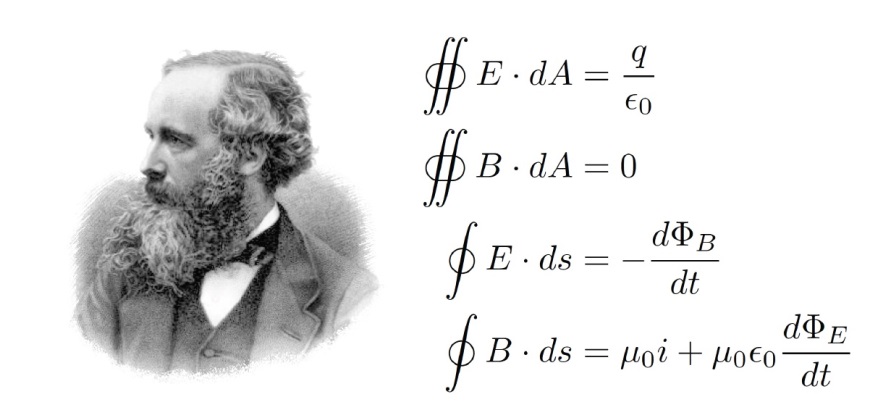
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**Maxwell’s Equations**

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Perhaps the greatest example demonstrating the effectiveness of mathematics in understanding the natural world would be a set of equations developed by James Clerk Maxwell, known as “Maxwell’s equations”. Appearing in the 19th century these four partial differential equations underpin our entire understanding of electromagnetic phenomena and are among the crown jewels of mathematical physics. By general consensus they are “four of the most influential equations in science: Gauss’s law for electric fields, Gauss’s law for magnetic fields, Faraday’s law, and the Ampere–Maxwell law” (Fleisch, forward). Despite their compact appearance Maxwell’s equations summarize all the fundamental properties of electricity and magnetism known to modern science. This simple way to state the complex nature of electromagnetism was due to Maxwell’s genius for formulating mathematical descriptions from his experimental work on the subject. The way these equations contain a wealth of content despite their relatively small size is almost unparalleled in the history of mathematics. It is fascinating to know that we were able to make the greatest technological leaps of the twentieth century thanks to the brilliant mathematical work done by Maxwell. The impact of his work on science and technology is exceptionally significant, just as Sir Isaac Newton laid the mathematical foundations for classical mechanics and gravity, James Clerk Maxwell created his own mathematical foundation for governing electricity and magnetism. (Fleisch)

The complete formulation of Maxwell’s equations did not happen right away; instead it was a long and painful journey that spanned centuries. The science of electricity and magnetism and their unification into a single force came to be known through a series of contributions made by a number of individual scientists and mathematicians. The great strides of the 18th century in understanding electric charges and currents were thanks to the pioneering work of great individuals such as Benjamin Franklin and Alessandro Volta; their work resulted in what we now know as the Coulomb Law. The law shows us that by varying the strength of an electric force with the inverse square of the distance to an electrically charged object we can find the exact force between them. In mathematical terms, this can be represented as:

where and are the two charges and is the distance between them, with being a constant. Coulomb Law was later generalized by the mathematician Carl Fredrick Gauss in the early 19th century leading to Gauss’ laws. The first Gauss’ law shows how the distribution of electrical charge relates to the acting electrical field, which can be portrayed as:

where is the vector field representing electric flux through a surface, is the total amount of charge enclosed within a surface, and is a constant. The second Gauss’ law states that the divergence of a magnetic field always equals to zero.

is a vector field representing magnetism. Both of these laws constitute the first and second equations out of a total of four. The third equation is perhaps the most important one; its deduction was directly influenced by Michael Faraday’s experiments with electromagnetic induction and is currently known as the Faraday’s Law. This law explains how a magnetic field will influence a circuit to produce a flowing current of electricity, putting this in an equation form we get

where again is an electric field, is a magnetic field, and is a unit of time. The last equation is known as the Ampere–Maxwell Law, which was derived from the work of Andre-Marie Ampere, who was a French mathematician and physicist and did extensive work on electrical currents. It states that a magnetic field formed around an enclosed electric current is proportional to the electric current that acts as its source, mathematically this would look like this:

where is the electric field, is the magnetic field, is the current density, and and are the two constants. By combining the four laws into four equations we get a complete set of Maxwell’s equations. With the help of his predecessors, James Clerk Maxwell would formulate a mathematical system that many today consider as the greatest equations ever invented in the history of science. (Guilmette)

To conclude his results, Maxwell was able to unify the theories of electricity, magnetism, and light into one coherent mathematical system, something that many of his contemporaries thought was impossible. Back then scientists thought that different natural phenomena had to be the result of independent events, not knowing that the same laws that explain one aspect of reality can affect other events that seemingly have no relation to each other. His greatest accomplishment was noticing the connection between electricity and magnetism, which at that time were considered as two different subjects. Maxwell was able to build a bridge between these two forces using some of the most brilliant mathematics ever conceived, from which electricity and magnetism came to be known as electromagnetism. The impact of this formulation was as grand as anything that came before, and was felt everywhere. New technologies and new areas of science have opened up due to the great leap made by understanding the mysterious new laws which eluded scientists for millennia, and that was all thanks to the introduction of Maxwell’s equations.

Each one of the four equations is quite dense, and just because they look simple it doesn’t mean that they are. The first Maxwell equation on our list is , known as Gauss’ law of electric field. In its differential form, the left hand side is a mathematical description of the divergence of vector fields; divergence is an operator from vector calculus that measures the rate in which vectors in a vector field move away from a certain point. They are presented as , where the gradient operator Del is dotted with some vector field . Since the gradient of a function points in the direction of greatest increase, intuitively we can see how taking a dot product between the gradient and will give us the divergence. The right hand side of the equation simply states that the only way we can have a divergence is by the presence of an electric charge giving off an electric field, otherwise the divergence would have to be zero. We can imagine electric fields as being a field of vectors coming out of some charged particles like rays of light, or going in as in the case of negatively charged particles. The rate at which this is happening can be measured by the divergence operator . The differential form of Gauss’ electric field law can be used “in any problem in which the spatial variation of the vector electric field is known at a specified location, you can find the volume charge density at that location using this form. And if the volume charge density is known, the divergence of the electric field may be determined.” (Fleisch pg. 30) The differential form of this law can be derived from Coulomb Law. First we rewrite the equation as an electric force field: , which measures the electric field inside some geometric sphere – produced by a charged particle at the center. Then we take the divergence of both sides and get the differential form of Gauss’ law, . (Fleisch)

One way to apply Gauss’ Law (differential form) for a given electric field is to compute the divergence of the field , . Using the definition of divergence, . This makes sense since the vector field has vectors swirling around the center and do not diverge at any point, proving that is not an electric field.

By writing in an integral form we get . Here, a surface integral is being used; they tend to be encountered when we want to integrate a vector field instead of regular scalar function. They are very useful in problems such as finding a mass of a particular object given that it has a non-uniform distribution of density, the density function being represented by a vector field. In our case, the integral equates the total amount of electric field lines coming out from a closed surface to the total amount of charge contained inside that surface divided by a constant called the “permittivity of free space” or “electric constant”, which has a meaning in physics but is irrelevant for this topic. The circle on the integral symbol signifies that we are integrating over a closed surface (surface that doesn’t have a boundary), the normal vector dotted with the vector field allows us to calculate the amount of electric field vectors passing through the entire surface while being parallel to . The operator specifies an infinitely small increment of a surface area, and is measured in (meters squared). (Fleisch)

The Gauss’ law is very useful in analyzing electric fluxes through any given closed surfaces. Here are the two basic problems where can be applied, “(1) Given information about a distribution of electric charge, you can find the electric flux through a surface enclosing that charge. (2) Given information about the electric flux through a closed surface, you can find the total electric charge enclosed by that surface.” (Fleisch pg. 2). These kind of problems are widely encountered in engineering applications. (Fleisch)

We can use integral form of Gauss’ Law to calculate the electric flux coming out from a closed surface containing 5 charges: , , , , and ; where . Using the equation we have . Notice that we didn’t have to evaluate the complicated double integral to get our result, showing the brilliance of the first Gauss’ Law.

The second equation is the “second Gauss’ Law” or “Gauss’ law for magnetic fields”. In differential form it is stated as , which is similar to the law of electric fields except that this time we have a magnetic field, and its divergence always equals to zero. The law describes a famous law of magnetism where the vector field of a magnet is structured in such a way that its vectors do not diverge at any given point, therefore the divergence of a magnetic field is . The reason for this is the fact that magnetic field vectors are going in and out of the source, unlike the vectors in an electric field that radiate outwards in straight lines; the physical reason for this is a question of physics and is not in the scope of this paper.

The differential form of Gauss’ law for magnetism is very helpful in determining whether the given vector field is generated by a magnetic phenomenon, or by using an incomplete information about some parts of a magnetic field to reestablish a complete relationship between individual components. If we were to use the vector field from our previous example problem where we had , we can then calculate its divergence and get , from which it can be concluded that the field is magnetic and not electric. (Fleisch)

For the integral form of we have , again notice the similarity to the integral form of Gauss’ law for electric fields. To describe this, we use a standard surface integral just like we did with the first law, then we integrate over a closed surface and equate it to zero, meaning the total magnetic flux passing through any closed surface is zero. It doesn’t mean that there are no magnetic field lines going through the surface, it means that the number of vector flowing out of the surface are equated to the number of vectors going inside the surface. So the inward magnetic flux must be equal to the outward magnetic flux. Knowing the fact that total magnetic flux through a closed surface needs to be zero may allow us to solve problems involving complex surfaces, which is very useful in differential geometry and in some areas of complex analysis. The example problems using the integral form are too complicated for this paper, so with this we finish Maxwell’s second equation. (Fleisch)

Our next equation is the Faraday’s law. The differential form is generally written as . The left hand side of this equation is a description of the of some electric field, stated by taking a cross product of the gradient operator Del and a vector field , . A is a mathematical operation done on vector fields that tells us the way the field vectors circulate around a given point, just like divergence is a way to measure the tendency of the field to flow away from a point. The direction of the , which is the axis of rotation, is determined by the famous right hand rule. A good analogy would be a paddle wheel spinning due to a current of water. The term “curl” was actually coined by James Clerk Maxwell himself for the vector operator that appears in his equations. For the right hand side of the differential form we have the rate of change of the magnetic field over time, and is represented by a partial derivative. Overall, the equation explains a circulating electric field that is produced by a magnetic field varying with time. One type of problem that can be solved using this equation is finding the electric field from nothing but the function of a magnetic field, and vice versa. (Fleisch)

For example, if we are given an electric field we can find the time rate of change of the magnetic field using . By taking a curl on the right hand side we get , . If we want to get all we have to do is integrate.

The integral form of Faraday’s law is written as . The equation defines how the changing magnetic flux through a surface induces an (a form of electromagnetic energy) in any boundary path of that surface, inversely, changing the magnetic field will induce a circulating electric field. On the left hand side we have a line integral of the electric field integrated over some path , which is represented by a line or a curve. The line integral is like a surface integral integrated over a path instead of a surface. A good example of a typical problem would be finding the total mass of a wire for which the density varies all along its length. The dot product between and tells us that we are only interested in those parts of that are parallel to along , where represents an incremental segment. The right hand side of the equation describes the flux of the magnetic field with respect to change in time. (Fleisch)

Faraday’s law can be used to solve a variety of problems involving changing magnetic flux and induced electric fields, there are two types of problems we can solve with this equation: “(1) Given information about the changing magnetic flux, find the induced . (2) Given the induced on a specified path, determine the rate of change of the magnetic field magnitude or direction or the area bounded by the path.” (Fleisch, pg. 61). The most famous application would probably be power generation for all of our electrical grids. (Fleisch)

To demonstrate the workings of Faraday’s Law (integral form) we have a given problem: we have a magnetic field , find the induced in a square loop of side lying in the with one corner at the origin. Using the equation, , . So we can see how to calculate the voltage (electric potential) using the given magnetic field . This concludes Maxwell’s third equation.

The final equation is the Ampere–Maxwell law. The differential form of this law can be written as . The left hand side of this equation is a mathematical description of the of the magnetic field. The two terms on the right hand side represent an electric current density and the time rate of change of the electric field. We add the current density vector to the rate of change of the electric field with respect to time and get the result. This equation shows the circulating magnetic field being produced by an electric current and by an electric field that changes with time. The most common applications of the differential form of the Ampere–Maxwell law are finding solutions to problems where we are given a vector magnetic field and we need to figure out the electric current density . The example problems for the differential form are too complicated and it’s best not to get overly technical. (Fleisch)

The integral form of the Ampere–Maxwell Law is . The left hand side of this equation is a description of the circulation of the magnetic field around some closed path . The right hand side has two sources for the given magnetic field, a steady conduction current and a changing electric flux through any surface along a path . Mathematically, the line integral on the left tells us to sum up each individual contribution from every portion of closed path C in the direction of moving charge, in our case it’s . On the right hand side we have the enclosed current plus the rate of change with respect to time of the electric flux coming out through a surface bounded by . This law is no doubt the most difficult to grasp, but there are many ways to use Ampere–Maxwell for solving real world problems, “You can use it to determine the circulation of the magnetic field if you’re given information about the enclosed current or the change in electric flux. Furthermore, in highly symmetric situations, you may be able to extract from the dot product and the integral and determine the magnitude of the magnetic field.” (Fleisch, pg. 84). This concludes Maxwell’s fourth and final equation. (Fleisch)

A Good problem for the integral form of Ampere–Maxwell Law is finding the magnetic field from an electric current in a wire. For example, a straight wire of radius carrying a steady current uniformly distributed throughout its cross-section. We need to find the magnitude of the magnetic field as a function of , where is the distance from the center of the wire. Using and the dot product property we get , then , where is some direction.

The four Maxwell’s equations represent the fundamental nature of electromagnetic field theory. The applications of these equations don’t stop with electricity and magnetism. Maxwell went on to use nothing but his own mathematics to deduce a theory about the true nature of light, which concludes that it is an electromagnetic wave. This was a total shock for the scientific community of his time. Having understood the theory of light he is credited for taking the first color photograph in history. The color triangle RGB (Red Green Blue) which allows us to synthesize all known colors by using just red, green, and blue is the basis of every pixel on every modern color display, from computer monitors to television sets. In this section we will see how Maxwell’s equations guide us directly to the wave equation. First we need to understand two important theorems of vector calculus, the divergence and Stokes’ theorems. Their main purpose is to help us make the transition from the integral form to the differential form of Maxwell’s Equations. (Hughes)

The divergence theorem is a relation in vector and multivariable calculus that equates the flux of a vector field to the triple integral of the field’s divergence, . The relationships between line, surface, and triple integrals have been researched by some of the greatest mathematicians of the eighteenth and nineteenth centuries, such as Joseph Luis Lagrange, George Green, and Carl Fredric Gauss. The formal definition of the divergence theorem is “The flux of a vector field through a closed surface S is equal to the integral of the divergence of that field over a volume for which is a boundary.” (Fleisch, pg. 114) It can be applied to the two differential form equations representing the two Gauss’ laws to transform them into integral forms. (Fleisch)

Just like how the divergence theorem relates a surface integral to a volume integral, Stokes’ theorem relates a line integral to a surface integral, “The circulation of a vector field over a closed path is equal to the integral of the normal component of the curl of that field over a surface for which is a boundary” (Fleisch, pg. 116). This can be expressed as . George Stokes first introduced this theorem as a question but it was solved by another mathematician, nonetheless the theorem still bears his name. We can use this theorem to transform the last two of Maxwell’s equations, Faraday’s Law and Maxwell-Ampere Law, from their differential form to the integral form. (Fleisch)

We are now ready to deduce the wave equation. First take the of both sides of the differential form of Faraday’s law, . For the next step we use an identity that states that the of the of any vector field equals the gradient of the divergence of the field minus the Laplacian of the field, where Laplacian is just the gradient operator to the second degree, . Then we get . Apply the curl of the magnetic field from the differential form of the Ampere–Maxwell law and get . After using Gauss’ law of electric fields the equation becomes . Doing some algebra on our last equation yields . Setting and to zero

finally we have a linear, second-order, homogeneous partial differential equation describing an electric field that travels from point to point , which is a propagating wave. (Fleisch)

The mastermind behind these elegant equations, James Clerk Maxwell, was born on 1831 in Edinburgh, Scotland. His parents were both fellows of the Royal Society, which was an organization for elite scientists of Great Britain; this could explain why his talent for mathematics came at an early age. When he was just fourteen years old, he published his first paper on “Oval Curves”, describing a number of oval curves that could be drawn with pins and thread. Young James Maxwell then went on to enroll at the prestigious Edinburgh University at the age of sixteen where he studied logic, mathematics, and natural philosophy. After finishing his graduate studies he became a Fellow of The Royal Society of Edinburgh at the age of 25, and Fellow of the Royal Society at 30. He would then go on to have a very productive career where he made numerous contributions to Mathematics, Physics, Astronomy, and engineering. Unfortunately, Maxwell had an untimely death at the age of 48 due to stomach cancer. Even though he lived a relatively short life, his discoveries made his name standout as one of the greatest in the history of science. Albert Einstein once acknowledged him by saying that “One scientific epoch ended and another began with James Clerk Maxwell”. (Guilmette)

The Impact of Maxwell’s Work was as significant as Newton’s Principia. After his publication of “A Dynamical Theory of the Electromagnetic Field” (1865), Maxwell produced a complete system of equations governing electricity, magnetism, and electromagnetic radiation; thus completely altering the course of science. His true gift was the ability to see and interpret physical phenomena in terms of relationships which can be translated to equations, even if it meant abandoning the physical analogy. Currently, Maxwell’s Equations are essential tools for physicists and electrical engineers, and are used to design all electronic devices from cell phones, satellites, televisions, computers, to transformers and power stations. His impact on theoretical physics allowed Einstein to develop his famous theories on relativity, which without Maxwell’s contribution would never have been made, thus showing that Maxwell’s equations are as relevant to modern technology as they are to unlocking the secrets of the universe.

Nowadays the name “James Clerk Maxwell” is not well known among the general public, but most scientists rank him along with Sir Isaac Newton and Albert Einstein as the most influential mathematicians/physicists of all time. The reason for this is perhaps that the average person would have a hard time understanding his work. Young school children can be taught to understand equations such as or , but to understand Maxwell’s equations you need a good grasp of advanced calculus.

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