Mechanics of a Deformable Lattice (Total Marks : 20)

Here we study a deformable lattice hanging in gravity which acts as a deformable physical pendulum. It has only one degree of freedom, i.e. only one way to deform it and the configuration is fully described by an angle α. Such structures have been studied by famous physicist James Maxwell in 19th century, and some surprising behaviors have been discovered recently.

As shown in the figure 1, N^2 identical triangular plates (red triangle) are freely hinged by identical rods and form an $N \times N$ lattice ($N > 1$). The joints at the vertices are denoted by small circles. The sides of the equilateral triangles and the rods have the same length l. The dashed lines in the figure represent four tubes; each tube confines N vertices (grey circles) on the edge and the N vertices can slide in the tube, i.e. the tube is like a sliding rail.

The four tubes are connected in a diamond shape with two angles fixed at 60° and another two angles at 120° as shown in Figure 1. Each plate has a uniform density with mass M, and the other parts of the system are massless. The configuration of the lattice is uniquely determined by the angle α, where $0^\circ \le \alpha \le 60^\circ$ (please see the examples of different angle α in Figure 1). The system is hung vertically like a "curtain" with the top tube fixed along the horizontal direction.

The coordinate system is shown in Figure 2. The zero level of the potential energy is defined at $y = 0$. A triangular plate is denoted by a pair of indices (m,n), where $m, n = 0, 1, 2, \cdots, N-1$ representing the order in the x and y directions respectively. A(m,n), B(m,n) and C(m,n) denote the positions of the 3 vertices of the triangle (m,n). The top-left vertex, A(0, 0) (the big black circle), is fixed.

The motion of the whole system is confined in the x-y plane. The moment of inertia of a uniform equilateral triangular plate about its center of mass is $I = Ml^2/12$. The free fall acceleration is g. Please use E_k and E_p to denote kinetic energy and potential energy respectively.

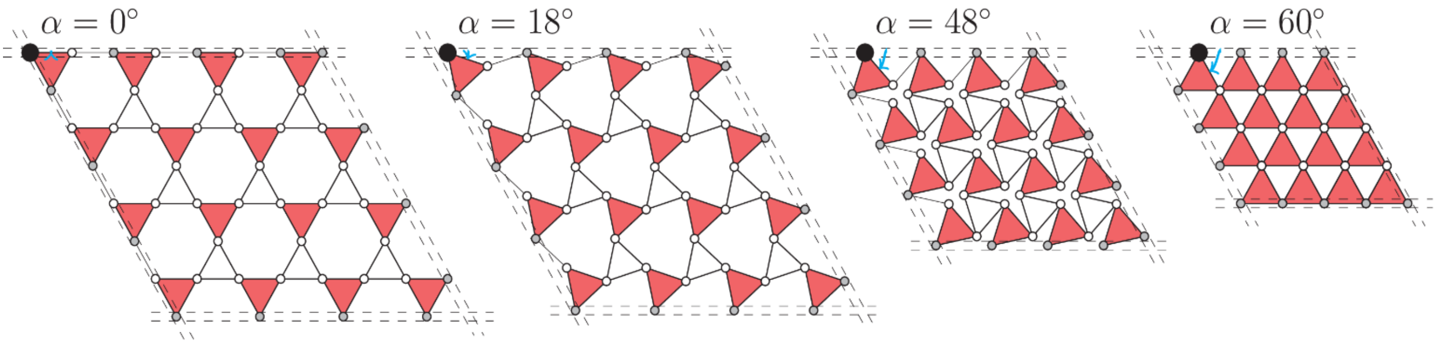


Figure 1

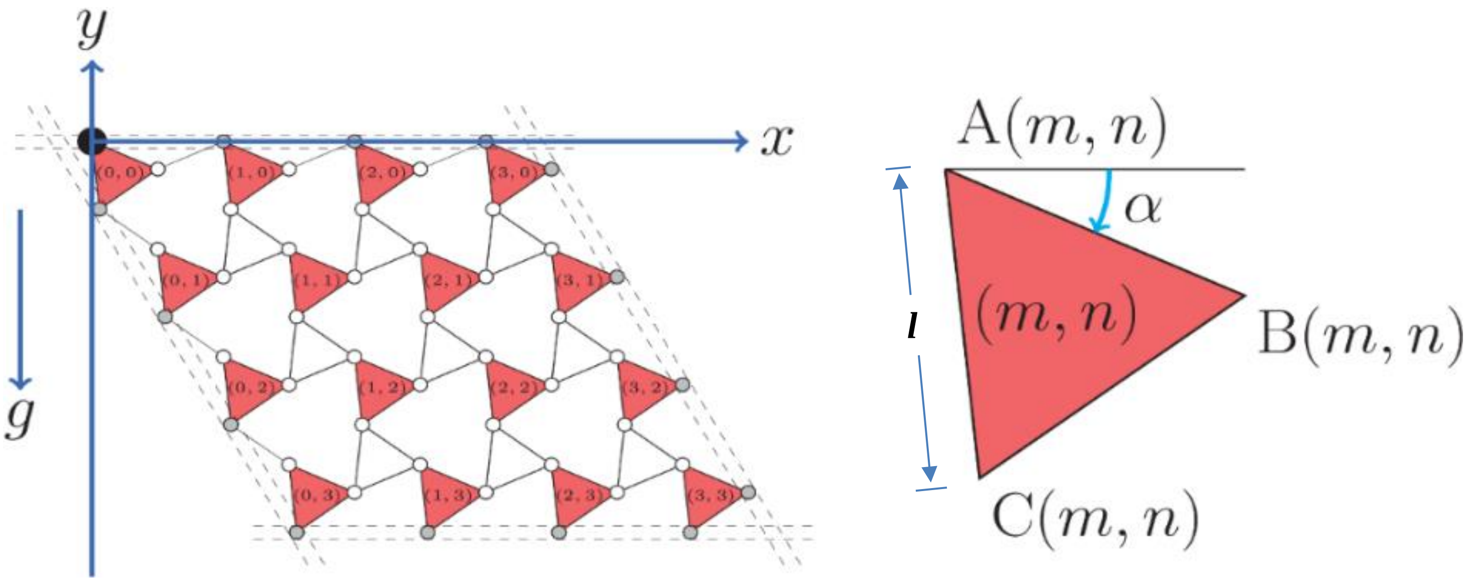


Figure 2

Section A: When N=2 (as shown in figure 3):

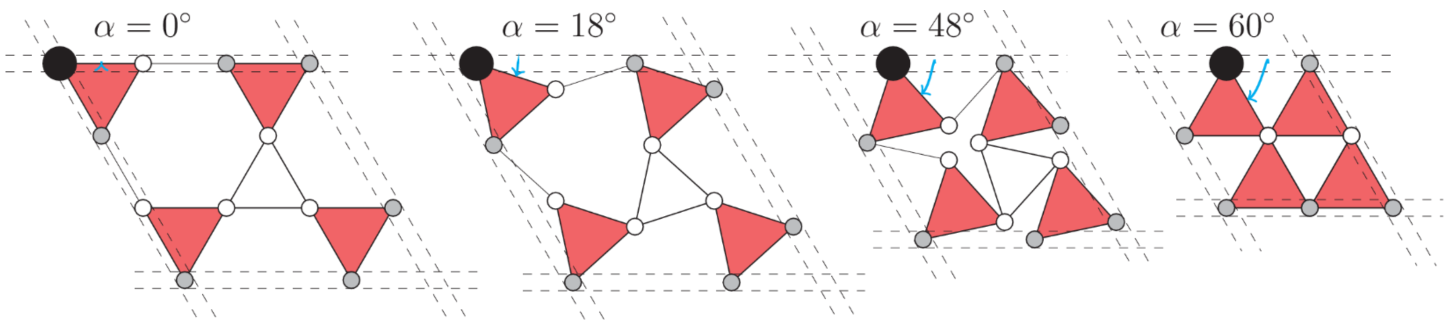


Figure 3

A1	What is the potential energy E_p of the system for a general angle α when $N = 2$?	**2 points**

A2	What is the equilibrium angle α_E of the system under gravity when $N = 2$?	**1 point**

A3	The system follows a simple harmonic oscillation under a small perturbation from equilibrium. Calculate the kinetic energy of this system in terms of $\Delta\dot{\alpha} \equiv d(\Delta\alpha)/dt$. Calculate the oscillation frequency f_E when $N = 2$.	**5 points**

Section B: For arbitrary N :

B1	What is the equilibrium angle α_E' under gravity when N is arbitrary?	**3 points**

B2	Consider the case when $N \to \infty$. Under a small perturbation of angle α, the change of potential energy of the system is $\Delta E_p \propto N^{\gamma_1}$, the kinetic energy of the system is $E_k \propto N^{\gamma_2}$, and the oscillation frequency is $f_E' \propto N^{\gamma_3}$. Find the values of γ_1, γ_2 and γ_3.	**3 points**

Section C: A force is exerted on one of the $3N^2$ triangle vertices so that the system maintains at $\alpha_m = 60°$.

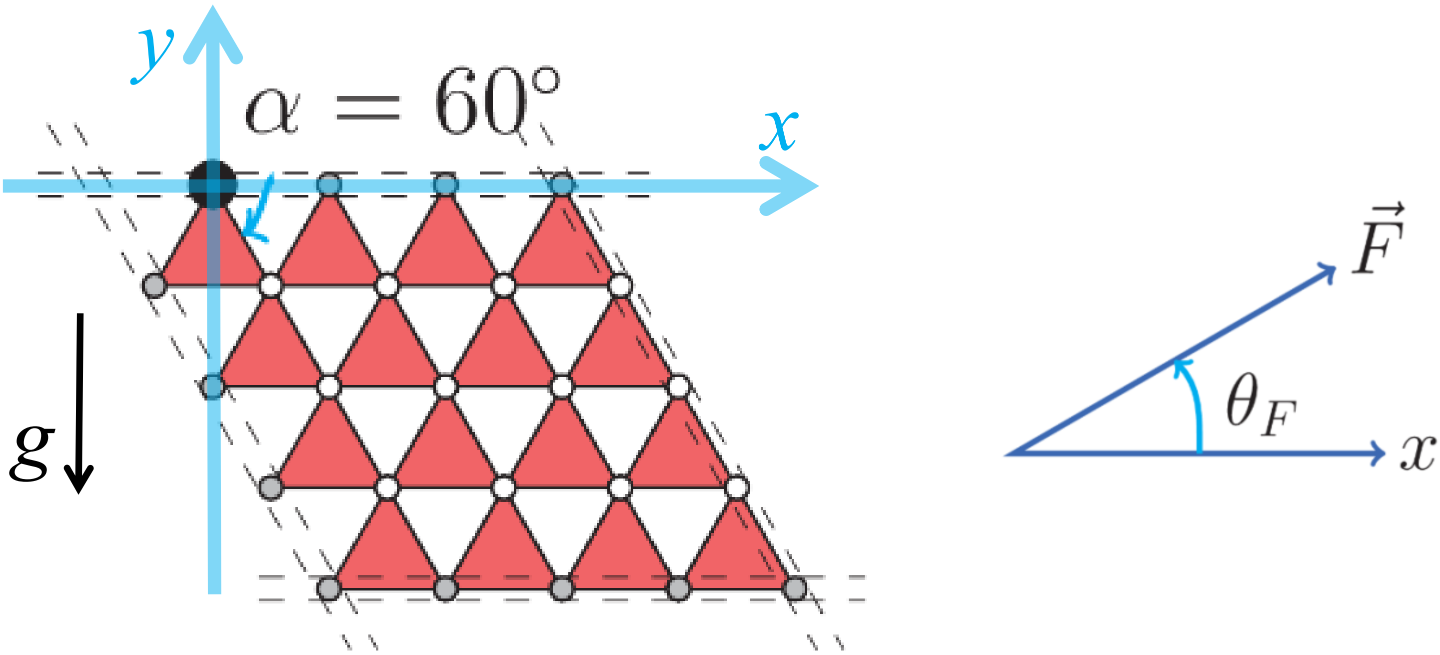


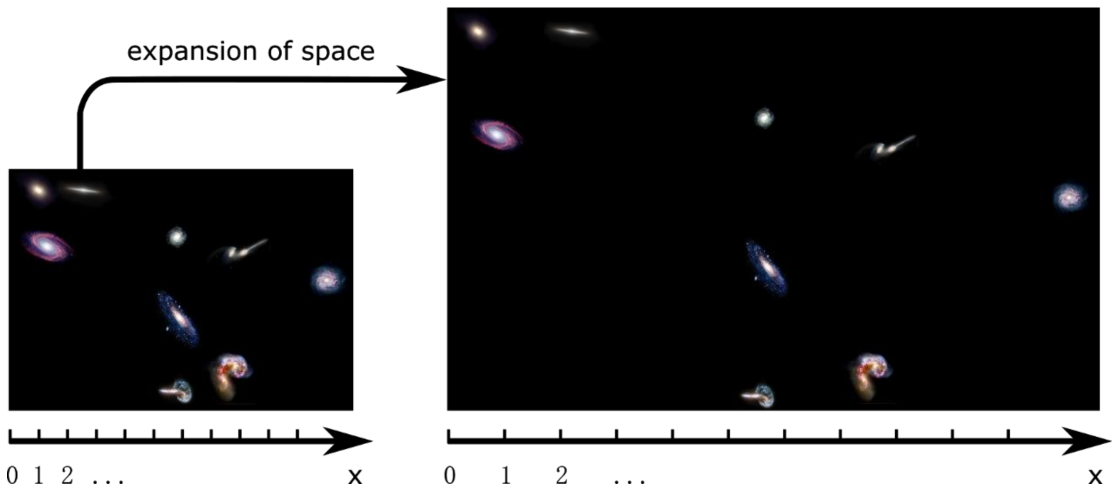
Figure 4

C1	Which vertex should we choose to minimize the magnitude of this force?	**1 point**

C2	What are the direction and magnitude of this minimum force? Describe the direction in terms of the angle θ_F defined in Figure 4.	**5 points**

The Expanding Universe (Total Marks : 20)

The most outstanding fact in cosmology is that our universe is expanding. Space is continuously created as time lapses. The expansion of space indicates that, when the universe expands, the distance between objects in our universe also expands. It is convenient to use "comoving" coordinate system $\vec{r} = (x, y, z)$ to label points in our expanding universe, in which the coordinate distance $\Delta r = |\vec{r}_2 - \vec{r}_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$ between objects 1 and 2 does not change. (Here we assume no peculiar motion, i.e. no additional motion of those objects other than the motion following the expansion of the universe.) The situation is illustrated in the figure below (the figure has two space dimensions, but our universe actually has three space dimensions).



The modern theory of cosmology is built upon Einstein's general relativity. However, under proper assumptions, a simplified understanding under the framework of Newton's theory of gravity is also possible. In the following questions, we shall work in the framework of Newton's gravity.

To measure the physical distance, a "scale factor" $a(t)$ is introduced such that the physical distance Δr_{p} between the comoving points $\vec{r}_1$ and $\vec{r}_2$ is

$$\Delta r_{\mathrm{p}} = a(t)\Delta r,$$

The expansion of the universe implies that $a(t)$ is an increasing function of time.

On large scales – scales much larger than galaxies and their clusters – our universe is approximately homogeneous and isotropic. So let us consider a toy model of our universe, which is filled with uniformly distributed particles. There are so many particles, such that we model them as a continuous fluid. Furthermore, we assume the number of particles is conserved.

Currently, our universe is dominated by non-relativistic matter, whose kinetic energy is negligible compared to its mass energy. Let $\rho_{\mathrm{m}}(t)$ be the physical energy density (i.e. energyper unit physical volume, which is dominated by mass energy for non-relativistic matter and the gravitational potential energy is not counted as part of the "physical energy density") of non-relativistic matter at time t. We use t_0 to denote the present time.

A	Derive the expression of $\rho_{\mathrm{m}}(t)$ at time t in terms of $a(t)$, $a(t_0)$ and $\rho_{\mathrm{m}}(t_0)$.	**2 points**

Besides non-relativistic matter, there is also a small amount of radiation in our current universe, which is made of massless particles, for example, photons. The physical wavelength of massless particles increases with the universe expansion as $\lambda_{\mathrm{p}} \propto a(t)$. Let the physical energy density of radiation be $\rho_{\mathrm{r}}(t)$.

B	Derive the physical energy density for radiation $\rho_{\mathrm{r}}(t)$ at time t in terms of $a(t)$, $a(t_0)$ and $\rho_{\mathrm{r}}(t_0)$.	**2 points**

Consider a gas of non-interacting photons which has thermal equilibrium distribution. In this situation, the temperature of the photon depends on time as $T(t) \propto [a(t)]^{\gamma}$.

C	Calculate the numerical value of γ.	**2 points**

Consider the thermodynamics of one type of non-interacting particle X. Note that the space expansion is slow enough and thermally isolated such that the entropy of X is a constant in time. Let the physical energy density of X be $\rho_{\mathrm{X}}(t)$, which includes mass energy and internal energy. Let the physical pressure be $p_{\mathrm{X}}(t)$.

D	Derive $\mathrm{d}\rho_{\mathrm{X}}(t)/\mathrm{d}t$ in terms of $a(t)$, $\mathrm{d}a(t)/\mathrm{d}t$,$\rho_{\mathrm{X}}(t)$,and $p_{\mathrm{X}}(t)$.	**4 points**

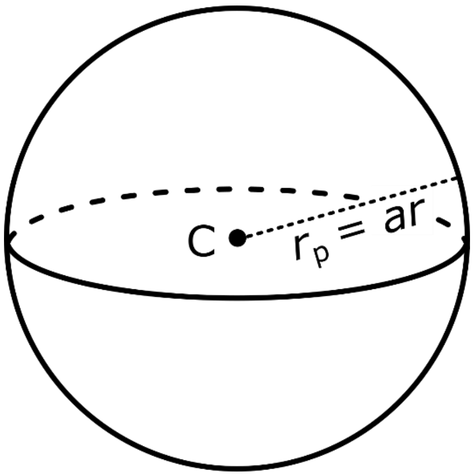
Consider a star S. At the present time t_0, the star is at a physical distance $r_\mathrm{p} = a(t_0)r$ away from us, where r is the comoving distance. Here we ignore the peculiar motion, i.e. assume that both the star and us just follow the expansion of the universe without additional motion.

The star is emitting energy in the form of light at power P_e, which is isotropic in every direction. We use a telescope to observe its starlight. For simplicity, assume the telescope can observe all frequencies of light with 100% efficiency. Let the area of the telescope lens be A.

E	Derive the power received by the telescope P_r from the star S, as a function of r, A, P_e, the scale factor $a(t_\mathrm{e})$ at the starlight emission time t_e, and the present (i.e. at the observation time) scale factor $a(t_0)$.	**4 points**

If there were no gravity, the expansion speed of the universe should be a constant. In Newton's framework, this can be understood as that, without force, matter just moves away from each other with constant speed and thus $\mathrm{d}a(t)/\mathrm{d}t$ is a constant depending on the initial condition.

Let us now consider how Newton's gravity affects the scale factor $a(t)$, in a universe filled with non-relativistic matter in a homogeneous and isotropic way.



As illustrated in the above figure, let us assume C is the center of our universe (this assumption can be removed in Einstein's general relativity, which is beyond the scope of this question). We slice matter into thin shells around C. Let us focus on one thin shell (the sphere in the above figure) whose comoving distance from the center is r (recall that this comoving distance is a constant in time).

F	Use the motion of the shell to find a relation between $\mathrm{d}a(t)/\mathrm{d}t$, $a(t)$ and the density of mass energy $\rho(t)$. (In the final relation, if you encounter a constant depending on the initial condition, it can be kept as it is.)	**5 points**

G	Based on the model described in Part (F), is the expansion of the universe (a) accelerating or (b) decelerating? Choose from (a) or (b).	**1 points**

For your information, in 1998, a new type of energy component of our universe is discovered. It actually changes the conclusion in Part (G).

Magnetic Field Effects on Superconductors (Total Marks: 20)

An electron is an elementary particle which carries electric charge and an intrinsic magnetic moment related to its spin angular momentum. Due to Coulomb interactions, electrons in vacuum are repulsive to each other. However, in some metals, the net force between electrons can become attractive due to the lattice vibrations. When the temperature of the metal is low enough, lower than some critical temperature T_c, electrons with opposite momenta and opposite spins can form pairs called Cooper pairs. By forming Cooper pairs, each electron reduces its energy by Δ compared to a freely propagating electron in the metal which has energy $\frac{p^2}{2m_e}$, where p is the momentum and m_e is the mass of an electron. The Cooper pairs can flow without resistance and the metal becomes a superconductor.

However, even at temperatures lower than T_c, superconductivity can be destroyed if the superconductor is under the influence of an external magnetic field. In this problem, you are going to work out how Cooper pairs can be destroyed by external magnetic fields through two effects.

The first is called the paramagnetic effect, in which all the electrons can lower their energy by aligning the electron magnetic moments parallel to the magnetic field instead of forming Cooper pairs with opposite spins.

The second is called the diamagnetic effect, in which increasing the magnetic field will change the orbital motion of the Cooper pairs and increase their energy. When the applied magnetic field is stronger than a critical value B_c, this increase in energy becomes higher than 2Δ. As a result, the electrons do not prefer forming Cooper pairs.

Recently, a type of superconductor called Ising superconductors was discovered. These superconductors can survive even when the applied magnetic field is as strong as 60 Tesla, comparable to the largest magnetic fields which can be created in laboratories. You will work out why Ising superconductors can overcome both the paramagnetic and the diamagnetic effects of the magnetic fields.

A. An Electron in a Magnetic Field

Let us consider a ring with radius r, charge $-e$ and mass m. The mass and the charge density around the ring are uniform (as shown in Figure 1).

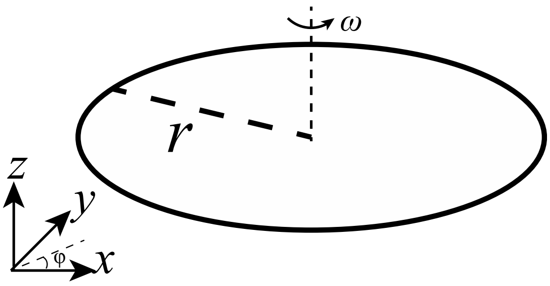
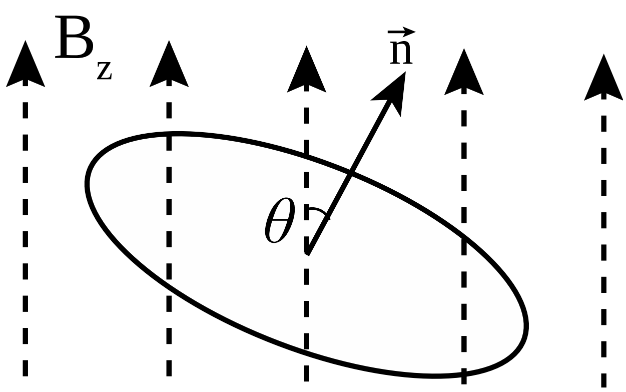


Figure 1

A1	What is the angular momentum $\vec{L}$ (magnitude and direction) of this ring if the ring is rotating with angular velocity $\vec{\omega}$?	**2 points**
A2	The magnitude of the magnetic moment is defined as $\lvert\vec{M}\rvert = IA$, where I is the current and A is the area of the ring. What is the relationship between the magnetic moment $\vec{M}$ and the angular momentum $\vec{L}$ of the ring?	**2 points**

Suppose the normal direction of the ring is $\vec{n}$ and it makes an angle θ with the applied magnetic field as shown in Figure 2.

A3	For the ring described in Part (A1), what is the potential energy U of this ring if the ring is placed in a uniform magnetic field B_z pointing to the z-direction? You should assume the potential energy to be zero when $\theta = \pi/2$.  Figure 2	**2 points**

A4	An electron carries an intrinsic angular momentum, which is called spin. We know that the magnitude of spin in a particular direction is $\frac{\hbar}{2}$, where $\hbar = h/2\pi$ and h is the Planck's constant. What are the values of the potential energy U_{up} and U_{down} for electrons with spins parallel and anti-parallel with the applied magnetic field respectively? Please express your results in terms of the Bohr magneton $\mu_B = \frac{e\hbar}{2m_e} = 5.788\times10^{-5}\,\text{eV}\cdot\text{T}^{-1}$ and the magnetic field strength B.	**1 point**
A5	According to quantum mechanics, the potential energy $\tilde{U}_{\text{up}}$ and $\tilde{U}_{\text{down}}$ are twice the values U_{up} and U_{down} found in Part (A4). Assuming that the applied magnetic field is 1 Tesla. What is the potential energy $\tilde{U}_{\text{up}}$ and $\tilde{U}_{\text{down}}$ for an electron with spin parallel and anti-parallel to the applied magnetic field respectively? In the rest of this question, you should use the expressions for $\tilde{U}_{\text{up}}$ and $\tilde{U}_{\text{down}}$ for your calculations.	**1 point**

B. Paramagnetic effect of the magnetic field on Cooper pairs

In the question below, we consider the paramagnetic effect of an external magnetic field on Cooper pairs (as shown in Figure 3).

Theoretical studies show that in superconductors, two electrons with opposite spins can form Cooper pairs so that the whole system saves energy. The energy of the Cooper pair can be expressed as $\frac{p_1^2}{2m_e} + \frac{p_2^2}{2m_e} - 2\Delta$, where the first two terms denote the kinetic energy of the Cooper pair and the last term is the energy saved for the electrons to form a Cooper pair. Here, Δ is a positive constant.

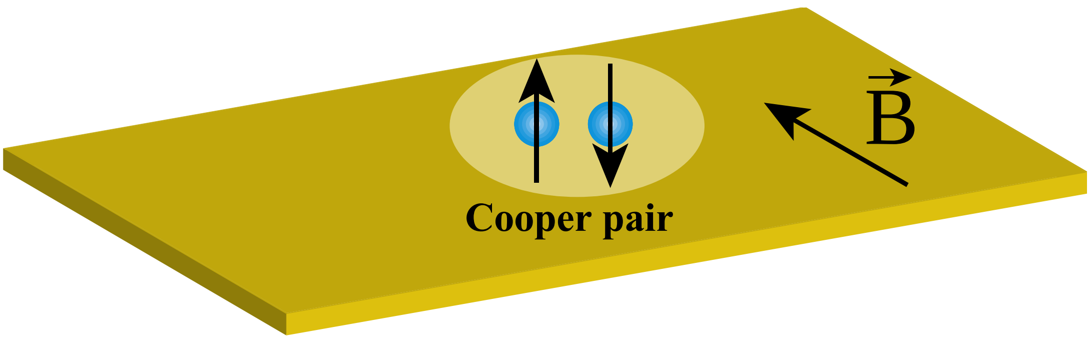


Figure 3

B1	Assuming that the effect of the external magnetic field is only on the spins of the electrons, not on the orbital motions of the electrons. What is the energy E_S of the Cooper pair under a uniform magnetic field $\vec{B}=\left(B_x,0,0\right)$? Recall that the electrons which form a Cooper pair must have opposite spins.	**1 point**
B2	In the normal state (non-superconducting state), electrons do not form Cooper pairs. What is the lowest energy E_N for the two electrons under a uniform in-plane magnetic field $\vec{B}=\left(B_x,0,0\right)$ pointing to the *x*-direction? Please use the $\tilde{U}_{\mathrm{up}}$ and $\tilde{U}_{\mathrm{down}}$ defined in Part (A5) in your calculations and ignore the effects of the magnetic field on the orbital motions of the electrons.	**1 point**
B3	At zero temperature, a system will favor the state with the lowest energy. What is the critical value B_{P} in terms of Δ, such that for $\left\|\vec{B}\right\|>B_{\mathrm{P}}$ superconductivity will disappear?	**1 point**

C. Diamagnetic effect of the magnetic field on Cooper pairs

In the question below, we are going to ignore the effects of magnetic fields on the spins of the electrons and consider the effects of external magnetic fields on the orbital motions of the Cooper pairs.

At zero temperature, the energy difference between the superconducting state and the normal state for a superconductor in a magnetic field $\vec{B}=(0,0,B_z)$ can be written as

$$F=\int_{-\infty}^{+\infty}\psi\left(-\alpha\psi-\frac{\hbar^2}{4m_{\mathrm{e}}}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2}+\frac{e^2B_z^2x^2}{m_{\mathrm{e}}}\psi\right)\mathrm{d}x.$$

Here $\psi(x)$ is a function of position x and independent of y. $\psi^2(x)$denotes the probability of finding a Cooper pair near x. Here, $\alpha>0$ is a constant and it is related to the energy saved by forming Cooper pairs. The second and the third terms in F are related to the kinetic energy of the Cooper pairs taking into account the effect of the magnetic field.

At zero temperature, the system prefers to minimize its energy F. In this case, $\psi(x)$ takes the form $\psi\left(x\right)=\left(\frac{2\lambda}{\pi}\right)^{\frac{1}{4}}\mathrm{e}^{-\lambda x^2}$, with $\lambda>0$.

C1	Find λ in terms of e, B_z, and $\hbar$. The following integrals may be useful: $$\int_{-\infty}^{+\infty}\mathrm{e}^{-ax^2}\mathrm{d}x=\sqrt{\frac{\pi}{a}},\qquad \int_{-\infty}^{+\infty}x^2\mathrm{e}^{-ax^2}\mathrm{d}x=\frac{1}{2a}\sqrt{\frac{\pi}{a}}.$$ Here a is a constant.	**3 points**

C2	Work out the critical value of B_z in terms of α, at which the superconducting state is no longer energetically favorable.	**2 points**

D Ising Superconductors

In materials with spin-orbit coupling (spin-spin couplings can be ignored), an electron with momentum $\vec{p}$ experiences an internal magnetic field $\vec{B}_{1\perp} = (0,0,-B_z)$. On the other hand, an electron with momentum $-\vec{p}$ experiences an opposite magnetic field $\vec{B}_{2\perp} = (0,0,B_z)$. These internal magnetic fields act on the spins of the electrons only as shown in Figure 4. Superconductors with this kind of internal magnetic fields are called Ising superconductors.

Cooper pair

Figure 4: Two electrons form a Cooper pair. Electron 1 with momentum $\vec{p}$ experiences internal magnetic field $\vec{B}_{1\perp} = (0,0,-B_z)$ but electron 2 with momentum $-\vec{p}$ experiences an opposite magnetic field $\vec{B}_{2\perp} = (0,0,B_z)$. The internal magnetic fields are denoted in dashed arrows.

D1	Then what is the energy E_I for a Cooper pair in an Ising superconductor?	**1 point**
D2	In the normal state of the material with spin orbit coupling, what is the energy $E_{\|\|}$ for the two electrons under a uniform in-plane magnetic field $\vec{B}_{\|\|} = (B_x, 0, 0)$? (Here the internal magnetic fields still exist and perpendicular to $\vec{B}_{\|\|}$. You should also ignore the effects of the in-plane magnetic field on the orbital motions of the Cooper pairs.)	**2 points**
D3	What is the critical value B_I such that for $\lvert\vec{B}_{\|\|}\rvert > B_I$, $E_{\|\|} < E_I$?	**1 point**