational kinetic energy, then

$$T = T_{He} + T_N = \frac{3}{2} N_{He} k_b T + \frac{3}{2} N_N k_b T$$

The total rotational energy, unlike the translational energy, comes entirely from the N atoms since they're diatomic. There are only 2 degrees of freedom here. Let R be the rotational kinetic energy, then

$$R = N_N k_b T$$

Taking the ratio of R to T , $\frac{R}{T}$, gives

$$\frac{R}{T} = \frac{N_N k_b T}{\frac{3}{2} N_{He} k_b T + \frac{3}{2} N_N k_b T} = \frac{2}{3} \frac{N_N}{N_{He} + N_N}$$

Note that this took into account that Helium is monatomic ($D = 3$), and Nitrogen is diatomic ($D = 5$, split among translation and rotation).

11. Suppose we place a proton in a magnetic field $B = 3$ Tesla. Like electrons, protons have two possible states for their magnetic moments, either aligned or anti-aligned with the magnetic field, and we can model it as a two-state system. The energy of a proton with its magnetic moment aligned with the magnetic field is $E_{align} = -\mu_p B$ where $\mu_p = 1.4 \times 10^{-26}$ J/Tesla. The energy of the anti-aligned state is $E_{anti} = +\mu_p B$. The proton in contact with a thermal reservoir at temperature $T = 0.04$ K.
- (a) At this temperature, what is the probability that the proton has its magnetic moment anti-aligned with the magnetic field, i.e., is in the state with higher energy?
- A) 0
 B) 0.5
 C) 0.538
 D) 0.462
 E) 0.481

This is an application of the Boltzmann distribution. To find the probability of the magnetic moment being anti-aligned, we must calculate

$$P(\text{anti}) = \frac{e^{-\frac{E_{anti}}{k_b T}}}{e^{-\frac{E_{anti}}{k_b T}} + e^{-\frac{E_{align}}{k_b T}}} = \frac{1}{1 + e^{\frac{E_{anti} - E_{align}}{k_b T}}}$$

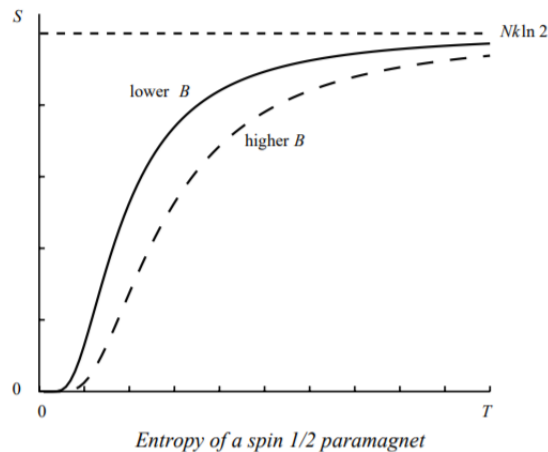
We factor out $e^{-\frac{E_{anti}}{k_b T}}$ from the numerator and denominator to get the final form of the equation.

- (b) At this temperature the entropy, S , of the system is
- A) Less than $k_b \ln(2)$
 B) Greater than $k_b \ln(2)$
 C) Equal to $k_b \ln(2)$

At extremely high temperature, the probability that the proton occupies either state approaches 50/50. In that limit, the entropy is simply $k_b \ln(2)$ as two states are equally likely. However, as we found out in the previous question, the proton is less likely to occupy the anti-aligned state which makes the two states not equally likely. Hence, the entropy is smaller than $k_b \ln(2)$ as the protons has less freedom.

- (c) If we increase the magnetic field to $B = 15$ Tesla, does the entropy of the proton system at this temperature
- A) Increase
 B) Stay the same
 C) Decrease

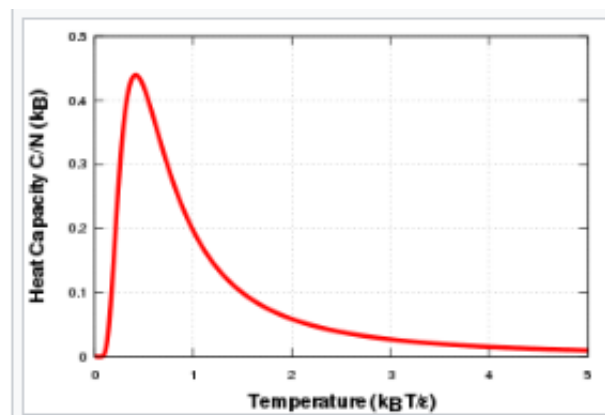
If we increase the magnetic field, the energy gap between the two energy levels becomes larger which makes it harder for the proton to occupy the anti-aligned state. This further limits proton's freedom and as a result reduces the entropy.



- (d) What is the heat capacity of the two-level system near $T = 0$? How about at extremely high temperatures ($T \rightarrow \infty$)?

The heat capacity of the two-level system at $T = 0$ is 0. This is because, theoretically, it requires no energy to increase the temperature of the system (since absolute zero isn't possible). As $T \rightarrow \infty$, the heat capacity again approaches zero because you run out of states for the protons to excite to and increase the temperature.

For a two-state system:



12. The heat capacity of a solid is linear with temperature. How does its entropy change with temperature?

The entropy will also increase linearly. To show this, let $C = \alpha T$

$$\Delta S = \int \frac{C}{T} dT = \int \alpha dT = \alpha T$$

Essentially, the T in the denominator reduces the heat capacity by a factor of T , but that factor is restored by the integral.

13. Consider a two-state oscillator with energies $E_0 = 0$ eV and $E_1 = 0.02$ eV. Calculate the average energy of this system if it is kept at 200 K.

This is another problem that requires using the Boltzman factor. But this time, instead of finding the probability of a certain state, we're finding an expected value.

$$E_0 P(E_0) + E_1 P(E_1) = 0.02 \cdot \frac{e^{-\frac{0.02}{200k_b}}}{e^{-\frac{0.02}{200k_b}} + 1} = 4.77 \times 10^{-3} \text{ eV}$$

Remember to use k_b in eV for this problem since that's what the values are given as!

14. A monatomic, ideal gas is contained at fixed volume under pressure p . Now suppose the pressure is tripled. What is the ratio of the initial v_{rms} to the final v'_{rms} ? (v_{rms}/v'_{rms})

In order to calculate v_{rms} , we must equate kinetic energy with the internal energy (of a single particle of the gas).

$$\frac{1}{2} m v_{rms}^2 = \frac{3}{2} k_b T = \frac{3 p V}{2 N}$$

Where the ideal gas law was used to make the last equation. Solving for v_{rms} , we get $\sqrt{\frac{3pV}{Nm}}$

If we then triple the pressure, we get $v'_{rms} = \sqrt{\frac{9pV}{Nm}}$

Taking the ratio, $\frac{v_{rms}}{v'_{rms}}$, we get

$$\frac{v_{rms}}{v'_{rms}} = \frac{\sqrt{\frac{3pV}{Nm}}}{\sqrt{\frac{9pV}{Nm}}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}$$