Theoretical Question 1
Particles and Waves

This question includes the following three parts dealing with motions of particles and waves:

Part A. Inelastic scattering of particles

Part B. Waves on a string

Part C. Waves in an expanding universe

Part A. Inelastic Scattering and Compositeness of Particles

A particle is considered ***elementary*** if it has no excitable internal degrees of freedom such as, for example, rotations and vibrations about its center of mass. Otherwise, it is ***composite***.

To determine if a particle is composite, one may set up a scattering experiment with the particle being the target and allow an *elementary* particle to scatter off it. In case that the target particle is composite, the scattering experiment may reveal important features such as *scaling*, i.e. as the forward momentum of the scattered particle increases, the scattering cross section becomes independent of the momentum.

For a scattering system consisting of an elementary particle incident on a target particle, we shall denote by Q the total translational kinetic energy *loss* of the system. Here the *translational kinetic energy* of a particle, whether elementary or composite, is defined as the kinetic energy associated with the translational motion of its center of mass. Thus we may write

$$Q = K_{\mathrm{i}} - K_{\mathrm{f}} \,,$$

where K_{i} and K_{f} are the *total* translational kinetic energies of the scattering pair before and after scattering, respectively.

In Part A, use non-relativistic classical mechanics to solve all problems. All effects due to gravity are to be neglected.

(a) As shown in Fig. 1, an *elementary* particle of mass m moves along the x axis with x-component of momentum $p_1 > 0$. After being scattered by a *stationary* target of mass M, its momentum becomes $\vec{p}_2$.

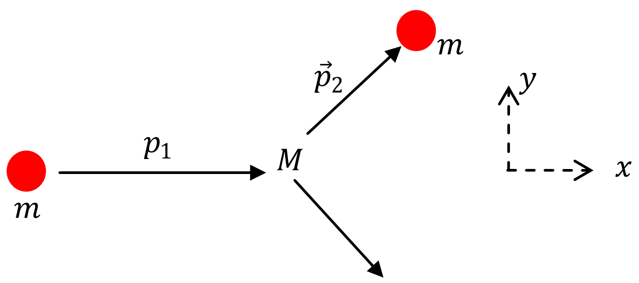


Fig. 1

From data on $\vec{p}_2$, one can determine if the target particle is elementary or composite. We shall assume that $\vec{p}_2$ lies in the x-y plane and that the x- and y-components of $\vec{p}_2$ are given, respectively, by p_{2x} and p_{2y}.

(i) Find an expression for Q in terms of m, M, p_1, p_{2x}, and p_{2y}. [0.2 point]

(ii) If the target particle is elementary, the momenta p_1, p_{2x}, and p_{2y} are related in a particular way by a condition.

For given p_1, plot the condition as a curve in the p_{2x}-p_{2y} plane. Specify the value of p_{2x} for each intercept of the curve with the p_{2x}-axis. In the same plot, locate regions of points of $\vec{p}_2$ corresponding to $Q < 0$, $Q = 0$, $Q > 0$, and label each of them as such. [0.7 point]

For a stationary composite target in its ground state before scattering, which region(s) of Q contains those points of $\vec{p}_2$ allowed? [0.2 point]

(b) Now, consider a composite target consisting of two elementary particles each with mass $\frac{1}{2}M$. They are connected by a spring of negligible mass. See Fig. 2. The spring has a force constant k and does not bend sideways. Initially, the target is *stationary* with its center of mass at the origin O, and the spring, inclined at an angle θ to the x-axis, is at its natural length d_0. For simplicity, we assume that only vibrational and rotational motions can be excited in the target as a result of scattering.

The incident elementary particle of mass m moves in the x-direction both before and after scattering with its momenta given, respectively, by p_1 and p_2. Note that p_2 is negative if the particle recoils and moves backward. A scattering occurs only if the incident particle hits one of the target particles and $p_2 \neq p_1$. We assume all three particles move in the same plane before and after scattering.

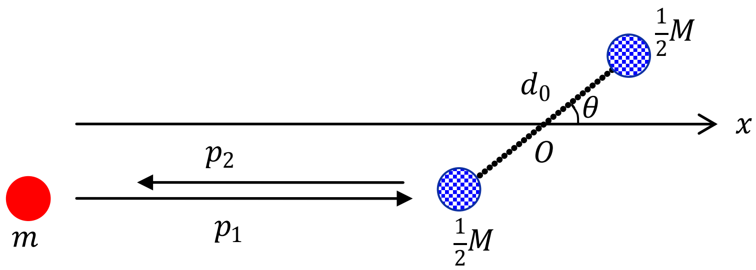


Fig. 2

(i) If the maximum length of the spring after scattering is $d_{\rm m}$, find an equation which relates the ratio $x = (d_{\rm m} - d_0)/d_0$ to the quantities $Q, \theta, d_0, m, k, M, p_1$ and p_2. [0.7 point]

(ii) Let $\alpha \equiv \sin^2\theta$. When the angle of orientation θ of the target is allowed to vary, the scattering cross section σ gives the effective target area, in a plane normal to the direction of incidence, which allows certain outcomes to occur as a result of scattering. It is known that for all outcomes which lead to the same value of p_2, the value of α must span an interval $(\alpha_{\min}, \alpha_{\max})$ and we may choose the unit of cross section so that σ is simply given by the numerical range $(\alpha_{\max} - \alpha_{\min})$ of the interval. Note that $\alpha_{\min}$, $\alpha_{\max}$, and, consequently, σ are dependent on p_2. Let p_c be the threshold value of p_2 at which σ starts to become independent of p_2.

In the limit of large k, give an estimate of p_c. Express your answer in terms of m, M, and p_1. [1.1 points]

Assume M=3m and in the limit of large k, plot σ as a function of p_2 for a given p_1. In the plot, specify the range of σ and p_2. [1.1 points]

Part B. Waves on a String

Consider an elastic string stretched between two fixed ends A and B, as shown in Fig. 3. The linear mass density of the string is μ. The speed of propagation for transverse waves in the string is c. Let the length $\overline{\mathrm{AB}}$ be L. The string is plucked sideways and held in a triangular form with a maximum height $h \ll L$ at its middle point. At time $t = 0$, the plucked string is released from rest. All effects due to gravity may be neglected.

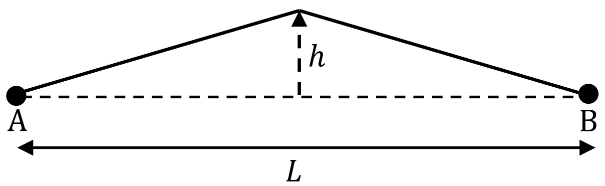


Fig. 3

(c) Find the period of vibration T for the string. [0.5 point]

Plot the shape of the string at $t = T/8$. In the plot, specify lengths and angles which serve to define the shape of the string. [1.7 points]

(d) Find the total mechanical energy of the vibrating string in terms of μ, c, h, and L. [0.8 point]

Part C. An Expanding Universe

Photons in the universe play an important role in delivering information across the cosmos. However, the fact that the universe is expanding must be taken into account when one tries to extract information from these photons. To this end, we normally express length and distance using a universal scale factor $a(t)$ which depends on time t. Thus the distance $L(t)$ between two stars stationary in their respective local frames is proportional to $a(t)$:

$$L(t) = ka(t), \tag{1}$$

where k is a constant and $a(\mathrm{t})$ accounts for the expansion of the universe. We use a dot above a symbol of a variable to denote its time derivative, i.e. $\dot{a}(\mathrm{t}) = da(t)/dt$, and let $v(t) \equiv \dot{L}(t)$. Taking time derivatives of both sides of Eq. (1), one obtains the Hubble law:

$$v(t) = H(t)L(t), \tag{2}$$

where $H(t) = \dot{a}(t)/a(t)$ is the *Hubble parameter* at time t. At the current time t_0, we have

$$H(t_0) = 72 \text{ km s}^{-1} \text{ Mpc}^{-1},$$

where 1 Mpc $= 3.0857 \times 10^{19}$ km $= 3.2616 \times 10^{6}$ light-year.

Assume the universe to be infinitely large and expanding in such a way that

$$a(t) \propto \exp(bt),$$

where b is a constant. In such a universe, the Hubble parameter is a constant equal to $H(t_0)$. Moreover, it can be shown that the wavelength λ of photons travelling in the universe will be stretched in proportion to the expansion of the universe, i.e.

$$\lambda(t) \propto a(t).$$

Now suppose that photons making up a Lyman-alpha emission line were emitted at t_e by a star that was stationary in its local frame and that we as observers are stationary in our local frame. When these photons were emitted, their wavelength was $\lambda(t_e) = 121.5$ nm. But when they reach us now at t_0, their wavelength is red-shifted to 145.8 nm.

(e) As these photons traveled, the universe kept expanding so that the star kept receding from us. Given that the speed of light in vacuum c has never changed, what *was* the distance $L(t_e)$ of the star from us when these photons were emitted at t_e? Express the answer in units of Mpc. [2.2 points]

(f) What is the receding velocity $v(t_0)$ of the star with respect to us *now* at t_0? Express the answer in units of the speed of light in vacuum c. [0.8 point]

Appendix

The following formula may be used when needed:

$$\int_a^b e^{\beta x}\, dx = \frac{1}{\beta}\left(e^{\beta b} - e^{\beta a}\right).$$

Theoretical Question 2

Strong Resistive Electromagnets

Resistive electromagnets are magnets with coils made of a normal metal such as copper or aluminum. Modern strong resistive electromagnets can provide steady magnetic fields higher than 30 tesla. Their coils are typically built by stacking hundreds of thin circular plates made of copper sheet metal with lots of cooling holes stamped in them; there are also insulators with the same pattern. When voltage is applied across the coil, current flows through the plates along a helical path to generate high magnetic fields in the center of the magnet.

In this question we aim to assess if a cylindrical coil (or *solenoid*) of many *turns* can serve as a magnet for generating high magnetic fields. As shown in Fig. 1, the center of the magnet is at O. Its cylindrical coil consists of N turns of copper wire carrying a current I uniformly distributed over the cross section of the wire. The coil's mean diameter is D and its length along the axial direction x is ℓ. The wire's cross section is rectangular with width a and height b. The turns of the coil are so tightly wound that the plane of each turn may be taken as perpendicular to the x axis and $\ell = Na$. In Table 1, data specifying physical dimensions of the coil are listed.

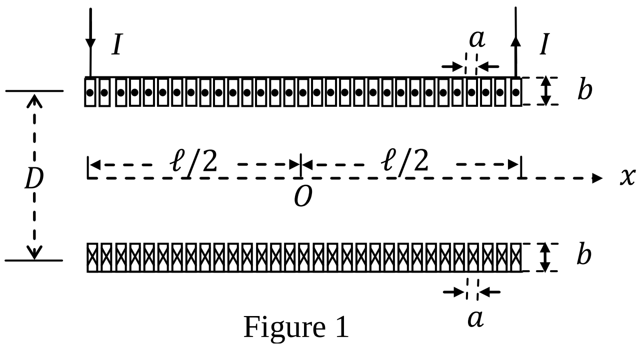


Figure 1

Table 1
$\ell = 12.0$ cm
$D = 6.0$ cm
$a = 2.0$ mm
$b = 5.0$ mm

In assessing if such a magnet can serve to provide high magnetic fields, two limiting factors must not be overlooked. One is the mechanical rigidity of the coil to withstand large Lorentz force on the field-producing current. The other is that the enormous amount of Joule heat generated in the wire must not cause excessive temperature rise. We shall examine these two factors using simplified models.

The Appendix at the end of the question lists some mathematical formulae and physical data which may be used if necessary.

Part A. Magnetic Fields on the Axis of the Coil

Assume $b \ll D$ so that one may regard the wire as a thin strip of width a. Let O be the origin of x coordinates. The direction of the current flow is as shown in Fig. 1.

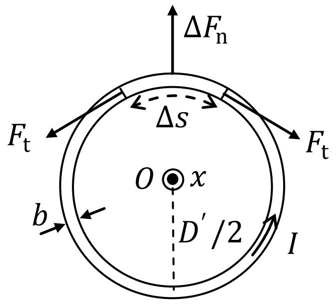
(a) Find the x-component $B(x)$ of the magnetic field on the axis of the coil as a function of x when the steady current passing through the coil is I. [1.0 point]

(b) Find the steady current I_0 passing through the coil if $B(0)$ is 10.0 T. Use data given in Table 1 when computing numerical values. [0.4 point]

Part B. The Upper Limit of Current

In Part B, we assume **the length ℓ of the coil is infinite** and $b \ll D$**.** Consider the turn of the coil located at $x = 0$. The magnetic field exerts Lorentz force on the current passing through the turn. Thus, as Fig. 2 shows, a wire segment of length Δs is subject to a normal force ΔF_{n} which tends to make the turn expand.

Figure 2



(c) Suppose that, when the current is I, the mean diameter of the expanded coil remains at a constant value D' larger than D, as shown in Fig. 2.

Find the outward normal force per unit length $\Delta F_{\mathrm{n}}/\Delta s$. [1.2 point]

Find the tension F_{t} acting along the wire. [0.6 point]

(d) Neglect the coil's acceleration during the expansion. Assume the turn will break when the wire's *unit elongation* (i.e. tensile strain or fractional change of the length) is 60 % and *tensile stress* (i.e. tension per unit cross sectional area of the unstrained wire) is $\sigma_{\mathrm{b}} = 455$ MPa. Let I_{b} be the current at which the turn will break and B_{b} the corresponding magnetic field at the center O.

Find an expression for I_{b} and then calculate its value. [0.8 point]

Find an expression for B_{b} and then calculate its value. [0.4 point]

Part C. The Rate of Temperature Rise

When the current I is 10.0 kA and the temperature T of the coil is 293 K, assume that the resistivity, the specific heat capacity at constant pressure, and the mass density of the wire of the coil are, respectively, given by $\rho_{\mathrm{e}} = 1.72 \times 10^{-8}\ \Omega \cdot \mathrm{m}$, $c_p = 3.85 \times 10^{2}\ \mathrm{J/(kg \cdot K)}$ and $\rho_m = 8.98 \times 10^{3}\ \mathrm{kg \cdot m^{-3}}$.

(e) Find an expression for the *power density* (i.e. power per unit volume) of heat generation in the coil and then calculate its value. Use data in Table 1. [0.5 point]

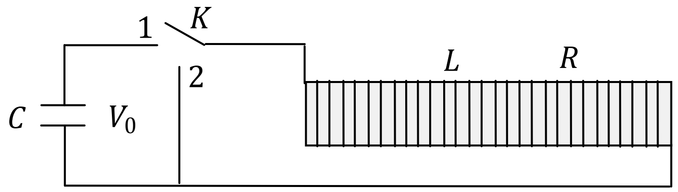
(f) Let $\dot{T}$ be the time rate of change of temperature in the coil. Find an expression for $\dot{T}$ and then calculate its value. [0.5 point]

Part D. A Pulsed-Field Magnet

If the large current needed for a strong magnet lasts only for a short time, the temperature rise caused by excessive Joule heating may be greatly reduced. This idea is employed in a *pulsed-field* magnet.

Thus, as shown in Fig. 3, a capacitor bank of capacitance C charged initially to a potential V_0 is used to drive the current I through the coil. The circuit is equipped with a switch K. The inductance L and resistance R of the circuit are assumed to be *entirely* due to the coil. The construct and dimensions of the coil are the same as given in Fig. 1 and Table 1. Assume R, L, and C to be independent of temperature and the magnetic field is the same as that of an infinite solenoid with $\ell \to \infty$.

Figure 3



(g) Find expressions for the inductance L and resistance R. [0.6 point]

Calculate the values of L and R. Use data given in Table 1. [0.4 point]

(h) At time $t = 0$, the switch K is thrown to position 1 and the current starts flowing. For $t \geq 0$, the charge $Q(t)$ on the positive plate of the capacitor and the current $I(t)$ entering the positive plate are given by

$$Q(t) = \frac{CV_0}{\sin\theta_0} e^{-\alpha t} \sin(\omega t + \theta_0), \tag{1}$$

$$I(t) = \frac{dQ}{dt} = \left(\frac{-\alpha}{\cos\theta_0}\right) \frac{CV_0}{\sin\theta_0} e^{-\alpha t} \sin \omega t, \tag{2}$$

in which α and ω are positive constants and θ_0 is given by

$$\tan\theta_0 = \frac{\omega}{\alpha}, \qquad 0 < \theta_0 < \frac{\pi}{2}. \tag{3}$$

Note that, if $Q(t)$ is expressed as a function of a new variable $t' \equiv (t + \theta_0/\omega)$, then $Q(t')$ and its time derivative $I(t)$ are identical in form except for an overall constant factor. The time derivative of $I(t)$ may therefore be obtained similarly without further differentiations.

Find α and ω in terms of R, L, and C. [0.8 point]

Calculate the values of α and ω when C is 10.0 mF. [0.4 point]

(i) Let I_{m} be the maximum value of $|I(t)|$ for $t > 0$. Find an expression for I_{m}. [0.6 point]
If $C = 10.0$ mF, what is the maximum value $V_{0\mathrm{b}}$ of the initial voltage V_0 of the capacitor bank for which I_{m} will not exceed I_{b} found in Problem (d)? [0.4 point]

(j) Suppose the switch K is moved instantly from position 1 to 2 when the absolute value of the current $|I(t)|$ reaches I_{m}. Let ΔE be the total amount of heat dissipated in the coil from $t = 0$ to ∞ and ΔT the corresponding temperature increase of the coil. Assume the initial voltage V_0 takes on the maximum value $V_{0\mathrm{b}}$ obtained in Problem (i) and the electromagnetic energy loss is only in the form of heat dissipated in the coil.
Find an expression for ΔE and then calculate its value. [1.0 point]
Find an expression for ΔT and then calculate its value. Note that the value for ΔT must be compatible with the assumption of constant R and L. [0.4 point]

Appendix

1. $\displaystyle \int_0^L \frac{dx}{(D^2 + x^2)^{3/2}} = \frac{1}{D^2}\left\{\frac{L}{(D^2 + L^2)^{1/2}}\right\}$

2. $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

3. permeability of free space $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{T \cdot m/A}$

-- END --

Theoretical Question 3

Electron and Gas Bubbles in Liquids

This question deals with physics of two bubble-in-liquid systems. It has two parts:

Part A. An electron bubble in liquid helium

Part B. Single gas bubble in liquid

Part A. An Electron Bubble in Liquid Helium

When an electron is planted inside liquid helium, it can repel atoms of liquid helium and form what is called an ***electron bubble***. The bubble contains nothing but the electron itself. We shall be interested mainly in its size and stability.

We use Δf to denote the uncertainty of a quantity f. The components of an electron's position vector $\vec{q} = (x, y, z)$ and momentum vector $\vec{p} = (p_x, p_y, p_z)$ must obey Heisenberg's uncertainty relations $\Delta q_\alpha \Delta p_\alpha \geq \hbar/2$, where $\hbar$ is the Planck constant divided by 2π and $\alpha = x, y, z$.

We shall assume the electron bubble to be isotropic and its interface with liquid helium is a sharp spherical surface. The liquid is kept at a constant temperature very close to $0\ \mathrm{K}$ with its surface tension σ given by $3.75 \times 10^{-4}\ \mathrm{N \cdot m^{-1}}$ and its electrostatic responses to the electron bubble may be neglected.

Consider an electron bubble in liquid helium with an equilibrium radius R. The electron, of mass m, moves freely inside the bubble with kinetic energy E_k and exerts pressure P_e on the inner side of the bubble-liquid interface. The pressure exerted by liquid helium on the outer side of the interface is P_He.

(a) Find a relation between P_He, P_e, and σ. [0.4 point]

Find a relation between E_k and P_e. [1.0 point]

(b) Denote by E_0 the smallest possible value of E_k consistent with Heisenberg's uncertainty relations when the electron is inside the bubble of radius R. Estimate E_0 as a function of R. [0.8 point]

(c) Let R_e be the equilibrium radius of the bubble when $E_\mathrm{k} = E_0$ and $P_\mathrm{He} = 0$. Obtain an expression for R_e and calculate its value. [0.6 point]

(d) Find a condition that R and P_He must satisfy if the equilibrium at radius R is to be locally stable under constant P_He. Note that P_He can be negative. [0.6 point]

(e) There exists a threshold pressure P_th such that equilibrium is not possible for the electron bubble when P_He is less than P_th. Find an expression for P_th. [0.6 point]

Part B. Single Gas Bubble in Liquid — Collapsing and Radiation

In this part of the problem, we consider a normal liquid, such as water.

When a gas bubble in a liquid is driven by an oscillating pressure, it can show dramatic responses. For example, following a large expansion, it can collapse rapidly to a small radius and, near the end of the collapse, emit light almost instantly. In this phenomenon, called ***single-bubble sonoluminescence***, the gas bubble undergoes cyclic motions which typically consist of three stages: expansion, collapse, and multiple after-bounces. In the following we shall focus mainly on the collapsing stage.

We assume that, at all times, the bubble considered is spherical and its center remains stationary in the liquid. See Fig 1. The pressure, temperature, and density are always uniform inside the bubble as its size diminishes. The liquid containing the bubble is assumed to be isotropic, nonviscous, incompressible, and very much larger in extent than the bubble. All effects due to gravity and surface tension are neglected so that pressures on both sides of the bubble-liquid interface are ***always equal***.

● *Radial motion of the bubble-liquid interface*

As the bubble's radius $R = R(t)$ changes with time t, the bubble-liquid interface will move with radial velocity $\dot{R} \equiv dR/dt$. It follows from the equation of continuity of incompressible fluids that the liquid's radial velocity $\dot{r} \equiv dr/dt$ at distance r from the center of the bubble is related to the rate of change of the bubble's volume V by

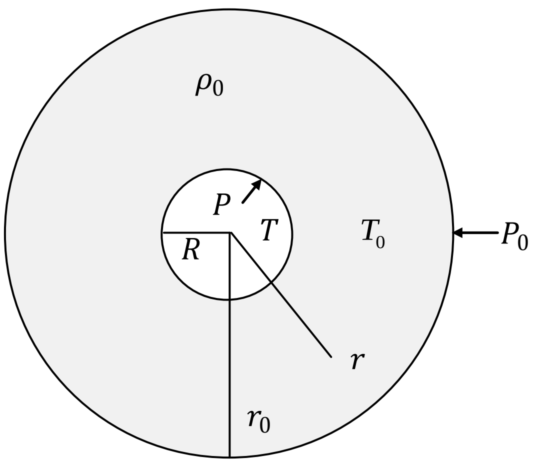
$$\frac{dV}{dt} = 4\pi R^{2}\dot{R} = 4\pi r^2 \dot{r}. \tag{1}$$

This implies that the total kinetic energy E_{k} of the liquid with mass density ρ_0 is

$$E_{\mathrm{k}} = \frac{1}{2}\int_R^{r_0} \rho_0\,(4\pi r^2 dr)\dot{r}^2 = 2\pi\rho_0 R^4 \dot{R}^2 \int_R^{r_0} \frac{1}{r^2}dr = 2\pi\rho_0 R^4\dot{R}^2\left(\frac{1}{R} - \frac{1}{r_0}\right) \tag{2}$$

where r_0 is the radius of the outer surface of the liquid.

Fig. 1



(f) Assume the ambient pressure P_0 acting on the outer surface $r = r_0$ of the liquid is constant. Let $P = P(R)$ be the gas pressure when the radius of the bubble is R.

Find the amount of work dW done on the liquid when the radius of the bubble changes from R to $R + dR$. Use P_0 and P to express dW. [0.4 point]

The work dW must be equal to the corresponding change in the total kinetic energy of the liquid. In the limit $r_0 \to \infty$, it follows that we have Bernoulli's equation in the form

$$\frac{1}{2}\rho_0\, d\left(R^{\mathrm{m}}\dot{R}^2\right) = (P - P_0)R^{\mathrm{n}}\, dR. \tag{3}$$

Find the exponents m and n in Eq. (3). Use dimensional arguments if necessary. [0.4 point]

● *Collapsing of the gas bubble*

From here on, we consider only the collapsing stage of the bubble. The mass density of the liquid is $\rho_0 = 1.0 \times 10^3\ \mathrm{kg \cdot m^{-3}}$, the temperature T_0 of the liquid is 300 K and the ambient pressure P_0 is 1.01×10^5 Pa. We assume that ρ_0, T_0, and P_0 remain constant at all times and the bubble collapses ***adiabatically*** without any exchange of mass across the bubble-liquid interface.

The bubble considered is filled with an ideal gas. The ratio of specific heat at constant pressure to that at constant volume for the gas is $\gamma = 5/3$. When under temperature T_0 and pressure P_0, the equilibrium radius of the bubble is $R_0 = 5.00\ \mu\mathrm{m}$.

Now, this bubble begins its collapsing stage at time $t = 0$ with $R(0) = R_{\mathrm{i}} = 7R_0$, $\dot{R}(0) = 0$, and the gas temperature $T_{\mathrm{i}} = T_0$. Note that, because of the bubble's expansion in the preceding stage, R_{i} is considerably larger than R_0 and this is necessary if sonoluminescence is to occur.

(g) Express the pressure $P \equiv P(R)$ and temperature $T \equiv T(R)$ of the ideal gas in the bubble as a function of R during the collapsing stage, assuming quasi-equilibrium conditions hold. [0.6 point]

(h) Let $\beta \equiv R/R_{\mathrm{i}}$ and $\dot{\beta} = d\beta/dt$. Eq. (3) implies a conservation law which takes the following form

$$\frac{1}{2}\rho_0\, \dot{\beta}^2 + U(\beta) = 0. \tag{4}$$

Let $P_{\mathrm{i}} \equiv P(R_{\mathrm{i}})$ be the gas pressure of the bubble when $R = R_{\mathrm{i}}$. If we introduce the ratio $Q \equiv P_{\mathrm{i}}/[(\gamma - 1)P_0]$, the function $U(\beta)$ may be expressed as

$$U(\beta) = \mu\beta^{-5}[Q(1 - \beta^2) - \beta^2(1 - \beta^3)]. \tag{5}$$

Find the coefficient μ in terms of R_{i} and P_0. [0.6 point]

(i) Let R_{m} be the minimum radius of the bubble during the collapsing stage and define $\beta_{\mathrm{m}} \equiv R_{\mathrm{m}}/R_{\mathrm{i}}$. For $Q \ll 1$, we have $\beta_{\mathrm{m}} \approx C_{\mathrm{m}}\sqrt{Q}$.

Find the constant C_{m}. [0.4 point]

Evaluate R_{m} for $R_{\mathrm{i}} = 7R_0$. [0.3 point]

Evaluate the temperature T_{m} of the gas at $\beta = \beta_{\mathrm{m}}$. [0.3 point]

(j) Assume $R_{\mathrm{i}} = 7R_0$. Let β_u be the value of β at which the dimensionless radial speed $u \equiv |\dot{\beta}|$ reaches its maximum value. The gas temperature rises rapidly for values of β near β_u.

Give an expression and then estimate the value of β_u. [0.6 point]

Let $\bar{u}$ be the value of u at $\beta = \bar{\beta} \equiv (\beta_{\mathrm{m}} + \beta_u)/2$. Evaluate $\bar{u}$. [0.4 point]

Give an expression and then estimate the duration Δt_{m} of time needed for β to diminish from β_u to the minimum value β_{m}. [0.6 point]

● *Sonoluminescence of the collapsing bubble*

Consider the bubble to be a surface black-body radiator of constant emissivity a so that the effective Stefan-Boltzmann's constant $\sigma_{\mathrm{eff}} = a\sigma_{\mathrm{SB}}$. If the collapsing stage is to be approximated as adiabatic, the emissivity must be small enough so that the power radiated by the bubble at $\beta = \bar{\beta}$ is no more than a fraction, say 20 %, of the power $\dot{E}$ supplied to it by the driving liquid pressure.

(k) Find the power $\dot{E}$ supplied to the bubble as a function of β. [0.6 point]

Give an expression and then estimate the value for an upper bound of a. [0.8 point]

Appendix

1. $\dfrac{d}{dx}x^n = nx^{n-1}$

2. Electron mass $m = 9.11 \times 10^{-31}$ kg

3. Planck constant $h = 2\pi\,\hbar = 2\pi \times 1.055 \times 10^{-34}\ \mathrm{J \cdot s}$

4. Stefan-Boltzmann's constant $\sigma_{\mathrm{SB}} = 5.67 \times 10^{-8}\ \mathrm{W \cdot m^{-2} \cdot K^{-4}}$

END
