Problem
Conductors in Conducting Liquid

A system consisted of two conductor bodies is immersed in a uniform dielectric and weakly conducting liquid. When a constant voltage difference is applied between both conductors, the system has both electric and magnetic fields. In this problem we will investigate this system.

1. **(0.4 pts)** First consider an infinitely long line with charge per unit length λ in vacuum. Calculate the electric field $\mathbf{E}(\mathbf{r})$ due to the line.

2. **(0.4 pts)** The potential due to the line charge could be written as
$$V(r) = f(r) + K,$$
where K is a constant. Determine $f(r)$.

3. **(0.7 pts)** Calculate the potential in all space $V(x,y,z)$due to an infinitely long line with charge per unit length λ at $x = -b, y = 0$ and another infinitely long line with charge per unit length $-\lambda$ at $x = b, y = 0$. Both lines are parallel to the z-axis. Take $V = 0$ at the origin. Sketch the equipotential surfaces.

   For the following questions, ignore any edge effects.

4. **(2.0 pts)** Now consider two identical conducting cylinders, both with radius $R = 3a$ in vacuum. The length of each cylinders are the same and much larger than its radius ($l \gg R$). The axis of both cylinders are on the xz-plane and parallel to the z-axis, one at $x = -5a$, $y = 0$ and the other at $x = 5a$, $y = 0$. An electrical potential difference of V_0 is applied between the two cylinders (the cylinder at $x = -5a$ has the higher potential) by connecting them to a battery. Calculate the potential in **all regions**. Take $V = 0$ at the origin.

5. **(0.5 pts)** Calculate the capacitance C of the system.

6. **(1.0 pts)** Now both cylinders are totally immersed in a weakly conducting liquid with conductivity σ. Calculate the total current that flows between both cylinders. Assume the permittivity of the liquid is equal to that of vacuum, $\epsilon = \epsilon_0$.

7. **(0.5 pts)** Calculate the resistance R of the system. Calculate RC of the system.

8. **(1.5 pts)** Calculate the magnetic field due to the current in question 6. Assume that the permeability of the liquid is equal to that of vacuum $\mu = \mu_0$.

   Notes $\int \frac{\alpha dx}{\alpha^2+x^2} = \arctan\frac{x}{a} + const$

Global Positioning System (GPS) is a navigation technology which uses signal from satellites to determine the position of an object (for example an airplane). However, due to the satellites high speed movement in orbit, there should be a special relativistic correction, and due to their high altitude, there should be a general relativistic correction. Both corrections seem to be small but are very important for precise measurement of position. We will explore both corrections in this problem.

First we will investigate the special relativistic effect on an accelerated particle. We consider two types of frame, the first one is the **rest frame** (called S or Earth's frame), where the particle is at rest initially. The other is the **proper frame** (called S'), a frame that instantaneously moves together with the accelerated particle. Note that this is not an accelerated frame, it is a constant velocity frame that at a particular moment has the same velocity with the accelerated particle. At that short moment, the time rate experienced by the particle is the **same** as the proper frame's time rate. Of course this proper frame is only good for an infinitesimally short time, and then we need to define a new proper frame afterward. At the beginning we synchronize the particle's clock with the clock in the rest frame by setting them to zero, $t = \tau = 0$ (t is the time in the rest frame, and τ is the time shown by particle's clock).

By applying **equivalence principle**, we can obtain general relativistic effects from special relavistic results which does not involve complicated metric tensor calculations. By combining the special and general relativistic effects, we can calculate the corrections needed for a GPS (global positioning system) satellite to provide accurate positioning.

Some mathematics formulas that might be useful

- $\sinh x = \frac{e^x - e^{-x}}{2}$
- $\cosh x = \frac{e^x + e^{-x}}{2}$
- $\tanh x = \frac{\sinh x}{\cosh x}$
- $1 + \sinh^2 x = \cosh^2 x$
- $\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$

- $\int \frac{dx}{(1-x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{1-x^2}} + C$
- $\int \frac{dx}{1-x^2} = \ln\sqrt{\frac{1+x}{1-x}} + C$

Part A. Single Accelerated Particle (2.8 points)

Consider a particle with a rest mass m under a constant and uniform force field F (defined in the rest frame) pointing in the positive x direction. Initially ($t = \tau = 0$) the particle is at rest at the origin ($x = 0$).

1. **(0.5 pts)** When the velocity of the particle is v, calculate the acceleration of the particle, a (with respect to the rest frame).
2. **(0.5 pts)** Calculate the velocity of the particle $\beta(t) = \frac{v(t)}{c}$ at time t (in rest frame), in terms of F, m, t and c.
3. **(0.3 pts)** Calculate the position of the particle $x(t)$ at time t, in term of F, m, t and c.
4. **(0.7 pts)** Show that the proper acceleration of the particle, $a' \equiv g = F/m$, is a constant. The proper acceleration is the acceleration of the particle measured in the instantaneous proper frame.
5. **(0.4 pts)** Calculate the velocity of the particle $\beta(\tau)$, when the time as experienced by the particle is τ. Express the answer in g, τ, and c.
6. **(0.4 pts)** Also calculate the time t in the rest frame in terms of g, τ, and c.

Part B. Flight Time (2.0 points)

The first part has **not** taken into account the flight time of the information to arrive to the observer. This part is the only part in the whole problem where the flight time is considered. The particle moves as in part A.

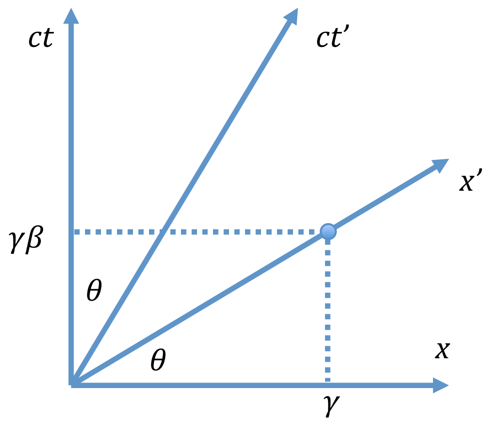
1. **(1.2 pts)** At a certain moment, the time experienced by the particle is τ. What reading t_0 on a stationary clock located at $x = 0$ will be observed by the particle? After a long period of time, does the observed reading t_0 approach a certain value? If so, what is the value?
2. **(0.8 pts)** Now consider the opposite point of view. If an observer at the initial point ($x = 0$) is observing the particle's clock when the observer's time

is t, what is the reading of the particle's clock τ_0? After a long period of time, will this reading approach a certain value? If so, what is the value?

Part C. Minkowski Diagram (1.0 points)

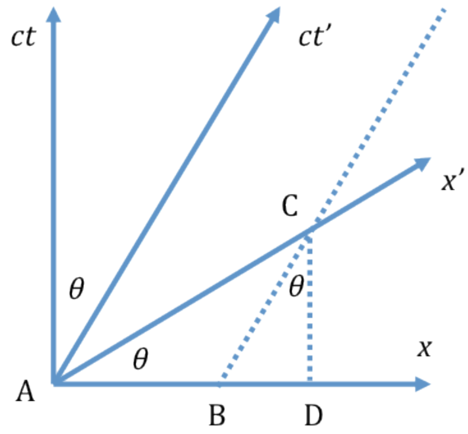
In many occasion, it is very useful to illustrate relativistic events using a diagram, called as Minkowski Diagram. To make the diagram, we just need to use Lorentz transformation between the rest frame S and the moving frame S' that move with velocity $v = \beta c$ with respect to the rest frame.

$$x = \gamma(x' + \beta ct'),$$
$$ct = \gamma(ct' + \beta x'),$$
$$x' = \gamma(x - \beta ct),$$
$$ct' = \gamma(ct - \beta x).$$
$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$



Let's choose x and ct as the orthogonal axes. A point (x', ct') = (1,0) in the moving frame S' has a coordinate (x, ct) = $(\gamma, \gamma\beta)$ in the rest frame S. The line connecting this point and the origin defines the x' axis. Another point (x', ct') = (0,1) in the moving frame S' has a coordinate (x, ct) = $(\gamma\beta, \gamma)$ in the rest frame S. The line connecting this point and the origin defines the ct' axis. The angle between the x *and* x' *axis is* θ, where $\tan\theta = \beta$. A unit length in the moving frame S' is equal to $\gamma\sqrt{1+\beta^2} = \sqrt{\frac{1+\beta^2}{1-\beta^2}}$ in the rest frame S.

To get a better understanding of Minkowski diagram, let us take a look at this example. Consider a stick of proper length L in a moving frame S'. We would like to find the length of the stick in the rest frame S. Consider the figure below.



The stick is represented by the segment AC. The length AC is equal to $\sqrt{\frac{1+\beta^2}{1-\beta^2}}L$ in the S frame. The stick length in the S frame is represented by the line AB.

$$\begin{aligned}\mathrm{AB} &= \mathrm{AD} - \mathrm{BD}\\ &= AC\cos\theta - AC\sin\theta\tan\theta\\ &= L\sqrt{1-\beta^2}\end{aligned}$$

1. **(0.5 pts)** Using a Minkowski diagram, calculate the length of a stick with proper length L in the rest frame, as measured in the moving frame.
2. **(0.5 pts)** Now consider the case in part A. Plot the time ct versus the position x of the particle. Draw the x' axis and ct' axis when $\frac{gt}{c} = 1$ in the same graph using length scale $x(c^2/g)$ and $ct(c^2/g)$.

Part D. Two Accelerated Particles (2.3 points)

For this part, we will consider two accelerated particles, both of them have the same proper acceleration g in the positive x direction, but the first particle starts from x = 0, while the second particle starts from x = L. Remember, **DO NOT** consider the flight time in this part.

1. **(0.3 pts)** After a while, an observer in the rest frame make an observation. The first particle's clock shows time at τ_A. What is the reading of the second clock τ_B, according to the observer in the rest frame.
2. **(1.0 pts)** Now consider the observation from the first particle's frame. At a certain moment, an observer that move together with the first particle

observed that the reading of his own clock is τ_1. At the same time, he observed the second particle's clock, and the reading is τ_2. Show that

$$\sinh\frac{g}{c}(\tau_2-\tau_1)=C_1\sinh\frac{g\tau_1}{c},$$

where C_1 is a constant. Determine C_1.

3. **(1.0 pts)** The first particle will see the second particle move away from him. Show that the rate of change of the distance between the two particles according to the first particle is

$$\frac{dL'}{d\tau_1}=C_2\frac{\sinh\frac{g\tau_2}{c}}{\cosh\frac{g}{c}(\tau_2-\tau_1)},$$

where C_2 is a constant. Determine C_2.

Part E. Uniformly Accelerated Frame (2.7 points)

In this part we will arrange the proper acceleration of the particles, so that the distance between both particles are constant according to each particle. Initially both particles are at rest, the first particle is at $x = 0$, while the second particle is at $x = L$.

1. **(0.8 pts)** The first particle has a proper acceleration g_1 in the positive x direction. When it is being accelerated, there exists a fixed point in the rest frame at $x=x_p$ that has a constant distance from the first particle, according to the first particle thoughout the motion. Determine x_p.
2. **(1.3 pts)** Given the proper acceleration of the first particle is g_1, determine the proper acceleration of the second particle g_2, so that the distance between the two particles are constant according to the first particle.
3. **(0.6 pts)** What is the ratio of time rate of the second particle to the first particle $\frac{d\tau_2}{d\tau_1}$, according to the first particle.

Part F. Correction for GPS (2.2 points)

Part E indicates that the time rate of clocks at different altitude will not be the same, even though there is no relative movement between those clocks.

According to the **equivalence principle** in general relativity, an observer in a small closed room could not tell the difference between a gravity pull g and the fictitious force from accelerated frame with acceleration g. So we can conclude that two clocks at different gravitational potential will have different rate.

Now let consider a GPS satellite that orbiting the Earth with a period of 12 hours.

1. **(0.6 pts)** If the gravitational acceleration on the Earth's surface is 9.78 m.s^{-2}, and the Earth's radius is 6380 km, what is the radius of the GPS satellite orbit? What is the velocity of the satellite? Calculate the numerical values of the radius and the velocity.
2. **(1.2 pts)** After one day, the clock reading on the Earth surface and the satellite will differ due to both special and general relativistic effects. Calculate the difference due to each effect for one day. Calculate the total difference for one day. Which clock is faster, a clock on the Earth's surface or the satellite's clock?
3. **(0.4 pts)** After one day, estimate the error in position due to this effect?

All matters in the universe have fundamental properties called spin, besides their mass and charge. Spin is an intrinsic form of angular momentum carried by particles. Despite the fact that quantum mechanics is needed for a full treatment of spin, we can still study the physics of spin using the usual classical formalism. In this problem, we are investigating the influence of magnetic field on spin using its classical analogue.

The classical torque equation of spin is given by

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = \boldsymbol{\mu} \times \boldsymbol{B}.$$

In this case, the angular momentum $\boldsymbol{L}$ represents the "intrinsic" spin of the particles, $\boldsymbol{\mu}$ is the magnetic moment of the particles, and $\boldsymbol{B}$ is magnetic field. The spin of a particle is associated with a magnetic moment via the equation

$$\boldsymbol{\mu} = -\gamma \boldsymbol{L}$$

where γ is the gyromagnetic ratio.

In this problem, the term "frequency" means angular frequency (rad/s), which is a scalar quantity. All bold letters represent vectors; otherwise they represent scalars.

Part A. Larmor precession (1.6 points)

1. **(0.8 pts)** Prove that the magnitude of magnetic moment μ is always constant under the influence of a magnetic field $\boldsymbol{B}$. For a special case of stationary (constant) magnetic field, also show that the angle between $\boldsymbol{\mu}$ and $\boldsymbol{B}$ is constant.
   (*Hint*: You can use properties of vector products.)
2. **(0.8 pts)** A uniform magnetic field $\boldsymbol{B}$ exists and it makes an angle ϕ with a particle's magnetic moment $\boldsymbol{\mu}$. Due to the torque by the magnetic field, the magnetic moment $\boldsymbol{\mu}$ rotates around the field $\boldsymbol{B}$, which is also known as Larmor precession. Determine the Larmor precession frequency ω_0 of the magnetic moment with respect to $\boldsymbol{B} = B_0\boldsymbol{k}$.

Part B. Rotating frame (3.4 points)

In this section, we choose a rotating frame S' as our frame of reference. The rotating frame $S' = (x', y', z')$ rotates with an angular velocity $\omega\boldsymbol{k}$ as seen by an observer in the laboratory frame $S = (x, y, z)$, where the axes x', y', z' intersect with x, y, z at time $t = 0$. Any vector $\boldsymbol{A} = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$ in a lab frame can be written as $\boldsymbol{A} = A_x{}'\boldsymbol{i}' + A_y{}'\boldsymbol{j}' + A_z{}'\boldsymbol{k}'$ in the rotating frame S'. The time derivative of the vector becomes

$$\frac{d\boldsymbol{A}}{dt} = \left(\frac{dA_x{}'}{dt}\boldsymbol{i}' + \frac{dA_y{}'}{dt}\boldsymbol{j}' + \frac{dA_z{}'}{dt}\boldsymbol{k}'\right) + \left(A_x{}'\frac{d\boldsymbol{i}'}{dt} + A_y{}'\frac{d\boldsymbol{j}'}{dt} + A_z{}'\frac{d\boldsymbol{k}'}{dt}\right)$$

$$\left(\frac{d\boldsymbol{A}}{dt}\right)_{lab} = \left(\frac{d\boldsymbol{A}}{dt}\right)_{rot} + (\omega\mathbf{k}\times\boldsymbol{A}),$$

where $\left(\frac{d\boldsymbol{A}}{dt}\right)_{lab}$ is the time derivative of vector $\boldsymbol{A}$ seen by an observer in the lab frame, and $\left(\frac{d\boldsymbol{A}}{dt}\right)_{rot}$ is the time derivative seen by an observer in the rotating frame. For all the following problems in this part, the answers are referred to the rotating frame S'.

1. **(0.8 pts)** Show that the time evolution of the magnetic moment follows the equation

$$\left(\frac{d\boldsymbol{\mu}}{dt}\right)_{rot} = -\gamma\, \boldsymbol{\mu} \times \boldsymbol{B}_{eff},$$

where $\boldsymbol{B}_{eff} = \boldsymbol{B} - \frac{\omega}{\gamma}\boldsymbol{k}'$ is the effective magnetic field.

2. **(0.4 pts)** For $\boldsymbol{B} = B_0\boldsymbol{k}$, what is the new precession frequency Δ in terms of ω_0 and ω?

3. **(1.2 pts)** Now, let us consider the case of a time-varying magnetic field. Besides a constant magnetic field, we also apply a rotating magnetic field $\boldsymbol{b}(t) = b(\cos\omega t\, \boldsymbol{i} + \sin\omega t\, \boldsymbol{j})$, so $\boldsymbol{B} = B_0\boldsymbol{k} + \boldsymbol{b}(t)$. Show that the new Larmor precession frequency of the magnetic moment is

$$\Omega = \gamma\sqrt{\left(B_0 - \frac{\omega}{\gamma}\right)^2 + b^2}.$$

4. **(1.0 pts)** Instead of applying the field $\boldsymbol{b}(t) = b(\cos\omega t\, \boldsymbol{i} + \sin\omega t\, \boldsymbol{j})$, now we apply $\boldsymbol{b}(t) = b(\cos\omega t\, \boldsymbol{i} - \sin\omega t\, \boldsymbol{j})$, which rotates in the opposite direction and hence $\boldsymbol{B} = B_0\boldsymbol{k} + b(\cos\omega t\, \boldsymbol{i} - \sin\omega t\, \boldsymbol{j})$. What is the effective magnetic field $\boldsymbol{B}_{eff}$ for this case (in terms of the unit vectors $\boldsymbol{i}', \boldsymbol{j}', \boldsymbol{k}'$)? What is its time average, $\overline{\boldsymbol{B}_{eff}}$ (recall that $\overline{\cos 2\pi t/T} = \overline{\sin 2\pi t/T} = 0$)?

Part C. Rabi oscillation (3.0 points)

For an ensemble of N particles under the influence of a large magnetic field, the spin can have two quantum states: "up" and "down". Consequently, the total population of spin up $N_\uparrow$ and down $N_\downarrow$ obeys the equation

$$N_\uparrow + N_\downarrow = N.$$

The difference of spin up population and spin down population yields the macroscopic magnetization along the z axis:

$$M = (N_\uparrow - N_\downarrow)\mu = N\mu_z.$$

In a real experiment, two magnetic fields are usually applied, a large bias field $B_0\boldsymbol{k}$ and an oscillating field with amplitude $2b$ perpendicular to the bias field ($b \ll B_0$). Initially, only the large bias is applied, causing all the particles lie in the spin up states ($\boldsymbol{\mu}$ is oriented in the z-direction at $t = 0$). Then, the oscillating field is turned on, where its frequency ω is chosen to be in resonance with the Larmor precession frequency ω_0, i.e. $\omega = \omega_0$. In other words, the total field after time $t = 0$ is given by

$$\boldsymbol{B}(t) = B_0\boldsymbol{k} + 2b\cos\omega_0 t\,\boldsymbol{i}.$$

1. **(1.2 pts)** In the rotating frame S', show that the effective field can be approximated by

   $$\boldsymbol{B_{eff}} \approx b\boldsymbol{i}',$$

   which is commonly known as rotating wave approximation. What is the precession frequency Ω in frame S'?
2. **(0.6 pts)** Determine the angle α that $\boldsymbol{\mu}$ makes with $\boldsymbol{B_{eff}}$. Also, prove that the magnetization varies with time as

   $$M(t) = N\mu(\cos\Omega t).$$
3. **(1.2 pts)** Under the application of magnetic field described above, determine the fractional population of each spin up $P_{\uparrow} = N_{\uparrow}/N$ and spin down $P_{\downarrow} = N_{\downarrow}/N$ as a function of time. Plot $P_{\uparrow}(t)$ and $P_{\downarrow}(t)$ on the same graph vs. time t. The alternating spin up and spin down population as a function of time is called Rabi oscillation.

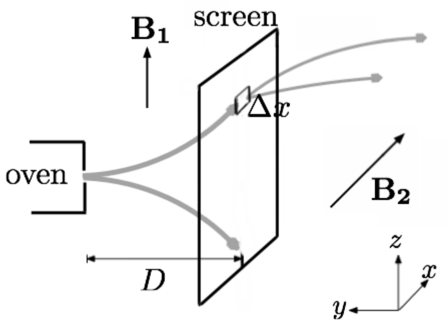
Part D. Measurement incompatibility (2.0 points)

Spin is in fact a vector quantity; but due to its quantum properties, we cannot measure each of its components simultaneously (i.e. we can know both $|\boldsymbol{\mu}|$ and μ_z as in above problems; but not all $|\boldsymbol{\mu}|$, μ_x, μ_y, and μ_z simultaneously). In this problem, we will do a calculation based on the Heisenberg uncertainty principle (using the relation $\Delta p_q \Delta q \geq \hbar$) to show how these measurements are incompatible with each other.

1. **(1.0 pts)** Let us consider an oven source of silver atoms, which has a small opening. The atoms stream out of the opening along $-y$ direction (see Figure below) and experience a spatial varying field $\boldsymbol{B}_1$. The field $\boldsymbol{B}_1$ has strong bias field component in the z direction, where the atoms with different magnetic moment $\mu_z = \pm\gamma\hbar$ are split in the z direction. At a distance D from the oven source, a screen SC_1 is put to allow only spin up atoms to pass (blocking spin down atoms). Thus, at the instant after passing the screen, the atoms are prepared in spin up states. After the screen, the atoms enter a region of non-homogenous field $\boldsymbol{B}_2$ where the atoms feel a force

   $$F_x = \mu_x C.$$

   The field $\boldsymbol{B}_2$ has strong bias field component in the x direction, where the atoms have magnetic moment $\mu_x = \pm\gamma\hbar$.



In order to determine μ_x by observing the splitting in x direction, show that the following condition must be fulfilled:

$$\frac{1}{\hbar}|\mu_x|\Delta x C t \gg 1,$$

where t is the duration after leaving the screen SC_1 and Δx is the opening width on SC_1.

2. **(1.0 pts)** The atoms are initially prepared in the spin up states right after leaving the screen, where $\mu_z = \gamma\hbar = |\mu_x|$. This means the atoms will precess at rates covering a range of values $\Delta\omega$ with respect to the x component of $\boldsymbol{B}_2$, specifically $B_{2x} = B_0 + Cx$. Prove that the spread in the precession angle $\Delta\omega t$ is so large and hence we cannot measure both μ_x and μ_z simultaneously. In other words, the measurement of μ_x destroys the information on μ_z.