Physics Competition

<u>Theoretical Competition</u>

Please read this first:

The time available is 5 hours for the theoretical competition.

1. Use only the pen provided.
2. Use only the one side of the paper.
3. Begin each part of the problem on a separate sheet.
4. For each question, in addition to the ***blank sheets*** where you may write, there is an ***answer sheet*** where you *must* summarize the results you have obtained. Numerical results should be written with as many digits as are appropriate to the given data.
5. Write on the blank sheets of paper whatever you consider is required for the solution of the question. Please use ***as little text as possible***; express your answers primarily in equations, numbers, figures, and plots.
6. Fill in the boxes at the top of each answer sheet of paper used by writing your student code as shown on your identification tag, and additionally on the "blank" sheets: your student code, the problem number, the progressive number of each sheet (**Page n**. from 1 to N) and the total number (N) of "blank" sheets that you use and wish to be evaluated (Page total) **for each problem**; If you use some blank sheets of paper for notes that you do not wish to be marked, put a large X across the entire sheet and do not include it in your numbering.
7. At the end of the competition, arrange all sheets for each problem *in the following order*:

   <u>(a) answer sheet</u>

   <u>(b) used sheets in order</u>

   <u>(c) the sheets you do not wish to be marked</u>

   <u>(d) unused sheets and the printed questions</u>

Place the papers inside the envelope provided and leave everything on your desk. You are not allowed to take *any* sheets of paper out of the room.

Theoretical problem 1

Back-and-Forth Rolling of a Liquid-Filled Sphere (10 points)

Consider a sphere filled with liquid inside rolling back and forth at the bottom of a spherical bowl. That is, the sphere is periodically changing its translational and rotational direction. Due to the viscosity of the liquid inside, the movement of the sphere would be very complicated and hard to deal with. However, a simplified model presented here would be beneficial to the solution of such a problem.

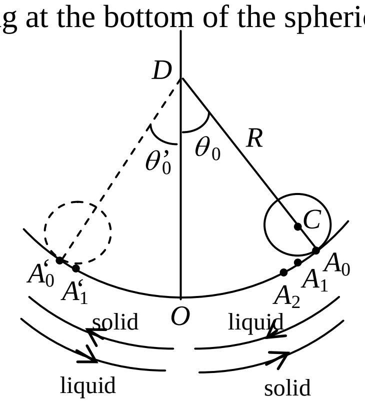
Assume that a rigid thin spherical shell of radius r and mass m is fully filled with some liquid substance of mass M, denoted as **W**. **W** has such a unique property that usually it behaves like an ideal liquid (i.e. without any viscosity), while in response to some special external influence (such as electric field) it transits to solid state immediately with the same volume; and once the applied influence removed, the liquid state recovers immediately. Besides, this influence does not give rise to any force or torque exerting on the sphere. This liquid-filled spherical shell (for convenience, called 'the sphere' hereafter) is supposed to roll back and forth at the bottom of a spherical bowl of radius R ($R > r$) without any relative slipping, as shown in the figure. Assume the sphere moves only in the vertical plane (namely, the plane of the figure), please study the movement of the sphere for the following three cases:

1. **W** behaves as in ideal solid state, meanwhile **W** contacts the inner wall of the spherical shell so closely that they can be taken as solid sphere as a whole of radius r with an abrupt density change across the interface between the inside wall of the shell and **W**.

   (1) Calculate the rotational inertia I of the sphere with respect to the axis passing through its center C. (You are asked to show detailed steps.)

   (1.0 points)

   (2) Calculate the period T_1 of the sphere rolling back and forth with a small amplitude without slipping at the bottom of the spherical bowl. (2.5points)



2. **W** behaves as an ideal liquid with no friction between **W** and the spherical shell. Calculate the period T_2 of the sphere rolling back and forth with a small amplitude without slipping at the bottom of the spherical bowl. (2.5 points)
3. **W** transits between ideal solid state and ideal liquid state.

   Assume at time $t=0$, the sphere is kept at rest, the line *CD* makes an angle θ_0 ($\theta_0 << 1\text{rad}$) with the plumb line *OD*, where *D* is the center of the spherical bowl. The sphere contacts the inner wall of the bowl at point A_0, as shown in the figure. Release the sphere, it starts to roll left from rest. During the motion of the sphere from A_0 to its equilibrium position *O*, **W** behaves as ideal liquid. At the moment that the sphere passes through point *O*, **W** changes suddenly into solid state and sticks itself firmly on the inside wall of the sphere shell until the sphere reaches the left highest position A_0'. Once the sphere reaches A_0', **W** changes suddenly back into the liquid state. Then, the sphere rolls right; and **W** changes suddenly into solid state and sticks itself firmly on the inside wall of the spherical shell again when the sphere passes through the equilibrium position *O*. When the sphere reaches the right highest position A_1, **W** changes into liquid state once again. Then the whole circle repeats time after time. The sphere rolls right and left periodically but with the angular amplitude decreased time after time. The motion direction of the sphere is shown by curved arrows in the figure, together with the words “solid” and “liquid” showing corresponding state of **W**. It is assumed that during such process of rolling back and forth, no any relative slide happens between the sphere and the inside wall of the bowl (or, alternatively, the bottom of the bowl can supply as enough friction as needed). Calculate the period T_3 of the sphere rolling right and left, and the angular amplitude θ_n of the center of the sphere, namely, the angle that the line *CD* makes with the vertical line *OD* when the sphere reaches the right highest position A_n for the n-th time (only A_2 is shown in the figure). （4.0 points）

Theoretical problem 2

2A. Optical properties of an unusual material (7 points)

The optical properties of a medium are governed by its relative permittivity (ε_r) and relative permeability (μ_r). For conventional materials like water or glass， which are usually optically transparent, both of their ε_r and μ_r are positive, and refraction phenomenon meeting Snell's law occurs when light from air strikes obliquely on the surface of such kind of substances. In 1964, a Russia scientist V. Veselago rigorously proved that a material with simultaneously negative ε_r and μ_r would exhibit many amazing and even unbelievable optical properties. In early 21st century, such unusual optical materials were successfully demonstrated in some laboratories. Nowadays study on such unusual optical materials has become a frontier scientific research field. Through solving several problems in what follows, you can gain some basic understanding of the fundamental optical properties of such unusual materials. **It should be noticed that a material with simultaneously negative ε_r and μ_r possesses the following important property. When a light wave propagates forward inside such a medium for a distance Δ, the phase of the light wave will decrease, rather than increase an amount of $\sqrt{\varepsilon_r\mu_r}k\Delta$ as what happens in a conventional medium with simultaneously positive ε_r and μ_r. Here, positive root is always taken when we apply the square-root calculation, while k is the wave vector of the light.** In the questions listed below, we assume that both the relative permittivity and permeability of air are equal to 1.

1. (1)According to the property described above, assuming that a light beam strikes from air on the surface of such an unusual material with relative permittivity $\varepsilon_r < 0$ and relative permeability $\mu_r < 0$, prove that the direction of the refracted light beam depicted in Fig.2-1 is reasonable. (1.2 points)

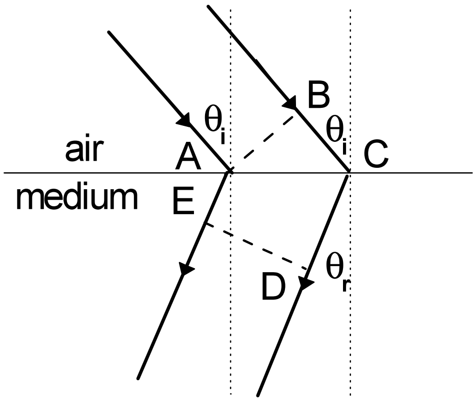


Fig. 2-1

(2) For Fig. 2-1, show the relationship between refraction angle θ_r (the angle that refracted beam makes with the normal of the interface between air and the material) and incidence angle θ_i. (0.8 points)

(3) Assuming that a light beam strikes from the unusual material on the interface between it and air, prove that the direction of the refracted light beam depicted in Fig.2-2 is reasonable. (1.2 points)

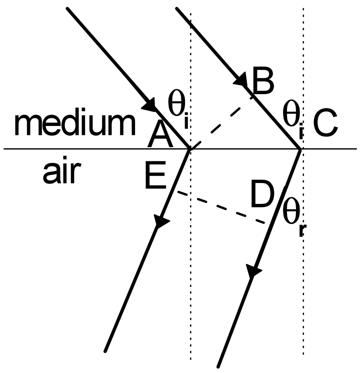


Fig. 2-2

(4) For Fig. 2-2, show the relationship between the refraction angle θ_r (the angle that refracted beam makes with the normal of the interface between two media) and the incidence angle θ_i. (0.8 points)

2. As shown in Fig. 2-3, a slab of thickness *d*, which is made of an unusual optical material with $\varepsilon_r = \mu_r = -1$, is placed in air, with a point light source located in

front of the slab separated by a distance of $\frac{3}{4}d$. Accurately draw the ray diagrams for the three light rays radiated from the point source. (Hints: under the conditions given in this problem, no reflection would happen at the interface between air and the unusual material). (1.0 points)

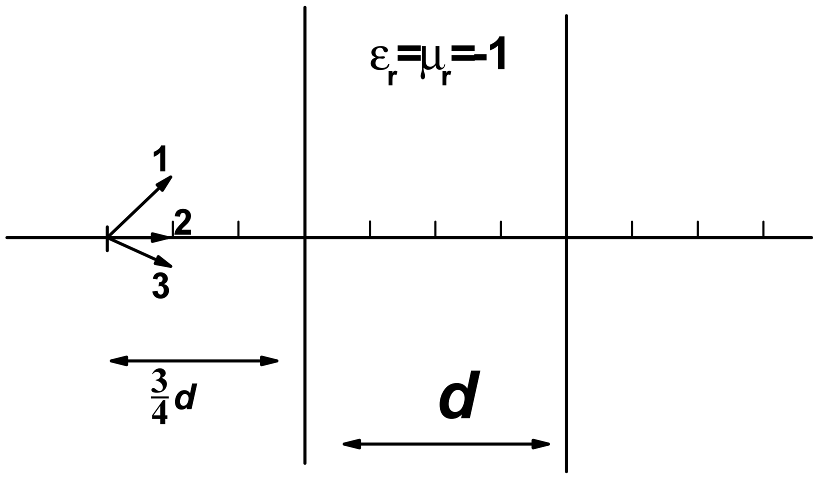


Fig. 2-3

3. As shown in Fig. 2-4, a parallel-plate resonance cavity is formed by two plates parallel with each other and separated by a distant *d*. Optically one of the plates, denoted as Plate 1 in Fig.2-4 , is ideally reflective (reflectance equals to 100%), and the other one, denoted by Plate 2, is partially reflective (but with a high reflectance). Suppose plane light waves are radiated from a source located near Plate 1, then such light waves are multiply reflected by the two plates inside the cavity. Since optically the Plate 2 is *not* ideally reflective, some light waves will leak out of Plate 2 each time the light beam reaches it (ray 1, 2, 3, as shown in Fig. 2-4), while some light waves will be reflected by it. If these light waves are in-phase, they will interfere with each other constructively, leading to resonance. We assume that the light wave gains a phase of π by reflection at either of the two plates. Now we insert a slab of thickness $0.4d$ (shown as the shaded area in Fig. 2-4), made of an unusual optical material with $\varepsilon_r = \mu_r = -0.5$, into the cavity parallel to the two plates. The remaining space is filled with air inside the cavity. Let us consider only the situation that the light wave travels along the direction perpendicular to the plates (the ray diagram depicted in Fig.2-4 is only a

schematic one), calculate all the wavelengths that satisfy the resonance condition of such a cavity. (Hints: under the condition given here, no reflection would occur at the interfaces between air and the unusual material). (1.0 points)

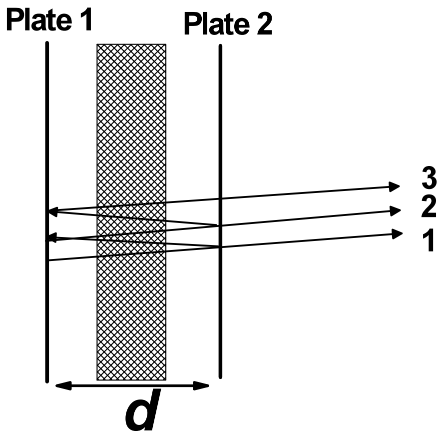


Fig. 2-4

4. An infinitely long cylinder of radius R, made of an unusual optical material with $\varepsilon_r = \mu_r = -1$, is placed in air, its cross section in XOY plane is shown in Fig. 2-5 with the center located on Y axis. Suppose a laser source located on the X axis (the position of the source is described by its coordinate x) emits narrow laser light along the Y direction. Show the range of x, for which the light signal emitted from the light source can *not* reach the infinite receiving plane on the other side of the cylinder. (1.0 points)

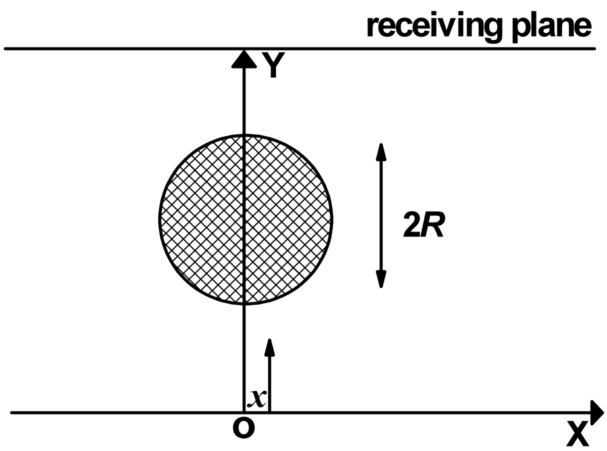


Fig. 2-5

2B. Dielectric spheres inside an external electric field (3 points)

By immersing a number of small dielectric particles inside a fluid of low-viscosity, you can get the resulting system as a suspension. When an external electric field is applied on the system, the suspending dielectric particles will be polarized with electric dipole moments induced. Within a very short period of time, these polarized particles aggregate together through dipolar interactions so that the effective viscosity of the whole system enhances significantly (the resulting system can be approximately viewed as a solid). This type of phase transition is called "electrorheological effect", and such a system is called "electrorheological fluid" correspondingly. Such an effect can be applied to fabricate braking devices in practice, since the response time of such a phase transition is shorter than conventional mechanism by several orders of magnitude. Through solving several problems in the following, you are given a simplified picture to understand the inherent mechanism of the electrorheological transition.

1. When there are many identical dielectric spheres of radius a immersed inside the fluid, we assume that the dipole moment of each sphere $\vec{p}$, is induced *solely* by the external field $\vec{E}_0$, independent of any of the other spheres (**Note**: $\vec{p} \parallel \vec{E}_0$).

   (1) When two identical small dielectric spheres exist inside the fluid and contact with each other, while the line connecting their centers makes an angle θ with the external field direction (see Fig. 2-6), write the expression of the energy of dipole-dipole interaction between the two small contacting dielectric spheres, in terms of p, a and θ. (**Note:** In your calculations each polarized dielectric sphere can be viewed as an electric dipole located at the center of the sphere) (0.5 points)

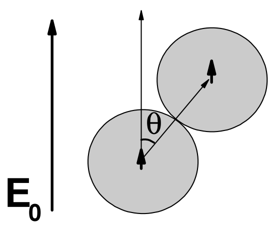


Fig. 2-6

   (2) Calculate the dipole-dipole interaction energies for the three configurations shown in Fig. 2-7. (0.75 points)

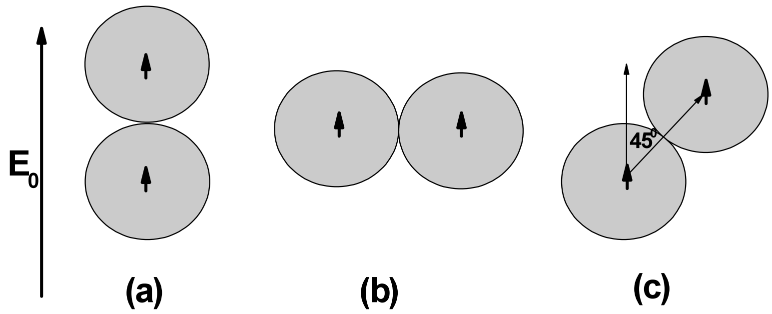


Fig. 2-7

(3) Identify which configuration of the system is the most stable one. (0.25 points)

(**<u>Note:</u>** In your calculations each polarized dielectric sphere can be viewed as an electric dipole located at the center of the sphere, and the energy of dipole-dipole interaction can be expressed in terms of p and a.)

2. In the case that three identical spheres exist inside the fluid, based on the same assumption as in question 1,

   (1) calculate the dipole-dipole interaction energies for the three configurations shown in Fig. 2-8; (0.9 points)

   (2) identify which configuration of the system is the most stable one; (0.3 points)

   (3) identify which configuration of the system is the most unstable one. (0.3 points)

   (**<u>Note:</u>** In your calculations each polarized dielectric sphere can be viewed as an electric dipole located at the center of the sphere, and the energy of dipole-dipole interaction can be expressed in terms of p and a.)

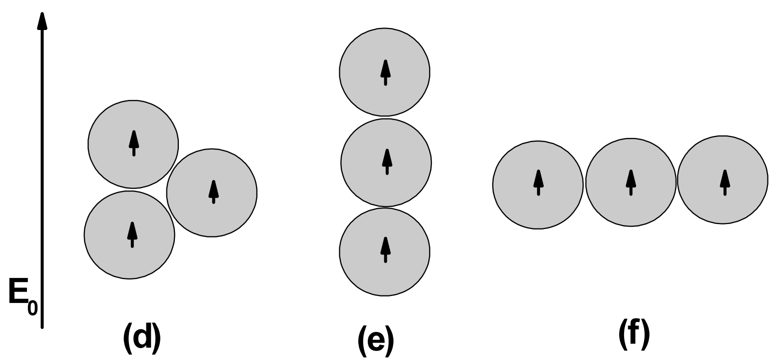


Fig. 2-8

Theoretical Problem 3

3A. Average contribution of each electron to specific heat of free electron gas at constant volume (5 points)

1. According to the classical physics the conduction electrons in metals constitute free electron gas like an ideal gas. In thermal equilibrium their average energy relates to temperature, therefore they contribute to the specific heat. The average contribution of each electron to the specific heat of free electron gas at constant volume is defined as

$$c_V = \frac{d\overline{E}}{dT} \quad , \tag{1}$$

where $\overline{E}$ is the average energy of each electron. However the value of the specific heat at constant volume is a constant, independent of temperature. Please calculate $\overline{E}$ and the average contribution of each electron to the specific heat at constant volume c_V.

(1.0 points)

2. Experimentally it has been shown that the specific heat of the conduction electrons at constant volume in metals depends on temperature, and the experimental value at room temperature is about two orders of magnitude lower than its classical counterpart. This is because the electrons obey the quantum statistics rather than classical statistics. According to the quantum theory, for a metallic material the density of states of conduction electrons (the number of electronic states per unit volume and per unit energy) is proportional to the square root of electron energy E, then the number of states within energy range $\mathrm{d}E$ for a metal of volume V can be written as

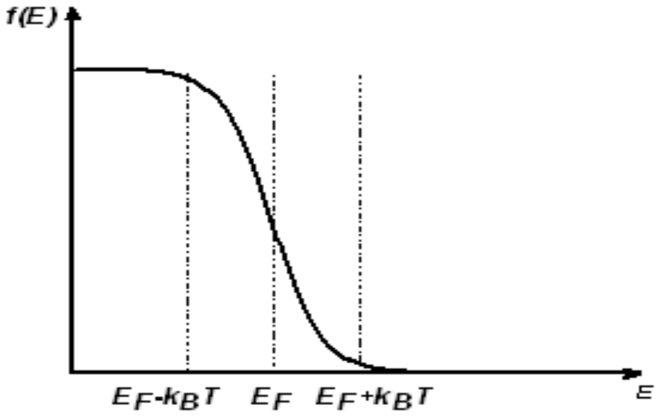
$$\mathrm{d}S = CVE^{1/2}\mathrm{d}E \tag{2}$$

where C is the normalization constant, determined by the total number of electrons of the system.

The probability that the state of energy E is occupied by electron is

$$f(E) = \frac{1}{1+\exp(\dfrac{E-E_F}{k_B T})} \quad , \tag{3}$$

where $k_B = 1.381\times10^{-23}\,\mathrm{J\cdot K^{-1}}$ is the Boltzmann constant and T is the absolute temperature, while E_F is called Fermi level. Usually at room temperature E_F is about several eVs for metallic materials (1eV=1.602×10^{-19} J). $f(E)$ is called Fermi distribution function shown schematically in the figure below.



(1) Please calculate c_V at room temperature according to $f(E)$ (3.5points)

(2) Please give a reasonable explanation for the deviation of the classical result from that of quantum theory. (0.5 points)

Note: In your calculation the variation of the Fermi level E_F with temperature could be neglected, i.e. assume $E_F = E_F^0$, E_F^0 is the Fermi level at 0K. Meanwhile the Fermi distribution function could be simplified as a linearly descending function within an energy range of $2k_BT$ around E_F, otherwise either 0 or 1, i.e.

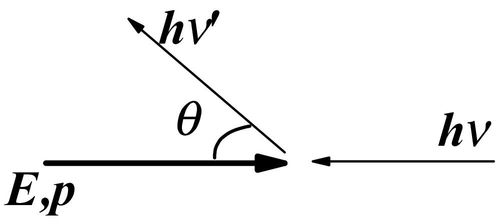
$$f(E)=\begin{cases} 1 & E<E_F-k_BT \\ \text{linearly descending function} & E_F-k_BT<E<E_F+k_BT \\ 0 & E>E_F+k_BT \end{cases}$$

At room temperature $k_BT << E_F$, therefore calculation can be simplified accordingly. Meanwhile, the total number of electrons can be calculated at 0K.

3B. The Inverse Compton Scattering (5 points)

By collision with relativistic high energy electron, a photon can get energy from the high energy electron, i.e. the energy and frequency of the photon increases because of the collision. This is so-called inverse Compton scattering. Such kind of phenomenon is of great importance in astrophysics, for example, it provides an important mechanism for producing X rays and γ rays in space.

1. A high energy electron of total energy E (its kinetic energy is higher than static energy) and a low energy photon (its energy is less than the static energy of an electron) of frequency ν move in opposite directions, and collide with each other. As shown in the figure below, the collision scatters the photon, making the scattered photon move along the direction which makes an angle θ with its original incident direction (the scattered electron is not shown in the figure). Calculate the energy of the scattered photon, expressed in terms of E, ν, θ and static energy E_0 of the electron. Show the value of θ, at which the scattered photon has the maximum energy, and the value of this maximum energy. (2.4 points)



2. Assume that the energy E of the incident electron is much higher than its static energy E_0, which can be shown as $E = \gamma E_0, \gamma >> 1$, and that the energy of the incident photon is much less than E_0 / γ, show the approximate expression of the maximum energy of the scattered photon. Taking $\gamma = 200$ and the wavelength of the incident visible light photon $\lambda = 500\text{nm}$, calculate the approximate maximum energy and the corresponding wavelength of the scattered photon.

   Parameters: Static energy of the electron $E_0 = 0.511\text{MeV}$, Planck constant $h = 6.63\times10^{-34}\ \text{J}\cdot\text{s}$, and $hc = 1.24\times10^{3}\ \text{eV}\cdot\text{nm}$, where c is the light speed in the vacuum. (1.2 points)

3. (1) A relativistic high energy electron of total energy E and a photon move in opposite directions and collide with each other. Show the energy of the incident photon, of which the photon can gain the maximum energy from the incident electron. Calculate the energy of the scattered photon in this case. (0.7 points)

   (2) A relativistic high energy electron of total energy E and a photon, moving in perpendicular directions respectively, collide with each other. Show the energy of the incident photon, of which the photon can gain the maximum energy from the incident electron. Calculate the energy of the scattered photon in this case. (0.7 points)