Class groups are essential in our life

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- Beautiful mathematical challenge
- Finite group $\implies$ Applications in Cryptology
- Make use of trendy structures

FROM ALEX GELIN'S

MATHEMATICAL WORLD

CLASS GROUPS

AND WHERE TO FIND THEM

Number fields

$\mathbf{K}$ number field $\Rightarrow$ finite-degree extension of $\mathbf{Q}$

Primitive Element Theorem $\Rightarrow$ $\exists T \in \mathbf{Z}[X]$ monic such that

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  Vector of $[\mathbf{K}:\mathbf{Q}]$ complex coordinates $\Longrightarrow$ Vector of $[\mathbf{K}:\mathbf{Q}]$ real coordinates

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Two interesting structures in number fields:

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$$1 \longrightarrow U(\mathbf{K}) \longrightarrow \mathbf{K}^* \longrightarrow P(\mathbf{K}) \longrightarrow 1$$

[Coh93], p.210

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & U(\mathbf{K}) \arrow[r] \arrow[d] & J_{S_\infty}(\mathbf{K}) \arrow[r] \arrow[d] & C_{S_\infty}(\mathbf{K}) \arrow[r] \arrow[d] & 1 \\
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Subexponential L-notation : $\qquad$ $L_{|\Delta_{\mathbf{K}}|}(0,c) \approx (\log|\Delta_{\mathbf{K}}|)^c \mid L_{|\Delta_{\mathbf{K}}|}(1,c) \approx |\Delta_{\mathbf{K}}|^c$

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Complexity history

1969 Shanks: quadratic number fields in $O\left(|\Delta_{\mathbf{K}}|^{\frac{1}{5}}\right)$

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how i
compute class
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Index Calculus Method

Well-known, used for discrete logarithms

1. **Factor base**
   Fix a factor base composed of small elements
2. **Relation collection**
   Collect some relations between those small elements, corresponding to linear equations
3. **Linear algebra**
   Deduce the sought result performing linear algebra on the system built

The factor base

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Practically

$$B = L_{|\Delta_{\mathbf{K}}|}\left(\beta, c_b\right)$$

Relation collection

$$\mathscr{B} = \left(\mathfrak{p}_1, \ldots, \mathfrak{p}_N\right)$$

Surjective morphism:

$$\begin{array}{ccccc} \mathbf{Z}^N & \longrightarrow & I & \longrightarrow & Cl(K) \\ (e_1, \ldots, e_N) & \longmapsto & \prod\limits_i \mathfrak{p}_i^{e_i} & \longmapsto & \left[\prod\limits_i \mathfrak{p}_i^{e_i}\right] \end{array}$$

$$Cl(\mathbf{K}) \quad \simeq \quad \mathbf{Z}^N \Big/ \left\{(e_1, \ldots, e_N) \in \mathbf{Z}^N \mid \prod \mathfrak{p}_i^{e_i} = (\alpha)\mathscr{O}_{\mathbf{K}}\right\}$$

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Idea:

1. Pick at random $\mathfrak{a} = \prod \mathfrak{p}_i^{a_i}$
2. Find a *reduced* ideal $\mathfrak{b}$ in the same class
3. If $\mathfrak{b}$ splits over $\mathscr{B}$ $\left(\iff \mathfrak{b} = \prod \mathfrak{p}_i^{b_i}\right)$ then

$$\mathfrak{a} \cdot \mathfrak{b}^{-1} = \prod \mathfrak{p}_i^{a_i - b_i} \quad \text{is principal}$$

Ideal Reduction

Correspondence between ideals and lattices:

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Biasse-Fieker's reduction:

- BKZ
- Trade-off between time spent and approximation factor
- Subexponential complexity

Linear algebra

- Relations stored in a matrix of size about $N \times N$
- Structure of the class group given by the *Smith Normal Form* of the matrix
- First compute *Hermite Normal Form* with a premultiplier because we need kernel vectors
- Storjohann and Labahn algorithm, runtime in $N^{\omega+1}$
  ($2 \leq \omega \leq 3$ exponent of matrix multiplication)

Verification

We find a tentative class group H, but the class group $Cl(\mathbf{K})$ may be only a quotient of H
$\Longrightarrow$ Need an approximation of the class number $h_{\mathbf{K}} = |Cl(\mathbf{K})|$

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$$h_{\mathbf{K}} \cdot Reg_{\mathbf{K}} \approx EP \cdot \frac{w_{\mathbf{K}} \cdot \sqrt{|\Delta_{\mathbf{K}}|}}{2^{r_1} \cdot (2\pi)^{r_2}}$$

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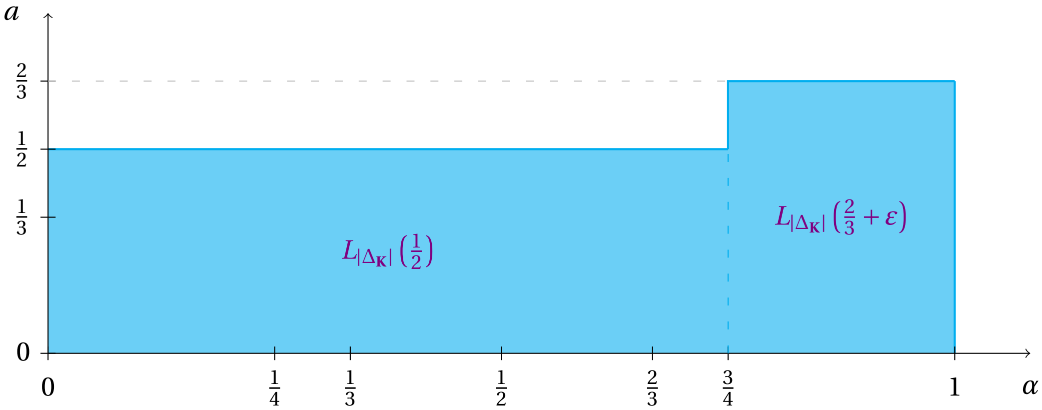
From the relations, we can also deduce a candidate for an approximation of $Reg_{\mathbf{K}}$ and perform the verification step

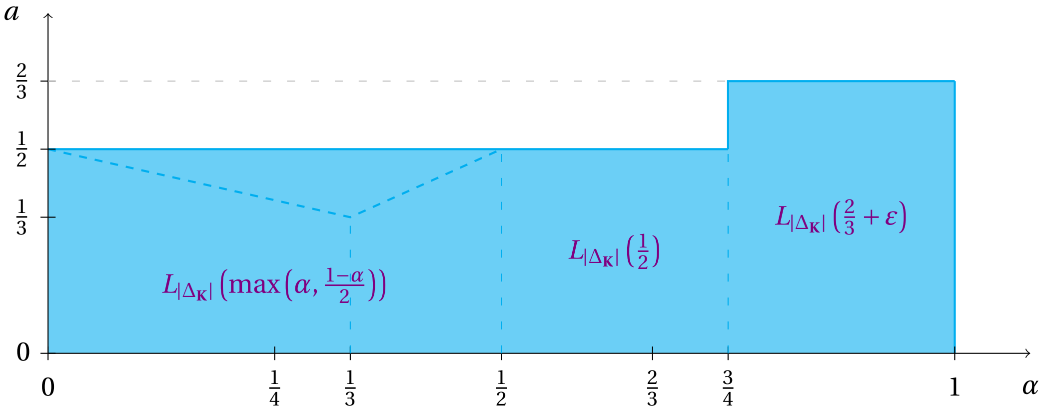
General case:



First general subexponential algorithm

State of the art [BF14]

Special case:



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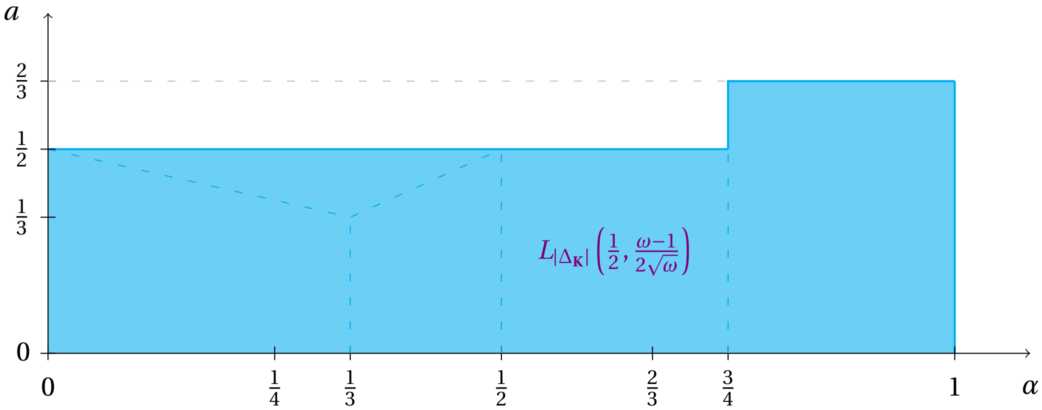

Refinement on the complexity analysis

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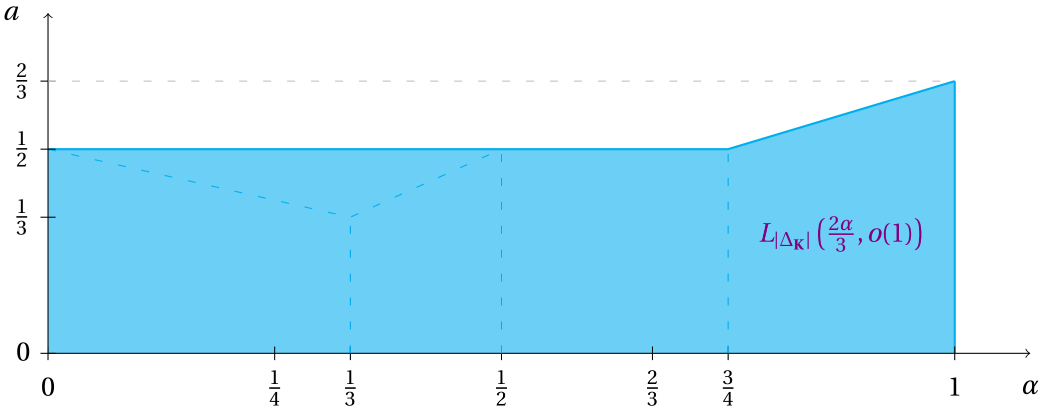

Improvement through a better parameters choice

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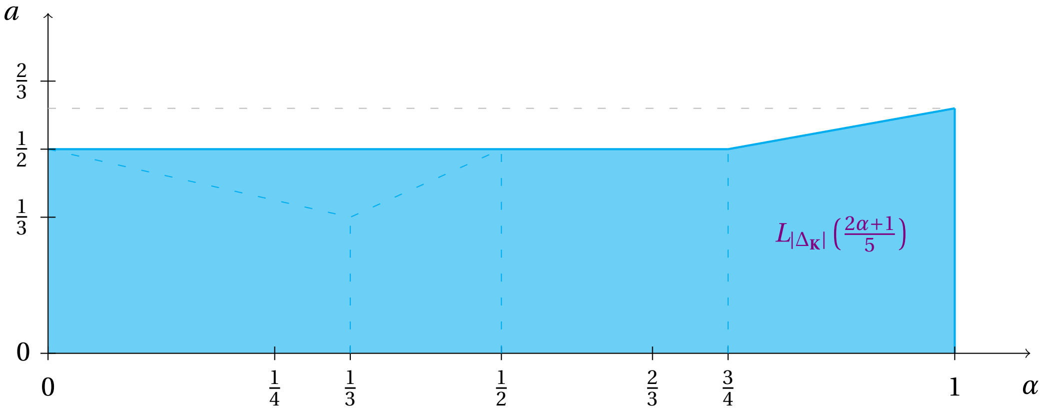

Improvement using special lattice-reduction algorithm

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$$H(T) = \max_k |a_k|$$

For every degree-n number field $\mathbf{K}$ and every defining polynomial T of $\mathbf{K}$,

$$|\Delta_{\mathbf{K}}| \leq n^{2n} H(T)^{2n-2}$$

Why minimizing the height?

Idea: Look at small algebraic integers $x \in \mathcal{O}_{\mathbf{K}}$ such that $x = A(\theta)$ with $\deg A \leq c_d$ and $H(A) \leq C_H$.

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- Classic reduction: same magnitude for all the coordinates
- Tricky reduction: one dominant coordinate

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Example: Number field defined by $T = x^5 - 5843635x^4 + 931633x^2 + 6577x - 8570$

- Rounded conjugates: $[-0.38, -0.10, 0.095, 0.39, 5843634.99999997]$
- Rounded shortest vector: $[-84411, -23707, -1315, 20616, 88819]$

$x^5 - 2x^4 - 8001397580x^3 - 31542753393650x^2 + 36366533024511131875x + 48185475294252800675 00$

State of the art [BF14]

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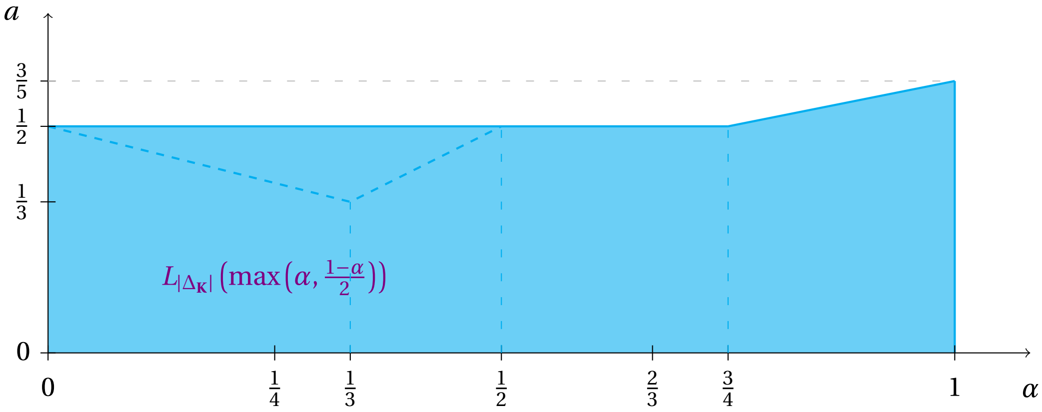

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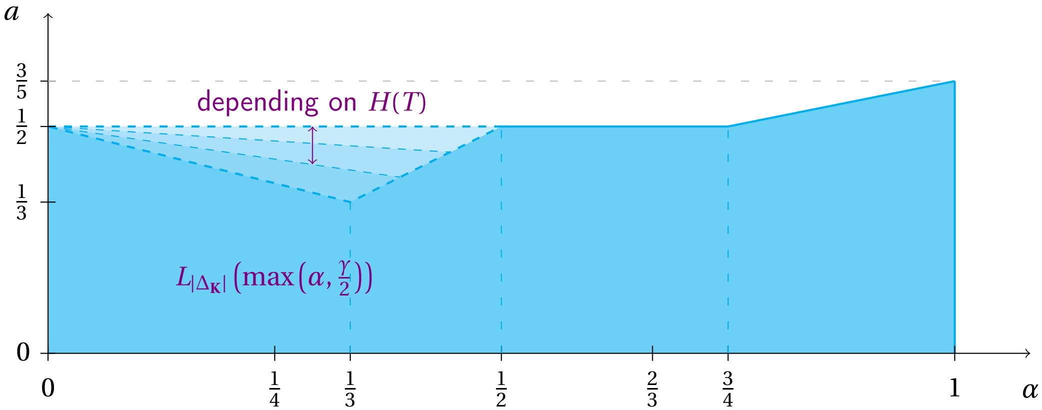

Without any condition

State of the art [G17]

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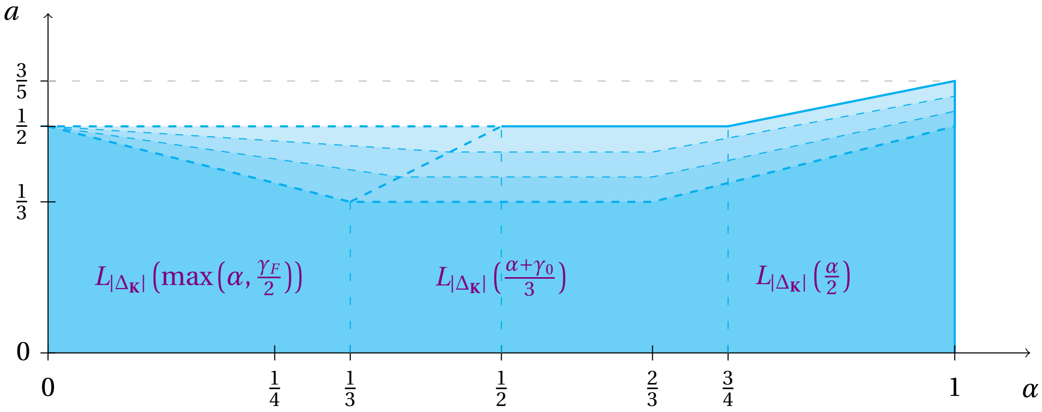


Only when it is better than the method based on ideal reductions

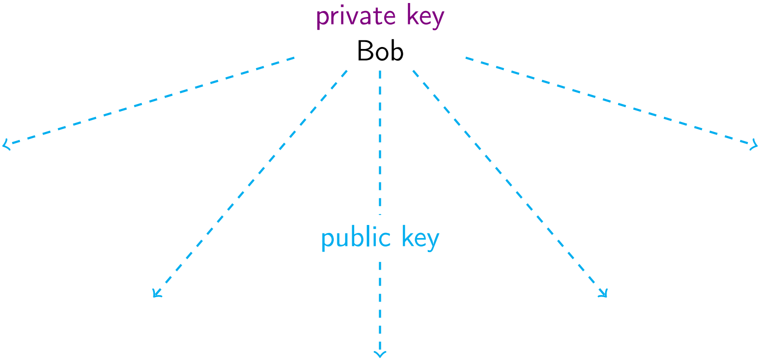
CLASS GROUPS AND CRYPTOLOGY

Public Key Cryptography

private key

Bob

Public Key Cryptography



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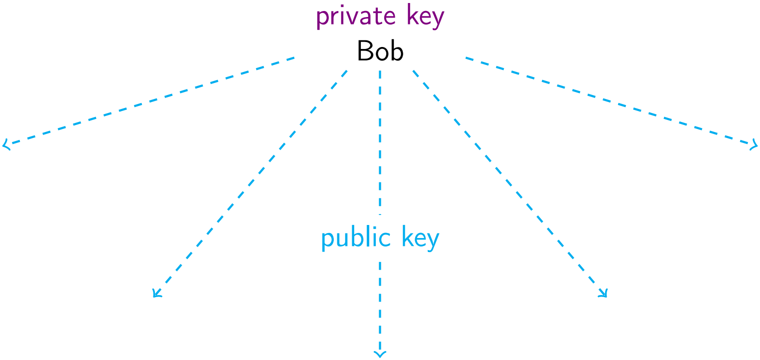

- Everyone uses the public key to encrypt

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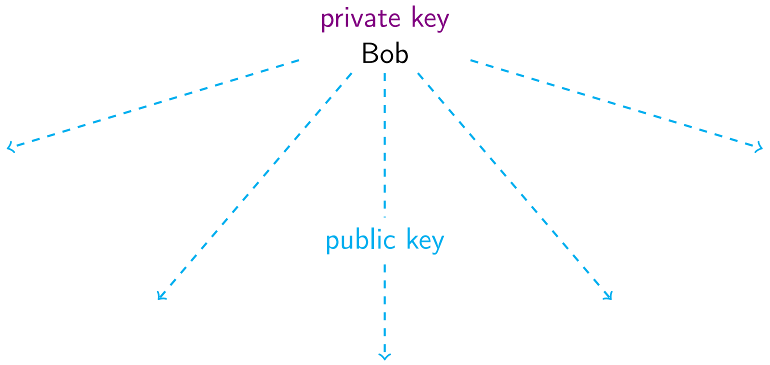

- Everyone uses the public key to encrypt
- Only Bob can decrypt thanks to his private key

The Principal Ideal Problem

Definition

The *Principal Ideal Problem* (PIP) consists in finding a generator of an ideal, assuming it is principal.

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- Two distinct phases:
  1. Given the $\mathbb{Z}$-basis of the ideal $\mathfrak{a} = \langle \boldsymbol{g} \rangle$, find a — not necessarily short — generator $\boldsymbol{g}' = \boldsymbol{g} \cdot \boldsymbol{u}$ for a unit $\boldsymbol{u}$
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2014 - Campbell, Groves, and Sheperd:
Reduction in polynomial time for power-of-two cyclotomic fields

2016 - Cramer, Ducas, Peikert, and Regev:
Proof and extension to prime-power cyclotomic fields

Key Generation:

1. Fix the security parameter $N = 2^n$
2. Let $F(X) = X^N + 1$ be the polynomial defining the cyclotomic field $\mathbf{K} = \mathbf{Q}(\zeta_{2N})$
3. Set $G(X) = 1 + 2 \cdot S(X)$,
   for $S(X)$ of degree $N - 1$ with coefficients in $\left[-2^{\sqrt{N}}, 2^{\sqrt{N}}\right]$,
   such that the norm $\mathcal{N}(\langle G(\zeta_{2N})\rangle)$ is prime
4. Set $\boldsymbol{g} = G(\zeta_{2N}) \in \mathcal{O}_{\mathbf{K}}$
5. Return the private key $\mathrm{sk} = \boldsymbol{g}$ and the public key $\mathrm{pk} = HNF(\langle \boldsymbol{g} \rangle)$

FHE scheme – Smart and Vercauteren PKC 2010

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Goal: Recover the private key from the public key

Outline of the algorithm [BEFGK17]

1. Perform a reduction from the cyclotomic field to its totally real subfield, allowing to work in smaller dimension
2. Then a descent makes the sizes of involved ideals decrease
3. Collect relations and run linear algebra to construct small ideals and a generator
4. Eventually run the derivation of the small generator from a bigger one

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All the complexities are expressed as a function of the field discriminant $\Delta_{\mathbf{Q}(\zeta_{2N})} = N^N$, for $N = 2^n$. For instance,

$$L_{|\Delta_{\mathbf{K}}|}(\alpha) = 2^{N^{\alpha+o(1)}}$$

1. Reduction to the totally real subfield

Goal: Halve the dimension of the ambient field

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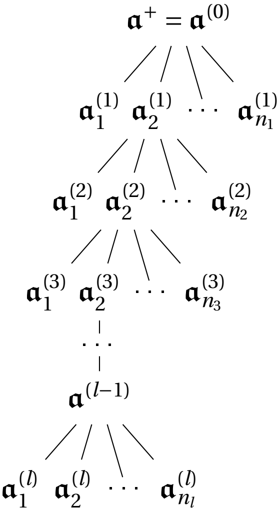
1. Reduction to the totally real subfield

Goal: Halve the dimension of the ambient field

- Based on the algorithm of Gentry and Szydlo
- Polynomial complexity
- **Input:** a $\mathbf{Z}$-basis of $\mathfrak{a} = \langle \boldsymbol{g} \rangle$
- **Output:** a $\mathbf{Z}$-basis of $\mathfrak{a}^+ = \langle \boldsymbol{g} + \bar{\boldsymbol{g}} \rangle \subset \mathbf{Q}(\zeta + \zeta^{-1})$ and $\boldsymbol{g} \cdot \bar{\boldsymbol{g}}^{-1}$ to recover $\boldsymbol{g}$ from $\boldsymbol{g} + \bar{\boldsymbol{g}}$

2. The descent



Input ideal – Norm arbitrary large

2. The descent

$$\mathfrak{a}^+ = \mathfrak{a}^{(0)}$$

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$$\mathfrak{a}_1^{(1)} \quad \mathfrak{a}_2^{(1)} \quad \cdots \quad \mathfrak{a}_{n_1}^{(1)}$$

Initial reduction – Norm: $L_{|\Delta_{\mathbf{K}}|}\left(\frac{3}{2}\right)$

$$\mathfrak{a}_1^{(2)} \quad \mathfrak{a}_2^{(2)} \quad \cdots \quad \mathfrak{a}_{n_2}^{(2)}$$

$$\mathfrak{a}_1^{(3)} \quad \mathfrak{a}_2^{(3)} \quad \cdots \quad \mathfrak{a}_{n_3}^{(3)}$$

$$\cdots$$

$$\mathfrak{a}^{(l-1)}$$

$$\mathfrak{a}_1^{(l)} \quad \mathfrak{a}_2^{(l)} \quad \cdots \quad \mathfrak{a}_{n_l}^{(l)}$$

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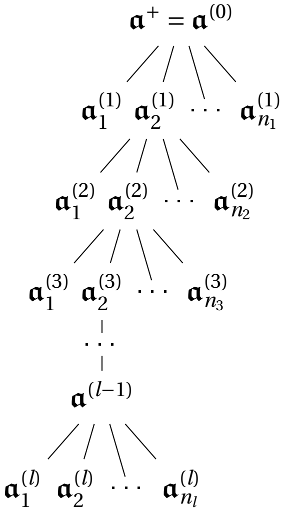

Input ideal – Norm arbitrary large

Initial reduction – $L_{|\Delta_{\mathbf{K}}|}(1)$-smooth

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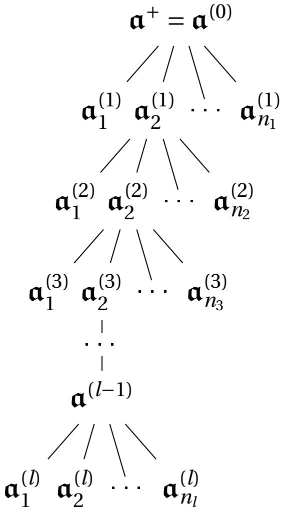

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First step – Norm: $L_{|\Delta_{\mathbf{K}}|}\left(\frac{5}{4}\right)$

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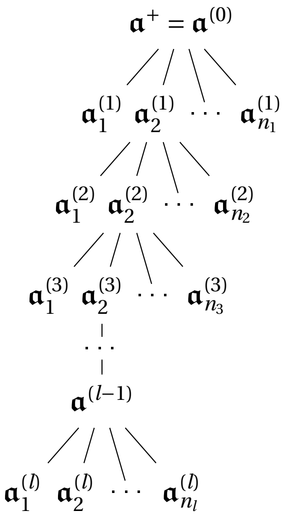

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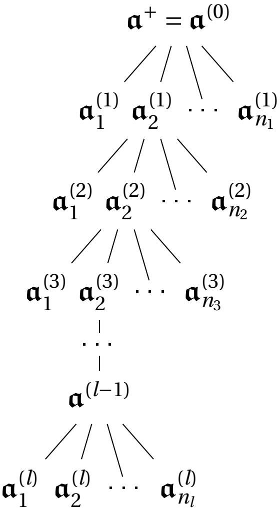
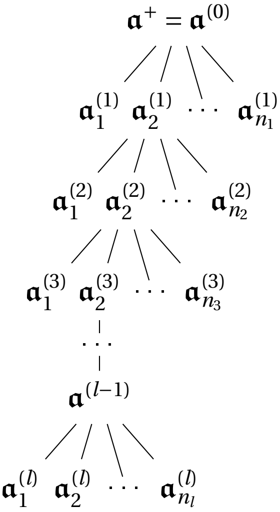


Input ideal – Norm arbitrary large

Initial reduction – $L_{|\Delta_{\mathbf{K}}|}(1)$-smooth

First step – $L_{|\Delta_{\mathbf{K}}|}\left(\frac{3}{4}\right)$-smooth

Second step – Norm: $L_{|\Delta_{\mathbf{K}}|}\left(\frac{9}{8}\right)$



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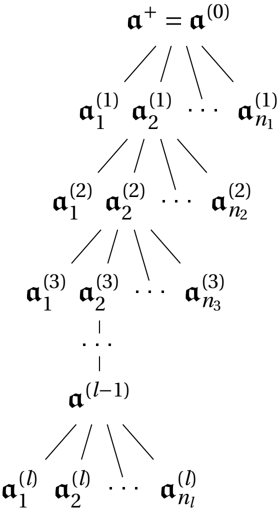

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Second step – $L_{|\Delta_{\mathbf{K}}|}\left(\frac{5}{8}\right)$-smooth

Last but one step – Norm: $\approx L_{|\Delta_{\mathbf{K}}|}(1)$

$\mathfrak{a}^+ = \mathfrak{a}^{(0)}$ Input ideal – Norm arbitrary large

$\mathfrak{a}_1^{(1)} \quad \mathfrak{a}_2^{(1)} \quad \cdots \quad \mathfrak{a}_{n_1}^{(1)}$ Initial reduction – $L_{|\Delta_\mathbf{K}|}(1)$-smooth

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$\cdots$

$\mathfrak{a}^{(l-1)}$ Last but one step – $\approx L_{|\Delta_\mathbf{K}|}\left(\frac{1}{2}\right)$-smooth

$\mathfrak{a}_1^{(l)} \quad \mathfrak{a}_2^{(l)} \quad \cdots \quad \mathfrak{a}_{n_l}^{(l)}$

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$\mathfrak{a}_1^{(1)} \quad \mathfrak{a}_2^{(1)} \quad \cdots \quad \mathfrak{a}_{n_1}^{(1)}$	Initial reduction – $L_{\lvert\Delta_{\mathbf{K}}\rvert}(1)$-smooth
$\mathfrak{a}_1^{(2)} \quad \mathfrak{a}_2^{(2)} \quad \cdots \quad \mathfrak{a}_{n_2}^{(2)}$	First step – $L_{\lvert\Delta_{\mathbf{K}}\rvert}\left(\frac{3}{4}\right)$-smooth
$\mathfrak{a}_1^{(3)} \quad \mathfrak{a}_2^{(3)} \quad \cdots \quad \mathfrak{a}_{n_3}^{(3)}$	Second step – $L_{\lvert\Delta_{\mathbf{K}}\rvert}\left(\frac{5}{8}\right)$-smooth
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$\mathfrak{a}_1^{(l)} \quad \mathfrak{a}_2^{(l)} \quad \cdots \quad \mathfrak{a}_{n_l}^{(l)}$	Last step – Norm: $L_{\lvert\Delta_{\mathbf{K}}\rvert}(1)$

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3. Solution for smooth ideals

Input: Bunch of prime ideals of norm below $B = L_{|\Delta_{\mathbf{K}}|}\left(\frac{1}{2}\right)$

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  Relation: principal ideal that splits on the factor base.
  Test ideals generated by $\boldsymbol{v} = \sum v_i(\zeta^i + \zeta^{-i})$ for $|v_i| \leq \log|\Delta_{\mathbf{K}}|$

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  Norm below $L_{|\Delta_{\mathbf{K}}|}(1) \implies L_{|\Delta_{\mathbf{K}}|}\left(\frac{1}{2}\right)$-smooth ideals in $L_{|\Delta_{\mathbf{K}}|}\left(\frac{1}{2}\right)$

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$$\begin{pmatrix} \boldsymbol{v}_1 \\ \boldsymbol{v}_2 \\ \vdots \\ \boldsymbol{v}_{Q|\mathscr{B}|} \end{pmatrix} \begin{matrix} \rightarrow \\ \rightarrow \\ \vdots \\ \rightarrow \end{matrix} \begin{pmatrix} M_{1,1} & \cdots & M_{1,|\mathscr{B}|} \\ M_{2,1} & \cdots & M_{2,|\mathscr{B}|} \\ \vdots & & \vdots \\ M_{Q|\mathscr{B}|,1} & \cdots & M_{Q|\mathscr{B}|,|\mathscr{B}|} \end{pmatrix} \Longrightarrow \forall\, i, \langle \boldsymbol{v}_i \rangle = \prod_{j=1}^{|\mathscr{B}|} \mathfrak{p}_j^{M_{i,j}}$$

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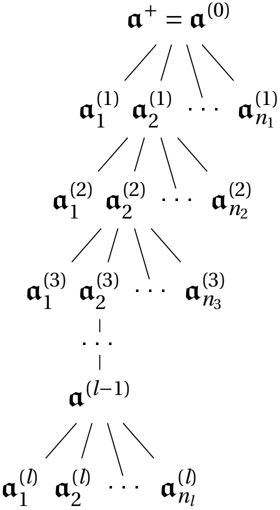
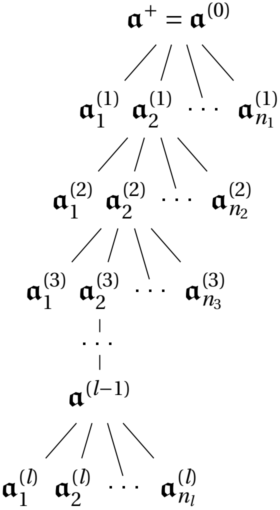
- A N-dimensional vector Y including all the valuations of the smooth ideals in the $\mathfrak{p}_i$
- A solution of $MX = Y$ provides a generator of the product of the $L_{|\Delta_{\mathbf{K}}|}\left(\frac{1}{2}\right)$-smooth ideals

4. The backtracking



A generator for the product



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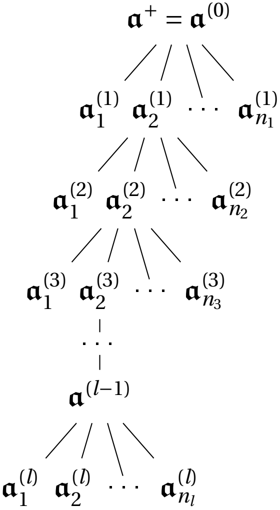

A generator for the initial ideal

$\Uparrow$

A generator for the product

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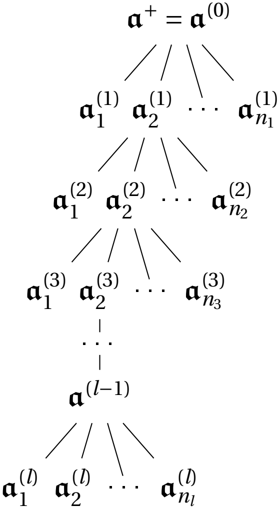

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Thanks

Kia ora

Illustrations by Alexia R. (@a_draw_r)